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Abstract. Given $\mathfrak{F}$ a coherent sheaf on a Noetherian integral algebraic stack $\mathfrak{P}$, we give two constructions of stacks $\widetilde{\mathfrak{P}}$, equipped with birational morphisms $p : \widetilde{\mathfrak{P}} \to \mathfrak{P}$ such that $p^*\mathfrak{F}$ is simpler: in the Rossi construction, the torsion free part of $p^*\mathfrak{F}$ is locally free; in the Hu–Li diagonalization construction, $p^*\mathfrak{F}$ is a union of locally free sheaves. We use these constructions to define reduced Gromov–Witten invariants of complete intersections in all genera.

Contents

1. Introduction

Overview of the problem. Let X be a smooth projective variety. We denote by $\overline{\mathcal{M}}_{g,n}(X,d)$ the moduli space of genus g, degree $d \in H_2(X;\mathbb{Z})$ stable maps to X (see [Kon95]). By [LT98, BF96] $\overline{\mathcal{M}}_{g,n}(X,d)$ has a virtual class $[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}} \in A_*(\overline{\mathcal{M}}_{g,n}(X,d))$. Gromov–Witten invariants of X are defined as intersection numbers against this virtual class. They are related to counts of curves in X of genus g and class d, but they often encode contributions from degenerate maps. These degenerate contributions can be explained by the geometry of the moduli space of stable maps.

We have little information about moduli spaces of stable maps to a variety X, even when X is complete intersection, but the moduli space of stable maps to projective spaces are better understood. The space of genus zero stable maps to a projective space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is a smooth Deligne–Mumford stack and the resulting genus zero Gromov–Witten invariants are enumerative. For $g>0$, the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ has several irreducible components and moreover, in genus one and two, we have explicit local equations for $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$: see [Zin09c, HL10, HLN12]. The existence of components consisting of maps with reducible domain is reflected by Gromov–Witten invariants: these components contribute lower genus stable maps.

In order to define invariants which do not have contributions from degenerate maps with reducible domains, we need to define a virtual class on the closure of the locus of maps with smooth domain. This is not possible directly, we need to replace this component with a birational one, which admits a virtual class. In genus one and two there are several such constructions [Zin09c, VZ08, HL10, RSPW19a, HLN12, BC23, HN19, HN20]. The resulting numbers are called *reduced* Gromov–Witten invariants.

Main result 1.0.1 (See Definition 7.3.10)**.** *We define reduced Gromov–Witten invariants in any genus for any complete intersection in a projective space.*

In genus one and two, our reduced Gromov–Witten invariants agree with the reduced invariants defined previously.

Genus one reduced invariants for varietes of any dimension are related to Gromov–Witten invariants [Zin08]. For three-folds the relation is much simpler. Let X be a threefold which is a complete intersection and let $\gamma \in H^*(X)^{\oplus n}$ be a collection of cohomology classes of X. Let $N_\beta^g(\gamma)$ be the genus g and degree β Gromov–Witten invariants of X with insertions given by γ, and let $r_\beta^g(\gamma)$ be the corresponding reduced invariants.

Conjecture 1.0.2. [Zin09a], [HL11, Conjecture 1.1] *Let X be a Calabi–Yau threefold. Then, there are universal constants $C_h(g) \in \mathbb{Q}$, such that for $\deg(\beta) > 2g-2$, we have*

$$N_\beta^g = \sum_{0\le h\le g} C_h(g) r_\beta^h.$$

When X is the quintic threefold, the above formula in genus one is the formula in [Zin09a, LZ09]

$$N_d^1 = \frac{1}{12} N_d^0 + r_d^1. \tag{1}$$

If X is a Fano threefold, then reduced invariants are expected to be equal to Gopakumar–Vafa invariants. Indeed, the Gopakumar–Vafa invariants are by definition related to Gromov–Witten invariants by a recursive formula, which takes into account degenerate lower genus and lower degree boundary contributions. For Fano varieties, there are no lower degree contributions. Boundary contributions were computed by Pandharipande in [Pan99]. The conjectural equality between reduced Gromov–Witten invariants and Gopakumar–Vafa invariants (see [Pan99, Section 0.3]) for Fano threefolds gives the following.

Conjecture 1.0.3. [Zin09a, Zin11] *Let X be a Fano threefold and let $C^X_{h,\beta}(g)$ be defined by the formula*

$$\sum_{g\geq 0} C^X_{h,\beta}(g)t^{2g} = \left(\frac{\sin(t/2)}{t/2}\right)^{2h-2-K_X\cdot\beta}.$$

Then, we have the following

$$N^g_\beta(\gamma) = \sum_{h=0}^{g} C^X_{h,\beta}(g-h)r^h_\beta(\gamma).$$

The above should also hold in the Calabi–Yau case, when $C^X_{h,\beta}(g)$ do not depend on X and β.

The above conjectures have been proved in genus one and two. These are the only cases in which a definition of reduced invariants existed prior to this work.

Approach. In a first step we use the geometry of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in the following way. The moduli space of stable maps admits a map $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathfrak{Pic}$, where $\mathfrak{Pic}$ denotes the stack which parameterises genus g curves with n marked points, together with a line bundle of degree d. One important observation is that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is an open substack in an abelian cone $\operatorname{Spec}\operatorname{Sym}\mathfrak{F}$, with $\mathfrak{F}$ a sheaf on $\mathfrak{Pic}$ (see (20) and (21)).

In a second step we use the sheaf $\mathfrak{F}$ to construct $\widetilde{\mathfrak{Pic}}$ (or $\widetilde{\mathfrak{Pic}}^{HL}$) together with a birational morphism $p : \widetilde{\mathfrak{Pic}} \to \mathfrak{Pic}$. By base change, this gives $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ allows us to define reduced Gromov–Witten invariants.

The stack $\widetilde{\mathfrak{Pic}}$ can be constructed in two ways: using the Rossi construction (see Section 3) or using the Hu–Li diagonalization construction (see Section 4). In both approaches we start with an atlas $U_i \to \mathfrak{Pic}$, and then we consider $\widetilde{U_i}$ as defined by Rossi (see Section 3), or by Hu–Li (see Section 4). In the end, we glue $\widetilde{U_i}$ to a global object $\widetilde{\mathfrak{Pic}}$ (see Theorem 6.2.1). The resulting stack $\widetilde{\mathfrak{Pic}}$ is called the Rossi construction or the Hu–Li construction, depending on the definition of $\widetilde{U_i}$.

In general, the Rossi construction is different from the Hu–Li construction (see 4.3.1). By Example 8.1.5 the Rossi construction gives a new moduli space which is different from the Vakil–Zinger blow-up. However, by Proposition 7.3.9 this does not change the reduced invariants: they are the same for all birational models of $\mathfrak{Pic}$.

The Rossi construction for $\widetilde{\mathfrak{Pic}}$ is enough to prove the Main Result 1.0.1, but the Hu–Li construction is better behaved in relation to Conjecture 1.0.2 and Conjecture 1.0.3.

Relation to previous approaches. The structure of this paper is different from the ones in [VZ08, HL10, HLN12, HN19, HN20]. In the mentioned papers, the authors have a three step strategy to constructing the stack $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$:

(1) they find equations of local embeddings $U_i \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in smooth spaces V_i;
(2) they blow up of U_i to obtain $\widetilde{U_i}$;
(3) they show that $\widetilde{U_i}$ glue to a (smooth) stack $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$.

The first step becomes involved in higher genus, due to the rather complicated geometry of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. Steps 2 and 3 are done by constructing an explicit blow of $\mathfrak{Pic}$ (or $\mathfrak{M}_{g,n}$). Finding a candidate for this blow-up is the hardest part of the construction.

We omit Step 1 completely. For us Step 2 is minimal in a suitable sense – it is given by a universal property. The main ingredient in Step 3 is that the (local) constructions proposed in Section 3 and in Section 4 commute with smooth pull-backs and this allows us to glue them. Explicit equations of the charts U_i are thus not necessary to construct $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$. The advantage of this approach is that the gluing is conceptual and straightforward. This is similar to what Hu and Li do in [HL11] – our construction heavily relies on their ideas.

Main technical result. In Sections 1–6 we work in the the following, completely general setup. Given $\mathfrak{P}$ a stack and $\mathfrak{F}$ a coherent sheaf on it, we want to construct $\widetilde{\mathfrak{P}}$ with a proper birational morphism $p : \widetilde{\mathfrak{P}} \to \mathfrak{P}$ such that $p^*\mathfrak{F}$ is better behaved. We present two general constructions:

(1) the Rossi construction and
(2) the Hu–Li diagonalization construction.

(1) Given an integral, Noetherian algebraic stack $\mathfrak{P}$, we construct $p : \widetilde{\mathfrak{P}} \to \mathfrak{P}$, such that the torsion free part of $p^*\mathfrak{F}$ is locally free. We show that the Rossi construction has a universal property in Theorem 6.2.4, in particular it is the minimal stack such that the torsion free part of $p^*\mathfrak{F}$ is locally free.

(2) The Rossi construction does not change torsion sheaves. The Hu–Li diagonalization construction also produces a sheaf $p^*\mathfrak{F}$ with the torsion free part of $p^*\mathfrak{F}$ locally free. In addition to this, $p^*\mathfrak{F}$ also has a well-behaved torsion in the sense of Definition 4.1.1. For schemes this is achieved by a construction by Hu and Li [HL11]. In theorem 6.2.1 we produce global object $\widetilde{\mathfrak{P}}^{HL}$. In Theorem 6.2.7 we show that $\widetilde{\mathfrak{P}}^{HL}$ satisfies a universal property and in Theorem 5.2.1 we show that the irreducible components of $\operatorname{Spec}\operatorname{Sym}\ p^*\mathfrak{F}$ are vector bundles.

Applying this construction with $\mathfrak{P} = \mathfrak{Pic}$, we obtain a well-behaved $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ as follows. In the following we fix $d > 2g - 2$ (see the explanation before 7.1.3 for details) and we consider X a hypersurface of degree k in $\mathbb{P}^r$. In Section 7 we define $\widetilde{\mathfrak{Pic}}$ and a proper and birational morphism $p_k : \widetilde{\mathfrak{Pic}} \to \mathfrak{Pic}$. We define

$$\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) := \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \times_{\mathfrak{Pic}} \widetilde{\mathfrak{Pic}},$$

which comes equipped with a morphism $\bar{p}_k : \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. We denote by $\overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ the closure in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of the locus of maps with smooth domain. The condition on d ensures that $\overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ is generically smooth and

unobstructed. As before, we consider the base-change

$$\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d) := \overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d) \times_{\mathfrak{Pic}} \widetilde{\mathfrak{Pic}}.$$

We have the following result in all genera.

Theorem 1.0.4 (See Theorem 7.3.4)**.** *Let $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) = \cup_{\theta\in\Theta}\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$, with $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$ irreducible components of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$. Then the following statements hold:*

1. *The stack $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ admits a virtual class.*
2. *The morphism $\bar{p}_k$ is proper, and we have $(\bar{p}_k)_*[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)]^{\mathrm{vir}}$.*
3. *For any $\theta\in\Theta$, $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$ is smooth over its image in $\widetilde{\mathfrak{Pic}}$; in particular, $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d)$ is smooth over $\widetilde{\mathfrak{Pic}}$.*
4. *Let $\tilde{\pi}^{\theta}:\tilde{\mathcal{C}}^{\theta}\to\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$ denote the universal curve. Then $\tilde{\pi}^{\theta}_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$; in particular $\tilde{\pi}^{\circ}_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d)$.*

The above theorem implies the Main result 1.0.1. See Section 7.3 for details.

History and related works. Reduced genus 1 invariants are the output of a long and impressive project. Reduced invariants were defined, using symplectic methods, and compared to Gromov–Witten invariants by Zinger [Zin08, Zin07, Zin09b, Zin09a]. Li–Zinger showed [LZ07, LZ09] that reduced Gromov–Witten invariants are the integral of the top Chern class of a sheaf over the main component of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$; this is an analog, for reduced genus 1 invariants, of the quantum Lefschetz hyperplane property [LZ07, LZ09]. In view of [Zin09b] this also gives a proof of the formula (1). The algebraic definition requires a blow-up construction for the moduli space of stable maps to projective space due to Vakil and Zinger [VZ08, VZ07]. See [Zin20] for a survey from the symplectic perspective.

Explicit local equations for the Vakil–Zinger blow-up in genus one are given in [Zin09c, HL10] and in genus two in [HLN12]. It is expected that the methods used in low genus could provide local equations for general moduli spaces of stable maps to projective spaces; the combinatorics is likely to be tedious.

In [HL10, HLN12, HN19, HN20] the authors give a modular interpretation of reduced invariants in terms of graphs of degenerate maps. A modular interpretation via log maps has been given by Ranganathan, Santos-Parker and Wise [RSPW19a, RSPW19b].

Hu and Li introduce the diagonalization construction in [HL11]. They use this construction to define an Euler class on the moduli space of stable maps to projective spaces. This gives a non-intrinsic definition of reduced invariants of complete intersections. Conjecture 1.0.3 is hard to approach with this definition. In this paper we rework their construction.

In a different direction, instead of replacing the moduli space of maps with a space which dominates $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)$, one can construct a space dominated by $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)$. This has been done by moduli spaces of maps from more singular curves, such as in [BCM20, BC23]. A modular interpretation comes for free with this approach, which makes these constructions particularly beautiful. A relationship between reduced invariants and invariants from maps with cusps was established in [BCM20]. Battistella and Carocci introduce a compactification of genus two maps to projective spaces [BC23]. An example of this compactification is given in [BC22].

More recently, reduced invariants for the quintic threefold have been compared to Gromov–Witten invariants using algebro-geometric methods by Chang and Li [CL15]. Chang–Li *define* reduced invariants as the integral against the top Chern class class of a sheaf but, as discussed above, this gives the same reduced invariants as [Zin09b]. The algebraic comparison relies on the construction of maps with fields due to Chang and Li [CL12], and on Kiem–Li's cosection localised virtual class [KL13]. This method has been employed in [LO22, LO21] to extend the genus one relation between absolute and reduced Gromov–Witten invariants of complete intersections. In genus two a similar work is done in [LLO22].

Zinger has computed reduced genus one invariants of projective hypersurfaces via localisation [Zin09a]. The computations in [Zin09c] and [Zin08] have been extended to complete intersections by Popa [Pop13].

Outline of the paper. In the following we give an outline of the paper and we highlight the main results.

In Section 2 we fix notation and briefly recall the background notions used, such as Fitting ideals and abelian cones.

In Section 3 we introduce the desingularization of a sheaf on a stack (Section 3.1) and we review the minimal desingularization, due to Rossi (Section 3.2). In Section 3.3 we give an algebraic desingularization in terms of Fitting ideals in the affine case, due to Villamayor. We show in Section 3.4 that the Rossi and Villamayor constructions agree. We show that the minimal desingularization of a sheaf has a universal property (see Theorem 3.3.4).

In Section 4 we introduce the notions of diagonal sheaf and of diagonalization of a sheaf on a scheme (see Definition 4.1.3). Construction 4.1.10, due to Hu and Li, gives the minimal diagonalization of a sheaf (see Theorem 4.1.15). In Theorem 4.1.14 we construct a filtration of a diagonal sheaf under certain conditions. The filtration will then be used in Section 5 to describe the irreducible components of the abelian cone of a diagonal sheaf. In Section 4.2, we collect properties of the Hu–Li blow-up, such as the existence of a morphism from the Hu–Li blow-up to the Rossi blow-up in Proposition 4.2.6. The two blow-ups are not isomorphic in general as shown in Example 4.3.1.

In Section 5 we study the irreducible components of the abelian cone $C(\mathfrak{F})$ of a diagonal sheaf. First, we show that $C(\mathfrak{F})$ is the pushout of its main component $C(\mathfrak{F}^{\mathrm{tf}})$ and a sheaf of generic rank 0. Then we deal with the remaining components, concluding with Theorem 5.2.1.

In Section 6 we generalize the Hu–Li and Rossi blow-ups to Artin stacks. These are constructed in Section 6.1, by first applying the Rossi and Hu–Li constructions for schemes to an atlas and then gluing. This works because the Hu–Li and Rossi constructions are local, they have a universal property and they commute with flat base-change by Propositions 3.5.3 and 4.2.1. We extend to stacks the results on the schematic version of these blowups, such as the universal properties of desingularization in Theorem 6.2.4 and of diagonalization in Theorem 6.2.7.

In Section 7 we use the notion of desingularization of a sheaf on a stack to define reduced Gromov Witten invariants in all genera: see Definition 7.3.10. In Section 7.1 we recall how $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ can be naturally embedded as an open substack in an abelian cone over $\mathfrak{Pic}$. In Section 7.2 we recall maps with fields and their moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$; this is an abelian cone over $\mathfrak{Pic}$. In Theorem 7.3.4 we show a refinement of Theorem 1.0.4 for maps with fields: after base changing

the moduli space of maps with fields, we obtain $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$ whose irreducible components are vector bundles over their support in $\mathfrak{Pic}$. In Proposition 7.3.9 we show that reduced invariants are independent of the birational model $\widetilde{\mathfrak{Pic}}$. In view of Proposition 7.3.9 many choices of $\widetilde{\mathfrak{Pic}}$ are possible. We choose the one which produces the simplest geometry of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$ and thus Theorem 7.3.4 explains the choices in Definition 7.3.10. We also use maps with fields to give an alternative definition of reduced Gromov–Witten invariants: see Proposition 7.3.13

In Section 8 we compare the moduli spaces obtained from the Rossi desingularization and the Vakil–Zinger blow-up. While reduced invariants are independent of the birational model of $\mathfrak{Pic}$, the induced moduli spaces can be different. We study charts of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ and we show that Rossi construction in genus one is different from the Vakil–Zinger blow-up.

How to read this paper. Sections 2–6 are self-contained and of independent interest. The schematic version of the results in Section 6 are explained in Sections 2–5. The reader interested in reduced Gromov–Witten invariants can take the results in Section 6 for granted and read Section 7 and Section 8 directly.

Further work. Our desingularizations do not come with a modular interpretation. It would be nice to have a modular interpretation of the resulting stack $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$, either in the spirit of [HL10, HLN12, HN19, HN20], or a log interpretation as in [RSPW19a]. It would be perhaps better to have a space of maps with more singular domains as in [BCM20, BC23].

While a modular interpretation would be very interesting from a theoretical point of view, higher genus computations as done by Zinger in [Zin09a] are likely to be hard. The genus two blow-up $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ already involves several rounds of blowups and a localisation computation would inherit the complexity of the blow-up. We hope that our construction sheds new light on this beautiful problem and will encourage more mathematicians to work on it.

On the positive side, we expect this construction to be enough for proving Conjecture 1.0.3. The main difference with [HL11] is that we blow up $\mathfrak{Pic}$, instead of blowing up $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$. The advantage of blowing up $\mathfrak{Pic}$ is that now we have the ingredients used by Chang–Li, Lee–Oh and Lee–Li–Oh to prove Conjecture 1.0.3 (and therefore Conjecture 1.0.2) in genus one and two. More precisely, we have fairly simple moduli spaces of maps with fields over $\widetilde{\mathfrak{Pic}}$, and these can be used to split the virtual class on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$. We hope to be able to prove Conjecture 1.0.3 without having explicit equations of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$, or a modular interpretation of $\widetilde{\mathfrak{Pic}}$. We will address this problem in future work.

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2. Background

In this section we recall several basic constructions and fix the notation used throughout the paper.

2.1. **The relative Grassmannian.** Let X be a scheme with a fixed quasi-coherent sheaf $\mathcal{E}$. The Grassmannian functor $\underline{\mathrm{Gr}}_X(\mathcal{E}, r) : ((\mathrm{Sch})/X)^{\mathrm{op}} \to (\mathrm{Set})$ is given on objects by

$$T \mapsto \{\mathcal{E}_T \twoheadrightarrow \mathcal{Q} \mid \mathcal{Q} \text{ is locally free of rank r } \} \tag{2}$$

with $\mathcal{E}_T \coloneqq \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_T$.

This functor is represented by a scheme $\mathrm{Gr}_X(\mathcal{E}, r)$ over X, which is projective if $\mathcal{E}$ is finitely generated. Moreover, the Grassmannian functor is compatible with base-change. In particular

$$\mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \cong \mathrm{Gr}(n, r) \times X$$

where $\mathrm{Gr}(n, r)$ is the usual Grassmannian of r-dimensional subspaces of $\mathbb{C}^n$ relative to a point. Since it represents a functor, the relative Grassmannian $\mathrm{Gr}_X(\mathcal{E}, r)$ comes with a universal sheaf and a universal quotient sheaf, which is locally free of rank r:

$$\mathcal{E}_{\mathrm{Gr}_X(\mathcal{E},r)} \twoheadrightarrow \mathcal{Q}_{\mathrm{Gr}_X(\mathcal{E},r)}.$$

As in the classical case, the relative Grassmannian admits a The Plücker embedding:

$$\lambda_{n,r} : \mathrm{Gr}_X(\mathcal{O}^{\oplus n}, r) \to \mathrm{Gr}_X\left(\bigwedge^r \mathcal{O}^{\oplus n}, 1\right) \cong \mathbb{P}_X^{m-1},$$

with $m = \binom{n}{r}$. For the last isomorphism, consider an X-scheme T. A point of $\mathrm{Gr}_X(\mathcal{O}^{\oplus m}, 1)(T)$ is a surjection

$$\mathcal{O}_T^{\oplus m} \twoheadrightarrow \mathcal{L}$$

with $\mathcal{L}$ a line bundle on T. This is a pair of a line bundle and an m-tuple of generating sections, which is an object of $\mathbb{P}_X^{m-1}(T)$.

When $\mathcal{E}$ is not trivial, we have

$$\lambda_{n,r} : \mathrm{Gr}_X(\mathcal{E}, r) \to \mathrm{Gr}_X\left(\bigwedge^r \mathcal{E}, r\right) \cong \mathbb{P}_X\left(\bigwedge^r \mathcal{E}^\vee\right).$$

2.2. **Fitting ideals.**

Definition 2.2.1. Let M be a finitely presented R-module. Let $F \xrightarrow{\varphi} G \to M \to 0$ be a presentation with F and G free modules and $\mathrm{rk}(G) = r$. Given $-1 \leq i < \infty$, the i-th Fitting ideal $F_i(M)$ of M is the ideal generated by all $(r-i)\times(r-i)$-minors of the matrix associated to φ after fixing basis of F and G. We use the convention that $F_i(M) = R$ if $r - i \leq 0$ and $F_{-1}(M) = 0$.

Intrinsically, $F_i(M)$ is the image of the map $\bigwedge^{r-i} F \otimes \bigwedge^{r-i} G^* \to R$ induced by $\bigwedge^{r-i} \varphi\colon \bigwedge^{r-i} F \to \bigwedge^{r-i} G$. The i-th Fitting ideal is well-defined in that it does not depend on the chosen presentation. Since determinants can be computed expanding by rows and columns, it follows that there are inclusions

$$0 = F_{-1}(M) \subset F_0(M) \subset F_1(M) \subset \ldots \subset F_k(M) \subset F_{k+1}(M) \subset \ldots$$

It follows from the definition and right-exactness of tensor product that Fitting ideals commute with base change. That is, given $R \to S$ ring homomorphism and M a finitely presented R-module, then

$$F_i(M \otimes_R S) = F_i(M) \cdot S.$$

Similarly, for a scheme X and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module of finite presentation, we have ideal sheaves

$$0 = F_{-1}(\mathcal{F}) \subset F_0(\mathcal{F}) \subset \cdots \subset F_n(\mathcal{F}) \subset \cdots \subset \mathcal{O}_X$$

which can be defined locally as described above. For $f : Y \to X$ a morphism of schemes, we have

$$f^{-1}F_i(\mathcal{F}) \cdot \mathcal{O}_Y = (f^*(\mathcal{F}))^{\mathrm{tf}} = F_i(f^*\mathcal{F}).$$

Fitting ideals describe the locus on X where the sheaf $\mathcal{F}$ is locally free of some rank. More precisely, we recall the following standard result (see for example [Sta22, Tag 05P8]).

Proposition 2.2.2. *For any n, the sheaf $\mathcal{F}$ is locally free of rank n on the locally closed subscheme $V(F_{n-1}(\mathcal{F})) \setminus V(F_n(\mathcal{F}))$ of X.*

2.3. **Abelian cones.** Let X be a Noetherian scheme.

Definition 2.3.1. A cone over X is a scheme $\pi\colon A \to X$ with a $\mathbb{G}_m$-action such that $\pi_*\mathcal{O}_A = \bigoplus_{d\geq 0} \mathcal{A}_d$ with

(1) $\mathcal{O}_X \to \mathcal{A}_0$ isomorphism,
(2) $\mathcal{A}_1$ coherent and
(3) $\operatorname{Sym} \mathcal{A}_1 \to \mathcal{A}$ surjective.

If, moreover, $\operatorname{Sym} \mathcal{A}_1 \to \mathcal{A}$ is an isomorphism then we say that $\pi\colon A \to X$ is an abelian cone.

Therefore, every cone over X is of the form $\operatorname{Spec}_X(\mathcal{A})$ for some graded sheaf of $\mathcal{O}_X$-algebras $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ with $\mathcal{A}_1$ a coherent $\mathcal{O}_X$-module and $\mathcal{A}_0 \simeq \mathcal{O}_X$. Here Spec_X denotes the relative spectrum over X. The cone $\operatorname{Spec}_X(\mathcal{A})$ is abelian if and only if $\mathcal{A} \simeq \operatorname{Sym} \mathcal{A}_1$. We denote by $C_X(\mathcal{F}) = \operatorname{Spec}_X(\operatorname{Sym}(\mathcal{F}))$ the abelian cone associated to any coherent sheaf $\mathcal{F}$. We will omit the subscript X in the formation of relative spectra and cones whenever it is possible to do so without introducing ambiguity.

Lemma 2.3.2. *For a cone $\pi\colon C = \operatorname{Spec}(\mathcal{A}) \to X$, the following are equivalent:*

(1) *C is a vector bundle over X,*
(2) *C is abelian and $\mathcal{A}_1$ is locally free over X,*
(3) *π is flat,*
(4) *π is smooth.*

3. Desingularizations of coherent sheaves

In this section we introduce several constructions which "desingularize" a coherent sheaf $\mathcal{F}$ on a base scheme X.

3.1. **Definition of desingularizations.** We define our notion of desingularization and prove that it behaves well with composition. This part can be formulated directly for algebraic stacks instead of schemes, which will be useful later.

Definition 3.1.1. Let $\mathfrak{F}$ be a coherent sheaf on an algebraic stack $\mathfrak{X}$ with finitely many irreducible components. A *desingularization* of $\mathfrak{F}$ is a morphism $p : \tilde{\mathfrak{X}} \to \mathfrak{X}$ such that

(1) p is birational and proper
(2) $(p^*\mathfrak{F})^{\mathrm{tf}}$ is a locally free sheaf.

Remark 3.1.2. If $\mathfrak{X}$ is a scheme and $\mathfrak{F}$ is an ideal sheaf, the usual blow-up of X at the closed subscheme defined by $\mathfrak{F}$ is a desingularization of $\mathfrak{F}$.

Lemma 3.1.3. *Let $\mathfrak{X}$ be an integral algebraic stack and $\mathfrak{F}$ a coherent sheaf on X. Let $p : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ be a desingularization of $\mathfrak{F}$ and let $q : \mathfrak{Y} \to \tilde{\mathfrak{X}}$ be a proper birational morphism. Then, the composition $p \circ q : \mathfrak{Y} \to \mathfrak{X}$ is a desingularization of $\mathfrak{F}$.*

Proof. The composition $r := p \circ q$ is birational and proper, so all we need to prove is that $(r^*\mathfrak{F})^{\mathrm{tf}}$ s locally free. In the following wee show that

$$(r^*\mathfrak{F})^{\mathrm{tf}} \simeq q^*((p^*\mathfrak{F})^{\mathrm{tf}}).$$

We have a commutative diagram of sheaves on $\mathfrak{Y}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{K}_1 & \longrightarrow & q^*p^*\mathfrak{F} & \longrightarrow & q^*((p^*\mathfrak{F})^{\mathrm{tf}}) & \longrightarrow & 0 \\ & & \vdots & & \downarrow & & \vdots & & \\ 0 & \longrightarrow & \mathfrak{K}_2 & \longrightarrow & q^*p^*\mathfrak{F} & \longrightarrow & (q^*p^*\mathfrak{F})^{\mathrm{tf}} & \longrightarrow & 0 \end{array}$$

where the map $q^*p^*\mathfrak{F} \to q^*((p^*\mathfrak{F})^{\mathrm{tf}})$ is the pull-back of the surjective map

$$p^*\mathfrak{F} \to (p^*\mathfrak{F})^{\mathrm{tf}},$$

$\mathfrak{K}_1$ and $\mathfrak{K}_2$ are the corresponding kernels and the solid vertical map is the identity. Since the image of the composition $\mathfrak{K}_1 \to (q^*p^*\mathfrak{F})^{\mathrm{tf}}$ is generically zero and $\mathfrak{X}$ is irreducible, we have that the morphism $\mathfrak{K}_1 \to (q^*p^*\mathfrak{F})^{\mathrm{tf}}$ is zero. By the universal property of cokernels this map factors through $q^*((p^*\mathfrak{F})^{\mathrm{tf}})$, which gives the right vertical map in the diagram. The universal property of kernels gives the left vertical map in the diagram. We have that $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are torsion sheaves. Let $\mathfrak{K}_3$ be the cokernel of $\mathfrak{K}_1 \to \mathfrak{K}_2$. By the Snake Lemma, we have an exact sequence

$$0 \to \mathfrak{K}_3 \to q^*((p^*\mathfrak{F})^{\mathrm{tf}}) \to (r^*\mathfrak{F})^{\mathrm{tf}} \to 0.$$

By assumption $(p^*\mathfrak{F})^{\mathrm{tf}}$ is locally free, so $q^*((p^*\mathfrak{F})^{\mathrm{tf}})$ is locally free. Since $\mathfrak{X}$ is irreducible and $\mathfrak{K}_3$ is a torsion sheaf, we get that $\mathfrak{K}_3 = 0$. This proves the claim. □

3.2. **Rossi's construction for affine schemes.** In the following we describe the desingularization construction proposed by Rossi in [Ros68], which is very geometric in nature. It gives a way of desingularizing coherent sheaves on integral Noetherian schemes by blowing up a certain Fitting ideal. In the following we present Rossi's construction in the particular case of affine schemes; a much more general gluing will be presented in Section 6. Our gluing recovers Rossi's original construction for schemes.

We have seen in Lemma 3.1.3 that desingularizations are not unique. The following construction is the minimal one.

Let $X = \operatorname{Spec}(R)$ be an integral affine Noetherian scheme and $\mathcal{F} = \widetilde{M}$ be the coherent sheaf associated to a coherent R-module M. This is a finitely generated module such that the kernel of any morphism $R^{\oplus n} \to M$ is finitely generated. Over the unique generic point $U \subset X$, $\mathcal{F}|_U$ is locally-free of rank r. We can define the *generic rank* of $\mathcal{F}$:

$$\operatorname{rk}(\mathcal{F}) \coloneqq \operatorname{rk}(\mathcal{F}|_U) = r.$$

Since R is affine, we can find for some n a surjective morphism of sheaves

$$f : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{F}. \tag{3}$$

Restricted to U, f gives a U-point of $\operatorname{Gr}_U(\mathcal{O}_U^{\oplus n}, r)$, which is a morphism

$$\Gamma_f : U \to \operatorname{Gr}_U(\mathcal{O}_U^{\oplus n}, r)$$

such that the composition of Γ_f with the structure map $\operatorname{Gr}(\mathcal{O}_U^{\oplus n}, r) \to U$ is id_U. Moreover, under this isomorphism, the locally-free sheaf $\mathcal{F}|_U$ is identified with the restriction to $\Gamma_f(U)$ of the universal quotient sheaf $\mathcal{Q}_{\operatorname{Gr}_U(\mathcal{O}_U^{\oplus n}, r)}$. This is automatic from the description of the relative Grassmannian as the fine moduli scheme of the functor in (2).

By compatibility of the relative Grassmannian with base-change we have an open embedding $\operatorname{Gr}_U(\mathcal{O}_U^{\oplus n}, r) \subset \operatorname{Gr}_X(\mathcal{O}_X^{\oplus n}, r)$. We give the following definition.

Definition 3.2.1. The blow-up of X at the coherent sheaf $\mathcal{F}$ is

$$\operatorname{Bl}_{\mathcal{F}}(X) = \overline{\Gamma_f(U)}$$

the closure of $\Gamma_f(U)$ in $\operatorname{Gr}_X(\mathcal{O}_X^{\oplus n}, r)$ with the reduced induced structure.

Proposition 3.2.2. *[cf.* [Ros68, Theorem 3.5]*] The blow-up* $\operatorname{Bl}_{\mathcal{F}}(X)$ *above is well-defined. Moreover,*

$$p : \operatorname{Bl}_{\mathcal{F}}(X) \to X$$

is a projective birational morphism such that the torsion-free part of $p^*\mathcal{F}$*,*

$$(p^*\mathcal{F})^{\mathrm{tf}} = p^*\mathcal{F}/\operatorname{Tors}(p^*\mathcal{F})$$

is locally-free of rank r*. That is,* p *is a desingularization of* $\mathcal{F}$*.*

Proof. Observe that $p : \operatorname{Bl}_{\mathcal{F}}(X) \to X$ is a projective morphism, since $\operatorname{Gr}_X(\mathcal{O}_X^{\oplus n}, r)$ is a projective variety. It is also clear by construction that

$$\begin{array}{ccc} \Gamma_f(U) & \hookrightarrow & \operatorname{Bl}_{\mathcal{F}}(X) \\ \downarrow \cong & & \downarrow p \\ U & \hookrightarrow & X \end{array}$$

is Cartesian, in particular p is birational. The claim that $(p^*\mathcal{F})^{\mathrm{tf}}$ is locally-free of rank r follows from the fact that it is just the restriction of the universal quotient bundle on the Grassmannian:

$$(p^*\mathcal{F})^{\mathrm{tf}} = \mathcal{Q}_{\mathrm{Gr}_X(\mathcal{O}_X^{\oplus n},r)}|_{\mathrm{Bl}_{\mathcal{F}}(X)}.$$

To check that the above is well-defined, we need to check that it is independent of the choice of f, n in equation (3). Indeed, suppose we have another surjection

$$f' : \mathcal{O}_X^{\oplus n'} \twoheadrightarrow \mathcal{F}.$$

We obtain

$$\Gamma_{f'} : U \to \mathrm{Gr}(\mathcal{O}_U^{\oplus n'}, r) \subset \mathrm{Gr}(\mathcal{O}_X^{\oplus n'}, r).$$

We can form

$$g = (f, f') : \mathcal{O}_X^{\oplus (n+n')} \twoheadrightarrow \mathcal{F}.$$

Projecting onto the first n coordinates by h gives a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X^{\oplus(n+n')} & \xrightarrow{g} & \mathcal{F} \\ \downarrow h & \nearrow f & \\ \mathcal{O}_X^{\oplus n} & & \end{array}.$$

The morphism h induces a morphism $\widetilde{h}$ of Grassmannians such that the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{\Gamma_f} & \mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \\ & \searrow^{\Gamma_g} & \downarrow \widetilde{h} \\ & & \mathrm{Gr}_X(\mathcal{O}_X^{\oplus(n+n')}, r). \end{array}$$

Moreover, $\tilde{h}$ is an isomorphism onto its image. Then $\tilde{h}$ induces an isomorphism on the closures of $\Gamma_f(U)$ and $\Gamma_g(U)$. Similarly, we have an isomorphism between the closures of $\Gamma_{f'}(U)$ and $\Gamma_g(U)$. So we obtain an isomorphism $\overline{\Gamma_f(U)} \cong \overline{\Gamma_{f'}(U)}$ as required. □

Proposition 3.2.3. *Let $\mathcal{L}$ be a line bundle on X. Then we have a unique isomorphism*

$$\begin{array}{ccc} \mathrm{Bl}_{\mathcal{F}}(X) & \xrightarrow{\exists!\tilde{\phi}} & \mathrm{Bl}_{\mathcal{F}\otimes\mathcal{L}}(X) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

which makes the diagram commute.

Proof. Let $f : \mathcal{O}_X^{\oplus n} \to \mathcal{F}$ be a surjective morphism, let S denote the kernel of f and let U an open subset of X such that S is a vector bundle. We thus obtain a short exact sequence

$$0 \to S \otimes \mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \otimes \mathcal{L}|_U \overset{f\otimes id}{\to} \mathcal{F} \otimes \mathcal{L}|_U \to 0.$$

By possibly shrinking U we may assume we have an isomorphism $g : \mathcal{L}|_U \simeq \mathcal{O}_U$. We thus obtain a commutative diagram

$$\begin{array}{ccc} \Gamma_f & \longrightarrow & \mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \\ \downarrow & & \downarrow \\ \Gamma_{f\otimes id} & \longrightarrow & \mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \end{array}$$

where the right vertical arrow is induced by g. This guves a morphism $\tilde{\phi} : \overline{\Gamma_f} \to \overline{\Gamma_{f\otimes id}}$ and proves the claim. □

Example 3.2.4. The following example is simple, but it captures much of the essence of Rossi's construction. We present it with full details.

Let $R = k[x, y]$ and let $I = (x, y)$ be the ideal of the origin. The sheaf $\mathcal{F} = \widetilde{I}$ on $\mathbb{A}^2 = \mathrm{Spec}\,(R)$ is torsion-free but not locally free. Since $\mathcal{F}$ is an ideal sheaf, $\mathrm{Bl}_{\mathcal{F}}\mathbb{A}^2 = \mathrm{Bl}_0\mathbb{A}^2$ is just the usual blow up of $\mathbb{A}^2$ along the origin. Let $p\colon \mathrm{Bl}_0\mathbb{A}^2 \to \mathbb{A}^2$ be the natural projection. Then $p^*\mathcal{F}$ is not torsion-free, but $(p^*\mathcal{F})^{\mathrm{tf}}$ is a line bundle. To see that, we start with the following resolution of $\mathcal{F}$.

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R \oplus R \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} I \longrightarrow 0.$$

Pulling back along p, we obtain a presentation of $p^*\mathcal{F}$:

$$\mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \xrightarrow{\begin{pmatrix} -ey' \\ ex' \end{pmatrix}} \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \oplus \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \xrightarrow{\begin{pmatrix} ex' & ey' \end{pmatrix}} p^*\mathcal{F} \longrightarrow 0.$$

Here e is a local coordinate for the exceptional divisor $E \subseteq \mathrm{Bl}_0\mathbb{A}^2$ and x' and y' correspond to the strict transforms of x and y. This induces a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & \longrightarrow & \mathrm{Coker}(e) & & \\ & & & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} & \xrightarrow{\cdot e} & \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2}(E) & \longrightarrow & \mathrm{Coker}(e) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \begin{pmatrix} -ey' \\ ex' \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} y' \\ -x' \end{pmatrix}} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \oplus \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} & \xrightarrow{\mathrm{id}} & \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \oplus \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \begin{pmatrix} x'e & y'e \end{pmatrix}} & & \downarrow & & \downarrow & & \\ & & p^*\mathcal{F} & \longrightarrow & \mathrm{Coker}(y', -x')^t & \longrightarrow & 0 & & \end{array}$$

Applying the snake lemma and using that $\mathrm{Coker}(e) \simeq \mathcal{O}_E(E)$ and that $\mathrm{Coker}(y', -x')$ is the ideal sheaf generated by x' and y', we get a short exact sequence

$$0 \to \mathcal{O}_E(E) \to p^*\mathcal{F} \to (x', y') \to 0.$$

It follows that $\mathrm{tor}(p^*\mathcal{F}) \simeq \mathcal{O}_E(E)$ and $(p^*\mathcal{F})^{\mathrm{tf}} \simeq (x', y')$. In particular, $p^*\mathcal{F}$ is not torsion-free.

We can also describe the geometry of the abelian cones $\mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F}$ and $\mathrm{Spec}\,\mathrm{Sym}\,(p^*\mathcal{F})$. We have

$$\pi_{\mathcal{F}}\colon \mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F} = \mathrm{Spec}\,(R[X, Y]/(xY - yX)) \to \mathbb{A}^2,$$

which is irreducible and singular.

Next we describe $\operatorname{Spec}\operatorname{Sym}(p^*\mathcal{F})$. Let $S = k[x, y, x', y']/(xy' - yx')$ where the variables x', y' are have degree 1 and x, y have degree 0. Then

$$\mathrm{Bl}_0\mathbb{A}^2 = \operatorname{Proj}(k[x, y, x', y']/(xy' - yx')).$$

A local equation for E is given by $e = x/x'$ or $e = y/y'$, depending on the chosen chart. Then

$$\pi_{p^*\mathcal{F}} \colon \operatorname{Spec}\operatorname{Sym}(p^*\mathcal{F}) = \operatorname{Spec} S[X, Y]/(e(x'Y - y'X)) \to \mathrm{Bl}_0\mathbb{A}^2$$

is reducible. It has two components:

$$\pi_{p^*\mathcal{F},\mathrm{main}} \colon C_{\mathrm{main}} = V(x'Y - y'X) \to \mathrm{Bl}_0\mathbb{A}^2,$$
$$\pi_{p^*\mathcal{F},\mathrm{tor}} \colon C_{\mathrm{tor}} = V(e) \to \mathrm{Bl}_0\mathbb{A}^2.$$

The main component C_{main} equals $\operatorname{Spec}\operatorname{Sym}(p^*\mathcal{F})^{\mathrm{tf}}$ and it is a vector bundle of rank 1. Meanwhile, C_{tor} corresponds to $\operatorname{tor}(\mathcal{F})$, it is supported over E and it is a vector bundle of rank 2 over its support.

3.3. **Villamayor's construction.** We now review an algebraic construction of Villamayor. In Section 3.4 we show that this is equivalent to Rossi's construction.

Fitting ideals (see Section 2.2) are related to ranks of modules and flatness. Indeed, the local rank of M at a prime ideal P of R is k if and only if $F_{-1}(M) \subseteq F_0(M) \subseteq \ldots \subseteq F_{k-1}(M) \subseteq P$ but $F_k(M) \not\subset P$. As a corollary, if R is a domain then the generic rank of M is k if and only if $F_k(M)$ is the first non-zero Fitting ideal, and moreover M is flat if and only if it is free, if and only if $F_k(M) = R$.

The relationship between Fitting ideals and local freeness of the torsion-free part comes from Lipman's theorem.

Theorem 3.3.1 (cf. [Lip69, Lemma 1])**.** *Let R be a local ring. Given a finitely presented module M and a non-negative integer r, the following are equivalent:*

1. *$F_0(M) = \ldots = F_{r-1}(M) = 0$ and $F_r(M)$ is invertible.*
2. *M has projective dimension at most one and $M/\operatorname{tor}(M)$ is free of rank r.*

It follows from theorem 3.3.1 that blowing up the first non-trivial Fitting ideal of M will make $(p^*M)^{\mathrm{tf}}$ locally free (with p the blow-up morphism). However, it is possible that M^{tf} is already locally free on $\operatorname{Spec} R$ even though its first non-trivial Fitting ideal is not principal, see remark 3.3.5. In order to find a minimal transformation of $\operatorname{Spec}(R)$ on which M^{tf} is locally free, Villamayor proposes in [Vil06] the following construction.

Construction 3.3.2 ([Vil06, Remark 2.1])**.** Let R be a domain and let M be a finitely presented R-module or rank r. Choose generators $m_1, \ldots, m_N$ for M. Then there is a short exact sequence

$$0 \to P \to R^N \to M \to 0.$$

Since M has rank r, there are elements $p_1, \ldots, p_{N-r}$ in P which induce a morphism $R^{N-r} \to R^N$ of rank $N - r$. Let $P_1 \simeq R^{N-r}$ be the free module generated by $p_1, \ldots, p_{N-r}$ and let $M_1 = R^N/P_1$, that is, the following is exact

$$0 \to P_1 \to R^N \to M_1 \to 0.$$

Then M_1 has projective dimension at most 1, $\operatorname{rk}(M_1) = \operatorname{rk}(M)$, there is a natural surjection $M_1 \to M$ and $M_1/\operatorname{tor}(M) = M/\operatorname{tor}(M)$.

Given a finitely presented module M of generic rank r over a domain R with fraction field K, define its norm to be the fractional ideal

$$[\![M]\!] = \mathrm{Im}(\bigwedge^r M \to K \simeq \bigwedge^r M \otimes_R K). \tag{4}$$

Then $F_r(M_1)$ and $[\![M]\!]$ are isomorphic fractional ideals over R by [Vil06, Proposition 2.5], in particular the isomorphism class of $F_r(M_1)$ as a fractional ideal does not depend on the choices made in construction 3.3.2. Note that any fractional ideal of R is isomorphic to an ideal of R and that given two ideals I, J which are isomorphic as fractional ideals, the blow-ups of $\mathrm{Spec}\,(R)$ along I and J are naturally isomorphic.

The same ideas apply to any (Noetherian) ring R if we restrict to finitely presented R-modules M such that $M \otimes_R Q(R)$ is a free $Q(R)$-module, where $Q(R)$ is the total quotient ring of R.

Definition 3.3.3. Let R be Noetherian. An R-module M is generically flat of generic rank r if $M \otimes_R Q(R) \simeq Q(R)^r$.

Theorem 3.3.4. *[Universal property of blow-up, affine case,* [Vil06, Theorem 3.3]*] Let R be a Noetherian ring, let M be a finitely presented R-module generically flat of generic rank r (in the sense of definition 3.3.3) and let $X = \mathrm{Spec}\,(R)$. There is a blow-up $p\colon \mathrm{Bl}_M X \to X$ satisfying the following universal property:*

1. *The sheaf $p^*M/\mathrm{tor}(p^*M)$ is locally free of rank r on $\mathrm{Bl}_M X$ and*
2. *for any morphism $q\colon Y \to X$ from an affine scheme Y for which $q^*M/\mathrm{tor}(q^*M)$ is locally free of rank r on Y, there is a unique morphism $q'\colon Y \to \mathrm{Bl}_M X$ such that $p \circ q' = q$.*

Proof. The fractional ideal $[\![M]\!]$ is isomorphic to some ideal I of R. Let $\mathrm{Bl}_M X = \mathrm{Bl}_I X$ be the blowup of X along I, which is independent of the choice of I. Since $q^*M/\mathrm{tor}(q^*M) = q^*M_1/\mathrm{tor}(q^*M_1)$ for any morphism $q\colon Y \to X$, we can replace M by M_1, which has projective dimension at most one. Conclude by theorem 3.3.1 and the universal property of (the usual) blow-up. □

Remark 3.3.5. In general, $\mathrm{Bl}_M X$ is not obtained by blowing up the first non-zero Fitting ideal of M. Indeed, let $I \subset R$ be an ideal and $M = R/I$. On the one hand, $M^{\mathrm{tf}} = 0$ is locally free of rank 0, which is the rank of M, so $\mathrm{Bl}_M X = X$. On the other hand, the first non-zero Fitting ideal of M is $F_0(M) = I$, so $\mathrm{Bl}_{F_0(M)} X = \mathrm{Bl}_M X$ if and only if I is principal.

However, if M has generic rank r and projective dimension ≤ 1, then $\mathrm{Bl}_M X = \mathrm{Bl}_{F_r(M)} X$.

3.4. **Equivalence of the two constructions.** We find it useful here to give a constructive comparison of the two approaches, which we find clarifies both.

Proposition 3.4.1. *Equivalence of Rossi–Villamayor constructions Let $X = Spec(R)$ be an affine, integral Noetherian scheme and $\mathcal{F} = \widetilde{M}$ be a coherent sheaf:*

$$\mathrm{Bl}_{\mathcal{F}} X = \mathrm{Bl}_M X.$$

The first is defined in definition 3.2.1 and the second in theorem 3.3.4.

Remark 3.4.2. Note that both constructions involve some choice, so they only define a blow-up up to isomorphism. The choice made in both constructions is that of an ordered finite generating set for M. We will show that different choices produce canonically isomorphic constructions.

The first observation, is that both constructions only depend on $\bigwedge^r M$, where r is the generic rank. We suppose here that $r \neq 0$. Otherwise, the sheaf is already locally free up to torsion, so no blow-ups are necessary.

Lemma 3.4.3. *Let X, M as above and $r = \mathrm{rk}M > 0$, we have canonical isomorphisms*

$$\mathrm{Bl}_{\mathcal{F}}X = \mathrm{Bl}_{\bigwedge^r \mathcal{F}}X \tag{5}$$

and

$$\mathrm{Bl}_{M}X = \mathrm{Bl}_{\bigwedge^r M}X. \tag{6}$$

Proof. For (5), fix a surjection $f : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{F}$. Then $\wedge^r f : \bigwedge^r \mathcal{O}_X^{\oplus n} \twoheadrightarrow \bigwedge^r \mathcal{F}$ defines an embedding

$$\Gamma_{\wedge^r f} : U \to \mathbb{P}^{m-1}_U \subset \mathbb{P}^{m-1}_X$$

of the generic open, where $m = \binom{n}{r}$. With the definition of the Plücker embedding from section 2.1, we have the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\Gamma_f} & \mathrm{Gr}_X(\mathcal{O}^{\oplus n}, r) \\ & \searrow^{\Gamma_{\wedge^r f}} & \downarrow{\scriptstyle \lambda_{n,r}} \\ & & \mathbb{P}^{m-1}_X. \end{array}$$

Since $\lambda_{n,r}$ is a closed embedding, it commutes with taking closures. Hence, it gives the required isomorphism.

For (6), recall the definition of the norm (4):

$$[\![M]\!] = \mathrm{Im}(\bigwedge^r M \to \bigwedge^r M \otimes_R K \cong K).$$

From this, it is clear that the Villamayor blow-up only depends on the top exterior power of M. Given a generating set $f_1, \dots f_n$ of M, the choice of the isomorphism $\bigwedge^r M \otimes_R K \cong K$ is given by picking a non-zero element of $\bigwedge^r M \otimes_R K$, which we can set as $f_1 \wedge \cdots \wedge f_r$. To obtain an ideal of R isomorphic to the fractional ideal $[\![M]\!]$, we clear denominators of the image in K of the other r-wedges of the generators by $b \in R$. This gives

$$\mathrm{Bl}_M X = \mathrm{Bl}_{b[\![M]\!]}X \subset \mathbb{P}^{m-1}_X.$$

Other choices of b will give a canonically isomorphic blow-up. □

So we reduced to $\mathcal{F}$ of rank 1. We can further reduce to the case of ideal sheaves.

Lemma 3.4.4. *Let M be a rank 1 module over R, then we can always choose an ideal I such that*

$$\mathrm{Bl}_M X = \mathrm{Bl}_I X$$

Proof. Choose $I \cong [\![M]\!]$. Or, to spell this out, we can see that the image of M in $M \otimes_R K(R)$ is a torsion-free rank 1 R-submodule of $K(R)$. Since it is finitely generated, we can clear denominators and obtain an ideal $I \subset R$ isomorphic to this image. □

For the Rossi construction, we reduce from rank 1 sheaves to ideal sheaves using the following lemma.

Lemma 3.4.5. *Let $\mathcal{F}$ and X as in proposition 3.4.1. If $\mathcal{F}$ has rank 1, then there is an ideal sheaf $\mathcal{I}$ on X such that*

$$\mathrm{Bl}_{\mathcal{F}}X = \mathrm{Bl}_{\mathcal{I}}X.$$

Proof. Let $\mathcal{K} = \widetilde{K(R)}$. Since $\mathrm{Bl}_{\mathcal{F}}X$ only depends on $\mathcal{F}\mid_U$ and since $\mathcal{F}$ and $\mathcal{F}^{\mathrm{tf}}$ are isomorphic over U, we can assume that $\mathcal{F}$ is torsion-free. In that case, arguing as in the proof of lemma 3.4.4, we can replace $\mathcal{F}$ by its image in $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{K}\simeq\mathcal{K}$ and we can find a section g of $\mathcal{K}$ such that $g\cdot\mathcal{F}\simeq\mathcal{I}$ is an ideal sheaf.

Remember that $\mathrm{Bl}_{\mathcal{F}}X = \overline{\Gamma_g(U)}$ where $\Gamma_g : U \to \mathbb{P}^{n-1}_U$ is induced by a surjection $f : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{F}$. Since g determines a section of $\mathcal{O}_X$ over U, we have another surjection

$$g\mid_U \cdot f\mid_U : \mathcal{O}_U^{\oplus n} \twoheadrightarrow \mathcal{I}.$$

By construction, $\Gamma_{g\cdot f} = \Gamma_f$, therefore $\mathrm{Bl}_{\mathcal{F}}X = \mathrm{Bl}_{\mathcal{I}}X$ follows. □

Lemma 3.4.6. *Let X an affine integral and noetherian scheme and let $\mathcal{F} = \widetilde{I}$ be an ideal sheaf. Then*

$$Bl_{\mathcal{F}}X = Bl_I X.$$

Proof. The right-hand side is the Villamayor blow-up of $[\![I]\!]$. As fractional ideals $[\![I]\!] = I$, so $Bl_I X$ is the usual blow-up of X along the ideal I. The left-hand side is the closure of the graph $\Gamma_f : U \to \mathbb{P}^{n-1}_U$ inside $\mathbb{P}^{n-1}_X$ where $f : \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$ is a choice of generators of the ideal sheaf $\mathcal{F}$ and U is the generic point of X. The lemma follows from the standard fact that the usual blow-up can be obtained as closure of a graph in projective space, see [EH99, Proposition IV-22]. □

3.5. **Properties.** We collect some properties of the blow-up $\mathrm{Bl}_{\mathcal{F}}X$ constructed in proposition 3.2.2. Next result is reformulation of theorem 3.3.4 in geometric terms.

Theorem 3.5.1 (Universal property of blowing up: affine version)**.** *Let X be an affine Noetherian integral scheme and $\mathcal{F}$ be a coherent sheaf of generic rank r. The blow-up $p : \mathrm{Bl}_{\mathcal{F}}X \to X$ from definition 3.2.1 satisfies*

1. *$(p^*\mathcal{F})^{\mathrm{tf}} = p^*\mathcal{F}/\mathrm{tor}(p^*\mathcal{F})$ is locally free of rank r on $\mathrm{Bl}_{\mathcal{F}}X$ and*
2. *for any morphism $f : Y \to X$ from an affine scheme Y for which $f^*M/\mathrm{tor}(f^*M)$ is locally free of rank r on Y, there is a unique morphism $f' : Y \to \mathrm{Bl}_{\mathcal{F}}X$ such that $p\circ f' = f$.*

$$\begin{array}{ccc} Y & \overset{\exists! f'}{\dashrightarrow} & \mathrm{Bl}_{\mathcal{F}}(X) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

The construction of section 3.2 glues to a definition of $\mathrm{Bl}_{\mathcal{F}}(X)$ on a reduced, irreducible, locally Noetherian scheme X. Moreover, $\mathrm{Bl}_{\mathcal{F}}X$ inherits the universal property in theorem 3.5.1, since in view of the uniqueness the question is local on X.

Corollary 3.5.2 (Universal property of blowing up)**.** *Let X be a Noetherian integral scheme and $\mathcal{F}$ be a coherent sheaf of generic rank r. There exists a blow-up $p : \mathrm{Bl}_{\mathcal{F}}X \to X$ which satisfies*

1. *The sheaf $(p^*\mathcal{F})^{\mathrm{tf}} = p^*\mathcal{F}/\mathrm{tor}(p^*\mathcal{F})$ is locally free of rank r on $\mathrm{Bl}_{\mathcal{F}}X$ and*

(2) *for any morphism $f\colon Y \to X$ from an scheme Y for which $f^*M/\mathrm{tor}(f^*M)$ is locally free of rank r on Y, there is a unique morphism $f'\colon Y \to \mathrm{Bl}_{\mathcal{F}}X$ such that $p \circ f' = f$.*

$$\begin{array}{ccc} Y & \overset{\exists! f'}{\dashrightarrow} & \mathrm{Bl}_{\mathcal{F}}(X) \\ & \searrow^{f} & \downarrow p \\ & & X \end{array}$$

Proposition 3.5.3. *Let $f : Y \to X$ be a morphism of Noetherian integral schemes and let $\mathcal{F}$ be a coherent sheaf on X of generic rank r. If $f^*\mathcal{F}$ has generic rank r then there is a unique morphism*

$$\begin{array}{ccc} \mathrm{Bl}_{f^*\mathcal{F}}(Y) & \overset{\exists! \widetilde{f}}{\dashrightarrow} & \mathrm{Bl}_{\mathcal{F}}(X) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

making the diagram commute. If, moreover, f is flat, then the square is Cartesian.

Proof. A unique morphism $\widetilde{f}$ making the diagram commute exists by the universal property of $\mathrm{Bl}_{\mathcal{F}}(X)$, which is corollary 3.5.2. To show that the diagram is Cartesian, we use [Sta22, Lemma 0805], which is the analogous result for blow-ups along ideal sheaves. This requires checking that $f^{-1}[\![\mathcal{F}]\!] \cdot \mathcal{O}_Y = [\![f^*\mathcal{F}]\!]$, which holds since $\mathcal{F}$ and $f^*\mathcal{F}$ have the same rank and the norm $[\![\cdot]\!]$ is a determinantal ideal.

Indeed, we can work locally. Then we have $X = \mathrm{Spec}\,(A)$, $Y = \mathrm{Spec}\,(B)$, a ring homomorphism $f^{\#}\colon A \to B$ and $\mathcal{F} = \widetilde{M}$ for some finitely presented A-module M. To compute $[\![\mathcal{F}]\!]$, we take a presentation

$$A^m \xrightarrow{\Gamma} A^n \longrightarrow M \longrightarrow 0,$$

we choose a submatrix Γ' of Γ consisting of $n-r$ columns of Γ and then $[\![\mathcal{F}]\!]$ is represented by the ideal generated by all the minors $\Delta_i(\Gamma')$ of size $(n-r)\times(n-r)$ of Γ'. The choice of Γ' must be so that this ideal is non-zero and such a choice exists because $\mathrm{rk}(\mathcal{F}) = r$. Then $f^{-1}[\![\mathcal{F}]\!] \cdot B$ is the ideal in B generated by $f^{\#}(\Delta_i(\Gamma'))$ for all i. On the other hand, tensoring by $\otimes_A B$ we get a presentation

$$B^m \xrightarrow{f^*\Gamma} B^n \longrightarrow M \otimes_A B \longrightarrow 0.$$

Since $f^*\mathcal{F} = \widetilde{M \otimes_A B}$ and since $\mathrm{rk}(f^*\mathcal{F}) = r$, we can compute $[\![f^*\mathcal{F}]\!]$ in the same manner, i.e., taking all the minors of size $(n-r)\times(n-r)$ of a submatrix $(f^*\Gamma)''$ consisting of $n-r$ columns of $f^*\Gamma$. This means that $[\![f^*\mathcal{F}]\!]$ is generated by $\Delta_i((f^{\#}\Gamma)'')$. We can actually choose Γ' and $(f^*\Gamma)''$ so that they consist of the same columns, and in that case we are done because $f^{\#}$ is a ring homomorphism. □

Proposition 3.5.4. *Let X be a noetherian integral scheme and $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that we have an exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$.*

(1) *If the sequence is locally split and $\mathcal{E}$ is locally free, then there is an isomorphism $\mathrm{Bl}_{\mathcal{F}}X \simeq \mathrm{Bl}_{\mathcal{G}}X$.*
(2) *If $\mathcal{G}$ is locally free, then there is an isomorphism $\mathrm{Bl}_{\mathcal{F}}X \simeq \mathrm{Bl}_{\mathcal{E}}X$.*

Proof. It is enough to prove the statement locally, so we may assume that we have $\mathcal{F} \simeq \mathcal{E} \oplus \mathcal{G}$. With this, we have that

$$\wedge^{\text{top}}\mathcal{F} = \wedge^{\text{top}}\mathcal{E} \otimes \wedge^{\text{top}}\mathcal{G}. \tag{7}$$

By Lemma 3.4.3 we have that $\text{Bl}_{\mathcal{G}}X \simeq \text{Bl}_{\wedge^{\text{top}}\mathcal{G}}X$ and $\text{Bl}_{\mathcal{F}}X \simeq \text{Bl}_{\wedge^{\text{top}}\mathcal{F}}X \simeq \text{Bl}_{\wedge^{\text{top}}\mathcal{E}\otimes\wedge^{\text{top}}\mathcal{G}}X$ using eq. (7). Suppose now that $\mathcal{E}$ is locally free, then $\wedge^{\text{top}}\mathcal{E}$ is a line bundle and we conclude that $\text{Bl}_{\mathcal{F}}X \simeq \text{Bl}_{\mathcal{G}}X$ by proposition 3.2.3.

If $\mathcal{G}$ is locally free, the sequence is locally split and a similar argument to the one above shows that $\text{Bl}_{\mathcal{F}}X \simeq \text{Bl}_{\mathcal{E}}X$. □

Remark 3.5.5. In general, $\text{Bl}_{\mathcal{F}}X$ is not isomorphic to $\text{Bl}_{\mathcal{F}^\vee}X$. For example let X be a normal scheme and $\mathcal{F}$ an ideal sheaf. Then $\mathcal{F}^\vee$ is reflexive and it has rank one, so it is an invertible sheaf. This shows that $\text{Bl}_{\mathcal{F}^\vee}X \simeq X$. If $\mathcal{F}$ is not locally free (see e.g. example 3.2.4), then we have $\text{Bl}_{\mathcal{F}}X \neq X$.

4. Diagonalization

The diagonalization process of a morphism $\varphi\colon E \to F$ of locally free sheaves on a scheme X is introduced by Hu and Li in [HL11]. The process requires to blow up the Fitting ideals of the morphism φ, and doing so desingularizes the kernel and cokernel of φ. Applied to the moduli space of maps, this construction can be used to desingularize all the components of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. We summarize the construction and its universal property and explore the possibility of finding a minimal blowup which also desingularizes all the components of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. In this section all schemes are assumed to be Noetherian.

4.1. **Construction of the Hu–Li blow up.**

Definition 4.1.1 (Diagonalizable morphism [HL11, Definition 3.2])**.** Let X be a scheme. A morphism $\varphi\colon \mathcal{O}_X^{\oplus p} \to \mathcal{O}_X^{\oplus q}$ is *diagonalizable* if there are direct sum decompositions by trivial sheaves

$$\mathcal{O}_X^{\oplus p} = G_0 \oplus \bigoplus_{i=1}^{\ell} G_i \quad \text{and} \quad \mathcal{O}_X^{\oplus q} = H_0 \oplus \bigoplus_{i=1}^{\ell} H_i \tag{8}$$

with $\varphi(G_i) \subseteq H_i$ for $0 \leq i \leq \ell$ such that

(1) $\varphi\,|_{G_0} = 0$;
(2) for every $1 \leq i \leq \ell$, there is an isomorphism $I_i : G_i \to H_i$;
(3) the morphism $\varphi\,|_{G_i}\colon G_i \to H_i$ is given by $f_i I_i$ for some $0 \neq f_i \in \Gamma(\mathcal{O}_X)$;
(4) $(f_{i+1}) \subsetneqq (f_i)$.

More generally, a morphism $\varphi\colon E_1 \to E_2$ of locally free sheaves on X is *locally diagonalizable* if X admits an open cover which trivializes E_1 and E_2 simultaneously and on which φ is diagonalizable.

We will be interested in the coherent sheaves arising as kernels and cokernels of such diagonalizable morphisms.

Proposition 4.1.2. *Let X be a Noetherian integral scheme and let $\varphi\colon E_1 \to E_2$ be a locally diagonalizable morphism between locally free sheaves on X. Then* $\ker(\varphi)$ *is locally free.*

Proof. The question is local, so we can assume that $E_1 = \mathcal{O}_X^{\oplus p}$ and $E_2 = \mathcal{O}_X^{\oplus q}$, and that they admit decompositions as in eq. (8). Then $\ker(\varphi) = G_0$ is free. □

Definition 4.1.3. (1) We say that a coherent sheaf $\mathcal{F}$ on a scheme X is *diagonal* if all of the Fitting ideal sheaves $F_i(\mathcal{F})$ are locally principal.
(2) Given a scheme X and a coherent sheaf $\mathcal{F}$, a *diagonalization* of $\mathcal{F}$ is a morphism $f : \widetilde{X} \to X$ such that $f^*\mathcal{F}$ is diagonal.
(3) Given a scheme X and a morphism of locally-free sheaves $\varphi : E_1 \to E_2$, a *diagonalization* of φ is a morphism $f : \widetilde{X} \to X$ such that $f^*\varphi$ is locally diagonalizable and $\mathrm{rk}(\mathrm{Coker}(\varphi)) = \mathrm{rk}(f^*\mathrm{Coker}(\varphi))$.

The two definitions are closely related, which justifies the choice of terminology.

Proposition 4.1.4. *A morphism $\varphi : E_1 \to E_2$ of locally free sheaves is locally diagonalizable if and only if the coherent sheaf* $\mathrm{Coker}(\varphi)$ *is diagonal.*

Proof. This result is contained in the proof of [HL11, Proposition 3.13]. Observe that the Fitting ideals $F_i(\mathcal{F})$ are just the determinantal ideals $\Delta_{(q-i)\times(q-i)}(\varphi)$, where $q = \mathrm{rk}(E_2)$. If φ is locally diagonalizable, take an open where it is of the form (8). Then the Fitting ideals of $\mathcal{F}$ are generated by products of the f_i's, so are principal in this open.

On the other hand, if $\mathcal{F}$ is diagonal, we can cover X by affine opens where all $F_i(\mathcal{F})$ are principal and where the E_i's are simultaneously trivialized. We quickly sketch how [HL11, Proposition 3.13] produces a decomposition as in (8), by possibly further restricting. The morphism φ is given by

$$\Gamma = (a_{i,j})$$

$i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$. The Fitting ideal $F_{q-1}(\mathcal{F}) = \Delta_{1\times 1}(\Gamma)$ is principal if and only if, after further localization, there is an entry a_{i_0,j_0} which divides every other entry $a_{i,j}$. In that case, one can perform row and column operations to put Γ in the following form

$$\left(\begin{array}{c|ccc} a_{i_0,j_0} & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \Gamma' & \\ 0 & & & \end{array}\right)$$

with Γ' a matrix of smaller size. The same argument works recursively since the remaining Fitting ideals of $\mathcal{F}$ and those of Γ' differ by the principal ideal (a_{i_0,j_0}). □

Remark 4.1.5. From proposition 4.1.4, we see that diagonalizing a coherent sheaf $\mathcal{F}$ is equivalent to diagonalizing any presentation $E_1 \to E_2 \twoheadrightarrow \mathcal{F}$ by locally free sheaves.

Remark 4.1.6. If $\varphi : E_1 \to E_2$ is a locally diagonalizable morphism on a scheme X, and $f : Y \to X$ is any morphism of Noetherian schemes, $f^*\varphi$ is locally diagonalizable.

Similarly, if $\mathcal{F}$ is diagonal, $f^*\mathcal{F}$ is diagonal.

Note that the generic ranks of $\mathcal{F}$ and $f^*\mathcal{F}$ will be different in general for non-dominant morphisms.

We give below a procedure which associates to X and $\mathcal{F}$ a diagonalization $\pi : \mathrm{Bl}^{HL}_{\mathcal{F}} X \to X$. We will show in theorem 4.1.16 that this is the minimal morphism which diagonalizes $\mathcal{F}$.

Definition 4.1.7 (Maximal rank)**.** Let X be an integral Noetherian scheme and $\mathcal{F}$ a coherent sheaf of generic rank r. The maximal rank $\mathrm{mrk}(\mathcal{F})$ is

$$\mathrm{mrk}(\mathcal{F}) = \max_{p \in X} \{\mathrm{rk}(\mathcal{F}|_p)\} \geq r$$

which is the maximum rank of $\mathcal{F}$ when restricted to a closed point of $p \in X$. Equivalently, $\mathrm{mrk}(\mathcal{F})$ is such that the Fitting ideals $F_{\mathrm{mrk}(\mathcal{F})}(\mathcal{F})$ is $\mathcal{O}_X$ and $F_{\mathrm{mrk}(\mathcal{F})-1}(\mathcal{F}) \neq \mathcal{O}_X$, with the convention that $F_{-1}(\mathcal{F}) = 0$.

Remark 4.1.8. The above $\mathrm{mrk}(\mathcal{F})$ is finite. Indeed, the ascending chain condition on the Fitting ideals

$$F_{-1}(\mathcal{F}) \subset F_0(\mathcal{F}) \subset \cdots \subset F_n(\mathcal{F})$$

guarantees that there is some $\mathrm{mrk}(\mathcal{F})$ such that $F_{\mathrm{mrk}(\mathcal{F})}(\mathcal{F}) = F_{\mathrm{mrk}(\mathcal{F})+1}(\mathcal{F}) = \ldots$.

Moreover, for any affine open $U \subset X$, the ascending chain of Fitting ideals stabilizes at $\mathcal{O}_U$, since $\mathcal{F}|_U = \widetilde{M}$ for a finitely generated module M. So the chain above must stabilize at $\mathcal{O}_X$.

Remark 4.1.9. There is a closed point $q \in X$ such that $\mathrm{rk}(\mathcal{F}|_q) = \mathrm{mrk}(\mathcal{F})$, and such that we have a resolution

$$\mathcal{O}_q^{\oplus p} \to \mathcal{O}_q^{\oplus \mathrm{mrk}(\mathcal{F})} \to \mathcal{F}|_q \to 0.$$

However, $\mathcal{F}$ may not be generated globally by $\mathrm{mrk}(\mathcal{F})$ sections. Indeed, it may not be globally generated at all!

Construction 4.1.10 (Hu–Li blow-up)**.** Let X be an integral Noetherian scheme and $\mathcal{F}$ a coherent sheaf of general rank r and maximal rank r_2. Recall from section 3.3 that the Fitting ideals of $\mathcal{F}$ satisfy a chain of inclusions $F_{-1}(\mathcal{F}) \subseteq F_0(\mathcal{F}) \subseteq \ldots$, that $F_{r_2}(\mathcal{F}) = \mathcal{O}_X$ and that $F_0(\mathcal{F}) = \ldots = F_{r-1}(\mathcal{F}) = 0$ because $\mathcal{F}$ has rank r.

Let

$$\mathrm{Bl}^{HL}_{\mathcal{F}} X = \mathrm{Bl}_{F_r(\mathcal{F}) \cdot \ldots \cdot F_{r_2-1}(\mathcal{F})} X.$$

By [Sta22, Lemma 080A], $\mathrm{Bl}^{HL}_{\mathcal{F}} X$ can also be constructed by successively blowing up X along (the total transforms of) the Fitting ideals of $\mathcal{F}$, that is

$$\mathrm{Bl}^{HL}_{\mathcal{F}} X = X_r \xrightarrow{p_r} \ldots \longrightarrow X_{r_2-2} \xrightarrow{p_{r_2-2}} X_{r_2-1} \xrightarrow{p_{r_2-1}} X$$

where

- $X_{r_2-1} = \mathrm{Bl}_{F_{r_2-1}(\mathcal{F})} X$,
- $X_{r_2-2} = \mathrm{Bl}_{F_{r_2-2}(p^*_{r_2-1}\mathcal{F})} X_{r_2-1} = \mathrm{Bl}_{p^{-1}_{r_2-1} F_{r_2-2}(\mathcal{F}) \mathcal{O}_{X_{r_2-1}}} X_{r_2-1}$ and
- $X_i = \mathrm{Bl}_{F_i(p_i^* p_{i+1}^* \cdots p_{r_2-1}^* \mathcal{F})} X_{i+1}$ for all i with $r \leq i \leq r_2 - 2$.

Each p_i is the natural morphism coming from the blow-up construction and we denote by p the composition $p_{r_2-1} \circ \ldots \circ p_r$.

The following two lemmas explain how, after applying the diagonalization construction, a diagonal coherent sheaf $\mathcal{F}$ has an increasing filtration $\mathcal{F}_\bullet$ with all the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ being vector bundles over some locally principal closed subscheme (i.e. a Cartier divisor).

Lemma 4.1.11 (c.f.[Sta22, Tag 0EST])**.** *Let $\mathcal{F}$ be a diagonal coherent sheaf of generic rank r and tor-dimension ≤ 1 on a Noetherian integral scheme X. Then there is a short exact sequence*

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}^{\mathrm{tf}} \to 0$$

where $\mathcal{F}^{\mathrm{tf}}$ is locally free of rank r on X and $\mathcal{K}$ is a coherent sheaf of generic rank 0 and tor-dimension ≤ 1.

Proof. Locally, we have a module M over a ring R with $F_r(M) = (f)$, $F_{r-1}(M) = 0$. Then the statement follows with $\mathcal{K}$ corresponding to the module $\ker\{M \xrightarrow{f} M\}$ by [Sta22, Tag 0F7M]. □

Lemma 4.1.12 (c.f. [Sta22, Tag 0ESU])**.** *Let $\mathcal{F}$ be a diagonal coherent sheaf of generic rank zero, $\mathrm{mrk}(\mathcal{F}) = n$, and tor dimension ≤ 1 on an integral scheme X. Then there is an increasing filtration, denote by $\mathcal{F}_\bullet$,*

$$\mathcal{F} = \mathcal{F}_n \supset \mathcal{F}_{n-1} \supset \cdots \supset \mathcal{F}_0 = 0$$

and effective Cartier divisors D_i such that for each i,

$$\mathcal{F}_i/\mathcal{F}_{i-1}$$

is locally free of rank i on the closed locally principal subscheme defined by D_i. The associated graded to this filtration is

$$\mathcal{E} = \bigoplus_i \mathcal{E}_i$$

where $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ is a vector bundle supported on a subscheme.

Proof. Our formulation differs from the one in the reference, so we present the construction of the filtration in our context. We can work locally and assume that $\mathcal{F}$ has a presentation which is diagonalizable in the sense of Definition 4.1.1 that is $\varphi : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n}$ where φ is the diagonal matrix

$$\varphi = \mathrm{Diag}\left(\overbrace{f_1, \ldots, f_1}^{n_1}, \overbrace{f_2, \ldots, f_2}^{n_2}, \ldots, \overbrace{f_k, \ldots, f_k}^{n_k}\right)$$

with $n_1 + \cdots + n_k = n$ and non-zero f_i's satisfying $(f_{i+1}) \subsetneq (f_i)$. Note that locally $\mathcal{F}$ may not attain its maximal rank n, but we can always choose f_1 to be a unit to obtain a presentation of the correct rank.

Note that we have to assume that the tor dimension of $\mathcal{F}$ is ≤ 1 to ensure we can obtain a presentation by a *square* matrix φ.

Since we are working over a domain, $(f_{i+1}) \subsetneq (f_i)$ is equivalent to $f_i | f_{i+1}$. We can define effective Cartier divisors $D_1, \ldots, D_n$ by taking ratios of successive entries of φ:

$$D_n = F_{n-1} = (f_1)$$
$$D_i = \left(\frac{\varphi_{n-i+1,n-i+1}}{\varphi_{n-i,n-i}}\right).$$

In other words, D_i is the ideal generated by the ratio of the entries in position $n-i+1$ and $n-i$ in φ. Note that, while the generators of the ideals are only well-defined up to a unit, the ideals themselves are well-defined and do not depend

on the chosen presentation of φ. In fact, they can be expressed as differences of the Fitting ideals of $\mathcal{F}$, which are independent of the chosen presentation.

The divisors D_i give closed locally principal subschemes of X, which are defined by (f_{k+1}/f_k) if $i = n - \sum_{j=1}^{k} n_k$ and are empty otherwise.

We define the increasing filtration of $\mathcal{F}_\bullet$ as follows. We set $\mathcal{F}_n := \mathcal{F}$, and define $\mathcal{F}_{n-1}$ as the cokernel of the morphism $\varphi' := \varphi/f_1$. That is,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & (\mathcal{O}_X/(f_1))^{\oplus n} & \xrightarrow{\sim} & \mathcal{F}_n/\mathcal{F}_{n-1} & \longrightarrow & 0 \\
 & & \uparrow\!\uparrow & & \uparrow\!\uparrow & & \uparrow\!\uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{\oplus n} & \xrightarrow{\varphi} & \mathcal{O}_X^{\oplus n} & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle f_1\cdot\mathrm{id}}\uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{\oplus n} & \xrightarrow{\varphi'} & \mathcal{O}_X^{\oplus n} & \longrightarrow & \mathcal{F}_{n-1} & \longrightarrow & 0.
\end{array}
\tag{9}
$$

As $\mathcal{O}_{D_1} = \mathcal{O}_X/(f_1)$, the graded piece $\mathcal{F}_n/\mathcal{F}_{n-1}$ is locally free of rank n on D_n. Now, φ' can be given by the diagonal matrix

$$
\varphi' = \mathrm{Diag}\left(\overbrace{1,\dots,1}^{n_1}, \overbrace{f_2/f_1,\dots,f_2/f_1}^{n_2}, \dots, \overbrace{f_k/f_1,\dots,f_k/f_1}^{n_k}\right).
$$

We can pass to $\varphi'' : \mathcal{O}_X^{\oplus n-1} \to \mathcal{O}_X^{\oplus n-1}$ by removing the first entry. Clearly, $\mathcal{F}_{n-1} = \mathrm{Coker}\varphi''$. Then we can iterate the construction in (9), factoring our multiplication by the first entry φ_1'' of φ''

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & (\mathcal{O}_X/(\varphi_1''))^{\oplus n-1} & \xrightarrow{\sim} & \mathcal{F}_{n-1}/\mathcal{F}_{n-2} & \longrightarrow & 0 \\
 & & \uparrow\!\uparrow & & \uparrow\!\uparrow & & \uparrow\!\uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{\oplus n-1} & \xrightarrow{\varphi''} & \mathcal{O}_X^{\oplus n-1} & \longrightarrow & \mathcal{F}_{n-1} & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \varphi_1''\cdot\mathrm{id}}\uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_X^{\oplus n-1} & \xrightarrow{\varphi'''} & \mathcal{O}_X^{\oplus n-1} & \longrightarrow & \mathcal{F}_{n-2} & \longrightarrow & 0.
\end{array}
$$

This defines the next subsheaf $\mathcal{F}_{n-2}$ in the filtration and the new morphism φ'''. If $n_1 > 1$, $(\varphi_1'') = (1)$, so we will have $\mathcal{F}_{n-2} = \mathcal{F}_{n-1}$ and D_{n-1} defining the empty subscheme. Note that the sub-schemes defined by $D_{n-1}, \dots, D_{n-n_1+1}$ are all empty, and the filtration is constant until $\mathcal{F}_{n-n_1-1}$, which is the cokernel of

$$
\mathcal{O}_X^{\oplus n-n_1} \xrightarrow{\psi} \mathcal{O}_X^{\oplus n-n_1}
$$

with

$$
\psi = \mathrm{Diag}\left(\overbrace{1,\dots,1}^{n_2}, \overbrace{f_3/f_2,\dots,f_3/f_2}^{n_3}, \dots, \overbrace{f_k/f_2,\dots,f_k/f_2}^{n_k}\right)
$$

and $D_{n-n_1} = (f_1/f_2)$. Iterating this construction clearly provides a filtration and a collection of effective divisors which satisfy the claims in the lemma.

The divisors D_i are defined globally in terms of Fitting ideals, and do not depend on the local expression of the matrix. In fact, unpacking the argument above we

can check that

$$\begin{pmatrix} F_0 \\ \vdots \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ & 1 & 2 & \cdots & n-1 \\ & & \ddots & \ddots & \vdots \\ & & & 1 & 2 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix}.$$

Then,

$$\begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} F_0 \\ \vdots \\ F_{n-1} \end{pmatrix}. \tag{10}$$

□

Example 4.1.13. Take $R = \mathbb{C}[x, y, z]$, $X = \operatorname{Spec} R$. Let $\mathcal{F} = \widetilde{M}$ be the diagonal sheaf defined by

$$0 \to R^{\oplus 4} \xrightarrow{\varphi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & xy & 0 \\ 0 & 0 & 0 & xyz \end{pmatrix}} R^{\oplus 4} \to M \to 0.$$

The divisors from the statement of lemma 4.1.12 are given by the ideals

$$\begin{aligned} D_4 &= (x) \\ D_3 &= (1) \\ D_2 &= (y) \\ D_1 &= (z). \end{aligned}$$

Now, all the elements of φ are divisible by D_4, which is the ideal generated by the first entry. We set $\mathcal{F}_4 = \mathcal{F}$. To obtain the next step in the filtration, $\mathcal{F}_3$, we consider the decomposition $\varphi = x \cdot \varphi'$ below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^{\oplus 4} & \xrightarrow{\varphi} & R^{\oplus 4} & \longrightarrow & M & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle x\cdot\mathrm{id}} & & \uparrow & & \\ 0 & \longrightarrow & R^{\oplus 4} & \xrightarrow{\varphi'} & R^{\oplus 4} & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

we set $\mathcal{F}_3 = \widetilde{M_3}$, the module defined by

$$\varphi' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & yz \end{pmatrix}$$

or equivalently as the cokernel of

$$R^{\oplus 3} \xrightarrow{\varphi' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & yz \end{pmatrix}} R^{\oplus 3}.$$

Similarly, $\mathcal{F}_2 = \widetilde{M_2}$ will be defined by

$$0 \to R^{\oplus 2} \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & yz \end{pmatrix}} R^{\oplus 2} \to M_2 \to 0$$

and $\mathcal{F}_1 = \widetilde{M_1}$ by

$$0 \to R \xrightarrow{(z)} R \to M_1 \to 0.$$

Finally, $M_0 = 0$. In conclusion, we obtain the filtration

$$M = M_4 = (R/(x))^{\oplus 2} \oplus R/(xy) \oplus R/(xyz) \xleftarrow{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ x & 0 \\ 0 & x \end{pmatrix}} M_3 = R/(y) \oplus R/(yz) \cong$$

$$\cong M_2 = R/(y) \oplus R/(yz) \xleftarrow{\begin{pmatrix} 0 & y \end{pmatrix}} M_1 = R/(z) \leftarrow 0 = M_0.$$

The graded pieces are

$$\begin{aligned} \mathcal{E}_4 &= \mathcal{F}_4/\mathcal{F}_3 = \widetilde{(R/(x))}^{\oplus 4} \\ \mathcal{E}_3 &= \mathcal{F}_3/\mathcal{F}_2 = 0 \\ \mathcal{E}_2 &= \mathcal{F}_2/\mathcal{F}_1 = \widetilde{(R/(y))}^{\oplus 2} \\ \mathcal{E}_1 &= \widetilde{R/(z)} \end{aligned}$$

and each of the $\mathcal{E}_i$ is locally free of rank i on the subscheme defined by D_i. Note that for $i = 3$, such subscheme is empty.

Theorem 4.1.14. *Let $\mathcal{F}$ be a diagonal coherent sheaf of generic rank r, maximal rank r_2 and tor-dimension ≤ 1 on a Noetherian integral scheme X. Then we have a filtration*

$$\mathcal{F} \supset \mathcal{K} = \mathcal{K}_{r_2 - r} \supset \mathcal{K}_{r_2 - r - 1} \supset \cdots \supset \mathcal{K}_0 = 0$$

such that

$$\mathcal{F}^{\mathrm{tf}} = \mathcal{F}/\mathcal{K}$$

is locally free of rank r and

$$\mathcal{E}_i = \mathcal{K}_i/\mathcal{K}_{i-1}$$

is locally free of rank i on the effective Cartier divisor D_i.

Proof. Immediate by lemma 4.1.11 and lemma 4.1.12.

□

We are now ready to state the minimality properties for the construction 4.1.10. We can can formulate a universal property for the morphism φ, or in light of remark 4.1.5 we can formulate it to only depend on the cokernel sheaf $\mathcal{F}$.

Theorem 4.1.15 (Universal property of diagonalization [HL11])**.** *Let $\varphi : E_1 \to E_2$ be a morphism of locally-free sheaves on a Noetherian integral scheme X. Let $\mathcal{F} = \mathrm{Coker}(\varphi)$. Define $\mathrm{Bl}^{HL}_{\mathcal{F}} X$ as in construction 4.1.10.*

The natural projection $q\colon \mathrm{Bl}^{HL}_{\mathcal{F}} X \to X$ is a diagonalization of φ. Moreover q satisfies the following universal property: for any morphism $f : Y \to X$ such that $f^*\varphi$ is locally diagonalizable and $\mathrm{rk}(f^*\mathcal{F}) = \mathrm{rk}(\mathcal{F})$, there is a unique morphism $f' : Y \to \mathrm{Bl}^{HL}_{\mathcal{F}} X$ factoring f.

$$\begin{array}{ccc} Y & \overset{\exists! f'}{\dashrightarrow} & \mathrm{Bl}^{HL}_{\mathcal{F}} X \\ & \searrow^{f} & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

Theorem 4.1.16 (Universal property of $\mathrm{Bl}^{HL}_{\mathcal{F}} X$ [HL11])**.** *Let X be a noetherian integral scheme and $\mathcal{F}$ a coherent sheaf on X of generic rank r. The natural projection $p\colon \mathrm{Bl}^{HL}_{\mathcal{F}} X \to X$ satisfies that*

1. *the sheaf $p^*\mathcal{F}$ has generic rank r and*
2. *the Fitting ideal $F_i(p^*\mathcal{F})$ is locally principal for all i.*

Moreover, $p\colon \mathrm{Bl}^{HL}_{\mathcal{F}} X \to X$ satisfies the following universal property: for any morphism $f\colon Y \to X$ of Noetherian integral schemes such that

1. *the sheaf $f^*\mathcal{F}$ has generic rank r and*
2. *the Fitting ideal $F_i(f^*\mathcal{F})$ is locally principal for all i,*

there is a unique morphism $f'\colon Y \to \mathrm{Bl}^{HL}_{\mathcal{F}} X$ factoring f.

$$\begin{array}{ccc} Y & \overset{\exists! f'}{\dashrightarrow} & \mathrm{Bl}^{HL}_{\mathcal{F}} X \\ & \searrow^{f} & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

4.2. **Properties of the Hu–Li blow up.** We collect properties of $\mathrm{Bl}^{HL}_{\mathcal{F}} X$.

Proposition 4.2.1. *Let $f : Y \to X$ be a morphism of Noetherian integral schemes and let $\mathcal{F}$ be a coherent sheaf on X of generic rank r. If $f^*\mathcal{F}$ has generic rank r then there is a unique morphism*

$$\begin{array}{ccc} \mathrm{Bl}^{HL}_{f^*\mathcal{F}} Y & \overset{\exists! \tilde{f}}{\dashrightarrow} & \mathrm{Bl}^{HL}_{\mathcal{F}} X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\quad f \quad} & X \end{array}$$

making the diagram commute.

If, moreover, f is flat, then the square is Cartesian.

Proof. A unique morphism $\tilde{f}$ making the diagram commute exists by the universal property theorem 4.1.15. To see that diagram is Cartesian we apply [Sta22, Lemma 0805] to each of the blow-ups defining $\mathrm{Bl}^{HL}_{\mathcal{F}} X$, using that the formation of Fitting ideals is compatible with pullbacks. □

Proposition 4.2.2. *Let X be a noetherian integral scheme, let $\mathcal{F}$ a coherent sheaf on X and let $\mathcal{L}$ be a line bundle on X. Then there is a unique isomorphism*

$$\begin{array}{ccc} \mathrm{Bl}^{HL}_{\mathcal{F}}(X) & \xrightarrow{\exists!\tilde{\phi}} & \mathrm{Bl}^{HL}_{\mathcal{F}\otimes\mathcal{L}}(X) \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

which makes the diagram commute.

Proof. By theorem 4.1.16, a unique factorization $\tilde{\phi}$ of p through q exists if and only if $F_i(p^*(\mathcal{F}\otimes\mathcal{L}))$ is locally principal for all i, and this holds because $F_i(p^*\mathcal{F})$ is locally free for all i. Indeed, choose an open cover of X trivializing $\mathcal{L}$. The preimage by p of this cover induces a cover of $\mathrm{Bl}_{\mathcal{F}}(X)$ where $p^*(\mathcal{F}\otimes\mathcal{L}) \simeq p^*\mathcal{F}$. This shows that $F_i(p^*(\mathcal{F}\otimes\mathcal{L})) \simeq F_i(p^*\mathcal{F})$ locally, so $\tilde{\phi}$ exists. The same argument shows there is a unique factorization of q through p, which must be the inverse of $\tilde{\phi}$ by uniqueness. □

Proposition 4.2.3. *Let X be a noetherian integral scheme and $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that we have an exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$.*

1. *If the sequence is locally split and $\mathcal{E}$ is locally free, then there is an isomorphism $\mathrm{Bl}^{HL}_{\mathcal{F}}X \simeq \mathrm{Bl}^{HL}_{\mathcal{G}}X$.*
2. *If $\mathcal{G}$ is locally free, then there is an isomorphism $\mathrm{Bl}^{HL}_{\mathcal{F}}X \simeq \mathrm{Bl}^{HL}_{\mathcal{E}}X$.*

Proof. It is enough to prove the statement locally, so we may assume that we have $\mathcal{F} \simeq \mathcal{E}\oplus\mathcal{G}$. With this, we have that

$$F_\ell(\mathcal{E}\oplus\mathcal{G}) = \sum_{k+k'=\ell} F_k(\mathcal{E})F_{k'}(\mathcal{G}) \tag{11}$$

by [Sta22, Lemma 07ZA]. If $\mathcal{G}$ is locally free, the sequence is locally split, therefore by symmetry it is enough to show one of the statements. Without loss of generality, suppose that $\mathcal{G}$ is locally free, therefore $F_{k'}(\mathcal{G}) = 0$ for all $k' < \mathrm{rk}(\mathcal{G})$ and $F_{k'}(\mathcal{G}) = \mathcal{O}_X$ for all $k' \geq \mathrm{rk}(\mathcal{G})$ by [Sta22, Lemma 07ZD]. Combining this fact with the chain of inclusions $F_0(\mathcal{E}) \subseteq F_1(\mathcal{E}) \subseteq \ldots$, it follows that

$$F_\ell(\mathcal{F}) = F_\ell(\mathcal{E}\oplus\mathcal{G}) = \begin{cases} 0 & \text{if } \ell < \mathrm{rk}(\mathcal{G}) \\ F_{\ell-\mathrm{rk}(\mathcal{G})}(\mathcal{E}) & \text{if } \ell \geq \mathrm{rk}(\mathcal{G}) \end{cases}$$

This means that the collection of Fitting ideals of $\mathcal{F}$ and $\mathcal{E}$ agree, so $\mathrm{Bl}^{HL}_{\mathcal{F}}X \simeq \mathrm{Bl}^{HL}_{\mathcal{E}}X$. □

Proposition 4.2.4. *Let X be a noetherian integral scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then for every positive integer n*

$$\mathrm{Bl}^{HL}_{\mathcal{F}}X = \mathrm{Bl}^{HL}_{\mathcal{F}^{\oplus n}}X.$$

Proof. There is a natural morphism $\mathrm{Bl}^{HL}_{\mathcal{F}}X \to \mathrm{Bl}^{HL}_{\mathcal{F}^{\oplus n}}X$ over X. To see this, let $p\colon \mathrm{Bl}^{HL}_{\mathcal{F}}X \to X$ be the natural projection. Then $p^*\mathcal{F}$ is diagonal and it follows from definition 4.1.1 that $p^*(\mathcal{F}^{\oplus n}) = (p^*\mathcal{F})^{\oplus n}$ is diagonalizable too. Then apply theorem 4.1.15 to get the desired morphism.

Conversely, we show that there is a natural morphism $\mathrm{Bl}^{HL}_{\mathcal{F}^{\oplus n}}X \to \mathrm{Bl}^{HL}_{\mathcal{F}}X$ over X, which is enough to conclude the proof by the universal properties of both blow-ups. Remember that

$$\mathrm{Bl}^{HL}_{\mathcal{F}}X = \mathrm{Bl}_{\prod_\ell F_\ell(\mathcal{F})}X$$

where the product is over all non-trivial Fitting ideals of $\mathcal{F}$, and similarly $\mathrm{Bl}^{HL}_{\mathcal{F}^{\oplus n}}X$ is the blowup of X along $\prod_\ell F_\ell(\mathcal{F}^{\oplus n})$. By [Moo01], it suffices to show that $\prod_\ell F_\ell(\mathcal{F})$ divides a power of $\prod_\ell F_\ell(\mathcal{F}^{\oplus n})$ as fractional ideals. Actually, we show that every Fitting ideal $F_\ell(\mathcal{F}^{\oplus n})$ is a product of certain Fitting ideals $F_k(\mathcal{F})$, with each k appearing at least once as ℓ varies, and this is clearly enough.

By [BV88, Lemma 10.10], if A is any $\mathbb{Q}$-algebra, if $M = (a_{i,j})$ is any matrix with coefficients in A and if Δ_i denotes the ideal generated by all minors of M of size $i \times i$, then

$$\Delta_i\Delta_j \subseteq \Delta_{i+1}\Delta_{j-1}$$

whenever $i \leq j-2$. From this, we can conclude that if $\ell = ds + r$ with $r \in \{0, \dots, s-1\}$

$$\sum_{j_1+\dots+j_s=\ell} \prod_i \Delta_{j_i} = \Delta_d^{s-r}\Delta_{d+1}^r. \tag{12}$$

To conclude, remember that locally $\mathcal{F}$ is the cokernel of a morphism $\varphi\colon E_1 \to E_2$, that $F_i(\mathcal{F})$ is the ideal $\Delta_{r_2-i}(\varphi)$ of minors in φ of size $r_2 - i$, where $r_2 = \mathrm{rk}(E_2)$, and the expression for Fitting ideals of direct sums eq. (11). □

Example 4.2.5. Take $n = 2$ in Proposition 4.2.4. Then eq. (12) is equivalent to

$$F_\ell(\mathcal{F}\oplus\mathcal{F}) = \sum_{k+k'=\ell} F_k(\mathcal{F})F_{k'}(\mathcal{F}) = \begin{cases} F_{r_2-k}^2 & \text{if } \ell = 2k \\ F_{r_2-k}F_{r_2-k-1} & \text{if } \ell = 2k+1 \end{cases}$$

where r_2 the maximal rank of $\mathcal{F}$ as in construction 4.1.10.

Proposition 4.2.6. *Let X be a Noetherian integral scheme and $\mathcal{F}$ a coherent sheaf on X. Then there is a natural morphism $\mathrm{Bl}^{HL}_{\mathcal{F}}X \to \mathrm{Bl}_{\mathcal{F}}X$.*

Proof. Let $\pi\colon \mathrm{Bl}^{HL}_{\mathcal{F}}X \to X$ be the natural projection and let $r = \mathrm{rk}(\mathcal{F})$. By theorem 3.3.4, it suffices to check that $(\pi^*\mathcal{F})^{\mathrm{tf}}$ is locally free of rank r. This can be checked locally. If $X = \mathrm{Spec}\,(R)$ for a local ring R, the result follows from theorem 3.3.1. □

4.3. **Remarks on minimality.** We saw in Remark 3.3.5 that blowing up the first non-zero Fitting ideal of $\mathcal{F}$ is, in general, not the minimal way to make $\mathcal{F}^{\mathrm{tf}}$ locally free. Similarly, blowing up all the Fitting ideals of $\mathcal{F}$ is not the minimal way to turn $(\mathcal{F}\,|_{D_i})^{\mathrm{tf}}$ into locally free sheaves for all i. This is illustrated in the following examples.

Example 4.3.1. Take $X = \mathrm{Spec}\,(R)$ and $\mathcal{F} = \widetilde{M}$ for $M = R/I$ with $I \subset R$ a non-principal ideal. The only non-trivial Fitting ideal of $\mathcal{F}$ is $F_0(\mathcal{F}) = I$. Note that $M^{\mathrm{tf}} = 0$ is locally free and $M\,|_{V(I)}$ is locally free of rank 1. This means that $\mathcal{F}$ already has the desired property on X. However, blowing up all the Fitting ideals of $\mathcal{F}$ results in $\mathrm{Bl}_I(X)$, which is isomorphic to X if and only if I is invertible.

This example also shows that given a coherent sheaf $\mathcal{F}$ on an integral scheme X, the blow up $\mathrm{Bl}_{\mathcal{F}}(X)$ from definition 3.2.1 and $\mathrm{Bl}^{HL}_{\mathcal{F}}X$ are different in general. Indeed, on the one hand, $\mathrm{Bl}_{\mathcal{F}}(X) = \mathrm{Bl}_M(X) = X$ because $M^{\mathrm{tf}} = 0$ is locally

free. On the other hand, the only non-trivial Fitting ideal of M is $F_1(M) = I$, so $\mathrm{Bl}^{HL}_{\mathcal{F}} X = \mathrm{Bl}_I X$. Therefore both blow ups agree if and only if I is invertible.

Example 4.3.2. Let P be the origin in $\mathbb{A}^2_{x,y}$ and consider the embedding $i\colon \mathbb{A}^2_{x,y} \to \mathbb{A}^3_{x,y,z}\colon (x,y) \mapsto (x,y,0)$. The image of i is the plane $\Pi = \{z = 0\}$. Let $I = (x,y)$ be the ideal of P in $\mathbb{A}^2$ and consider the module $M = i_* I$ in $\mathbb{A}^3$.

Note that M has generic rank 0, $M^{\mathrm{tf}} = 0$ and that $M\mid_\Pi$ is the ideal I, which is torsion-free. This means that $M^{\mathrm{tf}} = 0$ is already locally free, but $(M\mid_\Pi)^{\mathrm{tf}} = M\mid_\Pi = I$ is not locally free over Π.

To compute $\mathrm{Bl}^{HL}_M \mathbb{A}^3$, we start with the following resolution of M

$$R^3 \xrightarrow{\Gamma} R^2 \longrightarrow M \longrightarrow 0 \tag{13}$$

where $R = \mathbb{C}[x,y,z]$ and

$$\Gamma = \begin{pmatrix} y & z & 0 \\ -x & 0 & z \end{pmatrix}.$$

The Fitting ideals of M are

- $F_0(M) = z(x,y,z)$,
- $F_1(M) = (x,y,z)$,
- $F_n(M) = R$, for all $n \geq 3$

This reflects the fact that M has rank 0 on $\mathbb{A}^3 \setminus \Pi$, rank 1 on $\Pi \setminus P$ and rank 2 on P, as per proposition 2.2.2. Then

$$\mathrm{Bl}^{HL}_M \mathbb{A}^3 = \mathrm{Bl}_{F_0(M)\cdot F_1(M)} \mathbb{A}^3 = \mathrm{Bl}_{z(x,y,z)^2} \mathbb{A}^3 = \mathrm{Bl}_{(x,y,z)} \mathbb{A}^3 = \mathrm{Bl}_P \mathbb{A}^3$$

is simply the blowup of the origin in $\mathbb{A}^3$. Note that $\mathrm{Bl}^{HL}_M \mathbb{A}^3$ is distinct from $\mathrm{Bl}_M \mathbb{A}^3 = \mathbb{A}^3$ in this example as the latter does not flatten $(M\mid_\Pi)^{\mathrm{tf}}$.

Remark 4.3.3. If the sheaf $\mathcal{F}$ has projective dimension ≤ 1, then the Rossi construction is equal to the blow-up of the first non-zero Fitting ideal. In this case, blowing up all of the proper non-zero ideals as in the Hu–Li construction gives a minimal resolution with the property that $(\mathcal{F}\mid_{D_i})^{\mathrm{tf}}$ is locally free for all of the D_i's defined in terms of Fitting ideals by (10). For an ideal having projective dimension 1 is equivalent to being principal.

Remark 4.3.4 (Extension of sheaves)**.** Let X be a scheme, let Y be a closed subscheme an let $\mathcal{F}_Y$ be a coherent sheaf of rank r on Y. In order to find a minimal resolution of this torsion sheaf, one may try to extend $\mathcal{F}_Y$ to X as a sheaf which not a torsion sheaf and perform a repeated Rossi construction. One can find an open cover of X and blow-ups of the charts such that on the blow-up the torsion free part of the pull-back of $\mathcal{F}$ is locally free on the support. However, the blown up charts may not glue to a global construction. Below we explain that it is always possible to find *local* blow-ups.

Let X be an affine scheme and let Y be a closed subscheme. Let $\mathcal{F}_Y$ be a coherent sheaf of rank r on Y and assume that we have an exact sequence

$$\mathcal{O}_{U\cap Y}^{\oplus n-r} \xrightarrow{\widehat{M}} \mathcal{O}_{U\cap Y}^{\oplus n} \to \mathcal{F}_{U\cap Y} \to 0,$$

where $\widehat{M} \in M_{n,r-n}(\Gamma(O_Y))$. We may assume that $U = \operatorname{Spec} R$ and $Y \cap U = \operatorname{Spec} R/I$, where R is a ring and I is an ideal. Let $\widehat{M} = (\hat{f}_{ij})$, with $\hat{f}_{ij} \in R/I$. We

now choose $f_{ij} \in R$ a lift of $\hat{f}_{ij}$ and we denote by M the matrix (f_{ij}). Then we have a morphism $\mathcal{O}_U^{\oplus n-r} \xrightarrow{M} \mathcal{O}_U^{\oplus n}$ and an exact sequence

$$\mathcal{O}_U^{\oplus n-r} \xrightarrow{M} \mathcal{O}_U^{\oplus n} \to \mathcal{F}_U \to 0,$$

where $\mathcal{F}_U$ denotes the cokernel of the map induced by M. Then, we have that $\mathcal{F}_U|_Y = \mathcal{F}_Y$ and the resolution above induces a morphism

$$U \dashrightarrow U \times \mathrm{Gr}(r, n).$$

Since there is no canonical choice for the lift M, the above morphisms do not glue in general.

5. Components of abelian cones

Let $\mathcal{F}$ be a diagonal sheaf on an integral Noetherian scheme, we study the irreducible components of $C(\mathcal{F})$. We show that $C(\mathcal{F})$ has finitely many irreducible components, which we consider with their natural reduced structure. Each irreducible component is a vector bundle supported on a close integral subscheme. All of the cones in this section are taken over X, unless otherwise specified by the notation $C_{\text{base}}(\text{sheaf})$.

We start the study of the components of $C(\mathcal{F})$ by proving proposition 5.0.5, where we write the cone as a pushout, where one of the factors is its main component $C(\mathcal{F}^{\text{tf}})$. This decomposition is motivated by [AM98, Proposition 2.5], where it is shown (in the analytic category) that if $\pi\colon C = \operatorname{Spec}(\mathcal{A}) \to X$ is a cone with X integral and with $\mathcal{A}$ torsion-free outside of a closed $Z \subseteq X$, then the closure of $C \setminus \pi^{-1}(Z)$ inside C is equal to $\operatorname{Spec}(\mathcal{A}^{\text{tf}})$. In that case, $\operatorname{Spec}(\mathcal{A}^{\text{tf}})$ is an irreducible component of C, that we call the main component of C.

For technical reasons, we would like to ask $\mathcal{A}^{\text{tf}}$ to be locally free, rather than asking that $\mathcal{A}$ is torsion-free outside of Z. Also, since we are interested in the case of an abelian cone $C(\mathcal{F}) = \operatorname{Spec}(\operatorname{Sym}\mathcal{F})$, we would like to have a similar decomposition with assumptions depending on $\mathcal{F}$, rather than on $\operatorname{Sym}\mathcal{F}$. The strategy to do so is as follows.

Let $\mathcal{F}$ be a diagonal sheaf on an integral Noetherian scheme X. Remember that $\mathcal{F}^{\text{tf}}$ is locally free by proposition 4.2.6. Firstly, we show in lemma 5.0.1 that $(\operatorname{Sym}(\mathcal{F}))^{\text{tf}} = \operatorname{Sym}(\mathcal{F}^{\text{tf}})$. As pointed out in remark 5.0.2, symmetric algebra and torsion-free-part do not commute for a general coherent sheaf. The pushout we are looking for depends on a closed $Z \subseteq X$ (see proposition 5.0.5). One should not expect such a decomposition to exist unless $Z \subseteq \operatorname{supp}(\operatorname{tor}(\operatorname{Sym}\mathcal{F}))$. We relate $\operatorname{supp}(\operatorname{tor}(\operatorname{Sym}\mathcal{F}))$ and $\operatorname{supp}(\operatorname{tor}(\mathcal{F}))$ in lemma 5.0.3. The remaining piece is lemma 5.0.4. All the intermediate results are proven in more generality than needed.

Lemma 5.0.1. *Let X be a Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. If $\mathcal{F}^{\text{tf}}$ is locally free then*

$$(\operatorname{Sym}\mathcal{F})^{\text{tf}} = \operatorname{Sym}(\mathcal{F}^{\text{tf}}).$$

Proof. We have the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{tor}(\operatorname{Sym}\mathcal{F}) & \xrightarrow{e} & \ker(p) & \xrightarrow{e'} & \ker(p'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & & \\
0 & \longrightarrow & \operatorname{tor}(\operatorname{Sym}\mathcal{F}) & \xrightarrow{f} & \operatorname{Sym}\mathcal{F} & \xrightarrow{f'} & (\operatorname{Sym}\mathcal{F})^{\mathrm{tf}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} & & \\
0 & \longrightarrow & \operatorname{tor}(\operatorname{Sym}(\mathcal{F}^{\mathrm{tf}})) & \xrightarrow{g} & \operatorname{Sym}(\mathcal{F}^{\mathrm{tf}}) & \xrightarrow{g'} & (\operatorname{Sym}(\mathcal{F}^{\mathrm{tf}}))^{\mathrm{tf}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The last two rows are clearly exact. Moreover, since $\mathcal{F}^{\mathrm{tf}}$ is locally free, $\operatorname{tor}(\operatorname{Sym}(\mathcal{F}^{\mathrm{tf}})) = 0$ and g' is an isomorphism. The morphism i' is the identity on $\operatorname{tor}(\operatorname{Sym}\mathcal{F})$. The surjective morphism p comes from applying Sym to the surjection $\mathcal{F} \to \mathcal{F}^{\mathrm{tf}}$, because Sym preserves surjections. The morphism p'' is induced by p using that $\operatorname{tor}(\operatorname{Sym}(\mathcal{F}^{\mathrm{tf}})) = 0$, The first row is exact by the snake lemma. We want to show that $\ker(p'') = 0$ or, equivalently, that e is an isomorphism.

It follows from the above that we have

$$0 \longrightarrow \operatorname{tor}(\operatorname{Sym}\mathcal{F}) \xrightarrow{f\circ i'} \operatorname{Sym}\mathcal{F} \xrightarrow{p} \operatorname{Sym}(\mathcal{F}^{\mathrm{tf}}) \longrightarrow 0,$$

which is exact except possibly at $\operatorname{Sym}\mathcal{F}$. We conclude if we show exactness there. The inclusion $\operatorname{Im}(f \circ i') \subseteq \ker(p)$ is clear because $p \circ f = g \circ p' = 0$.

To show that $\ker(p) \subseteq \operatorname{Im}(f \circ i')$, we know that

$$\operatorname{tor}(\mathcal{F}) \otimes \operatorname{Sym}^{n-1}(\mathcal{F}) \to \operatorname{Sym}^{n}(\mathcal{F}) \to \operatorname{Sym}^{n}(\mathcal{F}^{\mathrm{tf}}) \to 0$$

is exact for all $n \geq 1$ by [Sta22, Lemma 01CJ]. Note that p is a morphism of graded algebras, therefore

$$\ker(p) = \bigoplus_n \ker(\operatorname{Sym}^{n}(\mathcal{F}) \to \operatorname{Sym}^{n}(\mathcal{F}^{\mathrm{tf}})).$$

It suffices to show that for each n, the morphism $\operatorname{tor}(\mathcal{F}) \otimes \operatorname{Sym}^{n-1}(\mathcal{F}) \to \operatorname{Sym}^{n}(\mathcal{F})$ factors through $\operatorname{tor}(\operatorname{Sym}(\mathcal{F}))$. Locally, $X = \operatorname{Spec}(R)$ and $\mathcal{F} = \widetilde{M}$ for some R-module M. Given $\lambda = \sum_j m_1^j \otimes \ldots \otimes m_n^j \in \operatorname{tor}(M) \otimes \operatorname{Sym}^{n-1}(M)$, we can choose for each j a non-zero divisor $r_j \in R$ such that $r_j m_1^j = 0$. Then $r = r_1 \cdots r_j$is a non-zero divisor and $r\lambda = 0$, so $\lambda \in \operatorname{tor}(\operatorname{Sym}(M))$. □

Remark 5.0.2. Note that lemma 5.0.1 does not holds in general if we do not assume that $\mathcal{F}^{\mathrm{tf}}$ is locally free. For example, let $\mathcal{F} = \mathcal{I}$ be the ideal sheaf of a closed point P on X. Then $(\operatorname{Sym}\mathcal{I})^{\mathrm{tf}} = \operatorname{Sym}\mathcal{I} = \bigoplus_{n\geq 0} \mathcal{I}^n$ if and only if P is regular.

Lemma 5.0.3. *Let R be a ring, I be an ideal in R and M be an R-module.*

If M^{tf} is locally free and $I \cdot \operatorname{tor}(M) = 0$ then $I \cdot \operatorname{tor}(\operatorname{Sym} M) = 0$.

Proof. Note that $\operatorname{tor}(\operatorname{Sym} M) = \bigoplus_{n\geq 0} \operatorname{tor}(\operatorname{Sym}^n M)$. In the proof of lemma 5.0.1 we show that $\operatorname{Sym}^n(M^{\mathrm{tf}}) \simeq (\operatorname{Sym}^n M)^{\mathrm{tf}}$. The following commutative diagram is exact by [Sta22, Lemma 01CJ].

$$\begin{array}{ccccccccc} & & \operatorname{tor}(M)\otimes \operatorname{Sym}^{n-1}(M) & \longrightarrow & \operatorname{Sym}^n M & \longrightarrow & \operatorname{Sym}^n(M^{\mathrm{tf}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{tor}(\operatorname{Sym}^n M) & \longrightarrow & \operatorname{Sym}^n M & \longrightarrow & \operatorname{Sym}^n(M)^{\mathrm{tf}} & \longrightarrow & 0 \end{array}$$

The first row is exact by by [Sta22, Lemma 01CJ], and the second row is also exact. By the snake lemma, $\operatorname{tor}(M)\otimes \operatorname{Sym}^{n-1}(M)$ surjects onto $\operatorname{tor}(\operatorname{Sym}^n M)$ and the claim follows. □

Lemma 5.0.4. *Let R be a commutative ring, A be an R-algebra and I be an ideal of R. If $I\cdot\operatorname{tor}(A) = 0$ and A^{tf} is locally free, then the following square is Cartesian in the category of R-algebras*

$$\begin{array}{ccc} A & \longrightarrow & A^{\mathrm{tf}} \\ \downarrow & & \downarrow \\ A\otimes R/I & \longrightarrow & A^{\mathrm{tf}}\otimes R/I. \end{array}$$

Proof. Firstly, we prove that the square is Cartesian in the category of R-modules. We have the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & IA & \xrightarrow{e'} & IA^{\mathrm{tf}} & & \\ & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & & \\ 0 & \longrightarrow & \operatorname{tor}(A) & \xrightarrow{f} & A & \xrightarrow{f'} & A^{\mathrm{tf}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} & & \\ & & \operatorname{tor}(A)\otimes R/I & \xrightarrow{g} & A\otimes R/I & \xrightarrow{g'} & A^{\mathrm{tf}}\otimes R/I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The three columns are exact because $N\otimes R/I \simeq N/IN$ for any R-module N and because $I\cdot\operatorname{tor}(A) = 0$.

Firstly, observe that g is injective. This is equivalent to $\operatorname{Tor}_1(R/I, A^{\mathrm{tf}}) = 0$, which holds because A^{tf} is locally free. By the snake lemma, the natural morphism $e'\colon IA \to IA^{\mathrm{tf}}$ induced by f' is an isomorphism.

In order to prove the lemma, we first need to show that the square in question is a Cartesian square of R-modules. Then we check that it is also a Cartesian diagram of R-algebras. Both can be achieved by routine diagram chasing.

□

Proposition 5.0.5. *Let X be a Noetherian scheme, $\mathcal{F}$ a coherent sheaf on X and let $\pi\colon C(\mathcal{F}) = \operatorname{Spec}(\operatorname{Sym}\mathcal{F}) \to X$ be the corresponding abelian cone. Let $i\colon Z \hookrightarrow X$*

be a closed subscheme in X with ideal sheaf $\mathcal{I}_Z$ such that $\mathcal{I}_Z \subseteq \mathrm{Ann}(\mathrm{tor}(\mathcal{F}))$. If $\mathcal{F}^{\mathrm{tf}}$ is locally free, then the following is a push-out of schemes

$$\operatorname{Spec}\operatorname{Sym}\mathcal{F} = \operatorname{Spec}(\operatorname{Sym}\mathcal{F}^{\mathrm{tf}}) \bigsqcup_{\operatorname{Spec} i_*(\operatorname{Sym}(\mathcal{F}^{\mathrm{tf}})|_Z)} \operatorname{Spec} i_*(\operatorname{Sym}(\mathcal{F})\,|_Z)$$

Proof. Locally, $X = \operatorname{Spec} R$ is affine, $\mathcal{F} = \widetilde{M}$ for some finitely presented module M over R such that M^{tf} is locally free and $\mathcal{I}_Z = I$ is an ideal with $I \subseteq \mathrm{Ann}(\mathrm{tor}(M))$. Let $A = \operatorname{Sym} M$. Then $I \cdot \mathrm{tor}(A) = 0$ by lemma 5.0.3 and lemma 5.0.1 ensures that $A^{\mathrm{tf}} = \operatorname{Sym}(M^{\mathrm{tf}})$ is locally free. The result follows from lemma 5.0.4. □

Remark 5.0.6. Remember that the support $\operatorname{supp}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ can be defined set-theoretically by locally looking at the prime ideals where the stalk of $\mathcal{F}$ is non-zero. A scheme structure on $\operatorname{supp}(\mathcal{F})$ is given by the sheaf $\mathrm{Ann}(\mathcal{F})$. Therefore the condition $\mathcal{I}_Z \subseteq \mathrm{Ann}(\mathrm{tor}(\mathcal{A}))$ in proposition 5.0.5 implies that the closed Z must contain $\operatorname{supp}(\mathrm{tor}(\mathcal{F}))$.

Another natural scheme structure in $\operatorname{supp}(\mathcal{F})$ is given by $F_0(\mathcal{F})$, the 0-th Fitting ideal of $\mathcal{F}$. There is an inclusion $F_0(\mathcal{F}) \subseteq \mathrm{Ann}(\mathcal{F})$ by [Sta22, Lemma 07ZA], thus in proposition 5.0.5 we can also take the particular case where $\mathcal{I}_Z = F_0(\mathrm{tor}(\mathcal{F}))$.

5.1. **The main component.** First, we reduce to the generic rank 0 case. Let $\mathcal{F}$ be of generic rank r. Since it is diagonal, $\mathcal{F}^{\mathrm{tf}}$ is locally free of rank r. Then

$$C(\mathcal{F}) = C(\mathcal{F}^{\mathrm{tf}}) \bigsqcup_{C(\iota_*\mathcal{F}^{\mathrm{tf}}|_{\mathrm{Supp}(\mathrm{tor}(\mathcal{F}))})} C(i_*\mathcal{F}|_{\mathrm{Supp}(\mathrm{tor}(\mathcal{F}))})$$

by proposition 5.0.5. Here both cones are taken over X. We call $C(\mathcal{F}^{\mathrm{tf}})$ the *main component* of $C(\mathcal{F})$. This is a rank r vector bundle on X. After replacing $\mathcal{F}$ by $i_*\mathcal{F}|_{\mathrm{Supp}(\mathrm{tor}(\mathcal{F}))}$, we may assume from now that $\mathrm{rk}(\mathcal{F}) = 0$.

5.2. **Decomposing the torsion.** Let $\mathcal{F}$ be a rank 0 diagonal sheaf. Recall that, by lemma 4.1.12, $\mathcal{F}$ has a filtration with quotients supported on some effective Cartier divisors D_i, $i = 1, \dots, n$. We can think, locally, of $\mathcal{F}$ as a direct sum of locally free sheaves on the D_i's, each of rank i. Consider the finite collection of closed integral subschemes $\{Z_i^j\}_j$, which are the irreducible components of D_i taken with reduced structure. These are in the support of $\mathcal{F}$ and $\mathcal{F}|_{Z_i^j}$ is locally free. Note that these collections are not necessarily disjoint for different i's. We denote the inclusion of Z_i^j in X simply by ι, without keeping track of the indices when it is not necessary.

Theorem 5.2.1. *The cone of $\mathcal{F}$ is topologically a union of finitely many irreducible components*

$$C(\mathcal{F}) = \bigcup_{i,j} C\left((\mathcal{F}\,|_{Z_i^j})^{\mathrm{tf}}\right) \cup X$$

where each $C\left((\mathcal{F}\,|_{Z_i^j})^{\mathrm{tf}}\right)$ a vector bundle supported on the integral subscheme Z_i^j.

Lemma 5.2.2. *The cone $C_{Z_i^j}\left((\mathcal{F}\,|_{Z_i^j})^{\mathrm{tf}}\right) \to Z_i^j$ is a vector bundle of rank r_i^j, where*

$$r_i^j = \max_k\{Z_i^j \subset D_k\}.$$

Proof. Since $\mathrm{tor}(\mathcal{F}|_U) = \mathrm{tor}(\mathcal{F})|_U$ for $U \subset X$ open, it is enough to prove it locally. We assume that $\mathcal{F}$ is the cokernel of a diagonal matrix

$$\mathrm{Diag}(f_1, \dots, f_1, f_2, \dots, f_2, \dots, f_s),$$

where f_k divides f_{k+1}. Observe that, if $f_k \mid_{Z_i^j} = 0$, then $f_\ell \mid_{Z_i^j}$ also vanishes for all $\ell > k$. Take r_i^j as in the statement of the theorem: Z_i^j is a component of $D_{r_i^j}$ and the latter divides the last r_i^j entries.

So that the matrix presentation of $\mathcal{F}$ on Z_i^j looks like $\mathrm{Diag}(f_1 \mid_{Z_i^j}, \dots, f_t \mid_{Z_i^j}, 0, \dots, 0)$ where $f_t \mid_{Z_i^j} \neq 0$. Since Z_i^j is not, by assumption, a component of $Z(f_t)$ we see that the cokernel of $\mathrm{Diag}(f_1 \mid_{Z_i^j}, \dots, f_t \mid_{Z_i^j})$ is a tosion sheaf and the torsion-free part of $\mathcal{F} \mid_{Z_i^j}$ is locally free of rank r_i^j. □

Proof of theorem 5.2.1. To check the claim set-theoretically, it suffices to argue that any closed point of $C(\mathcal{F})$ is contained in at least one of the cones. Let $v \in C(\mathcal{F})$, the projection to X is $x \in X$. Then v is specified by some section $x \to \mathcal{F}|_x$. If $x \notin \bigcup_{i,j} Z_i^j$, $\mathcal{F}|_x = 0$, so we are done. Otherwise, we need to argue that $\mathcal{F}|_x \cong ((\mathcal{F}|_{Z_i^j})^{\mathrm{tf}})|_x$ for some i, j.

Let i be such that $x \in Z_i^j$ for some j but $x \notin Z_k^\ell$ for all $k > i$ and all ℓ. Then $x \in D_i$ but $x \notin D_k$ for any $k > i$.

By the construction of the D_i's, we know that $\mathrm{supp}\,(\mathrm{tor}(\mathcal{F}|_{Z_i^j})) \subset \bigcup_{k>i,\ell} Z_k^\ell$. Then $(\mathcal{F}^{\mathrm{tf}}_{Z_i^j})|_x = \mathcal{F}|_x$, and we are done.

The morphism $\bigcup_{i,j} C((\mathcal{F}|_{Z_i^j})^{\mathrm{tf}}) \to C(\mathcal{F})$ of topological spaces, given by the universal property of push-outs, is continuous and closed for the Zariski topology. Since we have just checked that it is also bijective, it is a homeomorphism. □

6. Desingularization of coherent sheaves on stacks

In this section, we show that the constructions introduced so far in Section 3 and Section 4 make sense for algebraic stacks, since they are both local and they commute with flat base-change (see respectively Proposition 3.5.3 and Proposition 4.2.1). Thus we define the desingularization and the diagonalization of a coherent sheaf on a Noetherian integral Artin stack and we establish properties of both constructions.

6.1. **Construction of $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$.** Denote by $\mathfrak{P}$ a Noetherian, integral Artin stack. Consider a smooth presentation of $\mathfrak{P}$, i.e. a groupoid in algebraic spaces (U_0, U_1, s, t, m) whose associated quotient stack $[U_1 \rightrightarrows U_0]$ is $\mathfrak{P}$. Here $[U_1 \rightrightarrows U_0]$ denotes the stackyfication of a category fibered in groupoids $[U_1 \rightrightarrows U_0]^{\mathrm{pre}}$. Recall that U_0, U_1 are algebraic spaces $m : U_1 \,{}_s\!\times_t U_1 \to U_1$ is the composition of arrows, $s, t : U_1 \to U_0$ are respectively source and target morphism and they are smooth morphisms. They satisfy some compatibility conditions that we will not use explicitly here (See [LMB00, §(4.3)] or [Sta22, Definition 0441].

The reader can think of $\mathfrak{P}$ being the Picard stack $\mathfrak{Pic}_{g,n}$. Recall, that a S-point of $\mathfrak{Pic}_{g,n}$ is a couple (C, L) where C is a nodal curve of genus g with n distinct smooth marked points and L is a line bundle over it. It is well known that $\mathfrak{Pic}_{g,n}$ is a smooth Noetherian Artin stack which is not of finite type as we do not fix the degree of the line bundle.

Let $\mathfrak{F}$ be a coherent sheaf on $\mathfrak{P}$, i.e. we have a coherent sheaf $\mathcal{F}_0$ on U_0 and also a coherent sheaf $\mathcal{F}_1$ on U_1 with two fixed isomorphisms

$$s^*\mathcal{F}_0 \simeq \mathcal{F}_1 \simeq t^*\mathcal{F}_0 \tag{14}$$

that satisfy the cocycle condition on $U_1 \,{}_s\!\times_t U_1$. We refer to the article of Olsson [Ols07, Proposition 6.12] for the equivalent definitions of coherent sheaves on an Artin stack.

We now proceed to use the smooth presentation of $\mathfrak{P}$ to define a stack $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ desingularizing the coherent sheaf $\mathfrak{F}$. All of the following discussion holds formally identical when we consider the procedure that diagonalises $\mathfrak{F}$ instead. The stack we obtain with the second procedure is denoted $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$.

Later, we prove that the blow-up stacks obtained in both cases are algebraic and come equipped with a representable (by a scheme), proper and birational morphism to $\mathfrak{P}$.

With the theory developed in §3, we can construct $\mathrm{Bl}_{\mathcal{F}_1}U_1$ and $\mathrm{Bl}_{\mathcal{F}_0}U_0$. Note that to apply the results in that section we require U_0, U_1 to be integral Noetherian schemes and not merely algebraic spaces. However, it is routine to extend the construction to algebraic spaces and we omit that passage here.

Since the morphisms $s, t : U_1 \to U_0$ are smooth (hence flat), we apply flat base change for blowup of sheaves (see Proposition 3.5.3) to s, t and we get $\widetilde{s}, \widetilde{t} : \mathrm{Bl}_{\mathfrak{F}_1}U_1 \to \mathrm{Bl}_{\mathfrak{F}_0}U_0$. Using the fix isomorphisms (14), we obtain the following Cartesian diagrams

$$\begin{array}{ccc} \mathrm{Bl}_{\mathcal{F}_1}U_1 & \xrightarrow{\widetilde{s}} & \mathrm{Bl}_{\mathcal{F}_0}U_0 \\ \downarrow{\scriptstyle q} & \ulcorner & \downarrow{\scriptstyle p} \\ U_1 & \xrightarrow{s} & U_0 \end{array} \qquad \begin{array}{ccc} \mathrm{Bl}_{\mathcal{F}_1}U_1 & \xrightarrow{\widetilde{t}} & \mathrm{Bl}_{\mathcal{F}_0}U_0 \\ \downarrow{\scriptstyle q} & \ulcorner & \downarrow{\scriptstyle p} \\ U_1 & \xrightarrow{t} & U_0. \end{array} \tag{15}$$

In addition, using Cartesian diagram on a cube, we construct a map

$$\widetilde{m} : \mathrm{Bl}_{\mathcal{F}_1}U_1 \,{}_{\widetilde{s}}\!\times_{\widetilde{t}} \mathrm{Bl}_{\mathcal{F}_1}U_1 \to \mathrm{Bl}_{\mathcal{F}_1}U_1.$$

More precisely, we have

$$\begin{aligned} \mathrm{Bl}_{\mathcal{F}_1}U_1 \,{}_{\widetilde{s}}\!\times_{\widetilde{t}} \mathrm{Bl}_{\mathcal{F}_1}U_1 &\simeq \left(\mathrm{Bl}_{\mathcal{F}_0}U_0 \,{}_p\!\times_s U_1\right) \,{}_{\widetilde{s}}\!\times_{\widetilde{t}} \left(\mathrm{Bl}_{\mathcal{F}_0}U_0 \,{}_p\!\times_t U_1\right) \\ &\simeq \mathrm{Bl}_{\mathcal{F}_0}U_0 \,{}_p\!\times_s U_1 \,{}_s\!\times_t U_1 \\ &\to \mathrm{Bl}_{\mathcal{F}_0}U_0 \,{}_p\!\times_s U_1 \text{ by applying } m : U_1 \,{}_s\!\times_t U_1 \to U_1 \\ &\simeq \mathrm{Bl}_{\mathcal{F}_1}U_1 \text{ by the Cartesian diagram (15).} \end{aligned}$$

We obtain a smooth groupoid in algebraic spaces

$$(\mathrm{Bl}_{\mathcal{F}_0}U_0, \mathrm{Bl}_{\mathcal{F}_1}U_1, \widetilde{s}, \widetilde{t}, \widetilde{m})$$

with a morphism of groupoids to (U_0, U_1, s, t, m). This defines a 1-morphism

$$p : [\mathrm{Bl}_{\mathcal{F}_0}U_0 \rightrightarrows \mathrm{Bl}_{\mathcal{F}_1}U_1]^{\mathrm{pre}} \to [U_0 \rightrightarrows U_1]^{\mathrm{pre}}.$$

Let $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ denote the stackyfication of $[\mathrm{Bl}_{\mathcal{F}_0}U_0 \rightrightarrows \mathrm{Bl}_{\mathcal{F}_1}U_1]^{\mathrm{pre}}$. By universal property, the morphism discussed above lifts to a morphisms of stacks

$$\pi : \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}.$$

6.2. Properties of $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$.

Theorem 6.2.1. *Let $[U_1 \rightrightarrows U_0] \to \mathfrak{P}$ be an integral Noetherian Artin stack and $\mathfrak{F}$ be a coherent sheaf on it.*

1. *The stacks $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} = [\mathrm{Bl}_{\mathcal{F}_1}U_1 \rightrightarrows \mathrm{Bl}_{\mathcal{F}_0}U_0]$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} = [\mathrm{Bl}^{HL}_{\mathcal{F}_1}U_1 \rightrightarrows \mathrm{Bl}^{HL}_{\mathcal{F}_0}U_0]$ are integral Noetherian Artin stacks.*
2. *The morphisms $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ are representable proper and birational.*

Remark 6.2.2. Let S be a scheme and $f : S \to \mathfrak{P}$ be a flat morphism. Then we have

$$\begin{array}{ccc} \mathrm{Bl}_{f^*\mathfrak{F}}S & \longrightarrow & \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \\ \downarrow & \ulcorner & \downarrow \\ S & \xrightarrow{f} & \mathfrak{P} \end{array} \qquad \begin{array}{ccc} \mathrm{Bl}^{HL}_{f^*\mathfrak{F}}S & \longrightarrow & \mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} \\ \downarrow & \ulcorner & \downarrow \\ S & \xrightarrow{f} & \mathfrak{P} \end{array} \tag{16}$$

In this section, we choose to use an atlas of $\mathfrak{P}$ to define the blowups on Artin stacks but it is also possible to use the diagram above and fppf descent to define them.

Proof of Theorem 6.2.1. Once again, we will only discuss $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$, as the argument for $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$ is identical, mutatis mutandis.

Part (2), together with the properties of the respective constructions on schemes, implies part (1) of the theorem. To establish part (2), it suffices to compute the fiber $U_0 \times_{\mathfrak{P}} \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and show that it is $\mathrm{Bl}_{\mathcal{F}_0}U_0$. A priori, $\mathrm{Bl}_{\mathcal{F}_0}U_0 \to U_0$ is representable by an algebraic space, so the morphism $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ will be representable by an algebraic space [Sta22, Tag 045G]. However, running the argument below for a covering of an algebraic space by schemes, one can show in two steps that the morphism is in fact representable by a scheme. Now consider the 2-Cartesian diagram of categories fibered in groupoids

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & U_0 \\ \downarrow & \ulcorner & \downarrow \\ [\mathrm{Bl}_{\mathcal{F}_1}U_1 \rightrightarrows \mathrm{Bl}_{\mathcal{F}_0}U_0]^{\mathrm{pre}} & \xrightarrow{p} & [U_1 \rightrightarrows U_0]^{\mathrm{pre}} \end{array} \tag{17}$$

where by abuse of notation, U_0 is the category fibered in sets associated to this algebraic space. One computes (see the discussion around [Sta22, Tag 04Y4]) that the groupoid $\mathfrak{X}$ is given by (U_0', U_1', s', t', m') where

$$\begin{aligned} U_0' &= U_1 \times_{t,U_0,p} \mathrm{Bl}_{\mathcal{F}_0}U_0 \\ U_1' &= U_1 \times_{t,U_0,p\cdot s} \mathrm{Bl}_{\mathcal{F}_1}U_1 \\ s' &: (x,y) \mapsto (x, \widetilde{s}(y)) \\ t' &: (x,y) \mapsto (m(x,p(y)), \widetilde{t}(y)) \end{aligned}$$

By (15),

$$\begin{aligned} U_1' &= U_1 \times_{t,U_0,p\cdot s} (U_1 \times_{s,U_0,p} \mathrm{Bl}_{\mathcal{F}_0} U_0) \\ &= (U_1 \times_{t,U_0,s} U_1) \times_{t\cdot pr_2,U_0,p} \mathrm{Bl}_{\mathcal{F}_0} U_0 \\ s' &: ((x,y),z) \mapsto (x,z) \\ t' &: ((x,y),z) \mapsto (y,z) \end{aligned}$$

From this expression, $\mathfrak{X}$ is a banal groupoid whose stackyfication is equivalent to the scheme $\mathrm{Bl}_{\mathcal{F}_0} U_0$, as the relations s', t' identify all the points of the U_1 factor. Then the stackyfication of (17)gives us a 2-Cartesian diagram:

$$\begin{array}{ccc} \mathrm{Bl}_{\mathcal{F}_0} U_0 & \xrightarrow{\widetilde{\pi}} & U_0 \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Bl}_{\mathfrak{F}} \mathfrak{P} & \xrightarrow{\pi} & \mathfrak{P} \end{array} \tag{18}$$

This discussion proves that π is representable. Recall that a morphism of stacks is birational if there exists an isomorphism on dense open substacks on source and target (see [CMW12]). By proposition 3.2.2 we deduce that π is proper and birational. □

Proposition 6.2.3. *Let $\mathfrak{F}$ be a coherent sheaf on $\mathfrak{P}$.*

1. *For any line bundle $\mathfrak{L}$ on $\mathfrak{P}$, we have $\mathrm{Bl}_{\mathfrak{F}\otimes\mathfrak{L}}\mathfrak{P} = \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and also $\mathrm{Bl}^{HL}_{\mathfrak{F}\otimes\mathfrak{L}}\mathfrak{P} = \mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$.*
2. *Let $\mathfrak{E}$, $\mathfrak{F}, \mathfrak{G}$ be coherent $\mathcal{O}_{\mathfrak{P}}$-modules. Assume that we have an exact sequence $0 \to \mathfrak{E} \to \mathfrak{F} \to \mathfrak{G} \to 0$.*
   (a) *If the sequence is locally split and $\mathfrak{E}$ is locally free, then there are isomorphisms $\mathrm{Bl}_{\mathfrak{F}} X \simeq \mathrm{Bl}_{\mathfrak{G}} X$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}} X \simeq \mathrm{Bl}^{HL}_{\mathfrak{G}} X$.*
   (b) *If $\mathfrak{G}$ is locally free, then there are isomorphisms $\mathrm{Bl}_{\mathfrak{F}} X \simeq \mathrm{Bl}_{\mathfrak{E}} X$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}} X \simeq \mathrm{Bl}^{HL}_{\mathfrak{E}} X$.*

Proof. All the statements are local so they follow from the same statements on schemes with the two different blow-ups. For the Villamayor-Rossi blowup $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$, the schematic statements are Propositions 3.2.3 and 3.5.4 and for the Hu–Li blowup $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$, it follows from Propositions 4.2.2 and 4.2.3. □

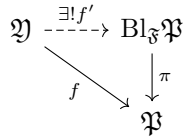
In Definition 4.1.1, we define the notion of diagonalizable morphism of sheaves on a scheme, we can extend it directly to algebraic stacks.

Theorem 6.2.4 (Universal property of the Rossi desingularization)**.** *Let $\pi : \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ be as above. Then*

1. *The sheaf $(\pi^*\mathfrak{F})^{\mathrm{tf}}$ is locally free of the same generic rank as $\mathfrak{F}$.*
2. *The morphism $\pi : \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ satisfies the following universal property: For any morphism of stacks $f : \mathfrak{Y} \to \mathfrak{P}$ such that $(f^*\mathfrak{F})^{\mathrm{tf}}$ is torsion-free of the same generic rank as $\mathfrak{F}$, there is a unique[1] morphism f', which makes the*

[1]To be precise, there exists a morphism f', unique up to a unique 2-morphism.

following diagram 2-commutative



Proof. The statement follows from corollary 3.5.2. □

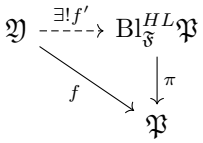
From the beginning of §4 (Definitions 4.1.1 4.1.3 4.1.3 and Proposition 4.1.4), we can define the notion of diagonalizable sheaves or diagonalizable morphism of sheaves on Artin stacks as follows.

Definition 6.2.5.
- A coherent sheaf $\mathfrak{F}$ on $\mathfrak{P}$ is *diagonalizable* if for any scheme S and morphism $f : S \to \mathfrak{P}$, the sheaf $f^*\mathfrak{F}$ is diagonalizable that is its Fitting ideals $F_i(f^*\mathfrak{F})$ are principal.
- A *diagonalization* of a coherent sheaf $\mathfrak{F}$ is a morphism $\pi : \widetilde{\mathfrak{P}} \to \mathfrak{P}$ such that $\pi^*\mathfrak{F}$ is locally diagonalizable.

Remark 6.2.6. Using the presentation of $\mathfrak{P}$, we could also define that $\mathfrak{F}$ is diagonilazable if $\mathcal{F}_0$ is.

Theorem 6.2.7 (Universal property of the diagonalization)**.** *Let $\pi : \mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ be as above. Then*

1. *The sheaf $\pi^*\mathfrak{F}$ is locally diagonalizable of the same generic rank as $\mathfrak{F}$.*
2. *The blow-up $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$ satisfies the universal property: For any morphism of stacks $f : \mathfrak{Y} \to \mathfrak{P}$ such that $f^*\mathfrak{F}$ is locally diagonalizable of the same generic rank as $\mathfrak{F}$, there is a unique morphism f', which makes the following diagram 2-commutative:*



Proof. The statement follows by Theorem 4.1.15. □

Remark 6.2.8. If $\mathfrak{P}$ has the resolution property in the sense of [Tot04], then we have that $\pi : \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ is projective. Indeed, if $\mathfrak{P}$ has the resolution property, then we have a global locally free sheaf $\mathfrak{E}$ with a surjective morphism

$$\mathfrak{E} \to \mathfrak{F} \to 0.$$

This allows us to define $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ via the graph construction and thus the resulting stack is projective over $\mathfrak{P}$. Note that projectivity is not local on the target, and thus, even though the local construction is projective, π may not be projective. By [Tot04] stacks which are not global quotient stacks do not have the resolution property. Many of the stacks that we work with are not global quotients. For more details on stacks which are not a global quotients see [Kre13].

7. Application to stable maps

In this section we apply the results in Section 6 to construct reduced Gromov–Witten invariants.

Given X a smooth subvariety in a projective space $\mathbb{P}^r$, there is an embedding of the moduli space of stable maps to X in the moduli space of stable maps to $\mathbb{P}^r$. The moduli space of genus *zero* stable maps to a projective space $\mathbb{P}^r$ is a smooth irreducible DM stack. If X is a hypersurface of degree k (or more generally a complete intersection) in $\mathbb{P}^r$, there is a locally free sheaf $\mathcal{E}_k$ on the moduli space of stable maps to $\mathbb{P}^r$, such that the moduli space of maps to X is cut out by the zero locus of a section of this sheaf. These statements are not true in higher genus. In general, the moduli space of stable maps to $\mathbb{P}^r$ has several irreducible components of different dimensions. We still have a natural sheaf $\mathcal{E}_k$ equipped with a section, but $\mathcal{E}_k$ is not locally free: its rank is different on different irreducible components.

There are several ways to use Section 6 to fix the above problem. We blow up the Picard stack along certain sheaves and we define reduced invariants via an obstruction theory of the main component of $\widetilde{\mathcal{M}}_{g,n}(X,d)$ relative to this blow-up of the Picard stack. We also recall maps with fields [CL12] and then we construct a blow-up of it which makes the resulting stack as simple as possible. The resulting stack gives an alternative definition of reduced invariants, which is not intrinsic; the relation between these two invariants is similar in spirit to a Quantum Lefschetz theorem. The definition we give is more intrinsic, but working with maps with fields instead of maps is more suited to approaching Conjecture 1.0.2 and Conjecture 1.0.3. See [CL15, LO22, LO21] for the proof of Conjecture 1.0.3 in genus one and two.

7.1. **Moduli spaces of stable maps.** Let $\mathfrak{M}_{g,n}$ denote the stack of genus g pre-stable curves with n marked points and let $\underline{\mathfrak{C}}_{g,n}$ denote its universal curve. Let $\mathfrak{Pic}_{g,n,d}$ denote the Artin stack which parameterises genus g pre-stable curves, with n marked points, together with a line bundle of degree d. Let $\mathfrak{Pic}^{\mathrm{st}}_{g,n,d}$ denote the open subset of $\mathfrak{Pic}_{g,n,d}$ consisting of $(C, p_1 \dots p_n, L)$ which satisfy the stability condition

$$L^{\otimes 3} \otimes \omega_C \left(\sum_{i=1}^{n} p_i \right)$$

is ample. Notice that $\mathfrak{M}_{g,n}$ and $\mathfrak{Pic}_{g,n,d}$ are not separated, but they are smooth and irreducible. The stack $\mathfrak{Pic}_{g,n,d}$ is locally Noetherian and the stack $\mathfrak{Pic}^{\mathrm{st}}_{g,n,d}$ is Noetherian. From now on we fix g, n, d and the stability condition and we drop all the indices.

We define $\mathfrak{C}$ the universal curve over $\mathfrak{Pic}$ by the Cartesian diagram. Notice that we also have a universal line bundle $\mathfrak{L}$ over $\mathfrak{C}$.

$$\begin{array}{ccc} \mathfrak{L} & & \\ \downarrow & & \\ \mathfrak{C} & \longrightarrow & \underline{\mathfrak{C}} \\ \downarrow{\scriptstyle \pi} & \ulcorner & \downarrow \\ \mathfrak{Pic} & \longrightarrow & \mathfrak{M}. \end{array}$$

We form the cone of sections of $\mathfrak{L}$ as in Chang-Li ([CL15], Section 2)

$$S(\pi_*\mathfrak{L}) := \operatorname{Spec}\operatorname{Sym}\left(R^1\pi_*(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})\right) \to \mathfrak{Pic}. \tag{19}$$

The definition requires further explanations as follows.

(1) We use the notation $S(\pi_*\mathfrak{L})$ for the cone of sections of $\mathfrak{L}$, which in [CL15] is denoted by $C(\pi_*\mathfrak{L})$; for us $C(\pi_*\mathfrak{L})$ denotes $\operatorname{Spec}\operatorname{Sym}\pi_*\mathfrak{L}$. In [CL15] the authors prove that $S(\pi_*\mathfrak{L})$ is a stack which parameterises (C, L, s), where $(C, L) \in \mathfrak{Pic}$ and $s \in H^0(C, L)$.

(2) Usually, the total space of a locally free sheaf $\mathfrak{E}$ is $\operatorname{Spec}\operatorname{Sym}(\mathfrak{E}^\vee)$. Here we would like to take $\mathfrak{E} := R^0\pi_*\mathfrak{L}$. By Serre duality, we have $\mathfrak{E}^\vee = R^1\pi_*(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})$.

For the rest of the section, let

$$\mathfrak{F} := R^1\pi_*(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}). \tag{20}$$

Note that since π is proper, we have that $\mathfrak{F}$ is a coherent sheaf on $\mathfrak{Pic}$. As defined in Section 2.3, we consider the stack $\operatorname{Spec}\operatorname{Sym}\mathfrak{F}$, which is an abelian cone stack over $\mathfrak{Pic}$.

Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ be the moduli space of genus g, degree d stable maps, with n marked points. As in [CFK10], we have that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is an open sub-stack of the stack

$$S(\pi_*\mathfrak{L}^{\oplus r+1}) = \operatorname{Spec}\operatorname{Sym}\left(\oplus_{i=0}^r \mathfrak{F}\right) \to \mathfrak{Pic}. \tag{21}$$

As before, a point of this cone over $(C, L) \in \mathfrak{Pic}$ is $(C, L, \underline{s})$ with $\underline{s} \in H^0(C, L)^{\oplus r+1}$. Note that

$$S(\pi_*\mathfrak{L}^{\oplus r+1}) = \overbrace{S(\pi_*\mathfrak{L}) \times_{\mathfrak{Pic}} \cdots \times_{\mathfrak{Pic}} S(\pi_*\mathfrak{L})}^{r+1 \text{ times}} \to \mathfrak{Pic}.$$

We define $\mathcal{L}, \mathcal{C}$ by the following Cartesian diagram

$$\begin{array}{ccc}
\mathcal{L} & \longrightarrow & \mathfrak{L} \\
\downarrow & \ulcorner & \downarrow \\
\mathcal{C} & \longrightarrow & \mathfrak{C} \\
\downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \pi} \\
S(\pi_*\mathfrak{L}^{\oplus r+1}) & \longrightarrow & \mathfrak{Pic}.
\end{array}$$

The complex $\oplus_{i=0}^r R^\bullet\overline{\pi}_*\mathcal{L}$ is a dual obstruction theory for the natural projection $S(\pi_*\mathfrak{L}^{\oplus r+1}) = \operatorname{Spec}\operatorname{Sym}\left(\oplus_{i=0}^r \mathfrak{F}\right) \to \mathfrak{Pic}$ (see [CL15]).

Lemma 7.1.1. *In notation as above we have an isomorphism of sheaves*

$$(\mathfrak{F})^\vee \simeq \pi_*\mathfrak{L}$$

over $\mathfrak{Pic}$.

Proof. The claim can be obtained by Verdier duality we have

$$R\mathcal{H}om(R\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}, \mathcal{O}) \simeq R\pi_*R\mathcal{H}om(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}, \omega_{\mathfrak{C}/\mathfrak{Pic}}[1]).$$

We can obtain an explicit resolution of $\mathfrak{F}$ as follows. Let A be a sufficiently high power of a very ample section of the morphism $\mathfrak{C} \to \mathfrak{Pic}$ that is we have $R^1\pi_*\mathfrak{L}(A) = 0$. Then, we have a morphism

$$0 \to \mathfrak{L} \to \mathfrak{L}(A) \to \mathfrak{L}(A)|_A \to 0. \tag{22}$$

This induces a long exact sequence

$$0 \to R^0\pi_*\mathfrak{L} \to R^0\pi_*\mathfrak{L}(A) \to R^0\pi_*\mathfrak{L}(A)|_A \to R^1\pi_*\mathfrak{L} \to 0. \tag{23}$$

Similarly, we can resolve $\mathfrak{F} = R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}$ considering the dual exact sequence

$$0 \to \mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to \mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to \mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A \to 0.$$

Pushing forward to $\mathfrak{Pic}$ we get

$$R^0\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A \to R^1\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to 0. \tag{24}$$

By Serre duality we have isomorphisms $R^0\pi_*\mathfrak{L}(A) \simeq (R^1\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})^\vee$ and $R^0\pi_*\mathfrak{L} \simeq (R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})^\vee$. Applying the functor $\mathcal{H}om(-, \mathcal{O})$ to 24 and using that it is left-exact, we get

$$0 \to (\mathfrak{F})^\vee \to R^0\pi_*\mathfrak{L}(A) \to R^0\pi_*\mathfrak{L}(A)|_A \tag{25}$$

Comparing (23) with (25) we get that $(\mathfrak{F})^\vee \simeq \pi_*\mathfrak{L}$.

The last claim follows from the fact that taking the dual of vector bundle commutes with base change. □

Remark 7.1.2. The proof of Lemma 7.1.1 shows that we have a resolution to the left given by (24). The isomorphisms after Equation (23) show that the morphism

$$R^0\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A \to R^1\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}$$

is dual to

$$R^0\pi_*\mathfrak{L}(A) \to R^0\pi_*\mathfrak{L}(A)|_A.$$

We will use this duality in Section 8. Hu and Li work with the resolution to the right we have in 23. In the previous sections we used the resolution to the left Equation (23). Since dual morphisms have the same fitting ideals, both morphisms give the same Hu–Li blow-up.

Definition 7.1.3. Consider the Zariski closure in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of the locus where the curve is smooth of genus g. We call this component the main component and we denote it by $\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$.

Note that on $\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$, the universal curve is generically smooth and $\pi_*\mathcal{L}$ is generically a vector bundle.

Proposition 7.1.4. *We have that $\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ is an open substack of*

$$\operatorname{Spec}(\operatorname{Sym} \oplus_{i=0}^r \mathfrak{F})^{\mathrm{tf}} = \prod_{\mathfrak{Pic}}^{r+1} \operatorname{Spec}(\operatorname{Sym} \mathfrak{F})^{\mathrm{tf}}.$$

Proof. We have a surjective morphism $\mu : \operatorname{Sym} \mathfrak{F} \to (\operatorname{Sym} \mathfrak{F})^{\mathrm{tf}}$. The torsion of $\operatorname{Sym} \mathfrak{F}$ is in the kernel of μ.

This gives a closed embedding

$$\operatorname{Spec}(\operatorname{Sym} \pi_*\mathfrak{F})^{\mathrm{tf}} \hookrightarrow \operatorname{Spec} \operatorname{Sym} \pi_*\mathfrak{F}.$$

$\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ is the open substack of $\operatorname{Spec}(\operatorname{Sym} \pi_*\mathfrak{F}^{\oplus r+1})^{\mathrm{tf}}$ given by imposing the stability condition for stable maps. Moreover, for a module over an integral domain R, we have that taking torsion commutes with direct sums. This gives the equality in the statement. □

Remark 7.1.5. For $g \geq 1$ the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ has several irreducible components (see for example [HL10]). While for $g = 1$ these components are understood, in general we do not have a complete picture. However, after desingularizing $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, theorem 5.2.1 describes its components.

7.2. **Maps with fields.** We recall the construction of maps with fields in [CL12] and its properties.

In the following we fix $k \in \mathbb{Z}$, $k > 1$ and we consider the sheaf

$$\pi_*(\mathfrak{L}^{\oplus r+1} \oplus (\mathfrak{L}^{\otimes -k} \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}))$$

on $\mathfrak{Pic}$ with corresponding abelian cone

$$S\left(\pi_*(\mathfrak{L}^{\oplus r+1} \oplus (\mathfrak{L}^{\otimes -k} \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})\right) = \operatorname{Spec}\operatorname{Sym} R^1\pi_*\left(\left(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^{\oplus r+1} \oplus \mathfrak{L}^{\otimes k}\right) \xrightarrow{\mu^p} \mathfrak{Pic}.$$

Imposing stable map stability we obtain the moduli space of maps with fields $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$. So an element of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ over $(C, L, \underline{s}) \in \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is given by a choice of a section $p \in H^0(C, L^{\otimes -k} \otimes \omega_C)$. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathcal{C}^p & \xrightarrow{\nu^p} & \mathcal{C} \\ \downarrow{\scriptstyle \overline{\pi}^p} & \ulcorner & \downarrow{\scriptstyle \overline{\pi}} \\ \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p & \longrightarrow & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d). \end{array} \tag{26}$$

The complex

$$\mathbb{E}^\bullet := R^\bullet \overline{\pi}^p_*(\oplus_{i=0}^r \mathcal{L} \oplus \mathcal{L}^{\otimes -k} \otimes \omega_{\overline{\pi}^p})$$

is a dual obstruction theory for the morphism μ^p.

The stack $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ is not proper, but the perfect obstruction theory admits a cosection σ. This data gives a cosection localised virtual class $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_\sigma$.

For X a smooth subvariety cut out by any regular section of $\mathcal{O}_{\mathbb{P}^r}(k)$ the authors of [CL12] prove that

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_\sigma = (-1)^{(r+1)d+1-g}[\overline{\mathcal{M}}_{g,n}(X, d)]^{\mathrm{vir}} \in A_{d^{\mathrm{vir}}(\overline{\mathcal{M}}_{g,n}(X,d))}(\overline{\mathcal{M}}_{g,n}(X, d)) \tag{27}$$

7.3. **Reduced Gromov–Witten invariants.** We start with a technical lemma below, which allows us to apply the results in Section 4.

Lemma 7.3.1 ([Sta22, Tag 0ESR])**.** *Let $\mathfrak{E}$ be any vector bundle on $\mathfrak{C}_\mathfrak{P}$, the universal curve on some locally Noetherian integral stack $\mathfrak{P}$ over $\mathfrak{M}$. Let $\mathfrak{F} = R^1\pi_{\mathfrak{P}*}\mathfrak{E}$. There exists a blow-up $b : \mathfrak{P}' \to \mathfrak{P}$ such that $b^*\mathfrak{F}$ is a perfect $\mathcal{O}_{\mathfrak{P}'}$ module of tor dimension ≤ 1.*

Proof. Consider $b : \mathfrak{P}' = \mathrm{Bl}^{HL}_\mathfrak{F}(\mathfrak{P}) \to \mathfrak{P}$. Locally on $\mathfrak{P}'$, $F_r(b^*\mathfrak{F})$, is a principal ideal defined by a non-zero divisor and $F_{r-1}(b^*\mathfrak{F}) = 0$. Then $b^*\mathfrak{F}$ has tor dimension ≤ 1 by [Sta22, Tag 0F7M]. Note that $b^*\mathfrak{F} = R^1\pi_{\mathfrak{P}'*}(\overline{b}^*\mathfrak{E})$, for $\overline{b} = \mathfrak{C}_{\mathfrak{P}'} \to \mathfrak{C}_\mathfrak{P}$. □

Remark 7.3.2. This allows us to apply theorem 4.1.14 to $\mathrm{Bl}^{HL}_{R^1\pi_*\mathfrak{E}}\mathfrak{Pic}$, since now $R^1\pi_*\mathfrak{E}$ is locally diagonalizable of tor dimension ≤ 1.

In the following we fix $k \geq 1$. Let

$$\mathfrak{E}_k := R^1\pi_*\mathfrak{L}^{\otimes k}$$

denote the sheaf on $\mathfrak{Pic}$ whose restriction to a point $(C, L) \in \mathfrak{Pic}$ is

$$H^1(C, L^{\otimes k}) \simeq (H^0(C, L^{\otimes -k} \otimes \omega_C))^\vee.$$

We define

$$\widetilde{\mathfrak{Pic}}_k := \mathrm{Bl}^{HL}_{\mathfrak{E}_k}\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{Pic}.$$

Let $p_k : \widetilde{\mathfrak{Pic}}_k \to \mathfrak{Pic}$ be the natural projection. Consider the Cartesian diagram

$$\begin{array}{ccc} \tilde{\mathfrak{C}} & \xrightarrow{q_k} & \mathfrak{C} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \widetilde{\mathfrak{Pic}}_k & \xrightarrow{p_k} & \mathfrak{Pic} \end{array}$$

where $\mathfrak{C}$ is the universal curve over $\mathfrak{Pic}$.

We define $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ as the following Cartesian diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) & \xrightarrow{\overline{p}_k} & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \\ \downarrow{\scriptstyle \widetilde{\mu}} & \ulcorner & \downarrow{\scriptstyle \mu} \\ \widetilde{\mathfrak{Pic}}_k & \xrightarrow{p_k} & \mathfrak{Pic}. \end{array} \tag{28}$$

One can see that we have an open embedding

$$\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \hookrightarrow \mathrm{Spec}\,\mathrm{Sym}\ \ \oplus_{i=0}^r p_k^*\mathfrak{F}.$$

We define the main component of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ as the fiber product

$$\begin{array}{ccc} \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) & \xrightarrow{\overline{p}_k} & \overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \\ \downarrow{\scriptstyle \widetilde{\mu}^\circ} & \ulcorner & \downarrow{\scriptstyle \mu^\circ} \\ \widetilde{\mathfrak{Pic}}_k & \xrightarrow{p_k} & \mathfrak{Pic}. \end{array} \tag{29}$$

Again, we have an open embedding

$$\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \subset \mathrm{Spec}\,\mathrm{Sym}\ \ \oplus_{i=0}^r (p_k^*\mathcal{F})^{\mathrm{tf}}.$$

By [CL12] the morphism μ has a perfect obstruction theory equal to $\oplus_{i=0}^r R^\bullet\overline{\pi}_*\mathcal{L}$. Let $\widehat{\pi} : \tilde{\mathcal{C}} \to \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ be the universal curve and let $\widehat{q_k} : \tilde{\mathcal{C}} \to \mathcal{C}$ the morphism induced by q_k. The morphism $\tilde{\mu}$ has a dual perfect obstruction theory equal to

$$\oplus_{i=0}^r R^\bullet\widehat{\pi}_*\widehat{q_k}^*\mathcal{L}.$$

These perfect obstruction theories induce virtual classes

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} := \mu^![\mathfrak{Pic}] \in A_*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d))$$

and

$$[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} := \widetilde{\mu}^![\widetilde{\mathfrak{Pic}}] \in A_*(\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)),$$

where $\mu^!$ and $\widetilde{\mu}^!$ are defined as in [Man11]. Similarly, we define $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ as the following Cartesian diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p & \longrightarrow & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p \\ \downarrow{\scriptstyle \widetilde{\mu}^p} & \ulcorner & \downarrow{\scriptstyle \mu^p} \\ \widetilde{\mathfrak{Pic}}_k & \xrightarrow{p_k} & \mathfrak{Pic}. \end{array} \tag{30}$$

This gives a localised virtual class $[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}$. See [CL12, Section 3] for details.

Remark 7.3.3. While $\mathfrak{Pic}$ is smooth, $\widetilde{\mathfrak{Pic}}_k$ does not need to be smooth. This is not a problem, all we need for a well-defined virtual class is that $\widetilde{\mathfrak{Pic}}$ has pure dimension. This is true, since $\mathfrak{Pic}$ has pure dimension and p_k is birational.

Theorem 7.3.4. *In notation as above, let*

$$\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p = \cup_{\lambda\in\Lambda}\,\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{p,\lambda}$$
$$\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) = \cup_{\theta\in\Theta}\,\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta},$$

with $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{p,\lambda}$ irreducible components of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$ and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$ irreducible components of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$. Let

$$\widehat{\pi}^{p,\lambda}:\tilde{\mathcal{C}}^{\lambda}\to\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{p,\lambda}$$
$$\tilde{\pi}^{\theta}:\tilde{\mathcal{C}}^{\theta}\to\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}.$$

The following statements hold.

1. *The morphism $\bar{p}_k$ is birational and proper.*
2. *The irreducible components $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{p,\lambda}$ and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$ are smooth over their image in $\widetilde{\mathfrak{Pic}}_k$. In particular, $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d)$ is smooth over $\widetilde{\mathfrak{Pic}}$.*
3. *The sheaf $\widehat{\pi}^{p,\lambda}_*ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{p,\lambda}$, the sheaf $\tilde{\pi}^{\theta}_*ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$. In particular $\tilde{\pi}^{\circ}_*ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d)$.*

Proof. 1. We have that $\bar{p}_k$ is proper, as p_k is proper.

2. Consider the following diagram

$$\begin{array}{ccc}\operatorname{Spec}\operatorname{Sym}p_k^*(\mathfrak{F}\oplus\mathfrak{E}_k) & \longrightarrow & \operatorname{Spec}\operatorname{Sym}\mathfrak{F}\oplus\mathfrak{E}_k\\ \downarrow & \ulcorner & \downarrow\\ \widetilde{\mathfrak{Pic}}_k & \xrightarrow{\quad p_k\quad} & \mathfrak{Pic}.\end{array}\tag{31}$$

We have that $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$ is an open sub-stack of $\operatorname{Spec}\operatorname{Sym}p_k^*(\mathfrak{E}_k\oplus\oplus_{i=0}^r\mathfrak{F})$ and by Theorem 5.2.1 we have that the irreducible components of the stacks

$$\operatorname{Spec}\operatorname{Sym}\ \oplus_{i=0}^r\mathcal{F}\oplus\mathfrak{E}_k\quad\text{and}\quad\operatorname{Spec}\operatorname{Sym}\ \oplus_{i=0}^r\mathfrak{F}$$

are smooth over their image in $\widetilde{\mathfrak{Pic}}_k$.

This shows that $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{p,\lambda}$ and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^{\theta}$ are smooth over their image in $\widetilde{\mathfrak{Pic}}_k$. In particular

$$\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d)=\operatorname{Spec}\operatorname{Sym}\left(\oplus_{i=0}^r p_k^*\mathfrak{F}\right)^{\mathrm{tf}}\text{ is smooth.}$$

3. Let $\mu^{\lambda}:\overline{M}_{g,n}(\mathbb{P}^r,d)^{p,\lambda}\to\mathfrak{Pic}$ be the restriction of μ^p. Let Z^{λ} be the image of μ^{λ}. Let $\pi^{\lambda}:\mathfrak{C}^{\lambda}\to Z^{\lambda}$ be the restriction of π. Let $\tilde{Z}^{\lambda}$ be the fibre product

$$\begin{array}{ccc}\tilde{Z}^{\lambda} & \xrightarrow{p_k^{\lambda}} & Z^{\lambda}\\ \downarrow & \ulcorner & \downarrow\\ \widetilde{\mathfrak{Pic}}_k & \xrightarrow{p_k} & \mathfrak{Pic}.\end{array}\tag{32}$$

Let $\tilde{\pi}^\lambda : \tilde{\mathfrak{C}}^\lambda \to Z^\lambda$ be the restriction of $\tilde{\pi}$ and let $q_k^\lambda : \tilde{\mathfrak{C}}^\lambda \to \mathfrak{C}^\lambda$ be the restriction of q_k. By commutativity of proper push-forwards with base-change we have that

$$(p_k^\lambda)^* R^\bullet \pi_*^\lambda \mathfrak{L} \simeq R^\bullet \tilde{\pi}_*^\lambda (q_k^\lambda)^* \mathfrak{L}$$

Again, cohomology and base change in the Cartesian diagram

$$\begin{array}{ccc} \tilde{\mathcal{C}}^\lambda & \xrightarrow{\nu^\lambda} & \tilde{\mathfrak{C}}^\lambda \\ \downarrow{\scriptstyle \widehat{\pi}^\lambda} & \ulcorner & \downarrow{\scriptstyle \tilde{\pi}^\lambda} \\ \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^\lambda & \xrightarrow{\mu^\lambda} & \tilde{Z}^\lambda. \end{array} \tag{33}$$

gives

$$R^\bullet \widehat{\pi}_*^\lambda ev^* \mathcal{O}(k) = (\mu^\lambda)^* R^\bullet \tilde{\pi}_*^\lambda \mathfrak{L}^k. \tag{34}$$

We have a short exact sequence

$$0 \to R^0 \tilde{\pi}_*^\lambda \mathfrak{L}^k \to E^0 \xrightarrow{\phi} E^1 \to R^1 \tilde{\pi}_*^\lambda \mathfrak{L}^k \to 0.$$

By construction we have that ϕ is locally diagonal. By eq. (34) we have that

$$R^\bullet \widehat{\pi}_*^\lambda ev^* \mathcal{O}(k) \simeq [(\mu^\lambda)^* E^0 \xrightarrow{(\mu^\lambda)^* \phi} (\mu^\lambda)^* E^1].$$

Since $(\mu^\lambda)^* \phi$ is locally diagonal, proposition 4.1.2 implies that $R^0 \widehat{\pi}_*^\lambda ev^* \mathcal{O}(k)$ is locally free.

A similar argument shows that $\tilde{\pi}_*^\theta ev^* \mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^\theta$ and in particular $\tilde{\pi}_*^\circ ev^* \mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}^\circ(\mathbb{P}^r, d)$. □

Remark 7.3.5. In [VZ08, HL10] the authors desingularize $\pi_* \mathfrak{L}$ for genus one and, as a consequence, the sheaf $p_k^*(\pi_* \mathfrak{L}^{\otimes k})^{\mathrm{tf}}$ becomes locally free on $\widetilde{\mathfrak{Pic}}_1$ for any $k \geq 1$. In the following we give a proof of this fact.

In a neighbourhood of $(\tilde{C}, \tilde{L}) \in \mathrm{Bl}_{\pi_* \mathfrak{L}} \mathfrak{Pic}$ we can choose Z a section of $\mathfrak{L}^{\otimes k-1}$. This gives an exact sequence

$$0 \to R^0 \tilde{\pi}_* \mathfrak{L} \to R^0 \tilde{\pi}_* (\mathfrak{L}^{\otimes k}) \to R^0 \tilde{\pi}_* (\mathfrak{L}^{\otimes k}|_Z) \to R^1 \tilde{\pi}_* \mathfrak{L} \to R^1 \tilde{\pi}_* (\mathfrak{L}^{\otimes k}) \to 0. \tag{35}$$

Notice that for $\tilde{C}$ a curve of genus one, and for any $k \geq 1$, we have an isomorphism

$$H^1(\tilde{C}, \tilde{L}) \simeq H^1(\tilde{C}, \tilde{L}^{\otimes k}). \tag{36}$$

This implies that the last arrow in sequence (35) is an isomorphism and thus we get a short exact sequence

$$0 \to R^0 \tilde{\pi}_* \mathfrak{L} \to R^0 \tilde{\pi}_* (\mathfrak{L}^{\otimes k}) \to R^0 \tilde{\pi}_* (\mathfrak{L}^{\otimes k}|_Z) \to 0.$$

Since $R^0 \tilde{\pi}_* (\mathfrak{L}^{\otimes k}|_Z)$ and $R^0 \tilde{\pi}_* \mathfrak{L}$ are locally free sheaves on $\mathrm{Bl}_{\pi_* \mathfrak{L}} \mathfrak{Pic}$, we get that $R^0 \tilde{\pi}_* (\mathfrak{L}^{\otimes k})$ is locally free.

Proposition 7.3.6. *We have the following equality*

$$(\overline{p}_k)_* [\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}}$$

Proof. Note that by cohomology and base change we have that $R^\bullet \tilde{\pi}_* \overline{p}_k^* \mathcal{L} = \overline{p}_k^* R^\bullet \overline{\pi}_* \mathcal{L}$. As p_k is birational and projective we have $(p_k)_* [\widetilde{\mathfrak{Pic}}_k] = [\mathfrak{Pic}]$. We now apply Costello's Pushforward theorem ([HW22]) to (28) and we get

$$(\overline{p}_k)_* [\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}}.$$ □

In the following we define reduced Gromov–Witten invariants.

Assumption 7.3.7. In the following we fix $d > 2g-2$. For C a smooth genus g curve, L a line bundle of degree d, and $d > 2g-2$, we have that $H^1(C, L) = 0$. This shows that for $d > 2g-2$, the locus of stable maps with smooth domain is smooth and irreducible, so its closure is an irreducible component of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.

Construction 7.3.8. Let $X \subset \mathbb{P}^r$ be a smooth complete intersection, given by a section

$$s : \mathcal{O}_{\mathbb{P}}(1)^{\oplus r+1} \to \mathcal{O}_{\mathbb{P}}(k).$$

We define

$$\begin{array}{ccc} \overline{\mathcal{M}}^{\circ}_{g,n}(X,d) & \longrightarrow & \overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d) \\ \downarrow & \ulcorner & \downarrow \\ \overline{\mathcal{M}}_{g,n}(X,d) & \longrightarrow & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d). \end{array} \tag{37}$$

The main component $\overline{\mathcal{M}}^{\circ}_{g,n}(X,d)$ does not have a perfect obstruction theory. In order to define a perfect obstruction theory on it, we define

$$\begin{array}{ccc} \widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d) & \longrightarrow & \widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d) \\ \downarrow & \ulcorner & \downarrow \\ \widetilde{\mathcal{M}}_{g,n}(X,d) & \xrightarrow{i} & \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d). \end{array} \tag{38}$$

By theorem 7.3.4 we get that $(\pi_*\mathcal{L}^{\otimes k})^{\mathrm{tf}}$ is locally free on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. We define

$$[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)]^{\mathrm{vir}} = i^![\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d)]. \tag{39}$$

Note that for any $1 \leq i \leq n$ we have morphisms

$$\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \xrightarrow{ev_i} \mathbb{P}^r$$

and

$$\widetilde{\mathcal{M}}_{g,n}(X,d) \to \overline{\mathcal{M}}_{g,n}(X,d) \xrightarrow{ev_i} X.$$

By abuse of notation we denote these compositions by ev_i.

Proposition 7.3.9. *Let assume $d > 2g-2$. Let $p' : \widetilde{\mathfrak{Pic}}' \to \mathfrak{Pic}$ and $p'' : \widetilde{\mathfrak{Pic}}'' \to \mathfrak{Pic}$ be birational projective maps such that $(p_i^*\pi_*\mathcal{F})^{\mathrm{tf}}$ and $p_i^*(\pi_*\mathfrak{L}^{\otimes k})^{\mathrm{tf}}$ are locally free. Consider $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)'$ and $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)''$ defined analogously to $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)$ above. Then we have*

$$\int_{[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)']^{\mathrm{vir}}} \prod ev^*\gamma_i = \int_{[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)'']^{\mathrm{vir}}} \prod ev^*\gamma_i$$

This proposition permits us to define the reduced Gromov–Witten invariants as they are independent of the blowing-up of $\mathfrak{Pic}$.

Definition 7.3.10. For $d > 2g-2$, we call *reduced* Gromov–Witten invariants of X, the following numbers

$$\int_{[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)]^{\mathrm{vir}}} \prod ev^*\gamma_i.$$

Proof of Proposition 7.3.9. Consider the fiber product

$$\begin{array}{ccc} \widehat{\mathfrak{Pic}} & \longrightarrow & \widetilde{\mathfrak{Pic}}' \\ \downarrow & & \downarrow{\scriptstyle p'} \\ \widetilde{\mathfrak{Pic}}'' & \xrightarrow{p''} & \mathfrak{Pic} \end{array}$$

This gives

$$\begin{array}{ccccc} \widehat{\mathcal{M}}(X)^\circ & \xrightarrow{q''} & \widetilde{\mathcal{M}}^\circ_{g,n}(X,d)' & \longrightarrow & \widetilde{\mathfrak{Pic}}' \\ {\scriptstyle q'}\downarrow & & \downarrow & & \downarrow{\scriptstyle p'} \\ \overline{\mathcal{M}}^\circ_{g,n}(X,d)'' & \longrightarrow & \overline{\mathcal{M}}^\circ_{g,n}(X,d) & \longrightarrow & \mathfrak{Pic} \\ \downarrow & & \downarrow & & \\ \widetilde{\mathfrak{Pic}}'' & \xrightarrow{p''} & \mathfrak{Pic} & & \end{array}$$

and thus q' and q'' are birational and projective.

Let $\mu' : \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)' \to \widetilde{\mathfrak{Pic}}'$, $\mu'' : \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)'' \to \widetilde{\mathfrak{Pic}}''$ and $\hat{\mu} : \widehat{\mathcal{M}}(\mathbb{P}^r) \to \widehat{\mathfrak{Pic}}$ denote the maps which forget sections. Then we have Cartesian diagrams

$$\begin{array}{ccccc} \widehat{\mathcal{M}}(X)^\circ & \longrightarrow & \widehat{\mathcal{M}}(\mathbb{P}^r)^\circ & \xrightarrow{\widehat{\mu}} & \widehat{\mathfrak{Pic}} \\ {\scriptstyle q''}\downarrow & & \downarrow{\scriptstyle r'} & & \downarrow \\ \widetilde{\mathcal{M}}^\circ_{g,n}(X,d)' & \xrightarrow{i'} & \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)' & \xrightarrow{\mu'} & \widetilde{\mathfrak{Pic}}' \end{array}$$

and

$$\begin{array}{ccccc} \widehat{\mathcal{M}}(X)^\circ & \longrightarrow & \widehat{\mathcal{M}}(\mathbb{P}^r)^\circ & \xrightarrow{\widehat{\mu}} & \widehat{\mathfrak{Pic}} \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle r''} & & \downarrow \\ \widetilde{\mathcal{M}}^\circ_{g,n}(X,d)'' & \xrightarrow{i''} & \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)'' & \xrightarrow{\mu''} & \widetilde{\mathfrak{Pic}}'' \end{array}$$

This implies that we have a virtual class

$$[\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} := (i')^![\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ]. \tag{40}$$

Diagram 38 implies that $[\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} = i^![\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ]$. Similarly we also get that $(i'')^![\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ] = i^![\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ]$ and so

$$[\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} = (i'')^![\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ]. \tag{41}$$

Since r' and r'' are birational and proper we have

$$r'_*[\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ] = [\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)'] \text{ and} \tag{42}$$

$$r''_*[\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ] = [\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)'']. \tag{43}$$

Using 40,41, 42, 43 and commutativity of pull-backs with push-forwards, we get

$$\begin{aligned} q''_*[\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} &= [\widetilde{\mathcal{M}}^\circ_{g,n}(X,d)']^{\mathrm{vir}} \text{ and} \\ q'_*[\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} &= [\widetilde{\mathcal{M}}^\circ_{g,n}(X,d)'']^{\mathrm{vir}}. \end{aligned} \tag{44}$$

Intersection both equations above with $\prod ev^*\gamma_i$ we get the conclusion. □

We also have the following.

Proposition 7.3.11. *The localised invariants do not depend on the blow-up of $\mathfrak{Pic}$, more precisely,*

$$\deg[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]^{\mathrm{vir}} = \deg[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]^{\mathrm{vir}}.$$

Proof. It follows from eq. (27) and proposition 7.3.9. □

Remark 7.3.12. The results of proposition 7.3.9 and proposition 7.3.11 can be stated at level of virtual classes. The statement of proposition 7.3.9 with virtual classes is given by eq. (44).

Let $\mathfrak{C}_{\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p/\widetilde{\mathfrak{Pic}}} = \cup_i \mathfrak{C}_i$ where $\mathfrak{C}_i$ denotes an irreducible component and let $\mathfrak{C}_0$ denote the component supported on the main component of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$. Let $[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]_i^{\mathrm{vir}}$ denote the class corresponding to the component $\mathfrak{C}_i$.

Proposition 7.3.13. *We have*

$$\deg[\widetilde{\mathcal{M}}^\circ_{g,n}(X,d)]^{\mathrm{vir}} = (-1)^{kd-g+1}\deg[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]_0^{\mathrm{vir}}.$$

Proof. This follows the lines of proof of Corollary 4.4 in [CL15]. Let

$$\mathbb{E}_1^\bullet := R^\bullet\widehat{\pi}^p_*(\oplus_{i=0}^r\mathcal{L}), \qquad \mathbb{E}_2^\bullet := R^\bullet\widehat{\pi}^p_*(\mathcal{L}^{\otimes -k}\otimes\omega_{\widehat{\pi}^p})$$

and let $\mathcal{E}_i = h^1/h^0(\mathbb{E}_i^\bullet)$ and $\mathcal{E} = h^1/h^0(\mathbb{E}^\bullet)$. We have that $\mathcal{E}_i$ is a vector bundle stack on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$ and $\mathcal{E}\simeq\mathcal{E}_1\oplus\mathcal{E}_2$. Let $\mathcal{U}$ be the open subset of the main component of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$, with consists of maps with fields with irreducible source. On $\mathcal{U}$ we have $R^1\widehat{\pi}_*f^*\mathcal{O}(k)=0$, and thus $\mathcal{U}$ is also an open subset of $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)$. Using that $\mathcal{U}$ is smooth and unobstructed, we see that $\mathfrak{C}_{\mathcal{U}/\widetilde{\mathfrak{Pic}}}$ is isomorphic to the vector bundle stack $\mathcal{E}_1|_{\mathcal{U}}$. Since the embedding $\mathfrak{C}_{\mathcal{U}/\widetilde{\mathfrak{Pic}}}\hookrightarrow h^1/h^0(\mathcal{E}|_{\mathcal{U}})$ is

$$(\mathcal{E}_1\oplus 0)|_{\mathcal{U}}\hookrightarrow(\mathcal{E}_1\oplus\mathcal{E}_2)|_{\mathcal{U}}$$

and $\mathfrak{C}_{\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p/\widetilde{\mathfrak{Pic}}}\hookrightarrow\mathcal{E}$ is a closed embedding, we get that $\mathfrak{C}_0\simeq\mathcal{E}_1$. By the definition of the localised cosection virtual class, we get

$$[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]_0^{\mathrm{vir}} = 0^!_{\sigma,loc}[\mathfrak{C}_0] = 0^![0_{\mathcal{E}_2}],$$

where $0_{\mathcal{E}_2}$ is the zero section of $\mathcal{E}_2|_{\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)}$. By Lemma 4.3 in [CL15] with the complex $R := R^\bullet\widehat{\pi}^{p,\circ}_*\mathcal{L}^{\otimes k}$ and Theorem 7.3.4, part 3 we get

$$[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]_0^{\mathrm{vir}} = c_{\mathrm{top}}(R^1\widehat{\pi}^{p,\circ}_*(\mathcal{L}^{\otimes-k}\otimes\omega_{\widehat{\pi}^{p,\circ}}))\cdot[\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)].$$

By Serre duality we have that

$$c_{1-g+kd}(R^1\widehat{\pi}^{p,\circ}_*(\mathcal{L}^{\otimes-k}\otimes\omega_{\widehat{\pi}^p})) = (-1)^{1-g+kd}c_{1-g+kd}(R^0\widehat{\pi}^p_*\mathcal{L}^{\otimes k})$$

and thus

$$[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]_0^{\mathrm{vir}} = (-1)^{1-g+kd}c_{1-g+kd}(R^0\widehat{\pi}^p_*\mathcal{L}^{\otimes k})\cdot[\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)].$$

This proves the claim. □

Conjecture 7.3.14. *Let X be a threefold. Then*

$$\deg[\widetilde{\mathcal{M}}_{g,n}(X,d)^p]_i^{\mathrm{vir}} = c_i\deg[\widetilde{\mathcal{M}}^\circ_{g_i,n}(X,d)]^{\mathrm{vir}},$$

for some $c_i\in\mathbb{Q}$ and $g_i<g$.

Remark 7.3.15. The conjecture has been proved for genus one [Zin09a, Zin09b, Zin08], [CL15], [LO21, LO22] and genus two [LLO22].

In genus $g = 1$ and X a Calabi–Yau three-fold, the conjecture translates into

$$\deg[\overline{\mathcal{M}}_{g,n}(X,d)] = \deg[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)]^{\mathrm{vir}} + \frac{1}{12}\deg[\overline{\mathcal{M}}_{0,n}(X,d)]^{\mathrm{vir}}.$$

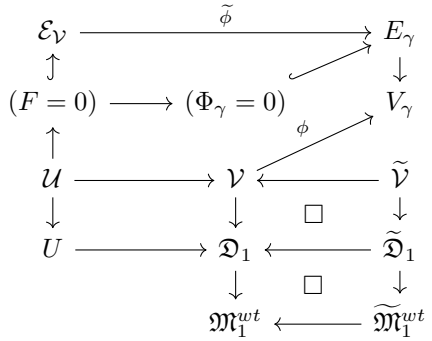
8. Desingularizations in genus one

In genus one, reduced Gromov–Witten invariants were originally defined using the desingularization constructed in [VZ08]. It consists of a sequence of blow-ups determined by the geometry of the moduli space $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$. In [HL10], local equations for the blowup are determined. We aim to compare this desingularization with the one obtained using the Rossi–Villamayor blow-up $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$, with $\mathfrak{F}$ as in eq. (20). In particular, we describe $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ in locally in the spirit of [HL10].

8.0.1. *Charts.* In genus one, the original definition of refined Gromov–Witten invariants comes from [VZ08]. The main is idea is to apply a sequence of blow ups to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in order to desingularize the main component. Strictly speaking, the sequence of blow ups takes place in the stack $\mathfrak{M}_1^{wt}$ of genus-1 prestable curves endowed with a weight. Let Θ_k denote the closure of the loci in $\mathfrak{M}_1^{wt}$ of curves with k trees of rational curves attached to the core. Then one should blow up $\mathfrak{M}_1^{wt}$ along the loci Θ_1, Θ_2, Θ_3 and so on in order to produce a stack $\widetilde{\mathfrak{M}}_1^{wt}$. This process induces a blowup $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ via fibre product.

Given a stratum $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)_\gamma$ corresponding to a weighted graph γ, local equations of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and the local description of Θ_k in that stratum are described explicitly in [HL10]. The purpose of this section is to summarize such local description, give coordinates for the new approach locally and compare both.

It may be helpful to keep in mind the following diagram, described below.



Fix a weighted graph γ with root o. Let $\mathrm{Ver}(\gamma)$, $\mathrm{Ver}(\gamma)^t$ and $\mathrm{Ver}(\gamma)^*$ denote the vertices, the terminal vertices (or leaves) and the non-rooted vertices of γ, respectively. We take the natural ordering in $\mathrm{Ver}(\gamma)$ making the root o the minimal element. We assume that the weight in γ is non-negative on every vertex and that γ is terminally weighted, meaning that the vertices with non-zero weights are exactly those in $\mathrm{Ver}(\gamma)^t$. Let $\mathfrak{M}_1^{wt}$ is the stack of genus-1 prestable curves endowed with a weight. Remember that every element C parametrized by $\mathfrak{M}_1^{wt}$ has a dual (weighted) graph γ, which can be made terminally weighted and rooted by first declaring the root to be the (contraction of) the core of C and then pruning along

all non-terminal positively weighted nodes. We will denote by o the root of any terminally weighted rooted tree, and by $a, b, \ldots$ the remaining vertices.

In the diagram, $\widetilde{\mathfrak{M}}_1^{wt}$ denotes the blow up of $\mathfrak{M}_1^{wt}$ described above and $\mathfrak{D}_1$ is the stack of stable pairs (C, D) with D an effective Cartier divisors supported in the smooth locus of C. Fix a point (C, D) in $\mathfrak{D}_1$ and a map in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ with underlying curve C. Then U is a small open around the fixed map in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, $\mathcal{V}$ is a smooth chart around the point (C, D) in $\mathfrak{D}_1$ containing the image of U and $\mathcal{E}_{\mathcal{V}}$ is the total space of the sheaf $\rho_*\mathcal{L}(\mathcal{A})^{\oplus n}$ on $\mathcal{V}$.

Let $V_\gamma = \prod_{v \in \mathrm{Ver}(\gamma)^*} \mathbb{A}^1$ be an affine space that serves as model for local equations. We denote by $z_a, z_b, \ldots$ the natural coordinates in V_γ. Similarly, $E_\gamma = V_\gamma \times \prod_{v \in \mathrm{Ver}(\gamma)^t} (\mathbb{A}^1)^r$ and the coordinates on the affine space $(\mathbb{A}^1)^r$ corresponding to $a \in \mathrm{Ver}(\gamma)^t$ will be denoted by $w_{a,1}, \ldots, w_{a,r}$. The ideal $\Phi_\gamma = (\Phi_{\gamma,1}, \ldots, \Phi_{\gamma,r})$ will be described explicitly in eq. (45). The smooth morphism ϕ comes from the natural coordinates on $\mathcal{V}$, associated to the smoothing of each of the disconnecting nodes in C (which are in natural bijection with $\mathrm{Ver}(\gamma)^*$). The map $\mathcal{U} \to (F = 0)$ is an open embedding. Finally, $\widetilde{\phi}$ is induced by ϕ and $F = \widetilde{\phi}^* \Phi_\gamma$. It is in this sense that we can think of Φ_γ as the equations of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)_\gamma$.

Following [HL10], given a terminally weighted rooted graph γ, the ideal $\Phi_\gamma = (\Phi_{\gamma,1}, \ldots, \Phi_{\gamma,r})$ inside V_γ can be described as

$$\Phi_{\gamma,i} = \sum_{v \in \mathrm{Ver}(\gamma)^t} z_{[v,o]} w_{v,i} \quad 1 \leq i \leq r, \tag{45}$$

where

$$z_{[v,o]} = \prod_{o \prec a \preceq v} z_a.$$

Note that, for fixed i, the variables $w_{a,i}$ only appear in the i-th equation $\Phi_{\gamma,i}$. Due to the symmetry of the equations and the fact that all blow ups take place in V_γ, which has coordinates $\{z_a\}_{a \in \mathrm{Ver}(\gamma)^*}$ (but not the $w_{a,i}$), in the examples below we will not write down the index i in the equations $\Phi_{\gamma,i}$ nor in the variables $w_{a,i}$. For example, in the study of the equations $\Phi_{\gamma,i}$ after blowing up, it will be clear that the index i is irrelevant, in the sense that the way that $\Phi_{\gamma,i}$ changes is independent of i.

8.1. **Local equations of desingularizations.** The local equations of the loci that must be blown up are described, following [HL10].

Firstly, we describe how to assign an ideal I_γ to any semistable terminally weighted rooted tree γ. Here, *semistability* of γ means that every non-root vertex with weight zero has at least two edges.

The *trunk* of γ is the maximal chain $o = v_0 \prec v_1 \prec \ldots \prec v_r$ of vertices in γ such that each vertex v_i with $1 \leq i < r$ has exactly one immediate descendant and v_r is called a *branch vertex* if is it not terminal. Note that γ is a path tree if and only if it has no branch vertex.

Definition 8.1.1. Let γ be a semistable terminally weighted rooted tree with branch vertex v and let $a_1, \ldots, a_k$ be the immediate descendants of v. To γ we associate the ideal

$$I_\gamma = (z_{a_1}, \ldots, z_{a_k})$$

in V_γ.

First, we must blow up V_γ along the ideal I_γ. To describe the remaining steps we need to introduce the following operations.

Definition 8.1.2. Let γ be a terminally weighted semistable rooted tree.

- The *pruning* of γ along a vertex v is the new tree obtained by removing all the descendants of v (and the corresponding edges) and declaring the weight of v to be the sum of the original weight of v plus the weights of all removed vertices.
- The *advancing* of a vertex v with immediate ascendant $\overline{v}$ in γ is a new tree obtained by replacing every edge $(\overline{v}, v')$ with $v' \neq v$ by an edge $(\overline{v}, v)$ and pruning along all positively weighted non-terminal vertices as long as possible. In section 8.1.4 we will denote by γ_v the advancing of v in γ and by γ'_v the same tree before pruning.
- Suppose γ has a branch vertex v. A *monoidal transform* of γ is a tree obtained by advancing one of the immediate descendants of v. The set of monoidal transforms of γ is $\mathrm{Mon}(\gamma)$.

It turns out that the ideal Φ_γ behaves nicely under monoidal transforms. Indeed, let γ be a semistable terminally weighted rooted tree with branch vertex v and let $a = a_1, a_2, \ldots, a_k$ be the immediate descendants of v. Let γ_a be the tree advancing of a in γ. Let $\pi\colon \widetilde{V_\gamma} \to V_\gamma$ be the blow-up of V_γ along the ideal $I_\gamma = (z_{a_1}, \ldots, z_{a_k})$. We view $\widetilde{V_\gamma}$ embedded inside $V_\gamma \times \mathbb{P}^{k-1}$. There is a natural way to associate to each generator z_{a_i} of I_γ one chart of $\mathbb{P}^{k-1}$, and thus also of $\widetilde{V_\gamma}$. We denote such chart by $\widetilde{V}_{\gamma,a_i}$. Let $\pi_a\colon \widetilde{V}_{\gamma,a} \to V_\gamma$ be the restriction of the natural projection, where $a = a_1$. Then, by the proof of [HL10, Lemma 5.14], one of the following must hold

- either γ_a is a path tree, and then the zero locus of $\pi_a^*(\Phi_\gamma)$ has smooth components;
- or γ_a is not a path tree and then

$$\pi_a^*(\Phi_\gamma) = \Phi_{\gamma_a}. \tag{46}$$

The whole blow-up process is summarized as follows. Fix γ. First blow up V_γ along I_γ. The pullback of Φ_γ is controlled by $\mathrm{Mon}(\gamma)$. If $\mathrm{Mon}(\gamma)$ consists only of path trees, we are done. Otherwise, for every element γ' in $\mathrm{Mon}(\gamma)$ which is not a path tree, blow up the chart of $\widetilde{V_\gamma}$ corresponding to γ' along $I_{\gamma'}$. Continue recursively. The process concludes by [HL10, Lemma 3.12].

Now we want to describe $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ locally. Namely, we want to describe which loci inside $\mathfrak{Pic}$ we are blowing up locally. We have a short exact sequence

$$0 \to \rho_*\mathcal{L} \to \rho_*\mathcal{L}(\mathcal{A}) \to \rho_*\mathcal{L}(\mathcal{A})\mid_{\mathcal{A}}$$

by eq. (23). The change of notation is due to the fact that eq. (23) was global in $\mathfrak{Pic}$, but we now work locally. After a careful study of the second morphism, [HL10, Theorem 4.16] concludes that $\rho_*\mathcal{L}$ is the direct sum of a trivial bundle with the kernel of the morphism

$$\bigoplus_{v \in \mathrm{Ver}(\gamma)^t} \varphi_v\colon \mathcal{O}_{\mathcal{V}}^{\oplus \ell} \to \mathcal{O}_{\mathcal{V}}, \tag{47}$$

where ℓ is the cardinality of $\mathrm{Ver}(\gamma)^t$ and $\varphi_v = \prod_{o \prec v' \preceq v} \zeta_v$, with ζ_v the smoothing parameter of the disconnecting node corresponding to the vertex v.

By lemma 7.1.1, the sheaf $\mathfrak{F}$ can be described locally as the dual of eq. (47). In particular, by Remark 8.1.6 we have that $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ agrees, locally, with the blowup along the ideal generated by the entries $(\varphi_v)_{v\in \mathrm{Ver}(\gamma)^t}$. In local coordinates, this ideal can be described as follows.

Definition 8.1.3. Let γ be a semistable terminally weighted rooted tree with $\mathrm{Ver}(\gamma)^t = \{v_1, \dots, v_t\}$. To γ we associate the ideal

$$J_\gamma = (z_{[v_1,o]}, \dots, z_{[v_t,o]}).$$

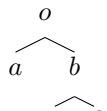
Similarly to eq. (46), in the same setup we have that

$$\pi_a^*(J_\gamma) = J_{\gamma_a},$$

independently of whether γ_a is has a branch vertex. This follows again from the proof of [HL10, Lemma 5.14].

8.1.4. *Examples.* For two concrete trees γ, we compute the equations Φ_γ as well as the ideals I_γ and J_γ. We describe the blow up process of Hu and Li and show that the result indeed desingularizes the main component of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ locally. Furthermore, we check that the ideal J_γ becomes locally principal in Hu–Li's blow-up.

Example 8.1.5. Consider the following labelled graph:

$\gamma =$ 

$$\Phi_\gamma = z_a w_a + z_b(z_c w_c + z_d w_d),$$
$$I_\gamma = (z_a, z_b),$$
$$J_\gamma = (z_a, z_b z_c, z_b z_d).$$

Let $\widetilde{V_\gamma}$ be the blow up along I_γ, that is the zero locus of $z_a z'_b - z_b z'_a$ inside $\mathbb{A}^4_{z_a,z_b,z_c,z_d} \times \mathbb{P}^1_{z'_a,z'_b}$. The chart associated to a is that where $z'_a \neq 0$. Dehomogenizing amounts to the change of variables $z_b = z'_b z_a$. By doing so, we get that

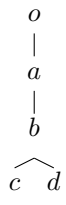
$$\pi_a^*(\Phi_\gamma) = z_a(w_a + z'_b(z_c w_c + z_d w_d))$$

and that

$$\pi_a^*(J_\gamma) = (z_a, z_a z'_b z_c, z_a z'_b z_d) = (z_a).$$

This means that the zero locus of $\pi_a^*(\Phi_\gamma)$ already has smooth components, so no further blowups are needed on this chart, and that $\pi_a^*(J_\gamma)$ is principal on this chart too.

Below are the trees γ'_a obtained by advancing a without pruning, and γ_a obtained by advancing a. We know that $\pi_a^* J_\gamma = J_{\gamma_a}$, but we check it in this example.

$\gamma'_a =$ 

$\gamma_a =$ 

$$J_{\gamma_a} = (z_a).$$

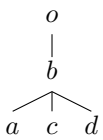
Similarly, we now look at the chart associated to b, where $z'_b \neq 0$. The change of variables is now $z_a = z_b z'_a$. It follows that

$$\pi_b^*(\Phi_\gamma) = z_b(z'_a w_a + z_c w_c + z_d w_d)$$

and that

$$\pi_b^*(J_\gamma) = (z_b z'_a, z_b z_c, z_b z_d) = z_b(z'_a, z_c, z_d).$$

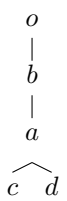
This means that we still need to blow up. This time the tree γ_b obtained by advancing b in γ (no pruning is needed) is not a path tree. We also check the identities $J_{\gamma_a} = \pi_a^* J_\gamma$ and $\pi_b^*(\Phi_\gamma) = \Phi_{\gamma_b}$.



$$\Phi_{\gamma_b} = z_b(z_a w_a + z_c w_c + z_d w_d),$$
$$I_{\gamma_b} = (z_a, z_c, z_d),$$
$$J_{\gamma_b} = z_b(z_a, z_c, z_d).$$

To conclude the example, we need to blow up along the ideal (z_a, z_c, z_d). We collect the result below.

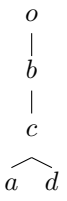
Advancing a, or equivalently looking at the chart $z'_a \neq 0$, we have

$\gamma'_{b,a} =$ 

$\gamma'_{b,a} =$

$$\pi_a^*\pi_b^*(\Phi_\gamma) = \pi_a^*(\Phi_{\gamma_b}) = z_a z_b(w_a + z_c w_c + z_d w_d),$$
$$J_{\gamma_{b,a}} = (z_a z_b).$$

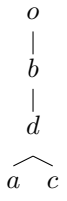
Advancing c, or equivalently looking at the chart $z'_c \neq 0$, we have

$\gamma'_{b,c} =$ 

$\gamma'_{b,c} =$ 

$$\pi_c^*\pi_b^*(\Phi_\gamma) = \pi_c^*(\Phi_{\gamma_b}) = z_b z_c(z_a w_a + w_c + z_d w_d),$$
$$J_{\gamma_{b,c}} = (z_b z_c).$$

And finally, advancing d, or equivalently looking at the chart $z'_d \neq 0$, we have

$\gamma'_{b,d} =$ 

$\gamma'_{b,d} =$ 

$$\pi_d^*\pi_b^*(\Phi_\gamma) = \pi_d^*(\Phi_{\gamma_b}) = z_b z_d(z_a w_a + z_c w_c + w_d),$$
$$J_{\gamma_{b,d}} = (z_b z_d).$$

Remark 8.1.6. Example 8.1.5 shows that the Rossi–Villamayor blow-up process of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is not equal to the Vakil–Zinger blow-up. This is compatible with Remark 4.4. in [HN19]. Indeed, the Rossi–Villamayor blowup around γ is given by $\mathrm{Bl}_{J_\gamma} V_\gamma$ and the Vakil–Zinger one is the iterated blow-up $\mathrm{Bl}_{I_{\gamma_b}} \mathrm{Bl}_{I_\gamma} V_\gamma$. We know there is a natural morphism

$$\mathrm{Bl}_{I_{\gamma_b}} \mathrm{Bl}_{I_\gamma} V_\gamma \to \mathrm{Bl}_{J_\gamma} V_\gamma$$

over V_γ, either by proposition 4.2.6 or because we checked that J_γ pulls back to a principal ideal in $\mathrm{Bl}_{I_{\gamma_b}} \mathrm{Bl}_{I_\gamma} V_\gamma$. By contradiction, if there is a morphism

$$f \colon \mathrm{Bl}_{J_\gamma} V_\gamma \to \mathrm{Bl}_{I_{\gamma_b}} \mathrm{Bl}_{I_\gamma} V_\gamma$$

over V_γ, then we get a morphism

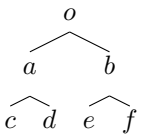
$$\tilde{f}\colon \mathrm{Bl}_{J_\gamma} V_\gamma \to \mathrm{Bl}_{I_\gamma} V_\gamma$$

over V_γ. By [Moo01], there is a fractional ideal K in V_γ and a positive integer α such that

$$I_\gamma \cdot K = J_\gamma^\alpha.$$

This is not true for $I_\gamma = (z_a, z_b)$ and $J_\gamma = (z_a, z_b z_c, z_b z_d)$ in $V_\gamma = \mathbb{A}^4_{z_a,z_b,z_c,z_d}$.

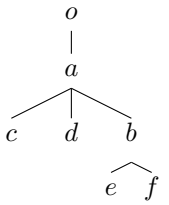
Example 8.1.7. We do a similar study for the following labelled graph γ:



$$\begin{aligned}\Phi_\gamma &= z_a(z_c w_c + z_d w_d) + z_b(z_e w_e + z_f w_f),\\ I_\gamma &= (z_a, z_b),\\ J_\gamma &= (z_a z_c, z_a z_d, z_b z_e, z_b z_f).\end{aligned}$$

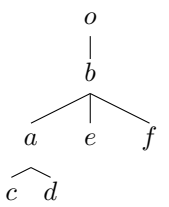
After blowing up, there are two charts, corresponding to the advacings of a and b respectively.

$\gamma_a =$



$$\begin{aligned}\Phi_{\gamma_a} &= z_a(z_c w_c + z_d w_d + z_b(z_e w_e + z_f w_f)),\\ I_{\gamma_a} &= (z_b, z_c, z_d),\\ J_{\gamma_a} &= z_a(z_c, z_d, z_b z_e, z_b z_f).\end{aligned}$$

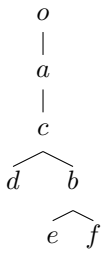
$\gamma_b =$



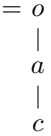
$$\begin{aligned}\Phi_{\gamma_b} &= z_b(z_a(z_c w_c + z_d w_d) + z_e w_e + z_f w_f),\\ I_{\gamma_b} &= (z_a, z_e, z_f),\\ J_{\gamma_b} &= z_b(z_a z_c, z_a z_d, z_e, z_f).\end{aligned}$$

By symmetry, it is enough to understand how to proceed in one of the charts. We choose the one corresponding to a. We get three new charts corresponding to the vertices c, d and b.

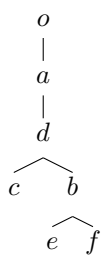
$\gamma'_{a,c} =$

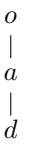


$\gamma_{a,c} =$

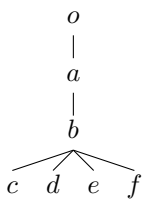


$$\begin{aligned}\pi_c^*\Phi_{\gamma_a} &= z_a z_c(w_c + z_d w_d + z_b(z_e w_e + z_f w_f)),\\ J_{\gamma_{a,c}} &= (z_a z_c).\end{aligned}$$

$\gamma'_{a,d} =$ 

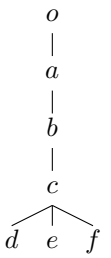
$\gamma_{a,d} =$ 

$$\pi_d^*\Phi_{\gamma_a} = z_a z_d(z_c w_c + w_d + z_b(z_e w_e + z_f w_f)),$$
$$J_{\gamma_{a,d}} = (z_a z_d).$$

$\gamma_{a,b} =$ 

$$\Phi_{\gamma_{a,b}} = z_a z_b(z_c w_c + z_d w_d + z_e w_e + z_f w_f),$$
$$I_{\gamma_{a,b}} = (z_c, z_d, z_e, z_f),$$
$$J_{\gamma_{a,b}} = z_a z_b(z_c, z_d, z_e, z_f).$$

To conclude, we need to blow up the last chart along $I_{\gamma_{a,b}}$. This produces four new charts corresponding to c, d, e and f. We will only write down one of them since the rest are very similar.

$\gamma'_{a,b,c} =$ 

$\gamma_{a,b,c} =$ 

$$\pi_c^*\Phi_{\gamma_{a,b}} = z_a z_b z_c(w_c + z_d w_d + z_e w_e + z_f w_f),$$
$$J_{\gamma_{a,b,c}} = (z_a z_b z_c).$$

8.1.8. *Smoothness.* In genus one, $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \times_{\mathfrak{Pic}} \mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ has simple normal crossings following the same argument as in [HL10, Theorem 5.24]. It is enough to show that the zero locus of the ideal Φ_γ becomes a simple normal crossing in the blow-up $\widetilde{V_\gamma}$ of V_γ along the ideal J_γ.

Remember that $\Phi_\gamma = (\Phi_{\gamma,1}, \ldots, \Phi_{\gamma,r})$ with

$$\Phi_{\gamma,i} = \sum_{v \in \mathrm{Ver}(\gamma)^t} z_{[v,o]} w_{b,i} \quad 1 \le i \le r,$$

where

$$z_{[v,o]} = \prod_{o \prec a \preceq v} z_a,$$

and that

$$J_\gamma = (z_{[v,o]})_{v \in \mathrm{Ver}(\gamma)^t}.$$

For given $v' \in \mathrm{Ver}(\gamma)^t$, the pullback of the equation $\Phi_{\gamma,i}$ on the chart corresponding to v' is equal to

$$z_{[v',o]} \left(w_{v',i} + \sum_{v' \neq v \in \mathrm{Ver}(\gamma)^t} z_{[v,o]} w_{b,i} \right)$$

by [HL10, Lemma 5.14]. This proves the claim.

8.1.9. *Maps between blowups.* By proposition 4.2.6 there is a morphism from Vakil–Zinger's blowup to Rossi-Villamayor's blowup. In genus one, we can check it locally: it is equivalent to the fact that the pullback of the ideal J_γ to each chart $\widetilde{V_\gamma}$ of the Hu–Li blowup of V_γ is principal. We have checked this in example 8.1.5 and example 8.1.7, More generally, we can give a proof for every γ as follows.

By eq. (46) if z_a is any of the generators of I_γ, then $\pi_a^*(J_\gamma) = J_{\gamma_a}$ where γ_a is the advancing of a in γ. In particular, it is enough to show that all the (natural) charts of $\widetilde{V_\gamma}$ correspond to path trees, which is proven in [HL10, Lemma 3.14].

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