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GAPS IN DYNAMICAL DEGREES FOR ENDOMORPHISMS AND RATIONAL MAPS

SERGE CANTAT

ABSTRACT. We study the ratio of dynamical degrees $\lambda_1(f)^2/\lambda_2(f)$ for regular, dominant endomorphisms of smooth complex projective surfaces, and obtain a gap property: for $\lambda_2(f) \leq D$, there is a uniform $\varepsilon(D) > 0$ such that this ratio is never contained in $]1, 1 + \varepsilon(D)[$. The proof is a simple variation on the main theorems of [6].

1. DYNAMICAL DEGREES

Let f be a dominant rational transformation of a smooth complex projective variety X . Let m denote the dimension of X . Let H be a hyperplane section of X . The dynamical degrees $\lambda_k(f)$ are defined for each dimension $0 \leq k \leq m$ by the following limits

$$\lambda_k(f) = \lim_{n \rightarrow +\infty} \left(((f^n)^* H^k) \cdot (H^{n-k}) \right)^{1/n} \quad (1.1)$$

where (\cdot) denotes the intersection product and $H^k = H \cdot H \cdots H$ (with k factors H). Thus, $\lambda_0(f) = 1$ and $\lambda_m(f)$ is the topological degree $\deg_{top}(f)$ (a positive integer since f is dominant). The sequence $k \mapsto \lambda_k(f)$ is log-concave, i.e.

$$\lambda_{k-1}(f)\lambda_{k+1}(f) \leq \lambda_k(f)^2 \quad (1.2)$$

for all $0 < k < m$. In particular, $\lambda_m(f)^{k/m} \leq \lambda_k(f) \leq \lambda_1(f)^k$. This proves the following well known result.

Theorem A. *There is a uniform lower bound*

$$\lambda_k(f) \geq \lambda_m(f)^{k/m} \geq 2^{k/m} > 1 \quad (\forall 1 \leq k \leq m)$$

for every variety X of dimension m and every dominant rational transformation f of X with topological degree $\lambda_m(f) > 1$.

For instance, when f is an endomorphism of the projective space \mathbb{P}^m defined by polynomial formulas of degree d , one gets $\lambda_k(f) = d^k$ and the previous inequality is indeed an equality. This paper discusses whether a further uniform gap

$\lambda_1(f)^m \geq \lambda_m(f)(1 + \varepsilon)$ is satisfied for maps with $\lambda_1(f)^m > \lambda_m(f)$. We focus on the first interesting case, that is when X is a surface.

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2. GAPS FOR SURFACES ?

When $\dim(X) = 2$, one gets $\lambda_1(f)^2 \geq \lambda_2(f)$. If $\lambda_2(f) = 1$, i.e. if f is a birational map of X , then either $\lambda_1(f) = 1 = \lambda_2(f)$, or $\lambda_1(f) \geq \lambda_L$, where λ_L is the Lehmer number: this is an important consequence of [5] proven in [1]. This inequality may be considered as a gap for dynamical degrees, since $\lambda_L \simeq 1.17628 > 1$.

With the Inequalities (1.2) in mind, one would like to compute the infimum $R(D)$ of $\lambda_1(f)^2/\lambda_2(f)$ over all dominant rational maps of a given surface X (resp. of any surface) with a given topological degree $\lambda_2(f) = D$ and a first dynamical degree $\lambda_1(f) > \sqrt{\lambda_2(f)}$. A less precise question is the following.

Question.– Fix an integer $D \geq 2$. Does there exist a constant $\varepsilon(D) > 0$ such that

$$\lambda_1(f)^2 \geq D(1 + \varepsilon(D)) \quad (2.1)$$

for all dominant rational maps of surfaces with $\lambda_2(f) = D$ and $\lambda_1(f)^2 > \lambda_2(f)$?

If the answer is positive for some family of rational maps, we say that this family satisfies the gap property for λ_1 . Theorem B below provides such a gap for *regular endomorphisms* of smooth complex projective surfaces.

3. MONOMIAL MAPS

Consider a monomial map $f: (x, y) \mapsto (\alpha x^a y^b, \beta x^c y^d)$, viewed as a rational transformation of the projective plane. Set $\tau = a + d$ and $\delta = ad - bc$, the trace and determinant of the 2×2 matrix

$$A_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.1)$$

associated to f . Then $|\delta| = \lambda_2(f)$, and the spectral radius of A_f is $\lambda_1(f)$; changing A_f into $-A_f$ does not change the dynamical degrees, so we assume $\tau \geq 0$. The characteristic polynomial of A_f is $\chi(t) = t^2 - \tau t + \delta$. If its eigenvalues are complex conjugate, then $\lambda_1^2(f) = |\delta| = \lambda_2(f)$. So, we now assume that χ has two real roots. The largest one is $\lambda_1(f) = \frac{1}{2} \left(\tau + \sqrt{\tau^2 - 4\delta} \right)$; it satisfies

$$\lambda_1(f)^2 = \frac{1}{2} \left(\tau^2 - 2\delta + \tau \sqrt{\tau^2 - 4\delta} \right). \quad (3.2)$$

Thus, with $a = d$ and $b = c = 1$ we obtain $\tau = 2a$, $\delta = a^2 - 1$, and

$$\frac{\lambda_1(f)^2}{\lambda_2(f)} = \frac{a+1}{a-1}. \quad (3.3)$$

As $a \rightarrow +\infty$, the limit is 1. Thus, if $D = |\delta|$ is not fixed there is no gap for λ_1 .

Now, if $D = |\delta|$ is fixed there is a gap:

Proposition 3.1. *Let D be an integer ≥ 1 . If $f: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a dominant monomial map with $\lambda_2(f) = D$ and $\lambda_1(f)^2 > \lambda_2(f)$, the ratio $\lambda_1(f)^2/\lambda_2(f)$ is bounded from below by $1 + (2D)^{-1}$.*

Proof. As explained above, we may assume $\tau \geq 0$. Since the eigenvalues are distinct, $\tau \neq 0$ and $\tau^2 - 4\delta \geq 1$; hence, $\tau \geq 1$. Using that $\lambda_2(f) = |\delta|$ and Equation (3.2), the lower bound $\lambda_1(f)^2/\lambda_2(f) \geq 1 + (2D)^{-1}$ is equivalent to

$$\tau^2 - 2\delta + \tau\sqrt{\tau^2 - 4\delta} \geq 2|\delta| + 1. \quad (3.4)$$

If $\delta < 0$, this follows from $\tau \geq 1$. If $\delta > 0$, we denote by α and β the two eigenvalues of A_f , and we remark that (3.4) is equivalent to

$$(\alpha - \beta)^2 + \tau\sqrt{\tau^2 - 4\delta} \geq 1. \quad (3.5)$$

This is always satisfied, because τ and δ are integers, and $\tau^2 - 4\delta \geq 1$. \square

Remark 3.2. Similar examples can be obtained on abelian surfaces. For instance, for any elliptic curve E , one gets linear endomorphisms of $X = E \times E$ with $\lambda_1(f)^2/\lambda_2(f) > 1$ but arbitrary close to 1.

4. REGULAR ENDOMORPHISMS

Let us look at regular endomorphisms of projective surfaces. As explained in Section 3, we need to fix $\lambda_2(f)$ to some value D in order to get a gap.

Theorem B. *Let D be a positive integer. There is a positive real number $\varepsilon(D)$ such that*

$$\frac{\lambda_1(f)^2}{\lambda_2(f)} \geq 1 + \varepsilon(D)$$

for every smooth complex projective surface X and every dominant endomorphism f of X with $\lambda_2(f) = D$ and $\lambda_1(f)^2 > \lambda_2(f)$.

The proof occupies the rest of this section. So, f will denote a dominant endomorphism of a smooth projective surface X . Since Theorem B is known for $D = 1$, we shall always assume $2 \leq \lambda_2(f) \leq D$ for some fixed integer D .

The main arguments, described in § 4.2, are taken from the very nice paper [6] of Noboru Nakayama (see also the companion paper [4]). For all necessary results on dynamical degree, we refer to [2].

4.1. When a bound on $\rho(X)$ is satisfied. Since f is dominant, f_*f^* is the multiplication by $\lambda_2(f)$, so f_* and f^* are isomorphisms of the Néron-Severi group $\text{NS}(X; \mathbf{Q})$. The dynamical degree $\lambda_1(f)$ is the spectral radius of f^* on $\text{NS}(X; \mathbf{R})$ and is the largest eigenvalue of f^* on $\text{NS}(X; \mathbf{R})$; as such, it is an algebraic integer.

Let $\rho(X)$ denote the Picard number of X , i.e. $\rho(X) = \dim_{\mathbf{Q}} \text{NS}(X; \mathbf{Q})$.

Lemma 4.1. *Let D and R be positive integers > 1 . There is a positive real number $\varepsilon(R, D)$ such that $\lambda_1(f)^2/\lambda_2(f) > 1 + \varepsilon(R, D)$ for every smooth projective surface X and every regular endomorphism f of X such that $\rho(X) \leq R$, $\lambda_2(f) \leq D$, and $\lambda_1(f)^2 > \lambda_2(f)$.*

Proof. We can assume $\lambda_1(f)^2 > \lambda_2(f)$ and $\lambda_1(f)^2/\lambda_2(f) \leq 2$. Thus, $\lambda_1(f)$ is bounded from above by $\sqrt{2D}$; since $\lambda_1(f)$ is the spectral radius of f^* , all eigenvalues of f^* on $\text{NS}(X; \mathbf{C})$ have modulus $\leq \sqrt{2D}$. So, the characteristic polynomial χ_{f^*} of $f^*: \text{NS}(X; \mathbf{Z}) \rightarrow \text{NS}(X; \mathbf{Z})$ is a polynomial with integer coefficients, of degree R , the coefficients of which are bounded from above by $C(R)(\sqrt{2D})^R$ for some constant $C(R)$. This gives only finitely many possibilities for χ_{f^*} , and the result follows. \square

4.2. Orbits of negative curves, following Nakayama. Consider the set $\text{Neg}(X)$ of irreducible curves $C \subset X$ with $C^2 < 0$ (negative curves).

Pick $C \in \text{Neg}(X)$ and set $C_1 = f(C)$; let $a > 0$ be the integer such that $f_*(C) = aC_1$ (a is the degree of f along C). If C' is another irreducible curve such that $f_*(C') = a'C_1$ for some $a' > 0$ then $aC' = a'C$ in $\text{NS}(X; \mathbf{Q})$ because f_* is injective. This implies that $C' = C$ because $C^2 < 0$ and C and C' are irreducible and reduced. Thus, $f^*C_1 = bC$ with $ab = \lambda_2(f)$; together with

$$f^*(C_1) \cdot C = bC \cdot C = C_1 \cdot f_*(C) = C_1 \cdot (aC_1) \quad (4.1)$$

this implies

$$ab = \lambda_2(f) \quad \text{and} \quad C_1^2 = (b/a)C^2 < 0. \quad (4.2)$$

In particular, f_* permutes the irreducible curves $C \subset X$ with negative self-intersection. This set of curves is, a priori, infinite, but we have

Lemma 4.2 (Nakayama, see Lem. 10 and Pro. 11 in [6]). *Let $R(f)$ be the ramification divisor of f . Let $\text{Neg}(X; R(f))$ be the set of irreducible components of $R(f)$ with negative self-intersection. Let C be an element of $\text{Neg}(X)$.*

(1) *There is an integer $0 \leq m \leq \log(|C^2|)$ such that $f^m(C) \in \text{Neg}(X; R(f))$.*

- (2) If $C^2 = -1$ then $C \in \text{Neg}(X; R(f))$, $f(C)^2 \leq -\lambda_2(f)$, and $f^m(C) \in R(f)$ for a positive integer $m \leq \log(\lambda_2(f))$.
- (3) The set $\text{Neg}(X)$ is finite.
- (4) There is an integer $N > 0$ such that $f^N(C) = C$ for every C in $\text{Neg}(X)$ and $\text{Neg}(X) = \text{Neg}(X; R(f^N))$.

Proof. It suffices to prove (1). With the above notation, the condition $C \subset R(f)$ is equivalent to $b \geq 2$. On the other hand, $b = 1$ means $a = \lambda_2(f)$; and then $C_1^2 = \lambda_2(f)^{-1}C^2 > C^2$, hence $f^m(C) \subset R(f)$ for an $m \leq \log(-C^2)/\log(a)$. \square

This lemma shows that one can contract a sequence of (-1) -curves in an f^N -equivariant way to reach a minimal model of X :

Theorem 4.3 (Nakayama). *If f is an endomorphism of a smooth projective surface X , there is an integer $N > 0$, a birational morphism $\pi: X \rightarrow X_0$ onto a minimal model X_0 of X , and an endomorphism f_0 of X_0 such that $\pi \circ f^N = f_0 \circ \pi$.*

Remark 4.4. Since the dynamical degrees are invariant under birational conjugacy, we have

$$\frac{\lambda_1(f)^2}{\lambda_2(f)} = \left(\frac{\lambda_1(f_0)^2}{\lambda_2(f_0)} \right)^{1/N}; \quad (4.3)$$

without any control on N , one can not deduce a gap for f from a gap for f_0 . But if N and $\rho(X_0)$ are bounded, then we automatically get a gap from Theorem 4.3 and Lemma 4.1. So, we shall either control N and $\rho(X_0)$, and for this we follow closely [6], or reduce the computation to the case of monomial maps (with the same value of D).

Let us replace f by $g := f^N$ to assume that $\text{Neg}(X) = \text{Neg}(X; R(g))$ and that g fixes each irreducible curve $C \subset R(g)$. From Equations (4.1) and (4.2) we obtain

- (1) $\lambda_2(g)$ is a square: there is an integer $a_g > 0$ such that $a_g^2 = \deg_{\text{top}}(g)$;
- (2) $g^*(C) = g_*(C) = a_g C$;
- (3) the multiplicity of C in $R(g)$ is $a_g - 1$.

Thus, if we set

$$N_X = \sum_{C \in \text{Neg}(X)} C, \quad (4.4)$$

we can write $R(g) = (a_g - 1)N_X + R^+(g)$ for some effective divisor $R^+(g)$, the components of which have non-negative self-intersection. With such a notation at hand, the main ingredient towards Theorem B is the following lemma.

Lemma 4.5 (Nakayama, Lem. 13 of [6]).

- (1) Let C be an element of $\text{Neg}(X)$. The arithmetic genus of C is ≤ 1 , and if it is equal to 1 then C is a connected component of the support of $R(g)$ (hence also of N_X), where $g = f^N$.
- (2) A connected component of the support of N_X is an irreducible curve, or a chain of rational curves, or a cycle of rational curves.

4.3. Rational surfaces. Assume that X is rational. We follow the proof of Theorem 17 in [6]. If $\rho(X) \leq 3$, § 4.1 shows that the endomorphisms of X satisfy a gap for λ_1 . Thus, we assume that $\rho(X) \geq 4$. Since X is the blow-up of a minimal rational surface (the plane, the quadric, or a Hirzebruch surface), there is a fibration $\pi: X \rightarrow B$ such that

- (i) B is the projective line \mathbb{P}^1 and the generic fiber of π is a projective line;
- (ii) there is at least one singular fiber F ;
- (iii) every singular fiber is a tree of smooth rational curves with negative self-intersection;
- (iv) there is at least one section S of π with self-intersection $S^2 < 0$.

Since X admits an endomorphism with $\lambda_2(f) > 1$, we also know that

- (iii') every singular fiber is a chain of smooth rational curves with negative self-intersections.

Indeed, such a fiber is entirely contained in N_X . Since S is also contained in N_X , we see that N_X is connected and contains at least three irreducible components. Thus, Lemma 4.5 implies that (iii') holds and that

- (v) π has at most 2 singular fibers and N_X is connected and is either a chain or a cycle of rational curves.

Case of a chain.— Assume that N_X is a chain of rational curves. Either f or f^2 fixes each irreducible component of N_X (because $f(C) \cap f(C') = f(C \cap C')$). Thus, when contracting (-1) -curves, we can do it f^2 -equivariantly up to a minimal model of X . Since a minimal rational surface satisfies $\rho(X_0) = 2$, the gap follows from Remark 4.4.

Case of a cycle.— Now, assume that N_X is a cycle of rational curves. There are two possibilities :

- (1) N_X is the union of two singular fibers F and F' and two sections S and S' ;
- (2) N_X is the union of the unique singular fiber F of π and two sections S and S' , with $S \cap S' = \{p\}$ for some $p \notin F$.

In the first case, we can contract (-1) -curves contained in the two singular fibers to reach a minimal model $\eta: X \rightarrow X_0$ on which $g := f^N$ induces an endomorphism g_0 and $\pi: X \rightarrow B$ induces a rational fibration $\pi_0: X_0 \rightarrow B$ such that

- $F_0 := \eta(F)$, $F'_0 := \eta(F')$ are two (smooth) fibers of π_0 , $S_0 := \eta(S)$ and $S'_0 := \eta(S')$ are two sections of π_0 ;
- f induces a rational transformation f_0 of X_0 such that $f|_{X_0 \setminus R(g_0)}$ is regular and $f_0^N = g_0$;
- $F_0 \cup F'_0 \cup S_0 \cup S'_0$ is f_0 -invariant and coincides with $R(g_0)$.

Then, the complement of $R(g_0)$ in X_0 is a torus $T \simeq \mathbb{G}_m \times \mathbb{G}_m$ on which f_0 and g_0 act as regular endomorphisms. The restriction of f_0 to $T \simeq \mathbb{G}_m \times \mathbb{G}_m \simeq \mathbb{C}^\times \times \mathbb{C}^\times$ is monomial: one can find integers a, b, c, d and elements α, β in \mathbb{C}^* such that $f_0(x, y) = (\alpha x^a y^b, \beta x^c y^d)$. From Section 3, we know that such transformations satisfy the gap property for λ_1 .

Let us show that the second case does not occur. We shall need the following:

Lemma 4.6 (Lemma 16 of [6]). *Let $U = \sum_{i=1}^k a_i C_i$ be an effective divisor on a smooth projective surface such that $a_i > 0$ for $1 \leq i \leq k$ and the C_i form a chain of smooth rational curves starting with C_1 and ending with C_k . If*

$$K_X \cdot U + 2 = 0 \quad \text{and} \quad U \cdot C_i = 0 \quad \text{for all } i$$

then $a_1 = a_k = 1$, $C_i^2 = -1$ for some $i < k$ and $C_j^2 = -1$ for some $j > 1$.

Let F be the singular fiber of π . Then $F = \sum_i a_i C_i$ for a chain of rational curves C_i , and moving F to a nearby smooth fiber we see that $F \cdot C_i = 0$ for each i , and $K_X \cdot F = -2$. Thus, Lemma 4.6 can be applied to $U = F$.

First, one applies this lemma to contract a (-1) -curve contained in F that does not intersect S' ; then we repeat this step until we reach a model X_1 of X in which the image S_1 of S satisfies $S_1^2 = 0$. This is always possible, at least after permutation of S and S' , since otherwise we would reach a minimal model X_0 with two sections of negative self-intersection, but no such surface exists.

Then, one applies Lemma 4.6 to contract (-1) curves of the singular fiber that do not intersect S_1 in order to reach a relatively minimal model X_0 of X in which S becomes a section S_0 with $S_0^2 = 0$ and S' provides a second section S'_0 .

The existence of a section S_0 with self-intersection 0 implies that X_0 is $\mathbb{P}^1 \times \mathbb{P}^1$. Since S'_0 intersects S_0 , $(S'_0)^2 \geq 2$. By construction, $q_0 := F_0 \cap S'_0$ is not contained in S_0 . Consider the section containing q_0 which is horizontal, i.e. linearly equivalent to S_0 . This section is not S_0 and, its self-intersection being 0, it is not equal to S'_0 either. Its proper transform in X is a negative curve; this proper transform should be in $\text{Neg}(X)$, and we get a contradiction.

4.4. Ruled surfaces. If X is ruled but not rational, the Albanese map $\alpha: X \rightarrow B$ is a surjective morphism onto a curve B of genus ≥ 1 ⁽¹⁾. There is an endomorphism f_B of B such that $\alpha \circ f = f_B \circ \alpha$; in particular, each fiber $X_b := \alpha^{-1}(b)$ is mapped to the fiber $X_{f_B(b)}$ by f . Then $\lambda_1(f_B)$ is an integer, the topological degree δ of $f|_{X_b}: X_b \rightarrow X_{f_B(b)}$ for a generic point $b \in B$ is also an integer, and we have

$$\lambda_1(f) = \max\{\lambda_1(f_B), \delta\} \quad \text{and} \quad \lambda_2(f) = \lambda_1(f_B)\delta; \quad (4.5)$$

see [3] for the general setting of rational maps permuting the fibers of a fibration. Thus, we obtain the gap property for $\lambda_1(f)$ with $\varepsilon(D) = \frac{1}{D-1}$, i.e. $\lambda_1(f)^2 \geq \lambda_2(f)(1 + \frac{1}{D-1})$ if $\lambda_2(f) \leq D$ and $\lambda_1(f)^2 > \lambda_2(f)$.

4.5. Surfaces with non-negative Kodaira dimension. Assume that $\text{kod}(X) \geq 0$. Since every dominant rational transformation of a surface X of general type is a birational transformation of finite order, we have $\text{kod}(X) \in \{0, 1\}$.

When $\text{kod}(X) = 1$ the Kodaira-Iitaka fibration $\Phi: X \dashrightarrow B$ maps X onto a smooth curve B and there is an automorphism f_B of B such that $\Phi \circ f = f_B \circ \Phi$; by a superb theorem of Noboru Nakayama and De-Qi Zhang, f_B has finite order (see [7]). Then, one easily shows that $\lambda_1(f) = \lambda_2(f)$. In particular, $\lambda_1(f)^2 = \lambda_2(f)^2$ and we have a gap property as in Theorem B with $\varepsilon(D) = D - 1$.

When $\text{kod}(X) = 0$, the unique minimal model X_0 of X must be a torus, a hyperelliptic surface, an Enriques or a K3 surface. Up to multiplication by an element of \mathbf{C}^\times , there is a unique non-zero section Ω of K_X , $f^*\Omega = \delta\Omega$ with $\delta^2 = \lambda_2(f)$, and the exceptional locus of the birational morphism $\pi: X \rightarrow X_0$ is the zero locus of Ω . Thus, f preserves this locus and induces a regular endomorphism of X_0 . Since K3 and Enriques surfaces do not admit endomorphisms with $\lambda_2(f) > 1$, we have $\rho(X_0) \leq 6$ (it would be ≤ 22 for K3 surfaces). Thus, the gap property follows when $\text{kod}(X) = 0$.

4.6. Conclusion. The last three subsections establish the gap property when X is rational, when X is ruled but not rational, and when $\text{kod}(X) \geq 0$. From the classification of surfaces, this covers all possible cases, and Theorem B is proven.

5. FINAL COMMENTS

5.1. It would be nice to determine the infimum of $\lambda_1(f)^2/\lambda_2(f)$ for dominant endomorphisms of complex projective surfaces with a fixed $\lambda_2(f) = D$, say for $D = 2, 3, 4$. The proof of Theorem B shows that this is a tractable problem.

¹Moreover, by a theorem of M. Segami, α endows X with the structure of a \mathbb{P}^1 -bundle, i.e. X is ruled and the ruling is relatively minimal (see Pro. 14 of [6]).

5.2. It seems reasonable to expect that Theorem B extends to projective surfaces over fields of positive characteristic, and to singular surfaces too.

5.3. As explained in § 2, the natural question is to decide whether a similar gap property holds for rational transformations of surfaces. This question was originally asked by Curtis T. McMullen, who also suggested Theorem B in a private communication. The difficult case is the one of rational transformations of the projective plane. I don't know what to expect in this more general context (see [1] for birational maps).

5.4. One can also ask similar questions for any fixed pair $(\dim(X), \deg_{top}(f)) = (m, D)$, the first ratios to consider being $\lambda_1(f)^m / \deg_{top}(f)$ and $\lambda_1^2 / \lambda_2(f)$.

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