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A note on the top Lyapunov exponent of linear cooperative systems

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Abstract

In a recent paper [5] P. Carmona gives an asymptotic formulae for the top Lyapunov exponent of a linear T -periodic cooperative differential equation, in the limit $T \rightarrow \infty$. This short note discusses and extends this result.

1 Notation and main results

Let $d \geq 1$ be an integer. Let \mathcal{M} denote the closed convex cone consisting of real $d \times d$ matrices having off diagonal nonnegative entries. Elements of \mathcal{M} are usually called *Metzler* or *cooperative* matrices. As usual, a matrix $M \in \mathcal{M}$ is called *irreducible* if for all $i, j \in \{1, \dots, d\}$ there exist $n \in \mathbb{N}$ and a sequence $i_1 = i, i_2, \dots, i_n = j$ such that $M_{i_l, i_{l+1}} > 0$ for $l = 1, \dots, n-1$. Equivalently e^M has positive entries. Throughout we let S denote a compact metric space and

$$A : S \mapsto \mathcal{M},$$

a continuous mapping. We consider the linear differential equation

$$\frac{dy}{dt} = A(\omega_t)y \tag{1}$$

with initial condition $y(0) = x \in \mathbb{R}_+^d \setminus \{0\}$, under the following assumptions:

- (i) The process $(\omega_t)_{t \geq 0}$ is a continuous time Feller Markov process¹ on S and is *uniquely ergodic*. By this we mean that $(\omega_t)_{t \geq 0}$ has a unique invariant probability measure denoted μ .
- (ii) The average matrix $\bar{A} = \int_S A(s)\mu(ds)$ is irreducible.

Remark 1 A sufficient (but non necessary) condition ensuring that \bar{A} is irreducible is that $A(s)$ is irreducible for some s in the topological support of μ . The (easy) proof is left to the reader.

Example 1 (Periodic case) Suppose $S = \mathbb{R}/\mathbb{Z}$ identified with the unit circle and

$$\omega_t = s + t \pmod{1}$$

for some $s \in S$. This is the case considered in [5]. Observe that here μ is the Lebesgue normalized measure on S .

Example 2 (Quasi-periodic case) A natural generalization of Example 1 is as follows. Suppose $S = (\mathbb{R}/\mathbb{Z})^n$ is the n -torus and

$$\omega_t = (s_1 + ta_1, s_2 + ta_2, \dots, s_n + ta_n) \pmod{1}$$

for some $s = (s_1, \dots, s_n) \in S$ and (a_1, \dots, a_n) rationally independent numbers. That is $\sum_{i=1}^n k_i a_i \neq 0$ for any integers k_1, \dots, k_n such that $(k_1, \dots, k_n) \neq (0, \dots, 0)$. Again $(\omega_t)_{t \geq 0}$ is uniquely ergodic with μ the Lebesgue measure on S .

Example 3 (Switching) Suppose $S = \{1, \dots, n\}$ for some $n \in \mathbb{N}^*$ and $(\omega_t)_{t \geq 0}$ is an irreducible continuous time Markov chain on S . In other words, the infinitesimal generator of $(\omega_t)_{t \geq 0}$ writes

$$Lf(i) = \sum_{j=1}^n a_{ij}(f(j) - f(i))$$

for all $f : S \mapsto \mathbb{R}$, where (a_{ij}) is an irreducible rate matrix. Then $(\omega_t)_{t \geq 0}$ is uniquely ergodic and μ is the unique probability vector solution to

$$\sum_{j=1}^n (\mu_j a_{ji} - \mu_i a_{ij}) = 0$$

for all $i = 1, \dots, n$. This situation has been considered in [4].

¹The precise definition will be recalled in the beginning of Section 2

Example 4 Suppose S is a compact connected riemannian manifold and $(\omega_t)_{t \geq 0}$ a Brownian motion (or an elliptic diffusion) on S . Then $(\omega_t)_{t \geq 0}$ is uniquely ergodic and μ is the normalized volume on S (or a measure absolutely continuous with respect to the volume, in the diffusion case).

Let $\Delta := \Delta^{d-1} = \{x \in \mathbb{R}_+^d : \sum_{i=1}^d x_i = 1\}$ be the unit $d - 1$ simplex. Every $y \in \mathbb{R}_+^d \setminus \{0\}$ can be written as

$$y = \rho\theta,$$

with $\rho = \langle y, \mathbf{1} \rangle = \sum_{i=1}^d y_i > 0$ and $\theta = \frac{y}{\langle y, \mathbf{1} \rangle} \in \Delta$. Here and throughout, $\mathbf{1}$ stands for the vector $(1, \dots, 1)^t$, and $\langle \cdot, \cdot \rangle$ is the usual Euclidean scalar product on \mathbb{R}^d .

Using this decomposition, the differential equation (1) rewrites

$$\frac{d\rho}{dt} = \rho \langle A(\omega_t)\theta, \mathbf{1} \rangle \quad (2)$$

and

$$\frac{d\theta}{dt} = F(\omega_t, \theta), \quad (3)$$

where for all $(s, \theta) \in S \times \Delta$

$$F(s, \theta) = A(s)\theta - \langle A(s)\theta, \mathbf{1} \rangle \theta. \quad (4)$$

The following proposition is proved in [4], Proposition 2.13, in the case corresponding to Example 3. It mainly relies on the Random Perron-Frobenius theorem as proved by Ruelle [14] and later by Arnold, Demetrius and Gundlach [1] (see also [12], [10], and the references therein). The proof given in [4] extends to the general situation considered here. Details are given in the next section.

Proposition 1 *Let (ρ_t, θ_t) be solution to ((2), (3)). The process $(\omega_t, \theta_t)_{t \geq 0}$ is a Feller Markov process uniquely ergodic on $S \times \Delta$.*

Let Π denotes its (unique) invariant probability and let

$$\Lambda = \int_{S \times \Delta} \langle A(s)\theta, \mathbf{1} \rangle \Pi(ds d\theta).$$

Then, for every initial conditions $\rho(0) > 0, \theta(0) \in \Delta$ and $\omega_0 = s$, with probability one,

$$\lim_{t \rightarrow \infty} \frac{\log(\rho_t)}{t} = \Lambda.$$

For further notice, we call Λ the *top Lyapunov exponent*² of the system given by (1). For periodic linear differential equations it corresponds to what is sometimes called the principal Lyapunov exponent [10], or the largest Floquet multiplier [5]. That is, the Floquet exponent with the largest real part. For further details we refer the reader to the Section II.2 of the excellent survey [10] by Mierczyński.

Corollary 2

$$\int_S [\min_{i=1,\dots,d} \sum_{j=1}^d A_{ji}(s)] \mu(ds) \leq \Lambda \leq \int_S [\max_{i=1,\dots,d} \sum_{j=1}^d A_{ji}(s)] \mu(ds)$$

Proof: for all $\theta \in \Delta$ and $s \in S$

$$\min_{i=1,\dots,d} \sum_{j=1}^d A_{ji}(s) \leq \langle A(s)\theta, 1 \rangle \leq \max_{i=1,\dots,d} \sum_{j=1}^d A_{ji}(s)$$

□

Remark 2 Let $C := \{\theta \in \mathbb{R}_+^d : \langle \theta, \theta \rangle = 1\}$. An alternative decomposition of $y \in \mathbb{R}_+^d \setminus \{0\}$ is to write $y = \rho\theta$ with $\rho = \sqrt{\langle y, y \rangle}$ and $\theta = y/\rho \in C$. This leads to another expression for Λ given as

$$\Lambda = \int_{S \times C} \langle A(s)\theta, \theta \rangle \tilde{\Pi}(dsd\theta)$$

where $\tilde{\Pi}$ is a probability over $S \times C$. As a consequence we get the other estimate:

$$\int_S \lambda_{\min} \left(\frac{A(s) + A(s)^t}{2} \right) \mu(ds) \leq \Lambda \leq \int_S \lambda_{\max} \left(\frac{A(s) + A(s)^t}{2} \right) \mu(ds)$$

where λ_{\min} (respectively λ_{\max}) stands for the smallest (largest) eigenvalue. Other estimates, mainly for periodic systems, can be found in [10] and in [11] for more general systems.

In the particular case of a periodic system (Example 1), more can be said.

²see the remark 3 for a justification of this terminology

Proposition 3 *Suppose $S = \mathbb{R}/\mathbb{Z} \simeq [0, 1[$ as in Example 1. There exists a continuous 1-periodic function $t \in \mathbb{R} \rightarrow \theta^*(t) \in \Delta$, such that: For all $s \in S$ and $\omega_t = s + t \pmod{1}$, $t \rightarrow \theta^*(s + t)$ is the unique 1-periodic solution to (3). It is globally asymptotically stable in the sense that*

$$\lim_{t \rightarrow \infty} \|\theta(t) - \theta^*(s + t)\| = 0$$

for every solution $(\theta(t))_{t \geq 0}$ to (3) with $\omega_t = s + t \pmod{1}$. In particular,

$$\Pi(ds d\theta) = ds \delta_{\theta^*(s)}(d\theta)$$

and

$$\Lambda = \int_0^1 \langle A(s) \theta^*(s), \mathbf{1} \rangle ds.$$

1.1 Slow and fast regimes

For all $T > 0$, let $\omega_t^T = \omega_{t/T}$. Like $(\omega_t)_{t \geq 0}$, $(\omega_t^T)_{t \geq 0}$ is a Feller Markov process on S , uniquely ergodic with invariant probability μ . The parameter $1/T$ can be understood as a velocity parameter. For instance, in the context of Example 1, $(\omega_t^T)_{t \geq 0}$ is a T -periodic signal. In the context of Example 3, its mean sojourn time in each state $i \in S$ is proportional to T .

Consider the differential equation (1) with $(\omega_t)_{t \geq 0}$ replaced by $(\omega_t^T)_{t \geq 0}$. We let Π^T and Λ^T denote the corresponding invariant probabilities on $S \times \Delta$ and top Lyapunov exponent as defined in Proposition 1. This section considers the fast and slow regimes obtained as $T \rightarrow 0$ and $T \rightarrow \infty$.

For a $d \times d$ real matrix M , we let $\lambda_{max}(M)$ denote the largest real part of its eigenvalues.

For $r > 0$ sufficiently large, $\bar{A} + rI$ has nonnegative entries and is irreducible. Hence, by Perron-Frobenius theorem (applied to $\bar{A} + rI$), $\lambda_{max}(\bar{A})$ is an eigenvalue and there exists a unique vector, the Perron-Frobenius vector of \bar{A} , $\theta^* \in \Delta$, such that

$$\bar{A}\theta^* = \lambda_{max}(\bar{A})\theta^*.$$

Proposition 4 (Fast regime)

$$\lim_{T \rightarrow 0} \Pi^T = \mu \otimes \delta_{\theta^*}$$

(for the weak* topology) and

$$\lim_{T \rightarrow 0} \Lambda^T = \lambda_{max}(\bar{A}).$$

Note that Proposition 4 has been proven for Example 3 in ([4], Corollary 2.15). The next result generalizes [5] beyond Example 1. Let $\text{supp}(\mu)$ be the topological support of μ . Assume that for all $s \in \text{supp}(\mu)$, $A(s)$ is irreducible. Then, under this assumption, there exists for all $s \in \text{supp}(\mu)$ a unique Perron-Frobenius vector for $A(s)$, $\theta^*(s) \in \Delta$ characterized by

$$A(s)\theta^*(s) = \lambda_{max}(A(s))\theta^*(s).$$

Proposition 5 (Slow regime) *Assume that for all $s \in \text{supp}(\mu)$, $A(s)$ is irreducible. Then*

$$\lim_{T \rightarrow \infty} \Pi^T = \mu(ds)\delta_{\theta^*(s)}$$

(for the weak* topology) and

$$\lim_{T \rightarrow \infty} \Lambda^T = \int_S \lambda_{max}(A(s))\mu(ds).$$

2 Proofs

Notation and Background

If X is a metric space (such as $S, \Delta, S \times \Delta$) we let $B(X)$ denote the space of real valued Borel bounded functions on X and $C(X) \subset B(X)$ the subspace of bounded continuous functions. For all $f \in B(X)$ we let $\|f\|_\infty = \sup_{x \in X} |f(x)|$. If ν is a probability on X and $f \in B(X)$ we write νf for $\int_X f d\nu$.

Our main assumption that $(\omega_t)_{t \geq 0}$ is a Feller Markov process on S means, as usual, that $(\omega_t)_{t \geq 0}$ is a Markov process whose transition semigroup $(P_t)_{t \geq 0}$ is Feller. That is:

- (a) $P_t(C(S)) \subset C(S)$;
- (b) $\lim_{t \rightarrow 0} P_t f(s) = f(s)$ for all $f \in C(S)$ and $s \in S$.

It turns out (see e.g [8], Theorem 19.6) that (a) and (b) make $(P_t)_{t \geq 0}$ strongly continuous in the sense that $\lim_{t \rightarrow 0} \|P_t f - f\|_\infty = 0$ for all $f \in C(S)$.

An *invariant probability* for $(\omega_t)_{t \geq 0}$ (or $(P_t)_{t \geq 0}$) is a probability μ on S such that for all $t \geq 0$ $\mu P_t = \mu$ (i.e $\mu P_t(f) = \mu(f)$ for all $f \in B(S)$). Feller continuity and compactness of S imply that such a μ always exist (see e.g [2], Corollary 4.21). Our assumption that $(\omega_t)_{t \geq 0}$ is *uniquely ergodic* means that μ is unique.

A useful consequence of Feller continuity is that we can assume without loss of generality that $(\omega_t)_{t \geq 0}$ is defined on the space Ω consisting of *cadlag* (right-continuous, left limit) paths $\omega : \mathbb{R}_+ \mapsto S$ equipped with the Skorohod topology and associated Borel sigma field (see e.g [8], Theorem 19.15). As usual, for all $s \in S$ we let \mathbb{P}_s denote the law of $(\omega_t)_{t \geq 0}$ starting from $\omega_0 = s$ and $\mathbb{P}_\mu = \int_S \mathbb{P}_s \mu(ds)$. The associated expectations are denoted \mathbb{E}_s and \mathbb{E}_μ .

For all $\omega \in \Omega$ and $t \geq 0$ we let $\Theta_t(\omega)$ denote the shifted path defined as $\Theta_t(\omega)(s) = \omega(t+s)$. Ergodicity of μ for the Markov process $(\omega_t)_{t \geq 0}$ makes \mathbb{P}_μ ergodic (but not uniquely ergodic) for the dynamical system $(\Theta_t)_{t \geq 0}$ on Ω (see e.g [2], Proposition 4.49).

2.1 Proof of Propositions 1 and 3

For $\omega \in \Omega$ the solution to (1) writes $y(t) = \Phi(t, \omega)x$ where $(\Phi(t, \omega))_{t \geq 0}$ is solution to the matrix valued differential equation

$$\frac{dM}{dt} = A(\omega_t)M, M(0) = Id.$$

Let $\mathcal{M}_+ \subset \mathcal{M}$ denote the set of $d \times d$ Metzler matrices having positive diagonal entries. Observe that

$$\Phi(t, \omega) \in \mathcal{M}_+$$

for all $t \geq 0$. Indeed, for R large enough and all $s \in S$, $A(s) + 2RIId \geq RIId$ so that $e^{2Rt}\Phi(t, \omega) \geq e^{Rt}Id$ (componentwise).

For all $\theta \in \Delta$, the solution to (3) with initial condition $\theta(0) = \theta$, writes

$$\theta(t) = \Psi(t, \omega)(\theta) := \frac{\Phi(t, \omega)\theta}{\langle \Phi(t, \omega)\theta, \mathbf{1} \rangle}.$$

Let $\mathcal{M}_{++} \subset \mathcal{M}_+$ the set of matrices having positive entries.

Lemma 6 For \mathbb{P}_μ almost all $\omega \in \Omega$:

- (i) There exists $N \in \mathbb{N}$ such that $\Phi(t, \omega) \in \mathcal{M}_{++}$ for all $t \geq N$;
(ii) For all $\theta, \theta' \in \Delta$

$$\lim_{t \rightarrow \infty} \|\Psi(t, \omega)\theta - \Psi(t, \omega)\theta'\| = 0.$$

Proof: (i). First observe that $\Phi(t, \omega) \in \mathcal{M}_{++} \Leftrightarrow e^{Rt}\Phi(t, \omega) \in \mathcal{M}_{++}$ for all $R > 0$. Therefore, replacing $A(s)$ by $A(s) + RId$ for $R > \|A\|_\infty$, we can assume without loss of generality that $A(s) \in \mathcal{M}_+$ for all $s \in S$.

Let $x(t) = \Phi(t, \omega)x$ with $x \in \mathbb{R}_+^d \setminus \{0\}$. Suppose $x_i(0) > 0$. Then $x_i(t) > 0$ because $\dot{x}_i(t) \geq A_{ii}(\omega_t)x_i(t) \geq 0$. By irreducibility of \bar{A} , for all $j \neq i$ there exists a sequence $i_0 = i, i_1, \dots, i_n = j$ such that $\bar{A}_{i_k i_{k-1}} > 0$ for $k = 1, \dots, n$. By ergodicity there exists a Borel set $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}_\mu(\tilde{\Omega}) = 1$ such that for all $\omega \in \tilde{\Omega}$

$$\frac{1}{t} \int_0^t A(\omega_u) du \rightarrow \bar{A}.$$

Therefore, for all $\omega \in \tilde{\Omega}$, there exists a sequence $t_1 > t_2 > \dots > t_n$ with

$$A_{i_k i_{k-1}}(\omega_{t_k}) > 0.$$

By right continuity of (ω_t) we also have $A_{i_k i_{k-1}}(\omega_t) > 0$ for $t_k \leq t \leq t_k + \varepsilon$ for some $\varepsilon > 0$. It follows that $\dot{x}_{i_1}(t) \geq A_{i_1, i_1}(\omega_t)x_{i_1}(t) > 0$ for all $t_1 \leq t \leq t_1 + \varepsilon$. Hence $x_{i_1}(t) > 0$ for all $t > t_1$. Similarly $x_{i_2}(t) > 0$ for all $t > t_2$ and, by recursion, $x_j(t) > 0$ for all $t > t_n$. In summary, we have shown that for all $i, j \in \{1, \dots, d\}$ and $\omega \in \tilde{\Omega}$, there exists a time t_n depending on i, j, ω such that for all $t \geq t_n$ $x_j(t) > 0$ whenever $x_i(0) > 0$. This proves (i).

(ii). Let $\mathbb{R}_{++}^d = \{x \in \mathbb{R}^d : x_i > 0, \text{ for all } i = 1 \dots d\}$ and $\dot{\Delta} = \Delta \cap \mathbb{R}_{++}^d$ be the relative interior of Δ . The *projective* or *Hilbert metric* d_H on \mathbb{R}_{++}^d (see Seneta [15]) is defined by

$$d_H(x, y) = \log \frac{\max_{1 \leq i \leq d} x_i/y_i}{\min_{1 \leq i \leq d} x_i/y_i}.$$

Note that for all $\alpha, \beta > 0$, $d_H(\alpha x, \beta y) = d_H(x, y)$ so that d_H is not a distance on \mathbb{R}_{++}^d . However its restriction to $\dot{\Delta}$ is. Furthermore, for all $\theta, \theta' \in \dot{\Delta}$,

$$\max_{1 \leq i \leq d} |\theta_i - \theta'_i| \leq e^{d_H(\theta, \theta')} - 1. \quad (5)$$

By a theorem of Garret Birkhoff (see e.g [15], Section 3.4), for all $M \in \mathcal{M}_+$,

$$\sup_{\{x, y \in \mathbb{R}_{++}^d : d_H(x, y) > 0\}} \frac{d_H(Mx, My)}{d_H(x, y)} = \tau[M] \quad (6)$$

where $0 \leq \tau(M) \leq 1$ is the number defined as $\tau(M) = \frac{1 - \sqrt{r(M)}}{1 + \sqrt{r(M)}}$ with $r(M) = \min_{i,j,k,l} \min \frac{M_{ik}M_{jl}}{M_{jk}M_{il}}$ if $M \in \mathcal{M}_{++}$ and $r(M) = 0$ if $M \in \mathcal{M}_+ \setminus \mathcal{M}_{++}$. In particular, for $M \in \mathcal{M}_+$, $\tau(M) < 1$ if and only if $M \in \mathcal{M}_{++}$.

For all $0 \leq s \leq t$, let

$$F_{s,t}(\omega) = \max\{\log(\tau[\Phi(t-s, \Theta_s(\omega))]), s-t\} \in [s-t, 0].$$

By the cocycle property (i.e $\Phi(t, \omega) = \Phi(t-s, \Theta_s(\omega)) \circ \Phi(s, \omega)$), $(F_{s,t})_{0 \leq s \leq t}$ is a subadditive process. That is

$$F_{s,u} \leq F_{s,t} + F_{t,u} \text{ and } F_{s,t} \circ \Theta_v = F_{s+v,t+v}$$

for all $s \leq t \leq u$ and $v \geq 0$. Note also that $t, s \rightarrow F_{s,t}(\omega)$ is continuous and that $\sup_{0 \leq s \leq t \leq 1} |F_{s,t}| \leq 1$, so that the integrability conditions required for the continuous time version of Kingman's subadditive ergodic theorem (as stated in [9], Theorem 5.6) are satisfied. Therefore, by this theorem,

$$\limsup_{t \rightarrow \infty} \frac{\log(\tau[\Phi(t, \omega)])}{t} \leq \lim_{t \rightarrow \infty} \frac{F_{0,t}(\omega)}{t} = \gamma,$$

\mathbb{P}_μ almost surely, where

$$\gamma = \inf_{t > 0} \mathbb{E}_\mu \frac{F_{0,t}}{t}.$$

Clearly $\gamma < 0$. For otherwise we would have that $\tau[\Phi(n, \omega)] = 1 \Leftrightarrow \Phi(n, \omega) \in \mathcal{M}_+ \setminus \mathcal{M}_{++}$ for all $n \in \mathbb{N}$, \mathbb{P}_μ almost surely, in contradiction with (i).

Let N be like in assertion (i) of the Lemma. Then, by what precedes, \mathbb{P}_μ almost surely,

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{\log(d_H(\Psi(t+N, \omega)(\theta), \Psi(t+N, \omega)(\theta'))) }{t} \\ & \leq \limsup_{t \rightarrow \infty} \frac{\log(\tau[\Phi(t, \Theta_N(\omega))])}{t} + \limsup_{t \rightarrow \infty} \frac{\log(d_H(\Psi(N, \omega)(\theta), \Psi(N, \omega)(\theta'))) }{t} = \gamma. \end{aligned}$$

By inequality (5), this concludes the proof. \square

Let $(Q_t)_{t \geq 0}$ denote the semigroup of the process $(\omega_t, \theta_t)_{t \geq 0}$. Then, for all $f \in B(S \times \Delta)$ $(s, \theta) \in S \times \Delta$,

$$Q_t f(s, \theta) = \mathbb{E}_s[f(\omega_t, \Psi(t, \omega)(\theta))].$$

Lemma 7 *The semigroup $(Q_t)_{t \geq 0}$ is Feller.*

Proof: We need to show that (a) $Q_t(C(S \times \Delta) \subset C(S \times \Delta)$ and (b) $\lim_{t \rightarrow 0} Q_t f(s, \theta) = f(s, \theta)$ for all $f \in C(S \times \Delta)$.

(a). It is easy to verify that there exist constants $c_1, c_2 \geq 0$ such that for all $s, s' \in S, \theta, \theta' \in \Delta$

$$\begin{aligned} \|F(s, \theta) - F(s, \theta')\| &\leq c_1 \|\theta - \theta'\| \\ \|F(s, \theta) - F(s', \theta)\| &\leq c_2 \|A(s) - A(s')\| \end{aligned} \quad (7)$$

where F is defined by (4). Fix $\varepsilon > 0$ and let $\tilde{\omega}$ be the path defined as $\tilde{\omega}_u = \omega_{k\varepsilon}$ for all $k\varepsilon \leq u < (k+1)\varepsilon$. Then, by Gronwall's lemma,

$$\|\Psi(t, \omega)(\theta) - \Psi(t, \tilde{\omega})(\theta)\| \leq c_t \int_0^t \|A(\omega(u)) - A(\tilde{\omega}(u))\| du \quad (8)$$

where $c_t = e^{c_1 t} c_2$. Thus, by Jensen inequality,

$$\begin{aligned} \mathbb{E}_s(\|\Psi(t, \omega)(\theta) - \Psi(t, \tilde{\omega})(\theta)\|)^2 &\leq \mathbb{E}_s(\|\Psi(t, \omega)(\theta) - \Psi(t, \tilde{\omega})(\theta)\|^2) \\ &\leq c_t^2 \int_0^t \mathbb{E}_s(\|A(\omega_u) - A(\tilde{\omega}_u)\|^2) du \end{aligned}$$

The choice of the norm being arbitrary we can assume that the norm on the right hand side of the preceding inequality is the Euclidean on \mathbb{R}^{d^2} . Then, for all $k\varepsilon \leq u < (k+1)\varepsilon$,

$$\begin{aligned} \mathbb{E}_s(\|A(\omega_u) - A(\tilde{\omega}_u)\|^2) &= \mathbb{E}_s(\mathbb{E}(\|A(\omega_u) - A(\tilde{\omega}_u)\|^2) | \mathcal{F}_{k\varepsilon}) \\ &= \mathbb{E}_s(P_{u-k\varepsilon}(\|A\|^2)(\omega_{k\varepsilon}) - 2\langle A(\omega_{k\varepsilon}), P_{u-k\varepsilon}(A)(\omega_{k\varepsilon}) \rangle + \|A(\omega_{k\varepsilon})\|^2) \\ &\leq \sup_{0 \leq h \leq \varepsilon} \|P_h(\|A\|^2) - \|A\|^2\|_\infty + 2\|A\|_\infty \|P_h A - A\|_\infty := \delta(\varepsilon). \end{aligned}$$

Observe that $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ by strong continuity of $(P_t)_{t \geq 0}$. Combining the two last inequalities, we get

$$\mathbb{E}_s(\|\Psi(t, \omega)(\theta) - \Psi(t, \tilde{\omega})(\theta)\|^2) \leq c_t^2 t^2 \delta(\varepsilon) \quad (9)$$

Let now $f \in C(S \times \Delta)$. Then, for every $\delta > 0$ there exists $\alpha > 0$, such that

$$|f(s, \theta) - f(s, \theta')| \leq \delta + 2\|f\| \mathbf{1}_{\|\theta - \theta'\| \geq \alpha}$$

Thus

$$|Q_t f(s, \theta) - \mathbb{E}_s(f(\omega_t, \Psi(t, \tilde{\omega})(\theta)))|$$

$$\leq \mathbb{E}_s (|f(\omega_t, \Psi(t, \omega)(\theta)) - f(\omega_t, \Psi(t, \tilde{\omega})(\theta))|) \leq \delta + 2 \frac{\|f\|}{\alpha^2} c_t^2 t^2 \delta(\varepsilon).$$

This shows that the left hand term goes to 0 uniformly in $(s, \theta) \in S \times \Delta$ as $\varepsilon \rightarrow 0$.

In order to conclude it suffices to show that $(s, \theta) \rightarrow \mathbb{E}_s (f(\omega_t, \Psi(t, \tilde{\omega})(\theta)))$ is continuous. For all $s \in S$, let $(\Psi_t^s)_{t \geq 0}$ denote the semi-flow on Δ induced by the autonomous differential equation $\frac{d\theta}{dt} = F(s, \theta)$. Then for $k\varepsilon \leq t < (k+1)\varepsilon$

$$f(\omega_t, \Psi(t, \tilde{\omega})(\theta)) = f(\omega_t, \Psi_{t-k\varepsilon}^{\omega_{k\varepsilon}} \circ \dots \circ \Psi_\varepsilon^{\omega_\varepsilon} \circ \Psi_\varepsilon^{\omega_0}(\theta))$$

Now, for every $h \in C(S^{k+2} \times \Delta)$, Feller continuity of $(P_t)_{t \geq 0}$, makes the map $(s, \theta) \rightarrow \mathbb{E}_s (h(\omega_t, \omega_{k\varepsilon}, \dots, \omega_0, \theta))$ continuous. This is immediate to verify when h is a product function (i.e $h(s_{k+1}, \dots, s_0, \theta) = h_{k+1}(s_{k+1}) \cdot h_0(s_0)g(\theta)$) and the general case follows by the density in $C(S^{k+2} \times \Delta)$ of the vector space span by product functions. This concludes the proof of (a).

(b). Let $f \in C(S \times \Delta)$ and $\delta > 0$. Because $\|\Psi(t, \omega)(\theta) - \theta\| \leq t\|F\|_\infty$, $\|Q_t f(s, \theta) - \mathbb{E}_s(f(\omega_t, \theta))\| \leq \delta$ for all t sufficiently small. By Feller continuity of (P_t) $\lim_{t \rightarrow 0} \mathbb{E}_s(f(\omega_t, \theta)) = \lim_{t \rightarrow 0} P_t(f(\cdot, \theta))(s) = f(s, \theta)$. \square

We can now conclude the proof of Proposition 1. It follows from Lemma 6 (ii) that for all $f : S \times \Delta \mapsto \mathbb{R}$ continuous (hence uniformly continuous) and all $\theta, \theta' \in \Delta$,

$$\lim_{t \rightarrow \infty} |f(\omega_t, \Psi(t, \omega)\theta) - f(\omega_t, \Psi(t, \omega)\theta')| = 0$$

\mathbb{P}_s almost surely, for μ almost all $s \in S$. Hence, for all $\theta, \theta' \in \Delta$,

$$\lim_{t \rightarrow \infty} |Q_t f(s, \theta) - Q_t f(s, \theta')| = 0, \quad (10)$$

for μ almost all $s \in S$. Let now Π be an invariant probability of $(Q_t)_{t \geq 0}$. Such a Π always exist because $(Q_t)_{t \geq 0}$ is Feller on $S \times \Delta$ compact. To prove that Π is unique, assume that Π' is another invariant probability. Then, writing Πf for $\int_{S \times \Delta} f(s, \theta) \Pi(ds d\theta)$,

$$\begin{aligned} \Pi f - \Pi' f &= \Pi Q_t f - \Pi' Q_t f \\ &= \int_S \left[\int_{\Delta \times \Delta} (Q_t f(s, \theta) - Q_t f(s, \theta')) \Pi(d\theta|s) \Pi(d\theta'|s) \right] \mu(ds) \end{aligned}$$

where for each $s \in S$, $\Pi(\cdot|s)$ (respectively $\Pi'(\cdot|s)$) is a conditional distribution of Π (respectively Π') (see [6], Section 10.2). It then follows

from (10) and dominated convergence that $\Pi f = \Pi' f$. Thus $\Pi = \Pi'$. This proves unique ergodicity.

Now, unique ergodicity and Feller continuity of $(\omega_s, \theta_s)_{s \geq 0}$ imply that for every continuous function $g : S \times \Delta \rightarrow \mathbb{R}$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t g(\omega_s, \theta_s) ds = \int g d\Pi$$

$\mathbb{P}_{s, \theta}$ almost surely for all $s, \theta \in S \times \Delta$. (see e.g. [2], Proposition 7.1 for discrete time chains combined with Proposition 4.58 to handle continuous time). This concludes the proof of Proposition 1 with $g(s, \theta) = \langle A(s)\theta, \mathbf{1} \rangle$.

Remark 3 By the multiplicative ergodic theorem, there exist numbers $\Lambda_1 < \dots < \Lambda_r, r \leq d$, called *Lyapunov exponents*, such that for \mathbb{P}_μ almost all ω and all $x \in \mathbb{R}^d \setminus \{0\}$,

$$\lim_{t \rightarrow \infty} \frac{\log \|\Phi(t, \omega)x\|}{t} := \Lambda(x, \omega) \in \{\Lambda_1, \dots, \Lambda_r\}.$$

The set of $x \in \mathbb{R}^d$ for which $\Lambda(x, \omega) < \Lambda_r$ is a vector space (depending on ω) having nonzero codimension. On the other hand, by what precedes, $\Lambda(x, \omega) = \Lambda$ for all $x \in \mathbb{R}_+^d \setminus \{0\}$. It follows that $\Lambda = \Lambda_r$.

Proof of Proposition 3

For $s \in S = \mathbb{R}/\mathbb{Z}$, let $\omega[s] \in \Omega$ be the path defined as

$$\omega_t[s] = s + t \pmod{1}.$$

By Brouwer fixed point theorem, the map $\Psi(1, \omega[0]) : \Delta \mapsto \Delta$ has a fixed point θ^* . Set $\theta^*(t) = \Psi(t, \omega[0])(\theta^*)$. Then

$$\theta^*(t+1) = \Psi(t, \omega[1]) \circ \Psi(1, \omega[0])(\theta^*) = \theta^*(t)$$

proving that $t \rightarrow \theta^*(t)$ is 1-periodic.

For all $s \in S$ and $\theta \in \Delta$

$$\lim_{t \rightarrow \infty} \|\Psi(t, \omega[s])(\theta) - \theta^*(t+s)\| = \lim_{t \rightarrow \infty} \|\Psi(t, \omega[s])(\theta) - \Psi(t, \omega[s])(\theta^*(s))\| = 0,$$

by Lemma 6 applied with $\omega = \omega[s]$ and $\theta' = \theta^*[s]$. Observe here, that the conclusions of Lemma 6 hold with $\omega = \omega[s]$ for all $s \in S$ simply because $\omega[s]$ is 1-periodic.

3 Proof of Propositions 4 and 5

For all $T > 0$, let $(Q_t^T)_{t \geq 0}$ denote the semigroup of $(\omega_t^T, \theta_t)_{t \geq 0}$ with $\omega_t^T = \omega_{t/T}$ and $(\theta_t)_{t \geq 0}$ is solution to (3) when ω_t is replaced by ω_t^T . Using the notation of the preceding section one sees that

$$Q_t^T(f)(s, \theta) = \mathbb{E}_s [f(\omega_{t/T}, \Psi(t, \omega^T)(\theta))]$$

for all $f \in B(S \times \Delta)$.

Proof of Proposition 4

For all $\theta \in \Delta$, let $\bar{F}(\theta) = \int_S F(s, \theta) \mu(ds)$, where F is defined by (4). Let $(\bar{\Psi}_t)_{t \geq 0}$ denote the semi-flow on Δ induced by the differential equation $\dot{\theta} = \bar{F}(\theta)$. The following lemma follows from the averaging principle as given in Freidlin and Wentzell [7] (Theorem 2.1, Chapter 7).

Lemma 8 *For all $\delta > 0$ and $t \geq 0$,*

$$\lim_{T \rightarrow 0} \mathbb{P}_\mu \left(\sup_{\theta \in \Delta, 0 \leq u \leq t} \|\Psi(u, \omega^T)(\theta) - \bar{\Psi}_u(\theta)\| \geq \delta \right) = 0.$$

In particular, for all $f \in C(\Delta)$ and $t \geq 0$,

$$\lim_{T \rightarrow 0} \mathbb{E}_\mu [\|f \circ \Psi(t, \omega^T) - f \circ \bar{\Psi}_t\|_\infty] = 0.$$

Proof: We claim that

$$\lim_{R \rightarrow \infty} \sup_{t \geq 0} \mathbb{P}_\mu \left(\left| \frac{1}{R} \int_t^{t+R} F(\omega_s, \theta) ds - \bar{F}(\theta) \right| \geq \delta \right) = 0. \quad (11)$$

Indeed, by stationarity (invariance of \mathbb{P}_μ for $(\Theta_t)_{t \geq 0}$),

$$\mathbb{P}_\mu \left(\left| \frac{1}{R} \int_t^{t+R} F(\omega_s, \theta) ds - \bar{F}(\theta) \right| \geq \delta \right) = \mathbb{P}_\mu \left(\left| \frac{1}{R} \int_0^R F(\omega_s, \theta) ds - \bar{F}(\theta) \right| \geq \delta \right)$$

for all $t \geq 0$; and the right hand term goes to 0, as $R \rightarrow \infty$, by ergodicity of μ .

By the averaging theorem (Theorem 2.1, Chapter 7 in [7]), condition (11) implies that for all $\delta > 0, t \geq 0$ and $\theta \in \Delta$,

$$\lim_{T \rightarrow 0} \mathbb{P}_\mu \left(\sup_{0 \leq u \leq t} \|\Psi(u, \omega^T)(\theta) - \bar{\Psi}_u(\theta)\| \geq \delta \right) = 0. \quad (12)$$

By Lipschitz continuity (see (7)) and Gronwall's lemma,

$$\sup_{0 \leq u \leq t} \|\Psi(u, \omega^T)(\theta) - \Psi(u, \omega^T)(\theta')\| + \|\bar{\Psi}_t(\theta) - \bar{\Psi}_t(\theta')\| \leq 2e^{c_1 t} \|\theta - \theta'\|$$

for all $\theta, \theta' \in \Delta$. Fix $\varepsilon < \frac{\delta}{4} e^{-c_1 t}$ and let $\{B(\theta_i, \varepsilon), i = 1, \dots, N\}$ be a finite covering of Δ by balls of radius ε . Then

$$\sup_{0 \leq u \leq t, \theta \in \Delta} \|\Psi(u, \omega^T)(\theta) - \bar{\Psi}_u(\theta)\| \leq \max_{i=1, \dots, N} \sup_{0 \leq u \leq t} \|\Psi(u, \omega^T)(\theta_i) - \bar{\Psi}_u(\theta_i)\| + \delta/2.$$

Hence

$$\mathbb{P}_\mu \left(\sup_{0 \leq u \leq t, \theta \in \Delta} \|\Psi(u, \omega^T)(\theta) - \bar{\Psi}_u(\theta)\| \geq \delta \right) \leq \sum_{i=1}^N \mathbb{P}_\mu \left(\sup_{0 \leq u \leq t} \|\Psi(u, \omega^T)(\theta_i) - \bar{\Psi}_u(\theta_i)\| \geq \delta/2 \right).$$

The right hand term goes 0 as $T \rightarrow 0$ by (12). \square

We now prove the proposition. Let Π^T be the invariant measure of $(Q_t^T)_{t \geq 0}$ and let Π^0 be a limit point of $(\Pi^T)_{T > 0}$ for the weak* topology, as $T \rightarrow 0$. That is: $\Pi^{T_n} f \rightarrow \Pi^0 f$ for some sequence $T_n \rightarrow 0$ and all $f \in C(S \times \Delta)$.

Let $p : S \times \Delta \rightarrow \Delta$ be the projection defined as $p(s, \theta) = \theta$ and let $\Pi_2^T = \Pi^T \circ p^{-1}$ be the second marginal of Π^T . Similarly, set $\Pi_2^0 = \Pi^0 \circ p^{-1}$.

For all $f \in C(\Delta)$ and $t \geq 0$,

$$\Pi_2^T f = \Pi^T(f \circ p) = \Pi^T Q_t^T(f \circ p) = \int_{S \times \Delta} \mathbb{E}_s[f(\Psi(t, \omega^T)(\theta))] \Pi^T(ds d\theta).$$

Thus,

$$\begin{aligned} |\Pi_2^T f - \Pi_2^T(f \circ \bar{\Psi}_t)| &= \left| \int_{S \times \Delta} \mathbb{E}_s[f(\Psi(t, \omega^T)(\theta)) - f(\bar{\Psi}_t(\theta))] \Pi^T(ds d\theta) \right| \\ &\leq \int_S \mathbb{E}_s[\|f \circ \Psi(t, \omega^T) - f \circ \bar{\Psi}_t\|_\infty] \mu(ds) = \mathbb{E}_\mu[\|f \circ \Psi(t, \omega^T) - f \circ \bar{\Psi}_t\|_\infty]. \end{aligned}$$

Here we have used the fact that the first marginal of Π^T is μ . Using Lemma 8, it comes that

$$\Pi_2^0 f = \Pi_2^0(f \circ \bar{\Psi}_t)$$

for all $t \geq 0$. This proves that Π_2^0 is invariant for $\{\bar{\Psi}_t\}_{t \geq 0}$, but since $\{\bar{\Psi}_t\}_{t \geq 0}$ has θ^* as globally asymptotically stable equilibrium, necessarily $\Pi_2^0 = \delta_{\theta^*}$. On the other hand, the first marginal of Π^0 is μ . Thus $\Pi^0 = \mu \otimes \delta_{\theta^*}$. This concludes the proof.

Proof of Proposition 5

Recall (see the proof of Lemma 7) that for all $s \in S$, we let $(\Psi_t^s)_{t \geq 0}$ denote the semi-flow on Δ induced by the differential equation $\dot{\theta} = F(s, \theta)$.

Let $(Q_t^\infty)_{t \geq 0}$ denote the Markov semigroup on $S \times \Delta$ defined as

$$Q_t^\infty f(s, \theta) = f(s, \Psi_t^s(\theta))$$

for all $f \in B(S \times \Delta)$.

Lemma 9 *For all $f \in C(S \times \Delta)$ and $t \geq 0$*

$$\lim_{T \rightarrow \infty} \|Q_t^T f - Q_t^\infty f\|_\infty = 0.$$

Proof: Let $f \in C(S \times \Delta)$. By uniform continuity of f , for every $t > 0$ and $\delta > 0$ there exists $\alpha > 0$ such that

$$|f(\omega_t^T, \Psi(t, \omega^T)\theta) - f(s, \Psi_t^s(\theta))| \leq \delta + 2\|f\|_\infty \mathbf{1}_{\{d(\omega_t^T, s) + \|\Psi(t, \omega^T)(\theta) - \Psi_t^s(\theta)\| \geq \alpha\}}.$$

Thus

$$\begin{aligned} \|Q_t^T f(s, \theta) - Q_t^\infty f(s, \theta)\| &\leq \mathbb{E}_s (|f(\omega_t^T, \Psi(t, \omega^T)(\theta)) - f(s, \Psi_t^s(\theta))|) \\ &\leq \delta + 2\|f\|_\infty \frac{\mathbb{E}_s(\|\Psi(t, \omega^T)(\theta) - \Psi_t^s(\theta)\|) + P_{t/T}(d(\cdot, s))(s)}{\alpha}. \end{aligned}$$

By Feller continuity, $P_{t/T}(d(\cdot, s))(s) \rightarrow 0$ uniformly in $s \in S$ as $T \rightarrow \infty$. This follows for example from Lemma 19.3 (F_3) in [8]. Now the estimate (9) applied with ω^T in place of ω , $(P_{t/T})_{t \geq 0}$ in place of $(P_t)_{t \geq 0}$ and $\varepsilon > t$ gives

$$\sup_{s \in S} \mathbb{E}_s(\|\Psi(t, \omega^T)(\theta) - \Psi_t^s(\theta)\|) \leq c^2 t^2 \delta(\varepsilon/T). \quad (13)$$

with $\delta(\varepsilon/T) \rightarrow 0$ as $T \rightarrow \infty$. This concludes the proof. \square

We can now prove Proposition 5. Let Π^T be the invariant measure of (ω_t^T, θ_t) and let Π^∞ be a limit point of $(\Pi^T)_{T > 0}$ for the weak* topology. That is $\Pi^{T_n} f \rightarrow \Pi^\infty f$ for some sequence $T_n \rightarrow \infty$ and all $f \in C(S \times \Delta)$. Then,

$$|\Pi^T(f) - \Pi^T(Q_t^\infty f)| = |\Pi^T(Q_t^T(f) - Q_t^\infty(f))| \leq \|Q_t^T(f) - Q_t^\infty(f)\|_\infty.$$

Thus, by Lemma 9, (ii), $\Pi^\infty(f) = \Pi^\infty(Q_t^\infty(f))$. Now for all $s \in \text{supp}(\mu)$

$$\lim_{t \rightarrow \infty} Q_t^\infty(f)(s, \theta) = \lim_{t \rightarrow \infty} f(s, \Psi_t^s(\theta)) = f(s, \theta^*(s)).$$

Thus, since $\Pi^\infty(\text{supp}(\mu) \times \Delta) = 1$, it comes that

$$\Pi^\infty(f) = \int_S \int_\Delta f(s, \theta^*(s)) \Pi^\infty(ds d\theta) = \int_S f(s, \theta^*(s)) \mu(ds).$$

This proves the first part of Proposition 5. The second part follows directly from the first one.

4 Concluding remarks

The results and proofs given here all rely on the assumption that (ω_t) is a Markov process. In particular, they do not apply to the case where $t \rightarrow \omega_t$ is a deterministic periodic signal with discontinuities. This situation is investigated in a forthcoming preprint [3]. In the particular case of Example 3, a first order expansion of Λ^T when T goes to 0 or ∞ will be given in another forthcoming preprint [13].

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