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Heterogeneous bequests and social inequalities

Kirill Borissov¹, Stefano Bosi², Thai Ha-Huy^{2, 3}, Mikhail Pakhnin^{4*}

¹Department of Economics, Non-government Educational Institution
“European University at St. Petersburg”, Russia.

²EPEE, Université Paris-Saclay, France.

³TIMAS, Thang Long University, Vietnam.

⁴Department of Applied Economics, University of the Balearic Islands,
Spain.

*Corresponding author(s). E-mail(s): m.pakhnin@uib.es;
Contributing authors: kirill@eu.spb.ru; stefano.bosi@univ-evry.fr;
thai.hahuy@univ-evry.fr;

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Corresponding author: Mikhail Pakhnin, Visiting Professor, Department of Applied Economics, University of the Balearic Islands

Address: Cra. de Valldemossa, km 7.5. 07122 Palma (Balearic Islands), Spain

Phone: +34 971 172 088

Email: m.pakhnin@uib.es

Abstract: We study a growth model with two types of agents who are heterogeneous in their degree of family altruism. We prove that every equilibrium path of consumption, bequests and capital converges to a unique steady state, and study the effect of altruism on the properties of steady-state equilibrium. We show that aggregate income is positively related to both level of altruism and altruism heterogeneity. When altruism heterogeneity is low or moderate, income inequality follows an inverse U-shaped pattern relative to the level of altruism. These observations are consistent with the cross-country Kuznets curve linking different steady-state levels of income to steady-state levels of inequality. When altruism heterogeneity is high, income inequality decreases with the level of altruism. Our results suggest that heterogeneous altruism is a possible mechanism linking economic growth and income inequality.

Keywords: Economic growth; Inequality; Altruism; Bequests

JEL Classification: D15, D64, E21, O40

1 Introduction

The relationship between economic growth and income inequality is intricate, as both income levels and inequality are endogenously shaped over the course of economic development. It is widely recognized that the growth-inequality nexus is influenced by various country-specific characteristics, including the degree of technological advancement, levels of education and human capital, political and institutional factors, and the design of public policies (see, e.g., Hellier and Lambrecht, 2013, for an overview). In this paper, we highlight the role of socio-cultural norms as an additional country-specific factor affecting both economic growth and income inequality. Specifically, we study the impact of heterogeneous altruism on income levels and inequality.

Parental altruism, a concern for the well-being of children as opposed to pure self-interest, is a natural determinant of the joint evolution of growth and inequality. It is generally acknowledged that altruism has a positive impact on economic development. For instance, Hatcher and Pourpourides (2018) report a positive correlation between country-level parental altruism and economic growth in a sample of 48 countries. Furthermore, altruistic bequests are a key driver of wealth accumulation, and differences in altruism are a significant factor contributing to inequality (see, e.g., Mankiw, 2000, for a discussion). Laitner (2002) argues that calibrated models with altruistic bequests are able to account for the empirical distribution of wealth in the US.

At the same time, there is substantial heterogeneity in the degree of altruism: some people and societies are more altruistic than others. Falk et al. (2018) find that the within-country altruism variation is much larger than the between-country variation: the former amounts to 87.7% in the total individual-level variation in altruism, while the latter explains only the remaining 12.3%.¹ In this paper, we develop and study a simple growth model in which agents differ in their degree of altruism. We show that within-country heterogeneity in altruism, as suggested by empirical evidence, leads to a non-monotonic relationship between income levels and inequality across countries.

Specifically, we consider a successive generations economy in which agents are motivated by *family altruism*, that is, they *care about the disposable income of their offsprings*. There are two types of agents who are heterogeneous in their degree of family altruism: *less altruistic* and *more altruistic* agents. Altruistic transfer is the only savings motive, and bequests left by agents become the capital involved in the production. We prove that when instantaneous utility functions are logarithmic and production technology is Cobb–Douglas, every equilibrium path of consumption, bequests and capital converges to a unique steady-state equilibrium.

We show that there are two types of steady-state equilibrium, depending on the difference between the more altruistic and the less altruistic agents. If both types of agents have similar degrees of altruism, then both types leave positive bequests. However, if the difference in the degrees of altruism between the two types is sufficiently high, then the less altruistic agents leave no bequests. We characterize the properties of a steady-state equilibrium and analyze the impact of the level of altruism and altruism heterogeneity on the steady state.

¹ Formally speaking, Falk et al. (2018) define altruism as a willingness to give to good causes without expecting anything in return. However, their measure is also a good proxy for parental altruism.

To study the role of the level of altruism, we employ the *share of the more altruistic agents* in total population as a measure of altruism. Given both degrees of altruism, the higher the share of the more altruistic agents, the more altruistic is society as a whole. We show that a higher level of altruism in society leads to a higher capital stock and higher aggregate income. Intuitively, the more altruistic agents are the main savers and leave higher bequests than the less altruistic agents. An increase in the share of the more altruistic agents leads to the two effects. First, the higher share of main savers drives down the interest rate and reduces their incentives to leave bequests, so each single more altruistic agent decreases the amount of bequest left to her offspring. Second, due to the increased presence of the main savers, the total amount of bequests (aggregate capital stock) is increasing. This result confirms the natural intuition that altruism positively affects aggregate income at the country level.

To study the impact of the level of altruism on the steady-state income inequality, we use the Gini index as a measure of inequality. For the two types of agents, the Gini index is determined by the shares of both types in the population and the difference in relative income between the types. As a consequence, our results essentially depend on the difference between the more altruistic and the less altruistic agents.

If both types have similar degrees of altruism, then an inverted U-shaped relationship is observed. Intuitively, when both types of agents leave positive bequests, changes in income are similar for both types. Therefore, an increase in the share of the more altruistic agents at first increases inequality: the direct effect of having more relatively rich people in the population leads to higher inequality. However, beyond a certain point, an increase in the share of the more altruistic agents decreases inequality due to the direct effect of having less relatively poor people in the population.

Since aggregate income increases with the level of altruism, there is an inverted U-shaped relationship between steady-state levels of income and the steady-state Gini indices. This result can be interpreted in the spirit of the *cross-country Kuznets curve*: an inverted U-shaped relationship between inequality and income in a cross-section of countries. Countries with low levels of altruism would have low aggregate income and low inequality; countries with midrange levels of altruism would be middle-income and have high inequality; while countries with high levels of altruism would have high aggregate income and low inequality.

If the difference in the degrees of altruism between the two types is sufficiently high, then, as the share of the more altruistic agents increases, the steady-state level of inequality decreases. Intuitively, in this case changes in income are different for different types of agents, which is especially pronounced when only the more altruistic agents leave bequests. In this case, an increase in the share of the more altruistic agents decreases the bequests of the rich and does not affect the bequests of the poor. Hence the difference in capital income is decreasing, which lowers the Gini index. Thus, for sufficiently heterogeneous altruistic societies there is no trade-off between economic growth and income inequality.

To study the role of altruism heterogeneity, we employ the *difference in the reciprocals of degrees of altruism* as a measure of heterogeneity. Given the harmonic mean of degrees of altruism and the share of the more altruistic agents, the higher is the

difference in the reciprocals, the higher is the variance in altruism, and the more heterogeneous is society as a whole. We show that a higher altruism heterogeneity leads to a higher capital stock and a higher aggregate income. A harmonic-mean-preserving increase in altruism heterogeneity increases the weighted arithmetic mean of degrees of altruism in the population, and increases the amount of total bequests. This result points out that not only the level of altruism, but also the diversity in the degree of altruism positively affects economic development.

The steady-state income inequality is affected by altruism heterogeneity only at low levels of heterogeneity. If both types of agents have similar degrees of altruism, then increasing altruism heterogeneity leads to higher inequality.² Intuitively, a harmonic-mean-preserving increase in altruism heterogeneity makes more altruistic agents richer while less altruistic agents become poorer. However, if the difference in the degrees of altruism between the two types is sufficiently high, then an increase in altruism heterogeneity does not affect inequality. In this case, the less altruistic agents leave no bequests, and a further decrease in their degree of altruism does not affect their relative position. This result can be interpreted as the existence of a maximum possible steady-state level of income inequality (in terms of altruism heterogeneity).

Furthermore, we analyze the impact of the level of altruism on the steady-state utility levels of both types of agents. We show that the utility of the more altruistic agents decreases in the level of altruism. Intuitively, for the more altruistic agents, capital income is more important than labor income, and their steady-state utility is mainly determined by bequests. An increase in the share of more altruistic agents lowers both the interest rate and the amount of bequests left by these agents. For the main savers, this decrease in capital income leads to lower utility.

At the same time, the shape of the steady-state utility for the less altruistic agents depends on altruism heterogeneity. If both types of agents have similar degrees of altruism, the less altruistic agents are almost identical to the more altruistic agents. Capital income plays a more important role in their utility, and due to the fall in capital income, the consumption and utility of the less altruistic agents decrease with the degree of altruism. However, if the difference in the degrees of altruism between the two types is sufficiently high, the bequests left by the less altruistic agents are zero or close to zero, and hence the utility of the less altruistic agents is determined mainly by labor income. Since wage rate increases with the level of altruism, the consumption and utility of the less altruistic agents increase as well.

If altruism heterogeneity is moderate, then the situation lies between these polar cases, and the utility of the less altruistic agents is U-shaped in the level of altruism. For low levels of altruism, the effect of decreasing capital income dominates, causing their steady-state utility to decrease with the level of altruism. However, for high levels of altruism, the effect of increasing labor income dominates, and the steady-state utility of the less altruistic agents increases with the level of altruism.

Our paper is related to a large theoretical literature on the links between parental altruism, growth and inequality. First, this paper contributes to the discussion of the role of altruism in economic development. The existing literature typically follows

² See also Krusell and Smith (1998) and Hendricks (2007), who find similar effect of discount rate heterogeneity on the wealth Gini index in stochastic growth models.

Barro (1974) and explores overlapping generations (OLG) models with *dynastic altruism* where agents care about their offspring’s welfare: each agent derives utility from her own consumption and the *utility* of her offspring.³ Barro (1974) shows that when the degree of altruism is sufficiently strong (so that the bequest motive is operative), the dynamics of an OLG model are analogous to the dynamics of the infinite-horizon Ramsey model, and Ricardian equivalence holds: government debt does not influence the steady-state capital stock.

Another strand of literature studies *paternalistic altruism* where agents care about their offspring’s consumption or about the bequests they leave: each agent derives utility from her own consumption and the *consumption level* of her offspring or the *amount of bequest*.⁴ In the case where agents care about their offspring’s consumption, each agent has a limited altruism towards only immediate successor, and there is a conflict of interests among different dynasty members about consumption schedule. Kohlberg (1976), Leininger (1986) and Bernheim and Ray (1987) study this conflict from a game-theoretic point of view, establish the existence of equilibria in a game between different altruistic dynasty members and characterize their properties. In the case where agents care about the amount of bequests they leave, Andreoni (1989) shows that government debt is not neutral, so that Ricardian equivalence fails to hold.

Our paper is different, as we follow Lambrecht et al. (2006) and assume that agents exhibit *family altruism*. In our setting, each agent derives utility from her own consumption and the *disposable income* of her offspring. This approach has several advantages over other types of altruism studied in the literature. In contrast to dynastic altruism, in our setting an agent can ignore the unknown preferences of her unborn offspring when making her decisions. Also, in our case all dynasty members are not equivalent to a single infinitely-lived agent which leads to a different long-run dynamics. Furthermore, in contrast to paternalistic altruism where agents care about the amount of bequests, in our setting agents can leave zero bequests which better fits empirical evidence (for instance, Hendricks, 2001, documents that 70% of households in the US receive no bequests). Thus, the assumption of family altruism allows one to gain new perspectives and understanding.⁵ Our contribution here is to clarify the mechanisms by which family altruism is positively related to economic growth.

Second, this paper contributes to the analysis of growth models with agents who differ in their degree of altruism. The existing literature typically considers *heterogeneous dynastic altruism*. Michel and Pestieau (1998; 2005), Smetters (1999) and Mankiw (2000) study the effectiveness of fiscal policy, and the very general result is that Ricardian equivalence also holds in heterogeneous agents models in the long run. However, government policies typically lead to a redistribution of income from the less altruistic agents (poor) to the more altruistic agents (rich) and an increase in inequality within society, which is not observed in representative agent models. Palivos (2005) shows that monetary policy under heterogeneous altruism also leads to substantial

³ Dynastic altruism is sometimes called *pure altruism* or *non-paternalistic altruism* (Ray, 1987).

⁴ The term “paternalistic” emphasizes that the altruist values the consumption of others, irrespective of their preferences. Andreoni (1989) calls the case where bequests are treated as a consumption good *impure altruism* and justifies it by assuming that the altruistic agent derives pleasure directly from the act of giving (“joy-of-giving” or “warm glow giving”).

⁵ See Borissov (2016) for a family altruism model with relative consumption concerns. For the analysis of fiscal policy in a family altruism model, see, among others, Borissov and Kalk (2020).

distributional effects. Reichlin (2020) highlights the difficulties with standard social welfare criteria in the OLG models with heterogeneous dynastic altruism.⁶

In a series of recent studies, Franks et al. (2018) and Franks and Edenhofer (2023) consider the model with *heterogeneous paternalistic altruism* where agents differ in their taste for leaving bequests. They calibrate the model to OECD data and show that generically bequest taxation has a higher potential to decrease the level of inequality without significantly reducing the output, as compared to capital taxation.

Our paper differs from previous contributions by focusing on *heterogeneous family altruism*. In our model, we observe two distinct types of steady-state equilibrium: one where all agents leave positive bequests and another where only the more altruistic agents leave bequests. In contrast, existing models feature only one type of equilibrium. In models with heterogeneous dynastic altruism (e.g., Michel and Pestieau, 2005), only agents with the highest degree of altruism leave bequests in the steady state.⁷ Conversely, in models with heterogeneous paternalistic altruism (e.g., Franks et al., 2018), all agents leave bequests in equilibrium, and empirically relevant zero-bequest outcomes are impossible. Thus, our approach offers a more flexible framework to analyze the impact of heterogeneous altruism on growth and inequality.

The paper is organized as follows. In Section 2 we present the model and define equilibria. Section 3 provides main results and their discussion. Section 4 concludes. All the proofs are relegated to the Appendix.

2 The model

We consider a closed market economy with households and firms. As usual, their fundamentals are given by preferences, technology and endowments. In this section, we describe the consumers' and the producers' programs at the individual level and define dynamic general equilibrium at the aggregate level.

2.1 Households and firms

Time is discrete and runs from $t = 0$ to infinity. The economy is populated by successive generations of agents. Each agent lives for one period, gives birth to one offspring, and supplies one unit of labor. The population is constant over time, and the population size is normalized to 1.

The population consists of two types of agents indexed by $i = L, H$. The share of type i in the population is π_i , with $\pi_L + \pi_H = 1$. Agents are identical within each type. The agent and her offspring are of same type, so population shares are constant over time. A disposable income of type i agent is defined as a sum of the wage bill, w_t , identical across types, and the current value of bequest left by her parent, b_t^i . Out of this, an agent consumes $c_t^i \geq 0$ and leaves $b_{t+1}^i \geq 0$ to her offspring as a bequest. Since the case of negative bequests is hard to justify on either a juridical or empirical

⁶ See also Pakhnin (2023) for the discussion of similar problems with social welfare under heterogeneous time preferences.

⁷ The long-run properties of a model with heterogeneous dynastic altruism are analogous to those of the many-agent Ramsey model in the spirit of Becker (1980), where the intertemporal utility function of a single dynasty is an infinite-horizon discounted sum of instantaneous utilities, and different dynasties have different discount factors.

ground, we assume that bequests are non-negative.⁸ Formally, the budget constraint of type i agent has the form

$$c_t^i + b_{t+1}^i \leq R_t b_t^i + w_t$$

where R_t is the gross interest rate.

Each agent cares about her consumption and the disposable income of her offspring. The relative preference for the offspring's disposable income with respect to own consumption is naturally interpreted as a degree of altruism. Formally, the preferences of type i agent are represented by the following utility function:

$$\ln c_t^i + \beta_i \ln (R_{t+1} b_{t+1}^i + w_{t+1})$$

where $\beta_i > 0$ is the degree of altruism of type i agent.

Throughout the paper, we assume that agents are heterogeneous in terms of altruism: type L agents are less altruistic, while type H agents are more altruistic.

Assumption 1. $\beta_H > \beta_L$.

Thus, type i agent living in period t solves the following maximization problem:

$$\begin{aligned} \max_{c_t^i, b_{t+1}^i} & \quad [\ln c_t^i + \beta_i \ln (R_{t+1} b_{t+1}^i + w_{t+1})] \\ \text{s. t.} & \quad c_t^i + b_{t+1}^i \leq R_t b_t^i + w_t \end{aligned} \quad (1)$$

with $c_t^i \geq 0$ and $b_{t+1}^i \geq 0$.

In every period, the economy produces a single good which is either consumed or invested. Technology is given by a neoclassical production function $F(K, N)$, where K is the stock of physical capital, N is the labor input, and function F is homogeneous of degree one. The production function in intensive form, f , is given by $f(k) = F(K/N, 1)$, where $k = K/N$ is the capital intensity. Capital fully depreciates each period, which is justified by the length of the period (the lifespan). Throughout the paper, we assume that the function $f(k)$ satisfies the standard assumptions.

Assumption 2. $f(0) = 0$, $f'(k) > 0$, $f''(k) < 0$, $\lim_{k \rightarrow 0} f'(k) = \infty$, $\lim_{k \rightarrow \infty} f'(k) = 0$.

In each period t producers maximize profits, so that the gross interest rate R_t (which coincides with the interest rate because of the complete capital depreciation) and the wage rate w_t are equal to the corresponding marginal products:

$$R_t = R(k_t) \equiv f'(k_t) \quad (2)$$

$$w_t = w(k_t) \equiv f(k_t) - k_t f'(k_t) \quad (3)$$

Since the size of population is constant and normalized to one, the aggregate capital K_t coincides with capital per capita k_t .

⁸Negative bequests would mean that offsprings have to pay parents' debts.

For future reference, we define the income ratio (labor income over capital income):

$$\gamma(k_t) \equiv \frac{w(k_t)}{k_t R(k_t)} = \frac{1 - \alpha(k_t)}{\alpha(k_t)}$$

where

$$\alpha(k_t) \equiv \frac{k_t f'(k_t)}{f(k_t)}$$

is the capital share in total income.

2.2 Temporary, intertemporal and steady-state equilibria

The definitions of equilibria in our model are fairly standard. First, we define a temporary equilibrium where each agent maximizes her utility, producers maximize profits, and the capital market clears, meaning that bequests become the capital involved in production.

Definition 1 (Temporary equilibrium). *Given the bequests $b_t^i \geq 0$ left by agents in period $t - 1$, and the capital stock $k_t = \pi_L b_t^L + \pi_H b_t^H$, a tuple $\left((c_t^i, b_{t+1}^i)_{i=L,H}, k_{t+1} \right)$ is a time- t temporary equilibrium if:*

(i) *for any i , (c_t^i, b_{t+1}^i) is a solution to the utility maximization problem (1) where $(R_t, w_t) = (R(k_t), w(k_t))$ and $(R_{t+1}, w_{t+1}) = (R(k_{t+1}), w(k_{t+1}))$, and the functions R and w are given by (2) and (3);*

(ii) $k_{t+1} = \pi_L b_{t+1}^L + \pi_H b_{t+1}^H$.

Second, we define an intertemporal equilibrium as a sequence of temporary equilibria.

Definition 2 (Intertemporal equilibrium). *A sequence $\left((c_t^i, b_{t+1}^i)_{i=L,H}, k_{t+1} \right)_{t=0}^{\infty}$ is an intertemporal equilibrium starting from (b_0^L, b_0^H) with $k_0 = \pi_L b_0^L + \pi_H b_0^H$ if $\left((c_t^i, b_{t+1}^i)_{i=L,H}, k_{t+1} \right)$ is a time- t temporary equilibrium for any $t \geq 0$.*

Definition 2 yields the dynamic system representing an intertemporal equilibrium.

Proposition 1 (Dynamic system). *The dynamics of bequests in an intertemporal equilibrium are given by*

$$b_{t+1}^i = \frac{1}{1 + \beta_i} \max \{ 0, \beta_i R(k_t) [b_t^i + k_t \gamma(k_t)] - k_{t+1} \gamma(k_{t+1}) \} \quad (4)$$

for $i = L, H$, with $k_t = \pi_L b_t^L + \pi_H b_t^H$, and the initial condition (b_0^L, b_0^H) .

Eq. (4) is a two-dimensional dynamic system in the variables (b_t^L, b_t^H) . We observe that these variables are predetermined because initial bequests b_0^L and b_0^H are given. Since $k_{t+1} = \pi_L b_{t+1}^L + \pi_H b_{t+1}^H$, we also have the dynamics of capital stock:

$$k_{t+1} = \sum_i \frac{\pi_i}{1 + \beta_i} \max \{ 0, \beta_i R(k_t) [b_t^i + k_t \gamma(k_t)] - k_{t+1} \gamma(k_{t+1}) \}$$

Here and in what follows, for notational simplicity, we denote $\sum_i = \sum_{i=L,H}$.

A steady-state equilibrium is naturally defined.

Definition 3 (Steady state). A tuple $\left((c^i, b^i)_{i=L,H}, k\right)$ is a steady-state equilibrium if the sequence $\left((c_t^i, b_{t+1}^i)_{i=L,H}, k_{t+1}\right)_{t=0}^{\infty}$ with $(c_t^i, b_{t+1}^i) = (c^i, b^i)$ and $k_{t+1} = k$ for any $i = L, H$ and any $t \geq 0$ is an intertemporal equilibrium starting from (b^L, b^H) .

The following proposition determines the steady state.

Proposition 2 (Steady-state bequests). Assume that $k > 0$. The steady-state bequests b^i are given by

$$b^i = k\gamma(k) \max\left\{0, \frac{\beta_i R(k) - 1}{1 + \beta_i - \beta_i R(k)}\right\} \quad (5)$$

where the steady-state capital stock k is a solution to the following equation:

$$\gamma(k) \sum_i \pi_i \max\left\{0, \frac{\beta_i R(k) - 1}{1 + \beta_i - \beta_i R(k)}\right\} = 1$$

Note that, for any k , b^i is non-decreasing in β_i . Therefore, if a steady state exists, the more altruistic agents leave higher steady-state bequests than the less altruistic agents: $b^H > b^L$.

2.3 Local dynamics

Let k be the steady-state capital stock. Consider the local dynamics of bequests in a neighborhood of a steady state. By (5), the steady-state bequests of type i agents are positive if and only if

$$\frac{1}{\beta_i} < R(k) < 1 + \frac{1}{\beta_i} \quad (6)$$

By Assumption 1, we have $1/\beta_H < 1/\beta_L$. Then it follows from (6) that there are two possible cases: (1) $1/\beta_L < R(k) < 1 + 1/\beta_H$, and (2) $1/\beta_H < R(k) < \min\{1/\beta_L, 1 + 1/\beta_H\}$.

Case (1) If $1/\beta_L < R(k) < 1 + 1/\beta_H$, then both the more and the less altruistic agents leave bequests. Local dynamics are given by

$$b_{t+1}^L = \frac{\beta_L R (\pi_L b_t^L + \pi_H b_t^H) [b_t^L + (\pi_L b_t^L + \pi_H b_t^H) \gamma_t] - (\pi_L b_{t+1}^L + \pi_H b_{t+1}^H) \gamma_{t+1}}{1 + \beta_L} \quad (7)$$

$$b_{t+1}^H = \frac{\beta_H R (\pi_L b_t^L + \pi_H b_t^H) [b_t^H + (\pi_L b_t^L + \pi_H b_t^H) \gamma_t] - (\pi_L b_{t+1}^L + \pi_H b_{t+1}^H) \gamma_{t+1}}{1 + \beta_H} \quad (8)$$

where $\gamma_t = \gamma(\pi_L b_t^L + \pi_H b_t^H)$.

Case (2) If $1/\beta_H < R(k) < \min\{1/\beta_L, 1 + 1/\beta_H\}$, then, by (6), $b^L = 0$, and only the more altruistic agents leave bequests. Local dynamics are given by $b_{t+1}^L = 0$ and

$$b_{t+1}^H = \frac{\beta_H R(\pi_H b_t^H) [b_t^H + \pi_H b_t^H \gamma(\pi_H b_t^H)] - \pi_H b_{t+1}^H \gamma(\pi_H b_{t+1}^H)}{1 + \beta_H}$$

3 Main results

In this section, we focus on the case of a Cobb–Douglas technology. Suppose that the production function is given by

$$F(K, N) = AK^\alpha N^{1-\alpha}$$

Then $f(k_t) = Ak_t^\alpha$, and the price functions (2)–(3) take the form

$$R(k_t) = \alpha Ak_t^{\alpha-1} \quad \text{and} \quad w(k_t) = (1 - \alpha) Ak_t^\alpha$$

The capital share in total income and the income ratio are constant:

$$\alpha(k_t) = \alpha \quad \text{and} \quad \gamma(k_t) = \gamma = \frac{1 - \alpha}{\alpha}$$

and the dynamic system (4) becomes

$$b_{t+1}^i = \frac{1}{1 + \beta_i} \max\{0, \beta_i R(k_t) (b_t^i + \gamma k_t) - \gamma k_{t+1}\}$$

with $k_t = \pi_L b_t^L + \pi_H b_t^H$. The transition dynamics of capital stock are given by

$$k_{t+1} = \sum_i \frac{\pi_i}{1 + \beta_i} \max\{0, \beta_i R(k_t) (b_t^i + \gamma k_t) - \gamma k_{t+1}\}$$

3.1 Steady state and convergence

Let $\pi \equiv \pi_H$ be the share of the more altruistic agents in total population, which is our measure of altruism. Let also

$$\delta \equiv \frac{1}{\beta_L} - \frac{1}{\beta_H}$$

be the altruism gap (the reciprocal of β_i captures the selfishness), which is our measure of altruism heterogeneity.

We introduce two critical interest rates:

$$R_1^* \equiv \frac{1}{2} \left(1 + \alpha + \frac{1}{\beta_L} + \frac{1}{\beta_H} - \sqrt{(\delta + \alpha - 1)^2 + 4\delta\pi(1 - \alpha)} \right) \quad (9)$$

$$R_2^* \equiv \frac{1}{\beta_H} + \frac{1}{1 + \gamma\pi} \quad (10)$$

Denote by

$$\bar{\delta} \equiv \frac{1}{1 + \gamma\pi} \quad (11)$$

a threshold value of altruism heterogeneity which will play an important role below. The following proposition characterizes the steady-state equilibrium.

Proposition 3 (Steady-state equilibrium). (1) Suppose that $\delta < \bar{\delta}$. Then there exists a unique steady-state equilibrium characterized by the interest rate R_1^* . The steady-state equilibrium is given by $(c_1^{L*}, b_1^{L*}, c_1^{H*}, b_1^{H*}, k_1^*)$, where

$$\begin{aligned} c_1^{L*} &= \frac{\gamma k_1^*}{\beta_L + 1 - \beta_L R_1^*} & \text{and} & & c_1^{H*} &= \frac{\gamma k_1^*}{\beta_H + 1 - \beta_H R_1^*} \\ b_1^{L*} &= \gamma k_1^* \frac{\beta_L R_1^* - 1}{\beta_L + 1 - \beta_L R_1^*} & \text{and} & & b_1^{H*} &= \gamma k_1^* \frac{\beta_H R_1^* - 1}{\beta_H + 1 - \beta_H R_1^*} \\ k_1^* &= \left(\frac{\alpha A}{R_1^*} \right)^{\frac{1}{1-\alpha}} \end{aligned} \quad (12)$$

(2) Suppose that $\delta \geq \bar{\delta}$. Then there exists a unique steady-state equilibrium characterized by the interest rate R_2^* . The steady-state equilibrium is given by $(c_2^{L*}, b_2^{L*}, c_2^{H*}, b_2^{H*}, k_2^*)$, where

$$\begin{aligned} c_2^{L*} &= \frac{\gamma k_2^*}{\beta_H} \left(1 + \frac{\beta_H}{1 + \gamma\pi} \right) & \text{and} & & c_2^{H*} &= \frac{k_2^*}{\beta_H} \frac{1 + \gamma\pi}{\pi} \\ b_2^{L*} &= 0 & \text{and} & & b_2^{H*} &= \frac{k_2^*}{\pi} \\ k_2^* &= \left(\frac{\alpha A}{R_2^*} \right)^{\frac{1}{1-\alpha}} \end{aligned} \quad (13)$$

Proposition 3 implies that our model admits two types of steady-state equilibrium which depend on the difference between the more and the less altruistic agents. First, if agents of both types have rather similar degrees of altruism (altruism heterogeneity δ does not exceed the threshold value $\bar{\delta}$), then both types of agents leave positive bequests in the steady-state equilibrium. Intuitively, in this case, the more altruistic agents, who are the main savers, do not leave high bequests, and the resulting interest rate is sufficiently high to allow the less altruistic agents to also leave bequests.

Second, if the altruism gap between the more and the less altruistic agents is sufficiently large ($\delta \geq \bar{\delta}$), then only the more altruistic agents leave positive bequests in the steady-state equilibrium: $b_2^{H*} > 0 = b_2^{L*}$. Intuitively, in this case the more altruistic agents are so altruistic that they leave substantial amounts of bequests which drive the interest rate down. The interest rate becomes too low, which induces the less altruistic agents to leave no bequests. Note that for any given set of parameters, the steady-state interest rate is given by $R^* = \min \{R_1^*, R_2^*\}$.

The role of a steady-state equilibrium is highlighted by the following result which shows that every intertemporal equilibrium converges to the steady state.

Proposition 4 (Global convergence). Let $\left((c_t^i, b_{t+1}^i)_{i=L,H}, k_{t+1} \right)_{t=0}^{\infty}$ be an intertemporal equilibrium.

(1) Suppose that $\delta < \bar{\delta}$. Then, $b_{t+1}^L > 0$ and $b_{t+1}^H > 0$ for all $t \geq 0$, and intertemporal equilibrium converges to the steady-state equilibrium $(c_1^{L*}, c_1^{H*}, b_1^{L*}, b_1^{H*}, k_1^*)$ defined in part (1) of Proposition 3.

(2) Suppose that $\delta \geq \bar{\delta}$. Then intertemporal equilibrium converges to the steady-state equilibrium $(c_2^{L*}, c_2^{H*}, b_2^{L*}, b_2^{H*}, k_2^*)$ defined in part (2) of Proposition 3. If $\delta = \bar{\delta}$, then either $b_{t+1}^L = 0$ for all $t \geq 0$ or b_{t+1}^L converges to 0. If $\delta > \bar{\delta}$, then there exists t_0 such that $b_{t+1}^L = 0$ for all $t \geq t_0$.

According to Proposition 4, the steady-state equilibrium is globally stable. To provide a numerical illustration of this property, we compute the speed of convergence which depends on the modulus of eigenvalues of the linearized dynamic system.

Case (1) of Propositions 3 and 4. Since $1/\beta_L < R_1^* < 1 + 1/\beta_H$, both the more and the less altruistic agents leave positive bequests and dynamics follow (7)–(8) with $\gamma_t = \gamma = (1 - \alpha)/\alpha$:

$$b_{t+1}^L = \frac{\beta_L R(k_t) (b_t^L + \gamma k_t) - \gamma k_{t+1}}{1 + \beta_L} \quad \text{and} \quad b_{t+1}^H = \frac{\beta_H R(k_t) (b_t^H + \gamma k_t) - \gamma k_{t+1}}{1 + \beta_H}$$

where $k_t = (1 - \pi)b_t^L + \pi b_t^H$. Linearizing this system around the steady state (12), we obtain the eigenvalues

$$\lambda_1 = \left(T - \sqrt{T^2 - 4D} \right) / 2 \quad \text{and} \quad \lambda_2 = \left(T + \sqrt{T^2 - 4D} \right) / 2 \quad (14)$$

where T and D are the trace and the determinant of the Jacobian matrix evaluated at the steady state:

$$T = R_1^* \left[1 - \frac{1 - \alpha\beta_L\beta_H - \pi(1 - \alpha)(\beta_H - \beta_L) \frac{1 + \alpha\beta_H - \beta_H R_1^*}{1 + \beta_H - \beta_H R_1^*}}{(1 + \alpha\beta_L)(1 + \beta_H) - \pi(1 - \alpha)(\beta_H - \beta_L)} \right] \quad (15)$$

$$D = (R_1^*)^2 \frac{\alpha\beta_L\beta_H}{(1 + \alpha\beta_L)(1 + \beta_H) - \pi(1 - \alpha)(\beta_H - \beta_L)} \quad (16)$$

Global convergence implies local convergence (both the eigenvalues are inside the unit circle in the Argand–Gauss plane). A constructive proof of local convergence can be also provided noticing that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if the pair (T, D) lie in the interior of the triangle defined by $D > -T - 1$, $D > T - 1$ and $D < 1$. Using expressions (15)–(16) for trace and determinant, it is possible to prove that when both types of agents leave bequests, these three inequalities are always jointly verified.⁹

In order to illustrate convergence in this case, we fix the parameter values as follows: $\alpha = 0.33$, $\pi = 0.5$, $\beta_L = 0.5$, $\beta_H = 0.6$. Using (9), we obtain $R_1^* = 2.1273$ and, thus, $2 = 1/\beta_L < R_1^* < 1 + 1/\beta_H = 2.667$. Using (15) and (16), we obtain $T = 1.0739$ and $D = 0.24685$. The eigenvalues given by (14), $\lambda_1 = 0.33333$ and $\lambda_2 = 0.74054$, are both

⁹See Proposition 11 in Borissov et al. (2023) for a detailed proof of local convergence.

inside the unit circle (sink). The smaller their modulus, the faster the convergence to the steady state, coherently with Proposition 4.

Case (2) of Propositions 3 and 4. Since $1/\beta_H < R_2^* < 1/\beta_L$, only the more altruistic agents leave bequests. Dynamics are given by

$$b_{t+1}^H = b_t^H \frac{R(\pi b_t^H)}{R_2^*}$$

and are locally approximated by the following equation:¹⁰

$$\frac{db_{t+1}^H}{b_2^{H*}} = \alpha \frac{db_t^H}{b_2^{H*}}$$

The eigenvalue is given by $\lambda = \alpha$. The trajectory locally converges to the steady state because $0 < \alpha < 1$. Moreover, the lower the capital share in total income, the faster the convergence.

3.2 Altruism and economic growth

Proposition 4 allows us to focus on the properties of the steady-state equilibrium. We now study the effect of altruism on economic growth in our model. For this, we analyze the dependence of the steady-state capital stock on the level of altruism and on altruism heterogeneity.

To study the impact of the level of altruism, we employ the share of more altruistic agents in the total population, π , as a measure of altruism. Given β_L and β_H , a higher π indicates a more altruistic society as a whole.

Denote by $\bar{\pi}$ the threshold value of altruism which corresponds to the threshold value of altruism heterogeneity (11):

$$\bar{\pi} \equiv \frac{\alpha}{1-\alpha} \frac{1-\delta}{\delta} \quad (17)$$

Note that for $\delta > 1$ we have $\bar{\pi} < 0$, while for $\delta < \alpha$ we have $\bar{\pi} > 1$. The impact of the level of altruism π on the steady-state capital stock k^* is characterized as follows.

Proposition 5 (Capital stock and altruism). *The steady-state capital stock k^* is continuous and strictly increases with π . For $\pi < \bar{\pi}$, we have $k^* = k_1^*$, while for $\pi \geq \bar{\pi}$, $k^* = k_2^*$.*

For any given set of parameters, the steady-state capital stock is given by $k^* = \max\{k_1^*, k_2^*\}$. Proposition 5 shows that the more altruistic a society is, the higher the steady-state capital stock. Since more altruistic agents leave higher bequests than less altruistic agents, an increase in π replaces some less altruistic agents, who leave relatively lower bequests, with more altruistic agents, who leave relatively higher bequests.

¹⁰For more details, the reader is referred to Proposition 12 in Borissov et al. (2023).

This change reduces the steady-state interest rate, and increases capital accumulation and output. Thus, greater altruism implies a higher level of aggregate income. Proposition 5 is illustrated in Figure 1 with $\alpha = 0.33$, $\beta_L = 0.5$, $\beta_H = 0.75$, $A = 1$.

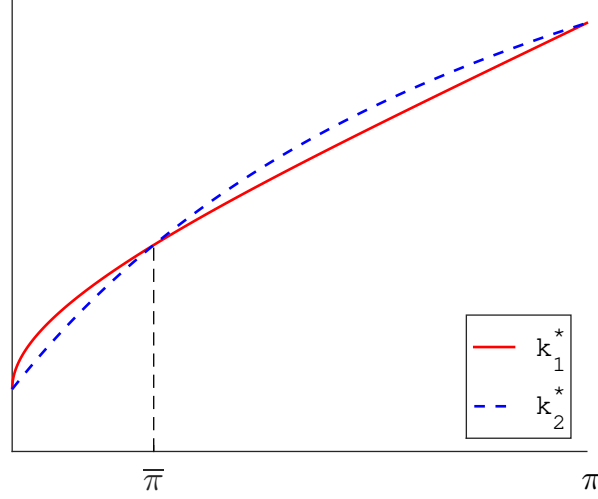


Fig. 1 Steady-state capital stock and the level of altruism

To study the impact of altruism heterogeneity, we employ altruism gap δ (the difference in the reciprocals of the degrees of altruism) as a measure of heterogeneity. Let $\bar{\beta}$ be the weighted harmonic mean of β_L and β_H :

$$\frac{1}{\bar{\beta}} = \frac{1-\pi}{\beta_L} + \frac{\pi}{\beta_H}$$

The variance of $1/\beta_L$ and $1/\beta_H$ is given by:

$$(1-\pi) \left(\frac{1}{\beta_L} - \frac{1}{\bar{\beta}} \right)^2 + \pi \left(\frac{1}{\beta_H} - \frac{1}{\bar{\beta}} \right)^2 = \pi(1-\pi) \left(\frac{1}{\beta_L} - \frac{1}{\beta_H} \right)^2 = \pi(1-\pi) \delta^2$$

Therefore, given π and the weighted average of $1/\beta_i$, the standard deviation of $1/\beta_i$ is directly proportional to the altruism gap δ , which can be used as a relevant indicator of altruism heterogeneity.¹¹

The impact of δ on the steady-state capital stock k^* is characterized as follows.

¹¹ The difference in the reciprocals of the degrees of altruism, δ , naturally appears as a measure of heterogeneity in our model. However, our qualitative results about the impact of heterogeneity on the steady-state values do not change if we consider the difference in the degrees of altruism as a measure of heterogeneity. It can be shown that an increase in δ keeping the weighted harmonic mean $\bar{\beta}$ constant is equivalent to an increase in $\beta_H - \beta_L$ keeping the weighted arithmetic mean $(1-\pi)\beta_L + \pi\beta_H$ constant.

Proposition 6 (Capital stock and heterogeneity). *The steady-state capital stock k^* is continuous and strictly increases with δ . For $\delta < \bar{\delta}$, $k^* = k_1^*$, while for $\delta \geq \bar{\delta}$, $k^* = k_2^*$.*

Thus, increasing altruism heterogeneity, that is, the standard deviation of $1/\beta_i$ around their mean $1/\bar{\beta}$, increases the steady-state capital stock. The reason is that a harmonic-mean-preserving increase implies an increase in the degree of altruism for the more altruistic agents and a decrease in the degree of altruism for the less altruistic agents. At the same time, the weighted arithmetic mean of degrees of altruism in the population is increasing. The resulting increase in bequests from the main savers, who become more altruistic, outweighs the reduction in bequests from the less altruistic agents, who become less altruistic. This effect is even more pronounced for $\delta \geq \bar{\delta}$ when the less altruistic agents do not leave any bequests. Thus, higher altruism heterogeneity promotes capital accumulation and leads to a higher level of aggregate income. Proposition 6 is illustrated in Figure 2 with $\alpha = 0.33$, $\bar{\beta} = 0.75$, $\pi = 0.4$, $A = 1$.

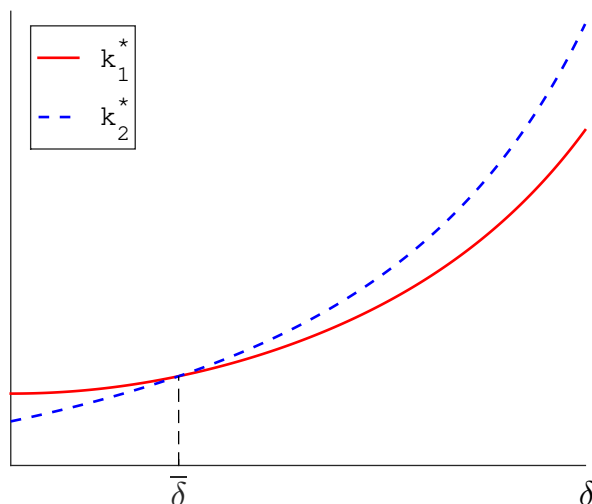


Fig. 2 Steady-state capital stock and altruism heterogeneity

Let us also address the question of the dynamic efficiency of an intertemporal equilibrium. According to the Malinvaud sufficiency theorem, an equilibrium is dynamically efficient if the steady-state capital stock is below the golden rule capital stock, defined by the condition $R^* = 1$ (see, e.g., Theorem 1 in Becker and Mitra, 2012). In our model, similarly to a canonical OLG model, dynamic inefficiency is theoretically possible. Intuitively, dynamic inefficiency arises when the degree of altruism of the more altruistic agents is high enough to leave excessively large bequests, leading to overaccumulation of capital.

Since R^* decreases with π , it follows from (9)–(10) that

$$R^* \geq R_1^* |_{\pi=1} = R_2^* |_{\pi=1} = \frac{1}{\beta_H} + \alpha$$

Therefore, under a simple sufficient condition $\beta_H < 1/(1-\alpha)$, we have $R^* > 1$ for all π , so that the steady-state capital stock is always less than the golden rule capital stock. Since an intertemporal equilibrium converges to a steady-state equilibrium, it follows that *when $\beta_H < 1/(1-\alpha)$, any intertemporal equilibrium is dynamically efficient.*

In both types of steady-state equilibrium, the more altruistic agents leave higher bequests than the less altruistic agents. Moreover, when $R^* > 1$, the more altruistic agents also have higher steady-state consumption levels. Thus, in the empirically plausible case of dynamic efficiency, the more altruistic agents have both higher income and higher consumption than the less altruistic agents, leading to a similar pattern of inequality in both income and consumption.

3.3 Altruism and income inequality

Consider now the effect of the level of altruism and altruism heterogeneity on steady-state income inequality. It is natural to represent the level of social inequality using the Gini index. The following proposition characterizes the Gini index in the steady-state equilibrium.

Proposition 7 (Gini index). (1) *Suppose that $\delta < \bar{\delta}$. The Gini index in the steady-state equilibrium $(c_1^{L*}, b_1^{L*}, c_1^{H*}, b_1^{H*}, k_1^*)$ is given by*

$$G_1^* = \frac{2\delta\pi(1-\pi)}{1-\alpha + (2\pi-1)\delta + \sqrt{(\delta+\alpha-1)^2 + 4\delta\pi(1-\alpha)}}$$

(2) *Suppose that $\delta \geq \bar{\delta}$. The Gini index in the steady-state equilibrium $(c_2^{L*}, b_2^{L*}, c_2^{H*}, b_2^{H*}, k_2^*)$ is given by $G_2^* = \alpha(1-\pi)$.*

Proposition 7 can be interpreted as follows. The Gini index for the society consisting of two types of agents is given by the product of the shares of both types (π and $1-\pi$) and the difference in relative income between them. If agents of both types have similar degrees of altruism, the difference in relative income depends on δ and on the interest rate, which is reflected in G_1^* . However, if the altruism gap is sufficiently large and the less altruistic agents leave no bequests, then it follows from (13) that the total amount of capital is divided only between the more altruistic agents, so the difference in relative incomes is inversely proportional to π . Therefore, the Gini index G_2^* depends only on the share of the less altruistic agents $1-\pi$.

The impact of the level of altruism π on the steady-state Gini index G^* , given β_L and β_H , is characterized as follows.

Proposition 8 (Gini index and altruism). *The steady-state Gini index G^* is continuous in π . For $\pi < \bar{\pi}$, $G^* = G_1^*$, while for $\pi \geq \bar{\pi}$, $G^* = G_2^*$.*

(1) Suppose that $\delta < \min\{1, 2(1 - \alpha)\}$. Then there exists a threshold share of the more altruistic agents, $0 < \tilde{\pi} \leq \bar{\pi}$, such that for $\pi < \tilde{\pi}$, G^* increases with π , while for $\pi \geq \tilde{\pi}$, G^* decreases with π . When $\delta < 1 - \alpha$, $G^*|_{\pi=0} = 0$, while, when $\delta > 1 - \alpha$, $G^*|_{\pi=0} = 1 - (1 - \alpha)/\delta > 0$.

(2) Suppose that $\delta \geq \min\{1, 2(1 - \alpha)\}$. Then G^* strictly decreases with π .

Note that for any given set of parameters, the steady-state Gini index is given by $G^* = \min\{G_1^*, G_2^*\}$. It is clear that G_2^* decreases with π , since it depends only on the share of the less altruistic agents, as explained above. Therefore, the non-monotonic impact of the level of altruism on the Gini index is shaped by the effect of π on G_1^* in a steady-state equilibrium where both types of agents leave positive bequests.

Proposition 8 suggests that there are three regimes of the steady-state Gini index, which are determined by the interplay between level of altruism and altruism heterogeneity. The first regime occurs when altruism heterogeneity is sufficiently low, $\delta < 1 - \alpha$. In this case, both types of agents have similar degrees of altruism, and the difference in their incomes is sufficiently small. Therefore, the direct effect on the Gini index comes from the shares of both types of agents in the population. When π is very low or very high, society is almost constituted by the same type of agents (less altruistic and more altruistic, respectively). Since the population is almost homogeneous up to a small minority of different agents, the social inequalities are close to zero. When π takes values in the middle of the range, the shares of the rich (the more altruistic agents who leave higher bequests) and the poor (the less altruistic agents) are similar, which drives up social inequality.

Thus, in this regime, the dependence of the steady-state level of inequality on the level of altruism has a rather symmetric inverted U-shape. The first regime is illustrated in Figure 3 with $\alpha = 0.33$, $\beta_L = 0.5$, $\beta_H = 0.75$.

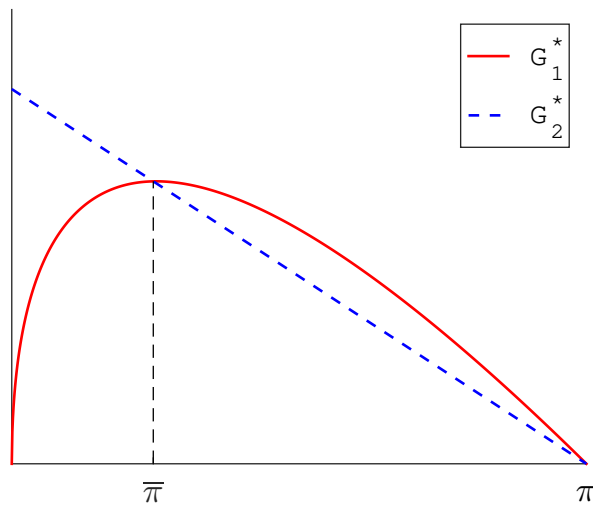


Fig. 3 Steady-state Gini index and the level of altruism: Low heterogeneity

The second regime occurs when altruism heterogeneity is moderate, $1 - \alpha < \delta < \min \{1, 2(1 - \alpha)\}$. In this case, when π is very low, the level of inequality is positive. Due to the altruism gap, the population is not homogeneous. Even though the share of the more altruistic agents is small, the difference in relative incomes is large enough to significantly affect inequality. Similar to the first regime, an increase in π increases the level of inequality. However, when π is already high, a further increase in π leaves fewer relatively poor people in the population, which reduces social inequalities.

Therefore, in this regime, the dependence of the steady-state Gini index on the level of altruism has an asymmetric inverted U-shape which is shifted upwards for low levels of altruism. The second regime is illustrated in Figure 4 with $\alpha = 0.33$, $\beta_L = 0.45$, $\beta_H = 0.75$.

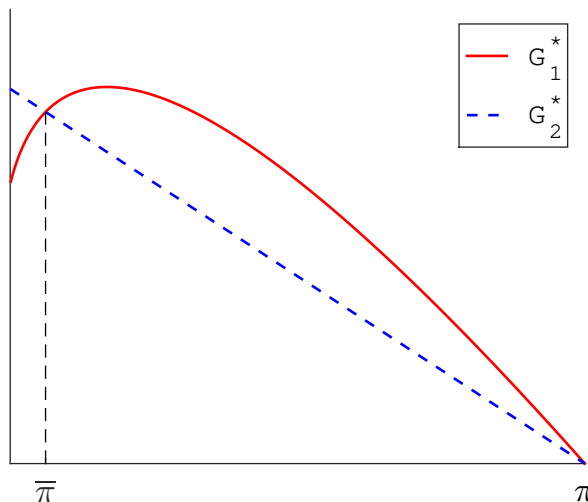


Fig. 4 Steady-state Gini index and the level of altruism: Moderate heterogeneity

The third regime occurs when altruism heterogeneity is sufficiently high, $\delta \geq \min \{1, 2(1 - \alpha)\}$. When the difference between the more and the less altruistic agents is high, then the highest possible level of inequality is observed in societies consisting of only the less altruistic agents, and inequality is decreasing with the level of altruism. Intuitively, in this regime, even if the less altruistic agents leave some bequests, the amount of these bequests is very small and has almost no impact on their income. An increase in the share of the more altruistic agents leads to equalization of relative incomes and decreases inequality. The third regime of the steady-state Gini index is illustrated in Figure 5, where we set $\alpha = 0.6$, $\beta_L = 0.45$, $\beta_H = 0.75$.

Consider also the impact of altruism heterogeneity on the steady-state Gini index. Fix π and $\bar{\beta}$. The following proposition characterizes the dependence of G^* on δ .

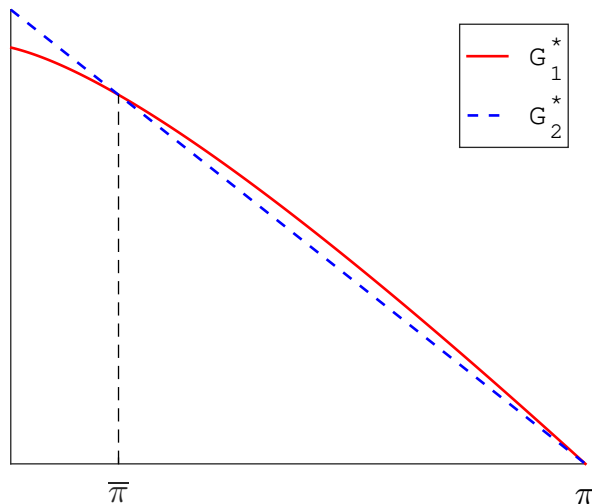


Fig. 5 Steady-state Gini index and the level of altruism: High heterogeneity

Proposition 9 (Gini index and heterogeneity). *The steady-state Gini index G^* is continuous and non-decreasing in δ . For $\delta < \bar{\delta}$, $G^* = G_1^*$ and G^* strictly increases with δ . For $\delta \geq \bar{\delta}$, $G^* = G_2^*$ and is independent of δ .*

It follows that when both types of agents have similar degrees of altruism, higher altruism heterogeneity leads to greater inequality. Intuitively, the more altruistic agents become even more altruistic and increase their bequests more than the less altruistic agents reduce their bequests after becoming even less altruistic. As a result, the rich become relatively richer, increasing social inequalities. However, when δ is sufficiently high, this effect disappears: an increase in altruism heterogeneity does not affect the steady-state Gini index. In terms of altruism heterogeneity, there exists a maximum possible steady-state Gini index which is equal to $\alpha(1 - \pi)$. Proposition 9 is illustrated in Figure 6 with $\alpha = 0.33$, $\pi = 0.65$.

3.4 Cross-country Kuznets curve

Comparing Proposition 5 and part (1) of Proposition 8, we observe that if altruism heterogeneity is low or moderate, then the resulting dependence of the steady-state level of inequality on the steady-state level of income is non-monotonic. This pattern is consistent with the *cross-country Kuznets curve*: an inverted U-shaped relationship between inequality and income in a cross-section of countries, confirmed in numerous empirical studies (among others, Campano and Salvatore, 1988; Bourguignon and Morrison, 1990; Jha, 1996; Milanovic, 2000; Savvides and Stengos, 2000).¹²

¹² The *cross-country Kuznets curve* differs from the originally proposed *within-country Kuznets curve*. Kuznets (1955) analyzed the evolution of the US and UK income distributions in the first half of the 20th century, and suggested that in the process of development within a single country, income inequality increases with the shift of labor force from traditional agricultural sector to modern industrial sector, but eventually declines as industrialization progresses. The recent rise in inequality in developed countries (see,

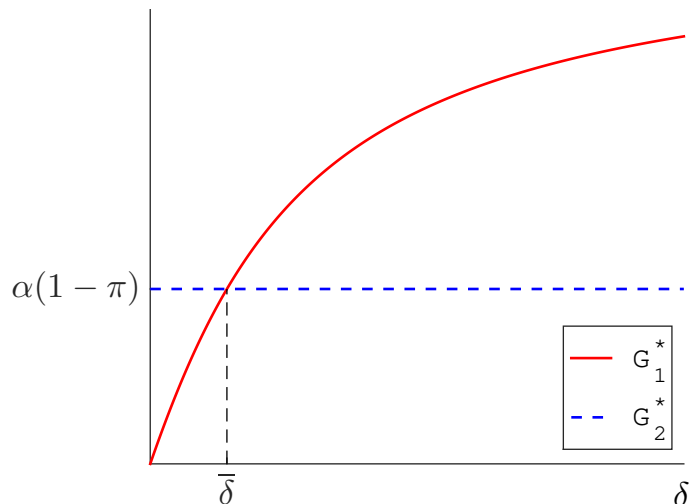


Fig. 6 Steady-state Gini index and altruism heterogeneity

Table 1 highlights this relationship using the World Bank 2019 open data on the Gini index for 105 countries divided into three income groups.¹³ The relationship between income and inequality is not absolute, and there are variations among individual countries — some high-income countries are very unequal, while some low-income countries are relatively equal.¹⁴ Nevertheless, it is clearly seen that high-income countries have much lower median and mean Gini indices compared to middle-income and low-income countries (see also OECD, 2011). Moreover, cross-country inequality is slightly higher in middle-income countries than in low-income countries.

Table 1 Gini indices for different income groups of countries in 2019

	Low income	Middle income	High income
Median	0.374	0.377	0.319
Mean	0.374	0.379	0.324
Maximum	0.513 (Angola)	0.535 (Brazil)	0.498 (Panama)
Minimum	0.260 (Moldova)	0.253 (Belarus)	0.232 (Slovakia)
N. of countries	37	30	38

Source: Authors' calculations based on the World Bank data.

e.g., Piketty, 2014; Bourguignon, 2018; Zucman, 2019) seems incompatible with the within-country pattern, but the relationship observed in a cross-section of countries is relatively stable.

¹³ We construct income groups based on the World Bank classification in 2019. High-income countries are World Bank's high-income economies (GNI per capita greater than \$12375); middle-income countries are upper middle-income economies (GNI per capita between \$3995 and \$12375); low-income countries correspond to World Bank's low-income economies and lower middle-income economies (GNI per capita less than \$3995). Gini indices refer to 2019, but also to 2017 and 2018 where data for 2019 are not available.

¹⁴ For instance, in 2018, the Guinea Gini index was 0.296, while, in 2019, the US Gini index was 0.415.

Table 1 can be illustrated in our model. Consider three countries (A, B, C) that differ in their levels of altruism ($\pi_A > \pi_B > \pi_C$) but are identical in every other respect. In the steady state, country A would have the highest aggregate income, and country C the lowest ($k_A^* > k_B^* > k_C^*$). At the same time, high-income country A would have the lowest income inequality, while middle-income country B would have the highest ($G_B^* > G_C^* > G_A^*$).

Moreover, for moderate altruism heterogeneity, the low-income country A and middle-income country B would have similar Gini indices (due to the asymmetric inverted U-shape). These observations are illustrated in Figure 7 with $\alpha = 0.33$, $\beta_L = 0.47$, $\beta_H = 0.75$, $A = 1$.

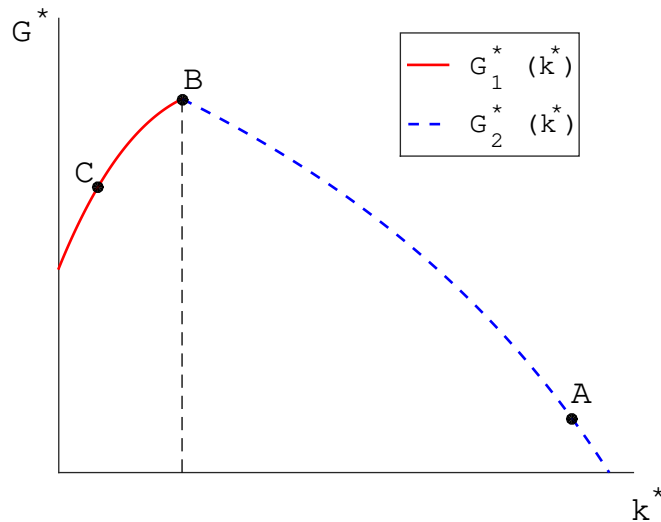


Fig. 7 Steady-state Gini index and steady-state capital stock: Cross-country Kuznets curve

Furthermore, comparing Proposition 5 and part (2) of Proposition 8, we observe that if altruism heterogeneity is high, there is no trade-off between economic growth and social inequality. The more altruistic society is as a whole, the higher the steady-state capital stock, and the lower the steady-state Gini index. Higher aggregate income is accompanied by a lower level of inequality.

Overall, these results suggest that heterogeneous altruism is a possible mechanism contributing to the tendency of high-income countries to have lower levels of income inequality.

3.5 Altruism and utility levels

Finally, we consider the effect of altruism on utility levels in the steady state. We analyze how the steady-state utilities of both types of agents depend on the level of altruism. Since social welfare criteria under heterogeneous preferences are in general

problematic, we focus on individual utilities and study separately the steady-state utility levels of the more and the less altruistic agents.¹⁵

According to (1), in the steady-state equilibrium $(c_j^{L*}, b_j^{L*}, c_j^{H*}, b_j^{H*}, k_j^*)$ with $j = 1, 2$, the utility of type $i = L, H$ agent is given by

$$U_j^{i*} = \ln c_j^{i*} + \beta_i \ln (c_j^{i*} + b_j^{i*})$$

Propositions 3 and 5 imply that the steady-state utility of type i agent, U^{i*} , is continuous, and for $\pi < \bar{\pi}$ we have $U^{i*} = U_1^{i*}$, while for $\pi \geq \bar{\pi}$ we have $U^{i*} = U_2^{i*}$. Let us introduce two additional threshold values of altruism heterogeneity:

$$\check{\delta} \equiv \frac{(1-\alpha)(1+\alpha\beta_L)}{(2-\alpha+\beta_L)\beta_L} \quad \text{and} \quad \hat{\delta} \equiv \frac{(1-\alpha)(1+\beta_L)^2}{(2-\alpha+\beta_L)\beta_L}$$

Note that $\check{\delta} < \hat{\delta}$. The following proposition shows how the steady-state utility levels depend on the level of altruism.

Proposition 10 (Agents' utilities and altruism). (1) *Less altruistic agents.*

(1.1) *If $\delta \leq \min\{\alpha, \check{\delta}\}$, then the utility U^{L*} strictly decreases with π .*

(1.2) *If $\min\{\alpha, \check{\delta}\} < \delta < \min\{1, \hat{\delta}\}$, then there exists a threshold share of the more altruistic agents, $0 < \tilde{\pi} < 1$, such that for $\pi < \tilde{\pi}$, U^{L*} decreases with π , while for $\pi \geq \tilde{\pi}$, U^{L*} increases with π .*

(1.3) *If $\delta \geq \min\{1, \hat{\delta}\}$, then U^{L*} strictly increases with π .*

(2) *More altruistic agents. The utility U^{H*} strictly decreases with π .*

Proposition 10 highlights the important difference between the more and the less altruistic agents. The steady-state utility of each more altruistic agent always decreases with π . At the same time, the shape of the steady-state utility of each less altruistic agent depends on altruism heterogeneity. When altruism heterogeneity is sufficiently low, U^{L*} decreases with π . When altruism heterogeneity is sufficiently high, U^{L*} increases with π . When altruism heterogeneity is moderate, the steady-state utility of less altruistic agents is U-shaped in the level of altruism: U^{L*} decreases for low π and increases for high π .

Intuitively, the difference in the behavior of steady-state utility between two types of agents reflects the difference in their roles. The more altruistic agents are the main savers, and their disposable income is determined primarily by capital income. Moreover, the amount of bequest left by a single more altruistic agent decreases with the share of more altruistic agents π . Indeed, the higher the π , the lower the interest rate, which reduces individuals' incentives to leave bequests and affects primarily the more altruistic agents.¹⁶

¹⁵ See also the discussion of social welfare criteria under dynastic altruism in Reichlin (2020), and under paternalistic altruism in Franks and Edenhofer (2023).

¹⁶ A similar effect can be observed in the many-agent Ramsey model (Becker, 1980) with agents heterogeneous in their discount factors. After an increase in the share of the most patient agents, the steady-state capital stock determined by the modified golden rule does not change. However, this stock is now divided among a larger number of the most patient agents, resulting in a decrease in savings for each of them. In our model, the situation is more complicated, because a higher π also leads to an increase in the aggregate steady-state capital stock. Nevertheless, the effect of decreasing bequests for more altruistic agents holds: the higher the π , the lower the bequest left by each more altruistic agent.

Since both the interest rate and the amount of bequests for the more altruistic agents decrease, the resulting fall in capital income tends to decrease their consumption and lower their steady-state utility. For the more altruistic agents, the optimal level of altruism is as close to $\pi = 0$ as possible: each more altruistic agent prefers to be the only rich individual in the population. Part (2) of Proposition 10 for the more altruistic agents is illustrated in Figure 8 with $\alpha = 0.64$, $\beta_L = 0.495$, $\beta_H = 0.75$, $A = 1$.

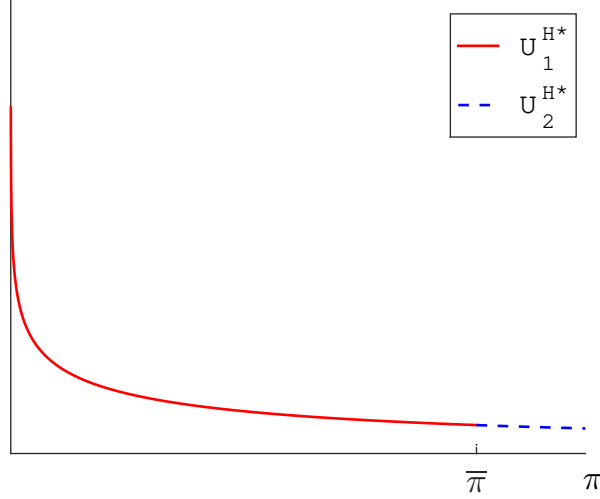


Fig. 8 More altruistic agents' utility and the level of altruism

In contrast, the role of capital income in the disposable income of the less altruistic agents depends on altruism heterogeneity. When both types of agents have similar degrees of altruism, the behavior of the less altruistic agents is almost identical to that of the more altruistic agents. Their bequests also decrease with π , and their disposable income significantly depends on capital income. As a result, the steady-state levels of consumption and utility of the less altruistic agents decrease with π . Thus, when altruism heterogeneity is low, the optimal level of altruism for the less altruistic agents is also $\pi = 0$.

However, when the difference in the degrees of altruism between the two types of agents is sufficiently high, the situation reverses. The role of bequests for the less altruistic agents becomes negligible, and their disposable income is primarily determined by labor income, which increases with π together with output. Therefore, an increase in π raises the steady-state levels of consumption and utility for the less altruistic agents. This effect is evident when altruism heterogeneity is very high, $\delta > 1$, where less altruistic agents leave no bequests and consume only their wages, but it can also be observed in steady states where less altruistic agents leave positive bequests. Thus, when altruism heterogeneity is high, the optimal level of altruism for the less altruistic agents is $\pi = 1$.

When altruism heterogeneity is moderate, the situation lies between the two polar cases described above. For low levels of altruism, wages are low and the effect of decreasing capital income dominates, so the steady-state utility of the less altruistic agents decreases with π . After a certain threshold level, the effect of increasing labor income dominates, and a further increase in π increases the steady-state levels of consumption and utility for the less altruistic agents. This case (part (1.2) of Proposition 10) is illustrated in Figure 9 with $\alpha = 0.64$, $\beta_L = 0.495$, $\beta_H = 0.75$, $A = 1$.

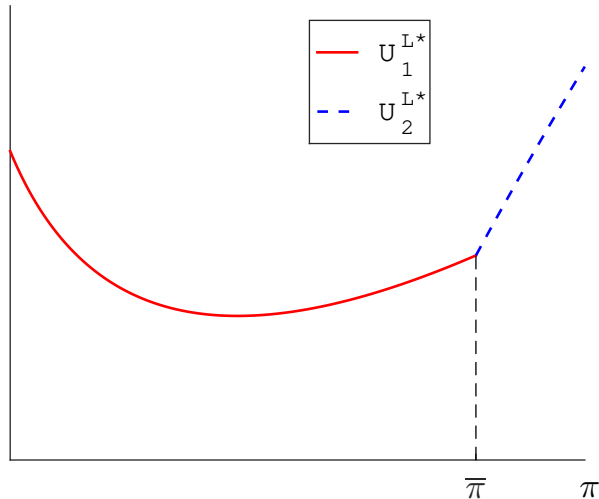


Fig. 9 Less altruistic agents' utility and the level of altruism

4 Concluding remarks

In this paper, we argue that altruism heterogeneity is a possible mechanism underlying the joint evolution of growth and inequality. We develop and analyze a simple growth model with agents who differ in their degree of altruism. The novelty of our approach rests on combining the assumption of family altruism (we consider agents who leave bequests taking care of the disposable income of their offsprings) with the assumption of agents' heterogeneity (we consider two types of agents: the one being more altruistic, the other less).

We prove that every path of consumption, bequests and capital converges to a unique steady-state equilibrium and study the properties of a steady state. Our results suggest that the effects of the level of altruism and altruism heterogeneity essentially depend on the difference between the more and the less altruistic agents.

When altruism heterogeneity is low, we observe a non-monotonic dependence of the steady-state level of inequality on the steady-state level of income, which is consistent with a cross-country Kuznets curve. Countries with low levels of altruism tend

to have low aggregate income and low level of inequality; countries with midrange levels of altruism are middle-income and highly unequal; while countries with high level of altruism tend to have high income and low level of inequality. Also, when altruism heterogeneity is low, an increase in altruism heterogeneity leads to both higher aggregate income and higher income inequality.

However, when altruism heterogeneity is sufficiently high, any trade-off between growth and inequality disappears. An increase in the level of altruism would increase aggregate income and decrease the level of inequality. Furthermore, an increase in altruism heterogeneity leads to a higher aggregate income and does not affect the steady-state level of inequality.

There are several opportunities for further theoretical research. First, altruism heterogeneity is an important factor for policy implications. When designing policies related to income redistribution or social welfare programs, one should take into account the empirically relevant fact that individuals have different degrees of altruism and hence respond differently to different incentives. Future research could introduce redistributive fiscal policies through bequest taxation, public debt and social security or more general social welfare programs.

Second, it is natural to assume that agents' degrees of altruism are not constant but change over time depending on the relative wealth of agents. This case of endogenous altruism has received considerable attention in the literature (see, among others, Das, 2007; Rapoport and Vidal, 2007). It is also interesting to understand the consequences of endogenous altruism in our framework. Overall, we believe that our approach and results contribute to the understanding of the role of heterogeneous altruism in economic growth and income inequality.

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Declarations

Competing interests

No funding was received for conducting this specific study. The authors have no relevant financial or non-financial interests to disclose. All authors declare that they have no conflicts of interest.

Appendix A Proofs

A.1 Proof of Proposition 1

Consider problem (1). Let ν_t^i and μ_t^i be the Lagrange multipliers of the budget constraint and non-negativity bequest constraint respectively. Maximizing the Lagrangian function of the Kuhn–Tucker program

$$\ln c_t^i + \beta_i \ln (R_{t+1} b_{t+1}^i + w_{t+1}) + \nu_t^i (R_t b_t^i + w_t - c_t^i - b_{t+1}^i) + \mu_t^i b_{t+1}^i$$

we get a system of first-order conditions:

$$\frac{1}{c_t^i} = \nu_t^i = \frac{\beta_i R_{t+1}}{R_{t+1} b_{t+1}^i + w_{t+1}} + \mu_t^i$$

jointly with $\nu_t^i \geq 0$, $R_t b_t^i + w_t - c_t^i - b_{t+1}^i \geq 0$, $\nu_t^i (R_t b_t^i + w_t - c_t^i - b_{t+1}^i) = 0$, and $\mu_t^i \geq 0$, $b_{t+1}^i \geq 0$, $\mu_t^i b_{t+1}^i = 0$. Since $\nu_t^i = 1/c_t^i > 0$, a non-negative pair (c_t^i, b_{t+1}^i) is a solution to (1) if and only if there exists $\mu_t^i \geq 0$ such that $\mu_t^i b_{t+1}^i = 0$, together with

$$\frac{1}{c_t^i} = \frac{\beta_i R_{t+1}}{R_{t+1} b_{t+1}^i + w_{t+1}} + \mu_t^i \quad \text{and} \quad c_t^i + b_{t+1}^i = R_t b_t^i + w_t$$

The reduced utility function $v(c_t^i) \equiv \ln c_t^i + \beta_i \ln [R_{t+1} (R_t b_t^i + w_t - c_t^i) + w_{t+1}]$ is strictly concave: $v''(c_t^i) = -(c_t^i)^{-2} - \beta_i (b_{t+1}^i + w_{t+1}/R_{t+1})^{-2} < 0$. Then the first-order conditions are necessary and sufficient to utility maximization.

Now, take into account equations (2)–(3) where $k_t = \pi_L b_t^L + \pi_H b_t^H$. If $\mu_t^i > 0$, then $b_{t+1}^i = 0$. If $b_{t+1}^i > 0$, then $\mu_t^i = 0$, so that

$$\frac{\beta_i R_{t+1}}{R_{t+1} b_{t+1}^i + w_{t+1}} = \frac{1}{c_t^i} = \frac{1}{R_t b_t^i + w_t - b_{t+1}^i}$$

and the dynamics of bequests are given by

$$b_{t+1}^i = \frac{1}{1 + \beta_i} (\beta_i R(k_t) [b_t^i + k_t \gamma(k_t)] - k_{t+1} \gamma(k_{t+1}))$$

A.2 Proof of Proposition 2

We observe that at the steady state,

$$(1 + \beta_i) b^i = \max \{0, \beta_i R(k) b^i + [\beta_i R(k) - 1] k \gamma(k)\}$$

with $i = L, H$ and $k = \pi_L b^L + \pi_H b^H$. In order to have positive bequests for type i , we need (6). In order to have positive bequests for both types, we need

$$\frac{1}{\beta_L} < R(k) < 1 + \frac{1}{\beta_H}$$

Then the steady-state capital stock is a solution to the following equation in k :

$$k = \sum_i \pi_i b^i = k\gamma(k) \sum_i \pi_i \max \left\{ 0, \frac{\beta_i R(k) - 1}{1 + \beta_i - \beta_i R(k)} \right\}$$

A.3 Proof of Proposition 3

Let $R(k) = R$ be the steady-state interest rate and $b^i \geq 0$ be the steady-state bequests for $i = L, H$, which are solutions to the following equation:

$$(1 + \beta_i) b^i = \max \{ 0, \beta_i R b^i + (\beta_i R - 1) \gamma k \} \quad (\text{A1})$$

If $R \leq 1/\beta_i$, then equation (A1) has a unique solution: $b^i = 0$.

If $R > 1/\beta_i$, then b^i is positive, and we have

$$b^i = \gamma k \frac{\beta_i R - 1}{1 + \beta_i - \beta_i R} \quad (\text{A2})$$

Therefore, if $1/\beta_i < R < 1 + 1/\beta_i$, equation (A1) has a unique solution given by (A2). If $R \geq 1 + 1/\beta_i$, equation (A1) has no solutions.

For $1/\beta_H < R < 1 + 1/\beta_H$, according to (5), we have

$$\frac{b^i}{k} = \gamma \max \left\{ 0, \frac{\beta_i R - 1}{1 + \beta_i - \beta_i R} \right\}$$

Observing that $\pi_L b^L/k + \pi_H b^H/k = 1$, we have that the steady-state interest rate R^* is a solution to the following equation in R :

$$\rho(R) \equiv \pi_L \max \left\{ 0, \frac{\beta_L R - 1}{1 + \beta_L - \beta_L R} \right\} + \pi_H \max \left\{ 0, \frac{\beta_H R - 1}{1 + \beta_H - \beta_H R} \right\} = \frac{1}{\gamma}$$

Note that $\rho(R)$ is a continuous function which increases with R , and $\rho(1/\beta_H) = 0$ and $\lim_{R \rightarrow (1+1/\beta_H)} \rho(R) = \infty$. Then there exists a solution R^* to the equation $\rho(R) = 1/\gamma$, and this solution is such that $1/\beta_H < R^* < 1 + 1/\beta_H$. Three cases are possible: (1) $1/\beta_L < R^* < 1 + 1/\beta_H$; (2) $1/\beta_H < R^* \leq 1/\beta_L < 1 + 1/\beta_H$; and (3) $1/\beta_H < R^* < 1 + 1/\beta_H \leq 1/\beta_L$.

Consider first Cases (2) and (3). Let $\pi \equiv \pi_H$. Since in both cases $R \leq 1/\beta_L$, we have

$$\rho(R) = \pi \frac{\beta_H R - 1}{1 + \beta_H - \beta_H R} \equiv \rho_2(R)$$

The solution to the equation $\rho_2(R) = 1/\gamma$ is given by

$$R^* = \frac{1}{\beta_H} + \frac{1}{1 + \gamma\pi} \equiv R_2^*$$

This solution is the steady-state interest rate if and only if $R_2^* \leq 1/\beta_L$, which is equivalent to $\delta \geq \bar{\delta}$. Since $f'(k) = R$, the capital stock corresponding to this steady

state is $k_2^* \equiv (\alpha A/R_2^*)^{1/(1-\alpha)}$. In this case, we also have $b_2^{L*} = 0$ and $b_2^{H*} = k_2^*/\pi$. Further,

$$c_2^{L*} = (1-\alpha)A(k_2^*)^\alpha = k_2^*(1-\alpha)A(k_2^*)^{\alpha-1} = \gamma k_2^* R_2^*$$

while

$$c_2^{H*} = (1-\alpha)A(k_2^*)^\alpha + \frac{(R_2^* - 1)k_2^*}{\pi} = \frac{k_2^*}{\pi}(\gamma\pi R_2^* + R_2^* - 1) = \frac{k_2^*}{\pi} \frac{1 + \gamma\pi}{\beta_H}$$

When $\rho(1/\beta_L) < 1/\gamma$, which is equivalent to $\delta < \bar{\delta}$, we are in the conditions of Case (1). We have

$$\rho(R) = (1-\pi) \frac{\beta_L R - 1}{1 + \beta_L - \beta_L R} + \pi \frac{\beta_H R - 1}{1 + \beta_H - \beta_H R} \equiv \rho_1(R)$$

The steady-state interest rate R^* is a solution to the equation $\rho_1(R) = 1/\gamma$, which can be written as

$$(1-\pi) \frac{\beta_L R - 1}{1 + \beta_L - \beta_L R} + \pi \frac{\beta_H R - 1}{1 + \beta_H - \beta_H R} = \frac{1}{\gamma}$$

or, equivalently, as

$$\begin{aligned} & \beta_L \beta_H (1 + \gamma) R^2 - [(1 + \gamma)(\beta_L + \beta_H) + (2 + \gamma)\beta_L \beta_H] R \\ & + (1 + \gamma + \beta_L + \beta_H + \beta_L \beta_H + \gamma[(1 - \pi)\beta_H + \pi\beta_L]) = 0 \end{aligned}$$

Noticing that $\gamma = (1 - \alpha)/\alpha$, we obtain

$$R_1^\pm = \frac{1}{2} \left(1 + \alpha + \frac{1}{\beta_L} + \frac{1}{\beta_H} \pm \sqrt{(\delta + \alpha - 1)^2 + 4\delta\pi(1 - \alpha)} \right)$$

Let $D(\pi) \equiv (\delta + \alpha - 1)^2 + 4\delta\pi(1 - \alpha)$. It is easily seen that $D(\pi)$ increases with π , and

$$D(0) = (\delta + \alpha - 1)^2, \quad D(\bar{\pi}) = (1 - \delta + \alpha)^2, \quad D(1) = (\delta + 1 - \alpha)^2 \quad (\text{A3})$$

where $\bar{\pi}$ is given by (17).

Since for all π , $D(\pi) \geq (\delta + \alpha - 1)^2$, it follows that

$$R_1^- \leq 1 + \frac{1}{\beta_H} \leq R_1^+$$

Denote $R_1^* \equiv R_1^-$. Then the corresponding steady-state capital stock is $k_1^* = (\alpha A/R_1^*)^{1/(1-\alpha)}$. Since $1/\beta_L < R_1^* < 1 + 1/\beta_H$, at the steady state we have for $i = L, H$,

$$b_1^{i*} = \gamma k_1^* \frac{\beta_i R_1^* - 1}{1 + \beta_i - \beta_i R_1^*}$$

and

$$\begin{aligned} c_1^{i*} &= (1 - \alpha) A(k_1^*)^\alpha + (R_1^* - 1) b_1^{i*} = \gamma k_1^* \alpha A(k_1^*)^{\alpha-1} + \gamma k_1^* \frac{(R_1^* - 1)(\beta_i R_1^* - 1)}{\beta_i + 1 - \beta_i R_1^*} \\ &= \gamma k_1^* \left[R_1^* + \frac{(R_1^* - 1)(\beta_i R_1^* - 1)}{\beta_i + 1 - \beta_i R_1^*} \right] = \frac{\gamma k_1^*}{\beta_i + 1 - \beta_i R_1^*} \end{aligned}$$

A.4 Proof of Proposition 4

(0) Consider a sequence $(b_t^L, b_t^H, k_t)_{t=0}^\infty$. Fix t . Let $\lambda_t^i \equiv b_t^i/k_t$. Let $\lambda_{t+1}^L, \lambda_{t+1}^H$ be such that for $i = L, H$,

$$\lambda_{t+1}^i k_{t+1} = \frac{\beta_i R(k_t) k_t (\lambda_t^i + \gamma) - \gamma k_{t+1}}{1 + \beta_i}$$

We observe that, at this stage of the proof, λ_{t+1}^i may lie outside of the interval $(0, 1/\pi_i)$. We have

$$\left(\lambda_{t+1}^i + \frac{\gamma}{1 + \beta_i} \right) k_{t+1} = \frac{\beta_i R(k_t) k_t (\lambda_t^i + \gamma)}{1 + \beta_i}$$

and, therefore,

$$\frac{\lambda_{t+1}^H + \frac{\gamma}{1 + \beta_H}}{\lambda_{t+1}^L + \frac{\gamma}{1 + \beta_L}} = \frac{\beta_H}{\beta_L} \frac{1 + \beta_L}{1 + \beta_H} \frac{\lambda_t^H + \gamma}{\lambda_t^L + \gamma}$$

Since we need $\pi_L \lambda_t^L + \pi_H \lambda_t^H = 1$, we study the following equation

$$\zeta(\lambda_{t+1}^H) \equiv \frac{\lambda_{t+1}^H + \frac{\gamma}{1 + \beta_H}}{\frac{1 - \pi_H \lambda_{t+1}^H}{\pi_L} + \frac{\gamma}{1 + \beta_L}} = \frac{\beta_H}{\beta_L} \frac{1 + \beta_L}{1 + \beta_H} \frac{\lambda_t^H + \gamma}{\frac{1 - \pi_H \lambda_t^H}{\pi_L} + \gamma} \equiv \xi(\lambda_t^H) \quad (\text{A4})$$

Functions ζ and ξ are increasing in the interval $(0, 1/\pi_H)$.

Notice that, given λ_t^H , equation (A4) has a solution $\lambda_{t+1}^H \in (0, 1/\pi_H)$ if and only if $b_{t+1}^L, b_{t+1}^H > 0$. More precisely, let λ_{t+1}^H be solution to (A4) in $(0, 1/\pi_H)$, $b_{t+1}^i = \lambda_{t+1}^i k_{t+1}$ and

$$\lambda_{t+1}^L = \frac{1}{\pi_L} - \frac{\pi_H}{\pi_L} \lambda_{t+1}^H$$

with

$$k_{t+1} = \sum_i \frac{\pi_i [\beta_i R(k_t) (b_t^i + \gamma k_t) - \gamma k_{t+1}]}{1 + \beta_i} = k_t R(k_t) \sum_i \frac{\pi_i \beta_i \left(\frac{b_t^i}{k_t} + \gamma \right)}{1 + \beta_i} - \gamma k_{t+1} \sum_i \frac{\pi_i}{1 + \beta_i}$$

that is

$$k_{t+1} = k_t R(k_t) \frac{\sum_i \frac{\pi_i \beta_i (\lambda_t^i + \gamma)}{1 + \beta_i}}{1 + \gamma \sum_i \frac{\pi_i}{1 + \beta_i}} \quad (\text{A5})$$

Let us prove that each solution to the equation $\zeta(\lambda) = \xi(\lambda)$ in $(0, 1/\pi_H)$ corresponds to a steady state where both bequests are strictly positive. Indeed, let λ^{H*} be a solution to equation $\zeta(\lambda) = \xi(\lambda)$ in this interval and

$$\lambda^{L*} = \frac{1}{\pi_L} - \frac{\pi_H}{\pi_L} \lambda^{H*}$$

Let the sequence $(\hat{b}_t^L, \hat{b}_t^H, \hat{k}_t)$ be such that $\hat{b}_0^L = \lambda^{L*} k_0$ and $\hat{b}_0^H = \lambda^{H*} k_0$. Let $\lambda_t^i = \hat{b}_t^i / \hat{k}_t$, for every i and $t \geq 0$. By induction, we obtain $\lambda_t^i = \lambda^{i*}$ and $\hat{b}_t^i = \lambda^{i*} \hat{k}_t$, for every $t \geq 0$.

From (A5), we have

$$\left(1 + \gamma \sum_i \frac{\pi_i}{1 + \beta_i}\right) \hat{k}_{t+1} = \alpha f(\hat{k}_t) \sum_i \frac{\pi_i \beta_i (\lambda^{i*} + \gamma)}{1 + \beta_i}$$

since $\hat{k}_t R(\hat{k}_t) = \alpha A \hat{k}_t^\alpha = \alpha f(\hat{k}_t)$.

Therefore,

$$\hat{k}_{t+1} = \alpha f(\hat{k}_t) \frac{\sum_i \frac{\pi_i \beta_i (\lambda^{i*} + \gamma)}{1 + \beta_i}}{1 + \gamma \sum_i \frac{\pi_i}{1 + \beta_i}} = \alpha S f(\hat{k}_t)$$

where

$$S \equiv \frac{\sum_i \frac{\pi_i \beta_i (\lambda^{i*} + \gamma)}{1 + \beta_i}}{1 + \gamma \sum_i \frac{\pi_i}{1 + \beta_i}}$$

is a constant.

Since $f(\hat{k}_t) = A \hat{k}_t^\alpha$ with $\alpha \in (0, 1)$ and $\hat{k}_{t+1} = \alpha S f(\hat{k}_t)$ with αS constant, it is known that \hat{k}_t monotonically converges to some capital level k^* .

(1) Consider part (1) of Proposition 3. We want to prove that:

(1.1) For any $0 \leq \lambda_t^H \leq 1/\pi_H$, there exists a unique $\lambda_{t+1}^H \in (0, 1/\pi_H)$ such that $\zeta(\lambda_{t+1}^H) = \xi(\lambda_t^H)$.

(1.2) There exists a unique $\lambda^{H*} \in (0, 1/\pi_H)$ which solves $\zeta(\lambda) = \xi(\lambda)$.

(1.1) To prove the first claim, we show that $\zeta(0) < \xi(0) \leq \xi(\lambda_t^H)$, and $\zeta(1/\pi_H) > \xi(1/\pi_H) \geq \xi(\lambda_t^H)$. Indeed, inequality $\zeta(0) < \xi(0)$ is equivalent to

$$\frac{\frac{\gamma}{1 + \beta_H}}{\frac{1}{\pi_L} + \frac{\gamma}{1 + \beta_L}} < \frac{\beta_H}{\beta_L} \frac{1 + \beta_L}{1 + \beta_H} \frac{\gamma}{\frac{1}{\pi_L} + \gamma}$$

that is to

$$1 + \gamma \pi_L < \frac{\beta_H}{\beta_L} (1 + \beta_L + \gamma \pi_L)$$

which is always true since $\beta_H > \beta_L > 0$.

The inequality $\zeta(1/\pi_H) > \xi(1/\pi_H)$ is equivalent to

$$\frac{\frac{1}{\pi_H} + \frac{\gamma}{1 + \beta_H}}{\frac{\gamma}{1 + \beta_L}} > \frac{\beta_H}{\beta_L} \frac{1 + \beta_L}{1 + \beta_H} \frac{\frac{1}{\pi_H} + \gamma}{\gamma}$$

that is to

$$\frac{1}{1 + \gamma\pi_H} > \frac{1}{\beta_L} - \frac{1}{\beta_H}$$

which is true because, in this case, $\delta < \bar{\delta}$. Therefore, $\zeta(0) < \xi(0) \leq \xi(\lambda_t^H)$ and $\zeta(1/\pi_H) > \xi(1/\pi_H) \geq \xi(\lambda_t^H)$.

Hence, there exists $0 < \lambda_{t+1}^H < 1/\pi_H$ such that $\zeta(\lambda_{t+1}^H) = \xi(\lambda_t^H)$. The strict monotonicity of function ζ ensures the uniqueness. Let $\lambda_{t+1}^H = \varphi(\lambda_t^H)$ be the unique solution to $\zeta(\lambda_{t+1}^H) = \xi(\lambda_t^H)$. The function φ is continuous in the interval $(0, 1/\pi_H)$ and is strictly increasing, with $\varphi(0) > 0$ and $\varphi(1/\pi_H) < 1/\pi_H$. This means that starting from any initial pair (b_0^L, b_0^H) with at least one positive bequest, bequests b_t^L and b_t^H are both strictly positive for any $t \geq 1$.

(1.2) Let us focus on the second claim, which is determinant in the proof of convergence. As a preliminary step, we observe that any solution in $(0, 1/\pi_H)$ to equation $\lambda = \varphi(\lambda)$ corresponds to a steady state, which, according to Proposition 3, is unique. We obtain also the uniqueness of λ^{H*} , a solution to $\lambda = \varphi(\lambda)$.

The uniqueness of the solution ensures that we have $\varphi(\lambda) > \lambda$ on $(0, \lambda^{H*})$ and $\varphi(\lambda) < \lambda$ on $(\lambda^{H*}, 1/\pi_H)$. Then, if $0 \leq \lambda_0^H < \lambda^{H*}$, the sequence $(\lambda_t^H)_{t=0}^\infty$ is increasing and converges to λ^{H*} , and, in the opposite case $\lambda_0^H > \lambda^{H*}$, this sequence is decreasing and converges to λ^{H*} . We can therefore ensure the convergence of $(\lambda_t^L, \lambda_t^H)$ to $(\lambda^{L*}, \lambda^{H*})$.

This implies also the convergence of (b_t^L, b_t^H) to (b_1^{L*}, b_1^{H*}) . Indeed, we observe that

$$k_{t+1} = k_t R(k_t) \frac{\sum_i \frac{\pi_i \beta_i (\lambda_t^i + \gamma)}{1 + \beta_i}}{1 + \gamma \sum_i \frac{\pi_i}{1 + \beta_i}} = \alpha f(k_t) \frac{\sum_i \frac{\pi_i \beta_i (\lambda_t^i + \gamma)}{1 + \beta_i}}{1 + \gamma \sum_i \frac{\pi_i}{1 + \beta_i}} = \alpha S_t f(k_t)$$

where

$$S_t \equiv \frac{\sum_i \frac{\pi_i \beta_i (\lambda_t^i + \gamma)}{1 + \beta_i}}{1 + \gamma \sum_i \frac{\pi_i}{1 + \beta_i}}$$

By the convergence of λ_t^i to λ^{i*} , we obtain

$$S_t \rightarrow S \equiv \frac{\sum_i \frac{\pi_i \beta_i (\lambda^{i*} + \gamma)}{1 + \beta_i}}{1 + \gamma \sum_i \frac{\pi_i}{1 + \beta_i}}$$

Fix any $0 < \varepsilon < S$. There exists T such that $S - \varepsilon < S_t < S + \varepsilon$ for any $t \geq T$. Let $(\bar{k}_t)_{t=T}^\infty$ and $(\underline{k}_t)_{t=T}^\infty$ be defined as

$$\bar{k}_T = \underline{k}_T = k_T, \quad \bar{k}_{t+1} = \alpha A(S + \varepsilon) \bar{k}_t^\alpha, \quad \underline{k}_{t+1} = \alpha A(S - \varepsilon) \underline{k}_t^\alpha$$

By induction, we have $\underline{k}_t \leq k_t \leq \bar{k}_t$, for every $t \geq T$. Clearly,

$$\lim_{t \rightarrow \infty} \bar{k}_t = [\alpha A(S + \varepsilon)]^{\frac{1}{1-\alpha}} \quad \text{and} \quad \lim_{t \rightarrow \infty} \underline{k}_t = [\alpha A(S - \varepsilon)]^{\frac{1}{1-\alpha}}$$

Hence we obtain

$$\limsup_{t \rightarrow \infty} k_t \leq [\alpha A (S + \varepsilon)]^{\frac{1}{1-\alpha}} \quad \text{and} \quad \liminf_{t \rightarrow \infty} k_t \geq [\alpha A (S - \varepsilon)]^{\frac{1}{1-\alpha}}$$

Since ε is arbitrary, we have $\lim_{t \rightarrow \infty} k_t = (\alpha A S)^{1/(1-\alpha)}$. The convergence of k_t implies the convergence of b_t^i and c_t^i . Therefore, the sequence $(c_t^L, c_t^H, b_t^L, b_t^H, k_t)_{t=0}^{\infty}$ converges to the values defined in part (1) of Proposition 3.

(2) Consider part (2) of Proposition 3 and suppose that $\delta > \bar{\delta}$.

(2.1) First, we prove the existence of some t such that $b_t^H \geq b_t^L$. Assume the contrary: $b_t^L > b_t^H$ for every $t \geq 0$.

(2.1.1) We prove that $b_{t+1}^H > 0$ for every $t \geq 0$. Indeed, assume the contrary, $b_{t+1}^H = 0$ for some t . Then, from $b_{t+1}^L > b_{t+1}^H = 0$, we have

$$\beta_L R(k_t)(b_t^L + \gamma k_t) \geq \gamma k_{t+1} \geq \beta_H R(k_t)(b_t^H + \gamma k_t)$$

for some t , that is

$$b_t^H + \gamma k_t \leq \frac{\beta_L}{\beta_H} (b_t^L + \gamma k_t)$$

or, equivalently,

$$\frac{\beta_L}{\beta_H} > \frac{\lambda_t^H + \gamma}{\lambda_t^L + \gamma}$$

We know that $b_{t+1}^H > 0$ if the equation $\zeta(\lambda_{t+1}^H) = \xi(\lambda_t^H)$ has a solution in the interval $(0, 1/\pi_H)$. We already have $\zeta(0) < \xi(0) < \xi(\lambda_t^H)$. We will verify that $\zeta(1/\pi_H) > \xi(\lambda_t^H)$. Indeed, we have

$$\zeta\left(\frac{1}{\pi_H}\right) = \frac{\frac{1}{\pi_H} + \frac{\gamma}{1+\beta_H}}{\frac{\gamma}{1+\beta_L}} > \frac{1 + \beta_L}{1 + \beta_H} = \frac{\beta_L}{\beta_H} \frac{\beta_H}{\beta_L} \frac{1 + \beta_L}{1 + \beta_H} > \frac{\lambda_t^H + \gamma}{\lambda_t^L + \gamma} \frac{\beta_H}{\beta_L} \frac{1 + \beta_L}{1 + \beta_H} = \xi(\lambda_t^H)$$

Hence, equation $\zeta(\lambda_{t+1}^H) = \xi(\lambda_t^H)$ has a solution in $(0, 1/\pi_H)$. Therefore, $\lambda_{t+1}^H > 0$ and $b_{t+1}^H > 0$, a contradiction. Then, $b_{t+1}^H > 0$ for every $t \geq 0$ under the assumption that $b_t^L > b_t^H$ for every $t \geq 0$.

(2.1.2) Since $b_{t+1}^L > b_{t+1}^H > 0$ for every $t \geq 0$, using the same arguments as in the preliminary part of the proof, we have $0 < \lambda_t^i < 1/\pi_H$ for any $t \geq 0$. Moreover, this sequence is monotonic and converges to a solution to equation $\zeta(\lambda) = \xi(\lambda)$. Hence, equation $\zeta(\lambda) = \xi(\lambda)$ has a solution in the interval $(0, 1/\pi_H)$. As proven above, this implies the existence of a steady state with positive bequests: a contradiction with the second part of Proposition 3. Hence, there exists some t_0 such that $b_{t_0}^H \geq b_{t_0}^L$.

(2.2) We will prove that $b_t^H \geq b_t^L$ for every $t \geq t_0$. Indeed, since $\beta/(1+\beta)$ increases with β and $1/(1+\beta)$ decreases, we have

$$\frac{\beta_H R(k_{t_0})(b_{t_0}^H + \gamma k_{t_0}) - \gamma k_{t_0+1}}{1 + \beta_H} \geq \frac{\beta_L R(k_{t_0})(b_{t_0}^L + \gamma k_{t_0}) - \gamma k_{t_0+1}}{1 + \beta_L}$$

This implies $b_{t_0+1}^H \geq b_{t_0+1}^L$. By induction, we have $b_t^H \geq b_t^L$ for every $t \geq t_0$.

(2.3) We prove the existence of some $t_1 \geq t_0$ such that $b_{t_1}^L = 0$. Assume the contrary: $b_t^H \geq b_t^L > 0$ for any $t \geq t_0$. This implies the existence of a steady state with strictly positives bequests: a contradiction. Therefore, there exists $t_1 \geq t_0$ such that $b_{t_1} = 0$.

(2.4) Now, we prove that $b_t^L = 0$ for every $t \geq t_1$. Assume the contrary: there is some $t \geq t_1$ such that $b_t^L = 0$ and $b_{t+1}^L > 0$. In this case, both b_{t+1}^L and b_{t+1}^H are strictly positive. Since $b_t^L = 0$, we have $\lambda_t^H = 1/\pi_H$. Using the same arguments as in part (1) with $b_{t+1}^L, b_{t+1}^H > 0$, we find that $\lambda_{t+1}^H = \varphi(\lambda_t^H) > \lambda_t^H = 1/\pi_H$, a contradiction.

Hence, $b_t^L = 0$ for every $t \geq t_1$. Therefore, the sequence (b_t^L, b_t^H) converges to $(0, b_2^{H*})$ in Proposition 3.

(3) Consider the cutting-edge case where $\delta = \bar{\delta}$. Consider functions ζ and ξ defined as in part (1) of the proof. We observe that, for any $\lambda \in [0, 1/\pi_H]$, we have $\zeta(\lambda) \leq \xi(\lambda)$, with equality if and only if $\lambda = 1/\pi_H$. Using the same arguments as in part (2) of the proof, we have $b_t^H > 0$ for every $t \geq 1$. Now, we consider two cases: either $b_0^L = 0$ or $b_0^L > 0$.

In the first case, following the same line of arguments as in part (2) of the proof, we have $b_t^L = 0$ for any $t \geq 1$ and the solution converges to the one described in Proposition 3.

In the second case, we have $0 < \lambda_0^H < 1/\pi_H$. Using arguments in part (1) of the proof, we have $0 < \lambda_0^L < \lambda_2^L < 1/\pi_H$. By induction we obtain that the sequence $(\lambda_t^H)_{t \geq 1}$ is strictly increasing and converges to the unique solution to $\zeta(\lambda) = \xi(\lambda)$, that is $1/\pi_H$. A direct consequence of this is that λ_t^L converges to 0. As in part (2), the convergence of λ_t^L to 0 and of λ_t^H to $1/\pi_H$ implies the convergence of (b_t^L, b_t^H, k_t) . It is easy to compute that they converge to the values defined in part (1) of Proposition 3.

A.5 Proof of Proposition 5

Denote

$$R_1^*(\pi) \equiv \frac{1+\alpha}{2} + \frac{1}{2} \left(\frac{1}{\beta_L} + \frac{1}{\beta_H} \right) - \frac{1}{2} \sqrt{(\delta + \alpha - 1)^2 + 4\delta\pi(1-\alpha)}$$

$$R_2^*(\pi) \equiv \frac{1}{\beta_H} + \frac{1}{1+\gamma\pi}$$

and consider the impact of π on the steady-state interest rate.

Condition $\delta < \bar{\delta}$ is equivalent to $\pi < \bar{\pi}$. Thus, there are three cases: (1) if $\bar{\pi} > 1$, which is equivalent to $\delta < \alpha$, then $R^*(\pi) = R_1^*(\pi)$; (2) if $0 < \bar{\pi} \leq 1$, then $R^*(\pi) = R_1^*(\pi)$ for $\pi < \bar{\pi}$ and $R^*(\pi) = R_2^*(\pi)$ for $\pi \geq \bar{\pi}$; and (3) if $\bar{\pi} \leq 0$, which is equivalent to $\delta \geq 1$, then $R^*(\pi) = R_2^*(\pi)$.

When $0 < \bar{\pi} \leq 1$ and $\pi = \bar{\pi}$, by (A3) and the fact that $\delta < 1$ we have

$$R_1^*(\bar{\pi}) = \frac{1}{2} \left[1 + \alpha + \frac{1}{\beta_L} + \frac{1}{\beta_H} - \sqrt{D(\bar{\pi})} \right] = \frac{1}{2} \left(\frac{1}{\beta_L} + \frac{1}{\beta_H} + \delta \right) = \frac{1}{\beta_L} = \frac{1}{\beta_H} + \delta = R_2^*(\bar{\pi})$$

Therefore, $R^*(\pi)$ is continuous. It follows that k^* is also continuous in π . Since both $R_1^*(\pi)$ and $R_2^*(\pi)$ decrease with π , k^* increases with π .

A.6 Proof of Proposition 6

Let $D(\delta) \equiv (\delta + \alpha - 1)^2 + 4\delta\pi(1 - \alpha)$. Denote

$$R_1^*(\delta) \equiv \frac{1 + \alpha}{2} + \frac{1}{\beta} + \frac{2\pi - 1}{2}\delta - \frac{1}{2}\sqrt{D(\delta)}$$

$$R_2^*(\delta) \equiv \frac{1}{\beta} - (1 - \pi)\delta + \frac{1}{1 + \gamma\pi}$$

and consider the impact of δ on the steady-state interest rate.

Since $\bar{\delta}$ given by (11) corresponds to $\bar{\pi}$ given by (17), it is easily seen from (A3) that

$$D(\bar{\delta}) = \left(1 + \alpha - \frac{1}{1 + \gamma\pi}\right)^2$$

Therefore, $R^*(\delta)$ is continuous, since

$$R_1^*(\bar{\delta}) = \frac{1}{\beta} + \frac{1}{2} \left[1 + \alpha + \frac{2\pi - 1}{1 + \gamma\pi} - \sqrt{D(\bar{\delta})}\right] = \frac{1}{\beta} + \frac{\pi}{1 + \gamma\pi} = R_2^*(\bar{\delta})$$

Notice that

$$(R_2^*)'(\delta) \equiv \pi - 1 < 0$$

$$(R_1^*)'(\delta) \equiv \frac{2\pi - 1}{2} - \frac{\delta + \alpha - 1 + 2\pi(1 - \alpha)}{2\sqrt{D(\delta)}} < 0$$

Indeed, $(R_1^*)'(\delta) < 0$ if $\delta > (2\pi - 1) \left(\sqrt{D(\delta)} - 1 + \alpha\right)$. When $2\pi - 1 > 0$, we have $\delta > (2\pi - 1)\delta$, and the above inequality holds if $\sqrt{D(\delta)} < \delta + 1 - \alpha$, which follows from (A3). When $2\pi - 1 < 0$, we have $\delta > (1 - 2\pi)\delta$, and the above inequality holds if $\sqrt{D(\delta)} > 1 - \alpha - \delta$, which also follows from (A3).

Since k^* is inversely related to R^* , it follows that k^* increases with δ .

A.7 Proof of Proposition 7

The level of social inequality in terms of income in the steady-state equilibrium is represented by the Gini index:

$$G = 2 \int_0^1 [x - g(x)] dx$$

where $g : [0, 1] \rightarrow [0, 1]$ is the Lorenz curve.

Consider a share π_L of less altruistic agents with income y^L , and a share π_H of more altruistic agents with income $y^H \geq y^L$. The continuous Lorenz curve is given by

$$g(x) = \frac{y^L}{\pi_L y^L + \pi_H y^H} x \quad \text{if} \quad 0 \leq x \leq \pi_L$$

$$g(x) = \frac{\pi_L y^L + (x - \pi_L) y^H}{\pi_L y^L + \pi_H y^H} \quad \text{if} \quad \pi_L < x \leq 1$$

We obtain

$$\begin{aligned} G &= 2 \int_0^1 [x - g(x)] dx = 2 \left(\int_0^{\pi_L} [x - g(x)] dx + \int_{\pi_L}^1 [x - g(x)] dx \right) \\ &= 1 - \pi_L^2 \frac{y^L}{\pi_L y^L + \pi_H y^H} - 2(1 - \pi_L) \frac{\pi_L y^L - \pi_L y^H}{\pi_L y^L + \pi_H y^H} - (1 - \pi_L^2) \frac{y^H}{\pi_L y^L + \pi_H y^H} \end{aligned}$$

and hence,

$$G = \frac{\pi_L \pi_H (y^H - y^L)}{\pi_L y^L + \pi_H y^H} \quad (\text{A6})$$

Consider two parts of Proposition 3.

(1) For $i = L, H$, we have $y_1^i = R_1^* b_1^{i*} + (1 - \alpha) A(k_1^*)^\alpha$. Observing that $R_1^* = \alpha A(k_1^*)^{\alpha-1}$ and using (12), it follows from (A6) that

$$G_1^* = \pi_L \pi_H \frac{R_1^* \frac{b_1^{H*}}{\gamma k_1^*} - R_1^* \frac{b_1^{L*}}{\gamma k_1^*}}{\alpha A(k_1^*)^{\alpha-1} + \pi_L R_1^* \frac{b_1^{L*}}{\gamma k_1^*} + \pi_H R_1^* \frac{b_1^{H*}}{\gamma k_1^*}} = \pi_L \pi_H \frac{\frac{1}{\beta_L} - \frac{1}{\beta_H}}{1 - R_1^* + \frac{\pi_H}{\beta_L} + \frac{\pi_L}{\beta_H}}$$

(2) We have $y_2^L = (1 - \alpha) A(k_2^*)^\alpha$ and $y_2^H = (1 - \alpha) A(k_2^*)^\alpha + R_2^* k_2^* / \pi_H$, and hence,

$$G_2^* = \frac{\pi_L \pi_H (y_2^H - y_2^L)}{\pi_L y_2^L + \pi_H y_2^H} = \frac{\pi_L R_2^* k_2^*}{(1 - \alpha) A(k_2^*)^\alpha + R_2^* k_2^*} = \frac{\pi_L \alpha A(k_2^*)^\alpha}{(1 - \alpha) A(k_2^*)^\alpha + \alpha A(k_2^*)^\alpha} = \alpha \pi_L$$

A.8 Proof of Proposition 8

Denote

$$\begin{aligned} G_1^*(\pi) &\equiv \frac{2\delta\pi(1 - \pi)}{1 - \alpha + (2\pi - 1)\delta + \sqrt{(\delta + \alpha - 1)^2 + 4\delta\pi(1 - \alpha)}} \\ G_2^*(\pi) &\equiv \alpha(1 - \pi) \end{aligned}$$

Using (A3), we have

$$\begin{aligned} G_1^*(\bar{\pi}) &= \frac{2\delta\bar{\pi}(1 - \bar{\pi})}{1 - \alpha + (2\bar{\pi} - 1)\delta + \sqrt{D(\bar{\pi})}} = \frac{2\delta\bar{\pi}(1 - \bar{\pi})}{1 - \alpha + (2\bar{\pi} - 1)\delta + 1 - \delta + \alpha} \\ &= \frac{\delta\bar{\pi}(1 - \bar{\pi})}{1 - \delta + \delta\bar{\pi}} = \frac{1}{1 + \gamma} (1 - \bar{\pi}) = \alpha(1 - \bar{\pi}) = G_2^*(\bar{\pi}) \end{aligned}$$

since $\delta\bar{\pi} = (1 - \delta)/\gamma$. Therefore, $G^*(\pi)$ is continuous.

To analyze the shape of $G_1^*(\pi)$, denote $G_1^*(\pi) = n(\pi)/d(\pi)$, where

$$n(\pi) \equiv 2\delta\pi(1 - \pi) \quad \text{and} \quad d(\pi) \equiv 1 - \alpha + \delta(2\pi - 1) + \sqrt{D(\pi)}$$

Since

$$n'(\pi) = 2\delta(1 - 2\pi) \quad \text{and} \quad d'(\pi) = 2\delta \left[1 + \frac{1 - \alpha}{\sqrt{D(\pi)}} \right]$$

we have

$$(G_1^*)'(\pi) = 2\delta \frac{(1 - 2\pi)d(\pi) - 2\delta\pi(1 - \pi) \left[1 + \frac{1 - \alpha}{\sqrt{D(\pi)}} \right]}{d(\pi)^2} \quad (\text{A7})$$

Now we have to distinguish between the two cases.

(1) Suppose that $\delta < 1 - \alpha$. Then $G_1^*(0) = G_1^*(1) = 0$. We show that there exists a unique $\hat{\pi}$ such that for $\pi < \hat{\pi}$, $G_1^*(\pi)$ strictly increases with π , while for $\pi \geq \hat{\pi}$, $G_1^*(\pi)$ strictly decreases with π .

Indeed, after some algebra, equation $(G_1^*)'(\pi) = 0$ can be written as $\mathcal{L}(\pi) = \mathcal{R}(\pi)$, where

$$\begin{aligned} \mathcal{L}(\pi) &\equiv (1 - \alpha - \delta) \frac{1 - 2\pi}{2\delta\pi} \\ \mathcal{R}(\pi) &\equiv \frac{\pi\sqrt{D(\pi)} + (1 - \alpha)(3\pi - 1)}{\sqrt{D(\pi)} + 1 - \alpha - \delta} = \frac{\sqrt{D(\pi)} + (1 - \alpha)(3 - \frac{1}{\pi})}{\frac{\sqrt{D(\pi)}}{\pi} + \frac{1 - \alpha - \delta}{\pi}} \end{aligned}$$

Since in this case $1 - \alpha - \delta > 0$, $\mathcal{L}(\pi)$ is convex and decreasing with $\mathcal{L}(0^+) = \infty$, $\mathcal{L}(1/2) = 0$ and $\mathcal{L}(1) = -(1 - \delta - \alpha)/(2\delta) < 0$.

Further, $\mathcal{R}(\pi)$ increases with π . Indeed, $\sqrt{D(\pi)} + (1 - \alpha)(3 - 1/\pi)$ increases with π , while $\sqrt{D(\pi)}/\pi + (1 - \alpha - \delta)/\pi$ decreases with π . By (A3),

$$\mathcal{R}(0) = -\frac{1}{2} \frac{1 - \alpha}{1 - \alpha - \delta} < 0 \quad \text{and} \quad \mathcal{R}\left(\frac{1}{2}\right) = \frac{1}{2} \frac{\sqrt{D(1/2)} + 1 - \alpha}{\sqrt{D(1/2)} + 1 - \alpha - \delta} > 0$$

Therefore, in this case there exists a unique value $0 < \hat{\pi} < 1/2$ such that $\mathcal{L}(\hat{\pi}) = \mathcal{R}(\hat{\pi})$, or $(G_1^*)'(\hat{\pi}) = 0$. Moreover, for $\pi < \hat{\pi}$, $\mathcal{L}(\pi) > \mathcal{R}(\pi)$, so that $G_1^*(\pi)$ is increasing, while for $\pi > \hat{\pi}$, $\mathcal{L}(\pi) < \mathcal{R}(\pi)$, and $G_1^*(\pi)$ is decreasing.

When $\delta < \alpha$, we have $\bar{\pi} > 1$, and hence, as in the proof of Proposition 5, we have $G^*(\pi) = G_1^*(\pi)$. In this case the threshold level of π , up to which G^* is increasing and after which G^* is decreasing, is $\hat{\pi}$. Similarly, when $\alpha < \delta \leq 1$, for $\pi < \bar{\pi}$ we have $G^*(\pi) = G_1^*(\pi)$, while for $\pi \geq \bar{\pi}$, we have $G^*(\pi) = G_2^*(\pi)$. In this case, the threshold level of π is $\tilde{\pi} \equiv \min\{\hat{\pi}, \bar{\pi}\}$.

(2) Suppose that $\delta > 1 - \alpha$. By Bernoulli's rule,

$$G_1^*(0) = \frac{n'(0)}{d'(0)} = \frac{\delta + \alpha - 1}{\delta} > 0 = G_1^*(1)$$

Moreover, in this case $G_1^*(0) < \alpha = G_2^*(0)$ if and only if $\delta < 1$.

The second case has two subcases.

(2.1) Suppose that $1 - \alpha < \delta < 2(1 - \alpha)$. Then we show that $(G_1^*)'(0^+) > 0$ and $(G_1^*)'(1^-) < 0$, so there exists an interior $\hat{\pi} = \arg \max_{0 \leq \pi \leq 1} G_1^*(\pi)$.

Indeed, using (A3) and (A7), we obtain

$$(G_1^*)'(1) = -\frac{\delta}{1 - \alpha + \delta} < 0$$

Applying Bernoulli's rule to (A7), we get

$$\begin{aligned} (G_1^*)'(0^+) &= 2\delta \lim_{\pi \rightarrow 0^+} \frac{-2d(\pi) + (1 - 2\pi)d'(\pi) - 2\delta(1 - 2\pi) \left[1 + \frac{1 - \alpha}{\sqrt{D(\pi)}} \right] + 4\delta^2(1 - \alpha)^2 \frac{\pi(1 - \pi)}{D(\pi)^{\frac{3}{2}}}}{2d(\pi)d'(\pi)} \\ &= \lim_{\pi \rightarrow 0^+} \frac{-d(\pi)\sqrt{D(\pi)} + 2\delta^2(1 - \alpha)^2 \frac{\pi(1 - \pi)}{D(\pi)}}{d(\pi) \left[1 - \alpha + \sqrt{D(\pi)} \right]} \end{aligned}$$

In this case, by (A3), $\sqrt{D(0)} = \delta + \alpha - 1 > 0$. Applying again Bernoulli's rule, we get

$$\begin{aligned} (G_1^*)'(0^+) &= \lim_{\pi \rightarrow 0^+} \frac{-d'(\pi)\sqrt{D(\pi)} - d(\pi)\frac{D'(\pi)}{2\sqrt{D(\pi)}} + 2\delta^2(1 - \alpha)^2 \frac{(1 - 2\pi)D(\pi) - \pi(1 - \pi)D'(\pi)}{D(\pi)^2}}{d'(\pi) \left[1 - \alpha + \sqrt{D(\pi)} \right] + d(\pi)\frac{D'(\pi)}{2\sqrt{D(\pi)}}} \\ &= \frac{2(1 - \alpha) - \delta}{\delta + \alpha - 1} \end{aligned}$$

Thus, for $1 - \alpha < \delta < 2(1 - \alpha)$, $(G_1^*)'(0^+) > 0$.

(2.2) Suppose that $\delta \geq 2(1 - \alpha)$. We have just seen that in this case $(G_1^*)'(0^+) \leq 0$. Let us show that $(G_1^*)'(\pi) < 0$ for all $\pi > 0$. By (A7), this is equivalent to

$$\pi(1 - \pi)d'(\pi) > (1 - 2\pi)d(\pi)$$

Since $d(0) = 0$, it is sufficient to check that for all $\pi > 0$,

$$[\pi(1 - \pi)d'(\pi)]' > [(1 - 2\pi)d(\pi)]'$$

that is $2d(\pi) > -\pi(1 - \pi)d''(\pi)$ or $2d(\pi) > \pi(1 - \pi)|d''(\pi)|$, since $d''(\pi) < 0$. Or, equivalently, it is sufficient to show that

$$\left[\sqrt{D(\pi)} - (\delta + \alpha - 1) + 2\delta\pi \right] [D(\pi)]^{\frac{3}{2}} > 2\pi(1 - \pi)\delta^2(1 - \alpha)^2$$

It is easily seen that

$$\sqrt{D(\pi)} - (\delta + \alpha - 1) > \frac{2\delta\pi(1 - \alpha)}{\sqrt{D(\pi)}}$$

and, since $\delta \geq 2(1 - \alpha) > 1 - \alpha$,

$$\sqrt{D(\pi)} = \sqrt{(\delta + \alpha - 1)^2 + 4\delta\pi(1 - \alpha)} > \delta - (1 - \alpha) \geq 1 - \alpha$$

Therefore,

$$\begin{aligned} & \left[\sqrt{D(\pi)} - (\delta + \alpha - 1) + 2\delta\pi \right] D(\pi)^{\frac{3}{2}} = \left[\sqrt{D(\pi)} - (\delta + \alpha - 1) \right] D(\pi)^{\frac{3}{2}} + 2\delta\pi D(\pi)^{\frac{3}{2}} \\ & > \frac{2\delta\pi(1 - \alpha)}{\sqrt{D(\pi)}} D(\pi)^{\frac{3}{2}} + 2\delta\pi D(\pi)^{\frac{3}{2}} = 2\delta\pi D(\pi) \left[1 - \alpha + \sqrt{D(\pi)} \right] \\ & > 2\delta\pi(1 - \alpha)^2 [1 - \alpha + \delta - (1 - \alpha)] = 2\pi\delta^2(1 - \alpha)^2 \geq 2\pi(1 - \pi)\delta^2(1 - \alpha)^2 \end{aligned}$$

Again, as in the proof of Proposition 5, in both cases (2.1) and (2.2), when $\delta < \alpha$, we have $G^*(\pi) = G_1^*(\pi)$. When $\alpha < \delta \leq 1$, for $\pi < \bar{\pi}$ we have $G^*(\pi) = G_1^*(\pi)$, while for $\pi \geq \bar{\pi}$, we have $G^*(\pi) = G_2^*(\pi)$. When $\delta > 1$, we have $\bar{\pi} \leq 0$, and hence $G^*(\pi) = G_2^*(\pi)$. Thus, if $1 - \alpha < \delta < \min\{1, 2(1 - \alpha)\}$, then $G^*(\pi)$ has an inverted-U shape. If $\delta \geq \min\{1, 2(1 - \alpha)\}$, then $G^*(\pi)$ decreases with π .

A.9 Proof of Proposition 9

Denote

$$G_1^*(\delta) \equiv \frac{2\delta\pi(1 - \pi)}{1 - \alpha + (2\pi - 1)\delta + \sqrt{(\delta + \alpha - 1)^2 + 4\delta\pi(1 - \alpha)}}$$

We have

$$\begin{aligned} G_1^*(\bar{\delta}) &= \frac{2\bar{\delta}\pi(1 - \pi)}{1 - \alpha + (2\pi - 1)\bar{\delta} + \sqrt{D(\bar{\delta})}} = \frac{2\bar{\delta}\pi(1 - \pi)}{1 - \alpha + (2\pi - 1)\bar{\delta} + 1 + \alpha - \bar{\delta}} \\ &= \frac{\bar{\delta}\pi(1 - \pi)}{1 - (1 - \pi)\bar{\delta}} = \frac{\pi(1 - \pi)}{1 + \gamma\pi - 1 + \pi} = \alpha(1 - \pi) = G_2^*(\bar{\delta}) \end{aligned}$$

since $\gamma = (1 - \alpha)/\alpha$. Therefore, $G^*(\delta)$ is continuous.

Let us show that $(G_1^*)'(\delta) > 0$. Indeed,

$$(G_1^*)'(\delta) = \frac{2\pi(1 - \pi)}{\left[1 - \alpha + \delta(2\pi - 1) + \sqrt{D(\delta)}\right]^2} \left[1 - \alpha + \sqrt{D(\delta)} - \frac{\delta D'(\delta)}{2\sqrt{D(\delta)}}\right]$$

and, since by (A3), $\sqrt{D(\delta)} \geq \delta + \alpha - 1$, we have

$$\begin{aligned} 1 - \alpha + \sqrt{D(\delta)} - \frac{\delta D'(\delta)}{2\sqrt{D(\delta)}} &= \frac{1 - \alpha}{\sqrt{D(\delta)}} \left[\sqrt{D(\delta)} + \frac{D(\delta) - \delta(\delta + \alpha - 1)}{1 - \alpha} - 2\delta\pi \right] \\ &= \frac{1 - \alpha}{\sqrt{D(\delta)}} \left[2\delta\pi + \sqrt{D(\delta)} - (\delta + \alpha - 1) \right] > 0 \end{aligned}$$

A.10 Proof of Proposition 10

In the steady state $j = 1, 2$, for agent $i = L, H$, we obtain

$$(U_j^{i*})'(\pi) = \frac{(c_j^{i*})'(\pi)}{c_j^{i*}(\pi)} + \beta_i \frac{(c_j^{i*})'(\pi) + (b_j^{i*})'(\pi)}{c_j^{i*}(\pi) + b_j^{i*}(\pi)} \quad (\text{A8})$$

In the following, for simplicity, we omit the argument π . We have $(U_j^{i*})' = 0$ if

$$\frac{(c_j^{i*})'}{c_j^{i*}} \left[1 + (1 + \beta_i) \frac{c_j^{i*}}{b_j^{i*}} \right] + \beta_i \frac{(b_j^{i*})'}{b_j^{i*}} = 0 \quad (\text{A9})$$

(1) Consider the steady-state equilibrium $(c_1^{L*}, b_1^{L*}, c_1^{H*}, b_1^{H*}, k_1^*)$ where both types of agents leave bequests. For $\pi < \bar{\pi}$, the utilities of the more and the less altruistic agents are given by U_1^{H*} and U_1^{L*} respectively. Recall that $(R_1^*)' < 0$ and

$$\frac{(k_1^*)'}{k_1^*} = -\frac{1}{1 - \alpha} \frac{(R_1^*)'}{R_1^*}$$

Furthermore, by part (1) of Proposition 3,

$$\frac{c_1^{i*}}{b_1^{i*}} = \frac{1}{\beta_i R_1^* - 1} \quad (\text{A10})$$

and hence

$$\frac{(b_1^{i*})'}{b_1^{i*}} = \frac{(c_1^{i*})'}{c_1^{i*}} + \frac{\beta_i (R_1^*)'}{\beta_i R_1^* - 1} \quad (\text{A11})$$

On the other hand,

$$\frac{(c_1^{i*})'}{c_1^{i*}} = \frac{(k_1^*)'}{k_1^*} + \frac{\beta_i (R_1^*)'}{\beta_i + 1 - \beta_i R_1^*} = \frac{(R_1^*)'}{R_1^*} \left(\frac{\beta_i R_1^*}{\beta_i + 1 - \beta_i R_1^*} - \frac{1}{1 - \alpha} \right) \quad (\text{A12})$$

Using (A8), (A9), (A10), (A11) and (A12), after some algebra we have

$$(U_1^{i*})' = \frac{(R_1^*)'}{R_1^*} \left[\frac{\beta_i (1 + \beta_i) R_1^*}{1 + \beta_i - \beta_i R_1^*} - \frac{1 + \alpha \beta_i}{1 - \alpha} \right]$$

Since $(R_1^*)' < 0$, we obtain that $(U_1^{i*})' < 0$ if and only if

$$\frac{\beta_i (1 + \beta_i) R_1^*}{1 + \beta_i - \beta_i R_1^*} - \frac{1 + \alpha \beta_i}{1 - \alpha} > 0$$

or, equivalently,

$$R_1^*(\pi) > \frac{1 + \beta_i}{\beta_i} \frac{1 + \alpha \beta_i}{2 - \alpha + \beta_i} \quad (\text{A13})$$

Note that the right-hand side of this inequality depends on β_i and hence is different for different types of agents.

(1.1) Consider the utility of the less altruistic agents. Denote

$$\bar{R} \equiv \frac{1 + \beta_L}{\beta_L} \frac{1 + \alpha\beta_L}{2 - \alpha + \beta_L} = 1 + \frac{1}{\beta_L} - \frac{(1 - \alpha)(1 + \beta_L)^2}{(2 - \alpha + \beta_L)\beta_L}$$

Let the critical value of the level of altruism $\hat{\pi}$ be a solution to the equation $R_1^*(\pi) = \bar{R}$. This solution exists only if $R_1^*(1) \leq \bar{R} \leq R_1^*(0)$. The first inequality holds when

$$\frac{1}{\beta_H} + \alpha \leq 1 + \frac{1}{\beta_L} - \frac{(1 - \alpha)(1 + \beta_L)^2}{(2 - \alpha + \beta_L)\beta_L} \iff \frac{1}{\beta_L} - \frac{1}{\beta_H} \geq \frac{(1 - \alpha)(1 + \alpha\beta_L)}{(2 - \alpha + \beta_L)\beta_L} \equiv \check{\delta}$$

For the second inequality, recall that $R_1^*(0) = 1/\beta_L + \alpha$ when $\delta < 1 - \alpha$, and $R_1^*(0) = 1/\beta_H + 1$ when $\delta > 1 - \alpha$. While it is always true that

$$1 + \frac{1}{\beta_L} - \frac{(1 - \alpha)(1 + \beta_L)^2}{(2 - \alpha + \beta_L)\beta_L} \leq \frac{1}{\beta_L} + \alpha \iff \frac{(1 - \alpha)(1 + \beta_L)^2}{(2 - \alpha + \beta_L)\beta_L} \geq 1 - \alpha \iff 1 + \alpha\beta_L > 0$$

the second inequality holds when

$$1 + \frac{1}{\beta_L} - \frac{(1 - \alpha)(1 + \beta_L)^2}{(2 - \alpha + \beta_L)\beta_L} \leq \frac{1}{\beta_H} + 1 \iff \frac{1}{\beta_L} - \frac{1}{\beta_H} \leq \frac{(1 - \alpha)(1 + \beta_L)^2}{(2 - \alpha + \beta_L)\beta_L} \equiv \hat{\delta}$$

Since $R_1^*(\pi)$ strictly decreases with π , it follows that when $\check{\delta} \leq \delta \leq \hat{\delta}$, there exists a unique solution $\hat{\pi}$ to the equation $R_1^*(\pi) = \bar{R}$, and $0 \leq \hat{\pi} \leq 1$.

Define also a critical value for the capital share in total income:

$$\bar{\alpha} \equiv \frac{1}{1 + \beta_L + \beta_L^2}$$

There are three cases.

(1.1.1) Suppose that $\delta < \check{\delta}$. Then we have $R_1^*(\pi) \geq R_1^*(1) > \bar{R}$ for all π , and hence, according to (A13), $(U_1^{L*})'(\pi) < 0$.

(1.1.2) Suppose that $\check{\delta} \leq \delta \leq \hat{\delta}$. Note that we have

$$\bar{\pi} \leq \hat{\pi} \iff R_1^*(\bar{\pi}) \geq \bar{R} \iff \frac{1}{\beta_L} \geq \frac{1 + \beta_L}{\beta_L} \frac{1 + \alpha\beta_L}{2 - \alpha + \beta_L} \iff \alpha \leq \bar{\alpha}$$

Therefore, since $R_1^*(\pi)$ strictly decreases, when $\alpha < \bar{\alpha}$, we have $R_1^*(\pi) > R_1^*(\bar{\pi}) > \bar{R}$ for all $\pi < \bar{\pi}$, and hence $(U_1^{L*})'(\pi) < 0$ for all $\pi < \bar{\pi}$.

Alternatively, when $\alpha > \bar{\alpha}$, we have $\hat{\pi} < \bar{\pi}$. Therefore, for all $\pi \leq \hat{\pi}$, we have $R_1^*(\pi) > \bar{R}$, and hence $(U_1^{L*})'(\pi) < 0$ for all $\pi \leq \hat{\pi}$. At the same time, for all $\pi \geq \hat{\pi}$, we have $R_1^*(\pi) < \bar{R}$, and hence $(U_1^{L*})'(\pi) > 0$ for all $\hat{\pi} < \pi < \bar{\pi}$.

(1.1.3) Suppose that $\delta > \hat{\delta}$. Then we have $R_1^*(\pi) \leq R_1^*(0) < \bar{R}$ for all π , and hence, according to (A13), $(U_1^{L*})'(\pi) > 0$.

(1.2) Consider now the more altruistic agents. Clearly, for all π ,

$$R_1^*(\pi) \geq R_1^*(1) = \frac{1}{\beta_H} + \alpha = \frac{1 + \alpha\beta_H}{\beta_H} > \frac{1 + \alpha\beta_H}{\beta_H} \frac{1 + \beta_H}{1 + \beta_H + 1 - \alpha} = \frac{1 + \beta_H}{\beta_H} \frac{1 + \alpha\beta_H}{2 - \alpha + \beta_H}$$

Therefore, by (A13), $(U_1^{H*})'(\pi) < 0$.

Let us also show that bequests of the more altruistic agents decrease with π . Substituting (A12) into (A11), we obtain

$$\frac{(b_1^{i*})'}{b_1^{i*}} = \frac{(R_1^*)'}{R_1^*} \left[\frac{\beta_i^2 R_1^*}{(\beta_i + 1 - \beta_i R_1^*)(\beta_i R_1^* - 1)} - \frac{1}{1 - \alpha} \right]$$

Since $(R_1^*)' < 0$, we have $(b_1^{i*})' < 0$ if and only if $(\beta_i R_1^* - 1)(\beta_i + 1 - \beta_i R_1^*) < (1 - \alpha)\beta_i^2 R_1^*$, which can be rewritten as

$$(R_1^*)^2 - 2R_1^* \left(\frac{\alpha}{2} + \frac{1}{\beta_i} \right) + \frac{1}{\beta_i} + \frac{1}{\beta_i^2} > 0 \quad (\text{A14})$$

The largest root of the quadratic equation $\beta_i^2 x^2 - (\alpha\beta_i^2 + 2\beta_i)x + 1 + \beta_i = 0$ is given by

$$x_i^+ = \frac{\alpha}{2} + \frac{1}{\beta_i} + \sqrt{\frac{\alpha^2}{4} - \frac{1 - \alpha}{\beta_i}} < \alpha + \frac{1}{\beta_i}$$

For $i = H$, we have $R_1^*(\pi) \geq R_1^*(1) = 1/\beta_H + \alpha > x_H^+$, and hence (A14) holds, meaning that $(b_1^{H*})' < 0$ for all π .

(2) Consider now the steady-state equilibrium $(c_2^{L*}, b_2^{L*}, c_2^{H*}, b_2^{H*}, k_2^*)$. For $\pi \geq \bar{\pi}$, the utilities of the less and the more altruistic agents are given by U_2^{L*} and U_2^{H*} respectively. Recall also that

$$(R_2^*)' = -\frac{\gamma}{(1 + \gamma\pi)^2} \quad \text{and} \quad \frac{(k_2^*)'}{k_2^*} = \frac{1}{(1 + \gamma\pi)^2} \frac{1}{\alpha R_2^*} > 0$$

(2.1) Less altruistic agents leave no bequests and consume their wages, so that

$$U_2^{L*} = \ln c_2^{L*} + \beta_L \ln (c_2^{L*} + b_2^{L*}) = (1 + \beta_L) \ln c_2^{L*} = (1 + \beta_L) \ln [(1 - \alpha)A(k_2^*)^\alpha]$$

Since k_2^* strictly increases with π , $U_2^{L*}(\pi)$ also strictly increases with π : $(U_2^{L*})'(\pi) > 0$ for all $\pi \geq \bar{\pi}$.

(2.2) For the more altruistic agents, we have

$$\frac{(b_2^{H*})'}{b_2^{H*}} = \frac{(k_2^*)'}{k_2^*} - \frac{1}{\pi} = -\frac{1}{\pi R_2^*} \left[R_2^* - \frac{\pi}{\alpha(1 + \gamma\pi)^2} \right] \quad (\text{A15})$$

Note that b_2^{H*} strictly decreases with π . Indeed, $(b_2^{H*})' < 0$ because

$$R_2^* - \frac{\pi}{\alpha(1+\gamma\pi)^2} = \frac{1}{\beta_H} + \frac{1}{1+\gamma\pi} - \frac{\pi(1+\gamma)}{(1+\gamma\pi)^2} = \frac{1}{\beta_H} + \frac{1-\pi}{(1+\gamma\pi)^2} > 0$$

Moreover,

$$\frac{c_2^{H*}}{b_2^{H*}} = \frac{1+\gamma\pi}{\beta_H} \quad (\text{A16})$$

and hence

$$\frac{(c_2^{H*})'}{c_2^{H*}} = \frac{(b_2^{H*})'}{b_2^{H*}} + \frac{\gamma}{1+\gamma\pi} = -\frac{1}{\pi(1+\gamma\pi)R_2^*} \left[R_2^* - \frac{\pi}{\alpha(1+\gamma\pi)} \right] \quad (\text{A17})$$

Therefore, using (A8), (A15), (A16) and (A17), and taking into account (10), we obtain

$$\begin{aligned} (U_2^{H*})' &= \frac{(c_2^{H*})'}{c_2^{H*}} + \beta_H \frac{(c_2^{H*})' + (b_2^{H*})'}{c_2^{H*} + b_2^{H*}} = \frac{b_2^{H*}}{c_2^{H*} + b_2^{H*}} \left(\frac{(c_2^{H*})'}{c_2^{H*}} \left[1 + (1+\beta_H) \frac{c_2^{H*}}{b_2^{H*}} \right] + \beta_H \frac{(b_2^{H*})'}{b_2^{H*}} \right) \\ &= -\frac{1}{\pi} \frac{b_2^{H*}}{c_2^{H*} + b_2^{H*}} \left[\frac{1}{\beta_H} + \frac{1+(1-\pi)(1+\beta_H)}{1+\pi\gamma} \right] < 0 \end{aligned}$$

Hence $(U_2^{H*})'(\pi) < 0$ for any $\pi \geq \bar{\pi}$.

Summing up the results for both steady states, we conclude the following.

(1) Less altruistic agents.

(1.1) Suppose that $\delta < \min\{\alpha, \check{\delta}\}$. Since $\delta < \alpha$, we have $\bar{\pi} > 1$, and hence only the steady state where both types of agents leave positive bequests is possible, so the utility is given by $U^{L*} = U_1^{L*}$. Since $\delta < \check{\delta}$, $(U_1^{L*})'(\pi) < 0$ for all π , and therefore U^{L*} strictly decreases with π for all π .

(1.2) Suppose that $\min\{\alpha, \check{\delta}\} < \delta < \min\{1, \hat{\delta}\}$. When $\check{\delta} < \delta < \alpha$, the utility is given by $U^{L*} = U_1^{L*}$ for all π , while when $\alpha < \delta < 1$, the utility is given by $U^{L*} = U_1^{L*}$ for $\pi < \bar{\pi}$, and $U^{L*} = U_2^{L*}$ for $\pi \geq \bar{\pi}$. In both cases, there is a threshold $\check{\pi}$ defined as follows:

$$\check{\pi} \equiv \bar{\pi} \quad \text{if} \quad \alpha \leq \bar{\alpha} \quad \text{and} \quad \check{\pi} \equiv \hat{\pi} < \bar{\pi} \quad \text{if} \quad \alpha > \bar{\alpha}$$

such that U^{L*} is continuous at $\pi = \check{\pi}$, strictly decreases with π for all $\pi < \check{\pi}$, and strictly increases with π for all $\pi > \check{\pi}$.

(1.3) Suppose that $\delta > \min\{1, \hat{\delta}\}$. When $\hat{\delta} < \delta < 1$, the utility is given by $U^{L*} = U_1^{L*}$ for $\pi < \bar{\pi}$, and $U^{L*} = U_2^{L*}$ for $\pi \geq \bar{\pi}$, while when $\delta > 1$, the utility is given by $U^{L*} = U_2^{L*}$ for all π . Since $\delta > \hat{\delta}$, we have $(U_1^{L*})'(\pi) > 0$, and it is always the case that $(U_2^{L*})'(\pi) > 0$. Therefore, U^{L*} strictly increases with π for all π .

(2) More altruistic agents.

In all possible cases, we have $(U_1^{H*})'(\pi) < 0$, and $(U_2^{H*})'(\pi) < 0$, and, therefore, U^{H*} strictly decreases with π for all π .

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