Kawasaki dynamics beyond the uniqueness threshold

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Abstract

Glauber dynamics of the Ising model on a random regular graph is known to mix fast below the tree uniqueness threshold and exponentially slowly above it. We show that Kawasaki dynamics of the canonical ferromagnetic Ising model on a random d-regular graph mixes fast beyond the tree uniqueness threshold when d is large enough (and conjecture that it mixes fast up to the tree reconstruction threshold for all $d \geqslant 3$). This result follows from a more general spectral condition for (modified) log-Sobolev inequalities for conservative dynamics of Ising models. The proof of this condition in fact extends to perturbations of distributions with log-concave generating polynomial.

1 Introduction and results

We first discuss our results for Kawasaki dynamics on random regular graphs, which are our primary motivation, in Section 1.1, and then the more general results for conservative dynamics of Ising models that these follow from, in Section 1.2.

1.1. Result for random regular graphs. Let $G = ([N], E)$ be a random d-regular graph on N vertices. Glauber dynamics of the ferromagnetic Ising model on G is known to mix fast when $\beta < \beta_c$ and exponentially slowly when $\beta > \beta_c$, where $\beta_c = \operatorname{artanh}(1/(d-1)) \sim 1/(d-1)$ is the uniqueness threshold for the Ising model on the infinite d-regular tree, see [38] for fast mixing (or [8, Example 6.15] for a perspective closer to that of this work) and [24, 29] for slow mixing. Different from the situation on finite-dimensional lattices, the Ising model on the infinite d-regular tree has a second phase transition which occurs at the reconstruction threshold $\beta_r = \operatorname{artanh}(1/\sqrt{d-1}) \sim 1/\sqrt{d-1}$, see for example [32] and the further discussion below. While the uniqueness transition of the ferromagnetic Ising model is related to the magnetisation (the average of all spins), which is an order parameter for this transition on random regular graphs, the magnetisation becomes irrelevant for the canonical Ising model, which is the Ising model conditioned on its magnetisation. The natural analogue of Glauber dynamics for the canonical Ising model is Kawasaki dynamics, which randomly swaps neighbouring spins, and thus conserves the total magnetisation.

Our following main result shows that, for $d \geqslant d_0$, Kawasaki dynamics of the ferromagnetic Ising model mixes fast on a random d-regular graph beyond the tree uniqueness threshold, at which Glauber dynamics slows down exponentially.

Theorem 1.1. *For $d \geqslant 3$ and $\beta < 1/(8\sqrt{d-1})$, the Kawasaki dynamics of the canonical Ising model on a random d-regular graph on N vertices (quenched, and ferromagnetic or antiferromagnetic) mixes in $O_{d,\beta}(N \log^6 N)$ steps, with high probability on the randomness of the graph.*

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In particular, since $\beta_c = \operatorname{artanh}(1/(d-1)) < 1/(8\sqrt{d-1})$ for $d \geqslant d_0$ large, it follows that indeed Kawasaki dynamics is fast beyond β_c. Our result also applies under a more general condition which we expect (but do not prove) holds for $\beta < \beta_r$, see the later discussion. Further we expect that the power of the logarithm is 1 instead of 6 in the optimal mixing time estimate. As part of the proof, we obtain this mixing time for a dynamics in which all exchanges of spins are permitted rather than only those of neighbours.

Interpretation of the result. Theorem 1.1 shows in some sense that, on random regular graphs, the magnetisation is the *only* bottleneck responsible for the critical slowdown at the uniqueness threshold β_c (at least for $d \geqslant d_0$). From the point of view of finite-dimensional lattices, where already at high temperatures Kawasaki dynamics is slower (and much more difficult to understand) than Glauber dynamics (with spectral gap of order L^{-2} on a d-dimensional cube of side length L, for all $d \geqslant 1$, proved in [34] and [19,20,44]), it might initially seem surprising that Kawasaki dynamics on random regular graphs can be much *faster* than Glauber dynamics. Furthermore, on finite-dimensional lattices, it is expected that Kawasaki dynamics slows down at the same critical temperature as Glauber dynamics (the unique critical temperature of the phase transition of the equilibrium measure); this is known at least in $d = 2$ [18]. Both points, that the critical β for slow down of Kawasaki dynamics is higher than that for Glauber dynamics, and that Kawasaki dynamics is fast (spectral gap of order 1) at small β, are related to the expansion properties of random regular graphs (and will be true on more general expander graphs). Indeed, that the magnetisation is the only bottleneck for random regular graphs beyond the uniqueness threshold is less surprising when compared with the behaviour of the Ising model on the complete graph, where at any temperature Kawasaki dynamics is simply the symmetric simple exclusion process. That Kawasaki dynamics on random regular graphs is not slowed down at high temperature is also intuitive: the slow behaviour of Kawasaki dynamics on finite-dimensional lattices results from the diffusive behaviour of the spin motion (viewing spins as particles and holes), while a simple random walk on a random regular graph is ballistic (and thus fast). From a different point of view, the two thresholds $\beta_c \sim 1/d$ and $\beta_r \sim 1/\sqrt{d}$ are parallel to the two largest eigenvalues $\lambda_1 = d$ and $\lambda_2 \sim 2\sqrt{d}$ of the adjacency matrix of the random regular graph and the existence of a second threshold is the analogue of random regular graphs being expanders.

Nonetheless, the hard constraint on the magnetisation is well known to make Kawasaki dynamics significantly more difficult to study than Glauber dynamics. Moreover, the established approaches to Kawasaki dynamics on lattices are based on comparison with the Ising model without constraint through mixing conditions [19,20,34,44], and one therefore cannot hope that these methods extend beyond the uniqueness threshold where the canonical and unconstrained models behave differently. Our perspective also provides a different approach to Kawasaki dynamics on lattices.

About the condition on β. The conclusion of Theorem 1.1 applies either under the stated assumption $\beta < 1/(8\sqrt{d-1})$, or under a more general condition on the covariance matrices of the canonical Ising model stated in (CC) below. We expect but do not prove that the latter condition holds for all β up to the reconstruction threshold β_r which we also expect to be a unique critical point for the canonical Ising model on the random regular graph, but are not aware of results that show this, see also the discussion below. A proof that Kawasaki dynamics remains fast up to β_r (either by establishing the covariance condition or in another way) would be very interesting.

Under essentially the same assumption as ours, namely $\beta < 1/(C\sqrt{d-1})$, and with a somewhat related motivation, it was recently shown that the Ising model on the infinite d-regular tree is a factor of IID [39].

About the canonical Ising model. Ising models on random regular graphs have been studied extensively, mostly due to their motivation in the spin glass context [36]. This motivation is even more relevant for the canonical Ising model as the constraint on the magnetisation can lead to frustration

while the randomness of the graph produces disorder, and indeed, it is expected that the canonical Ising model displays spin-glass like behaviour at low temperatures. In support of this, it has been proved that in the limit $d \to \infty$ and $\beta \to \infty$ the quenched free energy of the canonical Ising model (with magnetisation 0) coincides with that of the Sherrington–Kirkpatrik model given by the Parisi formula [25]. More generally, for finite β we expect that similar considerations show that the quenched free energy F_β of the canonical Ising model with $m = 0$ satisfies, as $d \to \infty$ with $\beta/\sqrt{d-1}$ of order 1 in our normalisation (discussed below),

$$F_{\beta/\sqrt{d-1}} = \mathsf{P}_\beta + O_{\beta/\sqrt{d-1}}(\frac{1}{\sqrt{d}}), \tag{1.1}$$

where P_β is the free energy of the Sherrington–Kirkpatrik model given by the Parisi solution. In particular, P_β is given by the replica symmetric solution for $\beta < 1$ and nontrivial for $\beta > 1$. Thus it seems natural to expect that the canonical Ising model has a related unique transition at $\beta/\sqrt{d-1} = \beta_r/\sqrt{d-1} \sim 1$, but we are not aware of results proving this. It would be especially interesting to construct a test function that shows that Kawasaki dynamics indeed becomes slow beyond this transition. The vanishing of the spectral gap of Glauber dynamics of the Sherrington–Kirkpatrik model at low temperature was proved in [10].

The above picture for the canonical Ising model on random regular graphs is different from that of the standard Ising model (without constraint of the magnetisation), whose behaviour we briefly summarise now. As a consequence of the local convergence of the random d-regular graph to the infinite d-regular tree, it has been shown at any temperature that the Ising model on a random regular graph converges weakly to the symmetric mixture of the plus and minus states of the Ising model on the infinite d-regular tree and further that the Ising model conditioned on positive magnetisation converges to the plus state on the infinite d-regular tree [37]. However, while the local correlations of the Ising model on the random regular graph are closely related to those of the infinite tree, its global behaviour is more subtle. It was shown [29] that the reconstruction problem for the Ising model on the random d-regular graph is solvable for $\beta > \beta_c = \operatorname{artanh}(1/(d-1))$, the tree uniqueness threshold, whereas the reconstruction threshold on the infinite d-regular tree is $\beta_r = \operatorname{artanh}(1/\sqrt{d-1})$. The solution of the reconstruction problem for the unconditioned Ising model on random d-regular graph is mediated through the magnetisation and its solvability amounts to showing that the latter is a bottleneck for Glauber dynamics. Therefore, for the unconditioned Ising model on a random regular graph, there is a standard ferromagnetic phase transition at the tree uniqueness threshold β_c (and this transition is well understood).

Further related literature. Glauber dynamics of Ising models on finite and infinite trees has been studied in detail [11, 35], but we emphasise again that the global behaviour on finite trees is quite different from that of random regular graphs. Results on Glauber dynamics of random regular graphs include [29, 38] which establish fast and slow mixing below respectively above the tree uniqueness threshold β_c and also [17] which investigates metastability properties. Swenden–Wang and block dynamics are also known to mix fast up to the tree uniqueness threshold [12].

For the spin-glass Ising model on a random regular graph, where each edge has an independent ± 1 coupling constant (but without the constraint on the magnetisation), it is known that Glauber dynamics mixes fast under the condition $\beta < 1/(4\sqrt{d-1})$, and again fast mixing is expected up to β_r which is, in fact, also the critical point for the spin glass Ising model on the infinite d-regular tree [21]. The fast mixing of Glauber dynamics for $\beta < 1/(4\sqrt{d-1})$ was observed in [27] as a consequence of the recent spectral criteria for the Glauber dynamics log-Sobolev inequality [6], spectral gap [27], or the modified log-Sobolev inequality [3], and is analogous to the simple criterion $\beta < 1/4$ for the Sherrington–Kirkpatrik model from [6]. Recent advances in sampling the Sherrington–Kirkpatrik model up to $\beta < 1$ (but not using Glauber dynamics) include [1].

1.2. General results for conservative dynamics. Let A be a symmetric matrix with constant eigenvector $\mathbf{1} = (1, \dots, 1)$, i.e., $A\mathbf{1} = \lambda_1 \mathbf{1}$, and remaining eigenvalues $\lambda_2 \leqslant \cdots \leqslant \lambda_N$. We write $\delta_\lambda = \lambda_N - \lambda_2$ for the length of the support of the nontrivial spectrum.[1] The canonical Ising measure with coupling matrix $A = (A_{ij})$ and external field $h = (h_i)$ is defined by

$$\mathbb{E}_\nu[F] \propto \sum_{\sigma \in \Omega_{N,m}} e^{-\frac{\beta}{2}(\sigma, A\sigma) + (h,\sigma)} F(\sigma), \tag{1.2}$$

where, for $m \in [-1, 1]$ such that Nm is an integer,

$$\Omega_{N,m} = \{\sigma \in \{\pm 1\}^N : \sum_{i=1}^N \sigma_i = Nm\}. \tag{1.3}$$

For $\sigma \in \Omega_{N,m}$ and $i, j \in [N]$, denote by $\sigma^{ij} \in \Omega_{N,m}$ the spin configuration with σ_i and σ_j exchanged. Given any measure ν and a function $F \geqslant 0$ the variance and relative entropy are defined by

$$\mathrm{Ent}_\nu(F) = \mathbb{E}_\nu[\Phi(F)] - \Phi(\mathbb{E}_\nu[F]), \qquad \Phi(x) = x \log x, \tag{1.4}$$

$$\mathrm{Var}_\nu(F) = \mathbb{E}_\nu[\Phi(F)] - \Phi(\mathbb{E}_\nu[F]), \qquad \Phi(x) = x^2. \tag{1.5}$$

In all of our results, we assume either of the following conditions:

Spectral condition (SC). Let $\delta_\lambda = \lambda_N - \lambda_2$ be the length of the support of the nontrivial spectrum of the coupling matrix A, assume $\beta < 1/(2\delta_\lambda)$ and set

$$C_\beta = \frac{1}{1 - 2\beta\delta_\lambda}. \tag{1.6}$$

Covariance condition (CC). Let $\bar{\chi}^0(\beta)$ be an upper bound on the largest eigenvalue of the covariance matrix $(\mathrm{Cov}_{\nu_{\beta,h}}(\sigma_i; \sigma_j))_{i,j}$ of the canonical Ising model on $\Omega_{N,m}$, uniformly in $h \in \mathbb{R}^N$, and set

$$C_\beta = \exp\left[\int_0^\beta \bar{\chi}^0(t)\, dt\right]. \tag{1.7}$$

The spectral condition (SC) is easy to verify and applies at sufficiently high temperatures. The Glauber dynamics analogue of the spectral condition appeared in [6] for the log-Sobolev inequality and other functional inequalities under this condition were later considered in [3, 22, 27]. Compared to the previously existing high temperature conditions the spectral condition on the coupling matrix for Glauber dynamics covers the SK model up to $\beta < 1/4$ (which is still the best known condition for fast relaxation of Glauber dynamics for the SK model, but see [1] for a different sampling strategy up to $\beta < 1$). The Glauber version of the spectral condition involves the *smallest* eigenvalue λ_1 instead of the *second smallest* eigenvalue λ_2 here.

The more general covariance condition (CC) is implied by (SC). The Glauber dynamics analogues (in which the uniform in external field condition is automatically guaranteed as a consequence of [26]) appeared in [9] and [22], see also the review [8]. For Glauber dynamics of the Ising model it applies up to the critical point, in a very general setting, and it moreover implies a polynomial dependence of the log-Sobolev constant near the critical temperature under the mean-field bound on the susceptibility which holds on $\Lambda \subset \mathbb{Z}^d$ if $d > 5$, see [9].

[1] We will sometimes refer to λ_1 as the smallest eigenvalue and to λ_2 as the second smallest one, because this is the situation for Laplacian matrices of regular graphs, but in principle the value of λ_1 has no significance for the following.

Theorem 1.2. *Assume either (SC) or (CC), and assume further* $\max_i \sum_{j\neq i} |A_{ij}| \leqslant \bar{A}$ *and* $\max_i |h_i| \leqslant \bar{h}$. *Then uniformly in size* N *and magnetisation* m, *the canonical Ising model satisfies*

$$\mathrm{Ent}_\nu(F) \leqslant C_\beta C(\beta\bar{A}, \bar{h}) D_\nu(F, \log F), \tag{1.8}$$

where

$$D_\nu(F,G) = \frac{1}{2N} \sum_{i,j=1}^{N} \mathbb{E}_\nu\Big[(F(\sigma) - F(\sigma^{ij}))(G(\sigma) - G(\sigma^{ij}))\Big]. \tag{1.9}$$

We expect that under the assumptions of the last theorem, and in addition $m \in [-1+\varepsilon, 1-\varepsilon]$, the corresponding (unmodified) log-Sobolev inequality is also true, i.e., that one can replace $D_\nu(F, \log F)$ by $D_\nu(\sqrt{F}) = D_\nu(\sqrt{F}, \sqrt{F})$ on the right-hand side of the inequality, but unfortunately we currently do not have a proof of this. Because of the much better comparison techniques available for log-Sobolev inequalities, this would strengthen our subsequent results. However, it was recently shown in a general setting that the modified log-Sobolev inequality implies the usual log-Sobolev inequality at the cost of a factor that is $O_{\beta,h}(\log N)$ in our situation [41]. Therefore these conclusions still apply at the cost of this additional logarithmic factor.

Corollary 1.3. *Under the same assumption as in Theorem 1.2 (and with the same Dirichlet form),*

$$\mathrm{Var}_\nu(F) \leqslant C_\beta C(\beta\bar{A}, \bar{h}) D_\nu(F), \tag{1.10}$$

and

$$\mathrm{Ent}_\nu(F) \leqslant C_\beta C(\beta\bar{A}, \bar{h}) (\log N) D_\nu(\sqrt{F}). \tag{1.11}$$

The constants $C(\beta\bar{A}, \bar{h})$ in the different inequalities can be different, but C_β is always the constant in either the condition (SC) or (CC).

Proof. It is a standard fact that the modified log-Sobolev inequality implies the spectral gap inequality (1.10). The log-Sobolev inequality (1.11) with the additional factor $O_{\beta,h}(\log N)$ follows from the main result of [41] since the minimal transition probability associated with the dynamics of the standard Dirichlet form is of order $1/N$ under our assumptions on A and h. □

At the cost of a nonoptimal (but still polynomial) dependence on N, one can also obtain versions of the above inequalities that do not require boundedness assumptions on the entries of A or h, see Remark 2.3 below. For the standard Dirichlet form (1.9), the log-Sobolev inequality will not be true with constant of order 1 without restrictions on m and h, see [33, Theorem 5] which shows a logarithmic correction for m very close to ± 1. However, it is possible that there exists a choice of rates that is equivalent to that of the standard Dirichlet form when A and h have bounded entries and $m \in [-1+\varepsilon, 1+\varepsilon]$ for which this inequality is true.

The above inequalities apply to the Dirichlet form (1.9) associated with a dynamics in which exchanges for all pairs of spins are permitted (also called the Bernoulli–Laplace Dirichlet form [33]). In Kawasaki dynamics only nearest-neighbour swaps are allowed. For the usual log-Sobolev inequality and the spectral gap, the change from the Bernoulli–Laplace Dirichlet form to the nearest-neighbour Dirichlet form can be achieved by comparison inequalities, called moving particle lemmas. The next corollaries therefore follow from Corollary 1.3 and versions of the moving particle lemma (combined with simple estimates of the geometry in the case of the random regular graph).

Corollary 1.4. *Let* $d \geqslant 3$. *For the canonical nearest-neighbour Ising model on a random* d*-regular graph on* N *vertices with* $\beta \leqslant 1/(8\sqrt{d-1})$, *the inverse log-Sobolev constant of Kawasaki dynamics is bounded by* $C(d, \beta, h) \log^5 N$. *Hence Kawasaki dynamics mixes in at most* $O_{d,\beta,h}(N \log^6 N)$ *steps.*

Corollary 1.5. *Let $d \geqslant 1$, let $\Lambda \subset \mathbb{Z}^d$ be a hypercube of side length L, and assume that either condition (SC) or (CC) holds. Assume further that $\max_i \sum_{j \neq i} |A_{ij}| \leqslant \bar{A}$ and $\bar{h} = \max_i |h_i|$ are bounded, and that the finite-range condition $A_{ij} = 0$ if $\operatorname{dist}(i,j) > R$ for some $R < \infty$ holds.*

Then, uniformly in the magnetisation m, the spectral gap of Kawasaki dynamics on Λ is of order L^{-2} and the log-Sobolev constant is at least of order $L^{-2}(\log L)^{-1}$.

Proof of Corollary 1.4. For the ferromagnetic (respectively antiferromagnetic) Ising model on a random regular graph on $[N]$, the coupling matrix A is minus the adjacency matrix (respectively the adjacency matrix) of the random regular graph. Thus $-A$ has Perron–Frobenius eigenvalue d with eigenvector $\mathbf{1}$, and it is known that the remaining eigenvalues are contained in $[-2\sqrt{d-1}-\varepsilon, 2\sqrt{d-1}+\varepsilon]$ for any $\varepsilon > 0$, with probability tending to 1 as $N \to \infty$, see [28]. The spectral condition (SC) therefore holds for $\beta < 1/(8\sqrt{d-1})$, with high probability. Thus Theorem 1.2 implies the uniform modified log-Sobolev inequality (1.8) with respect to the mean-field Dirichlet form, and by Corollary 1.3, the (unmodified) log-Sobolev inequality (still with respect to the mean-field Dirichlet form) holds with an additional factor $O_{\beta,h}(\log N)$:

$$\operatorname{Ent}_\nu(F) \leqslant C(d, \beta\bar{A}, \bar{h}) \frac{\log N}{N} \sum_{i,j} \mathbb{E}_\nu \left[(\sqrt{F}(\sigma) - \sqrt{F}(\sigma^{ij}))^2 \right]. \tag{1.12}$$

The comparison inequality from Corollary 4.3 now allows to replace the mean-field Dirichlet form by the Kawasaki (nearest-neighbour exchange) Dirichlet form:

$$\operatorname{Ent}_\nu(F) \leqslant C(d, \beta\bar{A}, \bar{h}) \log^5 N \sum_{i \sim j} \mathbb{E}_\nu \left[(\sqrt{F}(\sigma) - \sqrt{F}(\sigma^{ij}))^2 \right]. \tag{1.13}$$

Finally, it is well known that the mixing time is controlled by the log-Sobolev constant γ [42]:

$$t_{\mathrm{mix}}(1/e) \leqslant \frac{2}{\gamma} \left(1 + \frac{1}{4} \log\log \left(\max_\sigma \frac{1}{\nu(\sigma)} \right) \right) \leqslant \frac{2}{\gamma} \left(1 + \frac{1}{4} \log N + C(\beta\delta_\lambda, h) \right), \tag{1.14}$$

where we used that, for $\sigma \in \Omega_{N,m}$,

$$\log \frac{1}{\nu(\sigma)} \leqslant N \log 2 + (\beta\delta_\lambda + 2 \max_i |h_i|) N. \tag{1.15}$$

This completes the proof. □

Proof of Corollary 1.5. By Theorem 1.2 and Corollary 1.3, we again have

$$\operatorname{Var}_\nu(F) \leqslant C_\beta C(\beta\bar{A}, \bar{h}) \frac{1}{L^d} \sum_{i,j} \mathbb{E}_\nu \left[(\sqrt{F}(\sigma) - \sqrt{F}(\sigma^{ij}))^2 \right], \tag{1.16}$$

$$\operatorname{Ent}_\nu(F) \leqslant C_\beta C(\beta\bar{A}, \bar{h}) \frac{\log L}{L^d} \sum_{i,j} \mathbb{E}_\nu \left[(\sqrt{F}(\sigma) - \sqrt{F}(\sigma^{ij}))^2 \right], \tag{1.17}$$

and the proof is completed by the moving particle lemma, Corollary 4.2. □

1.3. Proof overview. The starting point for our results is the strategy of the proof of the Glauber log-Sobolev inequality under a spectral condition from [6] and much further developed in [3, 7, 9, 22, 27]. The set-up is recalled and adapted to the setting of conservative dynamics in Section 2. For a general introduction, also see [8].

The constraint on the magnetisation makes the analysis much more subtle, though, and several new ingredients are required. An important input is a (modified or unmodified) log-Sobolev inequality

for the infinite temperature case $\beta = 0$, for which we make use of the results for the down-up walk (or bases exchange walk) from [3,23] which are based on the deep log-concavity results of [4,15,16]. These are also recalled in Section 2, and a different presentation of the proof of the modified log-Sobolev inequality for the down-up walk is included in Appendix A.

A crucial new ingredient is a comparison estimate that connects the down-up walk and Kawasaki dynamics through the measure decomposition of Section 2. This estimate is given in Section 3.

In Section 4, we recall the moving particle lemma and apply it to the case of the random regular graph using well-known estimates on the geometry of these graphs.

Finally, in Appendix B, we state the extension of Theorem 1.2 to more general perturbations of log-concave distributions mentioned in the abstract.

1.4. Open questions. Our result leaves several natural questions open which we summarise here.

1. Can one extend the fast mixing result for Kawasaki dynamics on random regular graphs from $\beta < 1/(8\sqrt{d-1})$ to $\beta < \beta_r$, for example by establishing the covariance condition (CC)?

2. Can one show that Kawasaki dynamics on random regular graphs becomes slow for $\beta > \beta_r$ (or at least for β sufficient large) by choice of trial function in the spectral gap inequality? See [18] for such results on finite-dimensional lattices and [1] for hard sampling results for the related SK model.

3. Can one remove the logarithmic factor in the log-Sobolev inequality (1.11) for suitable choice of jump rates depending on h and m which are equivalent to those of the standard Dirichlet form when h is bounded and $m \in [-1+\varepsilon, 1-\varepsilon]$?

4. Can one remove the logarithmic factor in the Dirichlet form comparison estimate for random regular graphs, Corollary 4.3? This would imply that Kawasaki dynamics has spectral gap of order 1 on random regular graphs (rather than with a logarithmic correction), and therefore make this estimate suitable to deduce correlation decay [30].

5. Can one show that the covariance condition (CC) gives a polynomial gap near β_c on $\Lambda \subset \mathbb{Z}^d$ when $d \geqslant 5$, and in general that it applies up to $\beta < \beta_c$? Moreover, can one show that (CC) follows from spatial mixing assumptions?

1.5. Notation and conventions. Given a probability measure π, the relative entropy with respect to π is defined for any function $F \geqslant 0$ by

$$\mathrm{Ent}_\pi(F) = \mathbb{E}_\pi[\Phi(F)] - \Phi(\mathbb{E}_\pi[F]), \qquad \text{where } \Phi(x) = x \log x. \tag{1.18}$$

Given also a Dirichlet form D_π, we use the convention that the log-Sobolev constant γ, the modified log-Sobolev constant γ_m, and the spectral gap λ are given as the best constants in the inequalities

$$\mathrm{Ent}_\pi(F) \leqslant \frac{2}{\gamma} D_\pi(\sqrt{F}) \tag{1.19}$$

$$\mathrm{Ent}_\pi(F) \leqslant \frac{1}{2\gamma_m} D_\pi(F, \log F) \tag{1.20}$$

$$\mathrm{Var}_\pi(F) \leqslant \frac{1}{\lambda} D_\pi(F) \tag{1.21}$$

so that always $\gamma \leqslant \gamma_m \leqslant \lambda$, see for example [13].

2 Proof of Theorem 1.2

2.1. Initial decomposition of the measure. For $m \in [-1, 1]$ such that Nm is an integer, recall that

$$\Omega_{N,m} = \{\sigma \in \{\pm 1\}^N : \sum_{i=1}^N \sigma_i = Nm\}, \tag{2.1}$$

and also define the continuous analogue

$$X_{N,0} = \{\varphi \in \mathbb{R}^N : \sum_{i=1}^N \varphi_i = 0\}. \tag{2.2}$$

The proof of Theorem 1.2 starts from a decomposition of the canonical Ising measure ν on $\Omega_{N,m}$ into two measures: an infinite temperature part π with random external field on $\Omega_{N,m}$ and a renormalised measure ν_0 on $X_{N,0}$ which determines the statistics of the external field. This decomposition is analogous to the decompositions that we used in [6, 9] for Glauber dynamics of Ising models, which is also equivalent to the one implicit in [22], with the constraint on the magnetisation the only but crucial difference. All of the subsequent additional difficulties and improvements (in the high temperature condition to use the second smallest instead of smallest eigenvalue) are due to this constraint.

To define this decomposition, we may from now on assume that A regarded as an operator $X_{N,0} \to X_{N,0}$ has spectrum contained in $(0, 1)$. Indeed, by replacing A by $A + cP$ with $c > -\lambda_2$, where P is the orthogonal projection onto $X_{N,0}$, we may assume without loss of generality that A is positive definite on $X_{N,0}$ (the shift by cP does not affect the canonical Ising measure since $\sigma_i^2 = 1$ for all i), and by rescaling β we can further assume $\|A\| < 1$ where $\|A\|$ is the operator norm of A on $X_{N,0}$. By taking a limit, all estimates extend to the case in which A has spectrum in $[0, 1]$ rather than $(0, 1)$.

We then set

$$C_t = (tA + (\beta - t)P)^{-1} \tag{2.3}$$

where the inverse is taken on $X_{N,0}$ and note that C_t is strictly increasing as a quadratic form in $t \in [0, \beta]$. The following Gaussian convolution identity then holds: for any $\sigma \in \Omega_{N,m}$ and $t \in [0, \beta)$,

$$e^{-\frac{\beta}{2}(\sigma, A\sigma)} \propto e^{-\frac{1}{2}(\sigma, C_\beta^{-1}\sigma)} \propto \int_{X_{N,0}} e^{-\frac{1}{2}(\sigma-\varphi, C_t^{-1}(\sigma-\varphi))} e^{-\frac{1}{2}(\varphi, (C_\beta - C_t)^{-1}\varphi)}\, d\varphi, \tag{2.4}$$

where all inverses are as operators on $X_{N,0}$ and are extended to act trivially orthogonal to it, and $\propto$ denotes proportionality up to a constant independent of σ but dependent on t and β. For $t = 0$, in particular,

$$e^{-\frac{\beta}{2}(\sigma, A\sigma)} \propto \int_{X_{N,0}} e^{-\frac{\beta}{2}(\sigma-\varphi, \sigma-\varphi)} e^{-\frac{1}{2}(\varphi, C^{-1}\varphi)}\, d\varphi \tag{2.5}$$

where $C = C_\beta - C_0 = (\beta A)^{-1} - \beta^{-1}P$. The canonical Ising measure then decomposes as

$$\mathbb{E}_\nu[F] = \mathbb{E}_{\nu_0}[\mathbb{E}_{\pi_{N,m,h+\beta\varphi}}[F]]. \tag{2.6}$$

Here $\pi_{N,m,h}$ denotes the infinite temperature canonical Ising model with external field $h \in \mathbb{R}^N$, i.e., the probability measure on $\Omega_{N,m}$ with probabilities

$$\pi_{N,m,h}(\sigma) \propto \prod_{i=1}^N e^{\sigma_i h_i}. \tag{2.7}$$

The renormalised measure ν_0 on $X_{N,0}$ has density proportional to

$$\nu_0(\varphi) \propto e^{-\frac{1}{2}(\varphi, C^{-1}\varphi) - V_0(\varphi)}, \tag{2.8}$$

where the renormalised potential V_0 is given by

$$V_0(\varphi) = -\log \sum_{\sigma\in\Omega_{N,m}} e^{-\frac{\beta}{2}(\varphi-\sigma,\varphi-\sigma)}\pi_{N,m,h}(\sigma). \tag{2.9}$$

Using (2.4) instead of (2.5), the decomposition (2.6) generalises to

$$\mathbb{E}_\nu[F] = \mathbb{E}_{\nu_t}[\mathbb{E}_{\mu_t^\varphi}[F]], \tag{2.10}$$

where the renormalised measure ν_t on $X_{N,0}$ and the fluctuation measure μ_t^φ on $\Omega_{N,m}$, for $\varphi \in X_{N,0}$, are defined analogously by

$$\frac{d\nu_t}{d\varphi}(\varphi) \propto e^{-\frac{1}{2}(\varphi,(C_\beta-C_t)^{-1}\varphi)-V_t(\varphi)}, \qquad (\varphi \in X_{N,0}), \tag{2.11}$$

$$\mu_t^\varphi(\sigma) \propto e^{-\frac{1}{2}(\sigma-\varphi,C_t^{-1}(\sigma-\varphi))+(\sigma,h)} \propto e^{-\frac{1}{2}(\sigma,C_t^{-1}\sigma)+(C_t^{-1}\varphi+h,\sigma)}, \qquad (\sigma \in \Omega_{N,m}), \tag{2.12}$$

with $V_t(\varphi) = -\log\sum_{\sigma\in\Omega_{N,m}} e^{-\frac{1}{2}(\varphi-\sigma,C_t^{-1}(\varphi-\sigma))}$ the renormalised potential. Thus $\mu_0^\varphi = \pi_{N,m,\beta\varphi}$ and μ_t^φ is again a canonical Ising measure at inverse temperature t with general field $C_t^{-1}\varphi$.

2.2. Infinite temperature measure. We first collect properties of the infinite temperature measure $\pi_{N,m,h}$. One can equivalently view $\pi_{N,m,h}$ as a probability measure on subsets $I \subset [N]$ of cardinality $|I| = k = \frac{1}{2}N(m+1)$, where $I = \{i \in [N] : \sigma_i = +1\}$. It is well known that this measure satisfies the strong Rayleigh property [15]. In particular, it is negatively correlated for all $h \in \mathbb{R}^\Lambda$:

$$\mathrm{Cov}_{\pi_{N,m,h}}(\sigma_i,\sigma_j) \leqslant 0 \qquad (i \neq j), \tag{2.13}$$

and hence its covariance matrix satisfies (using also $2|f_if_j| \leqslant f_i^2 + f_j^2$ and $\sum_{j:j\neq i}\sigma_j = -\sigma_i + Nm$):

$$\begin{aligned}
\mathrm{Var}_{\pi_{N,m,h}}((f,\sigma)) &= \sum_i f_i^2 \,\mathrm{Var}_{\pi_{N,m,h}}(\sigma_i) + \sum_{i\neq j} f_if_j \,\mathrm{Cov}_{\pi_{N,m,h}}(\sigma_i,\sigma_j) \\
&\leqslant \sum_i f_i^2 \,\mathrm{Var}_{\pi_{N,m,h}}(\sigma_i) + \sum_i f_i^2 \sum_{j:j\neq i} \mathrm{Cov}_{\pi_{N,m,h}}(\sigma_i,-\sigma_j) \\
&= 2\sum_i f_i^2 \,\mathrm{Var}_{\pi_{N,m,h}}(\sigma_i) \leqslant 2|f|_2^2, \quad \text{with } |f|_2^2 = \sum_i f_i^2.
\end{aligned} \tag{2.14}$$

A related (but different) consequence of the strong Rayleigh property is the log-concavity of the generating polynomial $z \in [0,\infty)^N \mapsto g_\pi(z) = \mathbb{E}_\pi[z^I]$ where $z^I = \prod_{i\in I} z_i$, see [2, 16]. Note that the log-concavity property is also established for much more general measures. The log-concavity of $g_\pi(z)$ has been used to prove a modified log-Sobolev inequality for the so-called *down-up* or *bases-exchange walk* [3, 23] which we define next.

For any measure π on $\Omega_{N,m}$, we assume there are jump rates $(c(\sigma,\sigma'))_{\sigma,\sigma'} = (c_\pi(\sigma,\sigma'))_{\sigma,\sigma'}$ reversible with respect to π with corresponding Dirichlet form

$$D_\pi(F) = D_\pi(F,F) = \frac{1}{2}\sum_{\sigma'} \mathbb{E}_\pi\left[c_\pi(\sigma,\sigma')(F(\sigma)-F(\sigma'))^2\right]. \tag{2.15}$$

We will always assume that $c_\pi(\sigma,\sigma') = 0$ unless $\sigma' = \sigma^{ij}$ for some $i,j \in [N]$, where σ^{ij} is obtained from σ by swapping the spins at i and j (not necessarily neighbours).

A canonical choice of jump rates are those of the standard Dirichlet form which are $c_\pi(\sigma,\sigma') = \frac{1}{2N}(1+\pi(\sigma')/\pi(\sigma))$ and using which the Dirichlet form takes the simple form

$$D_\pi(F) = \frac{1}{2}\sum_{\sigma'} \mathbb{E}_\pi\left[c_\pi(\sigma,\sigma')(F(\sigma)-F(\sigma'))^2\right] = \frac{1}{2N}\sum_{\sigma'} \mathbb{E}_\pi\left[(F(\sigma)-F(\sigma'))^2\right]. \tag{2.16}$$

Another choice of jump rates we will use are those of the *down-up walk* (also called *bases-exchange walk*) studied in [3, 5, 23]. For $\sigma \in \Omega_{N,m}$, recall that $I(\sigma) = \{i \in [N] : \sigma_i = 1\}$ denotes the set of sites with + spins (particles). The down-up walk acts by removing a particle uniformly at random (down step) to obtain a distribution with $k-1$ particles and then selects a k particle distribution according to π from those containing the former distribution. In terms of spins, this corresponds to the jump rates:

$$c(\sigma, i \to j) = c_{\pi_{N,m,h}}(\sigma, i \to j) = \mathbf{1}_{\sigma_i=1}\mathbf{1}_{\sigma_j=-1}\frac{\pi(\sigma^{ij})}{\sum_{k \notin (I(\sigma)\setminus\{i\})}\pi(\sigma^{ik})}. \tag{2.17}$$

More precisely, we use this choice if $m \leqslant 0$ (i.e., the number of particles is at most $N/2$), and instead the down-up walk for $m > 0$ whose jump rates would be $c'(\sigma, i \to j) = c_{\pi_{N,m,-h}}(-\sigma, i \to j)$. Note that the number $N-k+1$ of terms in the sum in the denominator of the jump rate is always of order N (this is the reason for considering different jump rates depending on the sign of m).

We emphasise that the rates of the down-up walk are asymmetric in (i, j), i.e., $c(\sigma, i \to j) \neq c(\sigma, j \to i)$ – in fact if the left-hand side is nonzero the right-hand side is zero. This should simply be understood as the fact that, if a particle (+ spin) and a hole (− spin) exchange position, then we think of the particle as making the jump rather than the hole. Nonetheless the jump rates are reversible with respect to π since $\pi(\sigma)c(\sigma, i \to j) = \pi(\sigma^{ji})c(\sigma^{ji}, j \to i)$ where on the right-hand side we wrote $\sigma^{ji} = \sigma^{ij}$ to emphasise that the plus particle of σ^{ij} (among the sites i, j) is at j if the one of σ is at i. In the following the value of the jump rate will always be clear from the context, so we will use the more convenient notation:

$$c(\sigma, \sigma^{ij}) := c(\sigma, i \to j). \tag{2.18}$$

If $C^{-1}\pi(\sigma) \leqslant \pi(\sigma^{ij}) \leqslant C\pi(\sigma)$ for all $i, j \in [N]$, note moreover that

$$c(\sigma, \sigma^{ij}) \leqslant C^2\frac{\mathbf{1}_{\sigma_i=1}\mathbf{1}_{\sigma_j=-1}}{N-k+1} \leqslant \frac{2C^2}{N}\mathbf{1}_{\sigma_i=1}\mathbf{1}_{\sigma_j=-1}. \tag{2.19}$$

Thus under the condition $C^{-1}\pi(\sigma) \leqslant \pi(\sigma^{ij}) \leqslant C\pi(\sigma)$ the Dirichlet forms of the down-up walk and the standard Dirichlet form are equivalent, but the jump rates become inequivalent for large fields.

The following modified log-Sobolev inequality for $\pi_{N,m,h}$ has been proven as a consequence of the log-concavity of its generating polynomial [3, 23]. We have included an alternative presentation of the proof in Appendix A.

Theorem 2.1. *Let D_π be the Dirichlet form of the down-up walk (2.17). Then for any $F : \Omega_{N,m} \to \mathbb{R}$, the measure $\pi = \pi_{N,m,h}$ satisfies the modified log-Sobolev inequality (uniformly in N, m, h):*

$$\mathrm{Ent}_\pi(F) \leqslant D_\pi(F, \log F). \tag{2.20}$$

Remark 2.2. The choice (2.17) of normalisation of the jump rate differs from the one in [3, 23], in which (2.17) is further divided by the number m of particles. We prefer (2.17) as it corresponds to a + spin typically getting updated once per unit time.

Remark 2.3. We remark that we use the down-up walk as an ingredient to obtain an optimal constant in our modified log-Sobolev inequality for the canonical Ising model. For a nonoptimal rate, one could use the log-Sobolev inequalities for $\pi_{N,m,h}$ obtained in [31, Theorem 3] as a consequence of the strong Rayleigh property satisfied by $\pi_{N,m,h}$. Note that here the lower bound on the log-Sobolev (or modified log-Sobolev) constant is $1/N$ instead of 1: For any $F : \Omega_{N,m} \to \mathbb{R}$, the measure $\pi = \pi_{N,m,h}$ satisfies

$$\mathrm{Ent}_\pi(F) \leqslant 2ND_\pi(\sqrt{F}), \tag{2.21}$$

where the Dirichlet form is the standard Dirichlet form. The use of this inequality would simplify the arguments and yield results without assumptions on $\max_i \sum_{j\neq i} |A_{ij}|$ or $\max_i |h_i|$ but without the optimal rate obtained by using the down-up walk. To recover the optimal rate one could imagine using instead [31, Theorem 2]. However, the rates there are not explicit (though perhaps could be made explicit), while we need explicit rates for a comparison estimate to recover the original measure.

2.3. Proof of Theorem 1.2. Next we collect properties of the renormalised measure ν_t and the fluctuation measure μ_t^φ defined in (2.11) respectively (2.12). The following convexity is a key observation for the proof of Theorem 1.2.

Lemma 2.4. *For $\beta > 0$ and $\varphi \in X_{N,0}$, the renormalised potential*

$$V_0(\varphi) = -\log \sum_{\sigma\in\Omega_{N,m}} e^{-\frac{\beta}{2}(\varphi-\sigma,\varphi-\sigma)}\pi_{N,m,h}(\sigma) \tag{2.22}$$

has Hessian on $X_{N,0}$ bounded below by $\beta - 2\beta^2$.

Proof. For $f : [N] \to \mathbb{R}$ with $\sum_i f_i = 0$, the Hessian of V_0 is:

$$f \operatorname{Hess} V_0(\varphi) f = \beta|f|_2^2 - \beta^2 \operatorname{Var}_{\pi_{N,m,h+\beta\varphi}}((f,\sigma)). \tag{2.23}$$

Thus (2.14) implies $\operatorname{Hess} V_0(\varphi) \geqslant \beta - 2\beta^2$ as claimed. □

Lemma 2.5. *Assume $\|A\| \leqslant 1$ and let $\beta < 1/2$. Then the canonical Ising model with inverse temperature β and external field $h \in \mathbb{R}^N$ satisfies*

$$\operatorname{Var}_{\nu_{\beta,h}}((f,\sigma)) \leqslant (\tfrac{1}{2} - \beta)^{-1}|f|_2. \tag{2.24}$$

In particular, $\operatorname{Cov}(\mu_t^\varphi) \leqslant (\frac{1}{2} - t)^{-1}$.

Proof. Using the decomposition (2.6), with $\nu = \nu_{\beta,h}$ and $\pi = \pi_{N,m,h}$ with $h = \beta\varphi$,

$$\begin{aligned} \operatorname{Var}_\nu((f,\sigma)) &= \mathbb{E}_{\nu_0}[\operatorname{Var}_\pi((f,\sigma))] + \operatorname{Var}_{\nu_0}(\mathbb{E}_\pi(f,\sigma)) \\ &\leqslant 2|f|_2^2 + \frac{\beta^2}{\beta - 2\beta^2}\mathbb{E}_{\nu_0}\Big[|\nabla_h \mathbb{E}_\pi((f,\sigma))|_2^2\Big] \end{aligned} \tag{2.25}$$

where the bound of the first term is (2.14) and that of the second term the Bakry–Émery criterion for the spectral gap of ν_0 which is strictly log-concave for $\beta < 1/2$ by Lemma 2.4. The second term can be simplified by again using (2.14):

$$|\nabla_h \mathbb{E}_\pi((f,\sigma))|_2^2 = \sup_{|g|_2\leqslant 1} (g, \nabla_h \mathbb{E}_\pi[(f,\sigma)])^2 = \sup_{|g|_2\leqslant 1} \operatorname{Cov}_\pi((f,\sigma),(g,\sigma))^2 \leqslant 4|f|_2^2. \tag{2.26}$$

This gives the claim:

$$\operatorname{Var}_\nu((f,\sigma)) \leqslant \left(2 + \frac{4\beta^2}{\beta - 2\beta^2}\right)|f|_2^2 = \left(\frac{2\beta}{\beta - 2\beta^2}\right)|f|_2^2. \tag{2.27}$$

□

The following key estimate connects the Dirichlet forms of the down-up walk of the product measure (conditioned on its sum) with that of the canonical Ising model via the decomposition (2.6). Its proof is postponed to Section 3.

Lemma 2.6. *Let $D_\pi = D_{\pi_{N,m,\beta\varphi}}$ be the Dirichlet form with the jump rates of the down-up walk* (2.17) *and likewise for D_ν. Assume that $\max_j \sum_{k\neq j} |A_{jk}| \leqslant \bar{A}$. There is $K(\beta\bar{A}) > 0$ independent of N, m, h and the test function $F : \Omega_{N,m} \to \mathbb{R}_+$ such that:*

$$\mathbb{E}_{\nu_0}[D_{\pi_{N,m,\beta\varphi}}(F, \log F)] \leqslant C_\beta K(\beta\bar{A}) D_\nu(F, \log F). \tag{2.28}$$

The same bound also holds with $D_\pi(F), D_\nu(F)$ replacing $D_\pi(F, \log F), D_\nu(F, \log F)$.

Proof of Theorem 1.2. We first apply the entropic stability estimate from [22, Proposition 39]. In the notation and set-up we use, it is also reviewed in [8, Section 3.7]. It states that

$$\mathrm{Ent}_\nu(F) \leqslant C_\beta \mathbb{E}_{\nu_0}[\mathrm{Ent}_\pi(F)], \qquad C_\beta = e^{\int_0^\beta \alpha_t \, dt} \tag{2.29}$$

provided that the numbers α_t satisfy (recall that μ_t^φ is defined in (2.12))

$$\dot{C}_t C_t^{-1} \,\mathrm{Cov}(\mu_t^\varphi) C_t^{-1} \dot{C}_t \leqslant \alpha_t \dot{C}_t. \tag{2.30}$$

Since $\dot{C}_t C_t^{-2} = (1 - A)$ and $\|A\| \leqslant 1$, this condition follows from $\mathrm{Cov}(\mu_t^\varphi) \leqslant \alpha_t$ as in the covariance condition (CC), where we recall that μ_t^φ is again a canonical Ising measure at inverse temperature t. For $\beta < 1/2$, by Lemma 2.5, one can take $\alpha_t = (\frac{1}{2} - t)^{-1}$ and thus $C_\beta = \exp(\log(\frac{1}{2}) - \log(\frac{1}{2} - \beta)) = (1 - 2\beta)^{-1}$ which is as in the spectral condition (SC).

Next we apply the modified log-Sobolev inequality for the down-up walk for π from Theorem 2.1, followed by Lemma 2.6:

$$\mathrm{Ent}_\nu(F) \leqslant C_\beta \mathbb{E}_{\nu_0}[D_\pi(F, \log F)] \leqslant C_\beta K(\beta\bar{A}) D_\nu(F, \log F), \tag{2.31}$$

where D_ν is the Dirichlet form associated with rates of the down-up walk (2.17). As discussed below (2.19), the down-up walk Dirichlet form is equivalent to the standard Dirichlet form (which is the one in the statement of Theorem 1.2) provided that (2.19) holds. This is the case with constants depending on $\bar{A}$ and $\bar{h}$. □

3 Recovering the Dirichlet form of the down-up walk

In this section, we prove Lemma 2.6 which we restate below as Lemma 3.1 for convenience. Also recall that the jump rates of the down-up walk for a measure π on $\Omega_{N,m}$ with $m \leqslant 0$ are given by:

$$c(\sigma, \sigma^{ij}) = c_\pi(\sigma, \sigma^{ij}) = \mathbf{1}_{\sigma_i = 1} \mathbf{1}_{\sigma_j = -1} \frac{\pi(\sigma^{ij})}{\sum_{k \notin (I(\sigma) \setminus \{i\})} \pi(\sigma^{ik})}, \tag{3.1}$$

with $I(\sigma) \subset [N]$ the set of + spins in σ. If instead $m \geqslant 0$, the dynamics to consider has the jump rates of the up-down walk given by considering the above choice for $-\sigma$.

Lemma 3.1. *Let D_π be the Dirichlet form with the jump rates of the down-up walk* (3.1) *associated with the measure $\pi = \pi_{N,m,h+\beta\varphi}$. Assume that $\max_j \sum_{k\neq j} |A_{jk}| \leqslant \bar{A}$. There is $K(\beta\bar{A}) > 0$ independent of N, m, h and the test function $F : \Omega_{N,m} \to \mathbb{R}_+$ such that:*

$$\mathbb{E}_{\nu_0}[D_\pi(F)] \leqslant K(\beta\bar{A}) D_\nu(F) = K(\beta\bar{A}) \sum_{\sigma,\sigma' \in \Omega_{N,m}} \nu(\sigma) c_\nu(\sigma, \sigma') \big[F(\sigma') - F(\sigma)\big]^2, \tag{3.2}$$

where the canonical Ising model ν is defined in (1.2) *and the renormalised measure ν_0 in* (2.8).

The same bound also holds with $D_\pi(F), D_\nu(F)$ replaced by $D_\pi(F, \log F), D_\nu(F, \log F)$.

Proof. Fix $F : \Omega_{N,m} \to \mathbb{R}_+$. The jump rates for $\pi = \pi_{N,m,h+\beta\varphi}$ read:

$$\begin{aligned} c_\pi(\sigma, \sigma^{ij}) &= \mathbf{1}_{\sigma_i=1}\mathbf{1}_{\sigma_j=-1}\frac{e^{(\sigma_i-\sigma_j)\left(\beta(\varphi_j-\varphi_i)+h_j-h_i\right)}}{\sum_{k\notin(I(\sigma)\setminus\{i\})} e^{(\sigma_i-\sigma_k)\left(\beta(\varphi_k-\varphi_i)+h_k-h_i\right)}} \\ &= \mathbf{1}_{\sigma_i=1}\mathbf{1}_{\sigma_j=-1}\frac{e^{2(\beta\varphi_j+h_j)}}{\sum_{k\notin(I(\sigma)\setminus\{i\})} e^{2(\beta\varphi_k+h_k)}}. \end{aligned} \tag{3.3}$$

It suffices to prove that, for each $\sigma \in \Omega_{N,m}$, each $i \in I(\sigma)$, and each $j \notin I(\sigma)$, all henceforth fixed:

$$\mathbb{E}_{\nu_0}\left[\pi(\sigma)c_\pi(\sigma, \sigma^{ij})\right] \leqslant K\nu(\sigma)c_\nu(\sigma, \sigma^{ij}) \tag{3.4}$$

with a constant $K = K(\beta\bar{A})$. (We remark that if $\pi(\sigma)c_\pi(\sigma, \sigma^{ij})$ were a concave function of π, then this inequality would hold with constant $K = 1$. This is unfortunately not the case.)

Recall that $\mathbb{E}_{\nu_0}[\pi(\sigma)] = \nu(\sigma)$ by definition. From the expressions for ν_0 and $\pi_{N,m,h+\beta\varphi}$ in Section 2.1, the left-hand side of (3.4) reads:

$$\begin{aligned} \mathbb{E}_{\nu_0}\left[\pi(\sigma)c_\pi(\sigma, \sigma^{ij})\right] &\propto \int_{X_{N,0}} \exp\left[-\frac{(\varphi, C^{-1}\varphi)}{2} - \frac{\beta(\varphi-\sigma, \varphi-\sigma)}{2} + (h, \sigma)\right] \\ &\qquad \times \frac{e^{2(\beta\varphi_j+h_j)}}{\sum_{k\notin(I(\sigma)\setminus\{i\})} e^{2(\beta\varphi_k+h_k)}}\, d\varphi \\ &= \nu(\sigma)\mathbf{E}_\gamma\left[\frac{e^{2(\beta\varphi_j+h_j)}}{\sum_{k\notin(I(\sigma)\setminus\{i\})} e^{2(\beta\varphi_k+h_k)}}\right], \end{aligned} \tag{3.5}$$

where $\mathbf{E}_\gamma$ is the expectation with respect to a Gaussian distribution γ on $X_{N,0}$ with covariance matrix $(C^{-1} + \beta P)^{-1}$ and mean:

$$a^\sigma := \beta(C^{-1} + \beta P)^{-1}\sigma. \tag{3.6}$$

This quantity is well defined since C is a positive definite matrix. For later use, we note that in fact

$$a^\sigma = \sigma - A\sigma. \tag{3.7}$$

Indeed, from $C = (\beta A)^{-1} - \beta^{-1}P$ where $|A|_{X_{N,0}} < 1$ is the operator norm of A acting on $X_{N,0}$ and P is the identity on $X_{N,0}$, we get

$$(\beta C)^{-1} = (A^{-1} - P)^{-1} = \frac{A}{P - A} \tag{3.8}$$

as operators on $X_{N,0}$. This implies, again on $X_{N,0}$,

$$\beta(C^{-1} + \beta P)^{-1} = \left[\frac{A + (P - A)}{P - A}\right]^{-1} = P - A. \tag{3.9}$$

Writing γ_0 for the centred Gaussian distribution with covariance $(C^{-1} + \beta P)^{-1}$, we get:

$$\mathbb{E}_{\nu_0}\left[\pi(\sigma)c_\pi(\sigma, \sigma^{ij})\right] = \nu(\sigma)\mathbf{E}_{\gamma_0}\left[\frac{e^{2\beta(\psi_j+a^\sigma_j)+2h_j}}{\sum_{k\notin(I(\sigma)\setminus\{i\})} e^{2\beta(\psi_k+a^\sigma_k)+2h_k}}\right]. \tag{3.10}$$

To bound this last expectation by $c_\nu(\sigma,\sigma^{ij})$, we first rewrite $c_\nu(\sigma,\sigma^{ij})$. By (3.1), one has:

$$c_\nu(\sigma,\sigma^{ij}) = \mathbf{1}_{\sigma_i=1}\mathbf{1}_{\sigma_j=-1}\frac{\exp\Big[2h_j - \frac{\beta}{2}\nabla_{ij}(\sigma,A\sigma)\Big]}{\sum_{k\notin(I(\sigma)\setminus\{i\})}\exp\Big[2h_k - \frac{\beta}{2}\nabla_{ik}(\sigma,A\sigma)\Big]}, \tag{3.11}$$

where $\nabla_{ij}F(\sigma) = F(\sigma^{ij}) - F(\sigma)$. Therefore, defining

$$c_k = \frac{\beta}{2}\nabla_{ij}(\sigma,A\sigma) - \frac{\beta}{2}\nabla_{ik}(\sigma,A\sigma), \tag{3.12}$$

one has

$$c_\nu(\sigma,\sigma^{ij}) = \frac{e^{2h_j}}{\sum_{k\notin(I(\sigma)\setminus\{i\})} e^{2h_k+c_k}}. \tag{3.13}$$

Let $U_h = U_h^{i,j}$ denote the following probability measure on $[N]\setminus(I(\sigma)\setminus\{i\})$:

$$U_h(k) = \frac{1}{Z_h}e^{2h_k+c_k}, \qquad Z_h = Z_h(\sigma) = \sum_{k\notin(I(\sigma)\setminus\{i\})} e^{2h_k+c_k}. \tag{3.14}$$

Then, multiplying and dividing the right-hand side of (3.10) by $c_\nu(\sigma,\sigma^{ij})$ and using Jensen's inequality for the convex function $x\in(0,\infty)\mapsto 1/x$:

$$\begin{aligned}\mathbb{E}_{\nu_0}\big[\pi(\sigma)c_\pi(\sigma,\sigma^{ij})\big] &= \nu(\sigma)c_\nu(\sigma,\sigma^{ij})\mathbf{E}_{\gamma_0}\Big[\frac{e^{2\beta(\psi_j+a_j^\sigma)}}{\mathbb{E}_{U_h}\big[e^{2\beta(\psi_\cdot+a_\cdot^\sigma)-c_\cdot}\big]}\Big]\\ &\leqslant \nu(\sigma)c_\nu(\sigma,\sigma^{ij})\mathbb{E}_{U_h}\Big[\mathbf{E}_{\gamma_0}\Big[e^{2\beta(\psi_j-\psi_\cdot+a_j^\sigma-a_\cdot^\sigma)}\Big]e^{c_\cdot}\Big].\end{aligned} \tag{3.15}$$

Thus (3.4) holds with

$$K = \max_{i,j}\mathbb{E}_{U_h}\Big[\mathbf{E}_{\gamma_0}\Big[e^{2\beta(\psi_j-\psi_\cdot+a_j^\sigma-a_\cdot^\sigma)}\Big]e^{c_\cdot}\Big]. \tag{3.16}$$

Let us check that K is bounded in terms of $\beta\bar{A}$ uniformly in σ, where $\max_j\sum_{k\neq j}|A_{jk}|\leqslant\bar{A}$. For each $k\notin(I(\sigma)\setminus\{i\})$, the Gaussian expectation on the right-hand side is:

$$\mathbf{E}_{\gamma_0}\Big[e^{2\beta(\psi_j-\psi_k+a_j^\sigma-a_k^\sigma)}\Big] = e^{2\beta(a_j^\sigma-a_k^\sigma)}e^{2\beta^2\left(\delta_j-\delta_k,(C^{-1}+\beta P)^{-1}(\delta_j-\delta_k)\right)}. \tag{3.17}$$

By (3.6) and (3.9), the two terms in the exponent can be written as

$$2\beta^2\big(\delta_i-\delta_k,(C^{-1}+\beta P)^{-1}(\delta_i-\delta_k)\big) = 2\beta\big(\delta_j-\delta_k,(P-A)(\delta_j-\delta_k)\big) \tag{3.18}$$

$$2\beta(a_j^\sigma-a_k^\sigma) = 2\beta(\sigma,(P-A)(\delta_j-\delta_k)), \tag{3.19}$$

which are both bounded by $C\beta+C\beta\bar{A}$, and

$$-\frac{\beta}{2}\nabla_{ij}(\sigma,A\sigma) = \beta\sum_{\ell\notin\{i,j\}}\sigma_\ell(\sigma_i-\sigma_j)(A_{i\ell}-A_{j\ell}), \tag{3.20}$$

and analogously for $\frac{\beta}{2}\nabla_{ik}(\sigma,A\sigma)$, so that also $|c_k|\leqslant C\beta\bar{A}$. Hence $K\leqslant e^{C\beta+C\beta\bar{A}}$ as claimed. □

4 Moving particle lemmas

The following lemma was proved in [40, Section 5] and [43, Theorem 6.1].

Lemma 4.1 (Moving particle lemma)**.** *Let $\Lambda \subset \mathbb{Z}$ be an interval and assume the canonical Ising model ν on Λ has bounded couplings $\max_i \sum_{j \neq i} |A_{ij}| \leqslant \bar{A}$ and fields $\max_i |h_i| \leqslant \bar{h}$ and that it satisfies the finite range property $A_{ij} = 0$ if $\operatorname{dist}(i,j) > R$ for some $R < \infty$. Then for $i < j$ with $i, j \in \Lambda$,*

$$\mathbb{E}_\nu[(F(\sigma) - F(\sigma^{ij}))^2] \leqslant C(\beta\bar{A}, \bar{h}, R)|i-j| \sum_{k=i}^{j-1} \mathbb{E}_\nu[(F(\sigma) - F(\sigma^{k,k+1}))^2]. \tag{4.1}$$

Proof. It suffices to consider the case that $j = i_\ell$ where $i_k = i + k(R+1)$ and show that

$$\mathbb{E}_\nu[(F(\sigma) - F(\sigma^{ij}))^2] \leqslant C(\beta\bar{A}, \bar{h}, R)|\ell| \sum_{k=i}^{\ell-1} \mathbb{E}_\nu[(F(\sigma) - F(\sigma^{i_k,i_{k+1}}))^2]. \tag{4.2}$$

The general case then follows by bounding (with a different constant)

$$\mathbb{E}_\nu[(F(\sigma) - F(\sigma^{i_k,i_{k+1}}))^2] \leqslant C(\beta\bar{A}, \bar{h}, R) \sum_{m=0}^{R} \mathbb{E}_\nu[(F(\sigma) - F(\sigma^{i+m,i+m+1}))^2], \tag{4.3}$$

whose proof we omit (and which is easy since R is fixed). To verify (4.2), let $I = \{i_0, \dots, i_\ell\}$. By the finite-range property, conditional on the spins outside I, the canonical Ising measure restricted to $\sigma|_I$ is a product Bernoulli measure conditioned on its sum with marginal probabilities $\propto e^{\alpha_k \sigma_{i_k}}$ where $\alpha_k \in [-B, B]$ for $B = \bar{A} + \bar{h}$. Hence by [40, Lemma 5.2], the inequality (4.2) follows. □

A standard consequence of the above moving particle lemma is the following comparison estimate for the Kawasaki Dirichlet form on cubes $\Lambda \subset \mathbb{Z}^d$.

Corollary 4.2. *Let $d \geqslant 1$. For the canonical Ising model on a cube $\Lambda \subset \mathbb{Z}^d$ of side length L,*

$$\frac{1}{L^d} \sum_{i,j} \mathbb{E}_\nu[(F(\sigma) - F(\sigma^{ij}))^2] \leqslant C(d, \beta, h) L^2 \sum_{i \sim j} \mathbb{E}_\nu[(F(\sigma) - F(\sigma^{ij}))^2], \tag{4.4}$$

where the sum on the left runs over all vertices i and j and the sum on the right over pairs of nearest neighbours.

Proof. This is a standard consequence which we include for completeness and in preparation for the argument on random regular graphs. For each $i, j \in \Lambda$, let γ_{ij} be the nearest-neighbour path from i to j that moves first in the first coordinate direction, then in the second, and so on. By conditioning on the spins outside this path, Lemma 4.1 can be applied to the one-dimensional Ising model along the path (which has the same range with respect to the graph distance). This gives

$$\mathbb{E}_\nu[(F(\sigma) - F(\sigma^{ij}))^2] \leqslant C(d, \beta, h)|\gamma_{ij}| \sum_{k\ell \in \gamma_{ij}} \mathbb{E}_\nu[(F(\sigma) - F(\sigma^{k\ell}))^2], \tag{4.5}$$

and the assertion follows from the simple path counting argument

$$\max_e \frac{1}{L^d} \sum_{i,j} |\gamma_{ij}| 1_{e \in \gamma_{ij}} \leqslant C(d) L^2 \tag{4.6}$$

where e runs over the nearest-neighbour edges of Λ. □

The next corollary applies the moving particle lemma to compare Dirichlet forms for the canonical Ising model on a random regular graph. We expect that the logarithmic factor in the statement is technical since the adjacency matrix of the random regular graph has a uniform spectral gap. Our argument below effectively amounts to estimating this spectral gap using the path counting method which is too crude to obtain the optimal estimate.

Corollary 4.3. *Let $d \geqslant 3$. For the canonical Ising model on a random d-regular graph on N vertices,*

$$\frac{1}{N}\sum_{i,j}\mathbb{E}_\nu[(F(\sigma)-F(\sigma^{ij}))^2] \leqslant C(d,\beta,h)\log^4 N \sum_{i\sim j}\mathbb{E}_\nu[(F(\sigma)-F(\sigma^{ij}))^2], \tag{4.7}$$

with high probability under the randomness of the random regular graph, and again the sum on the left runs over all vertices i and j and the sum on the right over pairs of nearest neighbours.

Proof. For each pair of vertices $i, j \in [N]$, fix some geodesic nearest-neighbour path γ_{ij} of length $|\gamma_{ij}|$ between i and j. In particular, $\gamma_{ij} \neq \gamma_{k\ell}$ if $\{i,j\} \neq \{k,\ell\}$ and the canonical Ising model conditioned on the spins outside γ_{ij} again has the same range as the original one since γ_{ij} is a geodesic. Therefore it suffices to bound

$$\max_e \frac{1}{2N}\sum_{i,j}|\gamma_{ij}|1_{e\in\gamma_{ij}}. \tag{4.8}$$

For every edge e, one has the trivial bound

$$\frac{1}{2}\sum_{i,j}1_{e\in\gamma_{ij}} \leqslant \text{number of simple paths containing } e \text{ of length at most } D, \tag{4.9}$$

where D is the diameter of the graph. It is known [14, Theorem 10.14] that the diameter of a random regular graph satisfies (again with high probability)

$$D \leqslant \log_{d-1} N + \log_{d-1}\log N + C. \tag{4.10}$$

There are at most $(d-1)^k$ simple paths of length $k+1$ containing e at the m-th position, $1 \leqslant m \leqslant k$. Therefore there are at most $k(d-1)^k$ simple paths of length $k+1$ containing e and the above sum is crudely bounded by

$$\sum_{k=1}^{D-1} k(d-1)^k \leqslant D^2(d-1)^D. \tag{4.11}$$

Using (4.10) gives

$$\frac{1}{2N}\sum_{i,j}|\gamma_{ij}|1_{e\in\gamma_{ij}} \leqslant \frac{D}{2N}\sum_{i,j}1_{e\in\gamma_{ij}} \leqslant \frac{1}{N}D^3(d-1)^D \leqslant C_d\log^4 N \tag{4.12}$$

and the claim follows from Lemma 4.1. □

A Proof of Theorem 2.1

In this appendix, we include an alternative version of the proof of Theorem 2.1, i.e., we prove:

$$\mathrm{Ent}_\pi(F) \leqslant D_\pi(F, \log F), \qquad F : \Omega_{N,m} \to \mathbb{R}_+, \tag{A.1}$$

where $\pi = \pi_{N,m,h}$ for a field $h \in \mathbb{R}^N$, and where D_π is the Dirichlet form of the bases-exchange dynamics (down-up walk) defined in (2.17). It is convenient to see + spins as particles: we write $\Omega_{N,k}$ for $\Omega_{N,m}$ with $k = N(1+m)/2$ and

$$I = \{i \in [N] : \sigma_i = +1\}. \tag{A.2}$$

Our proof below uses the same entropy contraction argument as in [3, 23] for the bases-exchange walk. A main difference in our presentation is that we decouple this entropy contraction argument from the specific dynamics. In particular, we explain how to deduce log-Sobolev inequalities (rather than modified log-Sobolev) and modified log-Sobolev inequalities with respect to other dynamics.

A.1. Entropic independence estimate. The key ingredient is the following contraction estimate, referred to as entropic independence [3]: For a probability measure π on $\Omega_{N,k}$, introduce the probability vector in $[0,1]^N$ (equivalently viewed as a probability distribution on $\Omega_{N,1}$):

$$\pi D_{k\to 1} = \frac{1}{k}(\mathbb{E}_\pi[\mathbf{1}_{\sigma_i=1}])_{i\in[N]}. \tag{A.3}$$

The 1-entropic independence of π is the property that, for any probability measure μ on $\Omega_{N,k}$:

$$\mathbb{H}\big(\mu D_{k\to 1}|\pi D_{k\to 1}\big) \leqslant \frac{1}{k}\mathbb{H}\big(\mu|\pi\big), \tag{A.4}$$

where $\mathbb{H}(\mu_1|\mu_2)$ is the relative entropy between two probability measures μ_1, μ_2.

Proposition A.1. *If $g_\pi(z) = \mathbb{E}_\pi[z^I]$ is log-concave on $(0,\infty)^N$ then* (A.4) *holds (where $z^I = \prod_{i\in I} z_i$ and I was defined in* (A.2)*).*

Proof. This is shown in [3, Section 3]. Let $p = \pi D_{k\to 1}$ and $q = \mu D_{k\to 1}$. Then log-concavity implies

$$\frac{1}{k}\log g_\pi(z) \leqslant \log\left[\sum_i p_i z_i\right]. \tag{A.5}$$

The entropy inequality with $e^G = z^I$ implies, for any $z \in (0,\infty)^N$,

$$\mathbb{H}(\mu|\pi) = \sup_G \Big\{\mathbb{E}_\mu[G] - \log \mathbb{E}_\pi[e^G]\Big\} \geqslant \mathbb{E}_\mu[\log z^I] - \log \mathbb{E}_\pi[z^I] = k\sum_i q_i \log z_i - \log g_\pi(z). \tag{A.6}$$

Combining both estimates with the choice $z_i = q_i/p_i$ yields the claim:

$$\mathbb{H}(\mu|\pi) \geqslant k\sum_i q_i \log z_i - k\log\Big(\sum_i p_i z_i\Big) = k\sum_i q_i \log(q_i/p_i) = k\mathbb{H}(q|p). \tag{A.7}$$ □

A.2. Entropy contraction. We decompose π in terms of the number of particles as follows. For each $0 \leqslant \ell \leqslant k$, let ν_ℓ be the probability measure on $\varphi \in \Omega_{N,k-\ell}$ defined by

$$\nu_\ell(\varphi) = \frac{1}{\binom{k}{\ell}}\pi(\sigma \geqslant \varphi), \tag{A.8}$$

where $\sigma \geqslant \varphi$ means $\sigma_i \geqslant \varphi_i$ for each $i \in [N]$ and $\binom{k}{\ell}$ is the number of configurations in $\Omega_{N,k-\ell}$ smaller than a given $\sigma \in \Omega_{N,k}$. Let also $\mu_\ell = \mu_\ell^\varphi$ be the measure π on $\Omega_{N,k}$ conditioned on $\sigma \geqslant \varphi$:

$$\mu_\ell^\varphi(\sigma) = \pi(\sigma|\sigma \geqslant \varphi). \tag{A.9}$$

Then we have a decomposition analogous to (2.10): for any $0 \leqslant \ell \leqslant k$,

$$\mathbb{E}_\pi[F] = \mathbb{E}_{\nu_\ell}[\mathbb{E}_{\mu_\ell^\varphi}[F]]. \tag{A.10}$$

Let $P_{\ell,\ell+1} = P_{\ell,\ell+1}(\psi,\cdot)$ be the probability measure on $\Omega_{N,k-\ell}$ that adds a particle to $\psi \in \Omega_{N,k-(\ell+1)}$:

$$\mathbb{E}_{\nu_\ell}[F(\varphi)] = \mathbb{E}_{\nu_{\ell+1}}[P_{\ell,\ell+1}F(\psi)], \qquad P_{\ell,\ell+1}\mathbb{E}_{\mu_\ell}[F] = \mathbb{E}_{\mu_{\ell+1}}[F]. \tag{A.11}$$

Explicitly, for $\psi \in \Omega_{N,k-(\ell+1)}$:

$$P_{\ell,\ell+1}F(\psi) = \frac{1}{\ell+1}\sum_{i\in[N]} \frac{\pi(\sigma \geqslant \psi + \mathbf{1}_i)}{\pi(\sigma \geqslant \psi)} 1_{\psi_i=-1} F(\psi + 1_i), \tag{A.12}$$

and $P_{\ell,\ell+1}(\psi,\cdot)$ can equivalently be seen as a probability measure $(P^\psi_{\ell,\ell+1}(i))_i$ on $[N]$:

$$P^\psi_{\ell,\ell+1}(i) = P_{\ell,\ell+1}(\psi,\psi+\mathbf{1}_i) = \mathbf{1}_{\psi_i=-1}\frac{1}{\ell+1}\frac{\pi(\sigma\geqslant\psi+\mathbf{1}_i)}{\pi(\sigma\geqslant\psi)}. \tag{A.13}$$

Indeed, one then has:

$$\begin{aligned} P_{\ell,\ell+1}\mathbb{E}_{\mu_\ell}[F] &:= \frac{1}{\ell+1}\sum_{j\in[N]} \frac{\pi(\sigma\geqslant\psi+\mathbf{1}_j)}{\pi(\sigma\geqslant\psi)}\mathbf{1}_{\psi_j=-1}\mathbb{E}_{\pi(\cdot|\sigma\geqslant\psi+\mathbf{1}_j)}[F] \\ &= \frac{1}{\ell+1}\sum_{j\in[N]} \mathbf{1}_{\psi_j=-1}\mathbb{E}_{\pi(\cdot|\sigma\geqslant\psi)}[F\mathbf{1}_{\sigma_j=1}] \\ &= \mathbb{E}_{\pi(\cdot|\sigma\geqslant\psi)}[F] =: \mathbb{E}_{\mu^\psi_{\ell+1}}[F], \end{aligned} \tag{A.14}$$

where the third line comes from the fact that any $\sigma \in \Omega_{N,k}$ with $\sigma \geqslant \psi$ has exactly $\ell+1$ particles that are not present in ψ.

Proposition A.2. *Assuming that π has log-concave generating polynomial g_π, then for any $1 \leqslant p \leqslant k$,*

$$\mathrm{Ent}_\pi(F) = \mathbb{E}_{\nu_k}[\mathrm{Ent}_{\mu_k}(F)] \leqslant \frac{k}{p}\mathbb{E}_{\nu_p}[\mathrm{Ent}_{\mu_p}(F)]. \tag{A.15}$$

Proof. The first equality in (A.15) follows from the observation that $\mu_k = \pi$. The inequality in (A.15) will be derived by reducing recursively the number of particles and using the contraction estimate (A.4) which provides a bound on the loss of removing a particle. Writing $\mathrm{Ent}_{\ell,\ell+1}$ for the entropy associated with $P_{\ell,\ell+1}$, we are going to prove that:

$$\mathrm{Ent}_{\ell,\ell+1}(\mathbb{E}_{\mu_\ell}[F]) \leqslant \frac{1}{\ell+1}\,\mathrm{Ent}_{\mu_{\ell+1}}(F). \tag{A.16}$$

Admitting (A.16), let us conclude the proof of (A.15). Decomposing the measure using (A.10) with $\ell = k-1$, noticing that $\nu_{k-1}(\cdot) = P_{k-1,k}(-1,\cdot)$ with -1 the configuration with no particle and using (A.16),

$$\begin{aligned} \mathrm{Ent}_\pi(F) &= \mathbb{E}_{\nu_{k-1}}[\mathrm{Ent}_{\mu_{k-1}}(F)] + \mathrm{Ent}_{\nu_{k-1}}\left(\mathbb{E}_{\mu_{k-1}}[F]\right) \\ &\leqslant \mathbb{E}_{\nu_{k-1}}[\mathrm{Ent}_{\mu_{k-1}}(F)] + \frac{1}{k}\,\mathrm{Ent}_{\mu_k}(F). \end{aligned} \tag{A.17}$$

As $\mu_k = \pi$, we get:

$$\mathrm{Ent}_\pi(F) \leqslant \frac{k}{k-1}\mathbb{E}_{\nu_{k-1}}[\mathrm{Ent}_{\mu_{k-1}}(F)]. \tag{A.18}$$

One can then analogously decompose μ_{k-1} to obtain:

$$\mathrm{Ent}_\pi(F) \leqslant \frac{k}{k-1}\cdot\frac{k-1}{k-2}\mathbb{E}_{\nu_{k-2}}[\mathrm{Ent}_{\mu_{k-2}}(F)] = \frac{k}{k-2}\mathbb{E}_{\nu_{k-2}}[\mathrm{Ent}_{\mu_{k-2}}(F)]. \tag{A.19}$$

Iterating this procedure p times proves the proposition assuming (A.16), established next.

Fix $\psi \in \Omega_{N,k-(\ell+1)}$. By definition of $\mu_\ell^\varphi = \pi(\cdot|\sigma \geqslant \varphi)$ ($\varphi \in \Omega_{N,k-\ell}$) and $P_{\ell,\ell+1}$ (see (A.12)),

$$\mathrm{Ent}_{\ell,\ell+1}(\mathbb{E}_{\mu_\ell}[F]) = \sum_{i\in[N]} \mathbb{E}_{\mu_\ell^{\psi+\mathbf{1}_i}}[F]\log\Big(\frac{\mathbb{E}_{\mu_\ell^{\psi+\mathbf{1}_i}}[F]}{P_{\ell,\ell+1}\mathbb{E}_{\mu_\ell}[F]}\Big)P_{\ell,\ell+1}^\psi(i). \tag{A.20}$$

Recall that $P_{\ell,\ell+1}\mathbb{E}_{\mu_\ell}[F] = \mathbb{E}_{\mu_{\ell+1}}[F]$ and notice:

$$\mathbb{E}_{\mu_\ell^{\psi+\mathbf{1}_i}}[F] = \frac{\mathbb{E}_{\pi(\cdot|\sigma\geqslant\psi)}[\mathbf{1}_{\sigma_i=1}F]}{\mathbb{E}_{\pi(\cdot|\sigma\geqslant\psi)}[\mathbf{1}_{\sigma_i=1}]} = \frac{\mathbb{E}_{\mu_{\ell+1}^\psi}[\mathbf{1}_{\sigma_i=1}F]}{\mathbb{E}_{\mu_{\ell+1}^\psi}[\mathbf{1}_{\sigma_i=1}]}. \tag{A.21}$$

With $D_{\ell+1\to 1}$ defined in (A.3), we find:

$$\begin{aligned}\mathrm{Ent}_{\ell,\ell+1}(\mathbb{E}_{\mu_\ell}[F]) &= \frac{1}{\ell+1}\sum_{i\in[N]}\mathbf{1}_{\psi_i=-1}\mathbb{E}_{\mu_{\ell+1}^\psi}[F\mathbf{1}_{\sigma_i=1}]\log\left(\frac{\mathbb{E}_{\mu_{\ell+1}^{\psi,F}}[\mathbf{1}_{\sigma_i=1}]}{\mathbb{E}_{\mu_{\ell+1}^\psi}[\mathbf{1}_{\sigma_i=1}]}\right)\\ &= \mathbb{E}_{\mu_{\ell+1}^\psi}[F]\mathbb{H}\Big(\mu_{\ell+1}^{\psi,F}D_{\ell+1\to 1}\Big|\mu_{\ell+1}^\psi D_{\ell+1\to 1}\Big),\end{aligned} \tag{A.22}$$

where $\mu_{\ell+1}^{\psi,F} = F\mu_{\ell+1}^\psi/\mathbb{E}_{\mu_{\ell+1}^\psi}[F]$. The measure $\mu_{\ell+1}^\psi$ is strongly log-concave as it is just π conditioned on the position of some particles. The 1-entropic independence (A.4) therefore gives (A.16):

$$\mathrm{Ent}_{\ell,\ell+1}(\mathbb{E}_{\mu_\ell}[F]) \leqslant \frac{1}{\ell+1}\mathbb{E}_{\mu_{\ell+1}^\psi}[F]\,\mathbb{H}\Big(\mu_{\ell+1}^{\psi,F}\Big|\mu_{\ell+1}^\psi\Big) = \frac{1}{\ell+1}\,\mathrm{Ent}_{\mu_{\ell+1}^\psi}[F]. \tag{A.23}$$

□

A.3. Bases-exchange dynamics. The bases-exchange dynamics is very particular in the sense that it is designed to be compatible with the decomposition in terms of the number of particles. This can be understood from the following two observations. First, the bases-exchange jump rates are invariant under conditioning on the position of particles. Indeed, let $J \subset [N]$ with $|J| \leqslant k-2$, let $(i,j) \in ([N]\setminus J)^2$ with $i \neq j$. Write π^J for the conditional measure $\pi(\cdot|\sigma_\ell = 1 \text{ for } \ell \in J)$ and let $\sigma \in \Omega_{N,k}$ satisfy $\sigma_i = 1$ and $\sigma_\ell = 1$ for $\ell \in J$. Then:

$$c_\pi(\sigma,\sigma^{ij}) = \frac{\pi(\sigma^{ij})}{\pi(\sigma)+\sum_{k:\sigma_k=-1}\pi(\sigma^{ik})} = \frac{\pi^J(\sigma^{ij})}{\pi^J(\sigma)+\sum_{k:\sigma_k=-1}\pi^J(\sigma^{ik})} = c_{\pi^J}(\sigma,\sigma^{ij}). \tag{A.24}$$

This will be used to relate the jump rates for π and π conditional on having + spins in a certain set.

The second property is the following. Let $\tilde{P}$ denote the semigroup of the bases-exchange dynamics (A.1), i.e., with jump rates $\tilde{c}(\sigma,\sigma') = k^{-1}c(\sigma,\sigma')$ (the k^{-1} ensures that $\tilde{P}$ is a stochastic matrix). It has the important property that it can be decomposed as the following product of stochastic matrices:

$$\tilde{P} = D_{k\to k-1}U_{k-1\to k}, \tag{A.25}$$

where $D_{k\to k-1}$ is the operation that removes one particle uniformly at random, and $U_{k-1\to k}$ adds a particle with probability relative to π:

$$D_{k\to k-1} : (\sigma,\sigma') \in \Omega_{N,k}\times\Omega_{N,k-1} \longmapsto \frac{1}{k}\mathbf{1}\Big\{\sigma' = \sigma^i \text{ for some } i\in I(\sigma)\Big\},$$

$$U_{k-1\to k} : (\sigma',\sigma) \in \Omega_{N,k-1}\times\Omega_{N,k} \longmapsto \mathbf{1}_{I(\sigma')\subset I(\sigma)}\frac{\pi(\sigma)}{\sum_{i\notin I(\sigma')}\pi((\sigma')^i)}. \tag{A.26}$$

Lemma A.3. *The modified log-Sobolev inequality of Theorem 2.1 holds for $k = 2$:*

$$\mathrm{Ent}_\pi(F) \leqslant D_\pi(F,\log F). \tag{A.27}$$

Proof. For any $F : \Omega_{N,k}\to\mathbb{R}_+$, writing π^F for the probability measure $F\pi/\mathbb{E}_\pi[F]$, one has

$$\mathrm{Ent}_\pi(\tilde P F) = \mathbb{E}_\pi[F]\,\mathbb{H}(\pi^F\tilde P|\pi) \leqslant \mathbb{E}_\pi[F]\,\mathbb{H}(\pi^F D_{k\to k-1}|\pi D_{k\to k-1}). \tag{A.28}$$

The first equality follows from the reversibility of $\tilde P$ with respect to π :

$$\pi(\sigma)\tilde P F(\sigma) = \mathbb{E}_\pi(\tilde P F\,1_\sigma) = \mathbb{E}_\pi[F]\,\mathbb{E}_{\pi^F}(\tilde P 1_\sigma) = \mathbb{E}_\pi[F]\,\pi^F\tilde P(\sigma). \tag{A.29}$$

The inequality in (A.28) is the so-called data-processing inequality (i.e., the Jensen inequality) and the fact that $U_{k-1\to k}$ is a stochastic matrix.

For $k = 2$, combining (A.28) and the contraction estimate (A.4) implies for any $F : \Omega_{N,2}\to\mathbb{R}_+$

$$\mathrm{Ent}_\pi(\tilde P F) \leqslant \mathbb{E}_\pi[F]\mathbb{H}(\pi^F D_{2\to1}|\pi D_{2\to1}) \leqslant \frac{1}{2}\mathbb{E}_\pi[F]\mathbb{H}(\pi^F|\pi) = \frac{1}{2}\,\mathrm{Ent}_\pi(F). \tag{A.30}$$

This contraction of the entropy is equivalent to the modified log-Sobolev inequality (A.1) when $k = 2$ by standard arguments, see, e.g., [3, Lemma 16] or [13]. □

Proof of Theorem 2.1. To clarify the exposition, we restrict to configurations with $k \geqslant 2$ particles in the following proof. The case $k = 1$ is also accessible, see [23].

Applying the modified log-Sobolev inequality for the two-particle measure from Lemma A.3 to μ_2, we get:

$$\mathbb{E}_{\nu_2}[\mathrm{Ent}_{\mu_2}(F)] \leqslant \mathbb{E}_{\nu_2}[D_{\mu_2}(F,\log F)]. \tag{A.31}$$

Under the measure $\nu_2\otimes\mu_2$, the variable $(\varphi,\sigma)\in\Omega_{N,k-2}\times\Omega_{N,k}$ is distributed such that $\sigma\sim\pi$ and $I(\varphi) = \{i : \varphi_i = 1\}$ is uniform on subsets of $I(\sigma) = \{i : \sigma_i = 1\}$ of size 2:

$$\mathbb{E}_{\nu_2}\mathbb{E}_{\mu_2}[F] = \sum_{\varphi\in\Omega_{N,k-2}}\frac{2}{k(k-1)}\sum_{\sigma\in\Omega_{N,k}}\pi(\sigma)1_{\sigma\geqslant\varphi}F(\sigma). \tag{A.32}$$

From this fact and using the observation (A.24) that the bases-exchange jump rates for μ_2 are the same as for π, the right-hand side of (A.31) is:

$$\mathbb{E}_{\nu_2}[D_{\mu_2}(F,\log F)] = \mathbb{E}_{\nu_2}\Big[\sum_{\sigma\in\Omega_{N,k}}\pi(\sigma|\sigma\geqslant\varphi)\sum_{i\in I(\sigma)\setminus I(\varphi)}\sum_{j\in[N]}c(\sigma,\sigma^{ij})\nabla_{ij}F(\sigma)\nabla_{ij}\log F(\sigma)\Big]. \tag{A.33}$$

Note that the sum over j has been extended from $[N]\setminus I(\varphi)$ to $[N]$, as exchanges between two sites i,j occupied by particles do no contribute due to $\nabla_{ij}F(\sigma) = 0$. From (A.32), we then get

$$\mathbb{E}_{\nu_2}[D_{\mu_2}(F,\log F)] = \frac{2}{k(k-1)}\sum_{\sigma\in\Omega_{N,k}}\pi(\sigma)\sum_{i,j\in[N]}\nabla_{ij}F(\sigma)\nabla_{ij}\log F(\sigma)\sum_{\varphi\in\Omega_{N,k-2}}\mathbf{1}_{\sigma\geqslant\varphi}\mathbf{1}_{\sigma_i=1=-\varphi_i}$$

$$= \frac{2}{k}D_\pi(F,\log F), \tag{A.34}$$

where we used that there are exactly $k-1$ configurations $\varphi\in\Omega_{N,k-2}$ with $\varphi\leqslant\sigma$ and $\sigma_i = 1 = -\varphi_i$. Combining with (A.15) proves the claim. □

A.4. Other dynamics. The entropy contraction argument of Proposition A.2 is not related to the specific choice of dynamics. In particular, to prove a modified log-Sobolev inequality for other dynamics, it is enough to prove it with one (or a few) particles only as we now explain.

Let $1 \leqslant k \leqslant \lfloor N/2 \rfloor$ and consider any family of jump rates $(q(\sigma,\sigma'))_{ij}$ on $\Omega_{N,k}$ such that π is the stationary measure of the associated dynamics (which does not even have to be reversible). Proposition A.2 applies unchanged: for any $1 \leqslant p \leqslant k$,

$$\mathrm{Ent}_\pi(F) = \mathbb{E}_{\nu_k}\big[\,\mathrm{Ent}_{\mu_k}(F)\big] \leqslant \frac{k}{p}\mathbb{E}_{\nu_p}[\mathrm{Ent}_{\mu_p}(F)]. \tag{A.35}$$

Take for instance $p = 1$. In that case $\mu_1 = \pi(\cdot|\varphi)$ ($\varphi \in \Omega_{N,k-1}$) can be identified as a measure on a single particle jumping on the $N-k+1$ sites of $[N] \setminus I(\varphi)$. Write $q^\varphi(i,j) = \mathbf{1}_{\sigma_i=1} q(\sigma,\sigma^{ij})$ for the associated jump rates when $\sigma \geqslant \varphi$. It is convenient to assume $q^\varphi(i,j) = 0$ even when $\sigma_j = 1$, say $q^\varphi(i,j) = 1$. This is to avoid considering a one-particle dynamics on possibly disconnected subsets of $[N]$. For such jumps, the value of $q^\varphi(i,j)$ does not affect the Dirichlet form for the test functions we consider as $F(\sigma^{ij}) = F(\sigma)$ if $\sigma_j = 1 = \sigma_i$.

Let $\gamma(q,\varphi)$ be the corresponding log-Sobolev constant (the argument is identical with the modified log-Sobolev constant):

$$\mathrm{Ent}_{\mu_1}(G) \leqslant \frac{2}{\gamma(q,\varphi)} \sum_{i\in[N]} \mu_1(i)\Big[\frac{1}{2}\sum_{j\in[N]} q^\varphi(i,j)\big[\sqrt{G(i)} - \sqrt{G(j)}\big]^2\Big], \qquad G : [N] \to \mathbb{R}_+. \tag{A.36}$$

Then:

$$\begin{aligned}
\mathrm{Ent}_\pi(F) &\leqslant k\mathbb{E}_{\nu_1}\Big[\frac{1}{4\gamma(q,\varphi)} \sum_{i,j\in[N]} \mu_1(i) q^\varphi(i,j)\big[\sqrt{F}(\varphi+\mathbf{1}_j) - \sqrt{F}(\varphi+\mathbf{1}_i)\big]^2\Big] \\
&= \frac{1}{4}\sum_{\sigma\in\Omega_{N,k}} \pi(\sigma) \sum_{i,j\in[N]} q(\sigma,\sigma^{ij})[\nabla_{ij}\sqrt{F}(\sigma)]^2 \Big(\sum_{\varphi\in\Omega_{N,k-1}} \frac{1}{\gamma(q,\varphi)} \mathbf{1}_{\sigma_i=1=-\varphi_i}\mathbf{1}_{\sigma\geqslant\varphi}\Big) \\
&= \frac{1}{4}\sum_{\sigma\in\Omega_{N,k}} \pi(\sigma) \sum_{i,j\in[N]} \frac{1}{\gamma(q,\sigma-\mathbf{1}_i)} q(\sigma,\sigma^{ij})[\nabla_{ij}\sqrt{F}(\sigma)]^2.
\end{aligned} \tag{A.37}$$

Proving the log-Sobolev inequality for the k-particle dynamics is therefore reduced to a bound on the 1-particle log-Sobolev constant uniformly on subsets of $[N]$ with size $N-(k-1)$.

B Interacting measures on the bases of a matroid

The same proof as that of Theorem 1.2 also implies the following result.

Example B.1. Let π be the uniform distribution on the bases of a matroid, viewed as a probability measure on $\omega \in \{0,1\}^N$, and consider the perturbed probability measure

$$\mu(\omega) \propto \prod_{e,f}(1+\varepsilon_{ef}\omega_e\omega_f)\pi(\omega). \tag{B.1}$$

There is $\bar\varepsilon_0 > 0$ independent of π such that if $\bar\varepsilon = \max_e \sum_f |\varepsilon_{ef}| \leqslant \bar\varepsilon_0$ then the bases exchange walk (down-up walk) on μ satisfies a uniform modified log-Sobolev inequality:

$$\mathrm{Ent}_\mu(F) \leqslant C(\bar\varepsilon) D_\mu(F, \log F). \tag{B.2}$$

Moreover, a spectral condition involving the matrix $(\log(\frac{1-\varepsilon_{ef}}{1+\varepsilon_{ef}}))_{e,f\in[N]}$ can be formulated as well.

Tyler Helmuth pointed out that the following application of the above extension is of interest: let π be the uniform spanning tree measure on a graph so that $\omega_e = 1$ if e is an edge in the spanning tree and 0 otherwise. The set of spanning tree on a graph is a prototypical example of a matroid, and the above example gives a modified log-Sobolev inequality for weakly non-uniform spanning trees, e.g., measures on spanning trees where parallel edges are favoured.

Proof. The proof is identical to the proof for Kawasaki dynamics. Indeed, converting to Ising variables $\sigma_e \in \{\pm 1\}$ defined by $\omega_e = \frac{1}{2}(\sigma_e + 1)$, one has

$$\prod_{e,f}(1 + \varepsilon_{ef}\omega_e\omega_f) \propto \prod_{e,f}(1 + \varepsilon_{ef}(\sigma_e + 1)(\sigma_f + 1)) \propto e^{-\frac{\beta}{2}\sum_{e,f} A_{ef}\sigma_e\sigma_f + \sum_e h_e\sigma_e} \tag{B.3}$$

with $\beta A_{ef} = \log(\frac{1-\varepsilon_{ef}}{1+\varepsilon_{ef}}) = O(\varepsilon_{ef})$ and some $h_e \in \mathbb{R}$ depending on the (ε_{ef}). By adding a multiple of the identity and normalising, we can again assume that A has spectrum in $[0, 1]$ and the condition on the (ε_{ef}) then implies that $\beta = O(\bar{\varepsilon})$ is sufficiently small.

In the situation of Kawasaki dynamics, π was the product measure conditioned on its sum, but the only assumptions about π we actually used in the proof of (2.31) is that it is the uniform distribution over the bases of a matroid (so that its generating polynomial is homogeneous and log-concave) and that the covariance bound (2.14) holds. Both properties follow from the strong Rayleigh property for the measure π satisfied by the uniform distribution conditioned on its sum, see Section 2.2. More generally, the results of [4, 16] imply the log-concavity for the uniform distribution of any matroid, and a version of the covariance bound with the factor 2 replaced by 4 can also be deduced from the log-concavity. □

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