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# GENUS THEORY OF $p$ -ADIC PSEUDO-MEASURES TAME KERNELS & ABELIAN $p$ -RAMIFICATION

GEORGES GRAS

ABSTRACT. We consider the Birch–Tate formula, linking the order of the tame Hilbert kernel  $\mathbf{K}_2\mathbf{Z}_k$  to  $\zeta_k(-1)$ , for real abelian fields  $k$ , and give a method to compute  $\#\mathbf{K}_2\mathbf{Z}_k[p^\infty]$ ,  $p \in \{2, 3\}$ . We utilize the regular kernel  $\mathbf{R}_2\mathbf{Z}_k$  (of index  $2^{[k:\mathbb{Q}]}$  in  $\mathbf{K}_2\mathbf{Z}_k$ ), in relationship with the torsion group  $\mathcal{T}_{k,p}$  of abelian  $p$ -ramification theory, whose order is given by the residue of the  $p$ -adic  $\zeta$ -function  $\zeta_{k,p}(s)$  of  $k$  at  $s = 1$ . We intend to compute  $\zeta_{k,p}(s)$ , using Stickelberger’s  $p$ -adic pseudo-measures and to compare  $\#\mathbf{R}_2\mathbf{Z}_k[p^\infty]$  and  $\#\mathcal{T}_{k,p}$ , by means of the powerful “genus theory” of these  $p$ -adic pseudo-measures, which was inaugurated in the 1970/80’s, when  $k$  is cyclic of degree  $dp^e$ . As application, we prove a conjecture of Deng–Li dealing with  $\mathbf{K}_2\mathbf{Z}_k[2^\infty]$  of an interesting family of real quadratic fields. PARI programs and tables are given for quadratic and cyclic cubic fields.

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## 1. INTRODUCTION

The aim of this paper is to utilize the link between  $p$ -adic  $\mathbf{L}$ -functions  $\mathbf{L}_p(s, \chi)$ , of even Dirichlet's characters  $\chi = \varphi\psi$  ( $\varphi$  of prime-to- $p$  order,  $\psi$  of  $p$ -power order), and classical arithmetic  $p$ -invariants of the cyclic number field  $k$  fixed by  $\chi$ , and to show that these invariants have some connections due to the presence of the character  $\psi$ ; more precisely, genus theory, classically applied to these invariants, does exist for  $\mathbf{L}_p$ -functions, as this was initiated in the 1970/80's with the genus theory of  $p$ -adic abelian pseudo-measures issued from Stickelberger's elements.

These results [Gra1987, Théorème (0.1)] generalizing [Gra1986<sup>b</sup>, Théorème (0.3)] to the “split case”  $\theta^{-1}\varphi(p) = 1$ , claim that either  $\frac{1}{2}\mathbf{L}_p(s, \chi) > \mathbf{C}(s)$ ,  $\forall s \in \mathbb{Z}_p$ , or  $\frac{1}{2}\mathbf{L}_p(s, \chi) = \mathbf{C}(s)$ ,  $\forall s \in \mathbb{Z}_p$ , for an explicit  $\mathbf{C}(s)$  ( $\mathbf{C}(s)$  is a constant  $\mathbf{C}$  when  $\theta^{-1}\varphi(p) \neq 1$ ); recall that  $\theta(a)$  for  $a \in \mathbb{Z}_p^\times$  is the unique  $\xi \in \text{tor}_{\mathbb{Z}_p}(\mathbb{Z}_p^\times)$  such that  $a \equiv \xi \pmod{p}$  (resp.  $a \equiv \xi \pmod{4}$ ) for  $p \neq 2$  (resp.  $p = 2$ ).<sup>1</sup>

We will consider the torsion group  $\mathcal{T}_{k,p}$  of the Galois group of the maximal abelian  $p$ -ramified in the ordinary sense (i.e., non-complexified), pro- $p$ -extension  $H_{k,p}^{\text{pr}}$  of  $k$ . In the real abelian case,  $\mathcal{T}_{k,p} = \text{Gal}(H_{k,p}^{\text{pr}}/k_\infty)$ , where  $k_\infty$  is the cyclotomic  $\mathbb{Z}_p$ -extension of  $k$ ; it is easily computable from class field theory [Gra2017<sup>a</sup>]; then we will focus on the the Hilbert kernel  $\mathbf{K}_2\mathbf{Z}_k$  of the ring of integers  $\mathbf{Z}_k$  of  $k$ , and, for  $p = 2$ , we will replace it by the regular kernel  $\mathbf{R}_2\mathbf{Z}_k$  (see [GJ1989] for definitions and properties) in the ordinary sense (i.e., taking into account symbols of infinite real places), linked to the Hilbert one via the exact sequence [Gar1971]:

$$(1.1) \quad 1 \rightarrow \mathbf{R}_2\mathbf{Z}_k \rightarrow \mathbf{K}_2\mathbf{Z}_k \rightarrow (\mathbb{Z}/2\mathbb{Z})^{[k:\mathbb{Q}]} \rightarrow 1.$$

When  $\frac{1}{2}\mathbf{L}_p(s, \chi) = \mathbf{C}(s)$ ,  $\forall s \in \mathbb{Z}_p$ , we utilize the analytic formulas (and the correspondence between complex and  $p$ -adic  $\mathbf{L}$ -functions):

$$(1.2) \quad \begin{cases} \#\mathcal{T}_{k,p} \sim [k \cap \mathbb{Q}_\infty : \mathbb{Q}] \times \prod_{\chi \neq 1} \frac{1}{2}\mathbf{L}_p(1, \chi), \\ \#\mathbf{R}_2\mathbf{Z}_k = \left( \frac{1}{6} \prod_{[F:k]=2} \frac{\#\mu(F)}{2} \right) \times \prod_{\chi \neq 1} \frac{1}{2}\mathbf{L}(-1, \chi), \end{cases}$$

for  $p \in \{2, 3\}$ , where  $\mathbb{Q}_\infty$  is the cyclotomic  $\mathbb{Z}_p$ -extension and  $\mu(F)$  the group of roots of unity of  $F$ . Then, we use the fact that  $\#\mathcal{T}_{k,p}$  is more easily computable via class field theory giving (1.6), to deduce the order of the regular kernel  $\mathbf{R}_2\mathbf{Z}_k[p^\infty]$  then that of  $\mathbf{K}_2\mathbf{Z}_k[p^\infty]$ .<sup>2</sup>

We apply the case of equality to some families of real quadratic fields with  $p = 2$  that have been introduced by Deng–Li in [DeLi2023].

<sup>1</sup> Recall that  $\omega : \text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q}) \rightarrow \mu_{p-1}$ , called the cyclotomic (or Teichmüller) character, is defined by the Galois action on  $\mu_p$  (conductor  $p$ , order  $p-1$  for  $p \neq 2$ ); it is the unit character for  $p = 2$  since  $\mu_2 \subset \mathbb{Q}$ . For  $p \neq 2$ , we have in some sense  $\omega = \theta$ , but for  $p = 2$ , do not confuse  $\omega$  with the character  $\theta$  that may be considered as the cyclotomic character of  $\text{Gal}(\mathbb{Q}(\mu_4)/\mathbb{Q})$  (conductor 4, order 2).

For any Dirichlet's character  $\rho$ , of conductor  $f$ ,  $\rho(a)$  denotes the image of the corresponding Artin automorphism  $(\frac{\mathbb{Q}(\mu_f)/\mathbb{Q}}{a})$ ; for  $\text{gcd}(a, f) \neq 1$ , we put  $\rho(a) = 0$ .

<sup>2</sup> A finite  $p$ -invariant of  $k$  is denoted  $\mathbf{X}_k[p^\infty]$  when it is of the form  $\mathbf{X}_k \otimes \mathbb{Z}_p$ , for a finite global invariant  $\mathbf{X}_k$  (e.g.,  $\mathbf{H}_k[p^\infty]$  for  $p$ -class groups), otherwise it is denoted  $\mathcal{X}_{k,p}$ ; it is only conjectural (see [Gra2016, Section 8]) that  $\mathcal{T}_{k,p}$  can be written  $\mathbf{T}_k \otimes \mathbb{Z}_p$ . This conjecture is out of reach but may be coherent with the finiteness of  $\mathbf{H}_k$ ,  $\mathbf{K}_2\mathbf{Z}_k$ , among others, and is supported by many heuristics.

These families correspond, broadly speaking, to the case where class field theory gives invariants of minimal orders.

In a numerical approach, we compute these orders by means of explicit pseudo-measures yielding expressions of Theorem 3.1 and Corollary 3.2, then  $\#\mathcal{T}_{k,p}$  is also obtained by means of PARI computations of ray class groups [Gra2017<sup>a</sup>]; this allows safe verifications.

**1.1. Regular kernel and torsion group  $\mathcal{T}_{k,p}$ .** Let  $k$  be a real abelian number field. The regular kernel  $\mathbf{R}_2\mathbf{Z}_k$  fulfills, from reflection theorem and Tate's results [Tate1970, Tate1976], the  $p$ -rank formula ([Gra1986<sup>a</sup>, Theorem 1] then [Gra2005, Theorem II.7.7.3.1]):

$$\mathrm{rk}_p(\mathbf{R}_2\mathbf{Z}_k) = \mathrm{rk}_{\omega^{-1}}(\mathbf{H}_{k'}^{S_p^{\prime\mathrm{res}}}) + \#\{v \mid p, v \text{ totally split in } k'/k\} - \delta,$$

where  $k' = k(\mu_p)$ ,  $\mathbf{H}_{k'}^{S_p^{\prime\mathrm{res}}}$  is the  $S_p^{\prime\mathrm{res}}$ -class group (restricted sense) for the set  $S_p^{\prime}$  of  $p$ -places of  $k'$ ,  $\delta = 1$  or  $0$ , according as  $\mu_p \subset k$  or not.

Due to the reflection theorem between  $p$ -class groups and  $p$ -torsion groups of abelian  $p$ -ramification theory, we get with the same notations ([Gra2005, Proposition III.4.2.2]):

$$\mathrm{rk}_p(\mathcal{T}_{k,p}) = \mathrm{rk}_{\omega}(\mathbf{H}_{k'}^{S_p^{\prime\mathrm{res}}}) + \#\{v \mid p, v \text{ totally split in } k'/k\} - \delta.$$

Whence, from the two previous relations:

$$(1.3) \quad \mathrm{rk}_p(\mathbf{R}_2\mathbf{Z}_k) = \mathrm{rk}_p(\mathcal{T}_{k,p}),$$

as soon as  $\omega^2 = 1$ , that is to say, if  $k$  contains the maximal real subfield of  $\mathbb{Q}(\mu_p)$ , giving the notion of regular field stated in [GJ1989].

We note that  $\mathbb{Q}(\mu_p)^+ \subseteq k$  is always fulfilled for  $p \in \{2, 3\}$ , which will be the context for us in this paper.

For more history and classical properties of these fundamental invariants, see, e.g., [Ba1968, Tate1970, Gar1971, Tate1976, BS1982, Gra1986<sup>a</sup>, Jau1986, Ng1986, Hur1987, GJ1989, Keu1989, Kol1989, Ber1990, Ng1990, BGr1992, Qin1995, HuK1998, RØ2000, RWK2000, YF2000, Kol2002, Yue2002, KM2003, Qin2005, Yue2005, Gra2005] among many others, since we do not intend to write a survey about a so large subject; more precisely, our goal is to return to very simple effective  $p$ -adic methods allowing computations, by means of pseudo-measures.

These pseudo-measures are the Mellin transform (via a reflection principle) of Stickelberger elements, and give rise, after twisting, to  $p$ -adic measures of the form  $(\mathcal{S}_{L_n}^*(c)(s))_n$ ,  $s \in \mathbb{Z}_p$ , in the group algebras  $\mathbb{Z}_p[\mathrm{Gal}(L_n/\mathbb{Q})]$  of the layers  $L_n := k(\mu_{qp^n})$  of the cyclotomic  $\mathbb{Z}_p$ -extension of  $k(\mu_q)$  ( $q = 4$  for  $p = 2$ , otherwise  $q = p$ ), as stated in [Gra1978, Section II].

This was done after the main pioneering works of Kubota–Leopoldt [KL1964], then Fresnel [Fr1965], Amice–Fresnel [AF1972], Iwasawa [Iw1972], Coates [Co1975], Serre [Se1978], giving many results, as the value of the residue of Dedekind  $p$ -adic  $\zeta$ -functions at  $s = 1$  and some annihilation theorems for  $p$ -class groups and torsion groups  $\mathcal{T}_{k,p}$  [Gra1978, Gra1979, Or1981, Jau1990, BN2005, All2013, BM2014, Jau2021, Jau2023]; this was considerably generalized to a totally real base field by Deligne–Ribet in [DR1980].

The Iwasawa framework of these constructions is very similar and is detailed in [Wa1997, § 7.2].

For generalizations to the wide étale kernels  $\mathbf{WK}_{2i}(k)$ , see for instance [Ng1992, JaSo2001, JaMi2006, AAM2021].

**1.2.  $\mathbf{L}_p$ -functions and the Birch-Tate formula.** Let  $\zeta_k(s)$  be the Dedekind zeta function of the real abelian number field  $k$ . Let  $\mathcal{X}_k$  be the set of primitive Dirichlet's characters of  $k$  and let  $\mathbf{1}$  be the unit character; then

$$\zeta_k(s) = \prod_{\chi \in \mathcal{X}_k} \mathbf{L}(s, \chi),$$

as product of the complex  $\mathbf{L}$ -functions for which  $\mathbf{L}(\mathbf{1}, s) = \zeta_{\mathbb{Q}}(s)$ .

In the  $p$ -adic context, the definition of the  $\mathbf{L}_p$ -functions is as follows (see, e.g., [Fr1965, Section 5 (a), Remarque], Amice–Fresnel [AF1972, Section 1]):

$$(1.4) \quad \begin{cases} \mathbf{L}_p(1 - m, \chi) = (1 - p^{m-1} \chi(p)) \times \mathbf{L}(1 - m, \chi), \\ \text{for all } m > 0 \text{ such that:} \\ \quad m \equiv 0 \pmod{(p-1)} \text{ for } p > 2, \\ \quad m \equiv 0 \pmod{2} \text{ for } p = 2. \end{cases}$$

This allows to compute for instance  $\mathbf{L}(-1, \chi)$  in terms of  $\mathbf{L}_p$ -functions for  $p = 2$  and  $p = 3$  since  $m = 2$  fulfills the congruent conditions required in (1.4).

The Birch-Tate formula is the following equality (proved as consequence of the Mazur–Wiles “Main Theorem” in abelian theory and complements in the case  $p = 2$  [Kol1989]):

$$(1.5) \quad \begin{cases} \#\mathbf{K}_2\mathbf{Z}_k = w_2(k) \times \zeta_k(-1) = w_2(k) \times \zeta_{\mathbb{Q}}(-1) \prod_{\chi \neq \mathbf{1}} \mathbf{L}(-1, \chi), \\ w_2(k) = 4 \cdot \prod_{[F:k]=2} \frac{1}{2} \cdot \#\mu(F), \end{cases}$$

where  $\mu(F)$  is the group of roots of unity of  $F$ .

For instance, since  $w_2(\mathbb{Q}) = 24$  and  $\zeta_{\mathbb{Q}}(-1) = \frac{1}{12}$ , then  $\#\mathbf{K}_2\mathbb{Z} = 2$ , giving  $\#\mathbf{R}_2\mathbb{Z} = 1$  as expected since  $\mathcal{T}_{\mathbb{Q}, p} = 1$  for all  $p$ .

Thus, the computation of the  $p$ -Sylow subgroup  $\#\mathbf{K}_2\mathbf{Z}_k[p^\infty]$  of  $\mathbf{K}_2\mathbf{Z}_k$  is possible for  $p = 2$  and  $p = 3$  with the following formulas (using  $\zeta_{\mathbb{Q}}(-1) = \frac{1}{12}$  and (1.4) for  $n = 2$ ):

$$\begin{cases} \#\mathbf{K}_2\mathbf{Z}_k[2^\infty] \sim \frac{w_2(k)}{12} \times \prod_{\chi \neq \mathbf{1}} \frac{1}{1 - 2\chi(2)} \mathbf{L}_2(-1, \chi); \\ \#\mathbf{K}_2\mathbf{Z}_k[3^\infty] \sim \frac{w_2(k)}{12} \times \prod_{\chi \neq \mathbf{1}} \frac{1}{1 - 3\chi(3)} \mathbf{L}_3(-1, \chi), \end{cases}$$

where  $\sim$  means equality up to a  $p$ -adic unit factor for the considered  $p$ . Since the denominators  $1 - p\chi(p)$  are invertible in  $\mathbb{Z}_p$ , the formulas become with expression (1.5) of  $w_2(k)$ :

$$\begin{cases} \#\mathbf{K}_2\mathbf{Z}_k[2^\infty] \sim \prod_{[F:k]=2} \frac{1}{2} \cdot \#\mu_2(F) \times \prod_{\chi \neq \mathbf{1}} \mathbf{L}_2(-1, \chi); \\ \#\mathbf{K}_2\mathbf{Z}_k[3^\infty] \sim \frac{1}{3} \prod_{[F:k]=2} \#\mu_3(F) \times \prod_{\chi \neq \mathbf{1}} \mathbf{L}_3(-1, \chi). \end{cases}$$

**1.3. Remarks about the case  $p > 3$ .** For  $p > 3$  and  $k$  abelian real we may compute the following expression, for  $m = 2 + (p-3)p^N > 0$ ,  $N$  large enough:

$$\prod_{\chi \neq \mathbf{1}} \mathbf{L}_p(-1 - (p-3)p^N, \chi) \sim \prod_{\chi \neq \mathbf{1}} \mathbf{L}_p(-1, \chi);$$

this does not give  $\#\mathbf{K}_2\mathbf{Z}_k[p^\infty]$  but is in relation with the order of the  $p$ -invariant  $\mathbf{K}_{2m-2}\mathbf{Z}_k[p^\infty]$  of higher  $\mathbf{K}$ -theory, from Quillen–Lichtenbaum conjecture, via the  $\mathbf{L}(1-m, \chi)$ 's.

Indeed, the condition  $m \equiv 0 \pmod{p-1}$  of definition (1.4) is fulfilled for  $m = 2 + (p-3)p^N$ , so that relation between  $p$ -adic and complex  $\mathbf{L}$ -functions does exist, and formula of Theorem 3.1 for  $s = 1 - m = -1 - (p-3)p^N$  makes sense for the computation of  $\mathbf{L}(s, \chi)$ , giving that  $\#\mathbf{K}_{2m-2}\mathbf{Z}_k[p^\infty] = \mathbf{K}_{2+2(p-3)p^N}\mathbf{Z}_k[p^\infty]$  is constant for  $N \geq N_0$ , only depending on the computation of  $\mathbf{L}_p(-1, \chi)$  (formula of Theorem 3.1) and, in some sense, of  $\#\mathcal{T}_{k,p}$ , since  $\mathbf{K}_{2m-2}\mathbf{Z}_k[p^\infty]$  is non-trivial if and only if  $\mathcal{T}_{k,p} \neq 1$ .

For instance, for  $p = 5$  and cyclic quintic fields of prime conductor  $f$ , this gives the following examples:

|         |                  |            |                   |
|---------|------------------|------------|-------------------|
| f=101   | T_k=[5, 5]       | v_p(T_k)=2 | v_p(K_(2m-2)Zk)=2 |
| f=151   | T_k=[5]          | v_p(T_k)=1 | v_p(K_(2m-2)Zk)=1 |
| f=251   | T_k=[5]          | v_p(T_k)=1 | v_p(K_(2m-2)Zk)=1 |
| f=401   | T_k=[5, 5]       | v_p(T_k)=2 | v_p(K_(2m-2)Zk)=2 |
| f=5351  | T_k=[5, 5, 5]    | v_p(T_k)=3 | v_p(K_(2m-2)Zk)=3 |
| f=29251 | T_k=[5, 5, 5, 5] | v_p(T_k)=4 | v_p(K_(2m-2)Zk)=4 |

In the selected interval,  $\#\mathcal{T}_{k,5}$  (computed using ray class groups) and  $\prod_{\chi \neq 1} \mathbf{L}_5(-1, \chi)$  have same valuation, whereas  $\#\mathcal{T}_{k,5}$  is also obtained  $p$ -adically, replacing  $\langle a \rangle \cdot \theta^{-1} \chi(a)$  by  $a^{-1} \chi(a) = \langle a \rangle^{-1} \theta^{-1} \chi(a)$ .

However, we have found the minimal  $f$  giving distinct 5-valuations, with the following data:

```
f=56401 T_k=[5, 5, 5, 25] v_p(T_k)=5 v_p(K_(2m-2)Zk)=6
S0=173625530 S1=5807338145*Y S2=-7744983390*Y^2 S3=-4753111575/2*Y^3
S4=-8281263805/2*Y^3-8281263805/2*Y^2-8281263805/2*Y-8281263805/2
S=-6517187690*Y^3-23771230585/2*Y^2+3333412485/2*Y-7934012745/2
=5*(-1303437538*Y^3-4754246117/2*Y^2+666682497/2*Y-1586802549/2)
norm(S)=5^4*norm(-1303437538*Y^3-4754246117/2*Y^2+666682497/2*Y
-1586802549/2)
=5^4*539252420555567630791570203161974917625/16=5^7*u
R=norm(S)/p=5^6*u
```

Nevertheless, computation of  $\#\mathbf{K}_2\mathbf{Z}_k$  from Bernoulli polynomials is easy (see, e.g., [Wa1997, Theorem 4.2 and Proposition 4.1]); the following program, computing  $\prod_{\chi \neq 1} \mathbf{L}(-1, \chi)$ , gives the result for any cyclic field of degree  $p > 2$  of prime conductor  $f$ :

```
{p=7; forprime(f=1, 10^5, if(Mod(f, p) != 1, next); Q=polcyclo(p); X=Mod(x, Q);
g=znprimroot(f); B=f*polcoeff((x)*exp((x)*(y/f))/(exp(x)-1), 2);
L=0; for(k=1, f-1, a=lift(g^k); E=0; for(j=0, 2, E=E+polcoeff(B, j)*a^j);
L=L+E*X^k); v=valuation(norm(L), p);
print("p=", p, " f=", f, " v_p(K_2Zk)=" , v)}}}
```

|     |        |              |      |         |              |
|-----|--------|--------------|------|---------|--------------|
| p=5 | f=11   | v_p(K_2Zk)=1 | p=13 | f=53    | v_p(K_2Zk)=1 |
| p=5 | f=31   | v_p(K_2Zk)=2 | p=13 | f=1483  | v_p(K_2Zk)=2 |
| p=5 | f=181  | v_p(K_2Zk)=3 | p=13 | f=6761  | v_p(K_2Zk)=4 |
| p=5 | f=3001 | v_p(K_2Zk)=7 | p=17 | f=103   | v_p(K_2Zk)=2 |
| p=7 | f=29   | v_p(K_2Zk)=1 | p=17 | f=30091 | v_p(K_2Zk)=3 |
| p=7 | f=127  | v_p(K_2Zk)=2 | p=37 | f=149   | v_p(K_2Zk)=1 |
| p=7 | f=4159 | v_p(K_2Zk)=3 | p=37 | f=2887  | v_p(K_2Zk)=2 |
| p=7 | f=4229 | v_p(K_2Zk)=4 | p=37 | f=17909 | v_p(K_2Zk)=3 |

For  $p > 3$  all the kernels are non-trivial (certainly a genus theory giving the proof); when  $p$  increases, larger valuations become rare.

The direct computation of  $\mathbf{L}(1-m, \chi)$  is similar with the following program, with  $m = 2 + (p-3)p^N$ :

```
{p=17;N=2;m=2+(p-3)*p^N;n=8;
forprime(f=1,10^4,if(Mod(f,p)!=1,next);P=polsubcyclo(f,p);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);H=Kpn.cyc;T=List;
e=matsize(H)[2];valT=0;for(k=1,e-1,h=H[e-k+1];v=valuation(h,p);
if(v>0,valT=valT+v;listput(T,p^v));Q=polcyclo(p);X=Mod(x,Q);
g=znprimroot(f);Factm=1;for(i=1,m,Factm=i*Factm);
B=(x+0(x^(m+2)))*exp(x*y/f+0(x^(m+2)))/(exp(x+0(x^(m+2)))-1);
Bm=f^(m-1)/m*Factm*polcoeff(B,m);L=0;dm=poldegree(Bm);for(k=1,f-1,
a=lift(g^k);E=0;for(j=0,dm,cm=polcoeff(Bm,j);E=E+cm*a^j);L=L+E*X^k);
print("p=",p," f=",f," N=",N," n=",n);vm=valuation(norm(L),p);
print(" T_k=",T," v_p(T_k)=",valT," v_p(K_(2m-2)Zk)=",vm)}
```

|            |         |           |            |                   |
|------------|---------|-----------|------------|-------------------|
| p=7 f=29   | N=2 n=8 | T_k=[]    | v_p(T_k)=0 | v_p(K_(2m-2)Zk)=0 |
| p=7 f=43   | N=2 n=8 | T_k=[]    | v_p(T_k)=0 | v_p(K_(2m-2)Zk)=0 |
| p=7 f=197  | N=2 n=8 | T_k=[7,7] | v_p(T_k)=2 | v_p(K_(2m-2)Zk)=2 |
| p=7 f=491  | N=2 n=8 | T_k=[7]   | v_p(T_k)=1 | v_p(K_(2m-2)Zk)=1 |
| p=7 f=1471 | N=2 n=8 | T_k=[7,7] | v_p(T_k)=2 | v_p(K_(2m-2)Zk)=2 |

The non triviality occurs if and only if  $\mathcal{F}_k \neq 1$ ; in this degree  $p$  cyclic fields, this comes from the fixed points formula giving  $\#\mathcal{F}_k^G$  [Gra2005, Theorem IV.3.3 and Exercise IV.3.3.1]; in the present case of prime conductor, non triviality is equivalent to  $f \equiv 1 \pmod{p^2}$ .

**1.4. Main formulas for quadratic and cubic fields.** We deduce the following expressions applied with  $p \in \{2, 3\}$ .

a) Real quadratic fields.

(i) For real quadratic fields  $k = \mathbb{Q}(\sqrt{m})$ ,  $m \neq 2, 3$ ,  $p = 2$ :

$$\left\{ \begin{array}{l} \prod_{[F:k]=2} \frac{1}{2} \cdot \#\mu_2(F) = 2 \quad (F = k(\mu_4), F = k(\mu_6)); \\ \#\mathbf{K}_2\mathbf{Z}_k[2^\infty] \sim 2 \cdot \mathbf{L}_2(-1, \chi), \quad \#\mathbf{R}_2\mathbf{Z}_k[2^\infty] \sim \frac{1}{2} \cdot \mathbf{L}_2(-1, \chi). \end{array} \right.$$

(ii) For  $\mathbb{Q}(\sqrt{2})$ ,  $\prod_{[F:k]=2} \frac{1}{2} \cdot \#\mu_2(F) = 4$  ( $F = k(\mu_8)$ ,  $F = k(\mu_6)$ );

so,  $\#\mathbf{K}_2\mathbf{Z}_{\mathbb{Q}(\sqrt{2})}[2^\infty] \sim 4 \cdot \mathbf{L}_2(-1, \chi) \sim 4$ ,  $\#\mathbf{R}_2\mathbf{Z}_{\mathbb{Q}(\sqrt{2})}[2^\infty] \sim 1$ .

(iii) For  $\mathbb{Q}(\sqrt{3})$ ,  $\prod_{[F:k]=2} \frac{1}{2} \cdot \#\mu_2(F) = 4$  ( $F = k(\mu_4)$ ,  $F = k(\mu_6) =$

$k(\mu_4)$ ); so,  $\#\mathbf{K}_2\mathbf{Z}_{\mathbb{Q}(\sqrt{3})}[2^\infty] \sim 4 \cdot \mathbf{L}_2(-1, \chi) \sim 8$ ,  $\#\mathbf{R}_2\mathbf{Z}_{\mathbb{Q}(\sqrt{3})}[2^\infty] \sim 2$ .

(iv) For  $p = 3$ ,  $k \neq \mathbb{Q}(\sqrt{3})$ , the formulas are:

$$\left\{ \begin{array}{l} \frac{1}{3} \prod_{[F:k]=2} \#\mu_3(F) = 1 \quad (F = k(\mu_6)); \\ \#\mathbf{K}_2\mathbf{Z}_k[3^\infty] \sim \mathbf{L}_3(-1, \chi). \end{array} \right.$$

(v) For  $\mathbb{Q}(\sqrt{3})$ ,  $\frac{1}{3} \prod_{[F:k]=2} \#\mu_3(F) = 3$  ( $F = k(\mu_3)$ ,  $F = k(\mu_4) =$

$k(\mu_6)$ ); so,  $\#\mathbf{K}_2\mathbf{Z}_{\mathbb{Q}(\sqrt{3})}[3^\infty] \sim 3 \cdot \mathbf{L}_3(-1, \chi) \sim 3$ .

b) Cyclic cubic fields.

(i) For cubic fields  $k$  of conductor distinct from 9,  $p = 3$ :

$$\left\{ \begin{array}{l} \frac{1}{3} \prod_{[F:k]=2} \#\mu_3(F) = 1 \quad (F = k(\mu_6)); \\ \#\mathbf{K}_2\mathbf{Z}_k[3^\infty] \sim \mathbf{L}_3(-1, \chi) \times \mathbf{L}_3(-1, \chi'), \end{array} \right.$$

for the two conjugate characters of order 3 of  $k$ .

(ii) For the cubic field of conductor 9,  $\frac{1}{3} \prod_{[F:k]=2} \#\mu_3(F) = 3$

( $F = k(\mu_9)$ ); then  $\#\mathbf{K}_2\mathbf{Z}_k[3^\infty] = 3 \cdot \mathbf{L}_2(-1, \chi) \sim 1$ .

(iii) For  $p = 2$ , the formulas are, for the two conjugate characters of order 3 of  $k$ .

$$\begin{cases} \prod_{[F:k]=2} \frac{1}{2} \cdot \#\mu_2(F) = 2 \quad (F = k(\mu_4), F = k(\mu_6)); \\ \#\mathbf{K}_2\mathbf{Z}_k[2^\infty] \sim 2 \cdot \mathbf{L}_2(-1, \chi) \times \mathbf{L}_2(-1, \chi'), \\ \#\mathbf{R}_2\mathbf{Z}_k[2^\infty] \sim \frac{1}{2} \cdot \mathbf{L}_2(-1, \chi) \times \frac{1}{2} \cdot \mathbf{L}_2(-1, \chi') \end{cases}$$

c) For real quadratic fields with  $p = 3$  or cyclic cubic fields with  $p = 2$ , genus theory is empty; nevertheless, analytic properties of  $\mathbf{L}_p$ -functions leads to a residual congruence modulo  $2q$  (see more details in Section 6). In particular, one must verify, when genus theory does not apply, that the two invariant  $\mathcal{T}_{k,p}$  and  $\mathbf{R}_2\mathbf{Z}_k[p^\infty]$  are always simultaneously trivial or non-trivial.

**1.5. Torsion group of abelian  $p$ -ramification.** We recall in this part the properties of the group  $\mathcal{T}_{k,p}$ , the torsion group of the Galois group of the maximal abelian  $p$ -ramified, non complexified, pro- $p$ -extension  $H_{k,p}^{\text{pr}}$  of the real field  $k$ , of Galois group  $\mathcal{A}_{k,p}$ ; the definitions and notations are the following:

(i)  $\mathbf{H}_k[p^\infty]$  is the  $p$ -class group in the ordinary sense and  $H_{k,p}^{\text{nr}}$  the  $p$ -Hilbert class field,

(ii)  $\mathcal{R}_{k,p}$  is the *normalized*  $p$ -adic regulator, defined as a  $\mathbb{Z}_p$ -module in [Gra2018<sup>b</sup>, Section 5, Proposition 5.2]; its order is the classical  $p$ -adic regulator, up to an explicit factor,

(iii)  $\mathcal{U}_{k,p}$  is the group of principal local units at  $p$ ,  $\mathcal{E}_{k,p}$  the closure in  $\mathcal{U}_{k,p}$  of the group of units  $E_k$  and  $\mathcal{W}_{k,p} := \text{tor}_{\mathbb{Z}_p}(\mathcal{U}_{k,p})/\mu_p(k)$ ,

(iv)  $H_{k,p}^{\text{bp}}$  (resp.  $\mathcal{T}_{k,p}^{\text{bp}}$ ) is named in [Ng1986] the Bertrandias–Payan field (resp. module) and is issued from the paper [BP1972] ( $H_{k,p}^{\text{bp}}$  is the compositum of the  $p$ -cyclic extensions of  $k$  embeddable in  $p$ -cyclic extensions of arbitrary large degree; see [GJN2016] for bibliographical information and complements).

The corresponding diagram is valid for any number field  $k$  fulfilling the Leopoldt conjecture, replacing  $k_\infty$  by the compositum  $\tilde{k}$  of the  $\mathbb{Z}_p$ -extensions of  $k$ :

$$\begin{array}{ccccc} & & \mathcal{T}_{k,p} & & \\ & \tilde{k} & \xrightarrow{\tilde{k}H_k^{\text{nr}}} & H_k^{\text{bp}} & \xrightarrow{\simeq \mathcal{W}_k} & H_k^{\text{pr}} \\ & | & \searrow \mathcal{T}_{k,p}^{\text{bp}} & \simeq \mathcal{R}_k & \simeq \mathcal{U}_k/\mathcal{E}_k & | \\ H_k^{\text{nr}} \cap \tilde{k} & \xrightarrow{\simeq \mathbf{H}_k[p^\infty]} & H_k^{\text{nr}} & & & \\ & | & & & & \\ k & & & & & \end{array}$$

When  $H_{k,p}^{\text{nr}} \cap \tilde{k} = k$ , one obtains the formula:

$$(1.6) \quad \#\mathcal{T}_{k,p} = \#\mathbf{H}_k[p^\infty] \times \#\mathcal{R}_{k,p} \times \#\mathcal{W}_{k,p}.$$

The order of  $\mathcal{T}_{k,p}$  is given by the residue at  $s = 1$  of the  $p$ -adic  $\zeta$ -function  $\zeta_{k,p}$  [Co1975, Appendix] and  $\text{rk}_p(\mathbf{R}_2\mathbf{Z}_k) = \text{rk}_p(\mathcal{T}_{k,p})$ , as soon as  $k$  contains the maximal real subfield of  $\mathbb{Q}(\mu_p)$  [GJ1989, Gra1998]. But class field theory is more convenient to state the main properties



of this group (see, e.g., [Gra2005, § III.2.c, § III.4.b, § IV.3], after pioneering works as [BP1972, Gra1986<sup>a</sup>, Jau1986, Ng1986, GJ1989] and many others collected in [Gra2018<sup>a</sup>, Appendix]; then numerical aspects, with PARI [PARI] programs are given in [Gra2017<sup>a</sup>] and the order of magnitude of  $\mathcal{T}_{k,p}$  is studied in [Gra2019]).

Let's give the classical program computing the structure of  $\mathcal{T}_{k,p}$  as soon as any monic irreducible polynomial  $P$  is given in  $\mathbb{Z}[x]$ ; we apply it to the particular family of subfields studied in [DeLi2023, Table]. A remarkable fact is that the structure of  $\mathcal{T}_{k,2}$  is  $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$ , that of the 2-class group is  $(\mathbb{Z}/2\mathbb{Z})^3$ , and the computation is almost instantaneous (time = 217 ms):

```
{L=List
([7215,26455,77415,119535,142935,153735,166335,171015,196359,
226655,241215,243295,257335,283855,311415,315055,420135,430495,
447135,473415,475215,490295,504295,545415,550615,552695,553335,
563695,568815,592215,603655,606615,633399,657735,665223,673215,
685815,687895,727935,751335,755495,757055,790495,798135,803751,
807455,818935,833199,849615,878415,884455,886015,886335,896415,
905255,911495,934935,961935,973655,981695,990015]);
p=2;n=9;print("p=",p," n=",n);for(j=1,61,P=x^2-L[j];K=bnfinit(P,1);
HK=K.clgp[2];Kpn=bnrinit(K,p^n);Hn=Kpn.cyc;T=List;e=matsize(Hn)[2];
for(k=1,e-1,c=Hn[e-k+1];v=valuation(c,p);if(v>0,listput(T,p^v)));
print("P=",P," p-torsion group T_K=",T," class group=",HK)}
p=2 n=9
P=x^2-7215      p-torsion group T_K=[2,2,4]   class group=[2,2,2]
P=x^2-26455    p-torsion group T_K=[2,2,4]   class group=[2,2,2]
P=x^2-77415    p-torsion group T_K=[2,2,4]   class group=[2,2,2]
P=x^2-119535   p-torsion group T_K=[2,2,4]   class group=[2,2,2]
(...)
P=x^2-961935   p-torsion group T_K=[2,2,4]   class group=[2,2,2]
P=x^2-973655   p-torsion group T_K=[2,2,4]   class group=[6,2,2]
P=x^2-981695   p-torsion group T_K=[2,2,4]   class group=[10,2,2]
P=x^2-990015   p-torsion group T_K=[2,2,4]   class group=[18,2,2]
```

From (1.3), the 2-ranks of  $\mathbf{R}_2\mathbf{Z}_k$  and  $\mathcal{T}_{k,2}$  are equal to 3, this implies (since we have  $\mathbf{K}_2\mathbf{Z}_k[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}/8\mathbb{Z}$  from [DeLi2023, Theorem 1.2] assuming  $\delta = 3$ ) that:

$$\mathbf{R}_2\mathbf{Z}_k[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/2^2\mathbb{Z} \simeq \mathcal{T}_{k,2}.$$

More generally, for the Deng–Li family, we have conjecturally:

$$\mathbf{R}_2\mathbf{Z}_k[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^2\mathbb{Z} \simeq \mathcal{T}_{k,2}.$$

From the previous study of  $\mathcal{T}_{k,2}$ , we see that  $\mathcal{W}_{k,2} \simeq \mathbb{Z}/2\mathbb{Z}$ , which is coherent with the formula  $\#\mathcal{T}_{k,2} = \#\mathbf{H}_k[2^\infty] \times \#\mathcal{R}_{k,2} \times \#\mathcal{W}_{k,2}$  as soon as the normalized 2-adic regulator is trivial (we will prove arithmetically that  $\mathcal{R}_{k,2} = 1$ ).

Another invariant, quotient of  $\mathcal{T}_{k,p}$  and linked to the Hilbert kernel [Jau1994<sup>a</sup>, Théorème 9], is the Jaulent logarithmic class group  $\widetilde{\mathcal{T}}_{k,p}$ . The PARI computation of  $\widetilde{\mathcal{T}}_{k,2}$  for the Deng–Li family, gives always the structure [2, 2]; but we do not know if this invariant is given by values of  $\mathbf{L}_2$ -functions at some  $s \in \mathbb{Z}_2$ .

Moreover, the third component of  $\text{bnflog}(\mathbf{K}, 2)$  gives the  $S$ -class group  $\mathbf{H}_k[2^\infty]/\text{cl}(\mathfrak{p}_2) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , where  $\mathfrak{p}_2 \mid 2$ ; the results confirm that necessarily  $\mathfrak{p}_2$  is non-principal of order 2:

```
{L=List([7215,26455,77415,119535,142935,153735,166335,171015,
196359,226655,241215,243295,257335,283855,311415,315055,420135,
430495,447135,473415,475215,490295,504295,545415,550615,552695,
553335,563695,568815,592215,603655,606615,633399,657735,665223,
```

```

673215,685815,687895,727935,751335,755495,757055,790495,798135,
803751,807455,818935,833199,849615,878415,884455,886015,886335,
896415,905255,911495,934935,961935,973655,981695,990015]);
p=2;for(j=1,61,P=x^2-L[j];K=bnfinit(P,1);Hlog=bnflog(K,2)[1];
print("P=",P," ClassLog=", Hlog))}

```

```

P=x^2-7215   ClassLog=[2,2]   P=x^2-973655   ClassLog=[2,2]
P=x^2-26455  ClassLog=[2,2]   P=x^2-981695   ClassLog=[2,2]
P=x^2-77415  ClassLog=[2,2]   P=x^2-990015   ClassLog=[2,2]

```

## 2. DEFINITION OF A $p$ -ADIC PSEUDO-MEASURE $(\mathcal{S}_{L_n})_n$

As usual, put  $q = 4$  if  $p = 2$ , then  $q = p$  for  $p > 2$ . Let  $k$  be a real abelian field and put  $L_n := k(\mu_{qp^n})$ ,  $n \geq 0$ . To simplify, we assume that  $k \cap \mathbb{Q}_\infty = \mathbb{Q}$ ; in our context, this occurs except for  $k = \mathbb{Q}(\sqrt{2})$  ( $p = 2$ ) and for the cubic field of conductor 9 ( $p = 3$ ).

**2.1. The Stickelberger elements.** The conductor of  $L_n$  is of the form  $f_{L_n} = qp^n f$ , for  $f$  prime to  $p$ , taking  $n$  large enough (in formulas we shall abbreviate  $f_{L_n}$  by  $f_n$ ).

Then put (where all Artin symbols are taken over  $\mathbb{Q}$ ):

$$\mathcal{S}_{L_n} := - \sum_{a=1}^{f_n} \left( \frac{a}{f_n} - \frac{1}{2} \right) \left( \frac{L_n}{a} \right)^{-1},$$

as restriction to  $L_n$  of  $\mathcal{S}_{\mathbb{Q}(\mu_{f_n})} := - \sum_{a=1}^{f_n} \left( \frac{a}{f_n} - \frac{1}{2} \right) \left( \frac{\mathbb{Q}(\mu_{f_n})}{a} \right)^{-1}$ , where  $a$  runs through the integers  $a \in [1, f_n]$  prime to  $f_n$ .

This definition, using the partial  $\zeta$ -functions of  $\mathbb{Q}$ , is the canonical one and gives the most suitable algebraic properties defining  $p$ -adic pseudo-measures in the meaning of [Se1978].

**2.2. Norms of the Stickelberger elements.** Let  $f$  and  $m$  be such that  $m \mid f$ , consider  $\mathbb{Q}(\mu_f)$  and  $\mathbb{Q}(\mu_m)$  and let  $\mathbf{N}_{\mathbb{Q}(\mu_f)/\mathbb{Q}(\mu_m)}$  be the restriction map  $\mathbb{Q}[\text{Gal}(\mathbb{Q}(\mu_f)/\mathbb{Q})] \rightarrow \mathbb{Q}[\text{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})]$ . We have:

$$\mathbf{N}_{\mathbb{Q}(\mu_f)/\mathbb{Q}(\mu_m)}(\mathcal{S}_{\mathbb{Q}(\mu_f)}) = \prod_{\ell \mid f, \ell \nmid m} \left( 1 - \left( \frac{\mathbb{Q}(\mu_m)}{\ell} \right)^{-1} \right) \cdot \mathcal{S}_{\mathbb{Q}(\mu_m)}.$$

If  $L$  of conductors  $f$  is extension of  $K$  of conductor  $m$ , we define  $\mathcal{S}_L := \mathbf{N}_{\mathbb{Q}(\mu_f)/L}(\mathcal{S}_{\mathbb{Q}(\mu_f)})$ ; then:

$$\mathbf{N}_{L/K}(\mathcal{S}_L) = \prod_{\ell \mid f, \ell \nmid m} \left( 1 - \left( \frac{K}{\ell} \right)^{-1} \right) \cdot \mathcal{S}_K.$$

This implies that  $\mathbf{N}_{L/K}(\mathcal{S}_L) = 0$  as soon as a prime  $\ell$  is totally split in  $K$ . In particular, if  $K$  is real, the formula is valid for the infinite place and  $\mathbf{N}_{\mathbb{Q}(\mu_m)/K}(\mathcal{S}_{\mathbb{Q}(\mu_m)}) = 0$ .

**2.3. Twists of the Stickelberger elements.** Let  $c$  be an odd integer, prime to  $pf$ ; for  $L_n = k(\mu_{qp^n})$ , put:

$$\mathcal{S}_{L_n}(c) := \left( 1 - c \left( \frac{L_n}{c} \right)^{-1} \right) \cdot \mathcal{S}_{L_n}.$$

Then  $\mathcal{S}_{L_n}(c) \in \mathbb{Z}[\text{Gal}(L_n/\mathbb{Q})]$ . Indeed, we have:

$$\mathcal{S}_{L_n}(c) = \frac{-1}{f_n} \sum_a \left[ a \left( \frac{L_n}{a} \right)^{-1} - ac \left( \frac{L_n}{a} \right)^{-1} \left( \frac{L_n}{c} \right)^{-1} \right] + \frac{1-c}{2} \sum_a \left( \frac{L_n}{a} \right)^{-1}.$$

Let  $a'_c \in [1, f_n]$  be the unique integer such that  $a'_c \cdot c \equiv a \pmod{f_n}$ ; put  $a'_c \cdot c = a + \lambda_a^n(c)f_n$ ,  $\lambda_a^n(c) \in \mathbb{Z}$ ; then, using the bijection  $a \mapsto a'_c$  in the second summation and the fact that  $\binom{L_n}{a'_c} \binom{L_n}{c} = \binom{L_n}{a}$ :

$$(2.1) \quad \left\{ \begin{aligned} \mathcal{S}_{L_n}(c) &= \frac{-1}{f_n} \left[ \sum_a a \binom{L_n}{a}^{-1} - \sum_a a'_c c \binom{L_n}{a'_c}^{-1} \binom{L_n}{c}^{-1} \right] \\ &\quad + \frac{1-c}{2} \sum_a \binom{L_n}{a}^{-1} \\ &= \frac{-1}{f_n} \sum_a \left[ a - a'_c \cdot c \right] \binom{L_n}{a}^{-1} + \frac{1-c}{2} \sum_a \binom{L_n}{a}^{-1} \\ &= \sum_a \left( \lambda_a^n(c) + \frac{1-c}{2} \right) \binom{L_n}{a}^{-1} \in \mathbb{Z}[\text{Gal}(L_n/\mathbb{Q})]. \end{aligned} \right.$$

This twisted form gives the Stickelberger  $p$ -adic measure used to generate the  $\mathbf{L}_p$ -functions of  $k$  as explained in the next Section.

### 3. DEFINITION OF $\mathbf{L}_p(s, \chi)$ FROM THE MEASURE $(\mathcal{S}_{L_n}^*(c))_n$

From expression (2.1) of  $\mathcal{S}_{L_n}(c)$ , the image by the Mellin transform in  $\mathbb{Z}_p[\text{Gal}(L_n/\mathbb{Q})]$ , defined by (see Footnote 1):

$$(3.1) \quad \left( \frac{L_n}{a} \right) \mapsto \theta(a) \langle a \rangle^s \left( \frac{L_n}{a} \right)^{-1} = a \langle a \rangle^{s-1} \left( \frac{L_n}{a} \right)^{-1}, \quad s \in \mathbb{Z}_p,$$

yields the  $p$ -adic measure giving annihilation theorems [Gra2018<sup>a</sup>] for real invariants:

$$(3.2) \quad \mathcal{S}_{L_n}^*(c)(s) \equiv \sum_{a=1}^{f_n} \left( \lambda_a^n(c) + \frac{1-c}{2} \right) a^{-1} \langle a \rangle^{1-s} \left( \frac{L_n}{a} \right) \pmod{qp^n}.$$

**3.1. Arbitrary  $p$ -adic approximation of  $\mathbf{L}_p(s, \chi)$ .** The  $\mathbf{L}_p$ -functions, of even Dirichlet's characters, are obtained by means of the Mellin transform of the previous twisted pseudo-measures, so that the Mellin transform of  $\left(1 - c \left(\frac{L_n}{c}\right)^{-1}\right)$  is  $\left(1 - \langle c \rangle^{1-s} \left(\frac{L_n}{c}\right)\right)$  and is a factor of the  $p$ -adic measure  $\mathcal{S}_{K,n}^*(c)(s)$  that must be dropped when this makes sense (thus in any case, except when  $\chi = \mathbf{1}$ , the unit character, and  $s = 1$ ):

**Theorem 3.1.** *Let  $c$  be any odd integer prime to  $p$  and to the conductor  $f$  of  $k$ . For all  $n$  large enough, let  $f_n$  be the conductor of  $L_n = k(\mu_{qp^n})$ , and for all  $a \in [1, f_n]$ , prime to  $f_n$ , let  $a'_c$  be the unique integer in  $[1, f_n]$  such that  $a'_c \cdot c - a = \lambda_a^n(c)f_n$ ,  $\lambda_a^n(c) \in \mathbb{Z}$ . The  $p$ -adic measure generating the  $\mathbf{L}_p$ -functions is given by:*

$$\mathcal{S}_{k,n}^*(c)(s) \equiv \sum_{a=1}^{f_n} \left( \lambda_a^n(c) + \frac{1-c}{2} \right) a^{-1} \langle a \rangle^{1-s} \left( \frac{k}{a} \right) \pmod{qp^n}.$$

Whence the  $\mathbf{L}_p$ -functions of Dirichlet's characters  $\chi$ :

$$\mathbf{L}_p(s, \chi) = \lim_{n \rightarrow \infty} \frac{1}{1 - \chi(c) \langle c \rangle^{1-s}} \times \sum_{a=1}^{f_n} \left[ \lambda_a^n(c) + \frac{1-c}{2} \right] a^{-1} \langle a \rangle^{1-s} \chi(a).$$

**Corollary 3.2.** *Since for  $s = -1$ ,  $a^{-1} \langle a \rangle^{1-s} = a^{-1} \langle a \rangle^2 = a$ , in our context  $p \in \{2, 3\}$  for which  $\theta^2 = 1$ , we get:*

$$\mathbf{L}_p(-1, \chi) = \lim_{n \rightarrow \infty} \frac{1}{1 - \chi(c) \langle c \rangle^2} \times \sum_{a=1}^{f_n} \left[ \lambda_a^n(c) + \frac{1-c}{2} \right] a \chi(a)$$

**Corollary 3.3.** *Assume  $\chi \neq 1$  and  $\chi(c) \neq 1$ . For  $s = -1$ ,  $\langle c \rangle \equiv 1 \pmod{q}$  and  $\chi(c) \in \mu_{[k:\mathbb{Q}]} \setminus \{1\}$ ; so, in the quadratic case with  $p = 3$  and in the cubic case with  $p = 2$ ,  $1 - \chi(c)\langle c \rangle^2$  is invertible. Then:*

- $1 - \chi(c)\langle c \rangle^2 \sim 2$  for the quadratic case and  $p = 2$ ,
- $1 - \chi(c)\langle c \rangle^2 \sim 1 - j$  for the cubic case and  $p = 3$ ;

from Remark 1.4 one gets:

(i) For  $p = 2$  in the quadratic case  $k = \mathbb{Q}(\sqrt{m})$ ,  $m \neq 2, 3$  and a half summation giving  $\frac{1}{2} \mathbf{L}_2(s, \chi)$ , for  $\#\mathbf{R}_2\mathbf{Z}_k[2^\infty] = \frac{1}{4}\#\mathbf{K}_2\mathbf{Z}_k[2^\infty]$  one obtains (since  $1 - \chi(c)\langle c \rangle^2 \sim 2$ ):

$$\#\mathbf{R}_2\mathbf{Z}_k[2^\infty] \sim \frac{1}{2} \mathbf{L}_2(-1, \chi) \equiv \frac{1}{2} \sum_{a=1}^{f_n/2} \left[ \lambda_a^n(c) + \frac{1-c}{2} \right] a\chi(a) \pmod{2^n}.$$

(ii) For the cubic case with  $p = 3$ , the half summation does not matter,  $\#\mathbf{K}_2\mathbf{Z}_k[3^\infty] = \#\mathbf{R}_2\mathbf{Z}_k[3^\infty]$ ,  $(1 - \chi(c)\langle c \rangle^2)(1 - \chi'(c)\langle c \rangle^2) \sim (1 - j)(1 - j^2) \sim 3$ , giving, with  $\mathbf{N} = \mathbf{N}_{\mathbb{Q}(\mu_3)/\mathbb{Q}}$ :

$$\begin{aligned} \#\mathbf{R}_2\mathbf{Z}_k[3^\infty] &\sim \mathbf{L}_3(-1, \chi) \cdot \mathbf{L}_3(-1, \chi') \\ &\equiv \frac{1}{3} \mathbf{N} \left( \sum_{a=1}^{f_n/2} \left[ \lambda_a^n(c) + \frac{1-c}{2} \right] a\chi(a) \right) \pmod{3^n}. \end{aligned}$$

**3.2. Application to the results of Deng-Li.** It is interesting to compare to their analytic computations of  $\mathbf{K}_2\mathbf{Z}_k$  for a specific family of quadratic fields  $k = \mathbb{Q}(\sqrt{m})$ , for which  $\mathbf{K}_2\mathbf{Z}_k[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times (\mathbb{Z}/2\mathbb{Z})^\delta$  with  $\delta \geq 3$ , for  $n$  even, where  $m \equiv -1 \pmod{8}$  is the product of  $n$  odd primes fulfilling some conditions in terms of quadratic residues (see [DeLi2023, Theorem 1.2]).

One starts from the formula of Corollary 3.3 (i) to compute:

$$\#\mathbf{R}_2\mathbf{Z}_k[2^\infty] \equiv \frac{1}{2} \sum_{a=1}^{f_n/2} \left[ \lambda_a^n(c) + \frac{1-c}{2} \right] a\chi(a) \pmod{2^n}.$$

The following program runs from data of [DeLi2023, Table] but may be used for any list of square-free integers; all groups  $\mathbf{R}_2\mathbf{Z}_k[2^\infty]$  have order  $2^4$  as expected (the integer  $c$  is chosen odd, co-prime to the conductor, with  $\text{kroncker}(f, c) = -1$ , whence  $1 - \chi(c)\langle c \rangle^2 \sim 2$ ):

```
{L=List
([7215,26455,77415,119535,142935,153735,166335,171015,196359,
226655,241215,243295,257335,283855,311415,315055,420135,430495,
447135,473415,475215,490295,504295,545415,550615,552695,553335,
563695,568815,592215,603655,606615,633399,657735,665223,673215,
685815,687895,727935,751335,755495,757055,790495,798135,803751,
807455,818935,833199,849615,878415,884455,886015,886335,896415,
905255,911495,934935,961935,973655,981695,990015]);
p=2;q=4;n=4;for(j=1,61,m=L[j];f=quaddisc(m);fn=q*p^n*m;c=1;
while(gcd(c,2*m)!=1 || kroncker(f,c)!=-1,c=c+1);S=0;
forstep(a=1,fn/2,2,if(gcd(a,m)!=1,next);aa=lift(Mod(a/c,fn));
la=(aa*c-a)/fn;eps=kroncker(m,a);S=S+eps*(la+(1-c)/2));
v=valuation(S,2)-1;print("m=",m," c=",c," v_2(R_2Zk)=",v)}
```

|          |      |              |          |      |              |
|----------|------|--------------|----------|------|--------------|
| m=7215   | c=7  | v_2(R_2Zk)=4 | m=911495 | c=3  | v_2(R_2Zk)=4 |
| m=26455  | c=17 | v_2(R_2Zk)=4 | m=934935 | c=17 | v_2(R_2Zk)=4 |
| m=77415  | c=11 | v_2(R_2Zk)=4 | m=961935 | c=11 | v_2(R_2Zk)=4 |
| m=119535 | c=7  | v_2(R_2Zk)=4 | m=973655 | c=3  | v_2(R_2Zk)=4 |
| m=142935 | c=43 | v_2(R_2Zk)=4 | m=981695 | c=3  | v_2(R_2Zk)=4 |
| m=153735 | c=11 | v_2(R_2Zk)=4 | m=990015 | c=7  | v_2(R_2Zk)=4 |

**3.3. Genus theory of  $p$ -adic pseudo-measures.** Considering for instance [DeLi2023, Theorem 1.2] giving, for this specific family of real quadratic fields:

$$\mathbf{K}_2\mathbf{Z}_k[2^\infty] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^\delta\mathbb{Z}, \quad \delta \geq 3,$$

the Deng–Li conjecture is  $\delta = 3$ .

We suggest that the proof of this conjecture may be obtained by using the “genus theory of abelian  $p$ -adic pseudo-measures”, in *real cyclic extension*  $k/\mathbb{Q}$  of degree  $dp^e$ ,  $p \nmid d$ ,  $e \geq 1$ , that we have built in [Gra1986<sup>b</sup>, Gra1987].

We recall this result that will be applied in the cases  $p = 2$  (for quadratic fields) and  $p = 3$  (for cyclic cubic fields) since these primes allow an easy relation between complex and  $p$ -adic  $\mathbf{L}$ -functions.

**Notations 3.4.** (i) Let  $\chi =: \varphi\psi$  be such that  $k$  is the fixed field of  $\chi$ , where  $\varphi$  is of prime to  $p$  order  $d$  and  $\psi$  of  $p$ -power order  $p^e$ .

Let  $M$  be the subfield of  $\mathbb{Q}^{\text{ab}}$  fixed by the character  $\theta^{-1}\chi$ , where  $\theta$  will be considered as the cyclotomic character of  $\mathbb{Q}(\mu_q)$  ( $q = p$  or  $4$  as usual). Since  $k$  is real,  $M$  is always an imaginary field.

Let  $M_0$  be the maximal subfield of  $M$  of prime-to- $p$  degree:

- for  $p \neq 2$ , then  $M \subseteq k(\mu_p)$ , and  $M_0$  is the field fixed by  $\theta^{-1}\varphi$  since  $\theta$  is of order  $p-1$ ; if  $\varphi = 1$ ,  $M_0 = \mathbb{Q}(\mu_p)$ ;
- for  $p = 2$ , then  $M \subseteq k(\sqrt{-1})$ , and  $M_0$  is the fixed field of  $\varphi$  since  $\theta$  is of order 2. if  $\varphi = 1$ ,  $M_0 = \mathbb{Q}$ .

(ii) Let  $\mathfrak{m}$  be the maximal ideal of  $\mathbb{Q}_p(\mu_{dp^e})$  and let  $v_{\mathfrak{m}}$  be the corresponding valuation with image  $\mathbb{Z}$ .

(iii) For any prime number  $r$ , let  $p^{d_r}$  be the degree of the splitting field of  $r$  in  $M/M_0$ ; let  $\mathbf{D} = \sum_{\ell} p^{d_{\ell}}$  where  $\ell$  runs through the set of primes ramified in  $M/M_0$ ,  $\ell \neq p$  and  $\ell$  totally split in  $M_0/\mathbb{Q}$ , whence:

- for  $p \neq 2$ :  $\psi(\ell) = 0$  and  $\theta^{-1}\varphi(\ell) = 1$  ( $\ell \equiv 1 \pmod{p}$  if  $\varphi = \mathbf{1}$ );
- for  $p = 2$ :  $\psi(\ell) = 0$  and  $\varphi(\ell) = 1$ .

(iv) Set  $\epsilon = 0$  (resp.  $\epsilon = 1$ ) when  $\varphi \neq \mathbf{1}$  (resp.  $\varphi = \mathbf{1}$ ).

Then we have obtained in [Gra1987, Théorème 0.2] the following result which does not seem very known but states a property of “stability” probably linked to the existence (or not) of zeroes of  $\mathbf{L}_p$ -functions (for more examples about zeroes of  $\mathbf{L}_p$ -functions, one may refer to [Wa1980, Wa1981, Wa1995, Wa1996, Wa1997]):

**Theorem 3.5.** Under Notations 3.4, where  $\chi = \varphi\psi$  is even,  $\varphi$  is of prime-to- $p$  order  $d$  and  $\psi$  of order  $p^e$ ,  $e \geq 1$ , we have with respect to the constant  $\mathbf{D} = \sum_{\ell} p^{d_{\ell}}$ :

a)  $v_{\mathfrak{m}}\left(\frac{1}{2}\mathbf{L}_p(s, \chi)\right) \geq \mathbf{C}(s)$ , for all  $s \in \mathbb{Z}_p$ , where  $\mathbf{C}(s)$  has the following values:

(i<sub>p</sub>)  $p \neq 2$  and  $\theta^{-1}\varphi(p) \neq 1$  (i.e.,  $p$  not totally split in  $M_0$ ); then:

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} - \epsilon.$$

(i<sub>2</sub>)  $p = 2$  and  $\varphi(2) \neq 1$  (i.e., 2 not totally split in  $M_0$ ); then:

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D}.$$

(ii<sub>p</sub>)  $p \neq 2$ ,  $\theta^{-1}\varphi(p) = 1$  and  $\theta^{-1}\chi(p) \neq 1$  (i.e.,  $p$  totally split in  $M_0$  and not totally split in  $M$ ); then:

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} + p^{d_p}.$$

( $ii_2$ )  $p = 2$ ,  $\varphi(2) = 1$  and  $\theta^{-1}\chi(2) \neq 1$  (i.e., 2 totally split in  $M_0$  and not totally split in  $M$ ); then:

$$\mathbf{C}(s) = \mathbf{C} = \mathbf{D} + 2^{d_2} - 2\epsilon.$$

( $iii_p$ )  $p \neq 2$ ,  $\theta^{-1}\chi(p) = 1$  (i.e.,  $p$  totally split in  $M$ ); then:

$$\mathbf{C}(s) := \mathbf{D} + v_{\mathfrak{m}}(ps).$$

( $iii_2$ )  $p = 2$ ,  $\theta^{-1}\chi(2) = 1$  (i.e., 2 totally split in  $M$ ); then:

$$\mathbf{C}(s) := \mathbf{D} + v_{\mathfrak{m}}(qs) - 2\epsilon.$$

b) We have either the equality  $v_{\mathfrak{m}}(\frac{1}{2}\mathbf{L}_p(s, \chi)) = \mathbf{C}(s)$ , for all  $s \in \mathbb{Z}_p$ , or the strict inequality  $v_{\mathfrak{m}}(\frac{1}{2}\mathbf{L}_p(s, \chi)) > \mathbf{C}(s)$ , for all  $s \in \mathbb{Z}_p$  (with  $s \neq 0$  in cases ( $iii_p$ ) and ( $iii_2$ )).

So, under the case of stability  $v_{\mathfrak{m}}(\frac{1}{2}\mathbf{L}_p(s, \chi)) = \mathbf{C}(s)$ , for all  $s \in \mathbb{Z}_p$ , a computation may be done at a suitable value of the  $\mathbf{L}_p$ -function, e.g.,  $s = 1$  giving  $\#\mathcal{T}_{k,p} \sim [k \cap \mathbb{Q}_{\infty} : \mathbb{Q}] \times \prod_{\chi \neq 1} \frac{1}{2} \mathbf{L}_p(1, \chi)$ ; so, assuming that  $k \cap \mathbb{Q}_{\infty} = \mathbb{Q}$ , we get [Gra2019, §§ 3.3, 3.4]:

$$(3.3) \quad \#\mathcal{T}_{k,p} \sim \prod_{\chi \neq 1} \frac{1}{2} \mathbf{L}_p(1, \chi),$$

thus  $\#\mathcal{T}_{k,p} \sim \frac{1}{2} \mathbf{L}_p(1, \chi)$  for  $k$  real quadratic,  $p = 2$ ,  $k \neq \mathbb{Q}(\sqrt{2})$ , or  $\#\mathcal{T}_{k,p} \sim \frac{1}{2} \mathbf{L}_p(1, \chi) \times \frac{1}{2} \mathbf{L}_p(1, \chi')$  for  $k$  cyclic cubic,  $p = 3$ , except  $k$  of conductor 9.

Which gives the following process where  $\mathbf{C}(s) = \mathbf{C}$ , a constant, as soon as  $s \in \mathbb{Z}_p^{\times}$ , since  $v_{\mathfrak{m}}(s) = 0$  in cases (iii) (e.g.,  $s = \pm 1$ ):

**Corollary 3.6.** (i) Let  $k$  be a real quadratic field and  $p = 2$ . Assume that the knowledge of  $\#\mathcal{T}_{k,2}$  (for instance with an arithmetic study using formula (1.6)) implies  $v_2(\frac{1}{2}\mathbf{L}_2(1, \chi)) = \mathbf{C}$ ; then we get  $v_2(\frac{1}{2}\mathbf{L}_2(s, \chi)) = \mathbf{C}(s)$  for all  $s \in \mathbb{Z}_2$ , whence  $\#\mathbf{K}_2\mathbf{Z}_k[2^{\infty}] = 2^{\mathbf{C}+2}$ .

(ii) Let  $k$  be a cyclic cubic field and  $p = 3$ ; let  $\mathfrak{m}$  be the maximal ideal of  $\mathbb{Q}_3(\mu_3)$  (so that  $\mathfrak{m}^2 = (3)$ ). Assume that the knowledge of  $\#\mathcal{T}_{k,3}$  implies  $v_{\mathfrak{m}}(\mathbf{L}_3(1, \chi)) = \mathbf{C}$ ; then  $v_{\mathfrak{m}}(\mathbf{L}_3(s, \chi)) = \mathbf{C}(s)$  for all  $s \in \mathbb{Z}_3$  and  $\chi \neq \mathbf{1}$ , then in particular,  $\#\mathbf{R}_2\mathbf{Z}_k[3^{\infty}] = 3^{\mathbf{C}}$ .

(iii) If  $v_2(\frac{1}{2}\mathbf{L}_2(1, \chi)) > \mathbf{C}$  (resp.  $v_{\mathfrak{m}}(\mathbf{L}_3(1, \chi)) > \mathbf{C}$ ), then one obtains  $\#\mathbf{K}_2\mathbf{Z}_k[2^{\infty}] > 2^{\mathbf{C}+2}$  (resp.  $\#\mathbf{K}_2\mathbf{Z}_k[3^{\infty}] > 3^{\mathbf{C}}$ ).

**Remark 3.7.** Assume that for the Deng–Li family of real quadratic fields  $\mathbb{Q}(\sqrt{\ell_1 \cdots \ell_n})$ , we know that  $\#\mathbf{H}_k[2^{\infty}] = 2^{n-1}$ ,  $\mathcal{R}_{k,2} = 1$  and  $\#\mathcal{W}_{k,2} = 2$ ; then  $\#\mathcal{T}_{k,2} = 2^n$ , giving the valuation  $n$  for  $\frac{1}{2}\mathbf{L}_2(1, \chi)$ ; so, Theorem 3.5 ( $iii_2$ ) yields  $\mathbf{D} = n$ ,  $\mathbf{C} = \mathbf{D} + v_{\mathfrak{m}}(q) - 2\epsilon = n$ ; then the conjecture  $\delta = 3$  follows.

**3.4. Proof of the Deng–Li conjecture  $\delta = 3$ .** We consider the family defined in [DeLi2023, Theorem 1.2]. We will prove the conjecture  $\delta = 3$  in the writing  $\mathbf{K}_2\mathbf{Z}_k[2^{\infty}] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathbb{Z}/2^{\delta}\mathbb{Z}$  and some other properties of this family of real quadratic fields.

Recall that  $k = \mathbb{Q}(\sqrt{m})$ ,  $m = \ell_1 \ell_2 \cdots \ell_n$ ,  $n$  even, with:

- (i)  $\ell_1 \equiv 3 \pmod{8}$ ,  $\ell_i \equiv 5 \pmod{8}$  for  $i \geq 2$ ,
- (ii)  $(\frac{\ell_1}{\ell_2}) = -1$ ,  $(\frac{\ell_1}{\ell_j}) = 1$  for  $j \geq 3$ ,
- (iii)  $(\frac{\ell_i}{\ell_j}) = -1$  for  $2 \leq i < j \leq n$ .

Condition (i) implies  $m \equiv -1 \pmod{8}$  and  $k$  of discriminant  $4m$ ; then  $\left(\frac{\ell_i}{\ell_j}\right) = \left(\frac{\ell_j}{\ell_i}\right)$  for all  $i, j$  since for  $i \neq j$ , one of the two primes is congruent to 1 modulo 4. It implies also that the fundamental unit  $\varepsilon$  is of norm 1 since  $-1$  is not norm in  $k/\mathbb{Q}$ ; then, if  $\varepsilon = a + b\sqrt{m}$ , one has  $a^2 + b^2 \equiv 1 \pmod{8}$ , whence, either  $a = 4a'$  with  $b$  odd, or  $b = 4b'$  with  $a$  odd.

Let  $\mathfrak{p}$  be the prime ideal above (2) and let  $k_{\mathfrak{p}}$  be the completion of  $k$  at  $\mathfrak{p}$ ; thus  $k_{\mathfrak{p}} = \mathbb{Q}_2(\sqrt{-1})$  proving that 2 is local norm at 2 in  $k/\mathbb{Q}$  and that the norm group of local units is equal to  $1 + 4\mathbb{Z}_2$ .

The Hasse norm residue symbols, of the form  $\left(\frac{\ell_i, k/\mathbb{Q}}{\ell_j}\right)$ , characterizing the property “ $\ell_i$  local norm at  $\ell_j$  in  $k/\mathbb{Q}$ ”, are given by the quadratic residue symbols  $\left(\frac{\ell_i}{\ell_j}\right)$ ; indeed,  $\ell_i$  norm in  $\mathbb{Q}_{\ell_j}(\sqrt{m})/\mathbb{Q}_{\ell_j}$  is equivalent (for  $i \neq j$ ) to  $\ell_i$  square in  $\mathbb{Q}_{\ell_j}^{\times}$  since the norm group of local units of  $\mathbb{Q}_{\ell_j}$  is of index 2 in  $\mu_{\ell_j-1}$ . We can add the following:

(iv)  $\left(\frac{2}{\ell_i}\right) = (-1)^{\frac{\ell_i-1}{8}} = -1$ , for all  $i$ ,  $\left(\frac{\ell_1}{2}\right) = -1$ ,  $\left(\frac{\ell_i}{2}\right) = 1$  for  $i \geq 2$ .

(v)  $\left(\frac{\ell_i}{\ell_i}\right) = 1$  for all  $i \neq 2$  and  $\left(\frac{\ell_2}{\ell_2}\right) = -1$ , obtained from the product formula of the Hasse norm residue symbols of fixed  $\ell_i$  with  $n$  even. Whence the matrix with product formula taken on each line:

$$(3.4) \quad \begin{array}{c|cccccccc} & 2 & \ell_1 & \ell_2 & \ell_3 & \dots & \ell_i & \dots & \ell_n \\ \hline 2 & 1 & -1 & -1 & -1 & \dots & -1 & \dots & -1 \\ \hline \ell_1 & -1 & 1 & -1 & 1 & \dots & 1 & \dots & 1 \\ \hline \ell_2 & 1 & -1 & -1 & -1 & \dots & -1 & \dots & -1 \\ \hline \ell_3 & 1 & 1 & -1 & 1 & \dots & -1 & \dots & -1 \\ \hline \vdots & & & & & & & & \\ \hline \ell_i & 1 & 1 & -1 & -1 & \dots & 1 & \dots & -1 \\ \hline \vdots & & & & & & & & \\ \hline \ell_n & 1 & 1 & -1 & -1 & \dots & -1 & \dots & 1 \\ \hline -1 & -1 & -1 & 1 & 1 & \dots & 1 & \dots & 1 \end{array}$$

Put  $G := \text{Gal}(k/\mathbb{Q}) =: \langle \sigma \rangle$  and:

$$\Omega_k := \{(s_1, \dots, s_{n+1}) \in G^{n+1}, s_1 \cdots s_{n+1} = 1\} \simeq G^n;$$

let  $\omega_k$  be the map  $\mathbb{Q}^{\times} \rightarrow \Omega_k$  defined, by means of the Hasse norm residue symbols, by  $\omega_k(x) = \left(\left(\frac{x, k/\mathbb{Q}}{v}\right)\right)_v$ , where  $v$  runs through the  $n+1$  places ramified in  $k/\mathbb{Q}$ , and for which the product formula  $\prod_{v \text{ ramified}} \left(\frac{x, k/\mathbb{Q}}{v}\right) = 1$  holds as soon as  $(x)$  is the norm of an ideal in  $k/\mathbb{Q}$ , hence local norm at every non-ramified place.

3.4.1. *Proof of  $\mathbf{H}_k[2^{\infty}] \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$ .*

This is proven in [DeLi2023], but we can bring more informations and remarks. The first one is given by the Chevalley–Herbrand formula [Che1933, pp. 402-406] in  $k/\mathbb{Q}$ :

$$\#\mathbf{H}_k^G = \frac{2^{n+1}}{[k : \mathbb{Q}](E_{\mathbb{Q}} : E_{\mathbb{Q}} \cap \mathbf{N}_{k/\mathbb{Q}}(k^{\times}))},$$

since  $n+1$  primes are ramified; we shall write instead:

$$(3.5) \quad \#\mathbf{H}_k^G = \frac{\#\Omega_k}{\#\omega_k(E_{\mathbb{Q}})} = \frac{2^n}{\#\omega_k(\{\pm 1\})} = 2^{n-1},$$

with  $E_{\mathbb{Q}} = \{\pm 1\}$  and  $-1 \notin \mathbf{N}_{k/\mathbb{Q}}(k^{\times})$ .

**Lemma 3.8.** *The subgroup  $\mathbf{H}_k^G$  is elementary of 2-rank  $n - 1$ ; it is generated by the classes of the prime ideals  $\mathfrak{l}_i \mid \ell_i$ ,  $i = 1, \dots, n$ , and by the class of  $\mathfrak{p} \mid (2)$ . There are two independent relations of principality of the form  $\mathfrak{p}^{a_0} \prod_{i=1}^n \mathfrak{l}_i^{a_i} = (\alpha)$  between the ramified primes (where the exponents are 0 or 1), an obvious one being  $\prod_{i=1}^n \mathfrak{l}_i = (\sqrt{m})$ .*

*Proof.* We have the classical exact sequence, where  $E_k = \langle -1, \varepsilon \rangle$  is the group of units of  $k$  and where  $\mathbf{H}_k^{\text{ram}}$  is the subgroup of  $\mathbf{H}_k^G$  generated by the classes of the ramified primes:

$$1 \rightarrow \mathbf{H}_k^{\text{ram}} \rightarrow \mathbf{H}_k^G \rightarrow \{\pm 1\} \cap \mathbf{N}_{k/\mathbb{Q}}(k^\times) / \mathbf{N}_{k/\mathbb{Q}}(E_k) \rightarrow 1,$$

giving here  $\mathbf{H}_k^G = \mathbf{H}_k^{\text{ram}} \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$ ; so, there are exactly two independent relations of principality between the ramified primes.

Then, from [Gra2017<sup>b</sup>, § 4.4] or our old papers [Gra1973<sup>a</sup>, Gra1973<sup>b</sup>], we have for the second element of the filtration:

$$\# \left( (\mathbf{H}_k / \mathbf{H}_k^G)^G \right) = \frac{\#\Omega_k}{\#\omega_k(\Lambda)}, \quad \Lambda = \langle -1, 2, \ell_1, \dots, \ell_n \rangle_{\mathbb{Z}}$$

then,  $\#\omega_k(\Lambda) = (\Lambda : \Lambda \cap \mathbf{N}_{k/\mathbb{Q}}(k^\times)) = 2^n$ , computing norm residue symbols with the Rédei matrix (3.4) of rank  $n$ ; whence  $\mathbf{H}_k[2^\infty] = \mathbf{H}_k^G \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$ .  $\square$

### 3.4.2. Program of the relations.

The program gives the non-trivial relation whose writing does not contain  $\mathfrak{l}_1$  ( $\ell_1 \equiv 3 \pmod{8}$ ), then that obtained multiplying by  $(\sqrt{m})$ . The list  $L$  gives the exponents  $a_0, a_1, a_2, \dots$ , of ideals  $\mathfrak{p}, \mathfrak{l}_1, \mathfrak{l}_2, \dots$ , in this order especially for the particular primes  $\mathfrak{l}_1, \mathfrak{l}_2$  (which is not so immediate since the PARI factorization of  $m$  gives primes in the ascending order). The generator  $\alpha$  of the relation of principality is  $\alpha = u + v\sqrt{m}$ ; for checking, its norm  $N$  and the sign  $S$  of  $N$  are computed. We will prove that the relation is always  $\mathfrak{p}\mathfrak{l}_2 = (\alpha)$ .

```
{m=List
([7215,26455,77415,119535,142935,153735,166335,171015,196359,
226655,241215,243295,257335,283855,311415,315055,420135,430495,
447135,473415,475215,490295,504295,545415,550615,552695,553335,
563695,568815,592215,603655,606615,633399,657735,665223,673215,
685815,687895,727935,751335,755495,757055,790495,798135,803751,
807455,818935,833199,849615,878415,884455,886015,886335,896415,
905255,911495,934935,961935,973655,981695,990015]);p=2;
for(j=1,61,M=m[j];D=quaddisc(M);r=omega(D);L0=List;
for(i=1,r,listput(L0,0));P=x^2-M;K=bnfinit(P,1);print();Lel=List;
Div=component(factor(D),1);listput(Lel,2,1);h=3;
for(i=2,r,c=Div[i];
if(Mod(c,4)==-1,e11=c;listput(Lel,c,2)));
for(i=2,r,c=Div[i];
if(c!=e11 & kronecker(c,e11)==-1,e12=c;listput(Lel,c,3)));
for(i=2,r,c=Div[i];
if(c!=e12 & Mod(c,4)==1,h=h+1;listput(Lel,c,h)));
print("M=",M,"=",Lel);for(k=1,2^r-1,B=binry(k);t=#B;
LB=L0;for(i=1,t,listput(LB,B[i],r-t+i));F=idealfactor(K,D);
Fel=List;CFel=component(F,1);listput(Fel,CFel[1],1);h=3;
for(i=2,r,c=CFel[i][1];
if(Mod(c,4)==-1,e11=c;listput(Fel,CFel[i],2)));
for(i=2,r,c=CFel[i][1];
if(c!=e11 & kronecker(c,e11)==-1,e12=c;listput(Fel,CFel[i],3)));
for(i=2,r,c=CFel[i][1];
if(c!=e12 & Mod(c,4)==1,h=h+1;listput(Fel,CFel[i],h)));
A=1;for(j=1,r,e=LB[j];if(e==0,next);A=idealmul(K,A,Fel[j]));
Q=bnfisprincipal(K,A);u=Q[2][1];v=Q[2][2];
if(Q[1]==0 & u!=0,print("L=",LB);N=u^2-M*v^2;S=sign(N);
```



```
print(component(factor(abs(N)),1));
print("u=",u," v=",v," S=",sign(S))})
```

Let's give some excerpts showing all possible cases:

```
M=7215=List([2,3,5,13,37])      M=26455=List([2,11,13,5,37])
L=List([1,0,1,0,0])            L=List([1,0,1,0,0])
[2,5]                            [2,13]
u=-85 v=1 S=1                    u=86654841 v=-532769 S=1
L=List([1,1,0,1,1])            L=List([1,1,0,1,1])
[2,3,13,37]                      [2,5,11,37]
u=1443 v=17 S=-1                u=1084184915 v=-6665757 S=-1

M=504295=List([2,11,173,5,53])  M=665223=List([2,3,461,13,37])
L=List([1,0,1,0,0])            L=List([1,0,1,0,0])
[2,173]                          [2,461]
u=7970980451749 v=11224562253 S=1 u=10603 v=13 S=1
L=List([1,1,0,1,1])            L=List([1,1,0,1,1])
[-1,2,5,11,53]                  [2,3,13,37]
u=-32719598967495 v=46075031513 S=-1 u=18759 v=23 S=-1
```

### 3.4.3. The non-trivial relation for $n$ even.

Consider the relations:

$$(3.6) \quad \begin{cases} \mathfrak{p}^{a_0} \prod_{i=1}^n \ell_i^{a_i} =: \mathfrak{p}^{a_0} \prod_{i \in I} \ell_i = (\alpha), \quad \alpha \in k^\times, \\ \mathbf{N}_{k/\mathbb{Q}}(\alpha) = s 2^{a_0} \prod_{i \in I} \ell_i, \quad s \in \{\pm 1\}, \end{cases}$$

where  $I$  is a subset of  $[1, n]$ ; from the trivial relation  $(\sqrt{m}) = \prod_{i=1}^n \ell_i$  and  $\mathbf{N}_{k/\mathbb{Q}}(\sqrt{m}) = -\prod_{i=1}^n \ell_i$  we deduce  $(\alpha\sqrt{m}) = (c) \mathfrak{p}^{a_0} \prod_{i=1}^n \ell_i^{\bar{a}_i}$ , where  $\bar{a}_i = 1 - a_i$  and  $c = \prod_{i \in I} \ell_i$ ; taking  $\bar{\alpha} := \frac{\alpha\sqrt{m}}{c}$ , one gets the equivalent complementary relations, where  $\bar{I} = [1, n] \setminus I$ :

$$(3.7) \quad \begin{cases} \mathfrak{p}^{a_0} \prod_{i \in \bar{I}} \ell_i = (\bar{\alpha}), \\ \mathbf{N}_{k/\mathbb{Q}}(\bar{\alpha}) = -s 2^{a_0} \prod_{i \in \bar{I}} \ell_i. \end{cases}$$

**Lemma 3.9.** (i) In the relations (3.6) and (3.7), one has  $a_0 = 1$  and the two non-trivial equivalent relations may be written:

- $\mathfrak{p} \cdot \prod_{j \in J} \ell_j = (\beta)$ ,  $J \subseteq \{2, \dots, n\}$ ,  $\#J \equiv 1 \pmod{2}$ .
- $\mathfrak{p} \cdot \ell_1 \prod_{j \in \bar{J}} \ell_j = (\bar{\beta})$ ,  $\bar{J} \subseteq \{2, \dots, n\}$ ,  $\#\bar{J} \equiv 0 \pmod{2}$ ,

The relation is  $\mathfrak{p} \cdot \ell_2 = (\beta)$ .

(ii) These relations are given by the ideals  $(\varepsilon + 1)$  and  $(\varepsilon - 1)$  for which  $\frac{\varepsilon + 1}{\varepsilon - 1} = C \cdot \sqrt{m}$ , where  $C$  is an odd rational number.

*Proof.* (i) Assume that  $a_0 = 0$ ; let  $\prod_{j \in J} \ell_j = (\beta)$  be the non-trivial relation such that  $J \subseteq [2, n]$ . Then  $\prod_{j \in J} \ell_j = \pm \mathbf{N}_{k/\mathbb{Q}}(\beta)$ ; since  $\mathbf{N}_{k/\mathbb{Q}}(\beta) \equiv 1 \pmod{4}$  as well as the  $\ell_j$ 's, we get  $\prod_{j \in J} \ell_j = \mathbf{N}_{k/\mathbb{Q}}(\beta)$ .

Consider the symbols  $\left(\frac{\prod_{j \in J} \ell_j}{\ell_i}\right)$  for all  $i \in [1, n]$ , using (3.4):

- For  $i = 2$ , one gets  $\left(\frac{\prod_{j \in J} \ell_j}{\ell_2}\right) = 1$  if and only if  $\#J$  is even (which solves the case  $n = 2$ ).
- for  $i = 1$ , one obtains  $\left(\frac{\prod_{j \in J} \ell_j}{\ell_1}\right) = 1$  if and only if  $j = 2 \notin J$ ;
- for any  $i \geq 3$ ,  $\left(\frac{\prod_{j \in J} \ell_j}{\ell_i}\right) = 1$  if and only if  $j = i \notin J$ .

So we obtain  $J = \emptyset$  (absurd); whence the relation  $\mathfrak{p} \cdot \prod_{j \in J} \mathfrak{l}_j = (\beta)$  for  $J \subseteq [2, n]$ .

Then  $\left(\frac{2 \prod_{j \in J} \ell_j}{\ell_i}\right) = 1$  for all  $i \in [1, n]$  since  $\mathbf{N}_{k/\mathbb{Q}}(\beta) = -2 \prod_{j \in J} \ell_j$  is not possible ( $-1$  is not local norm at 2); so this is equivalent to  $\left(\frac{\prod_{j \in J} \ell_j}{\ell_i}\right) = -1$  for all  $i \in [1, n]$ . The case  $i = 2$  gives  $\#J$  odd; the case  $i \in J$ , when  $j = 2 \notin J$ , gives a contradiction, as well as  $2 \in J$  and  $i \notin J$ , so that one verifies that only the relation  $\mathfrak{p} \cdot \mathfrak{l}_2 = (\beta)$  holds (the  $(n+1) \times (n+1)$  matrix (3.4) is of rank  $n$  and the sum of the two lines corresponding to 2 and  $\ell_2$  gives again the relation).

(ii) We have  $(\varepsilon + s)^{1-\sigma} = s\varepsilon$ ; so  $(\varepsilon + s)$  is a principal invariant ideal necessarily of the form  $(c)\mathfrak{p}\mathfrak{m}$ ,  $c \in \mathbb{Q}^\times$ ,  $\mathfrak{p} \mid (2)$ ,  $\mathfrak{m} \mid (\sqrt{m})$ . Then,  $\frac{\varepsilon+1}{\varepsilon-1} = C \cdot \sqrt{m}$  since  $\left(\frac{\varepsilon+1}{\varepsilon-1} \cdot \frac{1}{\sqrt{m}}\right)^{1-\sigma} = 1$ .  $\square$

**Lemma 3.10.** *The case  $b$  even does not occur.*

*Proof.* Assume that  $b$  is even; then  $b = 4b'$ ; there exists a sign  $s = \pm 1$  such that  $a + s \equiv 2 \pmod{4}$ , so that  $\varepsilon + s = a + s + 4b'\sqrt{m}$  and the ideal  $(\varepsilon + s) = (a + s + 4b'\sqrt{m})$  is of the form  $(2)(a'' + 2b'\sqrt{m})$  with  $a''$  odd, which can not give the non-trivial relations of Lemma 3.9 since  $a'' + 2b'\sqrt{m}$  is “odd” (absurd).  $\square$

**Lemma 3.11.** *In the writing  $\varepsilon = a + b\sqrt{m}$ ,  $a = 4a'$  with  $a'$  odd.*

*Proof.* Assume that  $a' = 2a''$  and put  $\theta_{s'} := \varepsilon + s' = 8a'' + s' + b\sqrt{m}$ ,  $s' \in \{\pm 1\}$ . We have, using  $mb^2 = a^2 - 1$ :

$$\mathbf{N}_{k/\mathbb{Q}}(\theta_{s'}) = 2(8a''s' + 1).$$

From Lemma 3.9 the non-trivial relation may be written into the following forms (where  $c$  is an odd rational),  $(\theta_{s'}) = (c)\mathfrak{p} \prod_{j \in J} \mathfrak{l}_j$  or  $(\theta_{s'}) = (c)\mathfrak{p} \mathfrak{l}_1 \prod_{j \in \bar{J}} \mathfrak{l}_j$ ,  $\#J$  odd,  $\#\bar{J}$  even, giving by taking the norm:

$$8a''s' + 1 = s c^2 \prod_{j \in J} \ell_j \quad \text{or} \quad 8a''s' + 1 = s c^2 \ell_1 \prod_{j \in \bar{J}} \ell_j, \quad s \in \{\pm 1\}.$$

The second case gives  $3s \equiv 1 \pmod{8}$  since  $\ell_1 \equiv 3 \pmod{8}$  and  $\prod_{j \in \bar{J}} \ell_j \equiv 5^{\#\bar{J}} \equiv 1 \pmod{8}$ ; the first one gives  $5^{\#J} s \equiv 5s \equiv 1 \pmod{8}$  (absurd).  $\square$

#### 3.4.4. Structure of $\mathcal{T}_{k,2}$ and proof of the triviality of $\mathcal{R}_{k,2}$ .

**Lemma 3.12.** *The module  $\mathcal{T}_{k,2}$  is of  $\mathbb{Z}_2$ -rank  $n-1$  and the maximal subextension of  $H_{k,2}^{\text{pr}}/k_\infty$ , of exponent 2, is the Kummer extension  $k_\infty(\sqrt{\varepsilon}, \sqrt{\ell_1}, \dots, \sqrt{\ell_n})$ , under the existence of the two independent relations  $\sqrt{\ell_1} \cdots \sqrt{\ell_n} = \sqrt{m} \in k^\times$  and  $\sqrt{2} \prod_{i \in I} \sqrt{\ell_i} = \alpha\sqrt{\varepsilon}$ ,  $\alpha \in k^\times$ .*

*Proof.* The maximal subextension of  $H_{k,2}^{\text{pr}}/k$ , of exponent 2, is the Kummer extension  $k(\sqrt{2}, \sqrt{\varepsilon}, \sqrt{\ell_1}, \dots, \sqrt{\ell_n})$  (all these radicals are squares of ideals of  $k$  and are totally positive; there is no other radicals since  $\mathbf{H}_k[2^\infty] = \mathbf{H}_k^{\text{ram}}$ ). We utilize the fact that  $\sqrt{2} \in k_\infty$ , then the relations  $\ell_1 \cdots \ell_n = m \in k^{\times 2}$  and  $\mathfrak{p} \prod_{i \in I} \mathfrak{l}_i = (\alpha)$ ,  $\alpha \in k^\times$ , giving, by squaring,  $2 \prod_{i \in I} \ell_i = \alpha^2 \eta > 0$ ,  $\eta \in E_k$ ,  $\eta \neq 1$ ; then we may assume that  $\eta = \varepsilon$ , whence  $2 \prod_{i \in I} \ell_i = \alpha^2 \varepsilon$  (in fact  $2\ell_2 \varepsilon \in k^{\times 2}$ ).

Whence the result since  $\mathcal{T}_{k,2}$  is a direct factor in  $\text{Gal}(H_{k,2}^{\text{pr}}/k)$ .  $\square$

**Theorem 3.13.** *The normalized 2-adic regulator  $\mathcal{R}_{k,2}$  is trivial.*

*Proof.* We have  $\varepsilon = a + b\sqrt{m} = 4a' + b\sqrt{m}$  with  $a^2 - mb^2 = 1$ ; then:

$$\varepsilon^2 = a^2 + 2ab\sqrt{m} + mb^2 = 1 + 2mb^2 + 8a'b\sqrt{m}.$$

Recall that  $a'$  is odd, that  $b$  is odd and  $m \equiv -1 \pmod{8}$ ; thus  $\varepsilon^2 \equiv 1 - 2 + 8\sqrt{m} \pmod{16}$ ; whence  $\log_2(\varepsilon) \sim 4$ .

From [Gra2018<sup>b</sup>, Proposition 5.2], we have, in our context:

$$\#\mathcal{R}_{k,2} \sim \frac{1}{2} \cdot \frac{(\mathbb{Z}_2 : \log(\mathbf{N}_{k/\mathbb{Q}}(\mathcal{U}_{k,2})))}{\#\mathcal{W}_{k,2} \cdot \prod_{p|2} \mathbf{N}_p} \cdot \frac{R_k}{\sqrt{D_k}},$$

where  $R_k = \log_2(\varepsilon)$  is the usual 2-adic regulator [Wa1997, §5.5] and  $D_k$  the discriminant of  $k$ . Since  $(\mathbb{Z}_2 : \log(\mathbf{N}_{k/\mathbb{Q}}(\mathcal{U}_{k,2}))) = 4$ ,  $R_k \sim 4$ ,  $\#\mathcal{W}_{k,2} = 2$ ,  $\prod_{p|2} \mathbf{N}_p = 2$ ,  $\sqrt{D_k} \sim 2$ , this yields  $\#\mathcal{R}_{k,2} \sim 1$ .

Formula (1.6) gives  $\#\mathcal{T}_{k,2} = 2^{n-1} \times 1 \times 2 = 2^n$ , and Remark 3.7 ends the proof of the conjecture  $\delta = 3$ .  $\square$

**Remark 3.14.** *The other cases of [DeLi2023, Theorem 1.3] can be proved with the same method. Indeed, one obtains  $\mathbf{H}_k[2^\infty] = \mathbf{H}_k^G$  and  $\mathcal{R}_{k,2} = 1$ . Since  $\mathcal{W}_{k,2} = 1$  and  $\#\mathbf{H}_k[2^\infty] = 2^{n+1}$ ,  $\#\mathcal{T}_{k,2} = 2^{n-1}$ , giving  $\#\mathcal{T}_{k,2} \sim \frac{1}{2} \mathbf{L}_p(1, \chi) \sim 2^{n-1}$ . From Theorem 3.5,  $\mathbf{D} = n$ ,  $\mathbf{C} = n - 1$  yielding, in the two cases,  $\#\mathbf{R}_2\mathbf{Z}_k[2^\infty] = 2^{n-1}$ .*

#### 4. CASE OF REAL QUADRATIC FIELDS

**4.1. Computation of  $\mathcal{T}_{k,2}$  and  $\#\mathbf{R}_2\mathbf{Z}_k[2^\infty]$ .** The following program computes  $\#\mathbf{R}_2\mathbf{Z}_k[2^\infty]$  for any real quadratic field (so  $f$  is  $m$  or  $4m$ ) and the structure of  $\mathcal{T}_{k,2}$  for checking of Theorem 3.5 (cases (ii)<sub>2</sub>) and (iii)<sub>2</sub>). In that cases,  $\epsilon = 1$ ,  $\mathbf{D}$  is the number of odd ramified primes, and  $\mathbf{C} = \mathbf{D}$  (resp.  $\mathbf{C} = \mathbf{D} - 1$ ) in the case  $\theta^{-1}\chi(2) = 1$  (resp.  $\theta^{-1}\chi(2) \neq 1$ ).

Recall that  $n$  must be chosen large enough (below we take  $n = 10$  for  $m > 100$ ,  $n = 13$  for  $m > 10^3$ ).

We do not write the trivial cases  $\mathcal{T}_{k,2} = \mathbf{R}_2\mathbf{Z}_k[2^\infty] = 1$ .

```
{p=2;q=4;n=10;for(m=5,200,if(core(m)!=m,next);f=quaddisc(m);
P=x^2-m;K=bnfinit(P,1);Kpn=bnrinit(K,p^n);H=Kpn.cyc;T=List;
e=matsize(H)[2];valT=0;for(k=1,e-1,h=H[e-k+1];v=valuation(h,p);
if(v>0,valT=valT+v;listput(T,p^v));fn=q*2^n*f;c=1;
while(Mod(c,2)==0 || kronecker(f,c)!=-1,c=c+1);S=0;
forstep(a=1,fn/2,2,if(gcd(a,m)!=1,next);aa=lift(Mod(a/c,fn));
la=(aa*c-a)/fn;eps=kronecker(m,a);S=S+aa*eps*(1a+(1-c)/2));
valR=valuation(S,2)-1;D=omega(m);if(Mod(m,2)==0,D=D-1);
C=D;if(kronecker(quaddisc(-m),2)!=1,C=D-1);
if(valT==C,print("m=",m," c=",c," T_k=",T," valT=",valT,
" val(R_2Zk)=",valR," C=",C," Equality"));
if(valT>C,print("m=",m," c=",c," T_k=",T," valT=",valT,
" val(R_2Zk)=",valR," C=",C," Inequality"))}
```

|      |      |         |        |              |     |            |
|------|------|---------|--------|--------------|-----|------------|
| m=7  | c=5  | T_k=[4] | valT=2 | val(R_2Zk)=2 | C=1 | Inequality |
| m=14 | c=3  | T_k=[2] | valT=1 | val(R_2Zk)=1 | C=0 | Inequality |
| m=15 | c=13 | T_k=[4] | valT=2 | val(R_2Zk)=2 | C=2 | Equality   |
| m=17 | c=3  | T_k=[2] | valT=1 | val(R_2Zk)=1 | C=0 | Inequality |
| m=21 | c=11 | T_k=[2] | valT=1 | val(R_2Zk)=1 | C=1 | Equality   |
| m=23 | c=3  | T_k=[4] | valT=2 | val(R_2Zk)=2 | C=1 | Inequality |
| m=30 | c=11 | T_k=[2] | valT=1 | val(R_2Zk)=1 | C=1 | Equality   |
| m=31 | c=7  | T_k=[8] | valT=3 | val(R_2Zk)=3 | C=1 | Inequality |
| m=33 | c=5  | T_k=[2] | valT=1 | val(R_2Zk)=1 | C=1 | Equality   |
| m=34 | c=7  | T_k=[2] | valT=1 | val(R_2Zk)=1 | C=0 | Inequality |
| m=35 | c=3  | T_k=[2] | valT=1 | val(R_2Zk)=1 | C=1 | Equality   |
| m=39 | c=11 | T_k=[4] | valT=2 | val(R_2Zk)=2 | C=2 | Equality   |

```

m=41  c=3  T_k=[16]  valT=4  val(R_2Zk)=3  C=0  Inequality
m=42  c=5  T_k=[2]   valT=1  val(R_2Zk)=1  C=1  Equality
m=46  c=11 T_k=[2]   valT=1  val(R_2Zk)=1  C=0  Inequality
m=47  c=3  T_k=[8]   valT=3  val(R_2Zk)=3  C=1  Inequality
m=51  c=11 T_k=[2]   valT=1  val(R_2Zk)=1  C=1  Equality
(...)
m=1001 c=3  T_k=[2,2] valT=2  val(R_2Zk)=2  C=2  Equality
m=1002 c=5  T_k=[16]  valT=4  val(R_2Zk)=5  C=1  Inequality
m=1003 c=5  T_k=[4]   valT=2  val(R_2Zk)=2  C=1  Inequality
m=1005 c=29 T_k=[2,8]  valT=4  val(R_2Zk)=5  C=2  Inequality
m=1006 c=7  T_k=[2]   valT=1  val(R_2Zk)=1  C=0  Inequality
m=1007 c=3  T_k=[4]   valT=2  val(R_2Zk)=2  C=2  Equality
m=1009 c=11 T_k=[2]   valT=1  val(R_2Zk)=1  C=0  Inequality
m=1010 c=3  T_k=[2]   valT=1  val(R_2Zk)=1  C=1  Equality
m=1011 c=7  T_k=[4]   valT=2  val(R_2Zk)=2  C=1  Inequality
m=1015 c=17 T_k=[2,4]  valT=3  val(R_2Zk)=3  C=3  Equality
m=1022 c=3  T_k=[2,16] valT=5  val(R_2Zk)=9  C=1  Inequality
m=1023 c=5  T_k=[2,64] valT=7  val(R_2Zk)=7  C=3  Inequality

```

These results give an illustration of Theorem 3.5; some examples show that the equality  $v_2(\#\mathcal{T}_{k,2}) = \mathbf{C}$  often occur and that when  $v_2(\#\mathcal{T}_{k,2}) > \mathbf{C}$ , then  $v_2(\#\mathbf{R}_2\mathbf{Z}_k[2^\infty]) > \mathbf{C}$ , with some cases where the two valuations are distinct ( $m = 41, 65, 66, 114, \dots$ ).

Of course, the numerical examples given in [DeLi2023, Table], yield equalities with  $\mathbf{C} = \mathbf{D} = 4$  since 2 always splits in  $M = \mathbb{Q}(\sqrt{-m})$  (Theorem 3.5, case (iii)<sub>2</sub>),  $\mathbf{C} = \mathbf{D} + v_2(4s) - 2\epsilon = 4$  for  $s = -1$ ). So, we refer to Remark 3.7 for the continuation of the method.

**4.2. Computation of  $\mathcal{T}_{k,3}$  and  $\#\mathbf{K}_2\mathbf{Z}_k[3^\infty]$ .** For  $p = 3$ , since genus theory of  $p$ -adic pseudo-measures does not apply, we only compute the valuation of  $\#\mathbf{R}_2\mathbf{Z}_k[3^\infty]$  and the structure of  $\mathcal{T}_{k,3}$ ; the trivial cases  $v_3(\mathbf{K}_2\mathbf{Z}_k) = 0$   $T_k = [ ]$  are not written:

```

{p=3;q=3;n=5;for(m=2,10^2,if(core(m)!=m,next);P=x^2-m;f=quaddisc(m);
K=bnfinit(P,1);Kpn=bnrinit(K,p^n);H=Kpn.cyc;T=List;
e=matsize(H)[2];for(k=1,e-1,C=H[e-k+1];v=valuation(C,p);
if(v>0,listput(T,p^v));fn=q*p^n*f;c=1;while(Mod(c,p)==0 ||
kronecker(f,c)!=-1,c=c+1);S=0;forstep(a=1,fn/2,[1,2],
if(gcd(a,f)==1,next);aa=lift(Mod(a/c,fn));la=(aa*c-a)/fn;
eps=kronecker(f,a);S=S+a*eps*(la+(1-c)/2));print
("m=",m," c=",c," v_3(K_2Zk)=",valuation(S,3)," T_k=",T)}
m=6  c=7  v_3(K_2Zk)=1  T_k=[3]   m=69 c=2  v_3(K_2Zk)=1  T_k=[3]
m=15 c=13 v_3(K_2Zk)=1  T_k=[3]   m=74 c=11 v_3(K_2Zk)=1  T_k=[9]
m=29 c=2  v_3(K_2Zk)=1  T_k=[9]   m=77 c=2  v_3(K_2Zk)=1  T_k=[3]
m=33 c=5  v_3(K_2Zk)=1  T_k=[3]   m=78 c=5  v_3(K_2Zk)=1  T_k=[3]
m=42 c=5  v_3(K_2Zk)=3  T_k=[9]   m=79 c=11 v_3(K_2Zk)=1  T_k=[9]
m=43 c=5  v_3(K_2Zk)=2  T_k=[3]   m=82 c=5  v_3(K_2Zk)=4  T_k=[3]
m=51 c=11 v_3(K_2Zk)=1  T_k=[3]   m=83 c=5  v_3(K_2Zk)=1  T_k=[3]
(...)
m=10187 c=5 v_3(K_2Zk)=2 T_k=[81] m=10673 c=5 v_3(K_2Zk)=4 T_k=[27]
m=10239 c=7 v_3(K_2Zk)=2 T_k=[3,9] m=10771 c=7 v_3(K_2Zk)=6 T_k=[9]
m=10281 c=7 v_3(K_2Zk)=1 T_k=[27] m=10842 c=5 v_3(K_2Zk)=4 T_k=[3,3]
m=10297 c=5 v_3(K_2Zk)=4 T_k=[3]  m=10942 c=5 v_3(K_2Zk)=2 T_k=[3,243]
m=10351 c=7 v_3(K_2Zk)=1 T_k=[243] m=11062 c=5 v_3(K_2Zk)=3 T_k=[27]

```

In all the results of the tables, one checks that  $\mathbf{R}_2\mathbf{Z}_k[2^\infty] \neq 1$  is equivalent to  $\mathcal{T}_{k,2} \neq 1$ ; then the equality of the  $p$ -ranks, from (1.3), often gives the structure of  $\mathbf{R}_2\mathbf{Z}_k[2^\infty]$ ; that of  $\mathbf{K}_2\mathbf{Z}_k[2^\infty]$  depending on the exact sequence (1.1).

These computations show that, in the case  $\psi \neq 1$ , without any strong assumptions on the ramification in  $k/\mathbb{Q}$  and more precisely on the length of the filtrations of the invariants in the sub- $p$ -tower

of  $k/\mathbb{Q}$ , the results are as independent as possible and governed by standard heuristics of Cohen–Lenstra–Martinet–Malle type.

## 5. CASE OF CYCLIC CUBIC FIELDS

**5.1. Computation of  $\mathcal{T}_{k,3}$  and  $\#\mathbf{K}_2\mathbf{Z}_k[3^\infty]$ .** In Section 5.1.1, we consider all cyclic cubic fields, which requires calculating the Artin group giving  $\text{Gal}(\mathbb{Q}(\mu_f)/k)$ ; indeed, a composite conductor  $f$  gives rise to several fields, so that we must work from the defining polynomial depending on suitable integers  $a, b$  such that  $f = \frac{a^2+27b^2}{4}$  (another method would be to work in the group  $(\mathbb{Z}/f\mathbb{Z})^\times$ , giving other difficulties). We compare  $v_m(\#\mathbf{K}_2\mathbf{Z}_k[3^\infty])$  to  $\mathbf{C}$  to see if we are, or not, in case of equality  $v_m(\mathbf{L}_3(s, \chi)) = \mathbf{C}(s)$ , for all  $s \in \mathbb{Z}_3$ .

In Section 5.1.2, we limit ourselves to cyclic cubic fields of prime conductor, in which case  $\mathbf{C}(s) = 0$ , so that equality of the valuations for all  $s \in \mathbb{Z}_p$  occurs if and only if, for instance,  $\mathcal{T}_{k,p} = 1$ . The prime conductor case allows much larger  $f$ 's.

**5.1.1. Arbitrary conductors.** The first part of the program give the list of all cyclic cubic fields, of conductor  $f$ , with a defining polynomial  $P_k$ . We compute for  $p = 3$  the structures of the 3-class group (in  $\mathbf{H}_k$ ) that of the 3-torsion group (in  $\mathbf{T}$ ) and  $\#\mathbf{K}_2\mathbf{Z}_k[3^\infty]$  (in  $\mathbf{K}_2$ ), for any cyclic cubic field  $k$  of conductors  $f$  (in  $f$ ),  $f \in [\text{bf}, \text{Bf}]$ . Except if the first layer of the cyclotomic  $\mathbb{Z}_3$ -extension of  $k$  is non-ramified (e.g.,  $f = 657$   $\mathbf{P} = x^3 - 219 * x + 730$   $\mathbf{H}_K = [3, 3]$   $\mathbf{T}_K = [3]$ ), then  $\mathcal{T}_{k,3} = 1$  implies  $\mathbf{H}_k[3^\infty] = \mathcal{R}_{k,3} = 1$  since  $\mathcal{W}_{k,3} = 1$ :

```
{bf=7;Bf=10^3;Q=y^2+y+1;Y=Mod(y,Q);for(f=bf,Bf,
if(isprime(f)==1,next);h=valuation(f,3);if(h!=0 & h!=2,next);
F=f/3^h;if(core(F)!=F,next);F=factor(F);Div=component(F,1);
d=matsize(F)[1];for(j=1,d,Div[j]);if(Mod(D,3)!=1,break));
Df=component(factor(f),1);for(b=1,sqrt(4*f/27),
if(h==2 & Mod(b,3)==0,next);A=4*f-27*b^2;if(issquare(A,&a)==1,
if(h==0,if(Mod(a,3)==1,a=-a);PK=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27);
if(h==2,if(Mod(a,9)==3,a=-a);PK=x^3-f/3*x-f*a/27);
n=8;K=bnfinit(PK,1);HK=K.clgp;Kpn=bnrinit(K,3^n);HKn=Kpn.cyc;
T=List;w=0;e=matsize(HKn)[2];for(k=1,e-1,Ck=HKn[e-k+1];
v=valuation(Ck,3);if(v>0,w=w+v;listput(T,3^v));C=omega(f)-1;
if(Mod(f,3)==0,C=C-1);forprime(q=2,10^2*f,
if(Mod(3*f,q)!=0 & polisirreducible(PK+O(q))==1,q0=q;break));
A0=List;A1=List;A2=List;c=q0;n=3;fn=3^n*f;N=eulerphi(fn)/(2*3);
for(s=1,fn/2,if(gcd(s,f*3)!=1,next);for(k=0,fn,r=s+k*fn;
if(isprime(r)==0,next);if(polisirreducible(PK+O(r))==0,
listput(A0,s);break));S0=0;S1=0;S2=0;g=(1-c)/2;
for(j=1,N,s=A0[j];listput(A1,lift(Mod(s*q0,fn))));
for(j=1,N,s=A1[j];listput(A2,lift(Mod(s*q0,fn))));
for(j=1,N,s=A0[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S0=S0+s*(ls+g));
for(j=1,N,s=A1[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S1=S1+s*(ls+g));
for(j=1,N,s=A2[j];t=lift(Mod(s/c,fn));ls=(t*c-s)/fn;S2=S2+s*(ls+g));
S=S0+Y*S1+Y^2*S2;K2=norm(S)/3;v=valuation(K2,3);
if(v==C,print("f=",Df," P=",PK," c=",c," H_k=",HK[2],
" T_k=",T," #T_k=",3^w," #K_2Zk=",3^v," C(s)=",C," Equality"));
if(v>C,print("f=",Df," P=",PK," c=",c," H_k=",HK[2], " T_k=",T,
" #T_k=",3^w," #K_2Zk=",3^v," C(s)=",C," Inequality"))))}
```

```
f=[9,7] P=x^3-21*x-35 c=2
H_k=[3] T_k=[] #T_k=1 #K_2Zk=1 C(s)=0 Equality
f=[9,7] P=x^3-21*x+28 c=5
H_k=[3] T_k=[] #T_k=1 #K_2Zk=1 C(s)=0 Equality
f=[7,13] P=x^3+x^2-30*x-64 c=5
H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
```

```

f=[7,13] P=x^3+x^2-30*x+27 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[9,13] P=x^3-39*x+91 c=2
      H_k=[3] T_k=[] #T_k=1 #K_2Zk=1 C(s)=0 Equality
f=[9,13] P=x^3-39*x-26 c=5
      H_k=[3] T_k=[] #T_k=1 #K_2Zk=1 C(s)=0 Equality
f=[7,19] P=x^3+x^2-44*x+69 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[7,19] P=x^3+x^2-44*x-64 c=11
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[9,19] P=x^3-57*x-152 c=5
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=0 Inequality
f=[9,19] P=x^3-57*x+19 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=0 Inequality
f=[7,31] P=x^3+x^2-72*x+209 c=2
      H_k=[3] T_k=[3,3] #T_k=9 #K_2Zk=9 C(s)=1 Inequality
f=[7,31] P=x^3+x^2-72*x-225 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[13,19] P=x^3+x^2-82*x-311 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[13,19] P=x^3+x^2-82*x-64 c=5
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[7,37] P=x^3+x^2-86*x-211 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[7,37] P=x^3+x^2-86*x+48 c=5
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[9,31] P=x^3-93*x-341 c=2
      H_k=[3] T_k=[] #T_k=1 #K_2Zk=1 C(s)=0 Equality
f=[9,31] P=x^3-93*x+217 c=2
      H_k=[3] T_k=[] #T_k=1 #K_2Zk=1 C(s)=0 Equality
f=[7,43] P=x^3+x^2-100*x-379 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[7,43] P=x^3+x^2-100*x+223 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[9,37] P=x^3-111*x+370 c=7
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=0 Inequality
f=[9,37] P=x^3-111*x+37 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=0 Inequality
f=[9,43] P=x^3-129*x+559 c=2
      H_k=[3] T_k=[] #T_k=1 #K_2Zk=1 C(s)=0 Equality
f=[9,43] P=x^3-129*x-215 c=2
      H_k=[3] T_k=[] #T_k=1 #K_2Zk=1 C(s)=0 Equality
f=[13,31] P=x^3+x^2-134*x-597 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[13,31] P=x^3+x^2-134*x+209 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[7,61] P=x^3+x^2-142*x+601 c=2
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality
f=[7,61] P=x^3+x^2-142*x-680 c=11
      H_k=[3] T_k=[9,9] #T_k=81 #K_2Zk=9 C(s)=1 Inequality
f=[7,67] P=x^3+x^2-156*x-799 c=2
      H_k=[3] T_k=[3,3] #T_k=9 #K_2Zk=27 C(s)=1 Inequality
f=[7,67] P=x^3+x^2-156*x+608 c=5
      H_k=[3] T_k=[3] #T_k=3 #K_2Zk=3 C(s)=1 Equality

```

5.1.2. *Prime conductors.* We consider, now, cyclic cubic fields of prime conductor  $f$ , with  $p = 3$ . The program gives the three coefficients of  $\mathcal{S}_{L_n}^*(c)(s)$  at  $s = -1$ , after the choose of a primitive root modulo  $f$  giving  $c$  and an easier computation of the Artin group  $\text{Gal}(\mathbb{Q}(\mu_f)/k)$ ; this gives  $\#\mathbf{K}_2\mathbf{Z}_k[3^\infty] \sim \mathbf{L}_2(-1, \chi) \times \mathbf{L}_2(-1, \chi')$ ; then we compute  $\mathcal{T}_{k,3}$ . The outputs are limited to the non-trivial groups:

```

{p=3;q=3;n=5;Q=y^2+y+1;Y=Mod(y,Q);forprime(f=7,10^3,
if(Mod(f,3)!=1,next);for(b=1,sqrt(4*f/27),A=4*f-27*b^2;

```

```

if(issquare(A,&a)==1,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);H=Kpn.cyc;T=List;e=matsize(H)[2];
for(k=1,e-1,C=H[e-k+1];v=valuation(C,p);if(v>0,listput(T,p^v));
g=znprimroot(f);c=lift(g);if(Mod(c,p)==0,c=c+f);
ex=(f-1)/3;cex=Mod(c,f)^ex;S0=0;S1=0;S2=0;fn=q*p^n*f;
forstep(a=1,fn/2,[1,2],if(gcd(a,f)!=1,next);aa=lift(Mod(a/c,fn));
la=(aa*c-a)/fn;X=a*(la+(1-c)/2);aex=Mod(a,f)^ex;
if(aex==1,S0=S0+X);if(aex==cex,S1=S1+X);if(aex==cex^2,S2=S2+X));
A=lift(Mod(S0-S2,fn));B=lift(Mod(S1-S2,fn));
N=norm(A+Y*B);w=valuation(N,p)-1;print
("f=",f," P=",P," T_k=",T," A=",A," B=",B," #K_2Zk=",p^w))}

```

|       |                     |           |          |         |          |
|-------|---------------------|-----------|----------|---------|----------|
| f=19  | P=x^3+x^2-6*x-7     | T_k=[3]   | A=4608   | B=4611  | #K_2Zk=3 |
| f=37  | P=x^3+x^2-12*x+11   | T_k=[3]   | A=21     | B=21    | #K_2Zk=3 |
| f=73  | P=x^3+x^2-24*x-27   | T_k=[3]   | A=87     | B=429   | #K_2Zk=3 |
| f=109 | P=x^3+x^2-36*x-4    | T_k=[3]   | A=714    | B=219   | #K_2Zk=3 |
| f=127 | P=x^3+x^2-42*x+80   | T_k=[3]   | A=5829   | B=20535 | #K_2Zk=3 |
| f=163 | P=x^3+x^2-54*x-169  | T_k=[3]   | A=39573  | B=186   | #K_2Zk=3 |
| f=181 | P=x^3+x^2-60*x-67   | T_k=[3]   | A=93     | B=279   | #K_2Zk=3 |
| f=199 | P=x^3+x^2-66*x+59   | T_k=[3,3] | A=1410   | B=37542 | #K_2Zk=9 |
| f=271 | P=x^3+x^2-90*x+261  | T_k=[3,3] | A=40863  | B=61593 | #K_2Zk=9 |
| f=307 | P=x^3+x^2-102*x-216 | T_k=[3]   | A=72477  | B=3624  | #K_2Zk=3 |
| f=379 | P=x^3+x^2-126*x+365 | T_k=[3]   | A=91278  | B=91824 | #K_2Zk=3 |
| f=397 | P=x^3+x^2-132*x-544 | T_k=[3]   | A=6996   | B=2172  | #K_2Zk=3 |
| f=433 | P=x^3+x^2-144*x-16  | T_k=[3]   | A=100647 | B=97131 | #K_2Zk=3 |
| f=487 | P=x^3+x^2-162*x-505 | T_k=[3,3] | A=13458  | B=73185 | #K_2Zk=9 |

In the cubic case and  $p = 3$ , the two invariants  $\mathbf{K}_2\mathbf{Z}_k$  and  $\mathcal{T}_{k,3}$  have same 3-rank. This allows some information on the structure of  $\mathbf{K}_2\mathbf{Z}_k[3^\infty]$ ; for instance, for  $f = 199$ ,  $\mathcal{T}_{k,3} \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$  and  $\#\mathbf{K}_2\mathbf{Z}_k[3^\infty] = 9$ , whence  $\mathbf{K}_2\mathbf{Z}_k[3^\infty] \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ .

Some other examples allow to conclude or to limit the possible structures (for some, the value of  $n$  must be increased to  $n = 8$ ):

|        |                        |           |            |            |            |
|--------|------------------------|-----------|------------|------------|------------|
| f=1297 | P=x^3+x^2-432*x-1345   | T_k=[3,9] | A=115668   | B=39249    | #K_2Zk=27  |
| f=1747 | P=x^3+x^2-582*x-4141   | T_k=[3,9] | A=6147     | B=423369   | #K_2Zk=81  |
| f=2593 | P=x^3+x^2-864*x-2689   | T_k=[3,9] | A=52137    | B=16906509 | #K_2Zk=729 |
| f=3061 | P=x^3+x^2-1020*x+3968  | T_k=[9,9] | A=614961   | B=489852   | #K_2Zk=81  |
| f=4177 | P=x^3+x^2-1392*x-5724  | T_k=[9,9] | A=892755   | B=169866   | #K_2Zk=27  |
| f=4789 | P=x^3+x^2-1596*x-17737 | T_k=[3,9] | A=31385961 | B=31396383 | #K_2Zk=243 |
| f=6733 | P=x^3+x^2-2244*x+15461 | T_k=[9,9] | A=28053    | B=44140491 | #K_2Zk=243 |

**5.2. Computation of  $\mathcal{T}_{k,2}$  and  $\#\mathbf{K}_2\mathbf{Z}_k[2^\infty]$ .** We still compute  $\mathcal{T}_{k,2}$  and  $\#\mathbf{R}_2\mathbf{Z}_k[2^\infty] \sim \frac{1}{2} \cdot \mathbf{L}_2(-1, \chi) \times \frac{1}{2} \cdot \mathbf{L}_2(-1, \chi')$ .

```

{p=2;q=4;n=6;Q=y^2+y+1;Y=Mod(y,Q);forprime(f=7,10^3,
if(Mod(f,3)!=1,next);for(b=1,sqrt(4*f/27),A=4*f-27*b^2;
if(issquare(A,&a)==1,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);H=Kpn.cyc;T=List;e=matsize(H)[2];
for(k=1,e-1,C=H[e-k+1];v=valuation(C,p);if(v>0,listput(T,p^v));
g=znprimroot(f);c=lift(g);if(Mod(c,p)==0,c=c+f);ex=(f-1)/3;
cex=Mod(c,f)^ex;S0=0;S1=0;S2=0;fn=q*p^n*f;forstep(a=1,fn/2,2,
if(gcd(a,f)!=1,next);aa=lift(Mod(a/c,fn));la=(aa*c-a)/fn;
X=a*(la+(1-c)/2);aex=Mod(a,f)^ex;if(aex==1,S0=S0+X);
if(aex==cex,S1=S1+X);if(aex==cex^2,S2=S2+X));A=lift(Mod(S0-S2,fn));
B=lift(Mod(S1-S2,fn));N=norm(A+Y*B);w=valuation(N,p);print
("f=",f," P=",P," T_k=",T," A=",A," B=",B," #R_2Zk=",p^w))}

```

|       |                   |           |         |         |          |
|-------|-------------------|-----------|---------|---------|----------|
| f=31  | P=x^3+x^2-10*x-8  | T_k=[2,2] | A=24    | B=7902  | #R_2Zk=4 |
| f=43  | P=x^3+x^2-14*x+8  | T_k=[2,2] | A=64    | B=10978 | #R_2Zk=4 |
| f=109 | P=x^3+x^2-36*x-4  | T_k=[2,2] | A=26754 | B=17766 | #R_2Zk=4 |
| f=127 | P=x^3+x^2-42*x+80 | T_k=[2,2] | A=394   | B=32424 | #R_2Zk=4 |

|       |                       |             |         |          |           |
|-------|-----------------------|-------------|---------|----------|-----------|
| f=157 | $P=x^3+x^2-52*x+64$   | $T_k=[2,2]$ | A=1754  | B=418    | #R_2Zk=4  |
| f=163 | $P=x^3+x^2-54*x-169$  | $T_k=[2,2]$ | A=13702 | B=17908  | #R_2Zk=4  |
| f=223 | $P=x^3+x^2-74*x-256$  | $T_k=[2,2]$ | A=182   | B=56178  | #R_2Zk=4  |
| f=229 | $P=x^3+x^2-76*x-212$  | $T_k=[2,2]$ | A=5202  | B=4274   | #R_2Zk=4  |
| f=277 | $P=x^3+x^2-92*x+236$  | $T_k=[4,4]$ | A=3256  | B=70236  | #R_2Zk=16 |
| f=283 | $P=x^3+x^2-94*x+304$  | $T_k=[2,2]$ | A=71022 | B=71086  | #R_2Zk=4  |
| f=307 | $P=x^3+x^2-102*x-216$ | $T_k=[4,4]$ | A=2124  | B=74968  | #R_2Zk=16 |
| f=349 | $P=x^3+x^2-116*x-517$ | $T_k=[2,2]$ | A=31008 | B=6050   | #R_2Zk=4  |
| f=397 | $P=x^3+x^2-132*x-544$ | $T_k=[4,4]$ | A=94636 | B=99460  | #R_2Zk=16 |
| f=433 | $P=x^3+x^2-144*x-16$  | $T_k=[4,4]$ | A=4572  | B=8088   | #R_2Zk=16 |
| f=439 | $P=x^3+x^2-146*x-504$ | $T_k=[4,4]$ | A=22884 | B=48584  | #R_2Zk=16 |
| f=457 | $P=x^3+x^2-152*x-220$ | $T_k=[2,2]$ | A=23014 | B=77580  | #R_2Zk=4  |
| f=499 | $P=x^3+x^2-166*x+536$ | $T_k=[2,2]$ | A=16142 | B=121106 | #R_2Zk=4  |

The 2-rank of a  $\mathbb{Z}_2[\text{Gal}(k/\mathbb{Q})]$ -module is even and in most cases the structure of  $\mathbf{R}_2\mathbf{Z}_k[2^\infty]$  can be deduced from its order and  $\text{rk}_2(\mathcal{T}_{k,2})$  due to (1.3), as shown by the following complementary examples (up to the case  $f = 2689$ , the two modules  $\mathcal{T}_{k,2}$  and  $\mathbf{R}_2\mathbf{Z}_k[2^\infty]$  are isomorphic):

|         |                           |                 |            |
|---------|---------------------------|-----------------|------------|
| f=2689  | $P=x^3+x^2-896*x+5876$    | $T_k=[16,16]$   | #R_2Zk=64  |
| f=3163  | $P=x^3+x^2-1054*x-13472$  | $T_k=[16,16]$   | #R_2Zk=256 |
| f=3457  | $P=x^3+x^2-1152*x+13700$  | $T_k=[16,16]$   | #R_2Zk=64  |
| f=6079  | $P=x^3+x^2-2026*x-1576$   | $T_k=[2,2,2,2]$ | #R_2Zk=16  |
| f=6163  | $P=x^3+x^2-2054*x+17576$  | $T_k=[8,8]$     | #R_2Zk=256 |
| f=7687  | $P=x^3+x^2-2562*x-48969$  | $T_k=[2,2,2,2]$ | #R_2Zk=16  |
| f=9721  | $P=x^3+x^2-3240*x-39244$  | $T_k=[2,2,2,2]$ | #R_2Zk=16  |
| f=16561 | $P=x^3+x^2-5520*x+153956$ | $T_k=[2,2,4,4]$ | #R_2Zk=64  |
| f=16927 | $P=x^3+x^2-5642*x+161120$ | $T_k=[2,2,8,8]$ | #R_2Zk=256 |

For checking, one may also use the general formula of Theorem 3.1,  $\#\mathcal{T}_{k,2} \sim \frac{1}{2}\mathbf{L}_2(1, \chi) \times \frac{1}{2}\mathbf{L}_2(1, \chi')$ ; for this, the reader can run the following more complete program:

```
{p=2;q=4;n=8;Q=y^2+y+1;Y=Mod(y,Q);forprime(f=7,5*10^5,
if(Mod(f,3)!=1,next);for(b=1,sqrt(4*f/27),A=4*f-27*b^2;
if(issquare(A,&a)==1,if(Mod(a,3)==1,a=-a);
P=x^3+x^2+(1-f)/3*x+(f*(a-3)+1)/27;K=bnfinit(P,1);
Kpn=bnrinit(K,p^n);HKn=Kpn.cyc;T=List;e=matsize(HKn)[2];
for(k=1,e-1,Ck=HKn[e-k+1];v=valuation(Ck,p);if(v>0,listput(T,p^v)));
g=znprimroot(f);c=lift(g);if(Mod(c,p)==0,c=c+f);ex=(f-1)/3;
cex=Mod(c,f)^ex;R0=0;R1=0;R2=0;T0=0;T1=0;T2=0;fn=q*p^n*f;
forstep(a=1,fn/2,2,if(Mod(a,f)==0,next);aa=lift(Mod(a/c,fn));
am=lift(Mod(a,fn)^-1);la=(aa*c-a)/fn;XR=a*(la+(1-c)/2);XT=am*la;
aex=Mod(a,f)^ex;
if(aex==1,R0=R0+XR;T0=T0+XT);
if(aex==cex,R1=R1+XR;T1=T1+XT);
if(aex==cex^2,R2=R2+XR;T2=T2+XT));
AR=lift(Mod(R0-R2,fn));BR=lift(Mod(R1-R2,fn));
AT=lift(Mod(T0-T2,fn));BT=lift(Mod(T1-T2,fn));
NR=norm(AR+Y*BR);wR=valuation(NR,p);
NT=norm(AT+Y*BT);wT=valuation(NT,p);print();
print("f=",f," P=",P," T_k=",T," AR=",AR," BR=",BR," #R_2Zk=",p^wR);
print("2nd #T_k:",AT," AT=",AT," BT=",BT," #T_k=",p^wT))}
```

|      |                    |          |           |          |          |        |
|------|--------------------|----------|-----------|----------|----------|--------|
| f=7  | $P=x^3+x^2-2*x-1$  | $T_k=[]$ | AR=7163   | BR=6     | #R_2Zk=1 |        |
|      |                    |          | 2nd #T_k: | AT=2699  | BT=6894  | #T_k=1 |
| f=13 | $P=x^3+x^2-4*x+1$  | $T_k=[]$ | AR=13084  | BR=449   | #R_2Zk=1 |        |
|      |                    |          | 2nd #T_k: | AT=6420  | BT=12649 | #T_k=1 |
| f=19 | $P=x^3+x^2-6*x-7$  | $T_k=[]$ | AR=17687  | BR=437   | #R_2Zk=1 |        |
|      |                    |          | 2nd #T_k: | AT=15143 | BT=11317 | #T_k=1 |
| f=31 | $P=x^3+x^2-10*x-8$ |          |           |          |          |        |



```

T_k=[2,2]  AR=24 BR=31710      #R_2Zk=4
2nd #T_k:  AT=19488  BT=9630   #T_k=4
(...)
f=2689  P=x^3+x^2-896*x+5876
T_k=[16,16] AR=1733440 BR=461096 #R_2Zk=64
2nd #T_k:  AT=2255952 BT=2070288 #T_k=256
(...)
f=2731  P=x^3+x^2-910*x+10216
T_k=[4,4]  AR=2752976 BR=2756588 #R_2Zk=16
2nd #T_k:  AT=1941416 BT=2335796 #T_k=16
f=2749  P=x^3+x^2-916*x+1120
T_k=[8,8]  AR=1088880 BR=684872  #R_2Zk=64
2nd #T_k:  AT=2110728 BT=1518496 #T_k=64

```

## 6. MODULUS OF CONTINUITY OF THE $\mathbf{L}_p$ -FUNCTIONS

One may ask why  $\gcd(\#\mathcal{T}_{k,p}, \#\mathbf{R}_2\mathbf{Z}_k[p^\infty])$  may be large, regarding these orders, in the genus context given by the character  $\psi$  of order  $p^e$ , when  $p \in \{2, 3\}$ ; the explanation is in the existence, for  $\chi = \varphi\psi \neq \mathbf{1}$ , of a “modulus of continuity” of the form [Gra1987, Théorème 0.3] :

$$\frac{1}{2}\mathbf{L}_p(t, \chi) - \frac{1}{2}\mathbf{L}_p(s, \chi) \equiv a(t-s) \pmod{q\mathbf{m}^{\mathbf{V}}(t-s)},$$

with a computable constant  $a$  (equal to 0 when  $\varphi = \mathbf{1}$ ) and  $\mathbf{V}$  of the form  $\mathbf{V} = \mathbf{D} - \max_{\ell}(p^{d_\ell} - p^{d_p} + \epsilon, \epsilon)$  (with  $\epsilon = 1$  if  $\varphi = \mathbf{1}$ , 0 otherwise), where  $\ell$  runs through the set of primes ramified in  $M/M_0$ , totally split in  $M_0/\mathbb{Q}$  and such that  $\frac{1}{q}\log_p(\ell) \not\equiv 0 \pmod{p}$  (see Definitions 3.4).

We note that in the case  $\psi = \mathbf{1}$ ,  $\varphi \neq \mathbf{1}$ , the above formula becomes:

$$\frac{1}{2}\mathbf{L}_p(t, \varphi) - \frac{1}{2}\mathbf{L}_p(s, \varphi) \equiv 0 \pmod{q(t-s)}.$$

In other words, even if the genus principle is empty, there is a non-trivial congruence between the orders of the two corresponding invariants of  $k$  (much stronger when  $\psi \neq \mathbf{1}$ , but saying that the two invariants are always simultaneously trivial or non-trivial); for instance, we get  $\frac{1}{2}\mathbf{L}_p(1, \varphi) - \frac{1}{2}\mathbf{L}_p(-1, \varphi) \equiv 0 \pmod{2q}$ , whence  $\#\mathbf{R}_2\mathbf{Z}_k[3^\infty] \equiv \#\mathcal{T}_{k,3} \pmod{3}$  for quadratic fields, or  $\#\mathbf{R}_2\mathbf{Z}_k[2^\infty] \equiv \#\mathcal{T}_{k,2} \pmod{8}$  for cyclic cubic fields.

In the case of cubic fields for  $p = 2$ , we have for almost cases,  $\#\mathbf{R}_2\mathbf{Z}_k[2^\infty] = \#\mathcal{T}_{k,2}$ ; the exceptions up to  $10^4$  being:

```

f=2689  P=x^3+x^2-896*x+5876   #T_k=256 #R_2Zk=64
f=3457  P=x^3+x^2-1152*x+13700 #T_k=256 #R_2Zk=64
f=6163  P=x^3+x^2-2054*x+17576 #T_k=64   #R_2Zk=256
f=7963  P=x^3+x^2-2654*x+43944 #T_k=256 #R_2Zk=64
f=8167  P=x^3+x^2-2722*x+41440 #T_k=64   #R_2Zk=256

```

giving a difference  $\pm 2$  for the valuations.

In the case of quadratic fields for  $p = 3$ , the differences between the valuations takes more often a non-zero value; let's give some excerpts:

```

m=29  #T_k=9   #R_2Zk=3      m=122 #T_k=81  #R_2Zk=3
m=42  #T_k=9   #R_2Zk=27     m=247 #T_k=27  #R_2Zk=3
m=82  #T_k=3   #R_2Zk=81     m=406 #T_k=3   #R_2Zk=243
m=109 #T_k=3   #R_2Zk=27     m=3281 #T_k=3  #R_2Zk=729
m=3869 #T_k=3  #R_2Zk=2187  m=4794 #T_k=81  #R_2Zk=6561

```

## 7. CONCLUSION

This computational work is a confirmation of the properties of the  $\mathbf{L}_p$ -functions when  $s$  varies in  $\mathbb{Z}_p$  and takes the well-known values  $s \in \{-1, 0, 1\}$ , giving annihilation theorem, orders of isotypic components of suitable  $p$ -invariants (in the direction of the “Main Conjectures”). Other invariants than class groups, torsion groups and regular kernels exist, as for instance the Jaulent logarithmic class group [Jau1994<sup>b</sup>] in relation with Greenberg’s conjecture, and invariants of the higher  $K$ -theory (giving conjectural expressions suggested in [Gra1998, Section 12, Conjecture 12.2] and some results of [JaMi2006]).

An important problem is to find the complete structure of these invariants, it being understood that these structures fulfill, once genus part has been taken into consideration, standard densities in the spirit of the Cohen–Lenstra–Martinet–Malle distributions. A specific technique, of genus theory type, does exist in cyclic  $p$ -extensions, especially for  $p$ -class groups with determining the canonical filtration by means of a natural algorithm generalizing Chevalley–Herbrand fixed points formula (huge bibliography, synthesized in [Gra2017<sup>b</sup>]); a main question being to find such algorithms for the other  $p$ -invariants, which unfortunately does not exist to our knowledge.

The case of the most important groups  $\mathcal{T}_{k,p}$  being very interesting; indeed, class field theory (by means of PARI computation of suitable ray class groups) gives easily the group structure, but this computational aspect does not give theoretical informations on the filtration; for these groups in cyclic  $p$ -extensions, only the fixed points formula, is known [Gra2005, Theorem IV.3.3].

In a cohomological viewpoint, recall that  $\mathcal{T}_{k,p}$  is closely related to the Tate–Chafarevich group:

$$\text{III}_{k,p}^2 := \text{Ker} [\text{H}^2(\mathcal{G}_{k,S}, \mathbb{F}_p) \rightarrow \bigoplus_{\mathfrak{p}|p} \text{H}^2(\mathcal{G}_{k_{\mathfrak{p}}}, \mathbb{F}_p)],$$

and that  $\mathcal{T}_{k,p} \simeq \text{H}^2(\mathcal{G}_{k,S}, \mathbb{Z}_p)^*$  [Ng1986, Théorème 1.1], where  $\mathcal{G}_{k,S}$  is the Galois group of the maximal  $p$ -ramified pro- $p$ -extension of  $k$  and  $\mathcal{G}_{k_{\mathfrak{p}}}$  the local analogue over  $k_{\mathfrak{p}}$ ; generalizations of the same kind are done in the literature, as in [Kol2002, Theorem 0.4, 0.7, 0.11, ...], [AAM2021, Theorem 3.5, Example 4.3], with analogous cohomological framework, without any workable algorithm.

A major difficulty is to interpret, in terms of ideals, global units and so on, such cohomology groups; the previous example  $\mathcal{T}_{k,p} \simeq \text{H}^2(\mathcal{G}_{k,S}, \mathbb{Z}_p)^*$  is edifying since  $\mathcal{G}_{k,S}$  is inaccessible, while  $\mathcal{T}_{k,p}$  depends on the  $p$ -class group and on the normalized  $p$ -adic regulator, of  $k$ . Of course, cohomology gives important and non-obvious relations but, in the previous example, it is the knowledge of  $\mathcal{T}_{k,p}$  which yields important information on  $\mathcal{G}_{k,S}$ .

Maybe the link we made in [Gra2005, § III.2, Theorem III.2.6, Theorem III.3.3], using logarithms of ideals for an effectiveness of the arithmetic, may be generalized.

As a final remark, we point out the fact that these analytic genus theory aspects of  $p$ -adic pseudo-measures are stated in [Gra1986<sup>b</sup>] for any totally real base field replacing  $\mathbb{Q}$  (under Leopoldt’s conjecture for  $p$ , to get the important result of deployment [Gra2005, Theorem III.4.1.5]), and allowing similar technics with genus fields; these results being completed with for instance the article by Maire [Mai2018]

on the class field theory aspects and governing fields about the *existence* of prescribed  $p$ -cyclic ramified extensions for which an analog of Theorem 3.5 should apply.

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