



HAL
open science

Efficiency in Matching Markets: Application Costs in School Choice

Cyril Rouault

► **To cite this version:**

Cyril Rouault. Efficiency in Matching Markets: Application Costs in School Choice. 2023. hal-04238849v1

HAL Id: hal-04238849

<https://hal.science/hal-04238849v1>

Preprint submitted on 12 Oct 2023 (v1), last revised 24 Apr 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

Efficiency in Matching Markets: Application Costs in School Choice*

Cyril Rouault[†]

Abstract

We propose a strategic solution to the tradeoff between stability and efficiency in one-sided matching markets. We show that every assignment that Pareto dominates a stable outcome can be implemented in a Nash equilibrium using the deferred acceptance mechanism (DA). To incentivize students, we introduce application costs with a student's payoff lexicographic in the school they attend and the fee they pay. We establish that it is impossible to set identical costs for all students. Finally, we examine student incentives resulting from cost implementation in the Boston and Top Trading Cycle mechanisms showing that costs have no effect.

JEL Classification: C78, D47, D50, D82.

Keywords: School choice; Matching; Pareto efficiency; Nash equilibrium; Deferred acceptance; Application costs

1 Introduction

There exists a tradeoff between stability and efficiency in one-sided matching markets. Roth (1982) show that these two properties are incompatible. The deferred acceptance mechanism (DA) of Gale and Shapley (1962) produces a stable assignment that is the most efficient among all stable assignments. However, DA does not always yield a Pareto-efficient assignment and unstable assignments may Pareto dominate it. Numerous papers investigate the origins of DA's inefficiency (e.g., Ergin, 2002; Kesten, 2010). A question arises: Is it possible to consider that agents' strategies can lead to an equilibrium outcome that is Pareto-efficient and that Pareto dominates the students' optimal stable assignment? We show that the answer is yes.

*I am grateful to Olivier Bos, David Delacrétaz, Guillaume Hearinger, Victor Hiller, Vincent Iehlé, Morgan Patty, and Olivier Tercieux for their valuable comments.

[†]Université Paris-Saclay, ENS Paris-Saclay, CEPS. *E-mail address:* cyril.rouault@universite-paris-saclay.fr

There are many real-world examples of problems that fit into our framework, but perhaps the most important is public school choice.¹ Many cities around the world use centralized mechanisms to assign students to schools. Stability is a crucial concern for many schools because they must justify the acceptance and rejection of students. In one-sided markets, stability is a fairness criterion in the sense that a stable assignment eliminates justified envy (Abdulkadiroğlu and Sönmez, 2003). The fairness justification results from school priorities and student preferences. It is important to note that the standard approach in the literature analyzes efficiency only from the student’s perspective in school choice problems. The tradeoff is then selecting an appropriate mechanism that meets fairness objectives and maximizes student welfare.

This paper studies students’ strategic incentives in DA and the possible equilibria for a given problem. Since DA suggests the stable assignment that all students prefer, we focus on Nash equilibrium outcomes that Pareto dominate the students’ optimal stable assignment. We argue that these equilibria are crucial from a welfare perspective. We propose an approach based on implementing application costs to incentivize students to reach these equilibria.

Our first results show that an assignment that Pareto dominates a stable assignment can be implemented in a Nash equilibrium under DA (Theorem 1). This is, therefore, true for the students’ optimal stable assignment. However, due to uncertainty, students do not always play a Nash equilibrium in practice. For this reason, we study a solution that incentivizes students to consider desired equilibria. We propose an approach based on application costs.

There are different interpretations of application costs. Costs could be financial but not only, we can consider exams required for applications as costs. For example, in New York, students must pass an exam at each high school they apply. A centralized mechanism is then used to assign students to high schools. Similarly, preparing an application by writing a cover letter or practicing for an interview is costly.²

In our approach, we set the application costs to be low. The objective of these costs is to incentivize students without negatively impacting them. Thus, a school will not be valued less by a student because of its application cost.³ We consider a student’s payoff to be lexicographic in the school they attend and the fee they pay. We then show

¹Other problems of allocating doctors to hospitals or children to day-care centers can be considered.

²Several centralized platforms require specific forms to apply to some schools. In France, for instance, this is the case with the Parcoursup and Mon Master platforms: <https://www.parcoursup.fr/> and <https://www.monmaster.gouv.fr/>.

³The interpretation is that preferences are unchanged. Costs do not alter the order of schools in students’ preferences.

that there is always a cost profile that incentivizes students to reach an equilibrium such that the outcome Pareto dominates the students' optimal stable assignment. Costs incentivize students by modifying their dominant strategies (Theorem 2).

To study DA equilibria, we consider Kesten's (2010) Efficiency-Adjusted Deferred Acceptance Mechanism (EADAM).⁴ Proposition 1 illustrates the equilibrium structure that leads to a Pareto-efficient outcome equilibrium using DA. In this approach, we consider a strategy in which students apply to both the school they are assigned to using EADAM and their students' optimal stable assignment. We identify a condition where other schools must be added to the student's strategy to ensure equilibrium. The intuition is that all inefficiency in DA is attributable to rejection chains that worsen student outcomes. These chains can be avoided by directly proposing to the school obtained through EADAM. It is necessary to include the optimal stable assignment of students in their strategies; otherwise, students involved in a blocking pair may have an incentive to change their strategies. This incentive disappears when students list their optimal stable school because any deviation by a student triggers a rejection chain that ultimately results in the student's rejection.⁵ Additional schools can be added to the strategy, but they do not change the equilibrium outcome if students rank them according to their true preferences. Furthermore, we show that under a single condition, students can only apply to two schools to ensure equilibrium (Proposition 1).

We then study the structure that costs must have to ensure the existence of equilibria leading to an outcome that Pareto dominates the students' optimal stable assignment. Theorem 3 suggests that costs must be carefully designed. Suppose it is costly for a student to apply to her optimal stable assignment, and that student is assigned to another school using EADAM. In that case, there is no equilibrium outcome where the student prefers that assignment over her optimal stable assignment, and that outcome Pareto dominates the students' optimal stable assignment. Therefore, it is impossible to implement a Pareto-efficient assignment that Pareto dominates the students' optimal stable assignment under this cost profile. The consequence is the impossibility of treating students equally. Some students have to bear the application cost in a given school, while others do not. With such costs, students being assigned to preferred schools are incentivized not to apply to their optimal stable assignment. As a result, the rejection chain is not possible, and profitable deviations exist for some students.

⁴EADAM Pareto dominates DA. Therefore, we can identify the assignments that Pareto dominate the students' optimal stable assignment.

⁵A rejection chain is a succession of rejections of students tentatively assigned to a school in the DA procedure. Articles use this concept to introduce a form of fairness. Unstable assignments can be fair in this sense. See, for instance, Troyan et al. (2020).

We show that even low costs impact the equilibria when using DA. Costs can reinforce or hinder the existence of certain equilibria. This phenomenon raises a question about the impact of costs on other mechanisms and their possible equilibria. We find that costs do not have an impact on the equilibria outcomes of the Boston mechanism and the Top Trading Cycle mechanism (Propositions 3 and 4). Regarding the Boston mechanism, experimental studies show that, in practice, the outcome is more efficient than that of DA (Featherstone and Niederle, 2016). Thus, our study demonstrates that we can incentivize students in DA, whereas it is impossible to do so in Boston with costs. Therefore, we believe that application costs can incentivize students and enhance the efficiency of DA.

We provide in Appendix A additional results to study the tradeoff between our approach and stability. These results coincide with our efficiency results.

Related Literature

Our article is related to the literature on studying the sources of inefficiency in DA. We use preference and priority structures for this purpose. Ergin (2002) introduces the concept of cycles in school priorities which can lead to unstable outcomes in DA. Similarly, Kesten (2010) proposes the EADAM to identify sources of inefficiency in DA.⁶ In EADAM, the objective is to limit interrupters to improve student assignments.⁷ Kesten’s main contribution is that if all students consent, meaning they are willing to modify their reported preferences, EADAM produces a Pareto-efficient outcome that dominates DA. Kesten mainly uses rejection chains to show that the student interrupter is not assigned to the school with which it interrupts, thus preserving the student’s assignment. To the best of our knowledge, no paper studies student strategies leading to equilibrium outcomes that Pareto dominate the students’ optimal stable assignment under DA.

Our study is also related to the literature on constrained matching. Haeringer and Klijn (2009) investigate Nash equilibria under constraints in popular mechanisms such as DA, Boston, and Top Trading Cycles, focusing on quota games. In contrast, our study introduces constraints on students’ reports to schools and examines the impact of application costs on Nash equilibrium outcomes. Moreover, our analysis considers rational self-constraint by students at equilibrium, while Haeringer and Klijn assume an exogenous constraint. Our results coincide with those of Dur and Morrill (2020), but we

⁶Liu et al. (2014) propose a simplified version of EADAM. The two mechanisms are equivalent. The simplified version directly eliminates rejection chains. However, our approach cannot be implemented in their mechanism.

⁷The term interrupter describes students who cause the rejection of other students before being rejected themselves. We provide a formal definition in Section 4.

provide a characterization of equilibria that lead to outcomes that Pareto dominate the students' optimal stable assignment under DA.

Finally, we consider the inclusion of application costs. Chade et al. (2014) examine the impact of costs on students' application strategies in an incomplete information framework, while He and Magnac (2022) empirically study the implementation of application costs on student assignments. However, neither work explores equilibria. To our knowledge, our paper is the first to consider application costs in improving students' assignments within an equilibrium framework.

The remainder of the paper is organized as follows. We introduce the model in Section 2. We present our main result on implementing an assignment in a Nash equilibrium in Section 3. In Section 4, we introduce costs and EADAM. We design costs in Section 5. In Section 6, we study the impact of costs on the Boston and TTC mechanisms. Section 7 concludes. We provide additional results on stabilities in Appendix A. All proofs are collected in Appendix B.

2 Model

2.1 Assignment Problems

There is a nonempty finite set of *students* I with a typical element i and a nonempty finite set of *schools* S with a typical element s . For each $s \in S$, we denote the *school's capacity* by q_s , which is the number of students that can be assigned to it. We denote by $q \equiv (q_{s_1}, \dots, q_{s_n})$ the *capacity vector* of schools. Each student i is to be assigned to one school in $S \cup \{i\}$, where i is interpreted as some outside option.⁸ The outside option is not scarce, meaning that each student can remain unassigned.

The preference relation P_i of student i is a linear order (a complete, transitive, and antisymmetric relation) over $S \cup \{i\}$. We denote $sR_i s'$ if student i *weakly prefers* school s to school s' , meaning $sP_i s'$ or $s = s'$. A school s is *acceptable* to i if $sR_i i$. Let us denote the *strict preference profile* by $P = (P_i)_{i \in I}$. Let \succ_s denote the priority order (a complete, transitive, and antisymmetric relation) of school s over $I \cup \emptyset$. Being assigned to \emptyset is interpreted as having *no student assigned to the school*. Let us denote the *strict priority structure* by $\succ = (\succ_s)_{s \in S}$.

An assignment is a function $\mu : I \cup S \rightarrow S \cup 2^I$ that maps each student to either a school or a set of unassigned students. We denote $\mu(i)$ as the assignment of student i under assignment μ . Formally, an assignment is a function μ satisfying:

⁸This option can be to stay at home or attend a private school, for instance.

- (i) $\forall i \in I, \mu(i) \in S \cup \{i\}$;
- (ii) $\mu(s) \in 2^I$;
- (iii) $\forall s \in S, |\mu(s)| \leq q_s$;
- (iv) $\mu(i) = s$ if and only if $i \in \mu(s) \cup \{i\}$.

For $i \in I$, if $\mu(i) = s \in S$, then student i is assigned to s under μ . If $\mu(i) = i$, we say that i is *unassigned* under μ . For $s \in S$, if $\mu(s) \subseteq I$, then the subset of students $\mu(s)$ is assigned to s under μ . Student i is assigned to s under μ if $i \in \mu(s)$.

A school choice assignment problem is a tuple (I, S, P, \succ, q) . Because I, S , and q are fixed, a school choice assignment problem is given by (P, \succ) .

We now introduce standard properties of matchings. An assignment μ is stable if:

- it is *individually rational*, i.e., for each $i \in I, \mu(i)R_i i$,
- it is *non-wasteful*, i.e., for each $i \in I$ and each $s \in S, sP_i \mu(i)$ implies $|\mu(s)| < q_s$,
- there is no *justified envy*, i.e., for each $i, j \in I$ with $\mu(j) = s \in S, sP_i \mu(i)$ implies $j \succ_s i$.

Another desirable property for an assignment is Pareto efficiency. We say that μ' *Pareto dominates* μ if:

- $\forall i \in I, \mu'(i)R_i \mu(i)$ and for at least one $i \in I, \mu'(i)P_i \mu(i)$.

An assignment μ is *Pareto efficient* if it is not Pareto dominated by any other assignment.

2.2 Assignment Game

A mechanism φ is a function associating an assignment to each problem (P, \succ) . A mechanism φ naturally defines a game where students are the *players*. Let the *strategy* Q_i be an ordered list of school preferences of student i over the set of schools. For instance, we denote the strategy of i by $Q_i : s_1, s_2$. This is interpreted as, in her strategy, i ranks s_1 higher than s_2 , and she prefers being unassigned rather than being assigned to any other school. We denote by $A(Q_i)$ the set of schools reported by student i under strategy Q_i . Then, $s \notin A(Q_i)$ must be interpreted as student i ranks s as an unacceptable school under strategy Q_i . Let $Q \equiv (Q_i)_{i \in I}$ be a *strategy profile* of students. The outcome is the student assignment obtained with strategy profile Q as the preference profile. Preferences over outcomes are given by the preferences over assignments. For strategy profile, we use classic game-theoretic notations; for instance, $Q_{-i} \equiv (Q_{i'})_{i' \in I \setminus \{i\}}$. We denote the DA assignment for problem (P, \succ) by $DA(P, \succ)$, and we denote the students' optimal stable assignment by μ^{SOSM} .

A mechanism φ is *strategy-proof* if no student can ever benefit by unilaterally misrepresenting their preferences, meaning for any $P, i \in I$, and $Q_i \neq P_i$, we have for any

Q_{-i} :

- $\varphi_i((P_i, Q_{-i}), \succ) R_i \varphi_i((Q_i, Q_{-i}), \succ)$.

A strategy profile Q is a *Nash equilibrium* if for each $i \in I$, Q_i is student i 's best response to the other students' strategies Q_{-i} . Formally, for each $i \in I$, there is no strategy Q'_i such that $Q'_i \neq Q_i$, and we have:

- $\varphi_i((Q'_i, Q_{-i}), \succ) P_i \varphi_i((Q_i, Q_{-i}), \succ)$.

We now define the implementation in Nash equilibrium of an outcome, which is central to our paper.

Definition 1. Given an assignment mechanism φ and a preference profile P , an assignment μ can be *implemented* in a Nash equilibrium (implemented for short) if there exists a Nash equilibrium Q such that $\varphi(Q, \succ) = \mu$.

3 Equilibrium under DA and Pareto Domination

It is well-known from the literature that DA suggests the assignment $\mu^{SO\!SM}$ for a problem (P, \succ) .⁹ However, there exist assignments that Pareto dominate $\mu^{SO\!SM}$, and we consider them in this section.

Example 1. Let us consider a school choice problem (P, \succ) with four students $I := \{i_1, i_2, i_3, i_4\}$ and four schools $S := \{s_1, s_2, s_3, s_4\}$, each having one seat (for each $s \in S, q_s = 1$). The priorities and preferences are given in the following tables. The symbol (\cdot) indicates that the school does not have a priority over the student set and accepts all students who apply to it.

\succ_{s_1}	\succ_{s_2}	\succ_{s_3}	\succ_{s_4}	P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
i_3	i_4	i_1	\cdot	$s_1^{*\dagger}$	s_1	s_2^*	s_3^*
i_4	i_1	i_2	\cdot	s_2^\bullet	s_2	$s_3^{*\dagger}$	s_1^\bullet
i_1	i_2	i_3	\cdot	<u>s_3</u>	s_3	<u>s_1</u>	<u>s_2^\dagger</u>
i_2	i_3	i_4	\cdot	s_4	<u>$s_4^{*\bullet\dagger}$</u>	s_4	s_4

Table 1: Schools' priorities and students' preferences.

The table on the right shows four different assignments. An assignment $\mu^{SO\!SM}$ (underline), assignment μ^\bullet (denoted by bullet \bullet), assignment μ^\dagger (denoted by dagger \dagger) and μ^* (denoted by stars $*$). The DA assignment is $\mu^{SO\!SM}$ and is not Pareto efficient. It is easy to see that it is Pareto dominated by μ^\bullet, μ^\dagger and μ^* . In addition, μ^* is Pareto-efficient. However, these three assignments are unstable.

⁹See Gale and Shapley (1962).

Let us consider μ^\dagger for now. Suppose the strategy profile is such that for each student $i \in I$, we have $Q_i^\dagger : \mu^\dagger(i), \mu^{SO SM}(i)$ if $\mu^\dagger(i) \neq \mu^{SO SM}(i)$ and $Q_i^\dagger : \mu^{SO SM}(i)$ otherwise. We denote this strategy profile as Q^\dagger . The resulting assignment is shown in the table below.

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
s_1^\dagger			
		s_3^\dagger	
s_3		s_1	s_2^\dagger
	s_4^\dagger		

Table 2: A possible equilibrium to implement μ^\dagger .

Using DA for the problem (Q^\dagger, \succ) , the algorithm stops at the first round, and the obtained assignment is μ^\dagger . Furthermore, by considering each student i choosing their dominant strategies P_i , and with the strategies of other students Q_{-i}^\dagger fixed, it is easy to see that there are no profitable deviations for the students. Therefore, the strategy profile Q^\dagger is an equilibrium.

The same reasoning applies to the other assignments presented in this example. Our first theorem generalizes this point.

Theorem 1. Let (P, \succ) be a school choice problem, and let μ be an assignment that Pareto dominates a stable assignment μ' . Then μ can be implemented using DA.

Theorem 1 extends the observation of Example 1 to all stable assignments. As the proof suggests, replacing $\mu^{SO SM}$ with another stable assignment for each Q_i is sufficient. However, in the rest of the paper, we only consider assignments that Pareto dominate $\mu^{SO SM}$.

The intuition of Theorem 1 is that by reducing the number of applications, we reduce the number of cycles. In Example 1, we can observe a rejection chain: student i_1 was previously rejected from school s_1 due to the application of student i_4 , student i_4 was rejected from school s_3 due to the application of student i_3 , and student i_3 was rejected from school s_2 due to the application of student i_2 . Thus, the rejection chain is not initiated by considering a strategy for i_2 in which $s_2 \notin A(Q_{i_2})$. We use the same reasoning for the other rejection chains in the DA procedure.

For each student $i \in I$, it is necessary to have $\mu^{SO SM}(i) \in A(Q_i)$ in order to prevent profitable deviations of students causing rejection chains. If student i_2 applies first to school s_3 in her strategy, then student i_3 is rejected and applies to s_1 , which leads to

the rejection of student i_1 , who applies to s_3 , and ultimately results in student i_2 being rejected from school s_3 . The incentive for student i_2 to deviate disappears when, for each $i \in I$, $\mu^{SO SM} \in A(Q_i)$, as it triggers a rejection chain that cycles back to i_2 , making s_3 unachievable.

4 Application Costs and Kesten's EADAM

4.1 Application Costs

As mentioned in the introduction, it is well-known that equilibria in the DA algorithm are rarely implemented in practice.¹⁰ For this reason, we need to incentivize students to apply to specific schools to reach an equilibrium outcome that Pareto dominates the students' optimal stable assignment. Here, we suggest fixing low application costs. The idea is that these costs should not modify students' preferences.¹¹ The costs depend on the school and the student. For simplification, the costs are either strictly positive or zero. The cost profile is the $|I| \times |S|$ -dimensional matrix C , where each element $c_{i,s}$ represents the application cost of student i to school s . If the application cost for student i to school s is zero, then $c_{i,s} = 0$, and if the cost is positive, $c_{i,s} = 1$. Let $c_{Q_i} = \sum_{s \in A(Q_i)} c_{i,s}$ denote the sum of the application costs to be paid by i when using the strategy Q_i under the cost profile C .¹²

We use a lexicographic criterion to determine each student's payoff $(\mu(i), c_{Q_i})$. The primary criterion is the assigned school, and the secondary criterion is the sum of costs associated with their strategy. Thus, the student chooses the least costly strategy among those that result in the same assignment. We have $(\mu(i), c_{Q_i}) \succ (\mu(i)', c'_{Q_i})$ if and only if:

- $\mu(i) P_i \mu(i)'$ or $\mu(i) = \mu(i)'$ and $c_{Q_i} < c'_{Q_i}$.

We define a problem with costs. Using a cost profile C , a *problem with costs* is given by (P, \succ, C) . We consider a problem without costs by (P, \succ) to simplify notation.

¹⁰See for instance He and Magnac (2022) and Featherstone and Niederle (2016).

¹¹Considering such application costs is realistic because these costs are generally low. In addition, students prefer to pay a low cost to be assigned to a preferred school.

¹²This simplification assumes that costs are comparable. We consider the number of costly applications in the chosen strategy. In the article, students pay a maximum of one application fee at equilibrium. This illustrates the incentive nature of costs rather than their harmfulness.

4.2 Kesten Algorithm

To identify assignments that Pareto dominates the students' optimal stable assignment, we consider Kesten's EADAM. The algorithm focuses mainly on interrupters, which we define as follows:

Definition 2. In the DA procedure of a school choice problem, if student i is tentatively accepted by school s at some round t and is later rejected by school s at some round $t' > t$ and at least one other student is rejected by school s at some round l such that $t \leq l < t'$, then student i is an *interrupter* for school s , and the pair (i, s) is an interrupting pair of round t' .

Kesten (2010) show that the assignment is Pareto efficient by neutralizing specific interrupters, and EADAM Pareto dominates DA. The term *neutralize* means that a student who is an interrupter for a school no longer applies to it.¹³ However, if a student is an interrupter for school s , then she is indifferent between applying or not to s , given the fixed strategies of other students.¹⁴ Our contribution is considering the strategic aspect of the different equilibria.

For any school choice problem (P, \succ) , Kesten's EADAM operates as follows:¹⁵

Step 0: Run DA for the problem (P, \succ) .

Step $t, t \geq 1$: Identify the last round of the step- $(t - 1)$ DA procedure in which interrupter(s) are rejected. Identify all interrupting pairs of this round, and for each pair, remove the respective school from the interrupter's preference. After that, rerun DA (step- t DA) with the new preference profile.

Stop when there are no more interrupters that are rejected.

For each step $t \geq 0$ of EADAM, we denote by $\mu^{\kappa(t)}$ the temporary assignment, and μ^κ the final assignment. We denote $K(P, \succ) \equiv \bigcup_{t=0, \dots, n} \mu^{\kappa(t)}$ as the set of all temporary assignments. Similarly, for each step, we consider the set of interrupting pairs.

Definition 3. Let \tilde{I}_t be a *set of interrupting pairs* identified by EADAM at step t . For each pair $(i, s) \in \tilde{I}_t$, we denote s_i^t as the school for which i is an interrupter.

¹³Kesten differentiates between students who consent to have a modification of their preferences when they are interrupters for a school and those who do not. In our analysis, we assume that all students consent.

¹⁴This result is given by Proposition 3 of Kesten (2010). The formulation is different. Kesten does not consider indifference, but the assignment of a consenting student does not change.

¹⁵We use "step" for EADAM and "round" for DA.

Note that the order of the sets is essential. All assignments in $K(P, \succ)$ can be obtained by neutralizing interrupters. For example, $\mu^{\kappa(t)} \in K(P, \succ)$ can be obtained by neutralizing the interrupters identified in steps 0 to t .

Throughout the Kesten algorithm, students trade school positions to obtain more preferred assignments. We aim to identify applications that maintain equilibrium. To achieve this, we introduce the following definition:

Definition 4. Let I_t^T be a set of *trading students* in EADAM at step t , where $t \geq 1$. For any pair of students $i, i' \in I_t^T$ with $i \neq i'$, the following conditions hold:

- $\mu^{\kappa(t)}(i) = \mu^{\kappa(t-1)}(i')$, with $\mu^{\kappa(t)}(i) \neq \mu^{\kappa(t-1)}(i)$, and
- $\mu^{\kappa(t)}(i') = \mu^{\kappa(t-1)}(i'')$, with $i'' \in I_t^T$, $i'' \neq i'$, and $\mu^{\kappa(t)}(i') \neq \mu^{\kappa(t-1)}(i')$.

We refer to $\mu^{\kappa(t)}(i)$ as a *blocking application* if, at a later step $t' > t$, student i' is identified as an interrupter for school s , and $\mu^{\kappa(t')} \neq \mu^\kappa$.

In words, we consider all students who traded their assigned school from DA run at step $t - 1$ to step t . Note that several distinct sets I^T may exist at a given step t . This reflects the neutralization of several interrupters at the beginning of step t . Using this set, we can identify the application that will prevent the deviation of the interrupter identified at a later step. The condition $\mu^{\kappa(t')} \neq \mu^\kappa$ allows us to consider problems where the neutralization of $(i', s_{i'}^{t'})$ modifies the assignment.¹⁶ If this is not the case, then the application $\mu^{\kappa(t)}(i)$ is not blocking. Moreover, we can restrict this definition to cases where an interrupter i' is assigned to a different school under $\mu^{SO SM}$ and μ^κ .

4.3 Illustrative Example

In the rest of this section, we illustrate through two examples how EADAM proceeds and how to implement costs. That also highlight the use of our definitions.

Example 2. Consider the school choice problem (P, \succ) from Example 1. EADAM proceeds as follows:

- At step 0 of EADAM, the obtained assignment is $\mu^{SO SM}$ according to the preference profile. The strategy of i_2 is given by: $Q_{i_2} : s_1, s_2, s_3, s_4$.

¹⁶There are interrupters that, when neutralized, do not change the assignment. However, it is possible that after their neutralization, an interrupter is identified in a subsequent step, such that its neutralization improves the student's assignment. For this reason, we consider the final step of EADAM rather than step $t' + 1$.

- The identified set of interrupting pair is $\tilde{I}_1 = \{(i_2, s_3)\}$. In step 1, the obtained assignment is μ^\bullet according to the preference profile. The strategy of i_2 is given by:
 $Q'_{i_2} : s_1, s_2, s_4$.
- The identified set of interrupting pair is $\tilde{I}_2 = \{(i_2, s_2)\}$. In step 2, the obtained assignment is μ^* according to the preference profile. The strategy of i_2 is given by:
 $Q''_{i_2} : s_1, s_4$.

The DA outcome run at the end of step 2 is Pareto-efficient.

Let Q be a strategy profile such that for each $i \in I \setminus \{i_2\}$, $Q_i = P_i$, and $Q_{i_2} : s_1, s_4$, and let $Q'_{i_2} = P_{i_2}$. The DA outcome for the problem (Q, \succ) and the problem $((Q'_{i_2}, Q_{-i_2}), \succ)$ are underlined in the following tables.

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}	Q_{i_1}	Q'_{i_2}	Q_{i_3}	Q_{i_4}
<u>s_1</u>	s_1	<u>s_2</u>	<u>s_3</u>	s_1	s_1	s_2	s_3
s_2		s_3	s_1	s_2	s_2	s_3	s_1
s_3		s_1	s_2	<u>s_3</u>	<u>s_3</u>	<u>s_1</u>	<u>s_2</u>
s_4	<u>s_4</u>	s_4	s_4	s_4	<u>s_4</u>	s_4	s_4

Table 3: Outcomes of $DA(Q, \succ)$ and $DA((Q'_{i_2}, Q_{-i_2}), \succ)$.

We can see that $\mu^{SO\text{SM}}(i_2) = \mu^*(i_2)$, so Q and (Q'_{i_2}, Q_{-i_2}) are equilibria. Now let us consider the implementation of application costs. Let C be a cost profile such that for each $i \in I \setminus \{i_2\}$ and for each $s \in S$, $c_{i,s} = 0$. For i_2 , we have $c_{i_2, s_1} = c_{i_2, s_4} = 0$, and $c_{i_2, s_2} = c_{i_2, s_3} = 1$. Costs are positive only for interrupting pairs. We can compare the two equilibrium outcomes by considering the lexicographic payoff of i_2 . The outcomes for i_2 are $(s_4, 0)$ for $DA(Q, \succ, C)$ and $(s_4, 2)$ for $DA((Q'_{i_2}, Q_{-i_2}), \succ, C)$. The two assignments are identical, but the costs are different. Thus, i_2 prefers the outcome of $DA(Q, \succ, C)$, and (Q'_{i_2}, Q_{-i_2}) is no longer an equilibrium due to the implementation of the cost profile C .

In this example, we have set the costs to incentivize student i_2 not to apply to schools s_2 and s_3 . The next section investigates whether a more general cost profile design is possible.

Definition 4 cannot be illustrated in Example 2. This is because student i_2 , which is the only interrupter in this problem, is assigned to school s_4 in both DA and EADAM. There is, therefore, no assignment trading with student i_2 . Let us introduce a different example.

Example 3. Consider a school choice problem (P, \succ) with five students $I := \{i_1, i_2, i_3, i_4, i_5\}$ and five schools $S := \{s_1, s_2, s_3, s_4, s_5\}$, each having one seat (for each $s \in S, q_s = 1$).

The priorities and preferences are given in the following tables, and we describe the EADAM steps.

\succ_{s_1}	\succ_{s_2}	\succ_{s_3}	\succ_{s_4}	\succ_{s_5}	P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
i_3	i_4	i_2	i_1	\cdot	s_1^{\natural}	s_2	s_2^{\natural}	s_3^{\natural}	s_4
i_4	i_1	i_3	i_5	\cdot	s_2°	$s_4^{\circ \natural}$	s_3°	s_1°	$s_5^{\circ \natural}$
i_1	i_2	i_4	i_2	\cdot	<u>s_4</u>	<u>s_3</u>	<u>s_1</u>	<u>s_2</u>	
	i_3			\cdot					

Table 4: Schools' priorities and students' preferences.

- At step 0 of EADAM, the obtained assignment is μ^{SOSM} and is underlined in the preference profile.
- The identified set of interrupting pair is $\tilde{I}_1 = \{(i_5, s_4)\}$. In step 1, the obtained assignment is denoted by \circ (denoted μ°) in the preference profile. The strategy of i_5 is given by: $Q'_{i_5} : s_5$.
- The identified set of interrupting pair is $\tilde{I}_2 = \{(i_2, s_2)\}$. In step 2, the obtained assignment is denoted by \natural (denoted μ^\natural) in the preference profile. The strategy of i_2 is given by: $Q'_{i_2} : s_4, s_3$.

The DA outcome run at the end of step 2 is Pareto-efficient.

Following Definition 4, we have $I_1^T = I_2^T = I \setminus i_5$, and $I_3^T = i_1, i_3, i_4$. Student i_2 is in a set of trading students and is an interrupter for s_2 at step 2. Let us now consider the blocking applications. We have that the pairs (i_3, i_2) and (i_1, i_2) satisfy both conditions, and $\mu^\circ \neq \mu^\natural$. Hence, $\mu^{\kappa(1)}(i_1) = s_2$ and $\mu^{\kappa(1)}(i_3) = s_3$ are blocking applications.

5 Costs design

In this section, we determine the conditions under which the cost profile leads to a Pareto-efficient outcome at equilibrium using DA. Example 2 shows that the resulting assignment is Pareto-efficient by setting a positive cost for each interrupting pair. We must set identical costs for each student to generalize the cost profile.

An important observation in Example 2 is that the costs modified the set of possible equilibria. Indeed, (Q'_{i_2}, Q_{-i_2}) is no longer an equilibrium with the cost profile C . However, the equilibrium Q is preserved. Theorem 1 shows that all the assignments that Pareto-dominate μ^{SOSM} are implementable. We now determine that there is always a cost profile that preserves these equilibria.

Theorem 2. Let (P, \succ) be a school choice problem such that the students' optimal stable assignment is not efficient. Then there exists a cost profile C and a Nash equilibrium Q such that $DA(Q, \succ, C)$ Pareto dominates the students' optimal stable assignment. Additionally, truthful revelation is no longer a dominant strategy for some students.

Theorem 2 immediately states that if μ^{SOSM} is not efficient, then there is a cost profile that maintains an equilibrium such that the outcome is Pareto-efficient and Pareto dominates μ^{SOSM} . Furthermore, costs impact students' strategies, and certain equilibria are no longer possible. We now need to determine the structure of this cost profile. Theorem 1 of Kesten (2010) states that when all the interrupting pairs are neutralized, the resulting assignment is Pareto-efficient and dominates DA. Furthermore, Proposition 3 of Kesten (2010) states that students who are interrupters for a school are indifferent between applying to that school or not. To determine the structure of the cost profile, we need to identify the strategies that lead to an equilibrium with Pareto-efficient outcomes. We know that interrupting pairs must be neutralized, and we ensure that no interrupter has a profitable deviation. To do this, we must maintain rejection chains, guaranteeing their rejection from the school for which they are interrupters. Finally, each student applies to their assigned school in μ^κ . The following example illustrates this strategy profile.

Example 4. Consider the school choice problem (P, \succ) from Example 1. We can observe that students i_1, i_3 and i_4 are never assigned to s_4 . Thus, they are indifferent to including s_4 in their strategies. In this example, we illustrate that the assignment μ^* can be implemented with at most two schools in each student's strategy. To illustrate this, consider $Q_{i_1} : s_1, s_3$ and $Q_{i_3} : s_3, s_2$. Then let us study the possible strategies for i_4 and i_2 , an interrupter for s_2 and s_3 .

For student i_4 , we study the strategies denoted Q_{i_4} and Q'_{i_4} such that $Q_{i_4} : s_3, s_1, s_2$ and $Q'_{i_4} : s_3, s_2$. For student i_2 we study the strategies $Q_{i_2} : s_1, s_2, s_4$ and $Q'_{i_2} : s_1, s_4$.

$i_2 \backslash i_4$	Q_{i_4}	Q'_{i_4}
Q_{i_2}	$\mu(i_2) = s_4; \mu(i_4) = s_1$	$\mu(i_2) = s_4; \mu(i_4) = s_2$
Q'_{i_2}	$\mu(i_2) = s_4; \mu(i_4) = s_3$	$\mu(i_2) = s_4; \mu(i_4) = s_3$

Table 5: Assignment comparison of i_2 and i_4 according to their strategies.

In each outcome, i_2 is assigned to s_4 . Therefore, if i_4 does not apply to s_1 , student i_2 is still assigned to s_4 . By setting a positive cost to i_2 's application to s_2 , the strategy

is Q'_{i_2} , as suggested in Theorem 2. Knowing this, i_4 is indifferent between Q_{i_4} and Q'_{i_4} . The fact that i_2 has no interest in applying to s_2 when i_4 does not apply to s_1 is because i_4 applies to s_2 and $i_4 \succ_{s_2} i_2$. Following this argument, the application that blocks the deviation of i_2 is $s_2 \in A(Q_{i_4})$. We can use the same reasoning for other students. The strategy profiles can therefore be as follows.

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}		Q_{i_1}	Q'_{i_2}	Q_{i_3}	Q_{i_4}
s_1	s_1	s_2	s_3		<u>s_1</u>		<u>s_2</u>	<u>s_3</u>
	s_2							
<u>s_3</u>	s_3	<u>s_1</u>	<u>s_2</u>		s_3		s_1	s_2
	<u>s_4</u>					<u>s_4</u>		

Table 6: Strategy profiles with Q'_{i_2} as suggested by Theorem 2.

However, this is not always possible. Consider Example 3, using a strategy profile, such that for each $i \in I, Q_i : \mu^{\natural}, \mu^{SOSM}$. A profitable deviation exists for student i_2 , who can apply to s_2 . This deviation does not exist yet if we have $s_2 \in A(Q_{i_1})$ or $s_3 \in A(Q_{i_3})$. The following result generalizes Example 4 and provides the structure of possible strategies leading to equilibrium.

Proposition 1. Let Q be such that for each student $i \in I$,

- (i) for each $s \in S$, if $s = s_i^t$, for some t in the EADAM, $s_i^t \notin A(Q_i)$,
- (ii) $\mu^{SOSM}(i) \in A(Q_i)$ if $\mu^{SOSM}(i) = \mu^\kappa(i)$, and $\mu^{SOSM}(i), \mu^\kappa(i) \in A(Q_i)$ otherwise,
- (iii) for each t such that $\mu^{\kappa(t)}(i)$ is a blocking application, we have at least one student $i' \in I_t^T$ with $I_t^T \ni i$ such that $\mu^{\kappa(t)}(i')$ is a blocking application and $\mu^{\kappa(t)}(i') \in A(Q_{i'})$,
- (iv) for any school $s \in A(Q_i)$, the schools are ranked according to true preferences.

Then Q is a Nash equilibrium, and $DA(Q, \succ)$ Pareto dominates μ^{SOSM} and is Pareto-efficient.

We also generalize the result to each assignment in $K(P, \succ)$.¹⁷

Proposition 2. Let $\tilde{I}'_t \subseteq \bigcup_{t \in \{1 \dots n\}} \tilde{I}_t$ be a subset of interrupting pair. There exists k such that for each $h \leq k$, $(i, s_i^h) \in \tilde{I}'_t$ and $(i, s_i^{k+1}) \notin \tilde{I}'_t$. Let Q be a strategy profile such that for each student i' is $Q_{i'} = P_{i'}$ and only all the interrupting pair in \tilde{I}'_t are neutralized such that $s_i^t \notin A(Q_{i'})$. Then, the obtained assignment is $\mu^{\kappa(k)}$, and Q is an equilibrium using DA.

¹⁷The proof of Proposition 2 is immediate from Proposition 1.

The significant consequence of Proposition 1 is that if there are no blocking applications (as presented in (iii)), there is always an equilibrium where students apply to at most two schools. This result simplifies the design of the cost profile, making it more general. In the following, we analyze why the schools assigned under μ^{SOSM} must be included in each student's strategy and the implications of this for the cost profile design.

Example 5. Let (Q, \succ) be the problem presented in Table 6 (Example 4). Let C be a cost profile such that for each $i \in I$, and for each $s \in S$ we have $c_{i,s} = 1$ if $s \neq \mu^{SOSM}(i)$ and $c_{i,s} = 0$ if $s = \mu^{SOSM}(i)$. Considering the problem (Q, \succ, C) , the left-hand strategy profile is no longer an equilibrium. The right-hand strategy, by contrast, is an equilibrium. This illustrates that the cost can be positive for assigned schools in μ^κ . Using the lexicographic payoff, we have for each $i \in I \setminus \{i_2\}$ that $(\mu^\kappa(i), 1) \succ (\mu^{SOSM}(i), 0)$. For student i_2 , knowing that for each $i \in I \setminus \{i_2\}$, $\mu^{SOSM}(i) \in A(Q_i)$, and the cost is positive for each school except s_4 , the strategy of student i_2 is $Q_{i_2} : s_4$.

Let us now consider a new cost profile C' , such that there exists a student i where $\mu^{SOSM}(i) \neq \mu^\kappa(i)$ and $c'_{i,\mu^{SOSM}(i)} = 1$. As an illustration, consider that for each student $i \in I$, we have $c'_{i,s_3} = 1$. We have therefore $c'_{i_1,s_3} = 1$.

Q_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}		Q'_{i_1}	Q_{i_2}	Q_{i_3}	Q_{i_4}
s_1		s_2	s_3		s_1		s_2	s_3
	s_3		s_1	s_2			s_1	s_2
		s_4					s_4	

Table 7: Strategy profiles with C' .

We compare the strategy profiles Q and (Q'_{i_1}, Q_{-i_1}) . Knowing the strategy of other student Q_{-i_1} , and that $c'_{i_1,s_3} = 1$, student i_1 has a profitable deviation in problem (Q, \succ, C') . We have $DA(Q, \succ, C') = DA((Q'_{i_1}, Q_{-i_1}), \succ, C')$ and $(s_1, c_{Q'_{i_1}}) \succ (s_1, c_{Q_{i_1}})$. Hence, with Q_{-i_1} being fixed, i_1 chooses strategy Q'_{i_1} . The strategy profile Q is not an equilibrium. Note that student i_4 does not change her strategy. Indeed, even though it is now costly for i_4 to apply, she prefers s_3 to s_2 . Considering now the profile (Q'_{i_1}, Q_{i_2}) and C' . There is a profitable deviation for i_2 , for example $Q'_{i_2} : s_2$.

Therefore, (Q'_{i_1}, Q_{i_2}) is not an equilibrium when the cost profile is C' . Note that strategy profile $(Q'_{i_1}, Q'_{i_2}, Q_{i_3}, Q_{i_4})$ is not an equilibrium because student i_1 has a new profitable deviation, which is to apply to s_3 even if the cost is positive.

The reasoning is similar for blocking applications, as presented in Example 3, except that only one student is needed from each I_t^T to block deviations. Thus, considering

a cost profile such that $c_{i_1, s_2} = 1$ and $c_{i_3, s_3} = 1$, then no equilibrium leads to μ^{\natural} . We generalize these observations in the following results.

Theorem 3. Suppose there exists a student $i \in I$, such that $\mu^{SO SM}(i) \neq \mu^{\kappa}(i)$ and $c_{i, \mu^{SO SM}(i)} = 1$. Then, there is no equilibrium outcome μ that Pareto dominates $\mu^{SO SM}$, such that $\mu(i) P_i \mu^{SO SM}(i)$ with the cost profile.

Therefore, it only needs one student i with a different assignment in $\mu^{SO SM}$ and μ^{κ} , with a positive cost at $\mu^{SO SM}(i)$, to prevent any equilibrium with an outcome that Pareto dominates $\mu^{SO SM}$, such that the outcome is preferred by i to $\mu^{SO SM}(i)$, from existing. The consequence of Theorem 3 is, therefore, that it is necessary to discriminate between students when designing the cost profile. More precisely, some students may have a positive application cost for a given school, while others may not.

Corollary 1. Suppose there exists a student $i \in I$, such that $\mu^{SO SM}(i) \neq \mu^{\kappa}(i)$ and $c_{i, \mu^{SO SM}(i)} = 1$. Then, there is no equilibrium outcome that Pareto dominates $\mu^{SO SM}$, which is Pareto-efficient with the cost profile.

These results have two consequences. First, suppose there exists a school s such that $\mu^{SO SM}(s) \neq \mu^{\kappa}(s)$ and s has a positive application cost for every student, i.e., for each $i \in I, c_{i, s} = 1$. Then, there is no equilibrium such that the outcome Pareto dominates $\mu^{SO SM}$ and is Pareto efficient. This impossibility result outlines the importance of cost profile design. Second, suppose that application fees are positive for each application. Then only the stable assignments are implementable under DA. We, therefore, show that costs can have an incentive effect on students. This last result concerns the blocking applications presented in Example 3. Proposition 1 suggest they have a similar role to the schools obtained with $\mu^{SO SM}$.

Corollary 2. Suppose there exists t such that for each $i \in I_t^T$ such that $\mu^{\kappa(t)}(i)$ is a blocking application, and $\mu^{\kappa(t)}(i) \neq \mu^{\kappa}(i)$, we have $c_{i, \mu^{\kappa(t)}(i)} = 1$. Then, there is no equilibrium outcome that Pareto dominates $\mu^{SO SM}$, which is Pareto-efficient with the cost profile.

We propose a generalization of cost profiles. To ensure the existence of an equilibrium outcome that Pareto dominates $\mu^{SO SM}$, it is necessary for each student $i \in I$ to have $c_{i, \mu^{SO SM}(i)} = 0$ if $\mu^{SO SM}(i) \neq \mu^{\kappa}(i)$. We showed that a positive cost is sufficient to incentivize students who are interrupters to a school. Thus, for each student i and each school s such that for some t in the EADAM procedure, we have $(i, s_i^t) \in \tilde{I}^t$ with $s = s_i^t$, then $c_i, s_i^t = 1$. If there exist blocking applications for some t , then for at least one

student $i \in I_i^T$ such that $\mu^{\kappa(t)}(i)$ is a blocking application, we must have $c_{i, \mu^{\kappa(t)}(i)} = 0$. Our results show that other costs do not impact incentives. Thus, for any cost profile satisfying these three conditions, there is always an equilibrium outcome where students are incentivized to reach it.

Implementing a cost profile is difficult because of the difference in application costs required for a given school. One possibility would be for students to apply to some schools for free. These schools might be obtained under μ^{SOSM} and the blocking applications. However, it is difficult for students to know to which school they will be assigned. Moreover, students could be incentivized to apply only to these free schools, leading to inefficient assignments. Students need to be incentivized to apply to the right school. Grants could then be offered to cover the application fees.

6 The Boston Mechanism and the Top Trading Cycles Mechanism with Costs

We observe that in DA, costs have an impact on Nash equilibria. We have discussed the potential Pareto improvements that can be achieved by implementing application costs. This section examines whether similar effects can be expected in other mechanisms commonly discussed in the literature. Specifically, we study the implementation of application costs in the Boston mechanism and the Top Trading Cycles (TTC) mechanism.

6.1 The Boston Mechanism

The Boston mechanism is extensively studied in the literature.¹⁸ In this section, we investigate whether the introduction of application costs can incentivize students in this mechanism. Ergin and Sönmez (2006) demonstrate that every Nash equilibrium in the Boston mechanism leads to a stable outcome. Hence, we aim to determine if implementing application costs can create an equilibrium in this mechanism. Another contribution by Haeringer and Klijn (2009) show that every Nash equilibrium outcome in the Boston mechanism is achievable when each student can only apply to one school. Let Q^β denote a strategy profile using the Boston mechanism.

¹⁸The Boston mechanism is also known as the "immediate acceptance" mechanism. The term Boston mechanism is used in the context of school choice (Abdulkadiroğlu and Sönmez, 2003).

Proposition 3. Suppose μ^β is a Pareto-efficient assignment using the Boston mechanism. For every cost profile C , there exists a Nash equilibrium Q_C^β that leads to μ^β in the problem (P, \succ, C) if and only if there exists a Nash equilibrium Q^β that leads to μ^β in the problem (Q^β, \succ) .

In DA, students use their strategies to block deviations by applying to their obtained school in μ^{SOSM} . This generates a chain of rejections that guarantees equilibrium. However, blocking deviations is not possible in the Boston mechanism. Our definition of costs implies that they cannot alter the ranking of schools in students' preferences. Without a rejection chain, Proposition 3 follows directly.

Empirical and experimental studies indicate that, in practice, the Boston mechanism suggests assignments preferred by students over those suggested by DA. For example, Featherstone and Niederle (2016) compare the two mechanisms through experimental analysis and conclude that the Boston mechanism yields assignments that students prefer. Our findings differ for the two mechanisms. With DA, costs can influence equilibrium outcomes and provide incentives, while in the Boston mechanism, incentives through costs are not possible. Experimenting with application costs could then change the result of Featherstone and Niederle (2016). The assignment suggested by DA may then be preferred to that of Boston by the students.

6.2 The Top Trading Cycles Mechanism

From the literature, we know that there exist TTC equilibrium outcomes that are not Pareto-efficient. Haeringer and Klijn (2009) provide conditions under which all equilibrium outcomes are Pareto-efficient.¹⁹ Similar to the Boston mechanism, the set of equilibrium outcomes in the TTC mechanism does not depend on the number of schools reported in students' preferences (Haeringer and Klijn, 2009). All equilibrium outcomes are possible even when students rank only one school in their preferences. We have shown that in DA, some strategy profiles are no longer equilibria in the presence of costs. This raises the question of whether implementing costs can eliminate inefficient equilibrium outcomes in the TTC mechanism. The following result provides the answer. Let Q^τ denote a strategy profile using the TTC mechanism.

Proposition 4. Suppose μ^τ is not a Pareto-efficient assignment using the TTC mechanism. For every cost profile C , there exists a Nash equilibrium Q_C^τ that leads to μ^τ in

¹⁹They show that efficient Nash equilibrium outcomes can be guaranteed if, and only if, schools' priorities satisfy the X-acyclicity condition.

the problem (P, \succ, C) if and only if there exists a Nash equilibrium Q^τ that leads to μ^τ in the problem (Q^τ, \succ) .

Similar to the Boston mechanism, costs do not influence equilibrium outcomes in the TTC mechanism. Therefore, the introduction of costs cannot prevent inefficient equilibrium outcomes.

7 Conclusion

In this paper, we have examined the implementation of Nash equilibrium outcomes that Pareto dominate the students' optimal stable assignment in DA. We have shown that such implementation is possible for any assignment that Pareto dominates a stable assignment. Furthermore, we have proposed the implementation of application costs as a means to incentivize students. We have used Kesten's EADAM to determine the application costs that must be set. Our analysis has demonstrated that application costs can reinforce or hinder equilibria in the DA mechanism. However, a precise design of the cost profile is necessary to ensure the existence of Pareto-efficient equilibrium outcomes. Complete information is therefore required for the implementation of costs.

We have also investigated the impact of costs on the Boston mechanism and the TTC mechanism. Our results indicate that costs do not affect the equilibrium outcomes in these mechanisms. Consequently, introducing costs does not alter the equilibria outcomes in the Boston and TTC mechanisms. This article is the first to characterize equilibria in the presence of bid costs.

Future research could explore experimental and empirical comparisons of different mechanisms, considering application costs. The costs introduced as incentives in our theoretical framework could be used to enhance the efficiency of the DA mechanism in practice. The cost design is different from that proposed by He and Magnac (2022) and may produce new results.

The incorporation of incomplete information is another potential extension of our model. Ehlers and Massó (2015) study a many-to-one matching market with incomplete information. They show a link between stable mechanism equilibrium in incomplete information and complete information framework.²⁰ Investigating the impact of application costs on students in an incomplete information framework would provide valuable insights for policy implementation.

²⁰More precisely, they show that an equilibria characterization under complete information leads to the equilibria characterization under incomplete information.

In Appendix A, we explore the relationship between stability and efficiency in more detail. We identify conditions that guarantee the stability of an equilibrium outcome, which coincide with the concept of interrupters used in this paper. These results could be applied to a model with incomplete information.

Appendix A: Additional Results

In this Appendix A, we study equilibrium outcomes to identify conditions that guarantee stability. For this purpose, we define cycles according to Ergin (2002). Let $U_s(j)$ denote the set of students who have a higher priority at school s than student j , i.e., $U_s(j) = \{i \in I \mid i \succ_s j\}$.

Definition 5. Let \succ be a priority structure and q be a vector of quotas. A cycle is constituted by distinct schools $s, s' \in S$ and students $i, j, k \in I$ satisfying the following conditions:

- Cycle condition: $i \succ_s j \succ_s k \succ_{s'} i$.
- Scarcity condition: there exist (possibly empty) disjoint sets of agents $I_s, I_{s'} \subset I \setminus \{i, j, k\}$ such that $I_s \subset U_s(j)$, $I_{s'} \subset U_{s'}(i)$, $|I_s| = q_s - 1$, and $|I_{s'}| = q_{s'} - 1$.

A priority structure is acyclical if it has no cycles.

We know that in the presence of a cycle, there are outcomes that are not Pareto-efficient (Ergin, 2002). Equilibrium outcomes that are not stable can therefore exist in the presence of a cycle, as shown, for instance, by Haeringer and Klijn (2009) using DA. Let us illustrate with an example that considers cycles.

Example 6. Consider $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2, s_3\}$ with for each $s \in S$, $q_s = 1$. The priorities for the schools and the preferences of the students are given as follows:

\succ_{s_1}	\succ_{s_2}	\succ_{s_3}	P_{i_1}	P_{i_2}	P_{i_3}
i_1	i_2	i_1	<u>s_2</u>	s_3	<u>s_3</u>
i_2	i_1	i_3	s_3	<u>s_1</u>	s_2
i_3	i_3	i_2	s_1	s_2	s_1

There are two cycles in the schools' priorities:

- $i_2 \succ_{s_2} i_1 \succ_{s_2} i_3 \succ_{s_3} i_2$ and $i_1 \succ_{s_3} i_3 \succ_{s_3} i_2 \succ_{s_2} i_1$, for schools s_2 and s_3 .

Following the definition of a cycle, we refer to the student j as i_1 and i_3 .

The unique stable outcome is underlined in the students' preferences. Let us consider an equilibrium:

Q_{i_1}	Q_{i_2}	Q_{i_3}
<u>s_2</u>	<u>s_3</u>	s_2
s_3	s_2	<u>s_1</u>
s_1		

We can see that in this situation, there are no profitable deviations. Hence, this is an equilibrium, and the assignment is not stable.

In this equilibrium, if i_3 applies to the school s_3 , her assignment does not change, but the strategy of i_2 will be modified following a profitable deviation.

Q_{i_1}	Q_{i_2}	Q_{i_3}
s_2	s_3	s_3
<u>s_3</u>	<u>s_2</u>	s_2
s_1		<u>s_1</u>

The profitable deviation for i_2 is then to apply to s_1 . Hence, if the student i_3 applies to s_3 , the obtained assignment is stable at equilibrium.

Let us now consider another preference profile and an associated unstable equilibrium outcome:

P_{i_1}	P_{i_2}	P_{i_3}	Q_{i_1}	Q_{i_2}	Q_{i_3}
<u>s_2</u>	<u>s_3</u>	s_2	<u>s_1</u>	<u>s_3</u>	<u>s_2</u>
s_1	s_2	<u>s_1</u>	s_3	s_2	s_3
s_3	s_1	s_3		s_1	

Similarly, if i_1 applies to s_2 in her strategy Q_{i_1} , then i_3 will have a profitable deviation, which is to apply to s_1 . Following this deviation, the equilibrium outcome obtained will be stable. By routine computation, we obtain that for any preference profile, when the students i_1 and i_3 apply to s_2 and s_3 , respectively, using the correct ranking, all equilibrium outcomes are stable.

Note that we only consider equilibria. Strategy profiles may lead to an unstable outcome where students i_1 and i_3 apply to s_2 and s_3 , respectively. However, this is not an equilibrium.

The following theorem generalizes the result of this example.

Theorem 4. Suppose that for each cycle $i \succ_s j \succ_s k \succ_{s'} i$, we have $s \in A(Q_j)$, where s is ranked according to j 's true preferences in her strategy. Then each equilibrium outcome is stable.

EADAM identifies the interrupters to neutralize, and these interrupters involve the student j in the cycle. Thus, when j is an interrupter for school s , we have if $s \in A(Q_j)$, the assignment is stable at equilibrium.²¹ Still, if j does not apply, it is possible to have a Pareto-efficient equilibrium outcome under the condition that all other interrupters are neutralized.

The main consequence of Theorem 4 is that it is sufficient for each student j involved in a cycle to apply to school s to ensure outcome equilibrium stability. It is important to note that $s \notin A(Q_j)$ does not imply that it is impossible to have a stable equilibrium outcome. However, it is possible to guarantee stability by considering the preferences of students i and k . This is possible when i or k cannot block j within the cycle. We illustrate Proposition 5 with the following example:

Example 7. Consider the problem given in Example 6, and as presented, there are two cycles. In the first cycle, i is i_2 and k is i_3 , while in the second cycle, i is i_1 and k is i_2 . One consequence of our next result is that if preference profiles are such that $s_3 P_{i_2} s_2$ and $s_2 P_{i_3} s_3$ or $s_2 P_{i_1} s_3$ and $s_3 P_{i_2} s_2$, there exist unstable equilibrium outcomes. Consider, for instance, the following profile where (\cdot) can be any school:

P_{i_1}	P_{i_2}	P_{i_3}
·	·	·
·	s_2	·
·	s_3	·
	·	

Considering the preference of i_2 , all the Nash equilibrium outcomes are stable for each possible preference of other students P_{-i_2} . Similarly, if $s_3 P_{i_3} s_2$ and $s_3 P_{i_1} s_2$; then all the possible preferences of student i_2 lead to only stable equilibrium outcomes.

Proposition 5. Suppose that for each cycle $i \succ_s j \succ_s k \succ_{s'} i$, and for each student i and k , i ranks $s P_i s'$ or student k ranks $s' P_k s$, or both, in their true preferences. Then all equilibrium outcomes are stable.

The stability of equilibrium outcomes can be identified in two situations, and we can generalize these results as follows:

- If at least one student among i and k for each cycle has preferences such that $s P_i s'$ or $s' P_k s$, from Proposition 5, then all equilibrium outcomes are stable.

²¹This must be true for each cycle. Note that if s is not acceptable to j , then the school should not be included in j 's strategy.

- Otherwise, for the cycles concerned, if $s \in A(Q_j)$ ranked according to true preferences, then by Theorem 4, all equilibrium outcomes are stable.

We have identified the applications in the student strategies that guarantee the stability of the equilibrium outcomes. One of our conditions coincides with the interrupter notion mentioned above.

Appendix B: Proofs

Lemmata

Lemma 1. Using EADAM, if $\mu^\kappa(s) \neq \emptyset$, then $\mu^{SOSM}(s) \neq \emptyset$. Additionally, for each $s \in S$ where s is equal to s_i^t for some step t in EADAM and such that $(i, s_i^t) \in \tilde{I}_t$, then $\mu^\kappa(s_i^t) \neq \emptyset$.

Proof. By contradiction, suppose that $\mu^{SOSM}(s) = \emptyset$ and $\mu^\kappa(s) \neq \emptyset$. This implies $\mu^{SOSM}(s) \neq \mu^\kappa(s)$. Therefore, μ^{SOSM} is not stable, as there exists a student i who prefers s to their assigned school $\mu^{SOSM}(i)$. This violates the non-wastefulness property of the assignment. Hence, student i has a profitable deviation by being assigned to s . Therefore, if $\mu^\kappa(s) \neq \emptyset$, then $\mu^{SOSM}(s) \neq \emptyset$.

Consider an interrupter (j, s^*) . Suppose that there exists i such that $\mu^{SOSM}(i) = s^*$ and assume, by contradiction, that no student under EADAM is assigned to s^* , i.e., there is no $i' \in I$ such that $\mu^\kappa(i') = \mu^{SOSM}(i)$. From the definition of an interrupter, we know that j has been rejected by the application of another student. Then, either $\mu^\kappa(s^*) \neq \emptyset$ or (j, s^*) is not an interrupting pair, leading to a contradiction. ■

Lemma 2. Let Q be a strategy profile such that for each student $i \in I$ we have:

- $\mu^{SOSM}(i) \in A(Q_i)$,
- for each I_t^T such that a blocking application exists, at least one student $i \in I_t^T$, such that $\mu^{\kappa(t)}(i)$ is a blocking application, $\mu^{\kappa(t)}(i) \in A(Q_i)$, and
- schools are ranked according to true preferences.

Then, there exists a rejection chain that blocks the deviations of students involved in an interrupting pair.

Proof. Consider a student i who is assigned to school s at the end of EADAM, i.e., $\mu^\kappa(i) = s$, and suppose there exists $j \in I \setminus \{i\}$ such that (j, s) is an interrupting pair. We know that $j \succ_s i$ and $sP_j\mu^\kappa(j)$ (if the school was not involved in an interrupting pair, then there is no student i' such that $sP_{i'}\mu^\kappa(i')$ and $i' \succ_s i$). Consider that $s \in A(Q_j)$, thus j triggers a rejection chain leading to the rejection of i from school s .

Then, i applies to a school ranked after $\mu^\kappa(i)$.

- If this school is $\mu^{\kappa(t)}(i) \neq \mu^{SO SM}(i)$, then this leads to the rejection of a student i' who was assigned to $\mu^{\kappa(t)}(i)$ using EADAM, such that $\mu^\kappa(i') = \mu^{\kappa(t)}(i)$. Then, i' rejects i'' from the school ranks after $\mu^\kappa(i')$ in $Q_{i'}$, and this process continues until j is rejected from s .
- If this school is $\mu^{\kappa(t)}(i) = \mu^{SO SM}(i)$, then this leads to the rejection of a student i' who was assigned to $\mu^{SO SM}(i)$ using EADAM, such that $\mu^\kappa(i') = \mu^{SO SM}(i)$. Then, i' rejects i'' from the school ranks after $\mu^\kappa(i')$ in $Q_{i'}$, and this process continues until j is rejected from s .

By contradiction, suppose that i does not reject anyone or does not generate a rejection chain that allows j to be rejected from s .

Let us consider $DA(P, \succ)$ temporarily. We know that j is an interrupter for s , so there exists a student k such that $s \in A(Q_k)$ and causes j to be rejected at a later step. We know that k is assigned to s , such that $\mu^{SO SM}(k) = s$, otherwise k is also an interrupter (and k is the last interrupter for s to be rejected, and consequently, EADAM identifies k as the interrupter to be neutralized). If k is not in the rejection chain generated by j , then k will apply to s at the same step when j is rejected. Suppose that (j, s) is neutralized, then i is rejected at a later step by k and $\mu^{SO SM}(k) = s$. Since we have $k \succ_s j$ and $j \succ_s i$, it implies $k \succ_s i$. Therefore, i cannot be assigned to s if i does not generate the rejection chain that rejects j from s . This means that k is rejected from a school by j 's application to school s , and k rejects j from s .

Consequently, there exists a school, let us denote it as s^* , to which i applied such that $s^* P_i s^*$ and $s^* R_i \mu^{SO SM}(i)$, which generated a rejection chain that rejected j from s . We consider three cases:

- $s^* \neq \mu^{SO SM}(i)$ and $s^* \neq \mu^{\kappa(t)}(i)$: i is an interrupter for s^* because i is not assigned to s^* and leads to the rejection of a student from s^* (which then leads to the rejection of j from s). According to Lemma 1, i is indifferent when applying to s^* .
- $s^* \neq \mu^{SO SM}(i)$ and $s^* = \mu^{\kappa(t)}(i)$: i can generate the same rejection chain if $\mu^{\kappa(t)}(i) \in A(Q_i)$.
- $s^* = \mu^{SO SM}(i)$: i can generate the same rejection chain if $\mu^{SO SM}(i) \in A(Q_i)$.

Hence, we conclude that i generates its rejection chain (which leads to the rejection of j from s) by applying to $\mu^{\kappa(t)}(i)$ if such a school exists, and $\mu^{SO SM}(i)$. \blacksquare

Proof of Theorem 1

Proof. Let μ_S denote a stable assignment and μ^* denote an assignment that Pareto dominates μ_S . We know that there exists $I' \subset I$ such that for each $i \in I'$, $\mu^*(i)P_i\mu_S(i)$ by the definition of Pareto domination. If no student i strictly prefers $\mu^*(i)$ to $\mu_S(i)$, then the proof is immediate, and an equilibrium exists. Otherwise, there exist i and i' in I' such that $\mu^*(i) = \mu_S(i')$. Suppose that for each $i \in I$, Q_i is such that $\mu^*(i), \mu_S(i) \in A(Q_i)$ and all the schools in Q_i are ranked according to P_i . In addition, each student applies to her blocking application if such a school exists. Then, it is clear that by using DA, the algorithm stops at round 1, and for each student, the outcome is $\mu^*(i)$. We then need to show that this strategy is an equilibrium when using DA.

Suppose that Q is not a Nash equilibrium. Then, there exists $i_1 \in I$ such that $Q'_{i_1} \neq Q_{i_1}$ and $DA(Q'_{i_1}, Q_{-i_1})P_{i_1}DA(Q)$, and we denote s^\dagger as the school to which student i_1 is assigned under Q'_{i_1} . If i_1 applies only to s^\dagger , then the assignment does not change. Formally, $DA(\{s^\dagger\}, Q_{-i_1}) = DA(Q'_{i_1}, Q_{-i_1})$.

Using Lemma 1 of Alva and Manjunath (2019), we have $s^\dagger \neq \emptyset$. By the construction of DA and stability, there exists $i_2 \in I$ such that $i_2 \succ_{s^\dagger} i_1$:

- If $\mu_S(i_2) = s^\dagger$, then $\mu^*(i_2)P_{i_2}s^\dagger$. We know that there exists i_3 such that $\mu^*(i_2) = \mu_S(i_3)$. If $i_3 = i_1$, then the deviation is not profitable. If $i_3 \neq i_1$, there exists i_4 such that $\mu^*(i_3) = \mu_S(i_4)$.
- If $\mu_S(i_2) \neq s^\dagger$, then there exists $s \in A(P_{i_2})$ such that $s = s^\dagger$. Through the DA procedure, i_2 rejects i_1 from s^\dagger . We know that there exists i_3 such that $\mu^*(i_2) = \mu_S(i_3)$. If $i_3 = i_1$, then the deviation is not profitable. If $i_3 \neq i_1$, there exists i_4 such that $\mu^*(i_3) = \mu_S(i_4)$. Hence, $s^\dagger \in A(Q_{i_2})$ as s^\dagger is a blocking application.

By generalizing the reasoning, i_k cannot reject i_h , with $k > h > 1$; otherwise, μ_S is not stable. Then, i_k rejects i_1 . We conclude that there is no profitable deviation for student i_1 by applying to s^\dagger . Thus, Q is an equilibrium, and the outcome is μ^* . ■

Proof of Theorem 2

Proof. Let P_i denote the true preference list and Q_i denote any generic preference list for each student i . For any Q , let K_i^Q be the list Q_i where the schools s such that (i, s) is an interrupting pair have been dropped from Q_i . We know that under DA, we have:

- DA is strategy-proof by Gale and Shapley (1962): $DA_i(P_i, Q_{-i})R_iDA_i(Q_i, Q_{-i}), \forall Q_i, \forall Q_{-i}$,
- By Proposition 3 of Kesten (2010), $DA_i(K_i^Q, Q_{-i}) = DA_i(Q_i, Q_{-i}), \forall Q_i, \forall Q_{-i}$,

(iii) By Theorem 1 of Kesten (2010), $DA((K_i^P)_{i=1,\dots,n})$ Pareto dominates $DA((P_i)_i)$. Statements (i) and (ii) implies that $DA_i(K_i^P, K_{-i}^P) = DA_i(P_i, K_{-i}^P)R_iDA_i(Q_i, K_{-i}^P), \forall Q_i$. It follows that $(K_i^P)_{i=1,\dots,n}$ is an equilibrium, and the outcome Pareto dominates the DA outcome under P , as stated in (iii). According to our definition of costs, this equilibrium exists. Example 1 completes the proof by showing that costs have an impact on the dominant strategy. ■

Proof of Proposition 1

Proof. Let Q be a strategy profile that satisfies (i)-(iv). We need to show that there are no profitable deviations for students.

Let us first show that there are no profitable deviations for students identified as interrupters for a school at some step t . We know that for all $s \in S$ such that $\mu^{SO SM}(s) \neq \emptyset$, we have $\mu^\kappa(s) \neq \emptyset$ by Lemma 1. Thus, it is necessary to have a rejection chain that leads students to be rejected from the school for which they are an interrupter, as shown in Lemma 2. For the other students, we know that the obtained assignment is Pareto-efficient, and there are no profitable deviations because i obtains the same assignment using P_i .

From Lemma 1, we know that there exists a student $i' \in I$ such that $\mu^{SO SM}(i) = \mu^\kappa(i')$. By being rejected from $\mu^{SO SM}(i)$, student i' applies to the school ranks after $\mu^\kappa(i')$. If there is no blocking application, this school is $\mu^{SO SM}(i')$; otherwise, it is $\mu^\kappa(i')$. Using Lemma 2, we have the same reasoning; the rejection chain continues until j is rejected from s . Considering a strategy where each student $i \in I$ has, if it exists, such a school $\mu^\kappa(i) \in A(Q_i)$, and $\mu^{SO SM}(i) \in A(Q_i)$, is therefore sufficient to block j from being assigned to s .

Thus, j is indifferent between applying to school s or not if a strategy profile satisfies (i)-(iv). Therefore, the assignment is Pareto-efficient because, for each student $i \in I$, we have $\mu^\kappa(i) \in A(Q_i)$. Moreover, it is an equilibrium. ■

Proof of Theorem 3

Proof. By contradiction, suppose there exists an equilibrium outcome μ that Pareto dominates $\mu^{SO SM}$ such that $\mu(i)P_i\mu^{SO SM}(i)$ with $\mu^{SO SM}(i) \neq \mu^\kappa(i)$ and $c_{i,\mu^{SO SM}(i)}$. Let us denote that equilibrium by Q . We know that student i is assigned to $\mu(i)$ in $DA(Q, \succ, C)$. Then, if we have:

- $\mu^{SO SM}(i) \in A(Q_i)$, there exists a profitable deviation for i because $\mu(i)P_i\mu^{SO SM}(i)$ and $c_{i,\mu^{SO SM}(i)}$. Then we have $DA_i(\{\mu(i)\}, Q_{-i}, \succ, C) = \mu(i)$.

- $\mu^{SOSM}(i) \neq Q_i$, then by Proposition 1 and Theorem 1, we know that there exists a profitable deviation for a student $i' \in I \setminus \{i\}$.

Hence, Q is not an equilibrium. ■

Proof of Corollary 1

Proof. We denote by μ^{SOSM} the outcome of the equilibrium Q_T such that $Q_T = P$, and μ^{PE} the outcome of the equilibrium Q_{PE} , which is Pareto-efficient. Consider $i \in I$ such that $\mu^{SOSM}(i) \neq \mu^{PE}(i)$. We denote $s = \mu^{SOSM}(i)$, and we know that $\mu^{PE}(i)P_i s$ as preferences are strict. Using equilibrium Q_{PE} , i is assigned to $\mu^{PE}(i)$. We denote by Q_{PE_i} the strategy of i under the equilibrium Q_{PE} . If we set a positive cost such that $c_{i,s} = 1$, we know that considering $Q_{PE_{-i}}$, i prefers the strategy Q'_{PE_i} to the strategy Q_{PE_i} with $s \notin A(Q'_{PE_i})$, $\mu^{PE}(i) \in A(Q'_{PE_i})$, and $s, \mu^{PE}(i) \in A(Q_{PE_i})$. As the outcome of Q_T is stable and the outcome of Q_{PE} is unstable, we know that $\exists j \in I$ such that $j \succ_{\mu^{PE}(i)} i$ and $\mu^{PE}(i)P_j \mu^{PE}(j)$. Considering the strategy profile $(Q'_{PE_i}, Q_{PE_{-i}})$, it is no longer an equilibrium since j has a profitable deviation to be assigned to $\mu^{PE}(i)$ because the rejection chain is stopped since $s \notin A(Q'_{PE_i})$ (from Theorem 1, Lemma 1, and Lemma 2). Note that this is true even if the application cost $c_{j,\mu^{PE}(i)}$ is positive. The statement is valid as long as Q_T produces a stable assignment. Since the assignment suggested by DA is stable, according to reported preferences, we consider it in the proof. ■

Proof of Corollary 2

Proof. By contradiction, suppose there exists $i \in I_t^T$ such that i is an interrupter identified at a step $t' > t$. For each $i' \in I_t^T$ such that $\mu^{\kappa(t)}(i')$ is a blocking application, we have $c_{i',\mu^{\kappa(t)}(i')} = 1$ and $\mu^{\kappa(t)}(i') \neq \mu^{\kappa}(i')$. Let us denote by μ the assignment that is Pareto-efficient and Pareto dominates μ^{SOSM} . As $\mu^{\kappa(t)}(i') \neq \mu^{\kappa}(i')$, we know that for each $i' \in I_t^T$, $\mu(i')P_{i'} \mu^{\kappa(t)}(i')$. Then, knowing that $c_{i',\mu^{\kappa(t)}(i')} = 1$, there is a profitable deviation for each i' , which is $\mu^{\kappa(t)}(i') \in A(Q_{i'})$. The rest of the proof follows the proof of Corollary 1. ■

Proof of Proposition 3

Proof. Let us note $\mu^\beta(i)$ for each student i the assignment obtained with an equilibrium Q^β , and $\mu^{\beta PE}(i)$ the assignment obtained which is Pareto-efficient, using the Boston mechanism for problem (P, \succ) . We consider the following cases in the absence of costs. Proposition 5.2 of Haeringer and Klijn (2009) allows us to consider strategies for each

student with a single school.

Case 1: The assignment obtained at equilibrium provides a Pareto-efficient outcome. Then $\forall i \in I$, $\mu^\beta(i) = \mu^{\beta PE}(i)$. Then, according to our definition of costs, all the cost profiles maintain this equilibrium.

Case 2: The equilibrium outcome is not Pareto-efficient. Then $\exists i \in I$ such that $\mu^\beta(i) \neq \mu^{\beta PE}(i)$. From Theorem 1 in Ergin and Sönmez (2006), we know that the set of Nash equilibrium outcomes equals the set of stable assignments under the true preferences. Then we know that $\forall i \in I$, $\forall s$ such that $sP_i\mu^\beta(i)$, $\exists i' \in I \setminus \{i\}$ such that $i' \succ_s i$. By contradiction, let us suppose that there exists a cost profile, C , that allows an equilibrium Q^* with an outcome $\mu^{*\beta}$ such that $\forall i \in I$, $\mu^{*\beta}(i)R_i\mu^\beta(i)$ and $\exists i \in I$, $\mu^{*\beta}(i)P_i\mu^\beta(i)$. We denote by $Q_i : \mu^\beta(i)$ and $Q_i^* : \mu^{*\beta}(i)$ the student i strategies under the equilibrium Q and Q^* , respectively. Let us consider a student i such that $\mu^{*\beta}(i) \neq \mu^\beta(i)$. By definition, we have $\mu^{*\beta}(i)P_i\mu^\beta(i)$. By stability and the rural hospital theorem Roth (1986), we know that $\exists j \in I$ such that $\mu^{*\beta}(i) = \mu^\beta(j)$. It follows that, at Q^* , we have $\mu^{*\beta}(j)R_j\mu^\beta(j)$. Then, the obtained assignment is no longer stable, and a student can be assigned to the school by ordering the school to the first position in her strategy. Therefore, either Q^β or Q^* is not an equilibrium, which leads to a contradiction.

Hence, costs do not generally lead to a Pareto-efficient equilibrium outcome unless the equilibrium outcome is Pareto-efficient without costs. ■

Proof of Proposition 4

Proof. From Haeringer and Klijn (2009), we know that all equilibrium outcomes are possible with only one school using the TTC in the students' strategy. Then we consider that $\forall i \in I$, $Q_i^\tau : \mu^\tau(i)$, with $\mu^\tau(i)$ being the school to which i is assigned at equilibrium Q^τ . We have two cases:

Case 1: μ^τ is Pareto-efficient. We know that the costs are low, so the order of schools in students' preferences cannot be changed due to the implementation of costs. Then every cost profile maintains this equilibrium.

Case 2: μ^τ is not Pareto-efficient. Suppose that a cost profile C exists such that μ^τ is Pareto-efficient and it is the outcome of equilibrium $Q^{\tau'}$. We know that the costs are low, so the order of schools in students' preferences cannot be changed due to the implementation of costs. Then Q^τ is also an equilibrium, and the outcome is Pareto-efficient in the absence of costs, which is a contradiction. ■

Proof of Proposition 5

Proof. Suppose there is a cycle $i \succ_s j \succ_s k \succ_{s'} i$. Let us denote μ an equilibrium outcome.

Consider three cases:

Case 1: Suppose $sP_i s'$ and $sP_k s'$:

- If $\mu(i)P_i s$, then i does not block k . Then, if $\mu(k) = s$, we have $\mu(j)P_j s$; otherwise, $\mu(k) \neq s$.
- If $sP_i \mu(i)$, then i can block k , and i is blocked by a student, denoted as i' , such that $i' \succ_s i$ and therefore $i' \succ_s k$.
- If $\mu(i) = s$, then i blocks k if $sP_k \mu(k)$.

Case 2: Suppose $s'P_i s$ and $s'P_k s$:

- If $\mu(k)P_k s'$, then k does not block i . Then, if $\mu(i) = s'$, we have no student k' such that $k' \succ_{s'} i$ and $s'P_{k'} \mu(k')$.
- If $s'P_k \mu(k)$, then k can block i , and k is blocked by a student, denoted as k' , such that $k' \succ_{s'} k$ and therefore $k' \succ_{s'} i$.
- If $\mu(k) = s'$, then k blocks i if $s'P_i \mu(i)$.

Case 3: Suppose $sP_i s'$ and $s'P_k s$:

- If $\mu(i) = s$ and $\mu(k) = s'$. Then it is immediate, and μ is stable.
- If $\mu(i) = s$ and $\mu(k) \neq s'$. Then, if $\mu(k)P_k s'$, μ is stable, and if $s'P_k \mu(k)$, we know that there exists $i' \in I$ such that $\mu(i') = s'$ and $i' \succ_{s'} k$. We, therefore, have $i' \succ_{s'} i$; then i cannot block i' .
- If $\mu(i) \neq s$ and $\mu(k) = s'$. Then, if $\mu(i)P_i s$, μ is stable, and if $sP_i \mu(i)$, we know that there exists $i'' \in I$ such that $\mu(i'') = s$ and $i'' \succ_s i$. We, therefore, have $i'' \succ_s k$; then k cannot block i'' .
- If $\mu(i) \neq s$ and $\mu(k) \neq s'$. The reasoning is the same as in the previous cases.

In each case, μ is stable. ■

Proof of Theorem 4

Proof. From Theorem 1 in Ergin (2002), we know that if a cycle exists in schools' priorities, then a preference profile exists that leads to unstable outcomes at equilibrium. From Proposition 5, we can consider a specific case only such that $s'P_i s$ and $sP_k s'$. Suppose that $s \in A(Q_j)$ with the correct ranking.

Let us denote a stable outcome in this proof by $\mu^{SO SM}(i)$.

Case 1: If $\mu^{SO SM}(j)P_j s$: i and k can never benefit from the cycle, as the rejection chain that blocks j 's application is impossible.

Case 2: If $sR_j\mu^{SO\text{SM}}(j)$: by definition, $i \succ_s j \succ_s k$. As $s \in A(Q_j)$, we know that j prevents potential deviations of k using the rejection chain.

In case 2, k is rejected by the application of j , which guarantees the stability of the outcome. In case 1, j is indifferent between applying to s or not because, at equilibrium, she is assigned to a preferred school. Using the reasoning for each cycle completes the proof. ■

References

- Abdulkadiroğlu, A. and Sönmez, T. (2003). School choice: A mechanism design approach. *American economic review*, 93(3):729–747.
- Alva, S. and Manjunath, V. (2019). Stable-dominating rules. Technical report, Working paper, University of Ottawa.
- Chade, H., Lewis, G., and Smith, L. (2014). Student portfolios and the college admissions problem. *Review of Economic Studies*, 81(3):971–1002.
- Dur, U. M. and Morrill, T. (2020). What you don’t know can help you in school assignment. *Games and Economic Behavior*, 120:246–256.
- Ehlers, L. and Massó, J. (2015). Matching markets under (in) complete information. *Journal of economic theory*, 157:295–314.
- Ergin, H. (2002). Efficient resource allocation on the basis of priorities. *Econometrica*, 70(6):2489–2497.
- Ergin, H. and Sönmez, T. (2006). Games of school choice under the boston mechanism. *Journal of public Economics*, 90(1-2):215–237.
- Featherstone, C. R. and Niederle, M. (2016). Boston versus deferred acceptance in an interim setting: An experimental investigation. *Games and Economic Behavior*, 100:353–375.
- Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.
- Haeringer, G. and Klijn, F. (2009). Constrained school choice. *Journal of Economic theory*, 144(5):1921–1947.

- He, Y. and Magnac, T. (2022). Application costs and congestion in matching markets. *The Economic Journal*, 132(648):2918–2950.
- Kesten, O. (2010). School choice with consent. *The Quarterly Journal of Economics*, 125(3):1297–1348.
- Liu, Q., Mailath, G. J., Postlewaite, A., and Samuelson, L. (2014). Stable matching with incomplete information. *Econometrica*, 82(2):541–587.
- Roth, A. E. (1982). Incentive compatibility in a market with indivisible goods. *Economics letters*, 9(2):127–132.
- Roth, A. E. (1986). On the allocation of residents to rural hospitals: a general property of two-sided matching markets. *Econometrica: Journal of the Econometric Society*, pages 425–427.
- Troyan, P., Delacrétaz, D., and Kloosterman, A. (2020). Essentially stable matchings. *Games and Economic Behavior*, 120:370–390.