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Genus theory of p -adic pseudo-measures – Tame kernels and abelian p -ramification

Georges Gras

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GENUS THEORY OF p -ADIC PSEUDO-MEASURES HILBERT'S KERNELS & ABELIAN p -RAMIFICATION

GEORGES GRAS

ABSTRACT. We consider, for real abelian fields K , the Birch–Tate formula linking $\#\mathbf{K}_2(\mathbf{Z}_K)$ to $\zeta_K(-1)$; we compare, for quadratic and cyclic cubic fields with $p \in \{2, 3\}$, $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ to the order of the torsion group $\mathcal{T}_{K,p}$ of abelian p -ramification theory given by the residue of $\zeta_{K,p}(s)$ at $s = 1$. This is done via the “genus theory” of p -adic pseudo-measures, inaugurated in the 1970/80’s and the fact that $\mathcal{T}_{K,p}$ only depends on the p -class group and on the normalized p -adic regulator of K (Theorem 1.6 A). We apply this to prove a conjecture of Deng–Li giving the structure of $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$ for an interesting family of real quadratic fields (Theorem 1.6 B). Then, for $p \geq 5$, we give a lower bound of $\mathrm{rk}_p(\mathbf{K}_2(\mathbf{Z}_K))$ in cyclic p -extensions K/\mathbb{Q} (Theorem 1.6 C). Complements, PARI programs and tables are given in Appendices.

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1. INTRODUCTION AND PREREQUISITES

The aim of this paper is to utilize the link between p -adic \mathbf{L} -functions $\mathbf{L}_p(s, \chi)$, of even Dirichlet's characters $\chi = \varphi\psi$ (φ of prime-to- p order, ψ of *non-trivial* p -power order), and classical arithmetic p -invariants of the corresponding cyclic number field K , then to show that these invariants have some connections due to the character ψ (genus theory principle). This connection does exist because of reflection theorems of class field theory and this allows to get additional informations.

Let's give some notations for the description of these methods and the statement of the results:

1.1. Notations. (i) Let $\chi =: \varphi\psi$ be an even character, where φ is of prime-to- p order and ψ of p -power order p^e , $e \geq 1$. For $a \in \mathbb{Z}_p^\times$, let $\theta(a)$ be the unique $\xi \in \text{tor}_{\mathbb{Z}_p}(\mathbb{Z}_p^\times)$ such that $a \equiv \xi \pmod{p}$ for $p \neq 2$ (resp. $a \equiv \xi \pmod{4}$ for $p = 2$). The character θ will be considered as the cyclotomic character of $\mathbb{Q}(\mu_q)$ ($q = p$ or 4).

Let K be the real cyclic field fixed by χ and let M be the subfield of \mathbb{Q}^{ab} fixed by the character $\theta^{-1}\chi$; since K is real, M is an imaginary field. Let M_0 be the maximal subfield of M of prime-to- p degree:

- for $p \neq 2$, then $M \subseteq K(\mu_p)$, and M_0 is the fixed field of $\theta^{-1}\varphi$ since θ is of order $p-1$; if $\varphi = \chi_0$ (the unit character), $M_0 = \mathbb{Q}(\mu_p)$;
- for $p = 2$, then $M \subset K(\sqrt{-1})$, and M_0 is the fixed field of φ since θ is of order 2; if $\varphi = \chi_0$, $M_0 = \mathbb{Q}$.

(ii) Let \mathfrak{m} be the maximal ideal of $\mathbb{Q}_p(\mu_{p^e})$ and let $v_{\mathfrak{m}}$ be the corresponding valuation with image \mathbb{Z} (so, $v_{\mathfrak{m}} = (p-1)p^{e-1} \cdot v_p$).

(iii) For any prime number r , let p^{d_r} be the degree of the splitting field of r in M/M_0 ; let $\mathbf{D} = \sum_{\ell} p^{d_\ell}$, for $\ell \neq p$ ramified in M/M_0 and totally split in M_0/\mathbb{Q} ; whence for $\ell \neq p$ such that:

- for $p \neq 2$: $\psi(\ell) = 0$ and $(\theta^{-1}\varphi)(\ell) = 1$;
- for $p = 2$: $\psi(\ell) = 0$ and $\varphi(\ell) = 1$ (let \widehat{K} with $[K : \widehat{K}] = 2$; then M is the quadratic extension of \widehat{K} distinct from K and $\widehat{K}(\sqrt{-1})$).

1.2. Overview of the method. Genus theory does exist for \mathbf{L}_p -functions, as this was initiated in the 1970/80's with the genus theory of p -adic abelian pseudo-measures [Gra1986^b, Théorème (0.3)], [Gra1987, Théorème (0.1)] including the tricky split case $\theta^{-1}\varphi(p) = 1$. The results of genus theory under consideration claim that:

$$(1.1) \quad \begin{cases} \text{either} & v_m(\frac{1}{2}\mathbf{L}_p(s, \chi)) > \mathbf{C}(s), \forall s \in \mathbb{Z}_p, \\ \text{or} & v_m(\frac{1}{2}\mathbf{L}_p(s, \chi)) = \mathbf{C}(s), \forall s \in \mathbb{Z}_p, \end{cases}$$

for an explicit $\mathbf{C}(s)$, which is a constant \mathbf{C} as soon as $\theta^{-1}\varphi(p) \neq 1$ (see Theorem 4.4 defining $\mathbf{C}(s)$, essentially depending on \mathbf{D}).

This allows to assign, for $p \in \{2, 3\}$, similar properties to the following arithmetic p -invariants depending on values of \mathbf{L}_p -functions:

(a) We will consider, at first (Section 1.3), the torsion group $\mathcal{T}_{K,p}$ of the Galois group of the maximal abelian p -ramified (non-complexified) pro- p -extension $H_{K,p}^{\text{pr}}$ of K . In the real abelian case, $\mathcal{T}_{K,p} = \text{Gal}(H_{K,p}^{\text{pr}}/K_\infty)$, where K_∞ is the cyclotomic \mathbb{Z}_p -extension of K .

Analytically (where \sim means equality “up to a p -adic unit factor”):

$$\#\mathcal{T}_{K,p} \sim [K \cap \mathbb{Q}_\infty : \mathbb{Q}] \times \prod_{\chi \neq 1} \frac{1}{2} \mathbf{L}_p(1, \chi),$$

where χ runs through the set of primitive Dirichlet's characters of K [Co1975, Appendix], [Se1978], while class field theory elucidates it completely with Formula (1.3) [Gra2018^b], which explains that $\mathcal{T}_{K,p}$ plays a central role in this study.

(b) Then we will focus (Section 1.4) on the the Hilbert tame kernel $\mathbf{K}_2(\mathbf{Z}_K)$ of the ring of integers \mathbf{Z}_K of K and, for $p = 2$, we will replace it by the regular kernel $\mathbf{R}_2(\mathbf{Z}_K)$ in the ordinary sense, linked to the Hilbert one via the exact sequence [Gar1971]:

$$(1.2) \quad 1 \rightarrow \mathbf{R}_2(\mathbf{Z}_K) \longrightarrow \mathbf{K}_2(\mathbf{Z}_K) \longrightarrow (\mathbb{Z}/2\mathbb{Z})^{[K:\mathbb{Q}]} \rightarrow 1.$$

The class field theory information about the regular kernel comes in general from reflection theorems giving formulas (1.4) and (1.5); we have refrained from citing all the articles re-proving well-known reflection theorems between $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$, $\mathcal{T}_{K,p}$, $\mathcal{H}_{K,p}$, published in the 1980–90's by several authors.

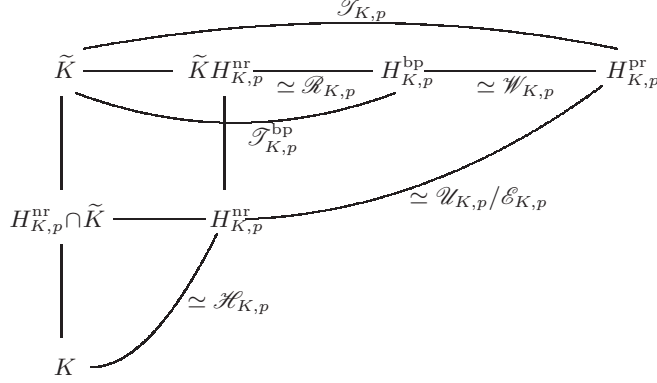
(c) Then $\#\mathbf{K}_2(\mathbf{Z}_K)$, being in relation with the complex $\mathbf{L}(-1, \chi)$'s, we use (Section 1.5), when it is possible, the classical link between p -adic and complex \mathbf{L} -functions for some comparison of the two p -adic invariants $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\mathcal{T}_{K,p}$ ¹, with the use of property (1.1) when it applies, thus giving new informations.

1.3. Torsion group of abelian p -ramification. Class field theory is especially convenient to state the arithmetic properties of $\mathcal{T}_{K,p}$ (e.g., [Gra2005, § III.2.c, § III.4.b, § IV.3], after some pioneering works [BP1972, Gra1986^a, Jau1986, Ng1986, GJ1989] and many others).

Numerical aspects are given in [Gra2017^a] with PARI [PARI] programs and the order of magnitude of $\#\mathcal{T}_{K,p}$ is studied in [Gra2019^b].

¹ A p -invariant may be denoted $\mathbf{X}_K[p^\infty]$ when it is of the form $\mathbf{X}_K \otimes \mathbb{Z}_p$, for a finite global invariant \mathbf{X}_K . Otherwise it is denoted $\mathcal{X}_{K,p}$; it is only conjectural [Gra2016, Section 8] that $\mathcal{T}_{K,p}$ can be written $\mathbf{T}_K \otimes \mathbb{Z}_p$; this conjecture is out of reach but may be coherent with the finiteness of \mathbf{H}_K (class group) and $\mathbf{K}_2(\mathbf{Z}_K)$, and is supported by many heuristics. For convenience, we will also use the notation $\mathcal{X}_{K,p}$ when \mathbf{X}_K does exist (e.g., $\mathcal{H}_{K,p} = \mathbf{H}_K[p^\infty]$).

We refer to Appendix A.1 for the program computing the structure of $\mathcal{T}_{K,p}$ in complete generality. The following diagram is valid for any number field K fulfilling Leopoldt's conjecture, replacing K_∞ by the compositum \tilde{K} of the \mathbb{Z}_p -extensions of K :



Definitions and notations are the following:

- (i) $\mathcal{H}_{K,p}$ is the p -class group and $H_{K,p}^{nr}$ the p -Hilbert class field;
- (ii) $\mathcal{R}_{K,p}$ is the *normalized* p -adic regulator, defined as a \mathbb{Z}_p -module in [Gra2018^b, Section 5, Proposition 5.2]; its order is the classical p -adic regulator, up to an explicit factor;
- (iii) $\mathcal{U}_{K,p}$ is the group of principal local units at p , $\mathcal{E}_{K,p}$ the closure in $\mathcal{U}_{K,p}$ of the group of units E_K and $\mathcal{W}_{K,p} := \text{tor}_{\mathbb{Z}_p}(\mathcal{U}_{K,p})/\mu_p(K)$;
- (iv) $H_{K,p}^{bp}$ (resp. $\mathcal{T}_{K,p}^{bp}$), named in [Ng1986] the Bertrandias–Payan field (resp. module), is issued from [BP1972]; $H_{K,p}^{bp}$ is the compositum of the p -cyclic extensions of K embeddable in p -cyclic extensions of arbitrary large degree; see [GJN2016] for complements. Whence:

$$(1.3) \quad \#\mathcal{T}_{K,p} = \frac{\#\mathcal{H}_{K,p}}{[H_{K,p}^{nr} \cap \tilde{K} : K]} \times \#\mathcal{R}_{K,p} \times \#\mathcal{W}_{K,p}.$$

In the real case, K_∞/K is in general totally ramified, so that the formula becomes $\#\mathcal{T}_{K,p} = \#\mathcal{H}_{K,p} \times \#\mathcal{R}_{K,p} \times \#\mathcal{W}_{K,p}$.

Remark 1.1. From [Gra2018^b, Proposition 5.2], we can give the following examples:

(i) For $p = 2$ and $K = \mathbb{Q}(\sqrt{m})$ ($m > 0$ square-free), this formula becomes $\#\mathcal{T}_{K,2} \sim h_K \times \frac{\log_2(\varepsilon_K)}{2\sqrt{m}}$, where h_K is the class number and ε_K the fundamental unit.

(ii) For $p = 3$ and a cyclic cubic field K , then $\#\mathcal{T}_{K,3} \sim h_K \times \frac{R_{K,3}}{3^u}$ with $u = 2$ (resp. $u = 1$) if 3 is unramified (resp. ramified) and where $R_{K,3}$ is the usual 3-adic regulator

1.4. Tame and regular Hilbert's kernels. Let K be any number field, let $p \geq 2$ be a prime number and let $S_p(K)$ be the set of p -places of K ; put $K' := K(\mu_p)$ and let $S_p(K')$ be the set of p -places of K' .

Recall that $\omega : \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \rightarrow \mu_{p-1}$, called the cyclotomic (or Teichmüller) character, is defined by the Galois action on μ_p (conductor p , order $p-1$ for $p \neq 2$); it is the unit character χ_0 for $p = 2$ since $\mu_2 \subset \mathbb{Q}$. For $p \neq 2$, we have in some sense $\omega = \theta$ (Notations 1.1 (i)), but for $p = 2$, do not confuse ω with θ as cyclotomic character of $\text{Gal}(\mathbb{Q}(\mu_4)/\mathbb{Q})$ (conductor 4, order 2).

1.4.1. *First reflection theorem.* The regular kernel $\mathbf{R}_2(\mathbf{Z}_K)$ (see (1.2)) fulfills, from Tate's results [Tate1970, Tate1976] and reflection theorem [Gra1998, Théorème 11.1] or [Gra2005, Theorem II.7.7.3.1], the following p -rank formula, where $\mathbf{H}_{K'}^{S_p(K')\text{res}}$ is the $S_p(K')$ -class group in the restricted sense, $\delta = 1$ or 0 , according as $\mu_p \subset K$ or not:

$$(1.4) \quad \begin{cases} \text{rk}_p(\mathbf{R}_2(\mathbf{Z}_K)) = \text{rk}_p((\mathbf{H}_{K'}^{S_p(K')\text{res}})_{\omega^{-1}}) \\ \quad + \#\{v \in S_p(K), v \text{ totally split in } K'/K\} - \delta, \\ \text{rk}_2(\mathbf{R}_2(\mathbf{Z}_K)) = \text{rk}_2(\mathbf{H}_K^{S_2\text{res}}) + \#S_2 - 1. \end{cases}$$

1.4.2. *Second reflection theorem.* Due to the reflection theorem between p -class groups and p -torsion groups of abelian p -ramification theory, we get similarly [Gra2005, Proposition III.4.2.2]:

$$(1.5) \quad \begin{cases} \text{rk}_p(\mathcal{T}_{K,p}) = \text{rk}_p((\mathbf{H}_{K'}^{S_p(K')\text{res}})_{\omega}) \\ \quad + \#\{v \in S_p(K), v \text{ totally split in } K'/K\} - \delta \\ \text{rk}_2(\mathcal{T}_{K,2}) = \text{rk}_2(\mathbf{H}_K^{S_2\text{res}}) + \#S_2 - 1. \end{cases}$$

From the two previous relations, we get:

$$(1.6) \quad \text{rk}_p(\mathbf{R}_2(\mathbf{Z}_K)) = \text{rk}_p(\mathcal{T}_{K,p}),$$

as soon as $\omega^2 = 1$, that is to say, if K contains the maximal real subfield of $\mathbb{Q}(\mu_p)$, a framework widely developed in [GJ1989]; these techniques yielding generalizations as that of Keune [Keu1989].

The relation was proven in [Gra1986^a, Theorems 1, 2] (in which $\mathbf{R}_2(\mathbf{Z}_K)$ was denoted $H_2^0 K$), to characterize, for $p \in \{2, 3\}$, the abelian p -extensions K/\mathbb{Q} such that $\mathbf{R}_2(\mathbf{Z}_K)[p^\infty] = 1$ (p -regular fields studied in [GJ1989, BGr1992, RØ2000]).

We note that $\mathbb{Q}(\mu_p)^+ \subseteq K$ is always fulfilled for $p \in \{2, 3\}$. More generally, [Gra1998, Remarque 11.5] gives, when K contains μ_p , the corresponding relation (1.6) with characters:

$$\text{rk}_\chi(\mathbf{R}_2(\mathbf{Z}_K)) = \text{rk}_{\omega^2\chi^{-1}}(\mathcal{T}_{K,p}).$$

Class field theory approach of the structures of $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\mathcal{T}_{K,p}$ has given a huge literature, especially for rank computations, often restricted to $p \in \{2, 3\}$, probably with (1.6) when $\omega^2 = \chi_0$, to get practical results (e.g., [BS1982, Gra1986^a, Keu1989, Ber1990, Br1992, Qin1995, YF2000, KM2003, Br2005, Yue2005, Qin2005, DeLi2023]).

For results in the case $p \geq 5$, see Section 6.

1.5. Arithmetic v.s. analytic properties. To complete these class field theory aspects, we will use the following (p -adic and complex) analytic formulas and the correspondence, for $p \in \{2, 3\}$, between p -adic and complex \mathbf{L} -functions of characters χ of K :

$$(1.7) \quad \begin{cases} \#\mathcal{T}_{K,p} \sim [K \cap \mathbb{Q}_\infty : \mathbb{Q}] \times \prod_{\chi \neq 1} \frac{1}{2} \mathbf{L}_p(1, \chi), \\ \#\mathbf{K}_2(\mathbf{Z}_K) = 2^{[K:\mathbb{Q}]} \times \left(\frac{1}{6} \prod_{[F:K]=2} \frac{\#\mu(F)}{2} \right) \times \prod_{\chi \neq 1} \frac{1}{2} \mathbf{L}(-1, \chi), \end{cases}$$

where $\mu(F)$ is the group of roots of unity of the field F .

For more history and theoretical contributions, one may refer to [Ba1968, Tate1970, Gar1971, Tate1976, Jau1986, Ng1986, Hur1987, GJ1989, Keu1989, Kol1989, Ng1990, HuK1998, RWK2000, Kol2002, Yue2002, Gra2005, Qin2010], among many others, dealing with the

more concrete results we have cited previously; for some generalizations to the wide étale kernels $\mathbf{WK}_{2i}(K)$, see for instance [Ng1992, JaSo2001, JaMi2006, AAM2021].

But our goal is to return to very simple effective p -adic methods allowing computations, by means of pseudo-measures. We intend to provide, simultaneously, survey and history parts about these various questions.

These pseudo-measures are the Mellin transform (via a reflection principle) of Stickelberger elements, and give rise, after twisting, to p -adic measures of the form $(\mathcal{S}_{L_n}^*(c)(s))_n$, $s \in \mathbb{Z}_p$, in the group algebras $\mathbb{Z}_p/qp^n\mathbb{Z}_p[\text{Gal}(L_n/\mathbb{Q})]$ of the layers $L_n := K(\mu_{qp^n})$ of the cyclotomic \mathbb{Z}_p -extension of $K(\mu_q)$ ($q = 4$ or p) [Gra1978, Section II].

This was done after the main pioneering works of Kubota–Leopoldt [KL1964], then Fresnel [Fr1965], Amice–Fresnel [AF1972], Iwasawa [Iw1972], Coates [Co1975], Serre [Se1978], giving many results, as the value of the residue of Dedekind p -adic ζ -functions at $s = 1$ and some annihilation theorems for p -class groups, torsion groups $\mathcal{T}_{K,p}$ and similar invariants [Gra1978, Gra1979, Jau1990, BN2005, Jau2021, Jau2023]; this was considerably generalized to a totally real base field by Colmez in [Col1988] and Deligne–Ribet in [DR1980].

The Iwasawa framework of these constructions is very similar and is detailed in [Wa1997, § 7.2].

1.6. Main results of the article. We describe three results about the study and the comparison of the modules $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\mathcal{T}_{K,p}$:

(a) From the properties of the constant \mathbf{C} described in § 1.2 and stated in Theorem 4.4, we obtain the following results for quadratic fields ($p = 2$) and cyclic cubic ones ($p = 3$):

Theorem A. (see Theorem 4.5 for the proof).

(i) Let $K = \mathbb{Q}(\sqrt{m})$ be a real quadratic field of conductor $\neq 8$.

Assume that $v_2(\#\mathcal{T}_{K,2}) = \mathbf{C}$ (equivalent to $v_2(\frac{1}{2}\mathbf{L}_2(1, \chi)) = \mathbf{C}$ from formula (1.7)); then $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] = 2^{\mathbf{C}+2}$.

(ii) Let K be a cyclic cubic field of conductor $\neq 9$.

Assume that $v_3(\#\mathcal{T}_{K,3}) = \mathbf{C}$ (so, $v_3(\mathbf{L}_3(1, \chi) \times \mathbf{L}_3(1, \chi^2)) = \mathbf{C}$); then $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] = 3^{\mathbf{C}}$.

(iii) If $v_2(\mathcal{T}_{K,2}) > \mathbf{C}$ (resp. $v_3(\mathcal{T}_{K,3}) > \mathbf{C}$), $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] > 2^{\mathbf{C}+2}$ (resp. $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] > 3^{\mathbf{C}}$).

We compute, in Appendix B.1, $\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty]$ for quadratic fields by means of explicit pseudo-measures yielding expressions of Theorem 4.1 and Corollary 4.2; this allows safe verifications.

(b) We apply the case of equality to a family of real quadratic fields introduced in [DeLi2023, Theorem 1.2] and prove the Deng–Li conjecture on $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$ in the following Theorem B (Section 5):

Theorem B. Let $K = \mathbb{Q}(\sqrt{m})$, $m = \ell_1\ell_2 \cdots \ell_n$, n even, with:

- $\ell_1 \equiv 3 \pmod{8}$, $\ell_i \equiv 5 \pmod{8}$ for $i \geq 2$,
- $(\frac{\ell_1}{\ell_2}) = -1$, $(\frac{\ell_1}{\ell_j}) = 1$ for $j \geq 3$, $(\frac{\ell_i}{\ell_j}) = -1$ for $2 \leq i < j \leq n$.

Then $\mathcal{H}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$, $\mathcal{T}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-2} \times \mathbb{Z}/4\mathbb{Z}$, and:

$$\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^3\mathbb{Z} \text{ (Deng–Li conjecture).}$$

(c) The cases $p \in \{2, 3\}$ being very specific (thus well studied in the literature) because of the relation (2.1) for $m = 2$ between p -adic and complex \mathbf{L} -functions, we have obtained the following Theorem C when p is distinct from 2 and 3 (see Theorem 6.1 for a much more general statement over a real base field k of prime-to- p degree):

Theorem C. *Let K/\mathbb{Q} be a cyclic p -extension for $p \geq 5$ and let S_{ta} be the set of primes $\ell \neq p$ ramified in K/\mathbb{Q} .*

Then $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \geq \#S_{\text{ta}}$. In particular, $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty] = 1$ if and only if K is contained in the cyclotomic \mathbb{Z}_p -extension \mathbb{Q}_∞ of \mathbb{Q} .

2. COMPUTATION OF $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$, $p \in \{2, 3\}$

2.1. Birch-Tate formula and \mathbf{L}_p -functions. Let $\zeta_K(s)$ be the Dedekind zeta-function of the real abelian number field K .

Then $\zeta_K(s) = \prod_{\chi} \mathbf{L}(s, \chi)$, as product of the complex \mathbf{L} -functions, where χ runs through the set of primitive Dirichlet's characters of K , for which $\mathbf{L}(s, \chi_0) = \zeta_{\mathbb{Q}}(s)$ for the unit character. In the p -adic context, definition of \mathbf{L}_p -functions is as follows (e.g., [Fr1965, Section 5 (a), Remarque], Amice-Fresnel [AF1972, Section 1]):

$$(2.1) \quad \begin{cases} \mathbf{L}_p(1-m, \chi) = (1-p^{m-1}\chi(p)) \mathbf{L}(1-m, \chi), \text{ for:} \\ m > 1 \ \& \ m \equiv 0 \pmod{p-1} \text{ if } p > 2, \\ m > 1 \ \& \ m \equiv 0 \pmod{2} \text{ if } p = 2. \end{cases}$$

This allows to compute for instance $\mathbf{L}(-1, \chi)$ in terms of \mathbf{L}_p -functions for $p = 2$ and $p = 3$ since $m = 2$ fulfills the congruent conditions required in (2.1).

The Birch-Tate formula is the following equality (proved as consequence of the Mazur-Wiles ‘‘Main Theorem’’ in abelian theory and complements in the case $p = 2$ [Kol1989]):

$$(2.2) \quad \begin{cases} \#\mathbf{K}_2(\mathbf{Z}_K) = w_2(K) \zeta_K(-1) = w_2(K) \zeta_{\mathbb{Q}}(-1) \prod_{\chi \neq \chi_0} \mathbf{L}(-1, \chi), \\ w_2(K) = 4 \cdot \prod_{[F:K]=2} \frac{\#\mu(F)}{2} \quad (\text{see (1.7)}). \end{cases}$$

For instance, $w_2(\mathbb{Q}) = 24$ and $\zeta_{\mathbb{Q}}(-1) = \frac{1}{12}$; then $\#\mathbf{K}_2(\mathbb{Z}) = 2$, giving $\#\mathbf{R}_2(\mathbb{Z}) = 1$ as expected since $\mathcal{T}_{\mathbb{Q},p} = 1$ for all p .

Thus, the computation of the p -Sylow subgroup $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ of $\mathbf{K}_2(\mathbf{Z}_K)$ is possible for $p = 2$ and $p = 3$ with the following formulas (using $\zeta_{\mathbb{Q}}(-1) = \frac{1}{12}$ and (2.1) for $m = 2$):

$$\begin{cases} \#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \sim \frac{w_2(K)}{12} \times \prod_{\chi \neq \chi_0} \frac{1}{1-2\chi(2)} \mathbf{L}_2(-1, \chi), \\ \#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \frac{w_2(K)}{12} \times \prod_{\chi \neq \chi_0} \frac{1}{1-3\chi(3)} \mathbf{L}_3(-1, \chi), \end{cases}$$

where \sim means equality up to a p -adic unit factor for the considered p . Since the denominators $1-p\chi(p)$ are invertible in \mathbb{Z}_p , the formulas become with expression (2.2) of $w_2(K)$:

$$\begin{cases} \#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \sim \prod_{[F:K]=2} \frac{\#\mu_2(F)}{2} \times \prod_{\chi \neq \chi_0} \mathbf{L}_2(-1, \chi), \\ \#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \frac{1}{3} \prod_{[F:K]=2} \#\mu_3(F) \times \prod_{\chi \neq \chi_0} \mathbf{L}_3(-1, \chi). \end{cases}$$

2.2. Formulas for quadratic and cubic fields. We deduce the following expressions with $p \in \{2, 3\}$; due to exact sequence (1.2) when $p = 2$, we do not write the corresponding results for $\mathbf{R}_2(\mathbf{Z}_K)$.

(a) Real quadratic fields.

(i) For real quadratic fields $K = \mathbb{Q}(\sqrt{m})$, $m \neq 2, 3$, $p = 2$:

$$\begin{cases} \prod_{[F:K]=2} \frac{1}{2} \cdot \#\mu_2(F) = 2 \quad (F = K(\mu_4), F = K(\mu_6)); \\ \#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \sim 2 \cdot \mathbf{L}_2(-1, \chi). \end{cases}$$

(ii) For $\mathbb{Q}(\sqrt{2})$, $\prod_{[F:K]=2} \frac{\#\mu_2(F)}{2} = 4$ ($F = K(\mu_8)$, $F = K(\mu_6)$);

so, $\#\mathbf{K}_2\mathbf{Z}_{\mathbb{Q}(\sqrt{2})}[2^\infty] \sim 4 \cdot \mathbf{L}_2(-1, \chi) \sim 4$.

(iii) For $\mathbb{Q}(\sqrt{3})$, $\prod_{[F:K]=2} \frac{\#\mu_2(F)}{2} = 4$ ($F = K(\mu_4)$, $F = K(\mu_6) =$

$K(\mu_4)$); so, $\#\mathbf{K}_2\mathbf{Z}_{\mathbb{Q}(\sqrt{3})}[2^\infty] \sim 4 \cdot \mathbf{L}_2(-1, \chi) \sim 8$.

(iv) For $p = 3$, $K \neq \mathbb{Q}(\sqrt{3})$, the formulas are:

$$\begin{cases} \frac{1}{3} \prod_{[F:K]=2} \#\mu_3(F) = 1 \quad (F = K(\mu_6)); \\ \#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \mathbf{L}_3(-1, \chi). \end{cases}$$

(v) For $\mathbb{Q}(\sqrt{3})$, $\frac{1}{3} \prod_{[F:K]=2} \#\mu_3(F) = 3$ ($F = K(\mu_3)$, $F = K(\mu_4) =$

$K(\mu_6)$); so, $\#\mathbf{K}_2\mathbf{Z}_{\mathbb{Q}(\sqrt{3})}[3^\infty] \sim 3 \cdot \mathbf{L}_3(-1, \chi) \sim 3$.

(b) Cyclic cubic fields.

(i) For cubic fields K of conductor distinct from 9, $p = 3$:

$$\begin{cases} \frac{1}{3} \prod_{[F:K]=2} \#\mu_3(F) = 1 \quad (F = K(\mu_6)); \\ \#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \mathbf{L}_3(-1, \chi) \times \mathbf{L}_3(-1, \chi^2), \end{cases}$$

for the two conjugate characters of order 3 of K .

(ii) For the cubic field of conductor 9, $\frac{1}{3} \prod_{[F:K]=2} \#\mu_3(F) = 3$

($F = K(\mu_9)$); then $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] = 3 \cdot \mathbf{L}_2(-1, \chi) \sim 1$.

(iii) For $p = 2$, the formulas are, for the two conjugate characters of order 3 of K :

$$\begin{cases} \prod_{[F:K]=2} \frac{1}{2} \cdot \#\mu_2(F) = 2 \quad (F = K(\mu_4), F = K(\mu_6)); \\ \#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \sim 2 \cdot \mathbf{L}_2(-1, \chi) \times \mathbf{L}_2(-1, \chi^2). \end{cases}$$

3. DEFINITION OF A p -ADIC PSEUDO-MEASURE $(\mathcal{A}_{L_n})_n$

Let K be a real abelian field and put $L_n := K(\mu_{qp^n})$, $q \in \{p, 4\}$, as usual, and $n \geq 0$.

3.1. The Stickelberger elements. The conductor of L_n is of the form $f_{L_n} = qp^n f$, for a prime-to- p integer f , taking n large enough if p ramifies in K (otherwise, f is the conductor of K) (in formulas we shall abbreviate f_{L_n} by f_n).

Then put, where all Artin symbols are taken over \mathbb{Q} :

$$\mathcal{A}_{L_n} := - \sum_{a=1}^{f_n} \left(\frac{a}{f_n} - \frac{1}{2} \right) \left(\frac{L_n}{a} \right)^{-1},$$

4.1. Approximations modulo qp^n of $\mathbf{L}_p(s, \chi)$. The \mathbf{L}_p -functions, of even Dirichlet's characters, are obtained by means of the Mellin transform of the previous twisted pseudo-measures, so that the Mellin transform of $\left(1 - c\left(\frac{L_n}{c}\right)^{-1}\right)$ is $\left(1 - \langle c \rangle^{1-s}\left(\frac{L_n}{c}\right)\right)$ and is a factor of the p -adic measure $\left(\mathcal{S}_{L_n}^*(c)(s)\right)_n$ that must be dropped when this makes sense (thus in any case, except $\chi = \chi_0$ and $s = 1$):

Theorem 4.1. *Let c be any odd integer, prime to p and to the conductor f of K . For all n large enough, let f_n be the conductor of $L_n = K(\mu_{qp^n})$, and for all $a \in [1, f_n]$, prime to f_n , let a'_c be the unique integer in $[1, f_n]$ such that $a'_c \cdot c - a = \lambda_a^n(c)f_n$, $\lambda_a^n(c) \in \mathbb{Z}$. The twisted p -adic measure generating the \mathbf{L}_p -functions is given by the restriction $\mathcal{S}_{K,n}^*(c)(s)$ of $\mathcal{S}_{L_n}^*(c)(s)$ to K , giving, for all $n \geq 0$:*

$$\mathcal{S}_{K,n}^*(c)(s) \equiv \sum_{a=1}^{f_n} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a^{-1} \langle a \rangle^{1-s} \left(\frac{K}{a} \right) \pmod{qp^n}.$$

Whence the \mathbf{L}_p -functions of Dirichlet's character χ of K :

$$\mathbf{L}_p(s, \chi) = \frac{1}{1 - \chi(c)\langle c \rangle^{1-s}} \times \lim_{n \rightarrow \infty} \sum_{a=1}^{f_n} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a^{-1} \langle a \rangle^{1-s} \chi(a).$$

This formula is used in Appendices B.1, B.3, C.1, C.3, D.3, for explicit numerical computations.

Corollary 4.2. *Assume that $p \in \{2, 3\}$; in this context, $\theta^2 = \chi_0$. Since for $s = -1$, $a^{-1}\langle a \rangle^{1-s} = a^{-1}\langle a \rangle^2 = a$, we get:*

$$\mathbf{L}_p(-1, \chi) = \frac{1}{1 - \chi(c)\langle c \rangle^2} \times \lim_{n \rightarrow \infty} \sum_{a=1}^{f_n} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a \chi(a).$$

Corollary 4.3. *Assume $\chi \neq \chi_0$ and $\chi(c) \neq 1$. For $s = -1$, $\langle c \rangle \equiv 1 \pmod{q}$ and $\chi(c) \in \mu_{[K:\mathbb{Q}]} \setminus \{1\}$; so, in the quadratic case with $p = 3$ and in the cubic case with $p = 2$, $1 - \chi(c)\langle c \rangle^2$ is invertible. Then:*

- $1 - \chi(c)\langle c \rangle^2 \sim 2$ for the quadratic case and $p = 2$,
- $1 - \chi(c)\langle c \rangle^2 \sim 1 - \zeta_3$ for the cubic case and $p = 3$.

From formulas of Section 2.2 one gets:

(i) For $p = 2$ in the quadratic case $K = \mathbb{Q}(\sqrt{m})$, $m \neq 2, 3$ and a half summation giving $\frac{1}{2} \mathbf{L}_2(s, \chi)$, for $\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] = \frac{1}{4} \#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$ one obtains (since $1 - \chi(c)\langle c \rangle^2 \sim 2$):

$$\left\{ \begin{array}{l} \#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \sim \frac{1}{2} \mathbf{L}_2(-1, \chi) \\ \equiv \frac{1}{2} \sum_{a=1}^{f_n/2} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a \chi(a) \pmod{2^n}. \end{array} \right.$$

(ii) For the cubic case with $p = 3$, $(1 - \chi(c)\langle c \rangle^2)(1 - \chi^2(c)\langle c \rangle^2) \sim (1 - \zeta_3)(1 - \zeta_3^2) \sim 3$, giving, with $\mathbf{N} = \mathbf{N}_{\mathbb{Q}(\mu_3)/\mathbb{Q}}$:

$$\left\{ \begin{array}{l} \#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \sim \mathbf{L}_3(-1, \chi) \cdot \mathbf{L}_3(-1, \chi^2) \\ \equiv \frac{1}{3} \mathbf{N} \left(\sum_{a=1}^{f_n/2} \left[\lambda_a^n(c) + \frac{1-c}{2} \right] a \chi(a) \right) \pmod{3^n}. \end{array} \right.$$

4.2. Genus theory of p -adic pseudo-measures. We have obtained in [Gra1987, Théorème 0.2], the following results which does not seem very known but states the property of ‘‘stability’’ mentioned in the Introduction and may be linked to the existence (or not) of zeroes of \mathbf{L}_p -functions (for examples about zeroes of \mathbf{L}_p -functions, one

may refer to [Wa1980, Wa1981, Wa1995, Wa1996, Wa1997] and see Appendix C.4 for some illustrations of this influence):

Theorem 4.4. *Under Notations 1.1, where $\chi = \varphi\psi$ is even, φ of prime-to- p order and ψ of order p^e , $e \geq 1$, we have, with $\mathbf{D} = \sum_{\ell} p^{d_{\ell}}$ and $\epsilon = 0$ (resp. $\epsilon = 1$) when $\varphi \neq \chi_0$ (resp. $\varphi = \chi_0$):*

(a) $v_m\left(\frac{1}{2}\mathbf{L}_p(s, \chi)\right) \geq \mathbf{C}(s)$, $\forall s \in \mathbb{Z}_p$, where $\mathbf{C}(s)$ is as follows:

(i_p) $p \neq 2$ and $\theta^{-1}\varphi(p) \neq 1$ (i.e., p not totally split in M_0); then:

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} - \epsilon.$$

(i₂) $p = 2$ and $\varphi(2) \neq 1$ (i.e., 2 not totally split in M_0); then:

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D}.$$

(ii_p) $p \neq 2$, $\theta^{-1}\varphi(p) = 1$ and $\theta^{-1}\chi(p) \neq 1$ (i.e., p totally split in M_0 and not totally split in M); then:

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} + p^{d_p}.$$

(ii₂) $p = 2$, $\varphi(2) = 1$ and $\theta^{-1}\chi(2) \neq 1$ (i.e., 2 totally split in M_0 and not totally split in M); then:

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} + 2^{d_2} - 2\epsilon.$$

(iii_p) $p \neq 2$, $\theta^{-1}\chi(p) = 1$ (i.e., p totally split in M); then:

$$\mathbf{C}(s) := \mathbf{D} + v_m(p s).$$

(iii₂) $p = 2$, $\theta^{-1}\chi(2) = 1$ (i.e., 2 totally split in M); then:

$$\mathbf{C}(s) := \mathbf{D} + v_m(q s) - 2\epsilon.$$

(b) We have either the equality:

$$v_m\left(\frac{1}{2}\mathbf{L}_p(s, \chi)\right) = \mathbf{C}(s), \forall s \in \mathbb{Z}_p,$$

or the strict inequality:

$$v_m\left(\frac{1}{2}\mathbf{L}_p(s, \chi)\right) > \mathbf{C}(s), \forall s \in \mathbb{Z}_p,$$

(with $s \neq 0$ in cases (iii_p) and (iii₂)).

So, under the case of stability $v_m\left(\frac{1}{2}\mathbf{L}_p(s, \chi)\right) = \mathbf{C}(s)$, $\forall s \in \mathbb{Z}_p$, a computation may be done at a suitable value s for $\mathbf{L}_p(s, \chi)$, e.g., $s = 1$ giving $\#\mathcal{T}_{K,p} \sim [K \cap \mathbb{Q}_{\infty} : \mathbb{Q}] \times \prod_{\chi \neq 1} \frac{1}{2}\mathbf{L}_p(1, \chi)$; thus $\#\mathcal{T}_{K,2} \sim \frac{1}{2}\mathbf{L}_2(1, \chi)$ for K real quadratic, except K of conductor 8, or $\#\mathcal{T}_{K,3} \sim \mathbf{L}_3(1, \chi) \times \mathbf{L}_3(1, \chi^2)$ for K cyclic cubic, except K of conductor 9.

Which gives the main process and Theorem 1.6 A, where $\mathbf{C}(s) = \mathbf{C}$, as soon as $s \in \mathbb{Z}_p^{\times}$, since $v_m(s) = 0$ in cases (iii) (e.g., $s = \pm 1$):

Theorem 4.5. *We obtain the following results:*

(i) Let K be a real quadratic field of character χ of conductor $\neq 8$, and set $p = 2$. Assume that the knowledge of $\#\mathcal{T}_{K,2}$ (e.g., using formula (1.3)) implies $v_2\left(\frac{1}{2}\mathbf{L}_2(1, \chi)\right) = \mathbf{C}$; then $v_2\left(\frac{1}{2}\mathbf{L}_2(s, \chi)\right) = \mathbf{C}$, $\forall s \in \mathbb{Z}_2$, whence in particular, $\#\mathbf{K}_2(\mathbf{Z}_K)[2^{\infty}] = 2^{\mathbf{C}+2}$.

We have $\mathbf{C} = \mathbf{D}$ (resp. $\mathbf{C} = \mathbf{D} - 1$) if 2 splits (resp. does not split) in $M = \mathbb{Q}(\sqrt{-m})$.

(ii) Let K be a cyclic cubic field of character χ of conductor $\neq 9$, and set $p = 3$. Assume that the knowledge of $\#\mathcal{T}_{K,3}$ implies

$v_m(\mathbf{L}_3(1, \chi)) = \mathbf{C}$; then $v_m(\mathbf{L}_3(s, \chi)) = \mathbf{C}, \forall s \in \mathbb{Z}_3$, then in particular, $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] = 3^{\mathbf{C}}$. We have $\mathbf{C} = \mathbf{D} - 1$.

(iii) If $v_2(\frac{1}{2}\mathbf{L}_2(1, \chi)) > \mathbf{C}$ (resp. $v_m(\mathbf{L}_3(1, \chi)) > \mathbf{C}$), then one obtains $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] > 2^{\mathbf{C}+2}$ (resp. $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] > 3^{\mathbf{C}}$).

Considering for instance [DeLi2023, Theorem 1.2] giving, for this specific family of real quadratic fields:

$$\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^\delta\mathbb{Z}, \quad \delta \geq 3,$$

the Deng–Li conjecture is $\delta = 3$; we intend to prove it by means of the previous process from showing that $\#\mathcal{T}_{K,2} = 2^n$:

Corollary 4.6. *Under the conditions: $\#\mathcal{H}_{K,2} = 2^{n-1}$, $\mathcal{R}_{K,2} = 1$ and $\#\mathcal{W}_{K,2} = 2$, then $\#\mathcal{T}_{K,2} = 2^n$, giving the valuation n for $\frac{1}{2}\mathbf{L}_2(1, \chi)$; so, Theorem 4.5 (i) yields $\mathbf{C} = \mathbf{D} = n$, since 2 splits in $\mathbb{Q}(\sqrt{-m})$, and the conjecture $\delta = 3$ follows.*

4.3. Modulus of continuity of $\mathbf{L}_p(s, \chi)$. Whatever p , if χ is of prime-to- p order, genus theory is empty and this raise the question of the ‘‘independence’’ (or not) of $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and $\#\mathcal{T}_{K,p}$. This is related to the rank formula (1.6) when K contains the maximal real subfield of $\mathbb{Q}(\mu_p)$, but is also a consequence of the existence of a non-trivial modulus of continuity for $\mathbf{L}_p(s, \chi)$, whatever $\chi = \varphi\psi$, as follows [Gra1987, Théorème 0.3]:

Theorem 4.7. *With Notations 1.1, we have, for all $s, t \in \mathbb{Z}_p$:*

$$\frac{1}{2}\mathbf{L}_p(t, \chi) - \frac{1}{2}\mathbf{L}_p(s, \chi) \equiv a(t - s) \pmod{q\mathbf{m}^{\mathbf{V}}(t - s)},$$

with a computable constant a ($a = 0$ when $\varphi \neq \chi_0$) and \mathbf{V} of the form $\mathbf{V} = \mathbf{D} - \max_{\ell} (p^{d_\ell} - p^{d_p} + \epsilon, \epsilon)$ ($\epsilon = 1$ if $\varphi = \chi_0$, 0 otherwise), where ℓ runs through the set of primes ramified in M/M_0 , totally split in M_0/\mathbb{Q} and such that $\frac{1}{q}\log_p(\ell) \not\equiv 0 \pmod{p}$.

In the case $\psi = \chi_0$, $\varphi \neq \chi_0$, the above formula becomes:

$$\frac{1}{2}\mathbf{L}_p(t, \varphi) \equiv \frac{1}{2}\mathbf{L}_p(s, \varphi) \pmod{q(t - s)}.$$

Thus, even if the genus principle is empty, there is a non-trivial congruence between the orders of the two corresponding invariants.

For instance, we get $\frac{1}{2}\mathbf{L}_p(1, \varphi) \equiv \frac{1}{2}\mathbf{L}_p(-1, \varphi) \pmod{2q}$, whence:

$$\begin{cases} \#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty] \equiv \#\mathcal{T}_{K,3} \pmod{3}, & \text{for quadratic fields,} \\ \#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \equiv \#\mathcal{T}_{K,2} \pmod{8}, & \text{for cyclic cubic fields.} \end{cases}$$

In the case of real quadratic fields for $p = 3$ (resp. of cyclic cubic fields for $p = 2$), see the numerical results given in Appendix B.3 (resp. Appendix C.3).

5. PROOF OF THE DENG–LI CONJECTURE $\delta = 3$

We consider the family defined in [DeLi2023, Theorem 1.2]. We will prove the conjecture $\delta = 3$ in the writing:

$$\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^\delta\mathbb{Z}$$

and some other properties of this family of real quadratic fields.

Recall that $K = \mathbb{Q}(\sqrt{m})$, $m = \ell_1\ell_2 \cdots \ell_n$, n even, with:

(i) $\ell_1 \equiv 3 \pmod{8}$, $\ell_i \equiv 5 \pmod{8}$ for $i \geq 2$,

$$(ii) \left(\frac{\ell_1}{\ell_2}\right) = -1, \left(\frac{\ell_1}{\ell_j}\right) = 1 \text{ for } j \geq 3,$$

$$(iii) \left(\frac{\ell_i}{\ell_j}\right) = -1 \text{ for } 2 \leq i < j \leq n.$$

Condition (i) implies $m \equiv -1 \pmod{8}$ and K of discriminant $4m$; then $\left(\frac{\ell_i}{\ell_j}\right) = \left(\frac{\ell_j}{\ell_i}\right)$ for all i, j since for $i \neq j$, one of the two primes is congruent to 1 modulo 4. It implies also that the fundamental unit ε is of norm 1 since -1 is not norm in K/\mathbb{Q} ; then, if $\varepsilon = a + b\sqrt{m}$, one has $a^2 + b^2 \equiv 1 \pmod{8}$, whence, either $a = 4a'$ with b odd, or $b = 4b'$ with a odd.

Let \mathfrak{p} be the prime ideal above (2) and let $K_{\mathfrak{p}}$ be the completion of K at \mathfrak{p} ; thus $K_{\mathfrak{p}} = \mathbb{Q}_2(\sqrt{-1})$ proving that 2 is local norm at 2 in K/\mathbb{Q} and that the norm group of local units is equal to $1 + 4\mathbb{Z}_2$.

The Hasse norm residue symbols, of the form $\left(\frac{\ell_i, K/\mathbb{Q}}{\ell_j}\right)$, characterizing the property “ ℓ_i local norm at ℓ_j in K/\mathbb{Q} ”, are given by the quadratic residue symbols $\left(\frac{\ell_i}{\ell_j}\right)$; indeed, ℓ_i norm in $\mathbb{Q}_{\ell_j}(\sqrt{m})/\mathbb{Q}_{\ell_j}$ is equivalent (for $i \neq j$) to ℓ_i square in $\mathbb{Q}_{\ell_j}^{\times}$ since the norm group of local units of $\mathbb{Q}_{\ell_j}(\sqrt{m})$ is of index 2 in $\mu_{\ell_j-1} \oplus (1 + \mathbb{Z}_{\ell_j})$.

We can add the properties:

$$(iv) \left(\frac{2}{\ell_i}\right) = (-1)^{\frac{\ell_i-1}{8}} = -1, \text{ for all } i, \left(\frac{\ell_1}{2}\right) = -1, \left(\frac{\ell_i}{2}\right) = 1 \text{ for } i \geq 2$$

(in the meaning ℓ_i norm (or not) in $\mathbb{Q}_2(\sqrt{-1})/\mathbb{Q}_2$).

$$(v) \left(\frac{\ell_i}{\ell_i}\right) = 1 \text{ for all } i \neq 2 \text{ and } \left(\frac{\ell_2}{\ell_2}\right) = -1, \text{ obtained from the product formula of the Hasse norm residue symbols of fixed } \ell_i \text{ with } n \text{ even.}$$

Put $G := \text{Gal}(K/\mathbb{Q}) =: \langle \sigma \rangle$ and:

$$\Omega_K := \{(s_1, \dots, s_{n+1}) \in G^{n+1}, s_1 \cdots s_{n+1} = 1\} \simeq G^n;$$

let h_K be the map $\mathbb{Q}^{\times} \rightarrow \Omega_K$ defined, by means of the Hasse norm residue symbols, by $h_K(x) = \left(\left(\frac{x, K/\mathbb{Q}}{v}\right)\right)_v$, where v runs through the $n+1$ places ramified in K/\mathbb{Q} , and for which the product formula:

$$\prod_{v \text{ ramified}} \left(\frac{x, K/\mathbb{Q}}{v}\right) = 1$$

holds as soon as (x) is the norm of an ideal in K/\mathbb{Q} , hence local norm at every non-ramified place (i.e., $x \in \{-1, 1\} \cdot \mathbb{Q}^{\times 2}$).

Whence the matrix of symbols, product formula taken on each line:

$$(5.1) \quad \begin{array}{c|cccccccc} & 2 & \ell_1 & \ell_2 & \ell_3 & \dots & \ell_i & \dots & \ell_n \\ \hline 2 & 1 & -1 & -1 & -1 & \dots & -1 & \dots & -1 \\ \hline \ell_1 & -1 & 1 & -1 & 1 & \dots & 1 & \dots & 1 \\ \hline \ell_2 & 1 & -1 & -1 & -1 & \dots & -1 & \dots & -1 \\ \hline \ell_3 & 1 & 1 & -1 & 1 & \dots & -1 & \dots & -1 \\ \hline \vdots & & & & & & & & \\ \hline \ell_i & 1 & 1 & -1 & -1 & \dots & 1 & \dots & -1 \\ \hline \vdots & & & & & & & & \\ \hline \ell_n & 1 & 1 & -1 & -1 & \dots & -1 & \dots & 1 \\ \hline -1 & -1 & -1 & 1 & 1 & \dots & 1 & \dots & 1 \end{array}$$

5.1. **Proof of $\mathcal{H}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$.** This is proven in [DeLi2023], but we can bring more informations and remarks. The first one is given

by the Chevalley–Herbrand formula [Che1933, pp. 402-406] in K/\mathbb{Q} :

$$\#\mathcal{H}_{K,2}^G = \frac{2^{n+1}}{[K:\mathbb{Q}](E_{\mathbb{Q}}:E_{\mathbb{Q}} \cap \mathbf{N}_{K/\mathbb{Q}}(K^{\times}))},$$

since $n+1$ primes ramify, with $E_{\mathbb{Q}} = \{\pm 1\}$; we shall write instead:

$$(5.2) \quad \#\mathcal{H}_{K,2}^G = \frac{\#\Omega_K}{\#h_K(E_{\mathbb{Q}})} = \frac{2^n}{\#h_K(\{\pm 1\})} = 2^{n-1},$$

since $-1 \notin \mathbf{N}_{K/\mathbb{Q}}(K^{\times})$.

Lemma 5.1. *The subgroup $\mathcal{H}_{K,2}^G$ is elementary of 2-rank $n-1$; it is generated by the classes of the prime ideals $\mathfrak{l}_i \mid \ell_i$, $i = 1, \dots, n$, and by the class of $\mathfrak{p} \mid (2)$. There are two independent relations of principality of the form $\mathfrak{p}^{a_0} \prod_{i=1}^n \mathfrak{l}_i^{a_i} = (\alpha)$ between the ramified primes (where the exponents are 0 or 1), an obvious one being $\prod_{i=1}^n \mathfrak{l}_i = (\sqrt{m})$.*

Proof. We have the classical exact sequence, where $E_K = \langle -1, \varepsilon \rangle$ is the group of units of K and where $\mathcal{H}_{K,2}^{\text{ram}}$ is the subgroup of $\mathcal{H}_{K,2}^G$ generated by the classes of the ramified primes:

$$1 \rightarrow \mathcal{H}_{K,2}^{\text{ram}} \rightarrow \mathcal{H}_{K,2}^G \rightarrow \{\pm 1\} \cap \mathbf{N}_{K/\mathbb{Q}}(K^{\times}) / \mathbf{N}_{K/\mathbb{Q}}(E_K) \rightarrow 1,$$

giving here $\mathcal{H}_{K,2}^G = \mathcal{H}_{K,2}^{\text{ram}} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$, the right term being trivial; so, there are exactly two independent relations of principality between the ramified primes.

Then, from [Gra2017^b, § 4.4] generalizing our old papers in “Annales de l’Institut Fourier”, we have for the second element of the filtration:

$$\#\left((\mathcal{H}_{K,2}/\mathcal{H}_{K,2}^G)^G\right) = \frac{\#\Omega_K}{\#h_K(\Lambda)}, \quad \Lambda = \langle -1, 2, \ell_1, \dots, \ell_n \rangle_{\mathbb{Z}};$$

then, $\#h_K(\Lambda) = (\Lambda : \Lambda \cap \mathbf{N}_{K/\mathbb{Q}}(K^{\times})) = 2^n$, by computing norm residue symbols with the Rédei matrix (5.1) which is of rank n ; whence $\mathcal{H}_{K,2} = \mathcal{H}_{K,2}^G = \mathcal{H}_{K,2}^{\text{ram}} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$. \square

A program computing these relations is given Appendix A.2.

5.2. The non-trivial relation for n even. Consider the relations:

$$(5.3) \quad \begin{cases} \mathfrak{p}^{a_0} \prod_{i=1}^n \mathfrak{l}_i^{a_i} =: \mathfrak{p}^{a_0} \prod_{i \in I} \mathfrak{l}_i = (\alpha), \quad \alpha \in K^{\times}, \\ \mathbf{N}_{K/\mathbb{Q}}(\alpha) = s 2^{a_0} \prod_{i \in I} \ell_i, \quad s \in \{\pm 1\}, \end{cases}$$

where I is a subset of $[1, n]$ and $s = \pm 1$; from the trivial relation $(\sqrt{m}) = \prod_{i=1}^n \mathfrak{l}_i$ and $\mathbf{N}_{K/\mathbb{Q}}(\sqrt{m}) = -\prod_{i=1}^n \ell_i$ we deduce $(\alpha\sqrt{m}) = (c) \mathfrak{p}^{a_0} \prod_{i=1}^n \mathfrak{l}_i^{\bar{a}_i}$, where $\bar{a}_i = 1 - a_i$ and $c = \prod_{i \in I} \ell_i$; taking $\bar{\alpha} := \frac{\alpha\sqrt{m}}{c}$, one gets the equivalent complementary relations, where $\bar{I} = [1, n] \setminus I$:

$$(5.4) \quad \begin{cases} \mathfrak{p}^{a_0} \prod_{i \in \bar{I}} \mathfrak{l}_i = (\bar{\alpha}), \\ \mathbf{N}_{K/\mathbb{Q}}(\bar{\alpha}) = -s 2^{a_0} \prod_{i \in \bar{I}} \ell_i. \end{cases}$$

Lemma 5.2. (i) *In the relations (5.3) and (5.4), one has $a_0 = 1$ and the two non-trivial equivalent relations may be written:*

- $\mathfrak{p} \cdot \prod_{j \in J} \mathfrak{l}_j = (\beta)$, $J \subseteq \{2, \dots, n\}$, $\#J \equiv 1 \pmod{2}$.
- $\mathfrak{p} \cdot \mathfrak{l}_1 \prod_{j \in \bar{J}} \mathfrak{l}_j = (\bar{\beta})$, $\bar{J} \subseteq \{2, \dots, n\}$, $\#\bar{J} \equiv 0 \pmod{2}$,

The relation without \mathfrak{l}_1 is $\mathfrak{p} \cdot \mathfrak{l}_2 = (\beta)$.

(ii) These relations are given by the ideals $(\varepsilon + 1)$ and $(\varepsilon - 1)$ for which $\frac{\varepsilon+1}{\varepsilon-1} = C \cdot \sqrt{m}$, where C is an odd rational number.

(iii) There exists a sign s_0 such that $(\varepsilon + s_0) = (c) \mathfrak{p} \cdot \mathfrak{l}_2$.

Proof. (i) Assume that $a_0 = 0$; let $\prod_{j \in J} \ell_j = (\beta)$ be the non-trivial relation such that $J \subseteq [2, n]$ (so, $J \neq \emptyset$). Then $\prod_{j \in J} \ell_j = \pm \mathbf{N}_{K/\mathbb{Q}}(\beta)$; since $\mathbf{N}_{K/\mathbb{Q}}(\beta) \equiv 1 \pmod{4}$ as well as the ℓ_j 's, we get $\prod_{j \in J} \ell_j = \mathbf{N}_{K/\mathbb{Q}}(\beta)$. Consider the symbols $\left(\frac{\prod_{j \in J} \ell_j}{\ell_i} \right)$, $i \in [1, n]$, using (5.1):

- For $i = 2$, one gets $\left(\frac{\prod_{j \in J} \ell_j}{\ell_2} \right) = 1$ if and only if $\#J$ is even (which solves the case $n = 2$ since $J = \{2\}$).
- for $i = 1$, one obtains $\left(\frac{\prod_{j \in J} \ell_j}{\ell_1} \right) = 1$ if and only if $j = 2 \notin J$;
- for any $i \geq 3$, $\left(\frac{\prod_{j \in J} \ell_j}{\ell_i} \right) = 1$ if and only if $j = i \notin J$.

So we obtain $J = \emptyset$ (absurd); whence the relation $\mathfrak{p} \cdot \prod_{j \in J} \ell_j = (\beta)$ for $J \subseteq [2, n]$.

Then $\left(\frac{2 \prod_{j \in J} \ell_j}{\ell_i} \right) = 1$ for all $i \in [1, n]$ since $\mathbf{N}_{K/\mathbb{Q}}(\beta) = -2 \prod_{j \in J} \ell_j$ is not possible (-1 is not local norm at 2); so this is equivalent to $\left(\frac{\prod_{j \in J} \ell_j}{\ell_i} \right) = -1$ for all $i \in [1, n]$. The case $i = 2$ gives $\#J$ odd; the case $i \in J$, when $j = 2 \notin J$, gives a contradiction, as well as $2 \in J$ and $i \notin J$, so that one verifies that only the relation $\mathfrak{p} \cdot \mathfrak{l}_2 = (\beta)$ holds (the $(n+1) \times (n+1)$ matrix (5.1) is of rank n and the sum of the two lines corresponding to 2 and ℓ_2 gives again the norm relation).

(ii) We have $(\varepsilon + s)^{1-\sigma} = s\varepsilon$; so $(\varepsilon + s)$ is a principal invariant ideal necessarily of the form $(c_s) \mathfrak{p} \mathfrak{m}_s$, $c_s \in \mathbb{Q}^\times$, $\mathfrak{p} \mid (2)$, $\mathfrak{m}_s \mid (\sqrt{m})$. Then, $\frac{\varepsilon+1}{\varepsilon-1} = C \cdot \sqrt{m}$ since $\left(\frac{\varepsilon+1}{\varepsilon-1} \cdot \frac{1}{\sqrt{m}} \right)^{1-\sigma} = 1$. \square

Lemma 5.3. *The case b even does not occur.*

Proof. Assume that b is even; then $b = 4b'$; there exists a sign $s = \pm 1$ such that $a + s \equiv 2 \pmod{4}$, so that $\varepsilon + s = a + s + 4b'\sqrt{m}$ and the ideal $(\varepsilon + s) = (a + s + 4b'\sqrt{m})$ is of the form $(2)(a'' + 2b'\sqrt{m})$ with a'' odd, which can not give the non-trivial relations of Lemma 5.2 since $a'' + 2b'\sqrt{m}$ is “odd” (absurd). \square

Lemma 5.4. *In the writing $\varepsilon = a + b\sqrt{m}$, $a = 4a'$ with a' odd.*

Proof. Assume that $a' = 2a''$ and put $\theta_{s'} := \varepsilon + s' = 8a'' + s' + b\sqrt{m}$, $s' \in \{\pm 1\}$. We have, using $mb^2 = a^2 - 1$, $\mathbf{N}_{K/\mathbb{Q}}(\theta_{s'}) = 2(8a''s' + 1)$. From Lemma 5.2 (iii) the non-trivial relation may be written (where c is an odd rational), $(\theta_{s'}) = (c) \mathfrak{p} \mathfrak{l}_2$ giving by taking the norm:

$$8a''s' + 1 = s c^2 \mathfrak{l}_2 \equiv 5s \pmod{8}, \quad s \in \{\pm 1\};$$

which is absurd. \square

5.3. Structure of $\mathcal{T}_{K,2}$ and triviality of $\mathcal{R}_{K,2}$. Let's give the Kummer generators of the maximal sub-extension, of $H_{K,2}^{\text{pr}}/K$, of exponent 2.

Lemma 5.5. *The module $\mathcal{T}_{K,2}$ is of \mathbb{Z}_2 -rank $n - 1$ and the maximal sub-extension of $H_{K,2}^{\text{pr}}/K_\infty$, of exponent 2, is the Kummer extension $K_\infty(\sqrt{\varepsilon}, \sqrt{\ell_1}, \dots, \sqrt{\ell_n})$, under the existence of the two independent relations $\sqrt{\ell_1} \cdots \sqrt{\ell_n} = \sqrt{m} \in K^\times$ and $\sqrt{2}\sqrt{\ell_2} = \alpha\sqrt{\varepsilon}$, $\alpha \in K^\times$.*

Proof. The maximal sub-extension of $H_{K,2}^{\text{pr}}/K$, of exponent 2, is the Kummer extension $K(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{\ell_1}, \dots, \sqrt{\ell_n})$ (all the radicals a in the \sqrt{a} 's are squares of ideals of K and are totally positive; there is no other radicals since $\mathcal{H}_{K,2} = \mathcal{H}_{K,2}^{\text{ram}}$). We utilize the fact that $\sqrt{2} \in K_\infty$, then the relations $\ell_1 \cdots \ell_n = m \in K^{\times 2}$ and $\mathfrak{p} \mid_2 = (\alpha)$, $\alpha \in K^\times$, giving, by squaring, $2\ell_2 = \alpha^2 \eta > 0$, $\eta \in E_K$, $\eta \neq 1$; then we may assume that $\eta = \varepsilon$, whence $2\ell_2 = \alpha^2 \varepsilon$.

Whence the result since $\mathcal{T}_{K,2}$ is a direct factor in $\text{Gal}(H_{K,2}^{\text{pr}}/K)$. \square

Theorem 5.6. *The normalized 2-adic regulator $\mathcal{R}_{K,2}$ is trivial.*

Proof. We have $\varepsilon = a + b\sqrt{m} = 4a' + b\sqrt{m}$ with $a^2 - mb^2 = 1$; then:

$$\varepsilon^2 = a^2 + 2ab\sqrt{m} + mb^2 = 1 + 2mb^2 + 8a'b\sqrt{m}.$$

Recall that a' is odd (Lemma 5.4), that b is odd and $m \equiv -1 \pmod{8}$; thus $\varepsilon^2 \equiv 1 - 2 + 8\sqrt{m} \pmod{16}$; whence $\log_2(\varepsilon) \sim 4$.

From [Gra2018^b, Proposition 5.2], we have, in our context:

$$\#\mathcal{R}_{K,2} \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_2 : \log(\mathbf{N}_{K/\mathbb{Q}}(\mathcal{U}_{K,2})))}{\#\mathcal{W}_{K,2} \cdot \prod_{\mathfrak{p} \mid 2} \mathbf{N}\mathfrak{p}} \cdot \frac{R_K}{\sqrt{D_K}},$$

where $R_K = \log_2(\varepsilon)$ is the usual 2-adic regulator [Wa1997, § 5.5] and D_K the discriminant of K . Since we have $(\mathbb{Z}_2 : \log(\mathbf{N}_{K/\mathbb{Q}}(\mathcal{U}_{K,2}))) = (\mathbb{Z}_2 : \log(1+4\mathbb{Z}_2)) = 4$, $R_K \sim 4$, $\#\mathcal{W}_{K,2} = 2$, $\prod_{\mathfrak{p} \mid 2} \mathbf{N}\mathfrak{p} = 2$, $\sqrt{D_K} \sim 2$, this yields $\#\mathcal{R}_{K,2} \sim 1$.

Formula (1.3) gives $\#\mathcal{T}_{K,2} = 2^{n-1} \times 1 \times 2 = 2^n$, and Corollary 4.6 ends the proof of the Deng-Li conjecture $\delta = 3$. \square

6. THE p -RANK OF $\mathbf{K}_2(\mathbf{Z}_K)$ WHEN $p \geq 5$

We consider a prime number $p \geq 5$ and a cyclic p -extension K/k fulfilling some conditions:

Theorem 6.1. *Let p be a prime ≥ 5 . Let k be a real abelian number field of prime-to- p degree and let K be any cyclic p -extension of k , abelian over \mathbb{Q} . We assume that $K \cap \mathbb{Q}(\mu_p) = \mathbb{Q}$ and that p does not totally split in $K(\mu_p)/K$. Let S_{ta} be the set of primes $\ell \neq p$, ramified in K/k , and let $t_\ell \geq 1$ be the number of prime ideals above ℓ in k/\mathbb{Q} . Then $\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \geq \sum_{\ell \in S_{\text{ta}}} t_\ell$.*

In particular, for all primes $\ell \equiv 1 \pmod{p}$ and for any non-trivial p -extension K/\mathbb{Q} , contained in $\mathbb{Q}(\mu_\ell)/\mathbb{Q}$, we have $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty] \neq 1$.

Proof. Let ζ_p be a generator of μ_p and let $\mathbb{Q}' := \mathbb{Q}(\zeta_p)$, $k' := k(\zeta_p)$ and $K' := K(\zeta_p)$; put $G := \text{Gal}(K'/k')$ and $g := \text{Gal}(K'/K)$. Recall that $\omega : g \rightarrow \mu_{p-1}(\mathbb{Z}_p)$ is the Teichmüller character of g (such that $\zeta_p^s =: \zeta_p^{\omega(s)}$, for all $s \in g$) and that any $\mathbb{Z}_p[g]$ -module X is the sum $\bigoplus_{j=1}^{p-1} X_{\omega^j}$ in an obvious meaning. Since $p \geq 5$, $\omega \neq \omega^{-1}$ and the comparison of $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty] = \mathbf{K}_2(\mathbf{R}_K)[p^\infty]$ with $\mathcal{T}_{K,p}$ made in [GJ1989] does not hold.

The primes of $S_p(K)$ being not totally split in K'/K , the ω^{-1} -component of the $\mathbb{Z}_p[G]$ -module $\langle S_p(K') \rangle \otimes \mathbb{Z}_p$ generated by $S_p(K')$ above $S_p(K)$ is trivial; thus, formula (1.4) becomes:

$$\text{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) = \text{rk}_p((\mathcal{H}_{K',p})_{\omega^{-1}}).$$

We note that a prime number ℓ , ramified in K/\mathbb{Q} with ramification index divisible by p , fulfills the condition $\ell \equiv 1 \pmod{p}$ (for generalizations to the non-Galois case, see [Gra2005, II, §(d)]); whence ℓ totally splits in \mathbb{Q}'/\mathbb{Q} and, for $\mathfrak{l} \mid \ell$ in k , the $p-1$ prime ideals \mathfrak{L} above \mathfrak{l} in k' ramify in K'/k' with the same ramification index; this gives $(p-1)t_\ell$ ramified ideals in K'/k' , whence $\#S_{\text{ta}}(k') = (p-1) \sum_{\ell \in S_{\text{ta}}} t_\ell$.

Let $G_v \subseteq G$ be the inertia group in K'/k' of a ramified place v of k' (so v is a prime ideal above $\ell \neq p$ or a prime ideal above p , but the ω^{-1} -component of $\langle S_p(k') \rangle \otimes \mathbb{Z}_p$ is trivial), and put $S(k') = S_{\text{ta}}(k') \cup S_p(k')$. Let $E_{k'}$ be the group of units of k' :

The Chevalley–Herbrand formula in K'/k' writes:

$$\#\mathcal{H}_{K',p}^G = \#\mathcal{H}_{k',p}^{\text{nr}} \times \frac{\#\Omega_{k'}}{\#h_{k'}(E_{k'})},$$

where $\mathcal{H}_{k',p}^{\text{nr}} \subseteq \mathcal{H}_{k',p}$ corresponds to $\text{Gal}(H_{k'}^{\text{nr}}/K' \cap H_{k'}^{\text{nr}})$ and $\Omega_{k'} = \left\{ (s_v)_v \in \bigoplus_{v \in S(k')} G_v, \prod_v s_v = 1 \right\}$, $h_{k'} : E_{k'} \rightarrow \Omega_{k'}$ being the map defined by the family of Hasse norm residue symbols $\left(\left(\frac{x, K'/k'}{v} \right) \right)_v$ which fulfill the “product formula” on units. This formula is associated to the following exact sequences:

$$(6.1) \quad \begin{cases} 1 \rightarrow \mathbf{J}_{K'/k'}(\mathcal{H}_{k',p}) \cdot \mathcal{H}_{K',p}^{\text{ram}} \longrightarrow \mathcal{H}_{K',p}^G \\ \quad \longrightarrow E_{k'} \cap \mathbf{N}_{K'/k'}(K'^{\times}) / \mathbf{N}_{K'/k'}(E_{k'}) \rightarrow 1, \\ 1 \rightarrow E_{k'}/E_{k'} \cap \mathbf{N}_{K'/k'}(K'^{\times}) \\ \quad \longrightarrow \Omega_{k'} \longrightarrow \text{Gal}(H_{K'/k'}^{\text{gen}}/K'H_{k'}^{\text{nr}}) \rightarrow 1, \end{cases}$$

where $\mathbf{J}_{K'/k'}$ is the extension of classes, $\mathcal{H}_{K',p}^{\text{ram}}$ the subgroup of $\mathcal{H}_{K',p}$ generated by the ramified primes, $H_{K'/k'}^{\text{gen}} \subseteq H_{K'}^{\text{nr}}$ the genus field, fixed by $\mathcal{H}_{K',p}^{1-\sigma}$, where σ generates G [Gra2005, Proposition IV.4.5].

The link between the two aspects (fixed points and genus exact sequences) is given by $\text{Gal}(H_{K'/k'}^{\text{gen}}/K') \simeq \mathcal{H}_{K',p} / \mathcal{H}_{K',p}^{1-\sigma}$ and the exact sequence $1 \rightarrow \mathcal{H}_{K',p}^G \rightarrow \mathcal{H}_{K',p} \rightarrow \mathcal{H}_{K',p}^{1-\sigma} \rightarrow 1$.

More precisely, we need the ω^{-1} -component of $\mathcal{H}_{K',p}^G$ and of the terms of the exact sequences. For $\mathcal{H}_{K',p}^G$, this computation has been done in Jaulent’s Thesis [Jau1986, Chapitre III] and reproduced in [Gra2023, §2.2, Theorem 2.1 (ii)]; the difficulty comes from the fact that $\mathcal{H}_{k',p}$ is not necessarily isomorphic to a sub-module of $\mathcal{H}_{K',p}$ (indeed, only $\mathbf{J}_{K'/k'}(\mathcal{H}_{k',p})$ makes sense as sub-module of $\mathcal{H}_{K',p}$ and $\mathbf{J}_{K'/k'}$ is not necessarily injective). It writes under our context:

$$(6.2) \quad \#(\mathcal{H}_{K',p}^G)_{\omega^{-1}} = \#(\mathcal{H}_{k',p}^{\text{nr}})_{\omega^{-1}} \times \frac{\#(\Omega_{k'})_{\omega^{-1}}}{\#(h_{k'}(E_{k'}))_{\omega^{-1}}}.$$

Since $E_{k'} = E_{k'+} \oplus \mu_p$ (up to a prime-to- p index), where the subgroup $E_{k'+}$ of real units of k' is of even character, and since μ_p is of character $\omega \neq \omega^{-1}$, the ω^{-1} -component of $h_{k'}(E_{k'})$ is trivial and exact sequences (6.1) and formula (6.2) reduce to:

$$(6.3) \quad \begin{cases} (\mathcal{H}_{K',p}^G)_{\omega^{-1}} \simeq (\mathbf{J}_{K'/k'}(\mathcal{H}_{k',p}) \cdot \mathcal{H}_{K',p}^{\text{ram}})_{\omega^{-1}}, \\ (\Omega_{k'})_{\omega^{-1}} \simeq (\text{Gal}(H_{K'/k'}^{\text{gen}}/K'H_{k'}^{\text{nr}}))_{\omega^{-1}}, \\ \#(\mathcal{H}_{K',p}^G)_{\omega^{-1}} = \#(\mathcal{H}_{k',p}^{\text{nr}})_{\omega^{-1}} \times \#(\Omega_{k'})_{\omega^{-1}}. \end{cases}$$

Whence:

$$(6.4) \quad \begin{cases} \mathrm{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) = \mathrm{rk}_p((\mathcal{H}_{K',p})_{\omega^{-1}}) \\ \geq \mathrm{rk}_p(\mathcal{H}_{K',p}/\mathcal{H}_{K',p}^{1-\sigma})_{\omega^{-1}} \geq \mathrm{rk}_p(\Omega_{k'})_{\omega^{-1}}. \end{cases}$$

Let $G' := \langle G_v \rangle_v \simeq \mathrm{Gal}(K'/K' \cap H_{k'}^{\mathrm{nr}})$; the product formula may be interpreted by means of the exact sequence:

$$1 \rightarrow \Omega_{k'} \longrightarrow \bigoplus_{v \in S(k')} G_v \longrightarrow G' \subseteq G \rightarrow 1,$$

in which the character of G' and that of $\bigoplus_{v \in S_p(k')} G_v$ is the unit character. Whence $(\Omega_{k'})_{\omega^{-1}} \simeq \bigoplus_{v \in S_{\mathrm{ta}}(k')} G_v$.

For $\ell \in S_{\mathrm{ta}}$, let p^{e_ℓ} , $e_\ell \geq 1$, be the ramification index of ℓ in K'/k' . The inertia groups G_v , for $v \mid \ell$ in k' , does not depend on v and by abuse may be denoted G_ℓ . Since each of the t_ℓ primes $\mathfrak{l} \mid \ell$ in k , totally splits in k'/k into $p-1$ primes $\mathfrak{L} \mid \mathfrak{l}$ of k' , one may write:

$$\bigoplus_{\mathfrak{L}} G_{\mathfrak{L}} = \bigoplus_{\ell \in S_{\mathrm{ta}}} \bigoplus_{\mathfrak{l} \mid \ell} \bigoplus_{\mathfrak{L} \mid \mathfrak{l}} G_\ell \simeq \bigoplus_{\ell \in S_{\mathrm{ta}}} \left((\mathbb{Z}/p^{e_\ell}\mathbb{Z})^{p-1} \right)^{t_\ell},$$

in which, each ℓ -component is, as Galois module, isomorphic to t_ℓ copies of the regular representation $\bigoplus_{j=1}^{p-1} (\mathbb{Z}[G_\ell]/p^{e_\ell}\mathbb{Z}[G_\ell])_{\omega^j}$ whose ω^{-1} -component is of order p^{e_ℓ} . Whence $(\Omega_{k'})_{\omega^{-1}} \simeq \bigoplus_{\ell \in S_{\mathrm{ta}}} (\mathbb{Z}/p^{e_\ell}\mathbb{Z})^{t_\ell}$.

Thus, one obtains, from (6.4), $\mathrm{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \geq \sum_{\ell \in S_{\mathrm{ta}}} t_\ell$. \square

Corollary 6.2. *We obtain the following consequences:*

- (i) $\#(\mathcal{H}_{k',p}^{\mathrm{nr}})_{\omega^{-1}} \times \#(\Omega_{k'})_{\omega^{-1}} = \#(\mathbf{J}_{K'/k'}(\mathcal{H}_{k',p}) \cdot \mathcal{H}_{K',p}^{\mathrm{ram}})_{\omega^{-1}}$.
- (ii) If $\mathbf{K}_2(\mathbf{Z}_k)[p^\infty] = 1$, then $(\mathcal{H}_{K',p}^{\mathrm{ram}})_{\omega^{-1}} \simeq \bigoplus_{\ell \in S_{\mathrm{ta}}} (\mathbb{Z}/p^{e_\ell}\mathbb{Z})^{t_\ell}$.
- (iii) If $k = \mathbb{Q}$, then:
 - $\mathrm{rk}_p(\mathbf{K}_2(\mathbf{Z}_K)) \geq \#S_{\mathrm{ta}}$;
 - $(\mathcal{H}_{K',p}^{\mathrm{ram}})_{\omega^{-1}} \simeq \bigoplus_{\ell \in S_{\mathrm{ta}}} \mathbb{Z}/p^{e_\ell}\mathbb{Z}$.

Proof. Cases (i), (ii) come from the fact that $\mathbf{K}_2(\mathbf{Z}_k)[p^\infty] = 1$ is equivalent to $(\mathcal{H}_{k',p})_{\omega^{-1}} = 1$, then to (6.3). If $k = \mathbb{Q}$, $\sum_{\ell \in S_{\mathrm{ta}}} t_\ell = \#S_{\mathrm{ta}}$; then $(\mathcal{H}_{\mathbb{Q},p})_{\omega^{-1}} = 1$ from Ribet's reciprocal of Herbrand's criterion, since the Bernoulli number $B_2 = \frac{1}{6}$ is prime to p [Rib2008, Theorem 5, p. 45]. Whence (iii). \square

Remarks 6.3. (i) For $p = 3$, $\omega^{-1} = \omega$, so that $(\mathcal{H}_{K',p}^G)_{\omega^{-1}}$ depends on $\frac{\#(\Omega_{\mathbb{Q}'})_{\omega^{-1}}}{\#h_{\mathbb{Q}'}(\mu_3)}$ which may be trivial, when $S_{\mathrm{ta}} = \{\ell\}$ and $[K : \mathbb{Q}] = 3$.

(ii) Replacing $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ by $\mathcal{T}_{K,p}$ (whatever the prime $p \geq 3$) and ω^{-1} by ω , $h_{\mathbb{Q}'}(\mu_p)$ may be non-trivial, being of character ω . So, if K/\mathbb{Q} is of prime conductor ℓ , $\mathcal{H}_{K,p} = 1$ and we may have $\mathcal{T}_{K,p} = 1$ but $\mathbf{K}_2(\mathbf{Z}_K)[p^\infty] \neq 1$.

(iii) For a table of numerical examples, see Appendix D.1 for the computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$, $p \geq 3$, and Appendix D.2 for the computation of $\mathrm{rk}_5(\mathbf{K}_2(\mathbf{Z}_K)[5^\infty])$. Numerical tables are given in [Br2005] for cyclic cubic fields; more on computations of p -ranks are given in Qin's papers [Qin1995, Qin2005, Qin2010].

7. CONCLUSION

These explicit calculations lead to a confirmation of the properties of the \mathbf{L}_p -functions when s varies in \mathbb{Z}_p and takes the values $s \in \{-1, 1\}$, giving annihilation theorems, orders of isotypic components of classical p -invariants.

Other invariants than class groups, torsion groups and regular kernels exist, as for instance the Jaulent logarithmic class group $\widetilde{\mathcal{H}}_{K,p}$ ² [Jau1994^b, Jau2021, Jau2023] linked to the wild Hilbert kernel [Jau1994^a, Théorème 9] and in relation with Greenberg’s conjecture [Jau2019, Jau2020] (see Appendices B.2, C.2 for some computations), and invariants of the higher K -theory (giving conjectural expressions suggested in [Gra1998, Section 12, Conjecture 12.2] and some results of [JaMi2006]).

An important problem is to find the structure of these invariants, it being understood that they fulfill, once genus part has been taken into consideration, standard densities in the spirit of the Cohen–Lenstra–Martinet–Malle distributions. A specific technique, of genus theory type, does exist in cyclic p -extensions, especially for p -class groups with determining the canonical filtration by means of a natural algorithm generalizing Chevalley–Herbrand fixed points formula (large bibliography, synthesized in [Gra2017^b]).

A main question being to find such algorithms for the other p -invariants, which unfortunately does not exist to our knowledge; the case of the most important groups $\mathcal{T}_{K,p}$ being very interesting; indeed, class field theory (by means of computation of suitable ray class groups) gives the group structure, but this computational aspect does not give a method to determine the filtration in cyclic p -extensions; only the fixed points formula, is known [Gra2005, Theorem IV.3.3].

In a cohomological viewpoint, $\mathcal{T}_{K,p}$ is closely related to the Tate–Chafarevich group $\text{III}_{K,p}^2 := \text{Ker}[\text{H}^2(\mathcal{G}_{K,S}, \mathbb{F}_p) \rightarrow \bigoplus_{\mathfrak{p}|p} \text{H}^2(\mathcal{G}_{K_{\mathfrak{p}}}, \mathbb{F}_p)]$, and that $\mathcal{T}_{K,p} \simeq \text{H}^2(\mathcal{G}_{K,S}, \mathbb{Z}_p)^*$ [Ng1986, Théorème 1.1], where $\mathcal{G}_{K,S}$ is the Galois group of the maximal p -ramified pro- p -extension of K and $\mathcal{G}_{K_{\mathfrak{p}}}$ the local analogue over $K_{\mathfrak{p}}$; generalizations of the same kind are done in the literature, as in [Kol2002, Theorem 0.4, 0.7, 0.11, ...], [AAM2021, Theorem 3.5, Example 4.3], with analogous cohomological framework, without any workable algorithm.

A major difficulty is to interpret, in terms of ideals, global units and so on, such cohomology groups; the case of $\mathcal{T}_{K,p} \simeq \text{H}^2(\mathcal{G}_{K,S}, \mathbb{Z}_p)^*$ is edifying since $\mathcal{G}_{K,S}$ is inaccessible, while $\mathcal{T}_{K,p}$ depends on the p -class group and on the normalized p -adic regulator, of K . Of course, cohomology gives important and non-obvious relations but, in the previous example, it is the knowledge of $\mathcal{T}_{K,p}$ which yields important information on $\mathcal{G}_{K,S}$ (e.g., case of the simplest structure giving the notion of p -rational fields issued from [Gra1986^a, Mov1988] and largely developed in a lot of papers; see [Gra2019^a, Appendix] for a survey about this notion and general abelian p -ramification theory).

² Also denoted, most often in the literature, $\widetilde{\mathcal{C}}_{K,p}$, $\widetilde{\mathcal{T}}_{K,p}$ as quotient of $\mathcal{T}_{K,p}$; but it looks more like the p -class group $\mathcal{H}_{K,p}$ and there exists a formula of the form $\#\mathcal{T}_{K,p} = \widetilde{\mathcal{H}}_{K,p} \cdot \widetilde{\mathcal{R}}_{K,p} \cdot \widetilde{\mathcal{W}}_{K,p}$, for a suitable “logarithmic regulator”, see [Jau2019, Schema § 2.3]. Moreover it may capitulates in real p -extensions of K (the key for Greenberg’s conjecture), while $\mathcal{T}_{K,p}$ never capitulates under Leopoldt’s conjecture.

Maybe the link we made in [Gra2005, § III.2, Theorem III.2.6, Theorem III.3.3] and [GJ1989, Appendice], using logarithms of ideals for an effectiveness of the arithmetic, may be generalized.

As a final remark, we point out the fact that these analytic genus theory aspects of p -adic pseudo-measures are stated in [Gra1986^b] for any totally real base field k replacing \mathbb{Q} (under Leopoldt's conjecture for p , to get the important result of deployment [Gra2005, Theorem III.4.1.5]), and allowing similar technics with genus fields from Deligne–Ribet pseudo-measures; these results being completed with for instance the article by Maire [Mai2018] on the class field theory aspects and governing fields about the *existence* of prescribed p -cyclic ramified extensions for which an analog of Theorem 4.4 should apply.

Acknowledgements. I thank with pleasure Litong Deng and Yongxiong Li for interesting exchanges about their paper, giving to me the idea of taking again into account my old papers on genus theory of p -adic pseudo-measures, hoping that arithmetic invariants of new families of real abelian fields can be elucidated via this method.

APPENDIX A. DATA RELATING TO THE DENG–LI FAMILY

This section is only concerned by the family of quadratic fields described and studied in Section 5; we illustrate and confirm various numerical aspects.

A.1. Computation of $\mathcal{H}_{K,p}$, $\mathcal{I}_{K,p}$ and $\widetilde{\mathcal{H}}_{K,p}$. We compute the three fundamental invariants for this family, it being understood that the structure of $\mathcal{H}_{K,p}$ and complex analytic properties are given in [DeLi2023] in order to study $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$.

A.1.1. General program for $\mathcal{H}_{K,p}$, $\mathcal{I}_{K,p}$. It may be convenient to recall the program computing the structures of $\mathcal{H}_{K,p}$ and $\mathcal{I}_{K,p}$ for a number field K , for any $p \in [\mathfrak{bp}, \mathfrak{Bp}]$, as soon as a defining polynomial $P \in \mathbb{Z}[X]$ (monic, irreducible) is given (in P); for the computation of $\mathcal{I}_{K,p}$, the parameter nu must be chosen such that p^{nu} be larger than the exponent of the result T [Gra2017^a, § 2.1]; the number r is the number $r_2 + 1$ of independent \mathbb{Z}_p -extensions of K :

```
{P=x^3-7*x+1;bp=2;Bp=5*10^5;K=bnfinit(P,1);r=K.sign[2]+1;
print("P=",P," H=",K.cyc);forprime(p=bp,Bp,nu=6;
Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];
R=0;for(k=1,e-r,c=HKn[e-k+1];v=valuation(c,p);if(v>0,R=R+1;
listinsert(T,p^v,1));if(R>0,print("p=",p," rk(T)=",R," T=",T)))}
```

P=x ³ -7*x+1	H=[]	P=x ⁸ -8*x+1	H=[4]		
p=7	rk(T)=1	T=[7]	p=2	rk(T)=1	T=[4]
p=701	rk(T)=1	T=[701]	p=3	rk(T)=1	T=[3]

A.1.2. Case of the Deng–Li family. The following program verifies the structure of $\mathcal{H}_{K,2}$ and $\mathcal{I}_{K,2}$ for the Deng–Li family of quadratic fields, for $n = 4$:

```
{L=List
([7215,26455,77415,119535,142935,153735,166335,171015,196359,
226655,241215,243295,257335,283855,311415,315055,420135,430495,
447135,473415,475215,490295,504295,545415,550615,552695,553335,
563695,568815,592215,603655,606615,633399,657735,665223,673215,
685815,687895,727935,751335,755495,757055,790495,798135,803751,
807455,818935,833199,849615,878415,884455,886015,886335,896415,
905255,911495,934935,961935,973655,981695,990015]);
```

```
p=2;nu=9;for(j=1,61,P=x^2-L[j];K=bnfinit(P,1);HK=K.cyc;
Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];
for(k=1,e-1,c=HKn[e-k+1];v=valuation(c,p);if(v>0,listput(T,p^v)));
print("P=",P," T=",T," H=",HK))}
```

```
P=x^2-7215 T=[2,2,4] H=[2,2,2] P=x^2-981695 T=[2,2,4] H=[10,2,2]
P=x^2-26455 T=[2,2,4] H=[2,2,2] P=x^2-990015 T=[2,2,4] H=[18,2,2]
```

From (1.6), the 2-ranks of $\mathbf{R}_2(\mathbf{Z}_K)$ and $\mathcal{T}_{K,2}$ are equal to 3, this implies, since $\mathbf{K}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}/2^3\mathbb{Z}$, that:

$$\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/2^2\mathbb{Z} \simeq \mathcal{T}_{K,2}.$$

More generally, for the Deng–Li family, we have:

$$\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \simeq \mathcal{T}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-2} \times \mathbb{Z}/2^2\mathbb{Z}, \quad \mathcal{H}_{K,2} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}.$$

From this, we verify that $\mathcal{W}_{K,2} \simeq \mathbb{Z}/2\mathbb{Z}$, which is coherent with the formula $\#\mathcal{T}_{K,2} = \#\mathcal{H}_{K,2} \times \#\mathcal{R}_{K,2} \times \#\mathcal{W}_{K,2}$ since the normalized 2-adic regulator is trivial from Theorem 5.6.

A.1.3. *Logarithmic class group.* The computation of $\widetilde{\mathcal{H}}_{K,2}$ (instruction `CLog = bnflog(K, 2)`) for the Deng–Li family, gives always the structure $(\mathbb{Z}/2\mathbb{Z})^2$ for $n = 4$; but we do not know if this invariant is given by the $\mathbf{L}_2(s, \chi)$'s at some $s \in \mathbb{Z}_2$ (see § B.2 for the computation of $\widetilde{\mathcal{H}}_{K,2}$ for quadratic fields and § C.2 for the cubic case):

```
P=x^2-7215 CLog=[[2,2],[],[2,2]] T=[2,2,4] H=[2,2,2]
P=x^2-26455 CLog=[[2,2],[],[2,2]] T=[2,2,4] H=[2,2,2]
(...)
P=x^2-981695 CLog=[[2,2],[],[2,2]] T=[2,2,4] H=[10,2,2]
P=x^2-990015 CLog=[[2,2],[],[2,2]] T=[2,2,4] H=[18,2,2]
```

Moreover, the third component of `bnflog(K, 2)` gives the S_2 -class group $\mathcal{H}_{K,2}/\text{cl}(\mathfrak{p}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where $\mathfrak{p} \mid 2$ is ramified; this result confirms that necessarily \mathfrak{p} is non-principal of order 2.

A.2. **The canonical non-trivial relation.** The program gives the non-trivial relation whose writing does not contain \mathfrak{l}_1 (with $\ell_1 \equiv 3 \pmod{8}$), then that obtained multiplying by (\sqrt{m}) . We have proven that this relation is always $\mathfrak{p}\mathfrak{l}_2 = (\alpha)$ (Lemma 5.2). The list L gives the exponents a_0, a_1, a_2, \dots , of ideals $\mathfrak{p}, \mathfrak{l}_1, \mathfrak{l}_2, \dots$, in this order especially for the particular primes $\mathfrak{l}_1, \mathfrak{l}_2$ (which is not so immediate since the PARI factorization of m gives primes in the ascending order). The generator α of the relation of principality is $\alpha = u + v\sqrt{m}$; for checking, its norm N and the sign S of N are computed:

```
{m=List
([7215,26455,77415,119535,142935,153735,166335,171015,196359,
226655,241215,243295,257335,283855,311415,315055,420135,430495,
447135,473415,475215,490295,504295,545415,550615,552695,553335,
563695,568815,592215,603655,606615,633399,657735,665223,673215,
685815,687895,727935,751335,755495,757055,790495,798135,803751,
807455,818935,833199,849615,878415,884455,886015,886335,896415,
905255,911495,934935,961935,973655,981695,990015]);p=2;
for(j=1,61,M=m[j];D=quaddisc(M);r=omega(D);L0=List;
for(i=1,r,listput(L0,0));P=x^2-M;K=bnfinit(P,1);print();Lel=List;
Div=component(factor(D),1);listput(Lel,2,1);h=3;for(i=2,r,c=Div[i];
if(Mod(c,4)==-1,e11=c;listput(Lel,c,2)));for(i=2,r,c=Div[i];
if(c!=e11 & kronecker(c,e11)==-1,e12=c;listput(Lel,c,3)));
for(i=2,r,c=Div[i];if(c!=e12 & Mod(c,4)==1,h=h+1;listput(Lel,c,h)));
print("M=",M," D=",Lel);for(k=1,2^r-1,B=binary(k);t=#B;
LB=L0;for(i=1,t,listput(LB,B[i],r-t+i));F=idealfactor(K,D);
Fel=List;CFel=component(F,1);listput(Fel,CFel[1],1);h=3;
```

```

for(i=2,r,c=CFel[i][1];if(Mod(c,4)==-1,e11=c;
listput(Fel,CFel[i],2));
for(i=2,r,c=CFel[i][1];if(c!=e11 & kronecker(c,e11)==-1,e12=c;
listput(Fel,CFel[i],3));
for(i=2,r,c=CFel[i][1];if(c!=e12 & Mod(c,4)==1,h=h+1;
listput(Fel,CFel[i],h));
A=1;for(j=1,r,e=LB[j];if(e==0,next);A=idealmul(K,A,Fel[j]);
Q=bnfisprincipal(K,A);u=Q[2][1];v=Q[2][2];
if(Q[1]==0 & u!=0,print("L=",LB);N=u^2-M*v^2;S=sign(N);
print(component(factor(abs(N)),1));
print("u=",u," v=",v," S=",sign(S))))}

```

Let's give some excerpts showing all possible cases:

M=7215 D=[2,3,5,13,37]	M=26455 D=[2,11,13,5,37]
L=[1,0,1,0,0]	L=[1,0,1,0,0]
[2,5]	[2,13]
u=-85 v=1 S=1	u=86654841 v=-532769 S=1
L=[1,1,0,1,1]	L=[1,1,0,1,1]
[2,3,13,37]	[2,5,11,37]
u=1443 v=17 S=-1	u=1084184915 v=-6665757 S=-1
M=504295 D=[2,11,173,5,53]	M=665223 D=[2,3,461,13,37]
L=[1,0,1,0,0]	L=[1,0,1,0,0]
[2,173]	[2,461]
u=7970980451749 v=11224562253 S=1	u=10603 v=13 S=1
L=[1,1,0,1,1]	L=[1,1,0,1,1]
[-1,2,5,11,53]	[2,3,13,37]
u=-32719598967495 v=46075031513 S=-1	u=18759 v=23 S=-1

APPENDIX B. QUADRATIC FIELDS

B.1. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$, $\mathcal{I}_{K,2}$ and \mathbf{C} . The following program computes, instead, $\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty]$ for any real quadratic field (so f is m or $4m$) and $\mathcal{I}_{K,2}$ for checking of Theorem 4.4 (cases (ii)₂ and (iii)₂) and property (b) about equality v.s. inequality, for all $s \in \mathbb{Z}_p$. In that cases, $\epsilon = 1$, \mathbf{D} is the number of odd ramified primes, and $\mathbf{C} = \mathbf{D}$ (resp. $\mathbf{C} = \mathbf{D} - 1$) if $\theta^{-1}\chi(2) = 1$ (resp. $\theta^{-1}\chi(2) \neq 1$).

Recall that nu and n must be chosen large enough (below we take $n = 10$ for $m > 100$, $n = 13$ for $m > 10^3$).

We do not write the trivial cases $\mathcal{I}_{K,2} = \mathbf{R}_2(\mathbf{Z}_K)[2^\infty] = 1$.

```

{p=2;q=4;nu=10;n=6;for(m=5,2000,if(core(m)!=m,next);f=quaddisc(m);
P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^nu);H=Kpn.cyc;T=List;
e=matsize(H)[2];valT=0;for(k=1,e-1,h=H[e-k+1];v=valuation(h,p);
if(v>0,valT=valT+v;listput(T,p^v));f=quaddisc(m);fn=q*p^n*f;c=1;
while(gcd(c,2*m)!=1 || kronecker(f,c)!=-1,c=c+1);S=0;
forstep(a=1,fn/2,2,if(gcd(a,m)!=1,next);aa=lift(Mod(a/c,fn));
la=(aa*c-a)/fn;eps=kronecker(m,a);S=S+aa*eps*(la+(1-c)/2));
valR=valuation(S,2)-1;D=omega(m);if(Mod(m,2)==0,D=D-1);
C=D;if(Mod(m,8)!=-1,C=D-1);
if(valT==C,print("m=",m," c=",c," T=",T," v_2(T)=",valT,
" val(R_2Z)=",valR," C=",C," Equality"));
if(valT>C,print("m=",m," c=",c," T=",T," v_2(T)=",valT,
" v_2(R_2Z)=",valR," C=",C," Inequality"))}

```

m=7	c=5	T=[4]	v_2(T)=2	v_2(R_2Z)=2	C=1	Inequality
m=14	c=3	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=0	Inequality
m=15	c=13	T=[4]	v_2(T)=2	v_2(R_2Z)=2	C=2	Equality
m=17	c=3	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=0	Inequality
m=21	c=11	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=1	Equality
m=23	c=3	T=[4]	v_2(T)=2	v_2(R_2Z)=2	C=1	Inequality
m=30	c=11	T=[2]	v_2(T)=1	v_2(R_2Z)=1	C=1	Equality

m=31	c=7	T=[8]	$v_2(T)=3$	$v_2(R_{2Z})=3$	C=1	Inequality
m=33	c=5	T=[2]	$v_2(T)=1$	$v_2(R_{2Z})=1$	C=1	Equality
m=34	c=7	T=[2]	$v_2(T)=1$	$v_2(R_{2Z})=1$	C=0	Inequality
m=35	c=3	T=[2]	$v_2(T)=1$	$v_2(R_{2Z})=1$	C=1	Equality
m=39	c=11	T=[4]	$v_2(T)=2$	$v_2(R_{2Z})=2$	C=2	Equality
m=41	c=3	T=[16]	$v_2(T)=4$	$v_2(R_{2Z})=3$	C=0	Inequality
(...)						
m=1001	c=3	T=[2,2]	$v_2(T)=2$	$v_2(R_{2Z})=2$	C=2	Equality
m=1002	c=5	T=[16]	$v_2(T)=4$	$v_2(R_{2Z})=5$	C=1	Inequality
m=1003	c=5	T=[4]	$v_2(T)=2$	$v_2(R_{2Z})=2$	C=1	Inequality
m=1005	c=29	T=[2,8]	$v_2(T)=4$	$v_2(R_{2Z})=5$	C=2	Inequality
m=1006	c=7	T=[2]	$v_2(T)=1$	$v_2(R_{2Z})=1$	C=0	Inequality
m=1007	c=3	T=[4]	$v_2(T)=2$	$v_2(R_{2Z})=2$	C=2	Equality
m=1009	c=11	T=[2]	$v_2(T)=1$	$v_2(R_{2Z})=1$	C=0	Inequality
m=1010	c=3	T=[2]	$v_2(T)=1$	$v_2(R_{2Z})=1$	C=1	Equality
m=1011	c=7	T=[4]	$v_2(T)=2$	$v_2(R_{2Z})=2$	C=1	Inequality
m=1015	c=17	T=[2,4]	$v_2(T)=3$	$v_2(R_{2Z})=3$	C=3	Equality
m=1022	c=3	T=[2,16]	$v_2(T)=5$	$v_2(R_{2Z})=9$	C=1	Inequality
m=1023	c=5	T=[2,64]	$v_2(T)=7$	$v_2(R_{2Z})=7$	C=3	Inequality

The equality $v_2(\#\mathcal{T}_{K,2}) = \mathbf{C}$ often occur and when $v_2(\#\mathcal{T}_{K,2}) > \mathbf{C}$, then $v_2(\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty]) > \mathbf{C}$, with some cases of distinct valuations ($m = 41, 65, 66, 114, \dots$).

Numerical examples given in [DeLi2023, Table], yield equalities with $\mathbf{C} = \mathbf{D} + v_2(4s) - 2\epsilon = 4$ for $s = -1$, since 2 splits in $M = \mathbb{Q}(\sqrt{-m})$ (Theorem 4.4 (iii)₂ and Corollary 4.6).

B.2. Computation of $\widetilde{\mathcal{H}}_{K,2}$. We give a program of computation of the logarithmic class group to compare with the invariants $\mathcal{H}_{K,2}$, $\mathcal{T}_{K,2}$. There are many structures, but the relation between $\mathcal{H}_{K,2}$ and $\mathcal{T}_{K,2}$ being well-known, we give only an excerpt of interesting examples concerning $\widetilde{\mathcal{H}}_{K,2}$. For $p = 2$, the first component of $\text{bnflog}(K, 2)$ gives $\widetilde{\mathcal{H}}_{K,2}^0$ of index 1 or 2 in $\widetilde{\mathcal{H}}_{K,2}$ [BeJa2016, § 3, Remark 3.1]:

```
{bf=10^4;Bf=10^4+10^3;for(m=bf,Bf,if(core(m)!=m,next);PK=x^2-m;
F=component(factor(m),1);nu=12;K=bnfinit(PK,1);HK=K.clgp;
Kpn=bnrinit(K,2^nu);HKn=Kpn.cyc;T=List;w=0;e=matsize(HKn)[2];
for(k=1,e-1,CK=HKn[e-k+1];v=valuation(CK,2);if(v>0,w=w+v;
listput(T,2^v));K=bnfinit(PK,1);HK=K.clgp;Kpn=bnrinit(K,2^nu);
HKn=Kpn.cyc;T=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,CK=HKn[k+1];
v=valuation(CK,2);if(v>0,w=w+v;listput(T,2^v));CLK=bnflog(K,2);
print("m=",F," P=",PK," H=",HK[2]," T=",T," CLog=",CLK)}
```

```
m=[3,5,23,29] x^2-10005 H=[2,2] T=[4,2,2] CLog=[[2,2],[2,2]]
m=[3,47,71] x^2-10011 H=[4,2] T=[4,2] CLog=[[4],[4]]
m=[3,5,11,61] x^2-10065 H=[2,2] T=[16,2,2] CLog=[[4],[2],[2]]
m=[2,3,23,73] x^2-10074 H=[8,2] T=[32,4] CLog=[[8],[8]]
m=[2,3,19,89] x^2-10146 H=[4,2] T=[8,2] CLog=[[4],[4,2]]
m=[7,31,47] x^2-10199 H=[4,2] T=[4,4,2] CLog=[[4,2],[4,2]]
m=[10337] x^2-10337 H=[] T=[8] CLog=[[4],[4]]
m=[2,5297] x^2-10594 H=[8] T=[32] CLog=[[4],[8]]
m=[3,11,17,19] x^2-10659 H=[2,2,2] T=[2,2,2] CLog=[[2,2,2],[2,2,2]]
m=[17,641] x^2-10897 H=[2] T=[64,2] CLog=[[2,2],[2],[2]]
m=[3,5,17,47] x^2-11985 H=[2,2] T=[1024,2,2]
CLog=[[128],[64],[2]]
m=[12161] x^2-12161 H=[11] T=[512] CLog=[[16],[16]]
m=[2,3,13,167] x^2-13026 H=[2,2] T=[128,2] CLog=[[2],[2]]
```

B.3. Computation of $\mathcal{T}_{K,3}$ and $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty]$. For $p = 3$, genus theory does not apply; we only give the valuation of $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty]$ and the structure of $\mathcal{T}_{K,3}$; trivial cases are not written:


```

{p=3;nu=5;n=6;for(m=2,10^2,if(core(m)!=m,next);P=x^2-m;
K=bnfinit(P,1);Kpn=bnrinit(K,p^nu);H=Kpn.cyc;T=List;e=matsize(H)[2];
for(k=1,e-1,C=H[e-k+1];v=valuation(C,p);if(v>0,listput(T,p^v));
f=quaddisc(m);fn=p^(n+1)*f;c=1;while(Mod(c,p)==0 ||
kronecker(f,c)!=-1,c=c+1);S=0;forstep(a=1,fn/2,[1,2],
if(gcd(a,f)!=1,next);aa=lift(Mod(a/c,fn));la=(aa*c-a)/fn;
eps=kronecker(f,a);S=S+a*eps*(1a+(1-c)/2));print
("m=",m," v_3(K_2Z)=",valuation(S,3)," T=",T)}

```

m=6	v_3(K_2Z)=1	T=[3]	m=69	v_3(K_2Z)=1	T=[3]
m=15	v_3(K_2Z)=1	T=[3]	m=74	v_3(K_2Z)=1	T=[9]
m=29	v_3(K_2Z)=1	T=[9]	m=77	v_3(K_2Z)=1	T=[3]
m=33	v_3(K_2Z)=1	T=[3]	m=78	v_3(K_2Z)=1	T=[3]
m=42	v_3(K_2Z)=3	T=[9]	m=79	v_3(K_2Z)=1	T=[9]
m=43	v_3(K_2Z)=2	T=[3]	m=82	v_3(K_2Z)=4	T=[3]
m=51	v_3(K_2Z)=1	T=[3]	m=83	v_3(K_2Z)=1	T=[3]
(...)					
m=10187	v_3(K_2Z)=2	T=[81]	m=10673	v_3(K_2Z)=4	T=[27]
m=10239	v_3(K_2Z)=2	T=[3,9]	m=10771	v_3(K_2Z)=6	T=[9]
m=10281	v_3(K_2Z)=1	T=[27]	m=10842	v_3(K_2Z)=4	T=[3,3]
m=10297	v_3(K_2Z)=4	T=[3]	m=10942	v_3(K_2Z)=2	T=[3,243]
m=10351	v_3(K_2Z)=1	T=[243]	m=11062	v_3(K_2Z)=3	T=[27]

APPENDIX C. CYCLIC CUBIC FIELDS

C.1. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty]$, $\mathcal{T}_{K,3}$ and \mathbf{C} . We consider all cyclic cubic fields, which requires calculating the Artin group giving $\text{Gal}(\mathbb{Q}(\mu_f)/K)$; indeed, a composite conductor f gives rise to several fields, so that we must work from the defining polynomial depending on suitable integers a, b such that $f = \frac{a^2+27b^2}{4}$ (another method would be to work in the group $(\mathbb{Z}/f\mathbb{Z})^\times$, giving other difficulties). We compare $v_m(\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty])$ to \mathbf{C} to see if we are, or not, in case of equality $v_m(\mathbf{L}_3(s, \chi)) = \mathbf{C}, \forall s \in \mathbb{Z}_3$.

C.1.1. General program. The program gives at first the list of all cyclic cubic fields, of conductor f (in \mathbf{f} , $\mathbf{f} \in [\mathbf{bf}, \mathbf{Bf}]$), with a defining polynomial P_K (in \mathbf{PK}). We compute for $p = 3$ the structures of the 3-class group (in \mathbf{H}) that of the 3-torsion group (in \mathbf{T}) and $\#\mathbf{K}_2(\mathbf{Z}_K)[3^\infty]$ (in $\#\mathbf{K}_2\mathbf{Z}$).

Except if the first layer of the cyclotomic \mathbb{Z}_3 -extension of K is non-ramified (e.g., $f = 657$, $P = x^3 - 219x + 730$, $\mathbf{H}_K = [3, 3]$, $\mathbf{T} = [3]$), then $\mathcal{T}_{K,3} = 1$ implies $\mathcal{H}_{K,3} = \mathcal{R}_{K,3} = 1$ since $\mathcal{W}_{K,3} = 1$:

```

{bf=7;Bf=10^3;Q=y^2+y+1;Y=Mod(y,Q);for(f=bf,Bf,
if(isprime(f)==1,next);h=valuation(f,3);if(h!=0 & h!=2,next);
F=f/3^h;if(core(F)!=F,next);F=factor(F);Div=component(F,1);
d=matsize(F)[1];for(j=1,d,D=Div[j];if(Mod(D,3)!=1,break));
Df=component(factor(f),1);for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
nu=8;K=bnfinit(PK,1);HK=K.clgp;Kpn=bnrinit(K,3^nu);HKn=Kpn.cyc;
T=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,CK=HKn[e-k+1];
v=valuation(CK,3);if(v>0,w=w+v;listput(T,3^v));
C=omega(f)-1;if(Mod(f,3)==0,C=C-1);forprime(q=2,10^2*f,
if(Mod(3*f,q)!=0 & polisirreducible(PK+0(q))==1,q0=q;break));
A0=List;A1=List;A2=List;c=q0;n=3;fn=3^n*f;N=eulerphi(fn)/(2*3);
for(s=1,fn/2,if(gcd(s,f*3)!=1,next);for(k=0,fn,r=s+k*fn;
if(isprime(r)==0,next);if(polisirreducible(PK+0(r))==0,
listput(A0,s);break));S0=0;S1=0;S2=0;g=(1-c)/2;
for(j=1,N,s=A0[j];listput(A1,lift(Mod(s*q0,fn)))));

```

```

for(j=1,N,s=A1[j];listput(A2,lift(Mod(s*q0,fn)))));
for(j=1,N,s=A0[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S0=S0+s*(ls+g));
for(j=1,N,s=A1[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S1=S1+s*(ls+g));
for(j=1,N,s=A2[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S2=S2+s*(ls+g));
S=S0+Y*S1+Y^2*S2;K2=norm(S)/3;v=valuation(K2,3);
if(v==C,print("f=",Df," P=",PK," H=",HK[2],
" T=",T," #K_2Z=",3^v," C(s)=",C," Equality"));
if(v>C,print("f=",Df," P=",PK," H=",HK[2],
" T=",T," #K_2Z=",3^v," C(s)=",C," Inequality"))))}}

```

C.1.2. *Numerical results.* We only write an excerpt of the results:

```

f=[9,7] P=x^3-21*x-35 H=[3] T=[] #K_2Z=1 C(s)=0 Equality
f=[9,7] P=x^3-21*x+28 H=[3] T=[] #K_2Z=1 C(s)=0 Equality
f=[7,13] P=x^3+x^2-30*x-64 H=[3] T=[3] #K_2Z=3 C(s)=1 Equality
f=[7,13] P=x^3+x^2-30*x+27 H=[3] T=[3] #K_2Z=3 C(s)=1 Equality
f=[9,19] P=x^3-57*x-152 H=[3] T=[3] #K_2Z=3 C(s)=0 Inequality
f=[9,19] P=x^3-57*x+19 H=[3] T=[3] #K_2Z=3 C(s)=0 Inequality
f=[7,31] P=x^3+x^2-72*x+209 H=[3] T=[3,3] #K_2Z=9 C(s)=1 Inequality
f=[7,31] P=x^3+x^2-72*x-225 H=[3] T=[3] #K_2Z=3 C(s)=1 Equality
f=[9,37] P=x^3-111*x+370 H=[3] T=[3] #K_2Z=3 C(s)=0 Inequality
f=[9,37] P=x^3-111*x+37 H=[3] T=[3] #K_2Z=3 C(s)=0 Inequality
f=[13,31] P=x^3+x^2-134*x-597 H=[3] T=[3] #K_2Z=3 C(s)=1 Equality
f=[13,31] P=x^3+x^2-134*x+209 H=[3] T=[3] #K_2Z=3 C(s)=1 Equality
f=[7,61] P=x^3+x^2-142*x+601 H=[3] T=[3] #K_2Z=3 C(s)=1 Equality
f=[7,61] P=x^3+x^2-142*x-680 H=[3] T=[9,9] #K_2Z=9 C(s)=1 Inequality
f=[7,67] P=x^3+x^2-156*x-799 H=[3] T=[3,3] #K_2Z=27 C(s)=1 Inequality
f=[7,67] P=x^3+x^2-156*x+608 H=[3] T=[3] #K_2Z=3 C(s)=1 Equality
(...)
f=[9,73] P=x^3-219*x-1241 H=[3,3] T=[3] #K_2Z=3 C(s)=0 Inequality

```

C.2. **Computation of $\widetilde{\mathcal{H}}_{K,3}$.** We use a part of the previous program, adding the computation of the logarithmic class group and we write some data obtained up to conductors $\leq 10^3$, then for larger conductors. The first component of $\text{bnflog}(K, 3)$ gives $\mathcal{H}_{K,3}$, the second one (often trivial) is the sub-group $\widetilde{\text{cl}}(\langle S_3 \rangle)$ generated by S_3 and the third component is $\mathcal{H}_{K,3}^{S_3} = \mathcal{H}_{K,3}/\widetilde{\text{cl}}(\langle S_3 \rangle)$; we have $\widetilde{\mathcal{H}}_{K,3}/\widetilde{\text{cl}}(\langle S_3 \rangle) \simeq \mathcal{H}_{K,3}^{S_3}$ [BeJa2016].

```

f=[19] x^3+x^2-6*x-7 H=[] T=[3] CLog=[[ ], [ ], [ ]]
f=[7,13] x^3+x^2-30*x-64 H=[3] T=[3] CLog=[[3], [ ], [3]]
f=[9,61] x^3-183*x-854 H=[3] T=[] CLog=[[ ], [ ], [3]]
f=[19,37] x^3+x^2-234*x-729 H=[2,6] T=[3,3] CLog=[[3], [3], [ ]]
f=[7,181] x^3+x^2-422*x+3191 H=[3,3] T=[3,3] CLog=[[3,3], [ ], [3,3]]
(...)
f=[7,19,73,103] P=x^3+x^2-333342*x+73964960
H=[3,3,3,3] T=[3,3,3,3] CLog=[[3,3,3,3], [ ], [3,3,3,3]]
f=[7,19,73,103] P=x^3+x^2-333342*x-69038901
H=[6,6,3] T=[3,3,3,3] CLog=[[3,3,3,3], [3], [3,3]]
f=[7,19,73,103] P=x^3+x^2-333342*x-49038361
H=[3,3,3,3] T=[3,3,3] CLog=[[3,3,3], [ ], [3,3,3]]
f=[397,251893] P=x^3+x^2-33333840*x+41296924413
H=[3] T=[9,3,3] CLog=[[3,3], [3,3], [ ]]
f=[7,14286007] P=x^3+x^2-33334016*x-56482638787
H=[54,18] T=[27,27] CLog=[[27,9], [ ], [27,9]]
f=[7,163,87643] P=x^3+x^2-33333554*x-214816239
H=[3,3] T=List([3,3,3]) CLog=[[3,3], [3], [3]]
f=[100001053] P=x^3+x^2-33333684*x-17437220649
H=[] T=List([9,3]) CLog=[[9,3], [9,3], [ ]]
f=[31,103,31321] P=x^3+x^2-33335984*x+68190607953
H=[6,6,3] T=List([9,9,3]) CLog=[[9,3], [3], [3,3]]

```

C.3. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[2^\infty]$ and $\mathcal{T}_{K,2}$. We still compute $\#\mathbf{R}_2(\mathbf{Z}_K)[2^\infty] \sim \frac{1}{2} \cdot \mathbf{L}_2(-1, \chi) \times \frac{1}{2} \cdot \mathbf{L}_2(-1, \chi^2)$, for prime conductors, noting that genus theory does not apply for cyclic cubic fields with $p = 2$. We compute $\mathcal{T}_{K,2}$, at first with the classical program using ray class groups, then we use formula of Theorem 4.1 in the following complete program (we only print an example of the six cases of structures that occur up to $f = 3500$):

```
{p=2;Q=y^2+y+1;Y=Mod(y,Q);forprime(f=7,5*10^5,
if(Mod(f,3)!=1,next);for(b=1,sqrt(4*f/27),A=4*f-27*b^2;
if(issquare(A,&a)==1,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27;K=bnfinit(P,1);nu=8;
Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];
for(k=1,e-1,CK=HKn[e-k+1];v=valuation(CK,p);if(v>0,listput(T,p^v)));
g=znprimroot(f);c=lift(g);if(Mod(c,p)==0,c=c+f);n=8;ex=(f-1)/3;
cex=Mod(c,f)^ex;R0=0;R1=0;R2=0;T0=0;T1=0;T2=0;fn=p^(n+2)*f;
forstep(a=1,fn/2,2,if(Mod(a,f)==0,next);aa=lift(Mod(a/c,fn));
am=lift(Mod(a,fn)^-1);la=(aa*c-a)/fn;XR=a*(la+(1-c)/2);XT=am*la;
aex=Mod(a,f)^ex;if(aex==1,R0=R0+XR;T0=T0+XT);if(aex==cex,
R1=R1+XR;T1=T1+XT);if(aex==cex^2,R2=R2+XR;T2=T2+XT));
AR=lift(Mod(R0-R2,fn));BR=lift(Mod(R1-R2,fn));
AT=lift(Mod(T0-T2,fn));BT=lift(Mod(T1-T2,fn));
NR=norm(AR+Y*BR);wR=valuation(NR,p);
NT=norm(AT+Y*BT);wT=valuation(NT,p);print();print("f=",f," P=",P);
print("AR=",AR," BR=",BR," #R_2Z=",p^wR);print("T=",T);
print("AT=",AT," BT=",BT," #T=",p^wT))}
```

f=7 P=x^3+x^2-2*x-1	f=739 P=x^3+x^2-246*x-520
AR=114683 BR=6	AR=7232 BR=4072
#R_2Z=1	#R_2Z=64
T=[]	T=[8,8]
AT=45707 BT=57070	AT=2395776 BT=8911992
#T=1	#T=64
f=31 P=x^3+x^2-10*x-8	f=2689 P=x^3+x^2-896*x+5876
AR=24 BR=507870	AR=1733440 BR=461096
#R_2Z=4	#R_2Z=64
T=[2,2]	T=[16,16]
AT=82976 BT=73118	AT=29791312 BT=37866256
#T=4	#T=256
f=277 P=x^3+x^2-92*x+236	f=3163 P=x^3+x^2-1054*x-13472
AR=3256 BR=4537692	AR=432 BR=51769200
#R_2Z=16	#R_2Z=256
T=[4,4]	T=[16,16]
AT=718480 BT=1119412	AT=11946416 BT=40520656
#T=16	#T=256
f=3457 P=x^3+x^2-1152*x+13700	f=6163 P=x^3+x^2-2054*x+17576
AR=286536 BR=3460600	AR=6140160 BR=6195120
#R_2Z=64	#R_2Z=256
T=List([16,16])	T=List([8,8])
AT=2216784 BT=1012224	AT=2754032 BT=3116408
#T=256	#T=64

We know from (1.6) that $\mathbf{R}_2(\mathbf{Z}_K)[2^\infty]$ and $\mathcal{T}_{K,2}$ have same 2-rank, but we see that most often the whole structures coincide (first exceptions $f = 2689, 3457$ and $f = 6163$ giving an interesting case).

C.4. Zeroes of p -adic \mathbf{L} -functions and $\mathbf{C}(s)$. A method, to study the influence of the zeroes, close to $s = 1$, of p -adic \mathbf{L} -functions, is to construct families of degree- p cyclic fields K_n such that the p -adic regulator tends to 0 as $n \rightarrow \infty$, so $\mathbf{L}_p(1, \chi_n) \rightarrow 0$ and $\#\mathcal{T}_{K_n,p} \rightarrow 0$ from (1.3); thus, invariants given by some $\mathbf{L}_p(s_0, \chi_n)$, $s_0 \in \mathbb{Z}_p^\times$, have the same behavior because of the role of $\mathbf{C}(s) = \mathbf{C}$ when $s \in \mathbb{Z}_p^\times$ (Theorem 4.4).

C.4.1. The Lecacheux–Washington cyclic cubic fields. The following family of cyclic cubic fields K illustrates, for $p = 3$, the fact that, in

a genus theory context, the strict inequality (Theorem 4.4 (b)):

$$v_m\left(\frac{1}{2}\mathbf{L}_3(s, \chi)\right) > \mathbf{C}, \forall s \in \mathbb{Z}_3,$$

can be largely exceeded because of $\mathbf{L}_3(1, \chi)$ very close to 0:

Using the family of cubic polynomials [Lec1993, Wa1996]:

$$P_n = x^3 - (n^3 - 2n^2 + 3n - 3)x^2 - n^2x - 1,$$

for some values of $n \rightarrow 1$ in \mathbb{Z}_3 , one obtains 3-adic \mathbf{L} -functions having a zero near $s = 1$, which of course gives large modules $\mathcal{T}_{K_n,3}$. The required conditions are given in [Wa1996, Theorem 3].

C.4.2. *Numerical examples.* The constant \mathbf{C} associated to these fields is always 2, so that we must obtain $v_3(\mathcal{T}_{K,3}) > 2$ as soon as n is close to 1 in \mathbb{Z}_3 . We give the program and some examples suggesting that $\#\mathcal{T}_{K_n,3}$ is unbounded in such families, as $n \rightarrow \infty$; but these fields have an huge discriminant D_n and the function $\mathcal{C}_p(K_n) := \frac{\log(\#\mathcal{T}_{K_n,p})}{\log(\sqrt{D_n})}$, that we have introduced in [Gra2019^b], is rather small and fulfills the various conjectures given in this article:

```
{M=3^5;forstep(n=1,10^6,M,if(isprime(n^2+3)==0,next);
A=n^2-3*n+3;if(numdiv(A)!=4,next);F=factor(A);
p=component(F,1)[1];q=component(F,1)[2];
if(Mod(p^2,9)==1 || Mod(q^2,9)==1,next);
P=x^3-(n^3-2*n^2+3*n-3)*x^2-n^2*x-1;K=bnfinit(P,1);
nu=18;HK=K.clgp;Kpn=bnrinit(K,3^nu);HKn=Kpn.cyc;
T=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,CK=HKn[e-k+1];
v=valuation(CK,3);if(v>0,w=w+v;listput(T,3^v)));
print("n=",n," p=",p," q=",q," T=",T," v_3(T)=",w)}
```

n=2674	p=43	q=166099	T=[3, 81, 243]	v_3(T)=10
n=40096	p=3271	q=491461	T=[3, 729, 729]	v_3(T)=13
n=43498	p=6547	q=288979	T=[3, 81, 243]	v_3(T)=10
n=50788	p=28111	q=91753	T=[3, 243, 243]	v_3(T)=11
n=56134	p=3613	q=872089	T=[3, 729, 2187]	v_3(T)=14
n=76546	p=31	q=189001951	T=[3, 2187, 2187]	v_3(T)=15
n=78490	p=7	q=880063519	T=[3, 81, 243]	v_3(T)=10
n=96958	p=79	q=118994467	T=[3, 9, 243, 729]	v_3(T)=14
n=124660	p=7	q=2219963089	T=[3, 6561, 6561]	v_3(T)=17
n=158194	p=135301	q=184957	T=[3, 243, 729]	v_3(T)=12
(...)				
n=570808	p=643	q=506718601	T=[3, 6561, 19683]	v_3(T)=18
n=649540	p=229	q=1842359227	T=[3, 3, 19683, 59049]	v_3(T)=21

Note that $n = 570808 \equiv 1 \pmod{3^9}$ and $n = 649540 \equiv 1 \pmod{3^{10}}$.

For $M = 3$, only giving $n \equiv 1 \pmod{3}$, we obtain many equalities:

n=340	p=7	q=16369	T=[3, 3]	v_3(T)=2
n=736	p=79	q=6829	T=[3, 3]	v_3(T)=2
n=970	p=7	q=133999	T=[3, 3]	v_3(T)=2
(...)				

For $M = 9$, giving $n \equiv 1 \pmod{9}$, we never obtain a valuation 2, but minimal valuations are 4:

n=874	p=7	q=108751	T=[3, 3, 9]	v_3(T)=4
n=2926	p=1777	q=4813	T=[3, 3, 9]	v_3(T)=4
n=4042	p=3067	q=5323	T=[3, 3, 9]	v_3(T)=4
(...)				

More generally, the minimal valuations are $2v_3(M)$.

APPENDIX D. NUMERICAL RESULTS FOR $p \geq 5$

When $p \geq 5$ and for a cyclic p -extension K/\mathbb{Q} , the trick used for $p \in \{2, 3\}$ does not exist and we only have available the complex analytic formulas of $\#\mathbf{K}_2(\mathbf{Z}_K)$ and the rank formula (1.4) giving Theorem 6.1.

D.1. Computation of $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$. The computation of the order of $\mathbf{K}_2(\mathbf{Z}_K)$ uses Bernoulli polynomials [Wa1997, Theorem 4.2 using Proposition 4.1].

The following program computes the p -valuation of the product $\prod_{\chi \neq \chi_0} \mathbf{L}(-1, \chi)$ for the $p-1$ characters χ of order p and prime conductor ℓ with the formula $\mathbf{L}(-1, \chi) \sim \sum_{a=1}^{\ell-1} \chi(a) (a^2 - \ell a)$, where $\chi(a)$ is computed from $\chi(g) := \zeta_p$, where g is a primitive root modulo ℓ and from the writing $a \equiv g^k \pmod{\ell}$ for $k \in [1, \ell-1]$; by comparison we compute the structure of $\mathcal{T}_{K,p}$. The case $p=3$ is included to recall that it is particular and may give trivial modules; moreover, for $p > 3$, equality (1.6) does not apply in general:

```
{forprime(p=3,100,Q=polcyclo(p);X=Mod(x,Q);print();
forprime(e11=1,10^4,if(Mod(e11,p)!=1,next);g=znprimroot(e11);
L=0;for(k=1,e11-1,a=lift(g^k);E=a^2-a*e11;L=L+E*X^k);
w=valuation(norm(L),p);P=polsubcyclo(e11,p);K=bnfinit(P,1);
nu=8;Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];
for(k=1,e-1,c=HKn[e-k+1];v=valuation(c,p);if(v>0,listput(T,p^v)));
print("p=",p," e11=",e11," T=",T," v(K_2Z)=",w))}
```

p=3 e11=7	T=[]	v(K_2Z)=0	p=11 e11=23	T=[]	v(K_2Z)=1
p=3 e11=19	T=[3]	v(K_2Z)=1	p=11 e11=727	T=[11]	v(K_2Z)=1
p=3 e11=199	T=[3,3]	v(K_2Z)=2	p=11 e11=1321	T=[]	v(K_2Z)=3
p=3 e11=4177	T=[9,9]	v(K_2Z)=3	p=11 e11=1453	T=[11]	v(K_2Z)=1
p=3 e11=2593	T=[3,9]	v(K_2Z)=6	p=11 e11=3631	T=[11]	v(K_2Z)=2
p=3 e11=21997	T=[27,81]	v(K_2Z)=7	p=11 e11=4357	T=[11,11]	v(K_2Z)=1
p=5 e11=11	T=[]	v(K_2Z)=1	p=13 e11=53	T=[]	v(K_2Z)=1
p=5 e11=101	T=[5,5]	v(K_2Z)=1	p=13 e11=677	T=[13]	v(K_2Z)=1
p=5 e11=181	T=[]	v(K_2Z)=3	p=13 e11=1483	T=[]	v(K_2Z)=2
p=5 e11=401	T=[5,5]	v(K_2Z)=2	p=13 e11=2029	T=[13]	v(K_2Z)=1
p=5 e11=3001	T=[5]	v(K_2Z)=7	p=13 e11=6761	T=[13]	v(K_2Z)=4
p=5 e11=5351	T=[5,5,5]	v(K_2Z)=1	p=13 e11=11831	T=[13]	v(K_2Z)=2
p=7 e11=29	T=[]	v(K_2Z)=1	p=17 e11=103	T=[]	v(K_2Z)=2
p=7 e11=127	T=[]	v(K_2Z)=2	p=17 e11=137	T=[]	v(K_2Z)=1
p=7 e11=197	T=[7,7]	v(K_2Z)=1	p=17 e11=3469	T=[17]	v(K_2Z)=1
p=7 e11=491	T=[7]	v(K_2Z)=1	p=17 e11=3571	T=[]	v(K_2Z)=1
p=7 e11=4159	T=[]	v(K_2Z)=3	p=17 e11=3673	T=[]	v(K_2Z)=1
p=7 e11=4229	T=[]	v(K_2Z)=4	p=17 e11=3911	T=[]	v(K_2Z)=1

For p large, the p -valuation is almost often 1 with a trivial $\mathcal{T}_{K,p}$.

D.2. Computations of $\text{rk}_5(\mathbf{K}_2(\mathbf{Z}_K))$. The next programs compute the ω^{-1} -component of the 5-class group \mathcal{H}_{K_2} of $K_2 := K(\mu_5)$, for degree-5 cyclic fields K , of prime conductor in the first section, then with $f = 11 \cdot 31$ in the second section, thus giving the 5-rank of $\mathbf{K}_2\mathbf{Z}_K[5^\infty]$ (formula (1.4), where $\langle S_5(K') \rangle$ is of character χ_0).

The program computes the 5-class group \mathcal{H}_{K_1} of $K_1 := K(\sqrt{5})$, so that $\mathcal{H}_{K_2} = \mathcal{H}_{K_1} \oplus (\mathcal{H}_{K_2})_{\omega^{-1}} \oplus (\mathcal{H}_{K_2})_{\omega}$, where $\text{rk}_5(\mathcal{H}_{K_2})_{\omega} = \text{rk}_5(\mathcal{T}_{K,5})$ (formula (1.5)), where the structure of $\mathcal{T}_{K,5}$ is computed as usual via the program of Section A.1. Whence:

$$\text{rk}_5(\mathbf{K}_2(\mathbf{Z}_K)) = \text{rk}_5(\mathcal{H}_{K_2}) - \text{rk}_5(\mathcal{H}_{K_1}) - \text{rk}_5(\mathcal{T}_{K,5}).$$

D.2.1. *Case of prime conductors.* So, f is a prime number $\ell \equiv 1 \pmod{5}$; thus, $\mathcal{H}_K = 1$. We write an excerpt of each structures obtained:

```
{p=5;forprime(e11=11,10^5,if(Mod(e11,p)!=1,next);
P=polsubcyclo(e11,p);print("conductor e11=",e11," P=",P);nu=8;
K=bnfinit(P,1);HK=K.clgp[2];Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;
TK=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,Ck=HKn[e-k+1];
v=valuation(Ck,p);if(v>0,listput(TK,p^v)));
R1=polcompositum(P,polsubcyclo(p,2))[1];LK1=bnfinit(R1,1);
R2=polcompositum(P,polcyclo(p))[1];LK2=bnfinit(R2,1);
print("TK=",TK," HK1=",LK1.clgp[2]," HK2=",LK2.clgp[2])}
```

```
conductor e11=11 P=x^5+x^4-4*x^3-3*x^2+3*x+1
TK=[] HK1=[] HK2=[5]
conductor e11=31 P=x^5+x^4-12*x^3-21*x^2+x+5
TK=[] HK1=[] HK2=[10,10,2,2]
conductor e11=101 P=x^5+x^4-40*x^3+93*x^2-21*x-17
TK=[5,5] HK1=[] HK2=[2005,5,5]
conductor e11=151 P=x^5+x^4-60*x^3-12*x^2+784*x+128
TK=[5] HK1=[] HK2=[5305,5]
conductor e11=251 P=x^5+x^4-100*x^3-20*x^2+1504*x+1024
TK=[5] HK1=[] HK2=[11810,10,2,2]
conductor e11=281 P=x^5+x^4-112*x^3-191*x^2+2257*x+967
TK=[] HK1=[5] HK2=[261005,5,5]
conductor e11=401 P=x^5+x^4-160*x^3+369*x^2+879*x-29
TK=[5,5] HK1=[] HK2=[42505,5,5,5]
conductor e11=421 P=x^5+x^4-168*x^3+219*x^2+3853*x-3517
TK=[] HK1=[5] HK2=[225915,15,3,3]
```

The rank 2 is obtained for $\ell = 31, 281, 401$.

D.2.2. *An example of composite conductor.* The next program computes the ω^{-1} -component for the four degree-5 cyclic fields K of conductor $f = 11 \cdot 31$ (for which $\mathcal{T}_{K,5} = 1$). One may replace $f = 11 \cdot 31$ by any product of primes $\ell_1 \equiv \ell_2 \equiv 1 \pmod{5}$):

```
{p=5;f=11*31;LP=polsubcyclo(f,p);d=matsize(LP)[2];
for(i=1,d,P=LP[i];D=nfdisc(P);if(D!=f^(p-1),next);
print();print(P);K=bnfinit(P,1);HK=K.clgp[2];
nu=8;Kpn=bnrinit(K,p^nu);HKn=Kpn.cyc;
TK=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,Ck=HKn[e-k+1];
v=valuation(Ck,p);if(v>0,listput(TK,p^v)));
R1=polcompositum(P,polsubcyclo(p,2))[1];LK1=bnfinit(R1,1);
R2=polcompositum(P,polcyclo(p))[1];LK2=bnfinit(R2,1);
print("HK=",HK," TK=",TK," HK1=",LK1.clgp[2]," HK2=",LK2.clgp[2])}
```

```
x^5-x^4-136*x^3+641*x^2-371*x-67
HK=[5] TK=[] HK1=[5,5] HK2=[218305,5,5,5,5]
x^5-x^4-136*x^3-723*x^2-1053*x-67
HK=[5] TK=[] HK1=[5,5] HK2=[145505,5,5,5,5]
x^5-x^4-136*x^3-41*x^2+3039*x-1431
HK=[5] TK=[] HK1=[5,5] HK2=[3805,5,5,5,5,5,5]
x^5-x^4-136*x^3+300*x^2+2016*x-3136
HK=[5] TK=[] HK1=[5,5] HK2=[12605,5,5,5,5,5]
```

The third case gives $\text{rk}_5((\mathbf{K}_2 \mathbf{Z}_K[5^\infty])_{\omega^{-1}}) = 4$ and the last one gives the rank 3. With $f = 11 \cdot 101$, one obtains interesting structures, giving the ranks 2, 2, 4, 3, respectively:

```
x^5-x^4-444*x^3+3644*x^2+80*x-32000
HK=[5] TK=[5] HK1=[5,5] HK2=[1196410,10,10,10,5]
x^5-x^4-444*x^3+311*x^2+32299*x+26883
HK=[5] TK=[5] HK1=[5,5] HK2=[15020305,5,5,5,5]
x^5-x^4-444*x^3-1911*x^2+12301*x-8669
```

```

HK=[5] TK=[5] HK1=[5,5] HK2=[1405,1405,5,5,5,5,5]
x^5-x^4-444*x^3+311*x^2+43409*x+4663
HK=[5] TK=[5] HK1=[5,5] HK2=[5559605,5,5,5,5,5]

```

These ranks may indeed be larger than the number of tame ramified primes; this illustrates Theorem 6.1.

D.3. Computation of $\mathbf{L}(-1 - (p-3)p^n, \chi)$. Let K/\mathbb{Q} be a degree- p cyclic extension. For $p \geq 5$ the conditions (2.1) ($m > 1$ and $m \equiv 0 \pmod{p-1}$) are not fulfilled for $m = 2$, which does not allow to express $\mathbf{L}(-1, \chi)$ by means of \mathbf{L}_p -function at $s = -1$. But we may compute, with:

$$m := 2 + (p-3)p^n > 1 \text{ and } (-1)_n := 1 - m = -1 - (p-3)p^n,$$

the following expression:

$$\begin{cases} \mathbf{L}_p(-1 - (p-3)p^n, \chi) = (1 - p^{1+(p-3)p^n}) \mathbf{L}((-1)_n, \chi) \\ \sim \mathbf{L}_p(-1, \chi), \text{ for } n \text{ large enough.} \end{cases}$$

This does not give $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ but is related to $\#\mathbf{K}_{2m-2}(\mathbf{Z}_K)[p^\infty]$ of higher \mathbf{K} -theory, from Quillen–Lichtenbaum conjecture, via the $\mathbf{L}((-1)_n, \chi)$'s [Kol2002, Ng1992]:

$$\zeta_K((-1)_n) = \zeta_K(1 - m) = \pm \frac{\#\mathbf{K}_{2m-2}(\mathbf{Z}_K)}{w_m(K)},$$

where $w_m(K)$ is the largest integer N such that $[K(\zeta_N) : K] \mid m$. One computes that for a degree- p cyclic extension K/\mathbb{Q} for $p \neq 2$, $w_m(K) \sim p$, which will give $v_p(\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]) = v_p(\zeta_K(1 - m)) + 1$.

Indeed, the condition $m \equiv 0 \pmod{p-1}$ of definition (2.1) is fulfilled for $m = 2 + (p-3)p^n$, so that relation between p -adic and complex \mathbf{L} -functions does exist and leads, for n large enough, by continuity of $\mathbf{L}_p(s, \chi)$, to $\prod_{\chi \neq \chi_0} \mathbf{L}((-1)_n, \chi) \sim \prod_{\chi \neq \chi_0} \mathbf{L}_p(-1, \chi)$, since the Euler factors $1 - p^{m-1}\chi(p)$ are p -adic units.

Thus, formula of Theorem 4.1 for $s = (-1)_n$ makes sense for computing $\mathbf{L}((-1)_n, \chi)$, of constant valuation for $n \geq n_0$, and only depending on the computation of $\mathbf{L}_p(-1, \chi)$ related to $\#\mathbf{K}_2(\mathbf{Z}_K)[p^\infty]$ and, in some sense, to $\#\mathcal{T}_{K,p}$.

The complex analytic computation of $\mathbf{L}((-1)_n, \chi)$ is done by the following program, with $m = 2 + (p-3)p^n$, but with less immediate Bernoulli polynomials; so we use their definition with series; we give also the computation of the structure of $\mathcal{T}_{K,p}$:

```

{n=2;forprime(p=5,50,m=2+(p-3)*p^n;print();forprime(e11=1,60000,
if(Mod(e11,p)!=1,next);Q=polcyclo(p);X=Mod(x,Q);g=znprimroot(e11);
P=polsubcyclo(e11,p);K=bnfinit(P,1);nu=8;Kpn=bnrinit(K,p^nu);
HKn=Kpn.cyc;TK=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,
Ck=HKn[e-k+1];v=valuation(Ck,p);if(v>0,listput(TK,p^v)));
Factm=1;for(i=1,m,Factm=i*Factm);
B=(x+0(x^(m+2)))*exp(x*y/e11+0(x^(m+2)))/(exp(x+0(x^(m+2)))-1);
Bm=e11^(m-1)/m*Factm*polcoeff(B,m);L=0;dm=poldegree(Bm);
for(k=1,e11-1,a=lift(g^k);E=0;for(j=0,dm,cm=polcoeff(Bm,j);
E=E+cm*a^j);L=L+E*X^k);vm=valuation(norm(L),p);
print("p=",p," e11=",e11," T=",TK," v(K_(2m-2)Z)=",vm+1))}

```

For instance, for $p = 5$ and cyclic quintic fields of prime conductor ℓ , this gives the following examples where $m = 52$:

e11=101 T=[5,5] v(K_(2m-2)Z)=3	e11=401 T=[5,5] v(K_(2m-2)Z)=3
e11=151 T=[5] v(K_(2m-2)Z)=2	e11=5351 T=[5,5,5] v(K_(2m-2)Z)=4
e11=251 T=[5] v(K_(2m-2)Z)=2	e11=29251 T=[5,5,5,5] v(K_(2m-2)Z)=5

In the selected interval, $\#\mathcal{T}_{K,5}$ and $\prod_{\chi \neq \chi_0} \mathbf{L}_5(-1, \chi)$ have same valuation, giving $v_5(\#\mathbf{K}_2(\mathbf{Z}_K)) = v_5(\#\mathcal{T}_{K,5}) + 1$, whereas $\#\mathcal{T}_{K,5}$ is also obtained p -adically, replacing $\langle a \rangle \cdot \theta^{-1}\chi(a)$ by $a^{-1}\chi(a) = \langle a \rangle^{-1}\theta^{-1}\chi(a)$ in formula of Theorem 4.1. However, the minimal ℓ giving distinct 5-valuations is given by the following data:

```
e11=56401 T=[5,5,5,25] v(T)=5 v(K_(2m-2)Z)=7
S0=173625530 S1=5807338145*Y S2=-7744983390*Y^2 S3=-4753111575/2*Y^3
S4=-8281263805/2*Y^3-8281263805/2*Y^2-8281263805/2*Y-8281263805/2
S=-6517187690*Y^3-23771230585/2*Y^2+3333412485/2*Y-7934012745/2
=5*(-1303437538*Y^3-4754246117/2*Y^2+666682497/2*Y-1586802549/2)
norm(S)=5^4*53925242055567630791570203161974917625/16=5^7*u
R=norm(S)/p=5^6*u
```

Computing $\mathbf{L}_5((-1)_n, \chi)$ by means of the formula of Theorem 4.1 gives again the results:

```
{n=2;p=5;m=2+(p-3)*p^n;Q=polcyclo(p);X=Mod(x,Q);
forprime(e11=1,5351,if(Mod(e11,p)!=1,next);fn=p*p^n*e11;
g=znprimroot(e11);c=lift(g);if(Mod(c,p)==0,c=c+e11);S=0;
for(a=1,fn/2,if(gcd(a,p*e11)!=1,next);aa=lift(Mod(a/c,fn));
la=(aa*c-a)/fn;A=lift(Mod(a,fn)^(m-1));u=znlog(Mod(a,e11),g);
S=S+(1a+(1-c)/2)*A*X^u);NS=norm(S);vm=valuation(NS,p)-1;
print("p=",p," e11=",e11," v(K_(2m-2)Z)=",vm+1))}
e11=101 v(K_(2m-2)Z)=3 e11=401 v(K_(2m-2)Z)=3
e11=151 v(K_(2m-2)Z)=2 e11=601 v(K_(2m-2)Z)=2
e11=251 v(K_(2m-2)Z)=2 e11=701 v(K_(2m-2)Z)=3
```

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VILLA LA GARDETTE, 4 CHEMIN CHÂTEAU GAGNIÈRE, 38520 LE BOURG D’OISANS
<http://orcid.org/0000-0002-1318-4414>

Email address: `g.mn.gras@wanadoo.fr`