



**HAL**  
open science

# The Airfoil Integral Equation over Disjoint Intervals: Analytic Solutions and Asymptotic Expansions

Leandro Farina, Marcos Ferreira, Victor Péron

► **To cite this version:**

Leandro Farina, Marcos Ferreira, Victor Péron. The Airfoil Integral Equation over Disjoint Intervals: Analytic Solutions and Asymptotic Expansions. 2023. hal-04237680

**HAL Id: hal-04237680**

**<https://hal.science/hal-04237680>**

Preprint submitted on 11 Oct 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Polynomial Solutions of the Airfoil Equation in Two Disjoint Intervals

Leandro Farina<sup>a,\*</sup>, Marcos R. S. Ferreira<sup>b</sup>, Victor Péron<sup>c</sup>

<sup>a</sup>*Institute of Mathematics and Statistics, Federal University of Rio Grande do Sul, Porto Alegre, RS, 91509-900, Brazil*

<sup>b</sup>*Federal Institute of Education, Science, and Technology of Southern Rio Grande (IFSul), Pelotas, RS, 96015-360, Brazil*

<sup>c</sup>*Université de Pau et des Pays de l'Adour, E2S UPPA, LMAP CNRS UMR 5142, 64013 Pau, France*

---

## Abstract

The airfoil integral equation over two disjoint intervals is considered. An analytic solution given in terms of polynomials satisfying a generalised three-term recurrence relation is obtained, when the input function is a Chebyshev polynomial of the first kind. A new class of polynomials is defined on two disjoint intervals and is shown to generalise a classical integral relationship previously satisfied by Chebyshev polynomials on a single interval. The solutions are calculated efficiently and comparisons with the original problem, defined on a continuous domain are presented. A spectral method for the generalised airfoil integral equation over two disjoint intervals is outlined.

*Keywords:* airfoil equation, singular integral equation, spectral method, Chebyshev polynomials, disjoint intervals

---

## 1. Introduction

The airfoil equation is given by

$$\int_a^b \frac{f(t)}{x-t} dt = g(x), \quad x \in (a, b), \quad (1)$$

where  $f$  is the unknown function,  $g$  is prescribed and the integral symbol must be understood in the sense of a *Cauchy principal value*, due to the strong singularity of the integrand. The equation receives its name because it models the two-dimensional flow over an infinitely thin airfoil in an inviscid fluid. In this case,  $f$  represents the pressure difference between the upper and lower sides in the plate and  $g$  is the downwash velocity. This equation has been extensively studied from the physical, computational and numerical analysis points of view. Several applications in acoustics, hydrodynamics and elastostatics are reduced to similar equation to (1) on a finite or infinity domain [1, 2]. In particular, the scattering of water waves by a thin vertical barrier can be modeled by an airfoil equation whose solution is unbounded at one end of the domain [1]. The Cauchy-type singularity in (1) can be associated with stronger singularities in finite part integrals, as discussed in previous works [3, 4]. Numerous algorithms have been proposed to address the numerical integration challenges posed by its full or nearly singular behavior, particularly in the context of boundary element methods [5, 6, 7].

A particular extension of the airfoil equation is the so called generalised airfoil equation, where the kernel function has an additional, not strongly singular, term. This more general equation governs the pressure across an airfoil oscillating in a closed tunnel [8, 9]. The stability and convergence properties of Galerkin and Collocation methods for the solution of this equation have been studied in [10, Chapter 5], [11] and [12, 13, 14, 15], respectively.

---

\*Corresponding author

*Email addresses:* farina@alum.mit.edu (Leandro Farina), mcsrond@gmail.com (Marcos R. S. Ferreira), victor.peron@univ-pau.fr (Victor Péron)

Despite the amount of publications and interest in this topic, very little is known about the airfoil equation on a discontinuous domain. Tricomi [16] was the first to present an analytic solution of the airfoil equation in two disjoint intervals. More recently, Dutta and Barneja [17] used Tricomi's solution to deduce a solution of a hypersingular integral equation on two disjoint intervals and Trogdon and Olver [18] analyse general Cauchy integral equations using their Riemann-Hilbert problem representations. Farina et al. [19] obtained approximate polynomial solutions of the airfoil equation over disjoint intervals, when the distance between them is small.

In the present paper, we obtain particular polynomial solutions of the airfoil equation defined over the domain  $(-1, -\kappa) \cup (\kappa, 1)$  for an arbitrary  $\kappa$ , when the input function,  $g$  is a Chebyshev polynomial of the first kind.

We outline the article as follows. We show how to analytically evaluate a Cauchy principal value over disjoint intervals when the integrand involves a polynomial, in section 2. When the input function is a Chebyshev polynomial, an explicit expression for the solution is given in section 3. In section 4, we show how to deduce simple recurrence relations for the solution. In section 5 we apply recurrence solutions to derive asymptotic solutions as  $\kappa \rightarrow 0$ . We exhibit also the first terms of an expansion in power series of  $\kappa$  for particular analytical solutions and we specify a functional framework in weighted Sobolev spaces, Section 5.1. Numerical results comparing the recurrence calculations and asymptotic solutions and also with solution of the classical airfoil equation are presented in section 6. In section 7 we discuss the work, perspectives and outline a spectral method for solving the generalised airfoil equation in two disjoint intervals.

## 2. Analytical solutions

In the next subsection, we present known results on the solution of the classic airfoil equation (defined on a single interval) and in subsection 2.2, we describe our new results on airfoil equation analytical solutions on two disjoint intervals.

### 2.1. The airfoil equation on a continuous interval

The simplest airfoil equation defined over a finite interval is

$$\frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{x-t} dt = g(x), \quad x \in (-1, 1). \quad (2)$$

Let  $g(x)$  be Hölder continuous for  $x \in [-1, 1]$ ,  $g \in C^{0,\alpha}([-1, 1])$ . We can write down the general solution of (2); it is (cf. [20, p. 173-180] or [21])

$$f(x) = -\frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-t^2}{1-x^2}} \frac{g(t)}{x-t} dt + \frac{C}{\sqrt{1-x^2}}, \quad (3)$$

where  $C$  is an arbitrary constant. This means that (3) has infinite solutions according to  $C$ . This constant can be physically interpreted as the circulation  $\Gamma_\infty$  around of the airfoil [22, p. 70]. For simplicity, we choose  $C = 0$ . Thus, the solution (3) is reduced to

$$f(x) = -\frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-t^2}{1-x^2}} \frac{g(t)}{x-t} dt. \quad (4)$$

Now, taking  $g(t) = T_n(t)$ , where  $T_n$  is the Chebyshev polynomial of the first kind, in (4), using [23, eq. 22.13.4]

$$\frac{1}{\pi} \int_{-1}^1 \frac{U_{n-1}(t)}{x-t} \sqrt{1-t^2} dt = T_n(x), \quad (5)$$

where  $U_{n-1}(t)$  is the Chebyshev polynomial of the second kind and degree  $n - 1$  and the relation [23]

$$T_n(t) = \frac{1}{2} [U_n(t) - U_{n-2}(t)],$$

we obtain

$$f(x) = \frac{-1}{2\sqrt{1-x^2}} [T_{n+1}(x) - T_{n-1}(x)]. \quad (6)$$

We will use the solution (6) for comparison with the solution of airfoil equation on two disjoint intervals, in section 6.2.

## 2.2. The airfoil equation in two disjoint intervals

We consider the airfoil over the discontinuous domain  $G_\kappa = (-1, -\kappa) \cup (\kappa, 1)$ . Writing it explicitly, on the two disjoint intervals of  $G_\kappa$ , we have

$$\frac{1}{\pi} \int_{-1}^{-\kappa} \frac{f_\kappa(t)}{x-t} dt + \frac{1}{\pi} \int_{\kappa}^1 \frac{f_\kappa(t)}{x-t} dt = -\psi(x), \quad x \in G_\kappa. \quad (7)$$

Here,  $f_\kappa$  is an unknown function and  $\psi$  is an input data of the problem. We assume the  $\psi$  is Hölder continuous (i.e.,  $\psi \in \mathcal{C}^{0,\alpha}(G_\kappa)$ ) and  $\kappa > 0$ . Tricomi [16] (see also [17]) obtained an explicit solution of this equation, as

$$f_\kappa(x) = \begin{cases} \frac{1}{\pi R_\kappa(x)} [C_{1,\kappa} + C_{2,\kappa}x + \Psi_\kappa(x)], & x \in (-1, -\kappa), \\ -\frac{1}{\pi R_\kappa(x)} [C_{1,\kappa} + C_{2,\kappa}x + \Psi_\kappa(x)], & x \in (\kappa, 1), \end{cases} \quad (8)$$

where

$$\Psi_\kappa(x) = \int_{-1}^{-\kappa} \frac{\psi(t)R_\kappa(t)}{x-t} dt - \int_{\kappa}^1 \frac{\psi(t)R_\kappa(t)}{x-t} dt, \quad (9)$$

$C_{1,\kappa}$  and  $C_{2,\kappa}$  are two arbitrary constants and

$$R_\kappa(x) = \sqrt{(1-x^2)(x^2-\kappa^2)}, \quad x \in G_\kappa. \quad (10)$$

We focus now on the case where  $C_{1,\kappa} = C_{2,\kappa} = 0^1$ ; the solution (8) reduces to

$$f_\kappa(x) = -\frac{\text{sgn}(x)}{\pi R_\kappa(x)} \Psi_\kappa(x), \quad x \in G_\kappa, \quad (11)$$

where  $\text{sgn}(x)$  denotes the sign of  $x$ . Note in (11) that the solution is given in terms of  $\Psi_\kappa(x)$  which is in turn, written in terms of Cauchy principal values. Aiming at approximate solutions for (7) obtained by efficient algorithms, we want to make progress now assuming the input function  $\psi(x)$  is a polynomial of degree  $n$ . We will use also the notation  $f_{n,\kappa}$ , when we want to emphasize the dependence on  $n$ .

The next proposition, due to Gakhov [24] (see also [25]), is key in this work: it will allow us to write  $\Psi_\kappa$  as a polynomial, when  $\psi(x)$  itself is also a polynomial.

**Proposition 1** ([24]). *Let  $L = \sum_{k=1}^m L_k$  be a contour consisting of  $m$  simple open curves having no common ends. The coordinates of the ends, taken in certain order, are denoted by  $c_1, c_2, c_3, \dots, c_{2m}$ . Let  $p$  be an integer,  $0 \leq p \leq 2m$ , and  $P(z)$  a polynomial.*

---

<sup>1</sup>In subsection 6.2 we present cases for other values of these constants.

Thus, the singular integral

$$I(t) = \frac{1}{\pi i} \int_L \frac{\prod_{k=1}^p (\tau - c_k)^{1/2}}{\prod_{j=p+1}^{2m} (\tau - c_j)^{1/2}} \frac{P(\tau)}{\tau - t} d\tau = -P^*(t), \quad (12)$$

where  $P^*(z)$  is a polynomial representing the principal part of the expansion of the Cauchy integral density, in the vicinity of infinity, i.e., obeys the condition

$$\lim_{z \rightarrow \infty} \left[ \frac{\prod_{k=1}^p (z - c_k)^{1/2}}{\prod_{j=p+1}^{2m} (z - c_j)^{1/2}} \cdot P(z) - P^*(z) \right] = 0. \quad (13)$$

**Remark 1.** In the integral (12),  $\frac{1}{\tau - t}$  is the Cauchy kernel and  $\frac{\prod_{k=1}^p (\tau - c_k)^{1/2}}{\prod_{j=p+1}^{2m} (\tau - c_j)^{1/2}} P(\tau)$  is the so-called density of the Cauchy integral. Thus, evaluating the integral  $I$  is accomplished calculating the principal part of the density function which is defined by (13). We denote the principal part by  $PP$ .

To evaluate (11) in terms of the principal part, we expand the density of the Cauchy integral in power series, in the vicinity of infinity; the details of such a procedure is shown in appendix Appendix A. But first, we need to put the integrals in  $\Psi_\kappa(x)$  in the same the form as in (12). For this, note that

$$\Psi_\kappa(x) = \int_\kappa^1 R_\kappa(t) \left[ \frac{x(\psi(-t) - \psi(t)) - t(\psi(-t) + \psi(t))}{x^2 - t^2} \right] dt \quad (14)$$

and separate two cases according with parity of  $\psi$ , as follows.

1. **Even polynomial  $\psi(x)$ .** In this case,

$$\Psi_\kappa(x) = \int_\kappa^1 R_\kappa(t) \left[ \frac{-2t\psi(t)}{x^2 - t^2} \right] dt. \quad (15)$$

Now, making the change of the variables  $x^2 = u$ ,  $t^2 = v$  and applying the proposition 1 (cf. appendix Appendix A), we get

$$\Psi_\kappa(x) = -\pi PP \left\{ \sqrt{(x^2 - 1)(x^2 - \kappa^2)} \psi(x) \right\}_{x \rightarrow \infty}. \quad (16)$$

Then, (11) is written as

$$f_\kappa(x) = \frac{\text{sgn}(x)}{R_\kappa(x)} PP \left\{ \sqrt{(x^2 - 1)(x^2 - \kappa^2)} \psi(x) \right\}_{x \rightarrow \infty}. \quad (17)$$

2. **Odd polynomial  $\psi(x)$ .** This is analogous to the case 1. Thus,

$$\Psi_\kappa(x) = -\pi x PP \left\{ \sqrt{(x^2 - 1)(x^2 - \kappa^2)} \frac{\psi(x)}{x} \right\}_{x \rightarrow \infty} \quad (18)$$

and the solution (11) is written as

$$f_\kappa(x) = \frac{\text{sgn}(x)x}{R_\kappa(x)} PP \left\{ \sqrt{(x^2 - 1)(x^2 - \kappa^2)} \frac{\psi(x)}{x} \right\}_{x \rightarrow \infty}. \quad (19)$$

To make progress in obtaining an explicit and simple expression and algorithm for  $f_\kappa$ , let us next assume  $\psi$  is a Chebyshev polynomial.

### 3. Chebyshev polynomials as input functions

We now make a specific choice for the input function in (7), namely

$$\psi(x) = -T_n(x),$$

where  $T_n$  is the Chebyshev polynomials of the first kind and degree  $n$ .

Consider the representation [26]

$$T_n(x) = \sum_{k=0}^{[n/2]} t_{n-2k}^k x^{n-2k}, \quad (20)$$

where  $[n/2]$  is the integer part of  $n/2$  and  $t_{n-2k}^k$ , the coefficients of the Chebyshev polynomial, which can be written as

$$t_i^j = 2^{i-1} (-1)^j \left\{ \begin{matrix} i+2j \\ i+j \end{matrix} \right\} \binom{i+j}{j}. \quad (21)$$

Equation (20) shows that  $T_n(x)$  has degree  $n$  and that it is odd or even according to whether  $n$  is odd or even, respectively.

Separating according with the parity, we get

$$T_{2m}(x) = \sum_{k=0}^m t_{2k}^{m-k} x^{2k} \quad (22)$$

and

$$T_{2m+1}(x) = \sum_{k=0}^m t_{2k+1}^{m-k} x^{2k+1}, \quad (23)$$

where  $m$  is the integer part of the  $n/2$ .

Substituting (22) and (23) in (17) and (19) respectively, we show in the appendix Appendix A that

$$f_{n,\kappa}(x) = \frac{-\text{sgn}(x)}{R_\kappa(x)} E_n(x), \quad (24)$$

where  $E_n(x)$  is a polynomials of degree  $n$ , given by

$$E_{2m}(x) = \sum_{l=0}^m \left[ \left( \sum_{k=m-l}^m t_{2k}^{m-k} \frac{s_\kappa^{(k-(m-l))}(0)}{(k-(m-l))!} \right) x^{2(m-(l-1))} + t_{2l}^{m-l} \frac{s_\kappa^{(l+1)}(0)}{(l+1)!} \right] \quad (25)$$

and

$$E_{2m+1}(x) = \sum_{l=0}^m \left[ \left( \sum_{k=m-l}^m t_{2k+1}^{m-k} \frac{s_\kappa^{(k-(m-l))}(0)}{(k-(m-l))!} \right) x^{2(m-(l-1))+1} + t_{2l+1}^{m-l} \frac{s_\kappa^{(l+1)}(0)}{(l+1)!} x \right], \quad (26)$$

$s_\kappa(w) = \sqrt{(1-w)(1-\kappa^2 w)}$ ,  $s_\kappa^{(n)}(w)$  is the  $n$ -th derivative of the  $s_\kappa(w)$  and  $s_\kappa^{(0)}(w) = s_\kappa(w)$ .

**Remark 2.** The square root term present in (24) could be regarded as a weight function in which the airfoil equation (7) would read

$$\frac{1}{\pi} \int_{G_\kappa} \frac{f_\kappa(t)}{x-t} w(t) dt = -\psi(x), \quad x \in G_\kappa, \quad (27)$$

where  $w(t) = \text{sgn}(t) \frac{1}{R_\kappa(t)}$ . We see clearly that  $E_n$  is a purely polynomial solution of the weighed airfoil equation: the following integral relationship is valid.

$$\frac{1}{\pi} \int_{G_\kappa} \frac{E_n(t)}{x-t} w(t) dt = T_n(x). \quad (28)$$

Equation (28) is possibly the main contribution of this work and generalises the classical relationship (5):

$$\frac{1}{\pi} \int_{-1}^1 \frac{U_{n-1}(t)}{x-t} \sqrt{1-t^2} dt = T_n(x).$$

In the next section, we show that  $E_n(x)$ , and consequently also  $f_{n,\kappa}(x)$ , can be easily and efficiently computed by means of a generalised three-term recurrence relation, meaning a three-term recurrence relation, when  $n$  is even, and a three-term recurrence relation with a residual term, when  $n$  is odd.

#### 4. Recurrence Relations

In order to get recurrence relations, we analyse two cases according with the parity of the input function  $\psi$ , as previously.

##### 1. **Even** $\psi(x) = -T_{2m}(x)$ .

We show in appendix Appendix A that

$$E_{2m} = \text{PP} \left\{ T_{2m}(x) \sum_{i=0}^{m+1} \frac{s_{\kappa}^{(i)}(0)}{i!} x^{2(1-i)} \right\}_{x \rightarrow \infty}. \quad (29)$$

Using the recurrence relation [23, Eq. 22.7.4]  $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ , in (29), we have

$$\begin{aligned} E_{2m} &= \text{PP} \left\{ (2x T_{2m-1}(x) - T_{2m-2}(x)) \sum_{i=0}^{m+1} \frac{s_{\kappa}^{(i)}(0)}{i!} x^{2(1-i)} \right\}_{x \rightarrow \infty} \\ &= \text{PP} \left\{ 2x T_{2m-1}(x) \sum_{i=0}^{m+1} \frac{s_{\kappa}^{(i)}(0)}{i!} x^{2(1-i)} - T_{2m-2}(x) \sum_{i=0}^{m+1} \frac{s_{\kappa}^{(i)}(0)}{i!} x^{2(1-i)} \right\}_{x \rightarrow \infty}. \end{aligned}$$

Incorporating  $T_{2m-1}(x)$  and  $T_{2m-2}(x)$  into the sums and discarding terms that tend to zero as  $x$  goes to infinity (see appendix Appendix B for the details), we get

$$E_{2m} = 2x(xE_{2m-1}) - E_{2m-2} + \left( 2 \sum_{i=0}^{m-1} t_{2i+1}^{m-1-i} \frac{s_{\kappa}^{(i+2)}(0)}{(i+2)!} \right). \quad (30)$$

Substituting (30) in (24) we obtain

$$f_{2m,\kappa}(x) = 2x f_{2m-1,\kappa}(x) - f_{2m-2,\kappa}(x) - \frac{\text{sgn}(x)}{R_{\kappa}(x)} \left( 2 \sum_{i=0}^{m-1} t_{2i+1}^{m-1-i} \frac{s_{\kappa}^{(i+2)}(0)}{(i+2)!} \right). \quad (31)$$

##### 2. **Odd** $\psi(x) = -T_{2m+1}(x)$ .

Following analogous steps for the odd case, we obtain the recurrence formula (see appendix Appendix B):

$$f_{2m+1,\kappa}(x) = 2x f_{2m,\kappa}(x) - f_{2m-1,\kappa}(x). \quad (32)$$

The initial terms are given by

$$f_{0,\kappa}(x) = -\frac{\text{sgn}(x)}{R_{\kappa}(x)} \left( x^2 - \frac{1 + \kappa^2}{2} \right) \quad (33)$$

and

$$f_{1,\kappa}(x) = -\frac{\text{sgn}(x)}{R_{\kappa}(x)} x \left( x^2 - \frac{1 + \kappa^2}{2} \right). \quad (34)$$

A straightforward implementation of (31) and (32) in Matlab showed the recurrence relations are numerically stables up to polynomials of order 45.

## 5. Asymptotic solutions when $\kappa \rightarrow 0$

In this section we derive asymptotic solutions for the Airfoil equation as  $\kappa \rightarrow 0$  when the input function is a Chebyshev polynomial. Then we exhibit the first terms of an asymptotic expansion of  $f_\kappa$  in power series of  $\kappa$  when the right hand side is a Chebyshev polynomial. Our general approach is based on the recurrence formula (31) and (32).

Firstly, note that using the initial term (33) we infer the formal limit:  $f_{0,\kappa}(x) \rightarrow f_{0,0}(x)$  as  $\kappa \rightarrow 0$ . Here  $f_{0,0}$  is defined by

$$f_{0,0}(x) = -\frac{x^2 - \frac{1}{2}}{x\sqrt{1-x^2}}.$$

This result coincides with Eq. (3.17a) in [12]. Secondly, note that using the initial term (34) we infer the formal limit:  $f_{1,\kappa}(x) \rightarrow f_{1,0}(x)$  as  $\kappa \rightarrow 0$ . Here  $f_{1,0}$  is defined by

$$f_{1,0}(x) = -\frac{x^2 - \frac{1}{2}}{\sqrt{1-x^2}}.$$

This result coincides with Eq. (3.17b) in [12]. Then when  $\psi = -T_2$ , we find the formal limit:  $f_{2,\kappa}(x) \rightarrow f_{2,0}(x)$  as  $\kappa \rightarrow 0$ . Here  $f_{2,0}(x)$  is given by :

$$f_{2,0}(x) = -\frac{2x^4 - 2x^2 + \frac{1}{4}}{x\sqrt{1-x^2}}. \quad (35)$$

The proof of this result is based on formula (31) when  $m = 1$ . Note that this result coincides with Eq. (3.17c) in [12]. When  $\psi = -T_3$ , we find the formal limit :  $f_{3,\kappa}(x) \rightarrow f_{3,0}(x)$  as  $\kappa \rightarrow 0$ . Here  $f_{3,0}(x)$  is given by :

$$f_{3,0}(x) = -\frac{4x^4 - 5x^2 + 1}{\sqrt{1-x^2}}. \quad (36)$$

The proof of this result is based on formula (32) when  $m = 1$ . This result coincides with Eq. (3.17d) in [12].

**Proposition 2.** *When  $\psi = -T_4$  we find the formal limit :  $f_{4,\kappa}(x) \rightarrow f_{4,0}(x)$  as  $\kappa \rightarrow 0$ . Here  $f_{4,0}(x)$  is given by :*

$$f_{4,0}(x) = (8x^3 - 4x)\sqrt{1-x^2}. \quad (37)$$

**Proof.** Let us prove this convergence result  $f_{4,\kappa}(x) \rightarrow f_{4,0}(x)$  as  $\kappa \rightarrow 0$ . The proof is based on formula (31) when  $m = 2$  :

$$f_{4,\kappa}(x) = 2xf_{3,\kappa}(x) - f_{2,\kappa}(x) - \frac{\text{sgn}(x)}{R_\kappa(x)} \left( 2 \sum_{i=0}^1 t_{2i+1}^{1-i} \frac{s_\kappa^{(i+2)}(0)}{(i+2)!} \right)$$

i.e.

$$f_{4,\kappa}(x) = 2xf_{3,\kappa}(x) - f_{2,\kappa}(x) - 2 \frac{\text{sgn}(x)}{R_\kappa(x)} \left( t_1^1 \frac{s_\kappa^{(2)}(0)}{2!} + t_3^0 \frac{s_\kappa^{(3)}(0)}{3!} \right). \quad (38)$$

First, by definition of  $R_\kappa$ , we have the following convergence result :  $R_\kappa(x) \rightarrow |x|\sqrt{1-x^2}$  as  $\kappa \rightarrow 0$ . Second according to (21) we have :  $t_1^1 = -3$  and  $t_3^0 = 4$ . Third, by definition of  $s_\kappa(\omega)$ , tedious calculi lead to the convergence results :  $s_\kappa^{(2)}(0) \rightarrow -\frac{1}{4}$  as  $\kappa \rightarrow 0$  and  $s_\kappa^{(3)}(0) \rightarrow -\frac{3}{8}$  as  $\kappa \rightarrow 0$ . We infer

$$\frac{\text{sgn}(x)}{R_\kappa(x)} \left( t_1^1 \frac{s_\kappa^{(2)}(0)}{2!} + t_3^0 \frac{s_\kappa^{(3)}(0)}{3!} \right) \rightarrow \frac{1}{x\sqrt{1-x^2}} \left( -3 \frac{-\frac{1}{4}}{2!} + 4 \frac{-\frac{3}{8}}{3!} \right) = \frac{\frac{1}{8}}{x\sqrt{1-x^2}} \quad \text{as } \kappa \rightarrow 0.$$

Finally, passing formally to the limit term by term in (38) as  $\kappa \rightarrow 0$ , and using equations (35)-(36) we infer successively

$$f_{4,\kappa}(x) \rightarrow -2x \frac{4x^4 - 5x^2 + 1}{\sqrt{1-x^2}} + \frac{2x^4 - 2x^2 + \frac{1}{4}}{x\sqrt{1-x^2}} - \frac{\frac{1}{4}}{x\sqrt{1-x^2}} \quad \text{as } \kappa \rightarrow 0, \quad (39)$$

and (37). ■



**Proposition 3.** When  $\psi = -T_5$  we find the formal limit :  $f_{5,\kappa}(x) \longrightarrow f_{5,0}(x)$  as  $\kappa \rightarrow 0$  where  $f_{5,0}(x)$  is defined by :

$$f_{5,0}(x) = (16x^4 - 12x^2 + 1)\sqrt{1-x^2}. \quad (40)$$

**Proof.** The proof is based on formula (32) when  $m = 2$  :

$$f_{5,\kappa}(x) = 2xf_{4,\kappa}(x) - f_{3,\kappa}(x). \quad (41)$$

Passing formally to the limit as  $\kappa \rightarrow 0$  and term by term in the previous equation, we infer

$$f_{5,\kappa}(x) \longrightarrow 2xf_{4,0}(x) - f_{3,0}(x) \quad \text{as } \kappa \rightarrow 0.$$

Substituting (36) and (37) into the above limit, then reducing the expression to the same denominator  $\sqrt{1-x^2}$ , and factoring the numerator by  $1-x^2$ , we deduce the result given by (40) by simplifying the fraction obtained.

■

### 5.1. First two terms of an asymptotic expansion for particular analytical solutions $f_\kappa$

In this section, we derive the first two terms of an asymptotic expansion for the solution  $f_\kappa$  of the airfoil equation set in  $G_\kappa$  when the right hand side  $\psi$  is a Chebyshev polynomial, Prop. 4. As a consequence, we infer weighted Sobolev spaces for the first terms of this expansion, Prop. 5.

**Proposition 4.** Let  $\psi \in \{-T_m, m \in \{4, 5\}\}$  (where  $T_n$  are the Chebyshev polynomials of the first types). Then, in the framework above, the solution of the airfoil equation set in  $G_\kappa$  admits the following asymptotic expansion

$$f_{m,\kappa} \approx f_{m,0} + \kappa^2 f_{m,2} \quad \text{as } \kappa \rightarrow 0. \quad (42)$$

Here, for all  $x > 0$ ,  $f_{m,0}(x)$  is given by (37) and (40) when  $m = 4$  and  $m = 5$ , respectively, and  $f_{m,2}(x)$  is given by :

$$f_{4,2}(x) = \frac{-2}{x\sqrt{1-x^2}} \quad \text{when } m = 4 \text{ (i.e. when } \psi = -T_4), \quad (43a)$$

$$f_{5,2}(x) = \frac{-8x^2 + 1}{2x^2\sqrt{1-x^2}} \quad \text{when } m = 5 \text{ (i.e. when } \psi = -T_5). \quad (43b)$$

**Proof.**

The proof of this result is based on the derivation of the first two terms of an asymptotic expansion in power series of  $\kappa$  for the analytical solution  $f_\kappa$  of the airfoil equation when  $\psi \in \{-T_m, m \in \{4, 5\}\}$ . This derivation is based on the following expressions that can be derived from (31) and (32) when  $m = 2$  after tedious calculi:

$$f_{4,\kappa}(x) = -\frac{\text{sgn}(x)}{R_\kappa(x)} \left( 8x^6 - 12x^4 + 4x^2 + \kappa^2(-4x^4 + 6x^2) + \kappa^4(-4x^2 + \frac{3}{2}) - \frac{\kappa^6}{2} \right) \quad (44)$$

$$f_{5,\kappa}(x) = -\frac{x\text{sgn}(x)}{R_\kappa(x)} \left( 16x^6 - 28x^4 + 13x^2 - 1 + \kappa^2(-8x^4 + 14x^2 - \frac{5}{2}) + \kappa^4(-8x^2 + \frac{7}{2}) - \kappa^6 \right) \quad (45)$$

To derive (44), we employed (38) together with the following expressions for  $f_{2,\kappa}$  and  $f_{3,\kappa}$  (see Eq. (3.20) and Eq. (3.21) in [12], respectively),

$$f_{2,\kappa}(x) = \frac{\text{sgn}(x)}{R_\kappa(x)} \left( -2x^4 + (2 + \kappa^2)x^2 - \frac{1}{4}(1 + 4\kappa^2 - \kappa^4) \right), \quad (46)$$

$$f_{3,\kappa}(x) = -\frac{x\text{sgn}(x)}{R_\kappa(x)} \left( 4x^4 - (5 + \kappa^2)x^2 + \frac{1}{2}(2 + 5\kappa^2 - \kappa^4) \right), \quad (47)$$

and we derived  $s_\kappa^{(2)}(0) = -\frac{1}{4}(1 - \kappa^2)^2$  and  $s_\kappa^{(3)}(0) = -\frac{3}{8}(1 - \kappa^2 - \kappa^4 + \kappa^6)$ . To derive (45), we employed (41) together with the expressions for  $f_{3,\kappa}(x)$  and  $f_{4,\kappa}(x)$  given by (47) and (44), respectively.

Then observe that the function  $R_\kappa^{-1}$  has the following asymptotic expansion when  $\kappa \rightarrow 0$  (see also Eq. (3.22) in [12]) : for all  $x \in G_\kappa$ ,

$$R_\kappa^{-1}(x) = \frac{1}{|x|\sqrt{1-x^2}} \left( 1 + \frac{1}{2} \frac{\kappa^2}{x^2} + \mathcal{O}(\kappa^4) \right). \quad (48)$$

Substituting the expansion (48) into (44) (and (45), respectively) and performing the identification of terms with the same power of  $\kappa$  we infer the expressions of  $f_{4,0}$  and  $f_{4,2}$  (and  $f_{5,0}$  and  $f_{5,2}$ , respectively) given by (43a) (and (43b), respectively). ■

This result completes an analogous result obtained during the previous study (see Prop 3.5 in [12]) when the right hand side  $\psi$  of the airfoil equation is a Chebyshev polynomial such that  $\psi \in \{T_m, m \in \{0, 1, 2, 3\}\}$ . As a consequence of Prop. 4, it is possible to specify a functional framework for  $f_{m,2}$ . We define two weight functions  $w_1$  and  $w_2$  as  $w_1(x) := |x|^{7/2}\sqrt{1-x^2}$  and  $w_2(x) := |x|^{3/2}\sqrt{1-x^2}$ , and we denote by  $L_{w_1}^2(G_0)$  and  $L_{w_2}^2(G_0)$  the associated  $L^2$ - weighted Sobolev spaces.

**Proposition 5.** *In the framework of Prop. 4, the solution of the airfoil equation satisfies at least formally the following expansion*

$$(f_{m,\kappa} - f_{m,0}) \approx \kappa^2 f_{m,2}, \quad \text{as } \kappa \rightarrow 0 \quad (49)$$

with

$$f_{m,2} \in L_{w_2}^2(G_0) \text{ when } m = 4 \text{ (i.e. when } \psi = -T_4)$$

and

$$f_{m,2} \in L_{w_1}^2(G_0) \text{ when } m = 5 \text{ (i.e. when } \psi = -T_5).$$

**Proof.**

As a consequence of (43a), the function  $f_{4,2}$  belongs to the space  $L_{w_2}^2(G_0)$ . As a consequence of (43b), the function  $f_{5,2}$  belongs to the space  $L_{w_1}^2(G_0)$ . ■

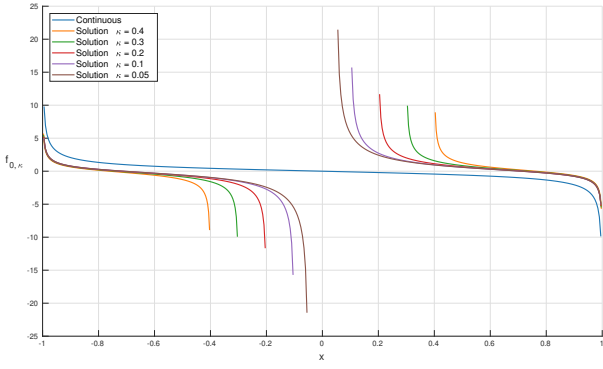
## 6. Computational Experiments

In this section, we present numerical solutions of the airfoil equation on two disjoint intervals and also compare it with the solution on a single interval.

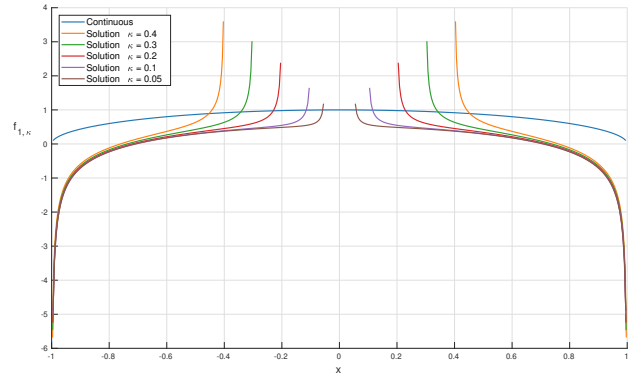
### 6.1. General behaviour

The solution  $f_{n,\kappa}$  obtained above is valid not only for small  $\kappa$ . The aim of this subsection is twofold: testing the accuracy of the numerical solutions and exhibiting the overall behaviour of  $f_{n,\kappa}$ , for several values of  $\kappa$ .

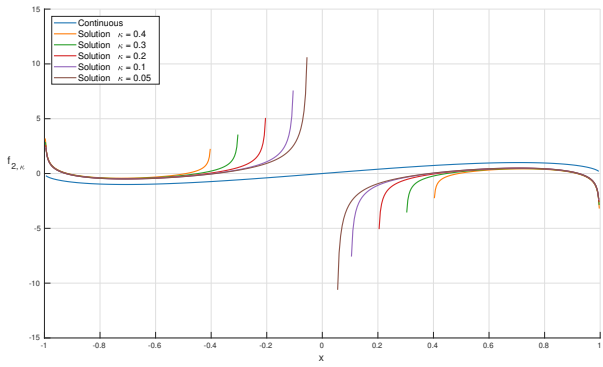
In figures 1–5, we see the graphs of  $f_{n,\kappa}$  for input functions  $T_0$  to  $T_5$ , respectively. We have computed the solutions using the recurrence relations (31) and (32) for all  $\kappa \in \{0.05, 0.1, 0.2, 0.3, 0.4\}$ . We have also included the graph of  $f(x)$ , the solution of the airfoil equation on a continuous domain, the single interval  $(-1, 1)$ , by evaluating formula (6). We clearly see that  $f_{n,\kappa}$  diverges from  $f(x)$  at the points  $x = -1, x = -\kappa, x = \kappa, x = 1$ . This behaviour is expected because of the choice  $C_1 = C_2 = 0$ .



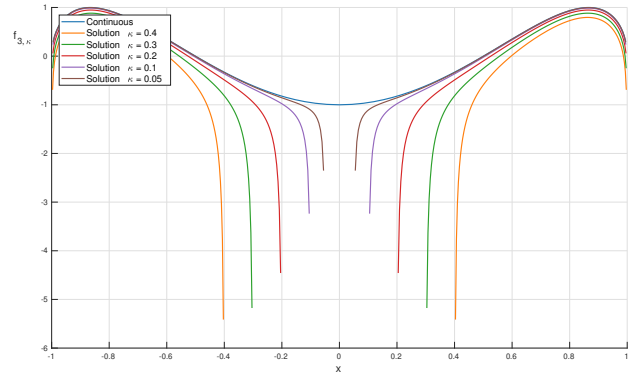
**Figure 1:** Solutions for  $\psi(x) = T_0(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$  and  $\kappa = 0.4$ ,  $\kappa = 0.3$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



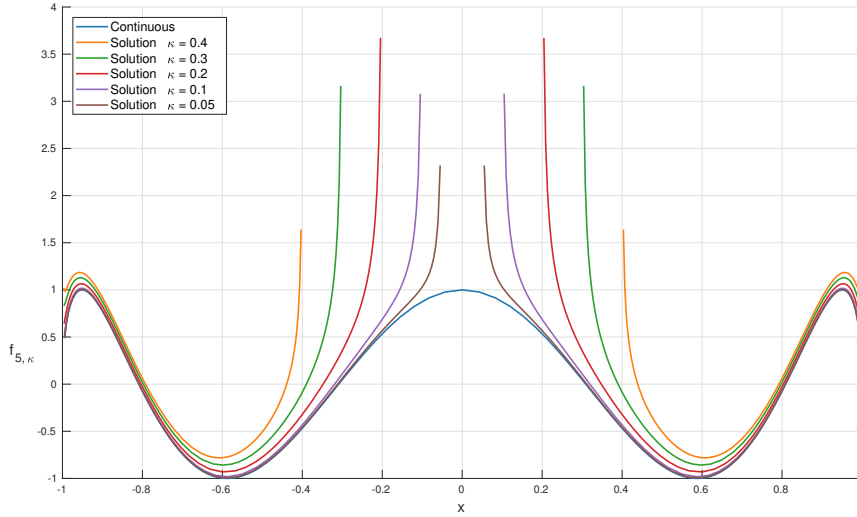
**Figure 2:** Solutions for  $\psi(x) = T_1(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$  and  $\kappa = 0.4$ ,  $\kappa = 0.3$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



**Figure 3:** Solutions for  $\psi(x) = T_2(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$  and  $\kappa = 0.4$ ,  $\kappa = 0.3$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



**Figure 4:** Solutions for  $\psi(x) = T_3(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$ ,  $\kappa = 0.4$ ,  $\kappa = 0.3$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



**Figure 5:** Solutions for  $\psi(x) = T_5(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$  and  $\kappa = 0.4$ ,  $\kappa = 0.3$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$

The solutions  $f_{n,\kappa}$  could be computed also by the series in (25) and (26). We did this and compared the with the numerical results obtained by recurrence relations. In table 1, we present the differences in these two forms of calculations, expressed by the error in the  $L_2$  norm. The fact that the error is very small shows that one can use the recurrence relations for computing the solutions, having as benefit, a fast and accurate result.

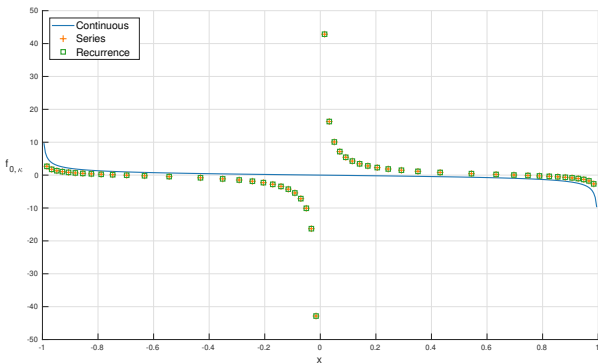
$\psi$	$\ \text{series} - \text{recurrence}\ _2$
$T_0$	0.00000000e+00
$T_1$	0.00000000e+00
$T_2$	9.80632415e-15
$T_3$	1.71891178e-14
$T_4$	2.83788189e-14
$T_5$	5.01361495e-14

**Table 1:** Error in the  $L_2$  norm of  $f_{n,\kappa}$ , computed by the series (25) and (26) and by the recurrence relations (31) and (32).

We have seen that  $f_{n,\kappa}$  diverges from  $f(x)$  at the endpoints of the two disjoint intervals. In the next subsection, we will look closer into this edge behaviour and conditions at which this divergence is attenuated.

## 6.2. Endpoints behaviour

We now examine in more detail the behaviour of  $f_{n,\kappa}$  at the endpoints of  $G_\kappa$ , for small  $\kappa$ . We show in the figures 6–10 the solutions for  $n \in \{0, 1, 2, 4\}$  and the solution,  $f$ , for airfoil equation on the continuous domain  $(-1, 1)$ . In these results,  $\kappa = 0.01$  and the solutions on the disjoint intervals have been computed both with the expansions (25) and (26), and with the recurrence relations (31) and (32). In order to make the differences between the  $f$  and  $f_{n,\kappa}$  explicit, we present in tables 2–6 the numerical data present in the graphs 6–10. We see that for  $n \in \{0, 1, 2, 3\}$  the divergence between the continuous and disjoint domain cases are very evident, particular near  $x = 0$ . As the degree  $n$  increases, for  $n > 3$  and we experimented with values up to  $n = 48$ , the solutions almost coincide, as shown in figure 10, for  $n = 4$ . In table 6, we can see, however, how  $f_{4,0.01}$  starts to diverge from  $f$  near  $x = 0$ . The behaviours above are expected due to the singularities present in the solutions for the disjoint intervals, with  $C_{1,\kappa} = C_{2,\kappa} = 0$ , because of the denominator term in (11).



**Figure 6:** Solutions for  $\psi(x) = T_0(x)$  and  $\kappa = 0, 01$

$x$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	9.7471668651	4.8220376332	4.8220376332
-0.9894	6.8258551221	3.3394986108	3.3394986108
-0.9840	5.5183033368	2.6683969374	2.6683969374
-0.9784	4.7307051583	2.2595310141	2.2595310141
-0.9726	4.1874612722	1.9742095759	1.9742095759
-0.9668	3.7820271604	1.7587011107	1.7587011107
-0.0442	0.0442336490	-11.5831024463	-11.5831024463
-0.0381	0.0381642652	-13.5578922160	-13.5578922160
-0.0322	0.0322509004	-16.2927283117	-16.2927283117
-0.0265	0.0264852341	-20.3764226611	-20.3764226611
-0.0209	0.0208595897	-27.3055463091	-27.3055463091
-0.0154	0.0153668699	-42.8503703562	-42.8503703562

**Table 2:** Comparisons of the solutions for  $\psi(x) = T_0(x)$  and  $\kappa = 0.01$

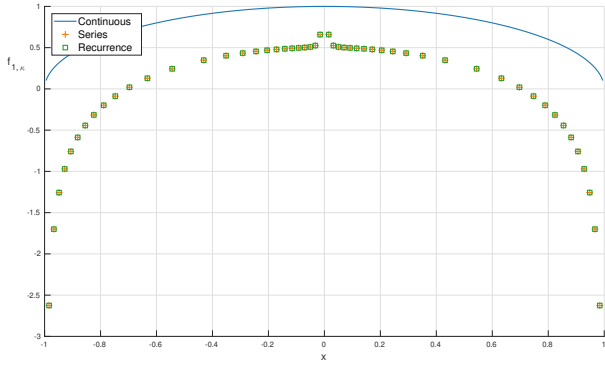


Figure 7: Solutions for  $\psi(x) = T_1(x)$  and  $\kappa = 0,01$

$x$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	0.1020582126	-4.7968590176	-4.7968590176
-0.9894	0.1449544865	-3.3042279082	-3.3042279082
-0.9840	0.1783109989	-2.6256336894	-2.6256336894
-0.9784	0.2068148543	-2.2106801749	-2.2106801749
-0.9726	0.2322767144	-1.9202144421	-1.9202144421
-0.9668	0.2556238435	-1.7002705857	-1.7002705857
-0.0442	0.9990231254	0.5118623741	0.5118623741
-0.0381	0.9992725390	0.5170505863	0.5170505863
-0.0322	0.9994803451	0.5251821021	0.5251821021
-0.0265	0.9996494506	0.5394851420	0.5394851420
-0.0209	0.9997825097	0.5694586149	0.5694586149
-0.0154	0.9998819506	0.6583983330	0.6583983330

Table 3: Comparisons of the solutions for  $\psi(x) = T_1(x)$  and  $\kappa = 0.01$

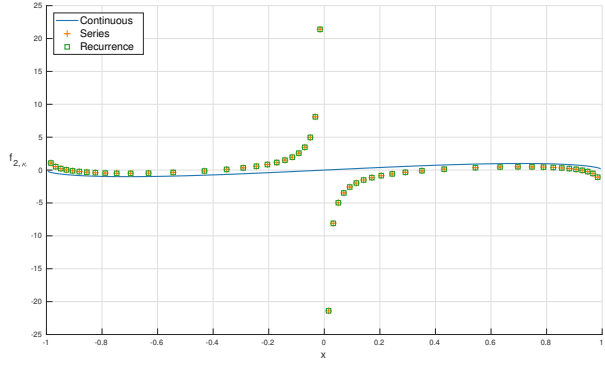


Figure 8: Solutions for  $\psi(x) = T_2(x)$  and  $\kappa = 0,01$

$x$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	-0.2030506166	2.2595139806	2.2595139806
-0.9894	-0.2868470483	1.4563311940	1.4563311940
-0.9840	-0.3509068381	1.0740463506	1.0740463506
-0.9784	-0.4046870748	0.8308999877	0.8308999877
-0.9726	-0.4518477743	0.6547780651	0.6547780651
-0.9668	-0.4942621568	0.5174008600	0.5174008600
-0.0442	-0.0882945396	5.7253505906	5.7253505906
-0.0381	-0.0762175188	6.7217744086	6.7217744086
-0.0322	-0.0644347808	8.0979928926	8.0979928926
-0.0265	-0.0529333372	10.1484029115	10.1484029115
-0.0209	-0.0417010344	13.6212345868	13.6212345868
-0.0154	-0.0307264840	21.4012621645	21.4012621645

Table 4: Comparisons of the solutions for  $\psi(x) = T_2(x)$  and  $\kappa = 0.01$

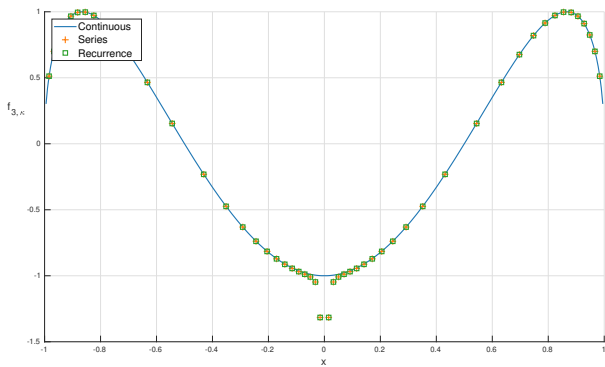
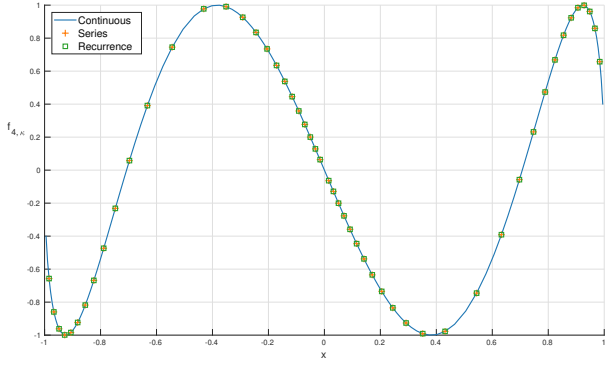


Figure 9: Solutions for  $\psi(x) = T_3(x)$  and  $\kappa = 0,01$

$x$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	0.3019225339	0.3014274852	0.3014274852
-0.9894	0.4226804389	0.4223281168	0.4223281168
-0.9840	0.5122555378	0.5119659343	0.5119659343
-0.9784	0.5850607054	0.5848081523	0.5848081523
-0.9726	0.6467025313	0.6464750051	0.6464750051
-0.9668	0.7000580535	0.6998487882	0.6998487882
-0.0442	-0.9912195766	-1.0178738782	-1.0178738782
-0.0381	-0.9934591999	-1.0297405153	-1.0297405153
-0.0322	-0.9953263454	-1.0472457920	-1.0472457920
-0.0265	-0.9968465299	-1.0768623526	-1.0768623526
-0.0209	-0.9980431552	-1.1376017527	-1.1376017527
-0.0154	-0.9989377223	-1.3160615091	-1.3160615091

Table 5: Comparisons of the solutions for  $\psi(x) = T_3(x)$  and  $\kappa = 0.01$



**Figure 10:** Solutions for  $\psi(x) = T_4(x)$  and  $\kappa = 0,01$

$x$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	-0.3976414309	-0.3976413811	-0.3976413811
-0.9894	-0.5495854017	-0.5495853661	-0.5495853661
-0.9840	-0.6571856078	-0.6571855784	-0.6571855784
-0.9784	-0.7401364258	-0.7401364000	-0.7401364000
-0.9726	-0.8061823310	-0.8061823076	-0.8061823076
-0.9668	-0.8593369390	-0.8593369174	-0.8593369174
-0.0442	0.1758993947	0.1759605027	0.1759605027
-0.0381	0.1519916367	0.1520884638	0.1520884638
-0.0322	0.1286017604	0.1287669002	0.1287669002
-0.0265	0.1057182544	0.1060320916	0.1060320916
-0.0209	0.0833295202	0.0840423342	0.0840423342
-0.0154	0.0614239517	0.0637703149	0.0637703149

**Table 6:** Comparisons of the solutions for  $\psi(x) = T_4(x)$  and  $\kappa = 0,01$

We can make use of the constants in (8) in order to modify the endpoints behaviour at  $x = \pm\kappa$ . In fact, for  $g(x) = T_0(x)$  we have from (4) that

$$f(x) = -\frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^1 \frac{\sqrt{1-t^2}}{x-t} dt, \quad (50)$$

which is

$$f(x) = \frac{-x}{\sqrt{1-x^2}}. \quad (51)$$

On the other hand, from (8), we have

$$f_{0,\kappa}(x) = \begin{cases} \frac{1}{\pi R_\kappa(x)} [C_{1,\kappa} + C_{2,\kappa}x] + \frac{1}{R_\kappa(x)} \left( x^2 - \frac{(1+\kappa^2)}{2} \right), & x \in (-1, -\kappa) \\ -\frac{1}{\pi R_\kappa(x)} [C_{1,\kappa} + C_{2,\kappa}x] - \frac{1}{R_\kappa(x)} \left( x^2 - \frac{(1+\kappa^2)}{2} \right), & x \in (\kappa, 1) \end{cases} \quad (52)$$

Taking  $C_{1,0} = \frac{\pi}{2}$  e  $C_{2,0} = 0$ , we get

$$f_{0,0}(x) = \begin{cases} \frac{x^2}{|x|\sqrt{1-x^2}}, & x \in (-1, 0) \\ -\frac{x^2}{|x|\sqrt{1-x^2}}, & x \in (0, 1), \end{cases} \quad (53)$$

and using  $\text{sgn}(x) = \frac{x}{|x|}$ , it follows that

$$f_{0,0}(x) = \begin{cases} \frac{\text{sgn}(x)x}{\sqrt{1-x^2}}, & x \in (-1, 0] \\ -\frac{\text{sgn}(x)x}{\sqrt{1-x^2}}, & x \in [0, 1). \end{cases} \quad (54)$$

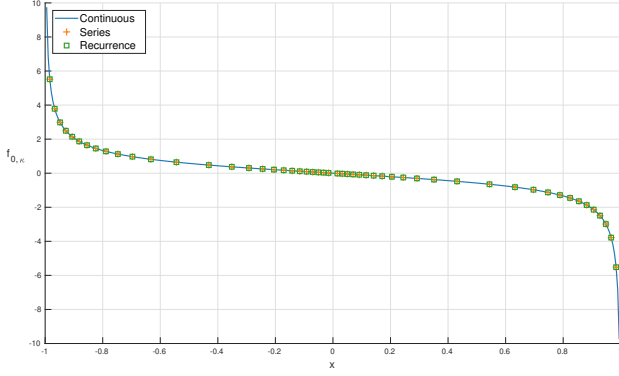
Then,

$$f_{0,0} = \frac{-x}{\sqrt{1-x^2}}, \quad x \in (-1, 1). \quad (55)$$

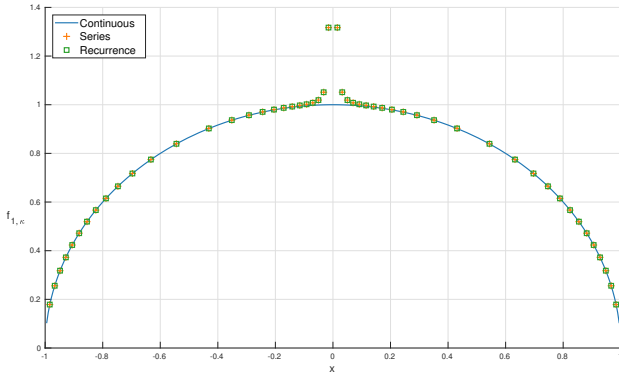
Thus, in general, we could choose  $C_{1,\kappa}$  and  $C_{2,\kappa}$  by comparing the solutions of the continuous and disjoint problems for small  $\kappa$ . In other words, this is done by assuming

$$\lim_{\kappa \rightarrow 0} f_{n,\kappa}(x) = f(x).$$

This procedure yields the values  $C_{1,\kappa} = 0$ ,  $C_{2,\kappa} = -\frac{\pi}{2}$  and  $C_{1,\kappa} = -\frac{\pi}{4}$ ,  $C_{2,\kappa} = 0$  for  $n = 1$  and  $n = 2$ , respectively. Figures 11, 12 and 13 show the results on the solutions with these nonzero constants and we see, in particular, that the solutions for the disjoint intervals get closer to the continuous interval solution at the endpoints  $x = \pm 1$ . Comparisons between tables 2 and 7, for instance, make this even more evident.



**Figure 11:** Solutions for  $\psi(x) = T_0(x)$ ,  $C_{1,\kappa} = \frac{1}{2}$ ,  $C_{2,\kappa} = 0$  and  $\kappa = 0,01$



**Figure 12:** Solutions for  $\psi(x) = T_1(x)$ ,  $C_{1,\kappa} = 0$ ,  $C_{2,\kappa} = -\frac{1}{2}$  and  $\kappa = 0,01$

$x$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	9.7471668651	9.7471668775	9.7471668775
-0.9894	6.8258551221	6.8258551310	6.8258551310
-0.9840	5.5183033368	5.5183033441	5.5183033441
-0.9784	4.7307051583	4.7307051648	4.7307051648
-0.9726	4.1874612722	4.1874612781	4.1874612781
-0.9668	3.7820271604	3.7820271658	3.7820271658
-0.0442	0.0442336490	0.0442489283	0.0442489283
-0.0381	0.0381642652	0.0381884757	0.0381884757
-0.0322	0.0322509004	0.0322921916	0.0322921916
-0.0265	0.0264852341	0.0265637054	0.0265637054
-0.0209	0.0208595897	0.0210378208	0.0210378208
-0.0154	0.0153668699	0.0159535547	0.0159535547

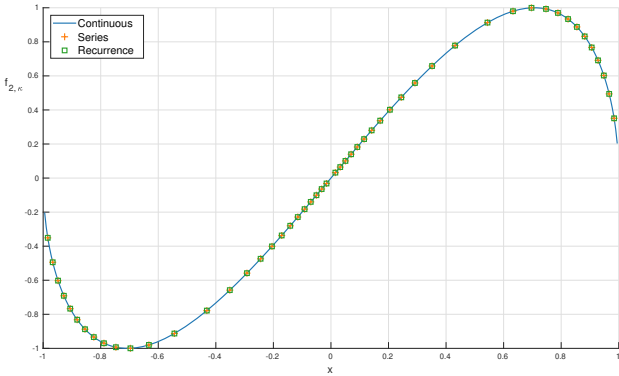
**Table 7:** Comparisons of the solutions for  $\psi(x) = T_0(x)$ ,  $C_{1,\kappa} = \frac{1}{2}$ ,  $C_{2,\kappa} = 0$  and  $\kappa = 0.01$

$x$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	0.1020582126	0.1025533109	0.1025533109
-0.9894	0.1449544865	0.1453068438	0.1453068438
-0.9840	0.1783109989	0.1786006314	0.1786006314
-0.9784	0.2068148543	0.2070674327	0.2070674327
-0.9726	0.2322767144	0.2325042634	0.2325042634
-0.9668	0.2556238435	0.2558331297	0.2558331297
-0.0442	0.9990231254	1.0256801277	1.0256801277
-0.0381	0.9992725390	1.0355575475	1.0355575475
-0.0322	0.9994803451	1.0514051153	1.0514051153
-0.0265	0.9996494506	1.0796735834	1.0796735834
-0.0209	0.9997825097	1.1393559746	1.1393559746
-0.0154	0.9998819506	1.3170417933	1.3170417933

**Table 8:** Comparisons of the solutions for  $\psi(x) = T_1(x)$ ,  $C_{1,\kappa} = 0$ ,  $C_{2,\kappa} = -\frac{1}{2}$  and  $\kappa = 0.01$

$x$	$f_{4,0}$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	-0.3968647346	-0.3976414309	-0.3976413811	-0.3976413811
-0.9894	-0.5504678107	-0.5495854017	-0.5495853661	-0.5495853661
-0.9840	-0.6567487601	-0.6571856078	-0.6571855784	-0.6571855784
-0.9784	-0.7398774665	-0.7401364258	-0.7401364000	-0.7401364000
-0.9726	-0.8066886684	-0.8061823310	-0.8061823076	-0.8061823076
-0.9668	-0.8591473374	-0.8593369390	-0.8593369174	-0.8593369174
-0.0442	0.1759370818	0.1758993947	0.1759605027	0.1759605027
-0.0381	0.1518472177	0.1519916367	0.1520884638	0.1520884638
-0.0322	0.1284662587	0.1286017604	0.1287669002	0.1287669002
-0.0265	0.1058139495	0.1057182544	0.1060320916	0.1060320916
-0.0209	0.0835087207	0.0833295202	0.0840423342	0.0840423342
-0.0154	0.0615634804	0.0614239517	0.0637703149	0.0637703149

**Table 10:** Comparisons of  $f_{4,0}$  with the solutions for  $\psi(x) = T_4(x)$  and  $\kappa = 0.01$



**Figure 13:** Solutions for  $\psi(x) = T_2(x)$ ,  $C_{1,\kappa} = -\frac{1}{4}$ ,  $C_{2,\kappa} = 0$  and  $\kappa = 0,01$

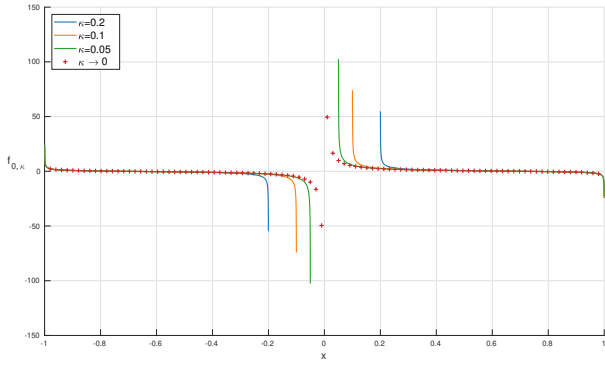
$x$	$(-1, 1)$	Series $G_\kappa$	Recurrence $G_\kappa$
-0.9948	-0.2030506166	-0.2030506415	-0.2030506415
-0.9894	-0.2868470483	-0.2868470661	-0.2868470661
-0.9840	-0.3509068381	-0.3509068528	-0.3509068528
-0.9784	-0.4046870748	-0.4046870877	-0.4046870877
-0.9726	-0.4518477743	-0.4518477860	-0.4518477860
-0.9668	-0.4942621568	-0.4942621676	-0.4942621676
-0.0442	-0.0882945396	-0.0883250967	-0.0883250967
-0.0381	-0.0762175188	-0.0762659372	-0.0762659372
-0.0322	-0.0644347808	-0.0645173590	-0.0645173590
-0.0265	-0.0529333372	-0.0530902718	-0.0530902718
-0.0209	-0.0417010344	-0.0420574782	-0.0420574782
-0.0154	-0.0307264840	-0.0318997909	-0.0318997909

**Table 9:** Comparisons of the solutions for  $\psi(x) = T_2(x)$ ,  $C_{1,\kappa} = -\frac{1}{4}$ ,  $C_{2,\kappa} = 0$  and  $\kappa = 0.01$

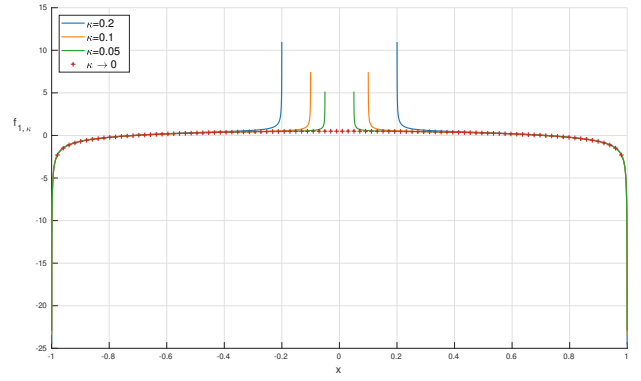
### 6.3. Asymptotic solutions

In Table 10 we make quantitative comparisons between the asymptotic solution  $f_{4,0}$  and the solutions for  $\psi(x) = T_4(x)$  and  $\kappa = 0,01$ . Table 10 shows that the asymptotic solution  $f_{4,0}$  provides a good approximation of the solutions for  $\psi(x) = T_4(x)$  and  $\kappa = 0,01$ . In figures 14–19, we see the graphs of  $f_{n,\kappa}$  and  $f_{n,0}$  for input functions  $T_0$  to  $T_5$ , respectively. We have computed the solutions using the recurrence relations (31) and (32) for all  $\kappa \in \{0.05, 0.1, 0.2\}$ . The convergence of  $f_{n,\kappa}$  to  $f_{n,0}$  is depicted in figures 14–19 for input functions  $T_0$  to  $T_5$ .

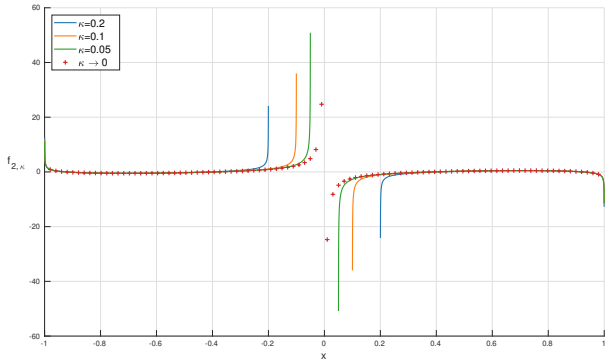




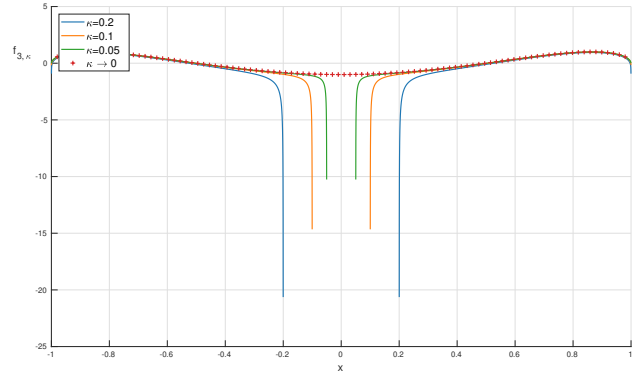
**Figure 14:** Solutions for  $\psi(x) = T_0(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



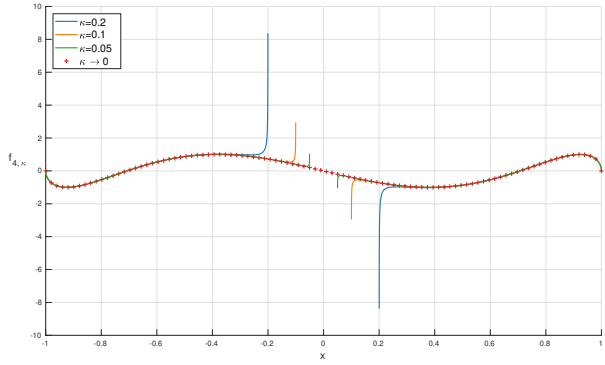
**Figure 15:** Solutions for  $\psi(x) = T_1(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



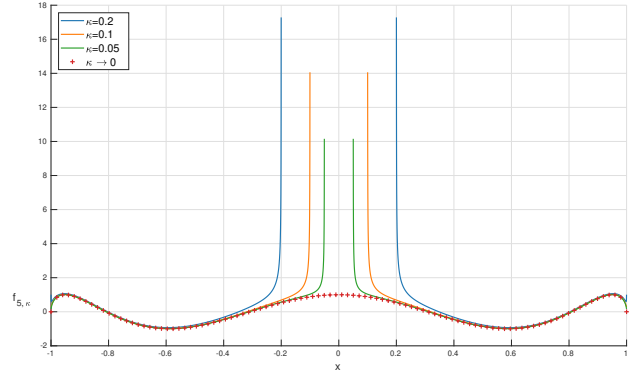
**Figure 16:** Solutions for  $\psi(x) = T_2(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



**Figure 17:** Solutions for  $\psi(x) = T_3(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



**Figure 18:** Solutions for  $\psi(x) = T_4(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$



**Figure 19:** Solutions for  $\psi(x) = T_5(x)$ ,  $C_{1,\kappa} = C_{2,\kappa} = 0$ ,  $\kappa = 0.2$ ,  $\kappa = 0.1$  and  $\kappa = 0.05$

## 7. Discussion and conclusion

We have found analytical closed-form solutions of the airfoil integral equation on the two disjoint intervals  $(-1, -\kappa) \cup (\kappa, 1)$  for an arbitrary  $\kappa$ , for input functions defined as Chebyshev polynomials of the first kind.

We have also generalised a classical relationship, given in (5), between Chebyshev polynomials of the first and second kind for the discontinuous domain  $G_\kappa$ , as

$$\frac{1}{\pi} \int_{G_\kappa} \frac{E_n(t)}{x-t} w(t) dt = T_n(x),$$

where  $E_n$  is a polynomials of degree  $n$ . We have derived recurrence formulas for  $E_n(x)$ .

Numerical results showing the behaviour of the solutions, computed by both an expansion and by recurrence formulas are shown and compared with the solution of the classical airfoil equation, for several values of  $\kappa$ . The singular endpoint behaviour of the solutions are analysed and it is shown are alternative choices of the constants in the solutions could modify this singular behaviour.

Further studies based on this work could provide solutions of the airfoil equation on discontinuous domain which are bounded at the endpoints; this could be done by appropriate choices of the adding constants to the solutions.

With the results obtained in this work, we can propose a spectral method for the solution of the integral equation

$$\frac{1}{\pi} \int_{G_\kappa} \left\{ \frac{1}{x-t} + K(x,t) \right\} f_\kappa(t) \omega(t) dt = h(x), \quad x \in G_\kappa, \quad (56)$$

where  $K$  is a regular (not strongly singular) kernel,  $h(x)$  is prescribed and  $w(t) = \text{sgn}(t)1/R_\kappa(t)$ . This equation is referred to as the generalised airfoil equation and its version defined on a continuous domain is a well studied integral equation from the physical and numerical points of view [8, 11, 27, 15].

Expanding the solution as

$$f_\kappa(t) \approx \sum_{n=0}^N a_n E_n(t), \quad (57)$$

substituting (57) in (56), and exchanging the orders of integration with sum we get

$$\frac{1}{\pi} \sum_{n=0}^N a_n \left( \int_{G_\kappa} \frac{1}{x-t} E_n(t) dt + \int_{G_\kappa} K(x,t) E_n(t) dt \right) = T_n(x), \quad x \in G_\kappa. \quad (58)$$

Using (28), it follows that

$$\sum_{n=0}^N a_n [T_n(x) + \mathcal{K}(E_n)(x)] = h(x), \quad (59)$$

where  $\mathcal{K}$  is the integral operator given by

$$\mathcal{K}(g) = \frac{1}{\pi} \int_{G_\kappa} K(x,t) g(t) \omega(t) dt. \quad (60)$$

Equation (59) can be solved by a Galerkin or a collocation method. The spectral method for the solution of (56) consists of (57) and (59).

The implementation and the convergence aspects of this solution method, as well as its physical applications are topics of research opened by the current work.

## Acknowledgments

The first author acknowledges financial support from CAPES, Brazil. This work has been partially funded by the Office of Naval Research Global, under the contract no. N629091812124.

## References

- [1] B. N. Mandal, A. Chakrabarti, Applied singular integral equations, Science Publishers, 2011.
- [2] P. Martin, N. Parsons, L. Farina, Interaction of water waves with thin plates, International Series on Advances in Fluid Mechanics 8 (1997) 197–230.

- [3] S. Mukherjee, Y. X. Mukherjee, W. Ye, Cauchy principal values and finite parts of boundary integrals—revisited, *Engineering analysis with boundary elements* 29 (9) (2005) 844–849.
- [4] C.-Y. Hui, D. Shia, Evaluations of hypersingular integrals using gaussian quadrature, *International Journal for Numerical Methods in Engineering* 44 (2) (1999) 205–214.
- [5] L. Farina, Evaluation of single layer potentials over curved surfaces, *SIAM Journal on Scientific Computing* 23 (1) (2001) 81–91.
- [6] E. Theotokoglou, G. Tsamasphyros, A modified gauss quadrature formula with special integration points for evaluation of quasi-singular integrals, *Engineering analysis with boundary elements* 30 (9) (2006) 758–766.
- [7] A. Klöckner, A. Barnett, L. Greengard, M. O’Neil, Quadrature by expansion: A new method for the evaluation of layer potentials, *Journal of Computational Physics* 252 (2013) 332–349.
- [8] D. Berthold, W. Hoppe, B. Silbermann, A fast algorithm for solving the generalized airfoil equation, *Journal of computational and applied mathematics* 43 (1-2) (1992) 185–219.
- [9] S. R. Bland, The two-dimensional oscillating airfoil in a wind tunnel in subsonic flow, *SIAM Journal on Applied Mathematics* 18 (4) (1970) 830–848.
- [10] M. A. Golberg, *Numerical solution of integral equations*, Vol. 42, Springer Science & Business Media, 2013.
- [11] G. Monegato, I. H. Sloan, Numerical solution of the generalized airfoil equation for an airfoil with a flap, *SIAM journal on numerical analysis* 34 (6) (1997) 2288–2305.
- [12] M. Golberg, J. Fromme, On the  $l_2$  convergence of collocation for the generalized airfoil equation, *Journal of Mathematical Analysis and Applications* 71 (1) (1979) 271–286.
- [13] T. Hartmann, E. Stephan, Rates of convergence for collocation with jacobi polynomials for the airfoil equation, *Journal of computational and applied mathematics* 51 (2) (1994) 179–191.
- [14] L. Scuderi, A collocation method for the generalized airfoil equation for an airfoil with a flap, *SIAM journal on numerical analysis* 35 (5) (1998) 1725–1739.
- [15] P. Junghanns, R. Kaiser, A note on Kalandiya’s method for a crack problem, *Applied Numerical Mathematics* 149 (2020) 52–64.
- [16] F. G. Tricomi, The airfoil equation for a double interval, *ZAMP* 2 (5) (1951) 402–406.
- [17] B. Dutta, S. Banerjea, Solution of a hypersingular integral equation in two disjoint intervals, *Appl. Math. Lett.* 22 (8) (2009) 1281–1285.
- [18] T. Trogdon, S. Olver, *Riemann–Hilbert problems, their numerical solution, and the computation of non-linear special functions*, SIAM, 2015.
- [19] L. Farina, M. R. Ferreira, V. Péron, The airfoil equation on near disjoint intervals: Approximate models and polynomial solutions, *Journal of Computational and Applied Mathematics* 298 (2016) 97–104.
- [20] F. G. Tricomi, *Integral equations*, Pure and Applied Mathematics. Vol. V, Interscience Publishers, Inc., New York, 1957.

- [21] P. A. Martin, Exact solution of a simple hypersingular integral equation, *J. Integral Equations Appl.* 4 (2) (1992) 197–204.
- [22] J. P. Breslin, P. Andersen, *Hydrodynamics of ship propellers*, Vol. 3 of Cambridge Ocean Technology Series, Cambridge University Press, 1994.
- [23] M. Abramowitz, I. A. Stegun (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover: New York, 1972.
- [24] F. D. Gakhov, *Boundary value problems*, Vol. 85 of International Series of Monographs in Pure and Applied Mathematics, Pergamon Press, 1966, translation by I. N. Sneddon.
- [25] B. Khvedelidze, The method of cauchy-type integrals in the discontinuous boundary-value problems of the theory of holomorphic functions of a complex variable, *Journal of Soviet Mathematics* 7 (3) (1977) 309–415.
- [26] M. A. Snyder, *Chebyshev Methods in Numerical Approximation*, Prentice-Hall, Inc., 1966.
- [27] G. M. Vainikko, I. K. Lifanov, Modeling of aerodynamic and wave diffraction problems and extensions of cauchy type integral operators on closed and open curves, *Differential Equations* 36 (9) (2000) 1310–1322.

## Appendix A. Appendix

Here we derive with details the analytical solution of the equation (7) for the case when  $\psi(x) = T_n(x)$ . For this aim, we evaluate the principal part of the density function in the Cauchy integral, from equations (16) (even case) and (18) (odd case).

**Case 1.**  $\psi(x) = T_{2m}(x)$  ( $m = 0, 1, 2, \dots$ ).

Applying the variable change  $x^2 = \frac{1}{w}$  in (16), we get

$$\Psi_\kappa \left( \sqrt{\frac{1}{w}} \right) = -\pi \text{PP} \left\{ \frac{s_\kappa(w)}{w} T_{2m} \left( \sqrt{\frac{1}{w}} \right) \right\}_{w \rightarrow 0}, \quad (\text{A.1})$$

where  $s_\kappa(w) = \sqrt{(1-w)(1-\kappa^2 w)}$ . Defining

$$E_{2m} := -\Psi/\pi \quad (\text{A.2})$$

and expanding  $s_\kappa(w)$  in Taylor series, we have

$$E_{2m} = \text{PP} \left\{ T_{2m}(x) \sum_{i=0}^{m+1} \frac{s_\kappa^{(i)}(0)}{i!} x^{2(1-i)} \right\}_{x \rightarrow \infty}. \quad (\text{A.3})$$

Using

$$T_{2m}(x) = \sum_{k=0}^m t_{2k}^{m-k} x^{2k} \quad (\text{A.4})$$

where

$$t_i^j = 2^{i-1} (-1)^j \binom{i+2j}{i+j} \binom{i+j}{j} \quad (\text{A.5})$$

are the coefficients from the Chebyshev polynomial (cf.[26, p. 14]), we express (A.3) as

$$E_{2m} = \text{PP} \left\{ \sum_{k=0}^m t_{2k}^{m-k} x^{2k} \sum_{i=0}^{m+1} \frac{s_\kappa^{(i)}(0)}{i!} x^{2(1-i)} \right\}_{x \rightarrow \infty}. \quad (\text{A.6})$$

Note that in the equation (A.6), we have expanded  $s_\kappa(\omega)$  only to degree  $m$ , because the higher order terms are canceled out as  $x \rightarrow \infty$  ( $\omega \rightarrow 0$ ). The remaining, non-zero terms correspond to the principal part. Reorganizing the terms in (A.6), we can determine the coefficients for each power of  $x$  as

$$\begin{aligned} E_{2m} = & t_{2m}^0 s_\kappa(0) x^{2m+2} + (t_{2m}^0 s'_\kappa(0) + t_{2m-2}^1 s_\kappa(0)) x^{2m} + \left( t_{2m}^0 \frac{s''_\kappa(0)}{2!} + t_{2m-2}^1 s'_\kappa(0) + t_{2m-4}^2 s_\kappa(0) \right) x^{2m-2} \\ & + \left( t_{2m}^0 \frac{s_\kappa^{(3)}(0)}{3!} + t_{2m-2}^1 \frac{s''_\kappa(0)}{2!} + t_{2m-4}^2 s'_\kappa(0) + t_{2m-6}^3 s_\kappa(0) \right) x^{2m-4} + \dots \\ & + \left( t_{2m}^0 \frac{s_\kappa^{(m-1)}(0)}{(m-1)!} + \dots + t_4^{m-2} s'_\kappa(0) + t_2^{m-1} s_\kappa(0) \right) x^4 \\ & + \left( t_{2m}^0 \frac{s_\kappa^{(m)}(0)}{m!} + t_{2m-2}^1 \frac{s_\kappa^{(m-1)}(0)}{(m-1)!} + t_{2m-4}^2 \frac{s_\kappa^{(m-2)}(0)}{(m-2)!} + \dots + t_4^{m-2} \frac{s''_\kappa(0)}{2!} + t_2^{m-1} s'_\kappa(0) + t_0^m s_\kappa(0) \right) x^2 \\ & + t_{2m}^0 \frac{s_\kappa^{(m+1)}(0)}{(m+1)!} + t_{2m-2}^1 \frac{s_\kappa^{(m)}(0)}{m!} + t_{2m-4}^2 \frac{s_\kappa^{(m-1)}(0)}{(m-1)!} + \dots + t_4^{m-2} \frac{s_\kappa^{(3)}(0)}{3!} + t_2^{m-1} \frac{s''_\kappa(0)}{2!} + t_0^m s'_\kappa(0). \end{aligned}$$

Rearranging terms, we get  $E_{2m}$  as

$$\begin{aligned} E_{2m} &= \left( \sum_{k=m}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k-m)}(0)}{(k-m)!} \right) x^{2m+2} + \left( \sum_{k=m-1}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k-(m-1))}(0)}{(k-(m-1))!} \right) x^{2m} \\ &+ \left( \sum_{k=m-2}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k-(m-2))}(0)}{(k-(m-2))!} \right) x^{2m-2} + \left( \sum_{k=m-3}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k-(m-3))}(0)}{(k-(m-3))!} \right) x^{2m-4} + \dots \\ &+ \left( \sum_{k=1}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k-1)}(0)}{(k-1)!} \right) x^4 + \left( \sum_{k=0}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k)}(0)}{k!} \right) x^2 + \left( \sum_{k=0}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k+1)}(0)}{(k+1)!} \right). \end{aligned}$$

In a more concise form, we have

$$E_{2m} = \sum_{l=0}^m \left[ \left( \sum_{k=m-l}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k-(m-l))}(0)}{(k-(m-l))!} \right) x^{2(m-(l-1))} \right] + \left( \sum_{k=0}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k+1)}(0)}{(k+1)!} \right),$$

or

$$E_{2m} = \sum_{l=0}^m \left[ \left( \sum_{k=m-l}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k-(m-l))}(0)}{(k-(m-l))!} \right) x^{2(m-(l-1))} + t_{2l}^{m-l} \frac{s_{\kappa}^{(l+1)}(0)}{(l+1)!} \right]. \quad (\text{A.7})$$

Therefore, the solution, from (11), when  $\psi(x) = T_{2m}(x)$ , is

$$f_{2m,\kappa}(x) = \frac{\text{sgn}(x)}{R_{\kappa}(x)} \left\{ \sum_{l=0}^m \left[ \left( \sum_{k=m-l}^m t_{2k}^{m-k} \frac{s_{\kappa}^{(k-(m-l))}(0)}{(k-(m-l))!} \right) x^{2(m-(l-1))} + t_{2l}^{m-l} \frac{s_{\kappa}^{(l+1)}(0)}{(l+1)!} \right] \right\} \blacksquare \quad (\text{A.8})$$

**Case 2.**  $\psi(x) = T_{2m+1}(x)$  ( $m = 0, 1, 2, \dots$ ).

Applying the change of the variable  $x^2 = \frac{1}{w}$  in (18), we get

$$\Psi_{\kappa} \left( \frac{1}{\sqrt{w}} \right) = -\pi \frac{1}{\sqrt{w}} \text{PP} \left\{ \frac{s_{\kappa}(w)}{w} T_{2m+1} \left( \frac{1}{\sqrt{w}} \right) \sqrt{w} \right\}_{w \rightarrow 0}, \quad (\text{A.9})$$

where  $s_{\kappa}(w) = \sqrt{(1-w)(1-\kappa^2 w)}$ . Analogously to the case 1, using

$$T_{2m+1}(x) = \sum_{k=0}^m t_{2k+1}^{m-k} x^{2k+1}. \quad (\text{A.10})$$

we expanding  $s_{\kappa}(w)$  in Taylor series centered in 0, we get

$$E_{2m+1} = \text{PP} \left\{ \sum_{k=0}^m t_{2k+1}^{m-k} x^{2k} \sum_{i=0}^{m+1} \frac{s_{\kappa}^{(i)}(0)}{i!} x^{2(1-i)} \right\}_{x \rightarrow \infty} \quad (\text{A.11})$$

Reorganizing the terms with the same exponents, we have

$$\begin{aligned} E_{2m+1} &= \left( \sum_{k=m}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k-m)}(0)}{(k-m)!} \right) x^{2m+2} + \left( \sum_{k=m-1}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k-(m-1))}(0)}{(k-(m-1))!} \right) x^{2m} \\ &+ \left( \sum_{k=m-2}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k-(m-2))}(0)}{(k-(m-2))!} \right) x^{2m-2} + \left( \sum_{k=m-3}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k-(m-3))}(0)}{(k-(m-3))!} \right) x^{2m-4} + \dots \\ &+ \left( \sum_{k=1}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k-1)}(0)}{(k-1)!} \right) x^4 + \left( \sum_{k=0}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k)}(0)}{k!} \right) x^2 + \left( \sum_{k=0}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k+1)}(0)}{(k+1)!} \right). \end{aligned}$$

We can rewrite  $E_{2m+1}$  as

$$E_{2m+1} = \sum_{l=0}^m \left[ \left( \sum_{k=m-l}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k-(m-l))}(0)}{(k-(m-l))!} \right) x^{2(m-(l-1))} + t_{2l+1}^{m-l} \frac{s_{\kappa}^{(l+1)}(0)}{(l+1)!} \right]. \quad (\text{A.12})$$

Therefore, the solution when  $\psi(x) = T_{2m+1}(x)$  is

$$f_{2m+1,\kappa}(x) = \frac{\text{sgn}(x)x}{R_{\kappa}(x)} \left\{ \sum_{l=0}^m \left[ \left( \sum_{k=m-l}^m t_{2k+1}^{m-k} \frac{s_{\kappa}^{(k-(m-l))}(0)}{(k-(m-l))!} \right) x^{2(m-(l-1))} + t_{2l+1}^{m-l} \frac{s_{\kappa}^{(l+1)}(0)}{(l+1)!} \right] \right\} \blacksquare \quad (\text{A.13})$$

## Appendix B. Appendix

Here we show in more detail how we get the recurrence formulae presented in the section 4. Let us take first the even input function case, as in the appendix Appendix A.

**Case 1.**  $\psi(x) = T_{2m}(x)$  ( $m = 0, 1, 2, \dots$ ).

Using the recurrence formula  $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$  for Chebyshev polynomials of the first kind, in (A.3), we have

$$E_{2m} = \text{PP} \left\{ (2xT_{2m-1}(x) - T_{2m-2}(x)) \sum_{i=0}^{m+1} s_{\kappa}^{(i)}(0) x^{2(1-i)} \right\}_{x \rightarrow \infty} \quad (\text{B.1})$$

Using formulae (A.4) and (A.10) for  $T_{2m-2}(x)$  and  $T_{2m-1}(x)$ , respectively, we get

$$\begin{aligned} E_{2m} = \text{PP} & \left\{ 2x (t_1^{m-1} x^1 + t_3^{m-2} x^3 + t_5^{m-3} x^5 + \dots + t_{2m-3}^1 x^{2m-3} + t_{2m-1}^0 x^{2m-1}) \times \right. \\ & \times \left( s_{\kappa}(0) x^2 + s'_{\kappa}(0) + \frac{s''_{\kappa}(0)}{2!} \frac{1}{x^2} + \dots + \frac{s_{\kappa}^{(m)}(0)}{m!} \frac{1}{x^{2m-2}} + \frac{s_{\kappa}^{(m+1)}(0)}{(m+1)!} \frac{1}{x^{2m}} \right) \\ & - (t_0^{m-1} + t_2^{m-2} x^2 + t_4^{m-3} x^4 + \dots + t_{2m-4}^1 x^{2m-4} + t_{2m-2}^0 x^{2m-2}) \times \\ & \left. \times \left( s_{\kappa}(0) x^2 + s'_{\kappa}(0) + \frac{s''_{\kappa}(0)}{2!} \frac{1}{x^2} + \dots + \frac{s_{\kappa}^{(m)}(0)}{m!} \frac{1}{x^{2m-2}} + \frac{s_{\kappa}^{(m+1)}(0)}{(m+1)!} \frac{1}{x^{2m}} \right) \right\}_{x \rightarrow \infty}. \quad (\text{B.2}) \end{aligned}$$

Rearranging terms,

$$\begin{aligned} E_{2m} = \text{PP} & \left\{ (2t_1^{m-1} x^2 + 2t_3^{m-2} x^4 + 2t_5^{m-3} x^6 + \dots + 2t_{2m-3}^1 x^{2m-2} + 2t_{2m-1}^0 x^{2m}) \times \right. \\ & \times \left( s_{\kappa}(0) x^2 + s'_{\kappa}(0) + \frac{s''_{\kappa}(0)}{2!} \frac{1}{x^2} + \dots + \frac{s_{\kappa}^{(m)}(0)}{m!} \frac{1}{x^{2m-2}} + \frac{s_{\kappa}^{(m+1)}(0)}{(m+1)!} \frac{1}{x^{2m}} \right) \\ & - (t_0^{m-1} + t_2^{m-2} x^2 + t_4^{m-3} x^4 + \dots + t_{2m-4}^1 x^{2m-4} + t_{2m-2}^0 x^{2m-2}) \times \\ & \left. \times \left( s_{\kappa}(0) x^2 + s'_{\kappa}(0) + \frac{s''_{\kappa}(0)}{2!} \frac{1}{x^2} + \dots + \frac{s_{\kappa}^{(m)}(0)}{m!} \frac{1}{x^{2m-2}} + \frac{s_{\kappa}^{(m+1)}(0)}{(m+1)!} \frac{1}{x^{2m}} \right) \right\}_{x \rightarrow \infty}. \quad (\text{B.3}) \end{aligned}$$

Performing the multiplications, applying the limit as  $x \rightarrow \infty$  and collecting terms with the same power, we get

$$E_{2m} = 2x \left( \sum_{k=m-1}^{m-1} t_{2k+1}^{m-1-k} \frac{s_{\kappa}^{(k-(m-1))}(0)}{(k-(m-1))!} \right) x^{2m+1} + 2x \left( \sum_{k=m-2}^{m-1} t_{2k+1}^{m-1-k} \frac{s_{\kappa}^{(k-(m-2))}(0)}{(k-(m-2))!} \right) x^{2m-1} \quad (\text{B.4})$$

$$+ 2x \left( \sum_{k=m-3}^{m-1} t_{2k+1}^{m-1-k} \frac{s_{\kappa}^{(k-(m-3))}(0)}{(k-(m-3))!} \right) x^{2m-3} + \dots + 2x \left( \sum_{k=0}^{m-1} t_{2k+1}^{m-1-k} \frac{s_{\kappa}^{(k)}(0)}{k!} \right) x^3 + \quad (\text{B.5})$$

$$+ 2x \left( \sum_{k=0}^{m-1} t_{2k+1}^{m-1-k} \frac{s_{\kappa}^{(k+1)}(0)}{(k+1)!} \right) x + 2 \left( \sum_{k=0}^{m-1} t_{2k+1}^{m-1-k} \frac{s_{\kappa}^{(k+2)}(0)}{(k+2)!} \right) - \left( \sum_{k=m-1}^{m-1} t_{2k}^{m-1-k} \frac{s_{\kappa}^{(k-(m-1))}(0)}{(k-(m-1))!} \right) x^{2m} \quad (\text{B.6})$$

$$- \left( \sum_{k=m-2}^{m-1} t_{2k}^{m-1-k} \frac{s_{\kappa}^{(k-(m-2))}(0)}{(k-(m-2))!} \right) x^{2m-2} - \left( \sum_{k=m-3}^{m-1} t_{2k}^{m-1-k} \frac{s_{\kappa}^{(k-(m-3))}(0)}{(k-(m-3))!} \right) x^{2m-4} - \quad (\text{B.7})$$

$$- \left( \sum_{k=m-4}^{m-1} t_{2k}^{m-1-k} \frac{s_{\kappa}^{(k-(m-4))}(0)}{(k-(m-4))!} \right) x^{2m-6} - \dots - \left( \sum_{k=1}^{m-1} t_{2k}^{m-1-k} \frac{s_{\kappa}^{(k-1)}(0)}{(k-1)!} \right) x^4 \quad (\text{B.8})$$

$$- \left( \sum_{k=0}^{m-1} t_{2k}^{m-1-k} \frac{s_{\kappa}^{(k)}(0)}{k!} \right) x^2 - \left( \sum_{k=0}^{m-1} t_{2k}^{m-1-k} \frac{s_{\kappa}^{(k+1)}(0)}{(k+1)!} \right). \quad (\text{B.9})$$

From (A.11) and (A.6),

$$E_{2m} = 2x(xE_{2m-1}) - E_{2m-2} + \left( 2 \sum_{k=0}^{m-1} t_{2k+1}^{m-1-k} \frac{s_{\kappa}^{(k+2)}(0)}{(k+2)!} \right).$$

Substituting the above equation in (A.2) and (17), we conclude that

$$f_{2m,\kappa}(x) = 2x f_{2m-1,\kappa}(x) - f_{2m-2,\kappa}(x) + \frac{\text{sgn}(x)}{R_{\kappa}(x)} \left( 2 \sum_{k=0}^{m-1} t_{2k+1}^{m-1-k} \frac{s_{\kappa}^{(k+2)}(0)}{(k+2)!} \right) \blacksquare \quad (\text{B.10})$$

**Case 2.**  $\psi(x) = T_{2m+1}(x)$  ( $m = 0, 1, 2, \dots$ ). From (A.10) and (A.11) we can write

$$E_{2m+1} = \text{PP} \left\{ \frac{T_{2m+1}(x)}{x} \sum_{i=0}^{m+1} s_{\kappa}^{(i)}(0) x^{2(1-i)} \right\}_{x \rightarrow \infty}. \quad (\text{B.11})$$

Substituting the recurrence formula  $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$  above, we have

$$E_{2m+1} = \text{PP} \left\{ \frac{2xT_{2m}(x) - T_{2m-1}(x)}{x} \sum_{i=0}^{m+1} s_{\kappa}^{(i)}(0) x^{2(1-i)} \right\}_{x \rightarrow \infty} \quad (\text{B.12})$$

$$= \text{PP} \left\{ 2T_{2m}(x) \sum_{i=0}^{m+1} s_{\kappa}^{(i)}(0) x^{2(1-i)} - \frac{T_{2m-1}(x)}{x} \sum_{i=0}^{m+1} s_{\kappa}^{(i)}(0) x^{2(1-i)} \right\}_{x \rightarrow \infty}. \quad (\text{B.13})$$

From (A.3) and (B.11), follows that

$$E_{2m+1} = 2E_{2m} - E_{2m-1} \quad (\text{B.14})$$

Substituting (B.14) in (A.2) and (19), we get

$$f_{2m+1,\kappa}(x) = 2x f_{2m,\kappa}(x) - f_{2m-1,\kappa}(x) \blacksquare$$