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► **To cite this version:**

Friedemann Brock, Francesco Chiacchio, Gisella Croce, Anna Mercaldo. Shape of extremal functions for weighted Sobolev-type inequalities. 2023. hal-04236918v2

HAL Id: hal-04236918

<https://hal.science/hal-04236918v2>

Preprint submitted on 1 Nov 2023

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SHAPE OF EXTREMAL FUNCTIONS FOR WEIGHTED SOBOLEV-TYPE INEQUALITIES

F. BROCK², F. CHIACCHIO¹, G. CROCE³, AND A. MERCALDO¹

ABSTRACT. We study the shape of solutions to some variational problems in Sobolev spaces with weights that are powers of $|x|$. In particular, we detect situations when the extremal functions lack symmetry properties such as radial symmetry and antisymmetry. We also prove an isoperimetric inequality for the first non-zero eigenvalue of a weighted Neumann problem.

Key words: Rayleigh quotient, foliated Schwarz symmetry, Bessel functions, eigenfunction, breaking of symmetry

2000 Mathematics Subject Classification: 49J40, 49K20, 49K30

1. INTRODUCTION

In this paper we study the shape of minimizers of some variational problems, defined with weights of power form, and we exhibit some breaking symmetry phenomena. These questions have attracted much interest in the literature. Consider for example the Rayleigh quotient

$$(1.1) \quad Q_{p,q,\gamma}(v) := \frac{\int_B |\nabla v|^p dx}{\left(\int_B |v|^q |x|^\gamma dx \right)^{p/q}}, \quad v \in W^{1,p}(B) \setminus \{0\},$$

where B denotes the unit ball, centered at the origin in \mathbb{R}^N , $N \geq 2$, $p > 1$ and $q \geq p$. Variational problems of the type

$$(1.2) \quad \inf \left\{ Q_{2,q,\gamma}(v) : v \in W_0^{1,2}(B) \setminus \{0\} \right\},$$

with $2 < q < 2^*$, have been studied in [9] and [29]. The authors have shown that the minimizers are not radial, if the parameter γ is large enough (see also [24], [1] for a different weight). Furthermore, much interest has been devoted to the shape of sign changing minimizers of integral

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functionals, see for example [16], [32], [6], [27] and [7]. In [16], Girao and Weth studied the symmetry properties of the minimizers of the problem

$$(1.3) \quad \inf \left\{ Q_{2,q,0}(v) \equiv \frac{\|\|\nabla v\|\|_2^2}{\|v\|_q^2} : v \in W^{1,2}(B) \setminus \{0\}, \int_B v \, dx = 0 \right\}$$

for $2 \leq q < 2^*$. They proved that the minimizers are foliated Schwarz symmetric. This means that they are symmetric with respect to reflection about some line $\mathbb{R}e$ and decreasing w.r.t. the angle $\arccos[\frac{x}{|x|} \cdot e] \in (0, \pi)$. Further, another interesting phenomenon related to the shape of the minimizers was pointed out for problem (1.3): if p is close to 2, then any minimizer is antisymmetric w.r.t. reflection about the hyperplane $\{x \cdot e = 0\}$. In contrast to this, the minimizers are not anymore antisymmetric if $N = 2$ and if p is sufficiently large. A similar break of symmetry was already observed in [1], [12], [8], [5], [19], [26] and [26] for the minimizers of a one-dimensional problem,

$$\inf \left\{ \frac{\|v'\|_p}{\|v\|_q}, \quad v \in W^{1,p}((0,1)) \setminus \{0\}, \quad v(0) = v(1), \quad \int_0^1 v \, dx = 0 \right\}.$$

More precisely, it has been shown that any minimizer is an antisymmetric function, if and only if $q \leq 3p$ (see also [11] and [15] for a more general constraint).

In this paper we study variational problems for similar Rayleigh quotients where the enumerator and the denominator carry a weight $|x|^\alpha$. Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 2$, with Lipschitz boundary, containing the origin, and define

$$(1.4) \quad R_{p,q,\alpha,\gamma}(v) := \frac{\int_\Omega |\nabla v|^p |x|^\alpha \, dx}{\left(\int_\Omega |v|^q |x|^\gamma \, dx \right)^{p/q}}, \quad v \in W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha),$$

where

$$(1.5) \quad p, q \in [1, +\infty)$$

and the numbers $\alpha, \gamma \in \mathbb{R}$ satisfy certain conditions. (The definitions of weighted function spaces, such as $W^{1,p}(\Omega, |x|^\alpha, |x|^\beta)$, will be given in Section 2). We focus on two variational problems, one with Dirichlet boundary conditions and one with a mean value condition:

$$(1.6) \quad (\mathbf{P}^D) \quad \inf \left\{ R_{p,q,\alpha,\gamma}(v) : v \in W_0^{1,p}(\Omega, |x|^\alpha, |x|^\alpha) \right\} =: \lambda^D,$$

$$(1.7) \quad (\mathbf{P}^M) \quad \inf \left\{ R_{2,q,\alpha,\alpha}(v) : v \in W^{1,2}(\Omega, |x|^\alpha, |x|^\alpha), \int_\Omega |x|^\alpha v \, dx = 0 \right\} =: \lambda^M.$$

We study the shape of solutions to these problems, and in particular, we detect situations when the extremal functions lack symmetry properties such as radial symmetry and antisymmetry.

In the case $p = q = 2$, we also prove an isoperimetric inequality for λ^M , thus generalizing a famous result of Szegő and Weinberger.

Let us outline the content of the paper. Some preliminary results are presented in Section 2: we obtain embedding properties for some weighted function spaces (Theorem 2.1, Corollary

2.2 and Lemma 2.3). These properties allow us to prove Theorem 2.4 which gives the existence of solutions to the variational problems (\mathbf{P}^D) and (\mathbf{P}^M) . Then we recall the definitions of the two-point rearrangement, foliated Schwarz symmetrization and foliated Schwarz symmetry given in Definitions 2.5, 2.7 and 2.8, respectively. Moreover we give some relations between these notions in Lemma 2.6 and Theorem 2.9. Note that a function is foliated Schwarz symmetric if it is axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis.

The following four Sections 3–6 deal with the problems (\mathbf{P}^D) and (\mathbf{P}^M) when Ω is a ball centered at the origin. In Section 3 we prove Theorem 3.1 which shows that the solution to Problem (\mathbf{P}^D) is not radially symmetric, if $q > p$ and the parameter γ is sufficiently large. Next we study Problem (\mathbf{P}^M) . We prove Theorem 4.1 in Section 4, which asserts that the minimizers of (\mathbf{P}^M) are foliated Schwarz symmetric, when $q \geq 2$. Next we study problem (\mathbf{P}^M) for $N = 2$. First we deal with the case $q = 2$ in Section 5. We prove that any minimizer of (\mathbf{P}^M) is, up to some rotation about the origin, symmetric (even) with respect to x_1 , and antisymmetric (odd) with respect to x_2 , as stated in Theorem 5.1 and the Remark following it. On the other hand, in Section 6 we prove Theorem 6.1 which shows that a breaking of antisymmetry occurs when $\alpha < 0$ and q is sufficiently large.

Finally, we study a shape optimization problem, for $q = 2$ and $\alpha > 0$, in Section 7. More precisely, let Ω be a bounded domain with Lipschitz boundary in \mathbb{R}^N , $N \geq 2$. Under an additional assumption on Ω we provide, in Theorem 7.1, a Szegő-Weinberger type inequality for the weighted Neumann eigenvalue

$$\mu(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla v|^2 |x|^\alpha dx}{\int_{\Omega} v^2 |x|^\alpha dx} : v \in W^{1,2}(B, |x|^\alpha, |x|^\alpha) \setminus \{0\}, \int_{\Omega} v |x|^\alpha dx = 0 \right\}.$$

Let us briefly describe how this result fits into the literature. We recall that Kornhauser and Stakgold conjectured in [20] that among all planar simply connected domains, with fixed Lebesgue measure, the first nonzero eigenvalue of the Neumann Laplacian, achieves its maximum value if and only if Ω is a disk. This conjecture was proved by Szegő in [30], by means of tools from complex analysis. Soon after Weinberger generalized this result to any bounded smooth domain Ω of \mathbb{R}^N (see [31]). Note that Weinberger's method turned out to be rather flexible. Indeed, similar inequalities have been proved, for examples in [4], [3] and [10]).

2. PRELIMINARIES

2.1. Embedding results and existence of solutions to (\mathbf{P}^D) and (\mathbf{P}^M)

In this subsection we assume that Ω is a bounded domain with Lipschitz boundary in \mathbb{R}^N , $N \geq 2$, containing the origin, $\alpha, \beta \in \mathbb{R}$ and $1 \leq p \leq N$. We denote by $L^p(\Omega, |x|^\alpha)$ the weighted Lebesgue space of all measurable functions $u : \Omega \rightarrow \mathbb{R}$ with

$$\|u\|_{p,\Omega,\alpha} := \left(\int_{\Omega} |u|^p |x|^\alpha dx \right)^{1/p} < \infty.$$

The weighted Sobolev space $W^{1,p}(\Omega, |x|^\alpha, |x|^\beta)$ is defined as the set of all functions $u \in L^p(\Omega, |x|^\alpha)$ having distributional derivatives $(\partial u / \partial x_i)$, $i = 1, \dots, N$, for which the norm

$$\|u\|_{1,p,\Omega,\alpha,\beta} := \left[\|u\|_{p,\Omega,\alpha}^p + \|\nabla u\|_{p,\Omega,\beta}^p \right]^{1/p}$$

is finite. It is well-known that, if

$$(2.1) \quad \alpha < N(p-1), \quad \beta < N(p-1), \quad \text{and}$$

$$(2.2) \quad \alpha > -N, \quad \beta > -N,$$

then $W^{1,p}(\Omega, |x|^\alpha, |x|^\beta)$ is a reflexive Banach space, and $C_0^\infty(\Omega) \subset W^{1,p}(\Omega, |x|^\alpha, |x|^\beta)$, (see e.g. [21], p. 240 ff., and [23], p. 1054).

Under the conditions (2.1) and (2.2) the space $W_0^{1,p}(\Omega, |x|^\alpha, |x|^\beta)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p,\Omega,\alpha,\beta}$.

One can find a variety of embedding theorems for weighted Sobolev spaces into weighted Lebesgue spaces or into spaces of continuous functions in the literature (see e.g. [21], [13], [18]). However, we could not find a reference to the following embedding result. It will be crucial in proving existence of solutions to our variational problems. We use the notation \hookrightarrow for continuous embedding and $\hookrightarrow\hookrightarrow$ for compact embedding.

Theorem 2.1. *Let $1 \leq p \leq N$, $-N < \alpha < N(p-1)$ and $\gamma \geq \alpha$. Then*

$$(2.3) \quad W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha) \hookrightarrow\hookrightarrow L^q(\Omega, |x|^\gamma) \quad \text{for every } q \in [1, q_0),$$

where q_0 is defined by

$$(2.4) \quad q_0 := \begin{cases} +\infty & \text{if } N = p \text{ and } -N < \alpha \leq 0 \\ \frac{Np}{N-p} & \text{if } N > p \text{ and } -N < \alpha \leq 0 \\ \frac{(N+\alpha)p}{N+\alpha-p} & \text{if } 0 < \alpha < N(p-1) \end{cases}.$$

Proof. In view of the classical Sobolev Embedding Theorem, we may assume that $\alpha \neq 0$.

In the following, let C, C', C'', \dots , denote positive constants that do not depend on the functions that are involved but may vary from line to line.

Our proof is in three steps.

Step 1: We claim that for every $q \in [1, q_0)$ there exists a constant $C = C(q) > 0$ such that

$$(2.5) \quad \|u\|_{1,p,\Omega,\alpha,\alpha} \geq C(q) \cdot \|u\|_{q,\Omega,\alpha} \quad \forall u \in W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha),$$

that is,

$$W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha) \hookrightarrow L^q(\Omega, |x|^\alpha).$$

We define

$$y := x \cdot |x|^{\alpha/N}, \\ \tilde{\Omega} := \{y = x \cdot |x|^{\alpha/N} : x \in \Omega\},$$

and for any measurable function $u : \Omega \rightarrow \mathbb{R}$ we set

$$\varphi(u)(y) := u(x), \quad (x \in \Omega).$$

A short computation shows, that there are constants C' and C'' , such that, if $u \in C^1(\Omega)$, then

$$(2.6) \quad C'|y|^{2\alpha/(N+\alpha)}|\nabla_y\varphi(u)(y)|^2 \leq |\nabla_x u(x)|^2 \leq C''|y|^{2\alpha/(N+\alpha)}|\nabla_y\varphi(u)(y)|^2 \quad \forall x \in \Omega \setminus \{0\}.$$

Now we split into two cases.

(i) Assume that $0 < \alpha < N(p-1)$. By (2.6) we have that

$$(2.7) \quad C'\|\varphi(u)\|_{1,p,\tilde{\Omega},0,p\alpha/(N+\alpha)} \leq \|u\|_{1,p,\Omega,\alpha,\alpha} \leq C''\|\varphi(u)\|_{1,p,\tilde{\Omega},0,p\alpha/(N+\alpha)}$$

if $u \in W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$, and

$$(2.8) \quad C\|\varphi(u)\|_{q,\tilde{\Omega},0} = \|u\|_{q,\Omega,\alpha} \quad \text{if } u \in L^q(\Omega, |x|^\alpha).$$

so that

$$(2.9) \quad u \in W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha) \iff \varphi(u) \in W^{1,p}(\tilde{\Omega}, 1, |y|^{p\alpha/(N+\alpha)}) \quad \text{and}$$

$$(2.10) \quad u \in L^q(\Omega, |x|^\alpha) \iff \varphi(u) \in L^q(\tilde{\Omega}, 1).$$

Note that $0 \leq p\alpha/(N+\alpha) < N(p-1)$, so that $W^{1,p}(\tilde{\Omega}, 1, |y|^{p\alpha/(N+\alpha)})$ is a Banach space, too. Since we have

$$|y|^{-\alpha/(N+\alpha)} \in L^t(\Omega) \quad \forall t \in \left[1, \frac{N(N+\alpha)}{p\alpha}\right)$$

we may apply Lemma 2.1 of [23] to conclude that

$$W^{1,p}(\tilde{\Omega}, 1, |y|^{p\alpha/(N+\alpha)}) \hookrightarrow L^q(\tilde{\Omega}, 1) \quad \text{for } 1 \leq q < p(N+\alpha)/(N-p+\alpha).$$

Now (2.5) follows from this and (2.7)-(2.10).

(ii) Next let $-N < \alpha < 0$.

First observe that, since

$$|x|^\alpha \geq C|x|^{\alpha(N-p)/N} \quad \forall x \in \Omega \setminus \{0\},$$

we also have

$$(2.11) \quad \|u\|_{1,p,\Omega,\alpha,\alpha} \geq C\|u\|_{1,p,\Omega,\alpha,\alpha(N-p)/N}.$$

Note that $W^{1,p}(\Omega, |x|^\alpha, |x|^{\alpha(N-p)/N})$ is a Banach space, too, since $-N < \alpha(N-p)/N \leq 0$.

In view of (2.6) we obtain

$$(2.12) \quad C'\|\varphi(u)\|_{1,p,\tilde{\Omega},0,0} \leq \|u\|_{1,p,\Omega,\alpha,\alpha(N-p)/N} \leq C''\|\varphi(u)\|_{1,p,\tilde{\Omega},0,0},$$

so that

$$(2.13) \quad u \in W^{1,p}(\Omega, |x|^\alpha, |x|^{\alpha(N-p)/N}) \iff \varphi(u) \in W^{1,p}(\tilde{\Omega}, 1, 1).$$

By the classical Sobolev Embedding Theorem, for every $q \in [1, p^*)$, where

$$p^* := \begin{cases} +\infty & \text{if } N = p \\ \frac{Np}{N-p} & \text{if } N > p \end{cases},$$

there is a constant $C = C(q)$, such that

$$(2.14) \quad \|v\|_{1,p,\tilde{\Omega},0,0} \geq C(q) \cdot \|v\|_{q,\tilde{\Omega},0,0} \quad \forall v \in W^{1,p}(\tilde{\Omega}, 1, 1).$$

Now (2.8) and (2.10)-(2.14) yield the claim (2.5) in this case.

Step 2: We claim that (2.5) implies

$$(2.15) \quad W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha) \hookrightarrow L^q(\Omega, |x|^\alpha) \quad \text{for every } q \in [1, q_0).$$

Let r_0 and R be positive numbers such that

$$B_{r_0} \subset \Omega \subset B_R.$$

Further, fix $q \in [1, q_0)$, and choose any number $q_1 \in (q, q_0)$. Assume that $\{f_n\}$ is a bounded sequence in $W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$. Then there exists a subsequence, that we still denote by $\{f_n\}$, and a function $f \in W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$, such that

$$f_n \rightharpoonup f \quad \text{weakly in } W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha).$$

Note that, by the structure of the weight and the classical Sobolev Embedding Theorem, we also may assume

$$f_n \rightarrow f \quad \text{a.e. in } \Omega.$$

Let $r \in (0, r_0)$. By Hölder's inequality and Step 1 we have

$$(2.16) \quad \begin{aligned} \int_{B_r} |f_n - f|^{q_1} |x|^\alpha dx &\leq \left(\int_{B_r} |f_n - f|^q |x|^\alpha dx \right)^{q/q_1} \left(\int_{B_r} |x|^\alpha dx \right)^{(q_1-q)/q_1} \\ &\leq C \cdot r^{(N+\alpha)(q_1-q)/q_1}. \end{aligned}$$

Since

$$\begin{aligned} \int_{\Omega \setminus B_r} (|\nabla f_n|^p + |f_n|^p) dx &\leq \max\{r^{-\alpha}; R^{-\alpha}\} \cdot \int_{\Omega \setminus B_r} (|\nabla f_n|^p + |f_n|^p) |x|^\alpha dx \\ &\leq \max\{r^{-\alpha}; R^{-\alpha}\} \cdot \int_{\Omega} (|\nabla f_n|^p + |f_n|^p) |x|^\alpha dx \\ &\leq C \cdot \max\{r^{-\alpha}; R^{-\alpha}\} \quad \forall n \in \mathbb{N}, \end{aligned}$$

the Sobolev Embedding Theorem ensures that, up to a subsequence,

$$\int_{\Omega \setminus B_r} |f_n - f|^q dx \rightarrow 0$$

and hence also

$$(2.17) \quad \int_{\Omega \setminus B_r} |f_n - f|^q |x|^\alpha dx \leq \max\{r^\alpha, R^\alpha\} \cdot \int_{\Omega \setminus B_r} |f_n - f|^q dx \rightarrow 0.$$

Now, since $r \in (0, r_0)$ was arbitrary, (2.15) follows from (2.16) and (2.17).

Step 3: It remains to show (2.3). Since Ω is bounded and $\gamma \geq \alpha$, there holds

$$|x|^\gamma \leq C|x|^\alpha \quad \forall x \in \Omega,$$

for some constant $C > 0$. Hence we have that

$$L^q(\Omega, |x|^\alpha) \hookrightarrow L^q(\Omega, |x|^\gamma).$$

Now the claim follows from this and (2.15). \square

Remark 2.1. *The result (2)(i) of Theorem 2.1 is optimal in the sense that there is no continuous embedding of $W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$ into $L^q(\Omega, |x|^\alpha)$ when $0 \leq \alpha < N(p-1)$ and $q > p(N+\alpha)/(N-p+\alpha)$.*

To see this, choose $B_R \subset \Omega$ and $u \in C_0^\infty(\Omega)$ with $\text{supp } u \subset B_R$. Setting

$$u_t(x) := u(t^{-1} \cdot x), \quad (0 < t \leq 1),$$

we have

$$\begin{aligned} \|u_t\|_{p,\Omega,\alpha}^p &= t^{N+\alpha} \cdot \|u\|_{p,\Omega,\alpha}^p, & \|u_t\|_{q,\Omega,\alpha}^q &= t^{N+\alpha} \cdot \|u\|_{q,\Omega,\alpha}^q \quad \text{and} \\ \|\nabla u_t\|_{p,\Omega,\alpha}^p &= t^{N+\alpha-p} \cdot \|\nabla u\|_{p,\Omega,\alpha}^p. \end{aligned}$$

It follows that

$$\frac{\|u_t\|_{1,p,\Omega,\alpha,\alpha}}{\|u_t\|_{q,\Omega,\alpha}} \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

and in particular,

$$\frac{\|\nabla u_t\|_{p,\Omega,\alpha}}{\|u_t\|_{q,\Omega,\alpha}} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

From this the claim follows. \square

By our assumptions, $W_0^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$ is a closed subspace of $W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$.

Hence we have the following

Corollary 2.2. *The assertion of Theorem 2.1 holds with $W_0^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$ in place of $W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$.*

We will also need the following Poincaré-type inequalities.

Lemma 2.3. *Let $1 \leq p \leq N$ and $-N < \alpha < N(p-1)$. Then there are positive constants C_1, C_2 , such that*

$$(2.18) \quad \|\nabla u\|_{p,\Omega,\alpha} \geq C_1 \|u\|_{p,\Omega,\alpha} \quad \forall u \in W_0^{1,p}(\Omega, |x|^\alpha, |x|^\alpha), \quad \text{and}$$

$$(2.19) \quad \|\nabla u\|_{p,\Omega,\alpha} \geq C_2 \|u - u_\Omega\|_{p,\Omega,\alpha} \quad \forall u \in W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha),$$

$$\text{where } u_\Omega := \frac{\int_\Omega u |x|^\alpha dx}{\int_\Omega |x|^\alpha dx}.$$

Proof. Since $-N < \alpha < N(p-1)$, the weight $|x|^\alpha$ belongs to the Muckenhoupt class A_p . Hence it is also p -admissible, which means that (2.18) holds, (see [18], Chapter 15 and formula (1.5)).

The proof of (2.19) can be carried out analogously as in the unweighted case $\alpha = 0$, using the compactness of the embedding of $W^{1,p}(\Omega, |x|^\alpha, |x|^\alpha)$ into $L^p(\Omega, |x|^\alpha)$, (compare [14], § 5.8.1, proof of Theorem 1). \square

We conclude this subsection with the following existence result.

Theorem 2.4. *Let $1 \leq p \leq N$, $-N < \alpha < N(p-1)$, $\gamma \geq \alpha$ and $q \in [1, p_0)$. Then the problems (\mathbf{P}^D) and (\mathbf{P}^M) have solutions and the corresponding minima λ^D and λ^M are positive.*

Proof: From Theorem 2.1 and Lemma 2.3 we deduce that there are positive constants C' and C'' such that

$$\begin{aligned} \|\nabla u\|_{p,\Omega,\alpha} &\geq C' \|u\|_{q,\Omega,\gamma}, & \forall u \in W_0^{1,p}(\Omega, |x|^\alpha, |x|^\alpha), \\ \|\nabla u\|_{2,\Omega,\alpha} &\geq C'' \|u\|_{q,\Omega,\alpha}, & \forall u \in W^{1,2}(\Omega, |x|^\alpha, |x|^\alpha) \text{ with } u_\Omega = 0, \end{aligned}$$

and the assertions follow by standard arguments. \square

2.2. Foliated Schwarz symmetry

In this subsection we assume that Ω is a domain that is radially symmetric w.r.t. the origin. In other words, Ω is either an annulus, a ball, or the exterior of a ball in \mathbb{R}^N . If $u : \Omega \rightarrow \mathbb{R}$ is a measurable function, we will for convenience always extend u onto \mathbb{R}^N by setting $u(x) = 0$ for $x \in \mathbb{R}^N \setminus \Omega$.

Definition 2.5. *Let \mathcal{H}_0 be the family of open half-spaces H in \mathbb{R}^N such that $0 \in \partial H$. For any $H \in \mathcal{H}_0$, let σ_H denote the reflection in ∂H . We write*

$$\sigma_H u(x) := u(\sigma_H x), \quad x \in \mathbb{R}^N.$$

The two-point rearrangement w.r.t. H is given by

$$u_H(x) := \begin{cases} \max\{u(x); u(\sigma_H x)\} & \text{if } x \in H, \\ \min\{u(x); u(\sigma_H x)\} & \text{if } x \notin H. \end{cases}$$

Note that one has $u = u_H$ if and only if $u(x) \geq u(\sigma_H x)$ for all $x \in H$. Similarly, $\sigma_H u = u_H$ if and only if $u(x) \leq u(\sigma_H x)$ for all $x \in H$.

We will make use of the following properties of the two-point rearrangement.

Lemma 2.6. *Let $H \in \mathcal{H}_0$.*

- (1) *If $A \in C([0, +\infty), \mathbb{R})$, $u : \Omega \rightarrow \mathbb{R}$ is measurable and $A(|x|, u) \in L^1(\Omega)$, then $A(|x|, u_H) \in L^1(\Omega)$ and $\int_{\Omega} A(|x|, u) dx = \int_{\Omega} A(|x|, u_H) dx$.*
- (2) *If $u \in W^{1,2}(B, |x|^\alpha)$, then $\int_B |\nabla u|^2 |x|^\alpha dx = \int_B |\nabla u_H|^2 |x|^\alpha dx$.*

Proof. We observe that $|\sigma_H x| = |x|$, we have for a.e. $x \in H \cap \Omega$. Therefore

$$A(|x|, u(x)) + A(|\sigma_H x|, u(\sigma_H x)) = A(|x|, u_H(x)) + A(|\sigma_H x|, u_H(\sigma_H x))$$

and

$$|x|^\alpha |\nabla u(x)|^2 + |\sigma_H x|^\alpha |\nabla u(\sigma_H x)|^2 = |x|^\alpha |\nabla u_H(x)|^2 + |\sigma_H x|^\alpha |\nabla u_H(\sigma_H x)|^2.$$

It is now sufficient to integrate these two equalities on $\Omega \cap H$. □

Now we recall the definition of foliated Schwarz symmetrization of a function. Such a function is axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis.

Definition 2.7. *If $u : \Omega \rightarrow \mathbb{R}$ is measurable, the foliated Schwarz symmetrization u^* of u is defined as the (unique) function satisfying the following properties:*

- (1) *there is a function $w : [0, +\infty) \times [0, \pi) \rightarrow \mathbb{R}$, $w = w(r, \theta)$, which is nonincreasing in θ , and*

$$u^*(x) = w(|x|, \arccos(x_1/|x|)), \quad (x \in \Omega);$$

- (2) *$\mathcal{L}^{N-1}\{x : a < u(x) \leq b, |x| = r\} = \mathcal{L}^{N-1}\{x : a < u^*(x) \leq b, |x| = r\}$ for all $a, b \in \mathbb{R}$ with $a < b$, and $r \geq 0$.*

Definition 2.8. *Let P_N denote the point $(1, 0, \dots, 0)$, the 'north pole' of the unit sphere \mathbb{S}^{N-1} . We say that u is foliated Schwarz symmetric w.r.t. P_N if $u = u^*$ - that is, u depends solely on r and on θ - the 'geographical width' -, and is nonincreasing in θ .*

We also say that u is foliated Schwarz symmetric w.r.t. a point $P \in \mathbb{S}^{N-1}$ if there is a rotation about the origin ρ such that $\rho(P_N) = P$, and $u(\rho(\cdot)) = u^(\cdot)$.*

In other words, a function $u : \Omega \rightarrow \mathbb{R}$ is foliated Schwarz symmetric with respect to P if, for every $r > 0$ and $c \in \mathbb{R}$, the restricted superlevel set $\{x : |x| = r, u(x) \geq c\}$ is equal to $\{x : |x| = r\}$ or a geodesic ball in the sphere $\{x : |x| = r\}$ centered at rP . In particular, u is axially symmetric with respect to the axis $\mathbb{R}P$.

Moreover a measurable function $u : \Omega \rightarrow \mathbb{R}$ is foliated Schwarz symmetric w.r.t. $P \in \mathbb{S}^{N-1}$ iff $u = u_H$ for all $H \in \mathcal{H}_0$ with $P \in H$.

The next result was proved in [7]. It will be used in Section 4.

Theorem 2.9. *Let $u \in L^p(\Omega)$ for some $p \in [1, +\infty)$, and assume that for every $H \in \mathcal{H}_0$ one has either $u = u_H$, or $\sigma_H u = u_H$. Then u is foliated Schwarz symmetric w.r.t. some point $P \in \mathbb{S}^{N-1}$.*

3. NON-RADIALITY FOR SOLUTIONS TO PROBLEM (\mathbf{P}^D)

In this section we study problem (\mathbf{P}^D) when $\Omega =: B$ is the unit ball centered at the origin. Let α , p and q be fixed. For any number $\gamma \geq \alpha$ we write for convenience

$$\begin{aligned} R_\gamma(v) &:= R_{p,q,\alpha,\gamma}(v), \quad v \in W_0^{1,p}(B, |x|^\alpha, |x|^\alpha), \\ (\mathbf{P}_\gamma) &:= (\mathbf{P}^D) \quad \text{and} \quad \lambda_\gamma := \lambda^D. \end{aligned}$$

Theorem 3.1. *Assume that Ω is a ball B , centered at the origin, $q \in (p, q_0)$, where q_0 is defined by (2.4) and $0 \leq \alpha < N(p-1)$. Then there exists a number $\gamma^* \geq \alpha$ such that the minimizer of (\mathbf{P}_γ) is not radially symmetric if $\gamma > \gamma^*$.*

Denote

$$(3.1) \quad \lambda_\gamma^{rad} := \inf \left\{ R_\gamma(v) : v \in W_0^{1,p}(B, |x|^\alpha, |x|^\alpha) \setminus \{0\}, v \text{ radial} \right\}.$$

We merely need to show that

$$(3.2) \quad \lambda_\gamma < \lambda_\gamma^{rad},$$

if γ is large enough. Our approach is similar as in [16]. For the proof of (3.2) we need two lemmata.

Lemma 3.3. *There exists a number $C_0 > 0$, independent of γ , such that for all $\gamma \geq 3$,*

$$(3.3) \quad \lambda_\gamma \leq C_0 \cdot \gamma^{-N+p+Np/q}.$$

Proof. Let $U \in W_0^{1,p}(B)$ be a positive first eigenfunction for the Dirichlet p -Laplacian in B , with eigenvalue $\underline{\lambda}$, that is

$$(3.4) \quad \begin{cases} -\Delta_p U \equiv -\nabla(|\nabla U|^{p-2} \nabla U) = \underline{\lambda} U^{p-1} & \text{in } B, \\ U = 0 & \text{on } \partial B. \end{cases}$$

We extend U by zero outside B and set $x^\gamma := (1 - \gamma^{-1}, 0, \dots, 0)$ and $U_\gamma(x) := U(\gamma(x - x^\gamma))$. Then $U_\gamma \in W_0^{1,p}(B_{1/\gamma}(x^\gamma))$ and

$$(3.5) \quad \int_B |\nabla U_\gamma|^p dx = \underline{\lambda} \gamma^p \int_B (U_\gamma)^p dx.$$

It follows that

$$(3.6) \quad \int_B |\nabla U_\gamma|^p |x|^\alpha dx \leq \int_B |\nabla U_\gamma|^p dx = \underline{\lambda} \gamma^p \int_B (U_\gamma)^p dx.$$

On the other hand, we have by the minimality property of λ_γ and in view of Hölder's inequality,

$$\begin{aligned}
(3.7) \quad \int_B |\nabla U_\gamma|^p |x|^\alpha dx &\geq \lambda_\gamma \left(\int_B (U_\gamma)^q |x|^\gamma dx \right)^{p/q} \\
&= \lambda_\gamma \left(\int_{B_{1/\gamma}(x^\gamma)} (U_\gamma)^q |x|^\gamma dx \right)^{p/q} \\
&\geq \lambda_\gamma (1 - 2\gamma^{-1})^{\gamma p/q} \cdot \left(\int_{B_{1/\gamma}(x^\gamma)} (U_\gamma)^q dx \right)^{p/q} \\
&\geq \lambda_\gamma (1 - 2\gamma^{-1})^{\gamma p/q} \cdot \left(\int_{B_{1/\gamma}(x^\gamma)} dx \right)^{(p/q)-1} \cdot \int_B (U_\gamma)^p dx \\
&= \lambda_\gamma (1 - 2\gamma^{-1})^{\gamma p/q} \cdot \gamma^{N-Np/q} \cdot (\omega_N)^{(p/q)-1} \cdot \int_B (U_\gamma)^p dx,
\end{aligned}$$

where ω_N denotes the Lebesgue measure of B . Now (3.6) and (3.7) yield

$$(3.8) \quad \lambda_\gamma \leq \underline{\lambda} \gamma^{-N+p+Np/q} (1 - 2\gamma^{-1})^{-\gamma p/q} (\omega_N)^{1-p/q} \leq C_0 \gamma^{-N+p+Np/q},$$

where C_0 does not depend on γ . □

Lemma 3.4. *There holds for all $\gamma \geq \alpha$,*

$$(3.9) \quad \lambda_\gamma^{rad} \geq \left(\frac{\gamma + N}{\alpha + N} \right)^{p-1+p/q} \cdot \lambda_\alpha^{rad}.$$

Proof. Let $u \in W_0^{1,p}(B, |x|^\alpha, |x|^\alpha)$ be a radial function, such that

$$(3.10) \quad \lambda_\gamma^{rad} = R_\gamma(u).$$

We write $u = u(r)$, where $r = |x|$. Setting $z := r^{(\gamma+N)/(\alpha+N)}$ and $w(z) := u(r)$, and taking into account that $\alpha + N - p > 0$ and $\gamma \geq \alpha$, we calculate

$$\begin{aligned}
(3.11) \quad \int_B |\nabla u|^p |x|^\alpha dx &= N\omega_N \int_0^1 r^{\alpha+N-1} |u'(r)|^p dr \\
&= N\omega_N \left(\frac{\gamma + N}{\alpha + N} \right)^{p-1} \int_0^1 z^{\alpha+N-1-(\gamma-\alpha)(N+\alpha-p)/(\gamma+N)} |w'(z)|^p dz \\
&\geq N\omega_N \left(\frac{\gamma + N}{\alpha + N} \right)^{p-1} \int_0^1 z^{\alpha+N-1} |w'(z)|^p dz,
\end{aligned}$$

and

$$(3.12) \quad \int_B |u|^q |x|^\gamma dx = N\omega_N \int_0^1 r^{\gamma+N-1} |u|^q dr = N\omega_N \cdot \frac{\alpha + N}{\gamma + N} \int_0^1 z^{\alpha+N-1} |w|^q dz.$$

From (3.10), (3.11) and (3.12) we obtain

$$\begin{aligned} \lambda_\gamma^{rad} &\geq \left(\frac{\gamma+N}{\alpha+N}\right)^{p-1+p/q} \frac{N\omega_N \int_0^1 z^{\alpha+N-1} |w'(z)|^p dz}{\left(N\omega_N \int_0^1 z^{\alpha+N-1} |w|^q dz\right)^{p/q}} \\ &\geq \left(\frac{\gamma+N}{\alpha+N}\right)^{p-1+p/q} \cdot \lambda_\alpha^{rad}. \end{aligned}$$

□

Proof of Theorem 3.1. One has from the inequalities (3.3) and (3.9)

$$\frac{\lambda_\gamma^{rad}}{\lambda_\gamma} \geq \frac{\lambda_\alpha^{rad} \cdot \left(\frac{\gamma+N}{\alpha+N}\right)^{p-1+p/q}}{C_0 \cdot \gamma^{-N+p+Np/q}}.$$

Since $q > p$, we have that

$$p-1 + \frac{p}{q} > -N + \frac{Np}{q} + p.$$

It follows that

$$\frac{\lambda_\gamma^{rad}}{\lambda_\gamma} \longrightarrow +\infty \text{ as } \gamma \rightarrow +\infty,$$

and (3.2) follows if γ is large enough. □

4. FOLIATED SCHWARZ SYMMETRY OF SOLUTIONS TO PROBLEM (\mathbf{P}^M)

The following result has been already obtained for the case $\alpha = 0$ in [16].

Theorem 4.1. *Assume $-N < \alpha < N$ and $q \in [2, q_0)$, where q_0 is given by (2.4). Then every minimizer of (\mathbf{P}^M) is foliated Schwarz symmetric w.r.t. some point $P \in \mathbb{S}^{N-1}$.*

Proof. We divide the proof into steps. We denote B any ball centered in the origin, and for convenience we write $\lambda^M = \lambda$.

Step 1: Let $H \in \mathcal{H}_0$, and let u be a minimizer of (\mathbf{P}^M) . Then by assuming that $\|u\|_{q,B,\alpha} = 1$, then u satisfies the following Neumann boundary value problem for the Euler equation given by

$$(4.1) \quad \begin{cases} -\nabla(|x|^\alpha \nabla u) = 2\lambda|x|^\alpha |u|^{q-2}u + \mu|x|^\alpha & \text{in } B \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B \end{cases},$$

for some $\mu \in \mathbb{R}$, where ν denotes the exterior unit normal.

By the assumption on q and classical regularity theory, we deduce that u is bounded in $B \setminus B_\epsilon$ for every $\epsilon > 0$, and then $u \in C^2(\overline{B} \setminus \{0\})$.

On the other hand, the following equalities hold by Lemma 2.6:

$$u_H \neq 0, \quad u_H \in W^{1,2}(B, |x|^\alpha, |x|^\alpha), \quad \int_\Omega u_H |x|^\alpha dx = 0, \quad \|u_H\|_{q,B,\alpha} = 1,$$

$$\int_B |u|^q |x|^\alpha dx = \int_B |u_H|^q |x|^\alpha dx, \quad \int_B u |x|^\alpha dx = \int_B u_H |x|^\alpha dx,$$

$$\int_B |\nabla u|^2 |x|^\alpha dx = \int_B |\nabla u_H|^2 |x|^\alpha dx.$$

and therefore we get

$$R_{2,q,\alpha,\alpha}(u) = R_{2,q,\alpha,\alpha}(u_H).$$

Hence, u_H is a minimizer, too, so that it satisfies the same Euler equation satisfied by u and boundary Neumann condition, i.e.

$$(4.2) \quad \begin{cases} -\nabla(|x|^\alpha \nabla u_H) = 2\lambda |x|^\alpha |u_H|^{q-2} u_H + \mu |x|^\alpha & \text{in } B \\ \frac{\partial u_H}{\partial \nu} = 0 & \text{on } \partial B. \end{cases}$$

Moreover u_H satisfies the same regularity properties of u , that is $u_H \in C^2(\overline{B} \setminus \{0\})$.

Step 2: Define $v := u - u_H$ and note that $v \geq 0$ in $B \cap H$. Then $v \in C^2(\overline{B} \setminus \{0\})$ satisfies the following linear elliptic equation

$$(4.3) \quad -\nabla(|x|^\alpha \nabla v) = 2\lambda |x|^\alpha m(x)v, \quad \text{in } B \setminus \{0\}$$

where

$$m(x) := \begin{cases} \frac{|u|^{q-2}u - |u_H|^{q-2}u_H}{v(x)} & \text{if } v(x) \neq 0, \\ 0 & \text{if } v(x) = 0 \end{cases}$$

Since $u, u_H \in C^2(\overline{B} \setminus \{0\})$, $m(x)$ is a bounded function in $B \setminus B_\epsilon$ for every $\epsilon > 0$.

We claim that for every half-space H with $0 \in \partial H$ there holds one of the following:

- (1) $\sigma_H u \equiv u_H$ on H ,
- (2) $u \equiv u_H$ on H .

If (1) holds, we are done. Note that (1) implies that $u(x) \leq \sigma_H u(x)$ on H . Hence, if (1) does not hold, then there is a point $x_0 \in H$ with $u(x_0) > \sigma_H u(x_0)$. Since u is continuous, there is a neighborhood W of x_0 with $W \subset H$, such that $u(x) > \sigma_H u(x)$ on W , which also implies $u(x) \equiv u_H(x)$ in W , that is, $v \equiv 0$ in W . We may apply the Principle of Unique Continuation to (4.3) to conclude that $v \equiv 0$, that is, $u \equiv u_H$ throughout H . In other words, (2) holds. This proves the claim.

Finally by Theorem 2.8 this implies that u is - up to a rotation about the origin - foliated Schwarz symmetric with respect to some point $P \in \mathbb{S}^{N-1}$. \square

Remark 4.1. *The above result holds in the case of an annulus centered at the origin, too.*

5. SHAPE OF SOLUTIONS TO PROBLEM (\mathbf{P}^M) FOR $q = 2$ AND $N = 2$

In this section we show the explicit expression of the solutions to problem (\mathbf{P}^M) in the case $q = 2$ and $N = 2$. This will be useful to prove symmetry properties of the minimizers.

First we recall some properties of Bessel functions (see, for example, [2]).

5.1. A few properties of Bessel functions.

It is well-known that Bessel functions J_ν , Y_ν of order ν of the first and second kind, are linearly independent for any value of ν (see, for example, [2] p. 358). The following relation holds true

$$(5.1) \quad Y_\nu(r) = \frac{J_\nu(r) \cos(\nu\pi) - J_{-\nu}(r)}{\sin(\nu\pi)},$$

for non integer α and where the right-hand side is replaced by its limiting value whenever ν is an integer. Moreover, J_ν satisfies the following fundamental recurrence relation

$$(5.2) \quad rJ'_\nu(r) - \nu J_\nu(r) = -rJ_{\nu+1}(r), \quad r \in \mathbb{R}.$$

If we denote by $j_{\nu,h}, j'_{\nu,h}$ the zeros of J_ν, J'_ν , respectively, then

$$\nu \leq j'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < \dots$$

and

$$(5.3) \quad j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < \dots$$

In [2] (Prop. 9.1.9, p. 360), the following identities can be found

$$(5.4) \quad \begin{aligned} J_\nu(r) &= \frac{\left(\frac{1}{2}r\right)^\nu}{\Gamma(\nu+1)} \prod_{h=1}^{\infty} \left(1 - \frac{r^2}{j_{\nu,h}^2}\right), \quad \nu \geq 0 \\ J'_\nu(r) &= \frac{\left(\frac{1}{2}r\right)^{\nu-1}}{2\Gamma(\nu)} \prod_{h=1}^{\infty} \left(1 - \frac{r^2}{(j'_{\nu,h})^2}\right), \quad \nu > 0. \end{aligned}$$

Finally, we will deal with the equation

$$(5.5) \quad -\frac{\alpha}{2}J_\nu(x) + xJ'_\nu(x) = 0 \quad x \geq 0.$$

The roots of this equation have been studied in [22]. We rewrite it as

$$(5.6) \quad F_\nu(x) = \frac{\alpha}{2},$$

where

$$(5.7) \quad F_\nu(x) = x \frac{J'_\nu(x)}{J_\nu(x)} = \nu - x \frac{J_{\nu+1}(x)}{J_\nu(x)} = -\nu + x \frac{J_{\nu-1}(x)}{J_\nu(x)},$$

for any positive x which is not a zero for J_ν . Here we used the property

$$(5.8) \quad zJ'_\nu(z) = \nu J_\nu(z) - zJ_{\nu+1}(z).$$

We emphasize that the positive zeros of $J_\nu(x)$ are not solutions of equation (5.6).

It is proved in [22] that, for any $\nu > -1$, the function $F_\nu(x)$ decreases from the value ν at $x = 0$ to $-\infty$ at $x = j_{\nu,1}$, the first positive zero of the function $J_\nu(x)$, jumping to $+\infty$ as x moves past $j_{\nu,1}$ and decreases to $-\infty$ at $x = j_{\nu,2}$ and so on (see Figures 4 in [22]).

Let $x_{\nu,k}$, $k = 1, 2, \dots$, be the positive roots of the equation

$$-\frac{\alpha}{2}J_\nu(x) + xJ'_\nu(x) = 0,$$

ordered in increasing order. In [22] the behaviour of $x_{\nu,k}$ is described as the order ν varies over the entire range of all real values. In particular, at page 196, it is proved that

$$\frac{d}{d\nu}x_{\nu,k} > 0 \text{ whenever } F'_\nu(x_{\nu,k}) < 0.$$

5.2. Explicit expression of the eigenfunctions in dimension 2, for $q = 2$.

For convenience we again write $\lambda := \lambda^M$ for the infimum in problem (\mathbf{P}^M) . The main result of this section is

Theorem 5.1. *Let $\Omega =: B$ be the unit ball in \mathbb{R}^2 , $q = 2$ and $|\alpha| < 2$. Then, if u is a minimizer of (\mathbf{P}^M) , there holds*

$$(5.9) \quad u(x) = \varphi_1(r)(A_1 \cos \theta + B_1 \sin \theta), \quad x = (x_1, x_2) = (r \cos \theta, r \sin \theta) \in B,$$

where

$$\varphi_1(r) = r^{-\alpha/2}J_{\nu_1}(\sqrt{2\lambda}r), \quad 0 \leq r \leq 1.$$

Here φ_1 is a solution to the problem

$$(5.10) \quad \begin{cases} r^2\varphi_1''(r) + (\alpha + 1)r\varphi_1'(r) + (2\lambda r^2 - 1)\varphi_1(r) = 0, \\ \varphi_1(r) > 0, \quad 0 < r \leq 1, \\ \varphi_1'(1) = 0, \end{cases}$$

$$\lambda = \frac{x_{\nu_1,1}^2}{2}, \quad \nu_1 = \sqrt{1 + \frac{\alpha^2}{4}},$$

$x_{\nu_1,1}$ is the first positive root of the equation

$$-\frac{\alpha}{2}J_{\nu_1}(x) + xJ'_{\nu_1}(x) = 0,$$

and $A_1, B_1 \in \mathbb{R}$ are arbitrary constants.

Remark 5.1. *Formula (5.9) can be rewritten as*

$$(5.11) \quad u(x) = C \cdot \varphi_1(r) \cdot \cos(\theta - \theta_0), \quad x \in B,$$

for some numbers $C \in \mathbb{R}$ and $\theta_0 \in [0, \pi]$, that is, u is foliated Schwarz symmetric.

Proof of Theorem 5.1. Let u be a minimizer to problem (\mathbf{P}^M) . Then u solves the Neumann boundary value problem for the Euler equation given by (4.1). It is easy to see that in this case $\mu = 0$. Indeed, one can use u as test function in the Euler equation and integrate on B . The constraint on the weighted average of u on the right-hand side gives the conclusion. By using polar coordinates, we can write u as

$$u(x_1, x_2) = v(r, \theta) = u(r \cos \theta, r \sin \theta).$$

An easy calculation shows that v solves the following equation

$$(5.12) \quad -\nabla(|x|^\alpha \nabla u) = -\alpha r^{\alpha-1} \frac{\partial v}{\partial r} - r^\alpha \left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial^2 v}{\partial \theta^2} \cdot \frac{1}{r^2} + \frac{\partial v}{\partial r} \cdot \frac{1}{r} \right] = 2\lambda r^\alpha v.$$

Assume that v has the following expression

$$v(r, \theta) = \varphi(r)w(\theta), \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

Equation (5.12) implies that $\varphi(r)$ and $w(\theta)$ are solutions to the following problems respectively:

$$(5.13) \quad \begin{cases} r^2 \varphi''(r) + (\alpha + 1)r \varphi'(r) + (2\lambda r^2 + C)\varphi(r) = 0, & 0 < r \leq 1, \\ \varphi'(1) = 0 \end{cases},$$

$$(5.14) \quad \begin{cases} w''(\theta) - Cw(\theta) = 0, & 0 < \theta \leq 2\pi, \\ w(0) = w(2\pi) \end{cases}.$$

Equation (5.14) has periodic continuous solutions for $C \equiv C_n = -n^2$, $n \in \mathbb{N} \cup \{0\}$. Consequently the solutions to (5.14) are given by

$$w_n(\theta) = \begin{cases} A_0, & n = 0 \\ A_n \cos(n\theta) + B_n \sin(n\theta), & n \geq 1 \end{cases}$$

for any constant $A_0, A_n, B_n \in \mathbb{R}$. The case $n = 0$ corresponds to a purely radial function.

Now we solve (5.13) with $C \equiv C_n = -n^2$. For any fixed $n \in \mathbb{N} \cup \{0\}$, the solutions φ_n to the equation

$$(5.15) \quad \varphi_n''(r) + (\alpha + 1) \frac{\varphi_n'(r)}{r} + \left(2\lambda - \frac{n^2}{r^2} \right) \varphi_n(r) = 0$$

are given by

$$\varphi_n(r) = r^{-\alpha/2} [c_1 J_{\nu_n}(\sqrt{2\lambda}r) + c_2 Y_{\nu_n}(\sqrt{2\lambda}r)],$$

where c_1, c_2 are arbitrary constants and $J_{\nu_n}(r), Y_{\nu_n}(r)$, with $\nu_n = \sqrt{n^2 + \frac{\alpha^2}{4}}$, are Bessel functions of first and second kind respectively.

Since the solution u must belong to the weighted space $L^2(B, |x|^\alpha)$, necessarily $c_2 = 0$. Indeed by (5.1) and (5.4), for any fixed $\nu > 0$ and $r \rightarrow 0^+$, it holds that

$$J_\nu(r) \sim c_\nu r^\nu \quad \text{and} \quad Y_\nu(r) \sim c_\nu r^{-\nu}.$$

Therefore the integral of $u_n = \varphi_n(r)w_n(\theta)$, that is,

$$\int_B |u_n|^2 |x|^\alpha dx = \int_0^{2\pi} w_n^2(\theta) d\theta \int_0^1 \varphi_n^2(r) r^{\alpha+1} dr,$$

is finite if, and only if,

$$-\alpha - 2\nu_n + \alpha + 1 > -1,$$

that is,

$$\nu_n < 1.$$

But such a condition is not verified if $n \geq 1$. This justifies the choice of $c_2 = 0$ for $n \geq 1$.

For $n = 0$, condition $\nu_n < 1$ is equivalent to $|\alpha| < 2$. Moreover, since

$$(5.16) \quad J_\nu(r) \sim r^\nu, \quad Y_\nu(r) \sim r^{-\nu}, \quad \frac{d}{dr} J_\nu(r) \sim r^{\nu-1}, \quad \frac{d}{dr} Y_\nu(r) \sim r^{-\nu-1},$$

an analogous argument shows that

$$\int_B |\nabla u_0|^2 |x|^\alpha dx = A_0 \int_0^1 |\varphi_0'(r)|^2 r^{\alpha+1} dr$$

is finite if, and only if $-\alpha - 2 - 2\nu_0 + \alpha + 1 > -1$. But such a condition is not verified. This justifies the choice of $c_2 = 0$ also for $n = 0$.

We now impose the Neumann condition $\varphi_n'(1) = 0$ in the expression

$$\varphi_n(r) = c_1 r^{-\alpha/2} J_{\nu_n}(\sqrt{2\lambda}r).$$

An easy calculation gives, for any $0 < r < 1$:

$$\varphi_n'(r) = -c_1 \frac{\alpha}{2} r^{-\frac{\alpha}{2}-1} J_{\nu_n}(\sqrt{2\lambda}r) + c_1 r^{-\frac{\alpha}{2}} \sqrt{2\lambda} J_{\nu_n}'(\sqrt{2\lambda}r).$$

Therefore the Neumann condition $\varphi_n'(1) = 0$ is equivalent to

$$(5.17) \quad -\frac{\alpha}{2} J_{\nu_n}(\sqrt{2\lambda}) + \sqrt{2\lambda} J_{\nu_n}'(\sqrt{2\lambda}) = 0.$$

This means that $\sqrt{2\lambda}$ is a positive root of the equation

$$(5.18) \quad -\frac{\alpha}{2} J_{\nu_n}(x) + x J_{\nu_n}'(x) = 0$$

or equivalently

$$(5.19) \quad F_{\nu_n}(x) = \frac{\alpha}{2},$$

according to (5.6). Let us consider now for any fixed $n \in \mathbb{N} \cup \{0\}$ the positive roots $x_{\nu_n, k}$, $k = 1, 2, \dots$ of the equation (5.18). For $n = 0$ the smaller positive root is

$$x_{\nu_0, 1}, \quad \text{with} \quad \nu_0 = \frac{|\alpha|}{2}.$$

For the value $\nu_0 = \frac{|\alpha|}{2}$ and definition (5.7) of function $F_{\nu_n}(x)$, equation (5.19) becomes

$$\frac{|\alpha|}{2} - x \frac{J_{\nu_0+1}(x)}{J_{\nu_0}(x)} = \frac{\alpha}{2}, \quad \text{if } \alpha > 0,$$

or

$$-\frac{|\alpha|}{2} + x \frac{J_{\nu_0-1}(x)}{J_{\nu_0}(x)} = \frac{\alpha}{2}, \quad \text{if } \alpha < 0.$$

This implies that the positive root $x_{\nu_0,1}$ of equation (5.19) coincides with the zero $j_{\nu_0+1,1}$ of the Bessel function J_{ν_0+1} , when $\alpha > 0$ and coincides with the zero $j_{\nu_0-1,1}$ of the Bessel function J_{ν_0-1} , when $\alpha < 0$.

Assume $\alpha > 0$. By previous described properties of $x_{\nu,k}$, we know that

$$x_{\nu_1,1} < j_{\nu_1,1}$$

and by properties of zero's Bessel functions, since $\nu_1 = \sqrt{1 + \frac{\alpha^2}{4}} < \nu_0 + 1 = \frac{|\alpha|}{2} + 1$, it results

$$j_{\nu_1,1} < j_{\nu_0+1,1} \equiv x_{\nu_0,1}.$$

Assume $\alpha < 0$. In such a way $x_{\nu_0,1} = j_{\nu_0-1,1}$ (with $\nu_0 - 1 > -1$) is the smallest positive root of equation (5.19) and therefore $\sqrt{2\lambda} = j_{\nu_0-1,1}$. But such a root cannot be considered. Indeed in this case we choose $n = 0$. Moreover the minimizer $u(x_1, x_2) = A_0 \varphi_0(r) = A_0 r^{-\alpha/2} J_{\nu_0}(j_{\nu_0-1,1} r)$ must have zero weighted mean value, while by the following equality (see [17], p.707 n.6.556 (9)) we get

$$\int_0^1 r^{1-\nu_0} J_{\nu_0}(j_{\nu_0-1,1} r) dr = \frac{(j_{\nu_0-1,1})^{\nu_0-2}}{2^{\nu_0-1} \Gamma(\nu_0)} - (j_{\nu_0-1,1})^{-1} J_{\nu_0-1}(j_{\nu_0-1,1}) = \frac{(j_{\nu_0-1,1})^{\nu_0-2}}{2^{\nu_0-1} \Gamma(\nu_0)} \neq 0.$$

We conclude that in both cases the smaller positive root of equation (5.19) is given by $x_{\nu_1,1}$. This implies that

$$(5.20) \quad \sqrt{2\lambda} = x_{\nu_1,1}, \quad \nu_1 = \sqrt{1 + \frac{\alpha^2}{4}}$$

and $\varphi_1(r)$ is the corresponding solution to problem (5.10).

Finally the uniqueness (up to rotations and multiples) of the function $u(x_1, x_2) = v(r, \theta)$ is a consequence of standard properties of completeness. \square

6. BREAK OF ANTI-SYMMETRY OF SOLUTIONS TO PROBLEM (\mathbf{P}^M) FOR $N = 2$ AND LARGE q

In this section we give conditions in the two-dimensional case, such that the minimizers of problem (\mathbf{P}^M) fail to be antisymmetric.

We recall that the foliated Schwarz symmetry proved in Section 4 implies that, up to a rotation about the origin, a minimizer $u(x_1, x_2)$ is symmetric (even) in the variable x_1 , for any $q \geq 2$. We are now going to analyse the behaviour of u with respect to the other variable, x_2 . Note that, for $q = 2$, formula (5.9) implies that, if u is even in the variable x_1 , then u is antisymmetric (odd) w.r.t. x_2 .

Readapting a technique of [16], we prove in this section that, for $-2 < \alpha < 0$ and sufficiently large q , if u is symmetric w.r.t. the variable x_1 , then u is *not* antisymmetric with respect to x_2 .

In the sequel let $B \subset \mathbb{R}^2$ denote the ball of radius 1 centered at the origin, and let $\lambda_{\alpha,q} := \lambda^M$ be the corresponding infimum in problem (\mathbf{P}^M) .

The main result of the section is the following

Theorem 6.1. *Let $-2 < \alpha < 0$ and $N = 2$. Then there is a number $\tilde{q} > 2$, such that, if $q > \tilde{q}$ and u is a corresponding minimizer which is symmetric (even) w.r.t. to x_1 , then u is not antisymmetric w.r.t. x_2 .*

The key point in this proof is a result by Ren and Wei (see Lemma 2.2 in [28]), where it is shown that if one considers the Rayleigh quotient $R_{0,q}$ in the space of $W_0^{1,2}(B)$ functions, the corresponding eigenvalue tends to 0 as the parameter q of the denominator goes to infinity. We prove that the same behaviour holds for our eigenvalue $\lambda_{\alpha,q}$.

Lemma 6.1. *Let Ω be a bounded domain in \mathbb{R}^2 containing 0 and $-2 < \alpha < 0$. Further, let*

$$\lambda_{\alpha,q}^0(\Omega) := \inf \left\{ \frac{\int_{\Omega} |\nabla v|^2 |x|^\alpha dx}{\left(\int_{\Omega} |v|^q |x|^\alpha dx \right)^{2/q}} : v \in W_0^{1,2}(\Omega, |x|^\alpha, |x|^\alpha) \setminus \{0\} \right\}, \quad q \geq 2.$$

Then $\lambda_{\alpha,q}^0(\Omega) \rightarrow 0$ as $q \rightarrow \infty$.

Proof. Choose $R > 0$ and $x_0 \in \Omega$ such that $B_{2R}(x_0) \subset \Omega$ and $0 \notin B_{2R}(x_0)$. For $q \geq 1$ we define $w_q : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$w_q(x) = \begin{cases} q & | 0 \leq |x| \leq Re^{-q} \\ \ln \frac{R}{|x|} & | Re^{-q} \leq |x| \leq R \\ 0 & | |x| \geq R. \end{cases}$$

It has been shown in [28], Lemma 2.2, that

$$(6.1) \quad \lim_{q \rightarrow +\infty} \frac{\int_{B_R(0)} |\nabla w_q|^2 dx}{\left(\int_{B_R(0)} |w_q|^q dx \right)^{2/q}} = 0.$$

Now let $u_q \in W_0(\Omega, |x|^\alpha, |x|^\alpha)$ be defined by

$$u_q(x) := w_q(x - x_0), \quad x \in \Omega.$$

In view of our assumptions there are positive constants C_1, C_2 such that

$$C_1 \leq |x|^\alpha \leq C_2, \quad \forall x \in B_R(x_0).$$

Together with (6.1) we finally obtain

$$\begin{aligned} \lambda_{\alpha,q}^0 &\leq \frac{\int_{B_R(x_0)} |\nabla u_q|^2 |x|^\alpha dx}{\left(\int_{B_R(x_0)} |u_q|^q |x|^\alpha dx \right)^{2/q}} \\ &\leq C_2 \cdot (C_1)^{-2/q} \cdot \frac{\int_{B_R(0)} |\nabla w_q|^2 dx}{\left(\int_{B_R(0)} |w_q|^q dx \right)^{2/q}} \rightarrow 0 \text{ as } q \rightarrow +\infty. \end{aligned}$$

□

A direct consequence of the above lemma is the following result

Corollary 6.1. *Let $-2 < \alpha < 0$ and*

$$\lambda_{\alpha,q}^{as}(B) := \inf \left\{ R_{\alpha,q}(v) : v \in W_0^{1,2}(B, |x|^\alpha, |x|^\alpha) \setminus \{0\}, v(x_1, -x_2) = -v(x_1, x_2) \right\}.$$

Then $\lambda_{\alpha,q}^{as}(B) \rightarrow 0$, as $q \rightarrow \infty$.

Proof. Let u be a function realizing $\lambda_{\alpha,q}^0(B^+)$, where B^+ is the upper half part of the unit ball in the plane. We define

$$w(x_1, x_2) = \begin{cases} u(x_1, x_2) & | (x_1, x_2) \in B^+ \\ -u(x_1, -x_2) & | (x_1, x_2) \in B \setminus B^+ \end{cases}$$

and use it as a test function. By Lemma 6.1 this gives

$$\lambda_{\alpha,q}^{as}(B) \leq \frac{\int_B |\nabla w|^2 |x|^\alpha dx}{\left(\int_B |w|^q |x|^\alpha dx \right)^{2/q}} = 2^{1-\frac{q}{2}} \lambda_{\alpha,q}^0(B^+) \rightarrow 0 \text{ as } q \rightarrow +\infty.$$

□

Now we prove the main result of this section.

Proof of Theorem 6.1. We define a particular test function \tilde{u}_q for $\lambda_{\alpha,q}(B)$ to prove that $\lambda_{\alpha,q}(B) < \lambda_{\alpha,q}^{as}(B)$. Let v_q be a function such that $v_q(x_1, -x_2) = -v_q(x_1, x_2)$ realizing $\lambda_{\alpha,q}^{as}(B)$, such that $\int_B |\nabla v_q|^2 |x|^\alpha dx = 1$. We define

$$\bar{u}_q(x_1, x_2) = \begin{cases} v_q, & (x_1, x_2) \in B^+ \\ 0 & (x_1, x_2) \in B \setminus B^+. \end{cases}$$

We observe that

$$(6.2) \quad \int_B |\nabla \bar{u}_q|^2 |x|^\alpha dx = \frac{1}{2},$$

and

$$(6.3) \quad \int_B |\bar{u}_q|^q |x|^\alpha dx = \frac{1}{2} \int_B |v_q|^q |x|^\alpha dx = \frac{1}{2} [\lambda_{\alpha,q}^{as}(B)]^{-q/2}.$$

We now use

$$\tilde{u}_q := \bar{u}_q - d, \quad \text{where } d := \frac{1}{\int_B |x|^\alpha dx} \int_B \bar{u}_q |x|^\alpha dx,$$

as a test function for $\lambda_{\alpha,q}(B)$. We have, by (6.2),

$$\lambda_{\alpha,q}(B) \leq \frac{\int_B |\nabla \bar{u}_q|^2 |x|^\alpha dx}{\left[\int_B |\bar{u}_q - d|^q |x|^\alpha dx \right]^{2/q}}.$$

By the triangle inequality and (6.3) we get

$$(6.4) \quad \begin{aligned} \lambda_{\alpha,q}(B) &\leq \frac{1/2}{\left[\left(\int_B |\bar{u}_q|^q |x|^\alpha dx \right)^{1/q} - d \left(\int_B |x|^\alpha dx \right)^{1/q} \right]^2} \\ &= \frac{1/2}{\left[\left[\frac{1}{2} [\lambda_{\alpha,q}^{as}(B)]^{-q/2} \right]^{1/q} - \left| \int_B \bar{u}_q |x|^\alpha dx \right| \left[\int_B |x|^\alpha dx \right]^{(1/q)-1} \right]^2}, \\ &= \frac{(1/2)^{-(2/q)+1}}{\left[1 - \left| \int_B \bar{u}_q |x|^\alpha dx \right| \left[\int_B |x|^\alpha dx \right]^{(1/q)-1} [\lambda_{\alpha,q}^{as}(B)]^{1/2} \right]^2} \cdot \lambda_{\alpha,q}^{as}(B). \end{aligned}$$

By Lemma 2.3, (2.19), and Theorem 2.1 there exists a positive constant C , independent of q , such that

$$\left| \int_B \bar{u}_q |x|^\alpha dx \right| \leq C \quad \forall q \geq 2.$$

Since $\lambda_{\alpha,q}^{as}(B) \rightarrow 0$ as $q \rightarrow \infty$ by Corollary 6.1, the denominator in the last line of (6.4) tends to 1, as $q \rightarrow \infty$. Therefore, for q sufficiently large

$$\lambda_{\alpha,q}(B) \leq \frac{2}{3} \cdot \lambda_{\alpha,q}^{as}(B) < \lambda_{\alpha,q}^{as}(B).$$

This shows the breaking of anti-symmetry. \square

7. A WEIGHTED SZEGŐ-WEINBERGER INEQUALITY

Throughout this section we will denote by B_R the ball in \mathbb{R}^N , with $N \geq 2$, centered at the origin with radius R and we will assume that $\alpha \in (0, N)$. Note that when α lies in this interval, Theorem 2.1 ensures that $W^{1,2}(\Omega, |x|^\alpha, |x|^\alpha)$ is compactly embedded in $L^2(\Omega, |x|^\alpha)$, for any Lipschitz bounded domain Ω in \mathbb{R}^N . Therefore

$$(7.1) \quad \mu(\Omega) := \inf \left\{ \frac{\int_{\Omega} |\nabla \varphi|^2 |x|^\alpha dx}{\int_{\Omega} \varphi^2 |x|^\alpha dx} : \varphi \in W^{1,2}(\Omega, |x|^\alpha, |x|^\alpha) \setminus \{0\}, \int_{\Omega} \varphi |x|^\alpha dx = 0 \right\}$$

coincides with the first nonzero eigenvalue of the problem

$$(7.2) \quad \begin{cases} -\nabla (|x|^\alpha \nabla u) = \mu(\Omega) |x|^\alpha u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν denotes the outer normal to $\partial\Omega$.

In this section we prove a Szegő-Weinberger type inequality, given in Theorem 7.1 below. For any bounded domain Ω in \mathbb{R}^N we will denote by Ω^\sharp the ball centered at the origin, whose radius r^\sharp is such that

$$\int_{\Omega} |x|^\alpha dx = \int_{\Omega^\sharp} |x|^\alpha dx = \frac{N\omega_N}{N+\alpha} (r^\sharp)^{N+\alpha}.$$

In other words we are assuming that the weighted measures of Ω and Ω^\sharp coincide.

Theorem 7.1. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^N , with $N \geq 2$, symmetric with respect to the origin. Let $\alpha \in (0, N)$ and let $\mu(\Omega)$ be defined in (7.1). Then*

$$(7.3) \quad \mu(\Omega) \leq \mu(\Omega^\sharp),$$

where equality holds if and only if $\Omega = \Omega^\sharp$.

The first step in proving the above-mentioned result is to show that $\mu(B_R)$ is an N -fold degenerate eigenvalue and a corresponding set of eigenfunctions is in the form

$$G(|x|) \frac{x_i}{|x|} \text{ for } i = 1, \dots, N,$$

for some suitable function G . To this aim it is convenient to rewrite problem (7.2), when $\Omega = B_R$, in polar coordinates as follows

$$(7.4) \quad \begin{cases} -\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial u}{\partial r} \right) - \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}}(u|_{\mathbb{S}_r^{N-1}}) - \frac{\alpha}{r} \frac{\partial u}{\partial r} = \mu(B_R) u & \text{in } B_R \\ \frac{\partial u}{\partial r} = 0 & \text{on } \partial B_R, \end{cases}$$

where $\mathbb{S}_r^{N-1} = \partial B_r$, $u|_{\mathbb{S}_r^{N-1}}$ is the restriction of u on \mathbb{S}_r^{N-1} and, finally, $\Delta_{\mathbb{S}^{N-1}}(u|_{\mathbb{S}_r^{N-1}})$ is the standard Laplace-Beltrami operator relative to the manifold \mathbb{S}_r^{N-1} .

It is well known that the solutions of the eigenvalue problem (7.4) can be found via separation of variables. Writing $u(x) = Y(\theta)f(r)$ and plugging it into the equation in (7.4), with $\theta \in \mathbb{S}_1^{N-1}$, we get

$$-\frac{Y}{r^{N-1}} (r^{N-1}f')' - \frac{f}{r^2} \Delta_{\mathbb{S}^{N-1}}(Y) - \frac{\alpha}{r} Y f' = \mu(B_R) Y f$$

and in turn

$$\frac{1}{f r^{N-3}} (r^{N-1}f')' + \alpha r \frac{f'}{f} + \mu(B_R) r^2 = -\frac{\Delta_{\mathbb{S}^{N-1}}(Y)}{Y} = \bar{k}$$

Since the last equality is fulfilled if and only if

$$\bar{k} = k(k + N - 2) \quad \text{with } k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

(see e.g. [25]), we have that

$$(7.5) \quad f'' + \frac{N-1+\alpha}{r} f' + \mu(B_R) f - \frac{k(k+N-2)}{r^2} f = 0 \quad \text{with } k \in \mathbb{N}_0.$$

Hence the eigenfunctions μ_i of problem (7.4) are either purely radial

$$(7.6) \quad u_i(r) = f_0(\mu_i; r), \quad \text{if } k = 0,$$

or in the form

$$u_i(r, \theta) = f_k(\mu_i; r) Y(\theta), \quad \text{if } k \in \mathbb{N}.$$

Denote $\mu := \mu(B_R)$. Let us explicitly remark that equation (7.5) can be rewritten as

$$(7.7) \quad f'' + \frac{\beta+1}{r} f' + \left[\mu - \frac{k(k+N-2)}{r^2} \right] f = 0,$$

with $\beta = N - 2 + \alpha$ and $k \in \mathbb{N}_0$. Therefore it coincides with equation (5.15) in Section 5 when α is replaced by β , 2λ by μ and n^2 by $k(k + N - 2)$. As in Section 5 we deduce that solutions to equation (7.5) are given by

$$f_k(r) = r^{-\frac{\beta}{2}} (c_1 J_{\nu_k}(\sqrt{\mu}r) + c_2 Y_{\nu_k}(\sqrt{\mu}r))$$

where c_1, c_2 are arbitrary constants and

$$\nu_k = \sqrt{\frac{\beta^2}{4} + k(k+N-2)} = \sqrt{\frac{(N-2+\alpha)^2}{4} + k(k+N-2)}.$$

Moreover the solutions f_k belonging to $W^{1,2}(B_R, |x|^\alpha, |x|^\alpha)$ are obtained by choosing $c_2 = 0$, i.e.

$$f_k(r) = c_1 r^{-\frac{\beta}{2}} J_{\nu_k}(\sqrt{\mu}r), \quad 0 < r < R.$$

In the sequel we will denote by $\tau_n(R)$, with $n \in \mathbb{N}_0$, the sequence of eigenvalues of (7.4) whose corresponding eigenfunctions are purely radial, i.e. in the form (7.6). Clearly in this case the first eigenfunction is constant and the corresponding eigenvalue $\tau_0(R) = 0$. We will denote by $v_n(R)$, with $n \in \mathbb{N}$, the remaining eigenvalues of (7.4). We finally arrange the eigenvalues in such a way that the sequences $\tau_n(R)$ and $v_n(R)$ are increasing.

Our weighted Szegő-Weinberger inequality relies on the following

Lemma 7.1. *The following inequality holds for every $R > 0$:*

$$v_1(R) < \tau_1(R).$$

Proof. We recall that $\tau_1 := \tau_1(R)$ is the first nonzero eigenvalue of

$$(7.8) \quad \begin{cases} g'' + \frac{N-1+\alpha}{r}g' + \tau g = 0 & \text{in } (0, R) \\ g'(0) = g'(R) = 0 \end{cases}.$$

Equation in (7.8) coincides with equation (7.5) by choosing $k = 0$ and $\mu = \tau$. Therefore the solutions to equation in (7.8) are given by

$$g(r) = f_0(r) = c_1 r^{-\frac{\beta}{2}} J_{\nu_0}(\sqrt{\tau_1} r)$$

with $\nu_0 = \frac{\beta}{2} = \frac{N-2+\alpha}{2}$ and moreover, as in Section 5, Neumann condition $g'(R) = 0$ is equivalent to

$$(7.9) \quad -\frac{\beta}{2} J_{\nu_0}(\sqrt{\tau_1} R) + \sqrt{\tau_1} R J'_{\nu_0}(\sqrt{\tau_1} R) = 0.$$

Furthermore we recall that $v_1 := v_1(R)$ is the first eigenvalue of

$$(7.10) \quad \begin{cases} w'' + \frac{N-1+\alpha}{r}w' + vw - \frac{N-1}{r^2}w = 0 & \text{in } (0, R) \\ w(0) = w'(R) = 0. \end{cases}$$

Equation in (7.10) coincides with equation (7.5) by choosing $k = 1$ and $\mu = v$. Therefore the solutions to equation in (7.10) are given by

$$w(r) = f_1(r) = c_1 r^{-\frac{\beta}{2}} J_{\nu_1}(\sqrt{v_1} r)$$

with $\nu_1 = \sqrt{N-1 + \frac{\beta^2}{4}} = \sqrt{N-1 + \frac{(N-2+\alpha)^2}{4}}$ and moreover, as in Section 5, Neumann condition $w'(R) = 0$ is equivalent to

$$(7.11) \quad -\frac{\beta}{2} J_{\nu_1}(\sqrt{v_1} R) + \sqrt{v_1} R J'_{\nu_1}(\sqrt{v_1} R) = 0.$$

By (7.9) and (7.11) we deduce that $\sqrt{\tau_1} R$ and $\sqrt{v_1} R$ are the smallest positive solution to the equations

$$(7.12) \quad -\frac{\beta}{2} J_{\nu_0}(x) + x J'_{\nu_0}(x) = 0$$

and

$$(7.13) \quad -\frac{\beta}{2} J_{\nu_1}(x) + x J'_{\nu_1}(x) = 0,$$

respectively. Therefore, arguing as in Section 5, since $\beta > 0$, the positive root $x_{\nu_0,1}$ of equation (7.12) coincides with the zero $j_{\nu_0+1,1}$ of the Bessel function J_{ν_0+1} and the positive root of (7.13) coincides with $x_{\nu_1,1}$. By properties of $x_{\nu,k}$, described in Section 5, we know that

$$(7.14) \quad x_{\nu_1,1} < j_{\nu_1,1}.$$

Moreover by properties of zero's Bessel functions (5.3), since for $N \geq 2$ and $\alpha > 0$, $\nu_1 = \sqrt{N-1 + \frac{\beta^2}{4}} < \nu_0 + 1 = \frac{\beta}{2} + 1$, it results

$$(7.15) \quad j_{\nu_1,1} < j_{\nu_0+1,1} \equiv x_{\nu_0,1}.$$

Combining (7.14) and (7.15), we get

$$x_{\nu_1,1} \equiv \sqrt{v_1}R < j_{\nu_1,1} < j_{\nu_0+1,1} \equiv x_{\nu_0,1} \equiv \sqrt{\tau_1}R.$$

This yields the conclusion. \square

We can now prove Theorem 7.1.

Proof. Lemma 7.1 ensures that $\mu(\Omega^\sharp)$ is a N -fold degenerate eigenvalue and a corresponding set of eigenfunctions is

$$w_1(|x|) \frac{x_i}{|x|}, \text{ for } i = 1, \dots, N,$$

where w_1 is the first eigenfunction of problem (7.10). As it is easy to verify we have

$$(7.16) \quad \mu(\Omega^\sharp) = \frac{\int_0^{r^\sharp} \left[\left(\frac{d}{dr} w_1 \right)^2 + \frac{N-1}{r^2} w_1^2 \right] r^{\alpha+N-1} dr}{\int_0^{r^\sharp} w_1^2 r^{\alpha+N-1} dr}.$$

By the assumptions on the symmetry of the set Ω , it holds that

$$\int_{\Omega} w_1(|x|) \frac{x_i}{|x|} |x|^\alpha dx = 0 \quad \forall i \in \{1, \dots, N\}.$$

Therefore we can use

$$w_1(|x|) \frac{x_i}{|x|}, \quad \forall i = 1, \dots, N,$$

as test functions for $\mu(\Omega)$, obtaining

$$(7.17) \quad \mu(\Omega) \leq \frac{\int_{\Omega} \left[(G'(|x|))^2 \frac{x_i^2}{|x|^2} + \frac{G^2(|x|)}{|x|^2} \left(1 - \frac{x_i^2}{|x|^2} \right) \right] |x|^\alpha dx}{\int_{\Omega} \left[G^2(|x|) \frac{x_i^2}{|x|^2} \right] |x|^\alpha dx} \quad \text{for } i = 1, \dots, N.$$

where

$$(7.18) \quad G(r) := \begin{cases} w_1(r) & \text{if } r \leq r^\sharp \\ w_1(r^\sharp) & \text{if } r > r^\sharp. \end{cases}$$

Summing over the index i inequalities (7.17), we get

$$\mu(\Omega) \leq \frac{\int_{\Omega} \left[(G'(|x|))^2 + \frac{N-1}{|x|^2} G^2(|x|) \right] |x|^\alpha dx}{\int_{\Omega} G^2(|x|) |x|^\alpha dx}.$$

Note that, since $w_1'(r) > 0$ in $(0, R)$, we have that $G^2(r)$ is a non decreasing function for $r \geq 0$. Therefore, since

$$(7.19) \quad \int_{\Omega} |x|^\alpha dx = \int_{\Omega^\sharp} |x|^\alpha dx,$$

Hardy-Littlewood inequality, with respect to the measure $|x|^\alpha dx$, implies

$$(7.20) \quad \int_{\Omega} G^2(|x|) |x|^\alpha dx \geq \int_{\Omega^\sharp} G^2(|x|) |x|^\alpha dx.$$

Now let us set

$$(7.21) \quad N(r) := \left(\frac{d}{dr} G(r) \right)^2 + \frac{N-1}{r^2} G^2(r).$$

Now we claim that the function $N(r)$ is strictly decreasing in $(0, +\infty)$. Indeed we have

$$\frac{d}{dr} N(r) = 2G'G'' + \frac{2(N-1)}{r^2} GG' - \frac{2(N-1)}{r^3} G^2.$$

Since $G'(r) = 0$ for any $r > r^\sharp$, we have

$$\frac{d}{dr} N(r) = -\frac{2(N-1)}{r^3} w_1^2(r^\sharp) < 0 \quad \text{for any } r > r^\sharp.$$

While for any $r \in (0, r^\sharp)$ it holds

$$\frac{d}{dr} N(r) = \frac{d}{dr} \left[\left(\frac{d}{dr} w_1 \right)^2 + \frac{(N-1)w_1^2}{r^2} \right] = 2w_1'w_1'' + \frac{2(N-1)w_1w_1'}{r^2} - \frac{2(N-1)}{r^3} w_1^2.$$

Using the equation for w_1 we get

$$\begin{aligned} \frac{d}{dr} N(r) &= 2w_1' \left(-\frac{N-1+\alpha}{r} w_1' - \mu(\Omega^\sharp) w_1 + \frac{N-1}{r^2} w_1 \right) + 2\frac{N-1}{r^2} w_1 w_1' - \frac{2}{r^3} (N-1) w_1^2 \\ &= -2\mu(\Omega^\sharp) w_1' w_1 - 2\frac{\alpha}{r} (w_1')^2 - 2\frac{N-1}{r} \left[(w_1')^2 - \frac{2}{r} w_1 w_1' + \frac{1}{r^2} w_1^2 \right] \\ &= -2\mu(\Omega^\sharp) w_1' w_1 - 2\frac{\alpha}{r} (w_1')^2 - 2\frac{(N-1)}{r} \left(w_1' - \frac{w_1}{r} \right)^2 < 0 \end{aligned}$$

Therefore

$$\frac{d}{dr} N(r) < 0 \quad \text{for any } r \in (0, r^\sharp),$$

since we are assuming that $\alpha \in (0, N)$ and we know, by Lemma 7.1, that $w_1' w_1 \geq 0$ in $(0, r^\sharp)$.

By repeating the same arguments used for (7.20), using the monotonicity of the function N , just proved, we get

$$(7.22) \quad \int_{\Omega} \left[(G'(|x|))^2 + \frac{N-1}{|x|^2} G^2(|x|) \right] |x|^\alpha dx \leq \int_{\Omega^\sharp} \left[(G'(|x|))^2 + \frac{N-1}{|x|^2} G^2(|x|) \right] |x|^\alpha dx.$$

Inequalities (7.22) and (7.20), taking into account equality (7.16), yield (7.3).

Finally by the proof follows also that if (7.3) holds as an equality then $\Omega \equiv \Omega^\sharp$. □

Remark 7.1. *Some numerics would suggest that if one drops the assumption on the sign of α , then the function $N(r)$, in general, is no longer decreasing.*

ACKNOWLEDGEMENTS

The research of F. Brock was supported by the University of Rostock. He wants to thank the University of Naples Federico II for a visiting appointment and their kind hospitality.

The research of F. Chiacchio was partially supported by the PRIN project 2017JPCAPN (Italy) Grant: “Qualitative and quantitative aspects of nonlinear PDEs” and by GNAMPA. The second author thanks the University of Leipzig for kind hospitality.

The research of G. Croce has been supported by the ANR Project SHAPO of the CNRS. Part of this work was done when the third author was hosted at “Institut de Mathématiques de Jussieu-Paris Rive Gauche, projet Combinatoire et Optimisation”. The author thanks this institution for the warm hospitality.

The research of A. Mercaldo was partially supported by Italian MIUR through research project PRIN 2017 “Direct and inverse problems for partial differential equations: theoretical aspects and applications” and GNAMPA. The author thanks Université de Le Havre and University of Rostock for their warm hospitality.

The authors thank Nikita Simonov for some useful references.

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