A verification of Wilf's conjecture up to genus 100

Manuel Delgado, Shalom Eliahou, Jean Fromentin

A VERIFICATION OF WILF'S CONJECTURE UP TO GENUS 100

M. DELGADO, S. ELIAHOU AND J. FROMENTIN

Abstract. For a numerical semigroup $S \subseteq \mathbb{N}$, let m, e, c, g denote its multiplicity, embedding dimension, conductor and genus, respectively. Wilf's conjecture (1978) states that $e(c-g) \geq c$. As of 2023, Wilf's conjecture has been verified by computer up to genus $g \leq 66$. In this paper, we extend the verification of Wilf's conjecture up to genus $g \leq 100$. This is achieved by combining three main ingredients: (1) a theorem in 2020 settling Wilf's conjecture in the case $e \geq m/3$, (2) an efficient trimming of the tree $\mathcal{T}$ of numerical groups identifying and cutting out irrelevant subtrees, and (3) the implementation of a fast parallelized algorithm to construct the tree $\mathcal{T}$ up to a given genus. We further push the verification of Wilf's conjecture up to genus 120 in the particular case where m divides c. Finally, we unlock three previously unknown values of the number n_g of numerical semigroups of genus g, namely for $g = 73, 74, 75$.

1. Introduction

A *numerical semigroup* is a cofinite submonoid S of $\mathbb{N}$, i.e. a subset containing 0, stable under addition and with finite complement $\mathbb{N} \setminus S$. Equivalently, it is a set of the form $S = \langle a_1, \ldots, a_n \rangle = \mathbb{N}a_1 + \cdots + \mathbb{N}a_n$ where $a_1, \ldots, a_n$ are positive integers with $\gcd(a_1, \ldots, a_n) = 1$, called *generators* of S. The least such n is usually denoted e and called the *embedding dimension* of S, see below.

Let S be a numerical semigroup and $S^* = S \setminus \{0\}$. A *primitive element* of S is an element $a \in S^* \setminus (S^* + S^*)$, i.e. a nonzero element of S which is not the sum of two nonzero elements of S. Let $P = P(S)$ denote the set of primitive elements of S. It is easy to see that P is finite and is the unique minimal generating set of S. The *embedding dimension* of S is $e = e(S) = |P|$, the *multiplicity* of S is $m = m(S) = \min S^*$, the *Frobenius number* of S is $F = F(S) = \max(\mathbb{Z} \setminus S)$ and the *conductor* of S is $c = c(S) = F + 1$, satisfying $c + \mathbb{N} \subseteq S$ and minimal with respect to that property. The *genus* of S is $g = g(S) = |\mathbb{N} \setminus S|$, its number of gaps. We partition S as

$$S = L \sqcup R,$$

where $L = L(S) = \{a \in S \mid a < F(S)\}$ and $R = R(S) = \{a \in S \mid a > F(S)\}$, the *left part* and *right part* of S, respectively.

Wilf's conjecture is the inequality

$$e|L| \geq c.$$

Date: October 10, 2023.

2010 *Mathematics Subject Classification.* 20M14.

Key words and phrases. Numerical semigroup; genus.

The first author was partially supported by CMUP, a member of LASI, which is financed by national funds through FCT – Fundação para a Ciência e a Tecnologia, I.P., under the projects with reference UIDB/00144/2020 and UIDP/00144/2020.

The authors gratefully acknowledge support from ULCO for visits of the first author to Calais in May 2019 and May 2023.

While this conjecture remains open since 1978, it has been verified in several cases. For convenience, we list some of them in a single statement together with their respective references.

Theorem 1.1. *Wilf's conjecture holds for all numerical semigroups satisfying one of the following conditions:*

(1) $e \leq 3$ [17]
(2) $c \leq 3m$ [12]
(3) $e \geq m/3$ [13]
(4) $|L| \leq 12$ [16]
(5) $m \leq 19$ [20]

See also [10] for an extensive recent survey of partial results on Wilf's conjecture, and [2, 5, 8, 11, 15, 18, 19, 21, 24, 25, 26, 27] for some other relevant papers.

The first significant verification by computer of Wilf's conjecture up to a given genus was accomplished in 2008 by Bras-Amorós [2]. There, Wilf's conjecture was announced to hold for all numerical semigroups of genus $g \leq 50$. This result was extended to genus $g \leq 60$ in 2016 by Fromentin-Hivert [18], to genus $g \leq 65$ in 2021 by Bras-Amorós and Rodríguez [3], and finally to genus $g \leq 66$ in 2023 by Bras-Amorós [4].

In this paper, we extend the verification of Wilf's conjecture up to genus $g \leq 100$. See Theorem 3.5 below. This is achieved by combining three main ingredients:

(1) The main result of [13] stating that Wilf's conjecture holds in case $e \geq m/3$.
(2) The method from [9] to exploit the above result by efficiently trimming the tree $\mathcal{T}$ of numerical semigroups (see below) and thereby drastically reduce the number of numerical semigroups up to a given large genus to test for Wilf's conjecture.
(3) A fast parallelisable algorithm to enumerate all numerical semigroups up to a given large genus [18].

For general reference books on numerical semigroups, see [22, 23].

2. The tree $\mathcal{T}$

The set of all numerical semigroups can be organised in a rooted tree $\mathcal{T}$, with root $\mathbb{N}$ of genus 0, such that all numerical semigroups of genus g lie at distance g from the root. Before recalling its construction, we introduce some terminology.

2.1. **Left and right primitive elements.** Let S be a numerical semigroup. Recall the above partition $S = L \sqcup R$ relative to the Frobenius number $F(S)$. Accordingly, we partition the set P of primitive elements of S as

$$P = (P \cap L) \sqcup (P \cap R).$$

We call *left primitive* the elements of $P \cap L$ and *right primitive* those of $P \cap R$. We denote

$$\begin{aligned} e &= |P|, \\ e_l &= |P \cap L|, \\ e_r &= |P \cap R|. \end{aligned}$$

Thus

$$e = e_l + e_r.$$

2.2. **The children of S.** We now briefly recall the construction of the tree $\mathcal{T}$. Let S be a numerical semigroup, and let $a \in P \cap R$ be a right primitive element of S, if any. Then the set $S' = S \setminus \{a\}$ is still a numerical semigroup, as easily seen.

A *child* of S in $\mathcal{T}$ is any numerical semigroup of the above form

$$S' = S \setminus \{a\}$$

where $a \in P \cap R$. Clearly, the number of children of S in $\mathcal{T}$ is equal to e_r.

The Frobenius number, genus and multiplicity of a child $S' = S\setminus\{a\}$ of S with $a \in P\cap R$ are easy to determine. Indeed, one has

$$\begin{cases} F(S') = a, \\ g(S') = g(S) + 1 \end{cases}$$

as easily seen. As for the multiplicity of S', one has

$$m(S') = m(S)$$

whenever $c(S) > m(S)$, see Proposition 3.2. The only numerical semigroups S for which $c(S) = m(S)$ are the so-called *ordinary* or *superficial* numerical semigroups O_m, defined for any $m \geq 1$ by

$$O_m = \langle m, m+1, \ldots, 2m-1 \rangle = \{0\} \sqcup (m + \mathbb{N}).$$

Then O_m is of multiplicity m and conductor $c = m$. It has exactly m children, namely $O_m \setminus \{m+i\}$ for $0 \leq i \leq m-1$. They still have multiplicity m for $1 \leq i \leq m-1$. The only case where the multiplicity differs is at $i = 0$, for which $O_m \setminus \{m\} = O_{m+1}$ is of multiplicity $m+1$.

Note finally that any numerical semigroup of multiplicity $m \geq 1$ is a descendant of O_m in the tree $\mathcal{T}$.

2.3. **Numerical semigroups of given genus.** For $g \in \mathbb{N}$, let n_g denote the number of numerical semigroups of genus g. It is well known that n_g is finite. The first few values of n_g are

$$(n_0, n_1, n_2, n_3, n_4, n_5, n_6) = (1, 1, 2, 4, 7, 12, 23).$$

In her famous paper [2], Maria Bras-Amorós conjectured that n_g behaves like the gth Fibonacci number F_g, with a growth rate tending to the golden ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ and satisfying

$$n_g \geq n_{g-1} + n_{g-2} \tag{1}$$

for all $g \geq 2$. The conjectured growth rate of n_g was subsequently confirmed by A. Zhai [28], implying $n_g \geq n_{g-1}$ for g large enough. Yet the conjectured inequality (1) remains widely open to this day, as is the case for the much weaker inequality $n_g \geq n_{g-1}$ for all $g \geq 1$.

Until now, the exact value of n_g had been computed up to $g \leq 72$. In this paper, with massive computations using a distributed version of the fast algorithms in [18], we unlock three new values, namely

$$\begin{aligned} n_{73} &= 6\,832\,823\,876\,813\,577, \\ n_{74} &= 11\,067\,092\,660\,179\,522, \\ n_{75} &= 17\,924\,213\,336\,425\,401. \end{aligned}$$

See Section 5 for more details.

3. Trimming $\mathcal{T}$

The main ideas proposed in [9] to trim the tree $\mathcal{T}$ so as to drastically reduce the verification of Wilf's conjecture up to a given genus G are the following ones.

Notation 3.1. *Let $G \geq 1$. We denote by $\mathcal{T}_G$ the subtree of $\mathcal{T}$ consisting of all numerical semigroups S of genus $g(S) \leq G$.*

Proposition 3.2. *Let $S \neq O_m$ be a numerical semigroup of multiplicity m. Let S' be a child of S. Then*

(1) $m(S') = m(S)$
(2) $e_l(S') \geq e_l(S)$
(3) $e(S) - 1 \leq e(S') \leq e(S)$

Proof. We have $S' = S \setminus \{a\}$ for some right primitive element $a \in P \cap R$. Denote m, c, P, L the multiplicity, conductor, primitive elements and left part of S, respectively, and m', c', P', L' the corresponding objects for S'.

Since $a \geq c > m$, it follows that $\min(S')^* = \min S^*$, i.e. $m' = m$. Moreover, since $c' = a + 1 > c$, it follows that $L \subset L'$, and any left primitive element $a \in P \cap L$ of S remains left primitive in S'. Thus $P \cap L \subseteq P' \cap L'$, implying $e_l(S') \geq e_l(S)$. Finally, as easily seen, one has either $P' = P \setminus \{a\}$ or $P' = P \setminus \{a\} \sqcup \{a + m\}$. The former occurs when $a + m = s_1 + s_2$ for some pair $\{s_1, s_2\} \subset S^*$ distinct from $\{a, m\}$, whereas the latter occurs when $a + m$ has no other representation as an element of $S^* + S^*$. Therefore $e(S) - 1 \leq e(S') \leq e(S)$. □

Corollary 3.3. *Let $S \neq O_m$ be a numerical semigroup of multiplicity m such that $e_l \geq m/3$. Then all descendants T of S in $\mathcal{T}$ satisfy Wilf's conjecture.*

Proof. Let T be a descendant of S. Then, by a repeated application of Proposition 3.2 (2) for each generation of children going down from S to T, we have

$$e(T) \geq e_l(T) \geq e_l(S) \geq m/3 = m(T)/3.$$

It then follows from Theorem 1.1 (3) that T satisfies Wilf's conjecture. □

Corollary 3.4. *Let $G \in \mathbb{N}^*$. Let S be a numerical semigroup of genus $g \leq G$ such that $e \geq m/3 + (G - g)$. Then all descendants T of S of genus $g(T) \leq G$ satisfy Wilf's conjecture.*

Proof. Let $h = g(T)$. Then $g \leq h \leq G$ by hypothesis, and T is an $(h-g)$th descendant of S. Now at each step from S down to T in $\mathcal{T}$, the number of primitive elements diminishes by at most 1 as stated in Proposition 3.2 (3). Hence

$$e(T) \geq e - (h - g) \geq e + g - G \geq m/3 + (G - g) + g - G = m(T)/3.$$

It then follows from Theorem 1.1 (3) that T satisfies Wilf's conjecture. □

Consequently, when exploring $\mathcal{T}_G$ to probe Wilf's conjecture up to a given maximal genus G, the subtree rooted at any numerical semigroup S satisfying Corollary 3.3 or 3.4 can be completely cut off from $\mathcal{T}_G$. What remains after this systematic trimming of $\mathcal{T}_G$ is a subtree $\mathcal{T}_G(3)$ all of whose numerical semigroups S satisfy

(1) $e_l(S) < m(S)/3$,
(2) $e(S) < m(S)/3 + (G - g(S))$.

Summarizing, to probe Wilf's conjecture up to genus G, we only need to test those numerical semigroups S in $\mathcal{T}_G(3)$. This is a significant reduction, as the subtree $\mathcal{T}_G(3)$ turns out to be much smaller than $\mathcal{T}_G$. For instance, for $G = 100$, we found that $\mathcal{T}_{100}(3)$ counts approximately 4.5×10^{15} nodes, as compared to the full tree $\mathcal{T}_{100}$ counting roughly $n_{75} \times \phi^{25} \approx 3.2 \times 10^{19}$ nodes. (See Section 5.) This level of reduction allowed us to reach one of the main computational results of this paper.

Theorem 3.5. *Wilf's conjecture holds for all numerical semigroups of genus $g \leq 100$.* □

Proof. By computer, we constructed the much smaller subtree $\mathcal{T}_G(3) \subset \mathcal{T}_G$ for $G = 100$, and checked that all of its nodes satisfy Wilf's conjecture. The claimed statement follows from Corollary 3.3 and 3.4. □

3.1. **Further trimming.** There are ways to further trim $\mathcal{T}_G(3)$ and thus further reduce the number of numerical semigroups of genus $g \leq G$ to test for Wilf's conjecture. They may be used whenever the added computational cost remains reasonable.

Here is a useful instance, exploiting the result that numerical semigroups with embedding dimension $e \leq 3$ satisfy Wilf's conjecture [17], as recalled in Theorem 1.1 (1).

Proposition 3.6. *Let S be a numerical semigroup such that $c \geq 4g/3$. Then S satisfies Wilf's conjecture.*

Proof. As $g = c - |L|$, the hypothesis yields $c \geq 4(c - |L|)/3$, i.e. $4|L| \geq c$. If $e \geq 4$ then $e|L| \geq 4|L| \geq c$, so S satisfies Wilf's conjecture. And if $e \leq 3$, the same conclusion holds by Theorem 1.1 (1). □

Corollary 3.7. *Let $G \in \mathbb{N}^*$. Let S be a numerical semigroup of genus $g \leq G$ such that $|L| \geq G/3$. Then all descendants T of S of genus $g(T) \leq G$ satisfy Wilf's conjecture.*

Proof. Let T be a descendant of S of genus $g(T) \leq G$. We have

$$\begin{aligned} c(T) &= |L(T)| + g(T) \\ &\geq |L| + g(T) \\ &\geq G/3 + g(T) \\ &\geq g(T)/3 + g(T) \\ &= 4g(T)/3. \end{aligned}$$

Proposition 3.6 then implies that T satisfies Wilf's conjecture. □

4. The case $c \in m\mathbb{N}$

Given a numerical semigroup S, consider the Euclidean division of its conductor c by its multiplicity m with nonpositive remainder:

$$c = qm - \rho, \quad 0 \leq \rho \leq m - 1. \tag{2}$$

As argued in [14], the particular case $c = qm$, i.e. with $\rho = 0$, might well be the heart of Wilf's conjecture. Indeed, the proofs of Wilf's conjecture in either case $c \leq 3m$ [12] or $e \geq m/3$ [13] can be significantly shortened in this particular case. Moreover, the first five instances of the rare occurrence $W_0(S) < 0$ all belong to this case [15].

Definition 4.1. *A numerical semigroup S is* special *if its multiplicity m divides its conductor c.*

For instance, the ordinary numerical semigroup $O_m = \{0\} \cup (m + \mathbb{N})$ is special since it satisfies $c = m$. The above discussion leads one to think that, if Wilf's conjecture is false, then some counterexamples might well be special. Thus, the special case should be given priority in related research work.

In this context, the validity of Wilf's conjecture has been extended to the case $e \geq m/4$ for special numerical semigroups. More generally, using the notation in (2), the result reads as follows [14].

Theorem 4.2. *Let S be a numerical semigroup satisfying $e \geq m/4$. Then $e|L| \geq c-m+\rho$. Morever, if S is special, then $e|L| \geq c$, i.e. S satisfies Wilf's conjecture.*

Let us see how we used here this theorem to push the verification of Wilf's conjecture up to genus $G = 120$ in the special case $c \in m\mathbb{N}$.

To start with, the method in Section 3 to trim $\mathcal{T}$ or its bounded version $\mathcal{T}_G$ to get the greatly reduced relevant subtree $\mathcal{T}_G(3)$ in case $e < m/3$ works as well to retain only those numerical semigroups such that $e < m/d$ for some fixed integer $d \geq 3$. This yields a subtree $\mathcal{T}_G(d)$ of $\mathcal{T}_G$ all of whose numerical semigroups S satisfy

(1) $e_l(S) < m(S)/d$,
(2) $e(S) < m(S)/d + (G - g(S))$.

By Theorem 4.2, the case of relevance to us here is $d = 4$. The subtree $\mathcal{T}_G(4)$ may be further trimmed by exploiting the added hypothesis $c = qm$, as explained below. For convenience, we introduce the following definition.

4.1. **Special trimming.** We start with a condition on a non-special numerical semigroup S ensuring that no descendant of S is special.

Proposition 4.3. *Let S be a non-special numerical semigroup. Assume further that S has no right primitive element $a \in P \cap R$ such that $a \equiv -1 \bmod m$. Then no descendant T of S is special.*

Proof. Let $S' = S \setminus \{a\}$ with $a \in P \cap R$ be a child of S. As usual, we denote by m, F, c, P the multiplicity, Frobenius number, conductor and primitive elements of S, and by m', F', c', P' the corresponding objects for S'. First of all, $S \neq O_m$ since O_m is special. Hence $m' = m$ by Proposition 3.2. We have $F' = a$ and so $c' = a + 1$. Since $a \not\equiv -1 \bmod m$ by hypothesis, and $m' = m$, we have $c' \not\equiv 0 \bmod m'$. Therefore S' is not special. Moreover, no right primitive element of S' can be congruent to $-1 \bmod m'$ since $P' = P \setminus \{a\}$ or $P \setminus \{a\} \sqcup \{a + m\}$ as seen in the proof of Proposition 3.2. Hence S' satisfies the same hypotheses as S and we are done by induction on the distance between S and any of its descendant T. □

We denote by $\mathcal{T}'_G(d)$ the subtree obtained from $\mathcal{T}_G(d)$ by the above special trimming.

4.2. **Outcome.** Having constructed the subtree $\mathcal{T}'_G(d)$ for $d = 4$ and maximal genus $G = 120$, we ended up with the following computational result.

Theorem 4.4. *Wilf's conjecture holds for all special numerical semigroups S of genus $g \leq 120$.*

Proof. For each numerical semigroup S in the subtree $\mathcal{T}'_{120}(4)$, we computed $W(S) = e|L| - c$ and found that $W(S) \geq 0$, i.e. that S satisfies Wilf's conjecture. All numerical semigroups of genus $g \leq 120$ outside $\mathcal{T}'_{120}(4)$ satisfy $e \geq m/4$ or are not special. Combining the computational results on $\mathcal{T}'_{120}(4)$ and Theorem 4.2, we conclude that all special numerical semigroups of genus $g \leq 120$ satisfy Wilf's conjecture. □

5. Computational results

Our experiments were carried out on the computational platform CALCULCO [1] using an adapted and distributed version of algorithms given in [18]. Source codes are available on GitHub [6]. These experiments were run for two weeks during summer 2023, when the platform CALCULCO, shared by the whole university, had a much better availability. The distributed computation ran in parallel on up to 1500 cores, with a special-purpose finely tuned dynamic balancing of the tasks of the respective cores. Early experiments were conducted using the GAP package *NumericalSgps* [7].

5.1. **Exploration of $\mathcal{T}_{100}(3)$.** The following table gives the number t_g of numerical semigroups of genus g in the trimmed subtree $\mathcal{T}_{100}(3)$.

g	t_g	g	t_g	g	t_g
0	1	34	183 029	68	78 371 434 661
1	1	35	268 072	69	114 677 728 452
2	1	36	392 646	70	167 759 612 028
3	1	37	575 237	71	245 327 971 537
4	2	38	842 632	72	358 502 883 157
5	3	39	1 234 294	73	523 268 737 918
6	4	40	1 808 003	74	762 512 542 535
7	6	41	2 648 088	75	1 108 797 952 894
8	9	42	3 878 863	76	1 608 029 199 893
9	13	43	5 681 044	77	2 323 793 898 612
10	19	44	8 320 312	78	3 343 540 732 459
11	28	45	12 184 995	79	4 786 270 172 173
12	41	46	17 844 810	80	6 811 932 935 500
13	60	47	26 134 470	81	9 633 271 340 874
14	88	48	38 275 824	82	13 524 365 031 892
15	129	49	56 052 677	83	18 835 200 708 312
16	189	50	82 079 784	84	26 006 592 640 071
17	277	51	120 191 188	85	35 586 447 144 420
18	406	52	176 010 965	86	48 235 329 094 317
19	595	53	257 743 713	87	64 707 333 203 651
20	872	54	377 377 331	88	85 854 587 472 809
21	1 278	55	552 530 112	89	112 592 214 454 082
22	1 870	56	809 003 680	90	145 836 255 324 616
23	2 741	57	1 184 568 132	91	186 464 879 487 116
24	4 019	58	1 734 367 942	92	234 882 687 403 501
25	5 888	59	2 539 101 162	93	290 865 320 646 279
26	8 622	60	3 717 160 466	94	353 167 513 519 152
27	12 634	61	5 441 979 825	95	419 043 410 131 476
28	18 513	62	7 967 290 270	96	483 141 727 918 288
29	27 128	63	11 663 422 314	97	534 768 932 735 380
30	39 749	64	17 072 801 062	98	557 018 016 635 015
31	58 192	65	24 990 316 134	99	522 447 041 258 147
32	85 285	66	36 581 421 194	100	389 883 092 218 470
33	124 928	67	53 548 048 989		

The number of nodes in $\mathcal{T}_{100}(3)$ is equal to $4\,554\,895\,996\,302\,538 \approx 4.5\times10^{15}$. We checked that all of these numerical semigroups satisfy Wilf's conjecture, whence Theorem 3.5.

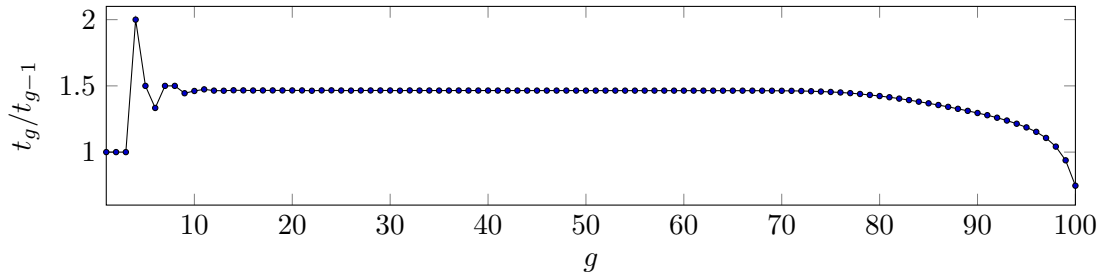
5.2. **Exploration of $\mathcal{T}'_{120}(4)$.** The following table gives the number t'_g of numerical semigroups of genus g in the subtree $\mathcal{T}'_{120}(4)$.

g	t'_g	g	t'_g	g	t'_g
0	1	41	200 122	82	98 068 856 236
1	1	42	278 371	83	136 816 899 688
2	1	43	369 269	84	188 124 954 369
3	1	44	510 693	85	246 090 486 177
4	1	45	699 711	86	336 614 411 642
5	2	46	975 178	87	446 053 184 686
6	3	47	1 342 072	88	606 309 920 447
7	4	48	1 876 236	89	801 094 605 082
8	5	49	2 608 650	90	1 075 828 933 040
9	7	50	3 458 914	91	1 412 830 185 991
10	10	51	4 794 003	92	1 734 561 883 001
11	14	52	6 551 846	93	2 241 022 409 143
12	19	53	9 147 280	94	2 760 341 832 836
13	26	54	12 582 317	95	3 516 881 984 622
14	36	55	17 614 571	96	4 299 648 368 842
15	49	56	24 483 065	97	5 388 292 247 874
16	67	57	32 457 488	98	6 544 140 003 541
17	93	58	45 063 305	99	7 117 478 807 043
18	128	59	61 488 392	100	8 451 858 568 006
19	177	60	85 953 600	101	9 261 220 874 872
20	245	61	118 226 772	102	10 843 978 899 677
21	340	62	165 696 926	103	11 904 592 718 137
22	455	63	230 209 288	104	13 736 474 753 272
23	624	64	305 247 236	105	15 172 362 267 910
24	863	65	424 326 522	106	13 564 852 076 075
25	1 194	66	578 281 772	107	14 519 416 932 134
26	1 647	67	809 156 311	108	13 448 628 571 779
27	2 286	68	1 112 860 410	109	14 268 658 755 506
28	3 180	69	1 561 100 560	110	13 799 851 102 125
29	4 234	70	2 168 306 879	111	14 508 263 526 352
30	5 823	71	2 876 214 827	112	14 534 198 939 692
31	8 035	72	4 002 364 427	113	11 099 577 260 819
32	11 135	73	5 449 537 905	114	10 699 792 288 594
33	15 341	74	7 630 005 823	115	9 284 396 042 115
34	21 369	75	10 492 890 135	116	8 422 462 308 726
35	29 722	76	14 726 118 585	117	6 954 667 530 694
36	39 491	77	20 444 255 810	118	5 013 091 736 917
37	54 511	78	27 121 425 859	119	2 599 964 149 312
38	74 910	79	37 736 682 161	120	289 298 823 487
39	104 183	80	51 267 633 069		
40	143 431	81	71 625 262 707		

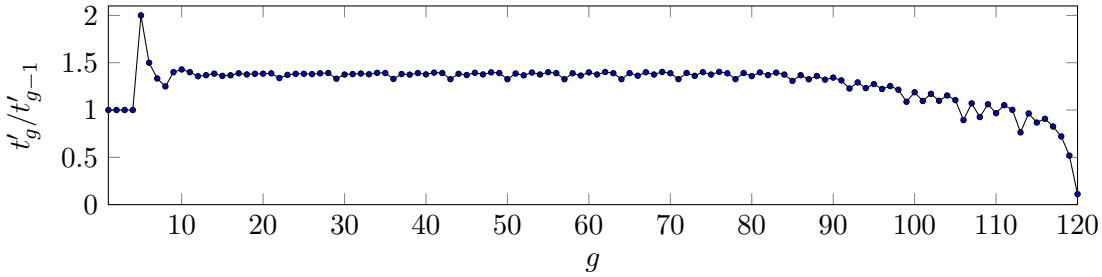
The number of nodes in $\mathcal{T}'_{120}(4)$ is equal to $261\,588\,966\,883\,192 \approx 2.6 \times 10^{14}$. We checked that all of these numerical semigroups satisfy Wilf's conjecture, whence Theorem 4.4.

5.3. **Growth rates.** Recall that t_g and t'_g denote the number of numerical semigroups of genus g in $\mathcal{T}_{100}(3)$ and $\mathcal{T}'_{120}(4)$, respectively. Recall also that the growth rate of n_g tends to $\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$ [28].

As illustrated by the following figure, the growth rate of t_g seems to stabilize around 1.46 for $g \in [10, 70]$.



The growth rate of t'_g is a little more chaotic but seems to have the same behavior, as illustrated by the following figure.



5.4. **New values of n_g.** We took advantage of our distributed version of the algorithms in [18] to compute the values of n_g for all $g \leq 75$. See Table 1. The total number of numerical semigroups of genus $g \leq 75$ is equal to $46\,844\,766\,834\,597\,649 \approx 4.6 \times 10^{16}$.

g	n_g	g	n_g	g	n_g
0	1	26	770 832	52	266 815 155 103
1	1	27	1 270 267	53	433 317 458 741
2	2	28	2 091 030	54	703 569 992 121
3	4	29	3 437 839	55	1 142 140 736 859
4	7	30	5 646 773	56	1 853 737 832 107
5	12	31	9 266 788	57	3 008 140 981 820
6	23	32	15 195 070	58	4 880 606 790 010
7	39	33	24 896 206	59	7 917 344 087 695
8	67	34	40 761 087	60	12 841 603 251 351
9	118	35	66 687 201	61	20 825 558 002 053
10	204	36	109 032 500	62	33 768 763 536 686
11	343	37	178 158 289	63	54 749 244 915 730
12	592	38	290 939 807	64	88 754 191 073 328
13	1 001	39	474 851 445	65	143 863 484 925 550
14	1 693	40	774 614 284	66	233 166 577 125 714
15	2 857	41	1 262 992 840	67	377 866 907 506 273
16	4 806	42	2 058 356 522	68	612 309 308 257 800
17	8 045	43	3 353 191 846	69	992 121 118 414 851
18	13 467	44	5 460 401 576	70	1 607 394 814 170 158
19	22 464	45	8 888 486 816	71	2 604 033 182 682 582
20	37 396	46	14 463 633 648	72	4 218 309 716 540 814
21	62 194	47	23 527 845 502	73	**6 832 823 876 813 577**
22	103 246	48	38 260 496 374	74	**11 067 092 660 179 522**
23	170 963	49	62 200 036 752	75	**17 924 213 336 425 401**
24	282 828	50	101 090 300 128		
25	467 224	51	164 253 200 784		

TABLE 1. Number n_g of numerical semigroups of genus $g \leq 75$. Previously unknown values are in bold font.

References

[1] CALCULCO platform. `https://www-calculco.univ-littoral.fr`, 2023.

[2] Maria Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. *Semigroup Forum*, 76(2):379–384, 2008.

[3] Maria Bras-Amorós and César Marín Rodríguez. New Eliahou semigroups and verification of the Wilf conjecture for genus up to 65. 12898:17–27, [2021] ©2021.

[4] Maria Bras-Amorós. On the seeds and the great-grandchildren of a numerical semigroup, 2023. To appear in Mathematics of Computation.

[5] Winfried Bruns, Pedro García-Sánchez, Christopher O'Neill, and Dane Wilburne. Wilf's conjecture in fixed multiplicity. *Internat. J. Algebra Comput.*, 30(4):861–882, 2020.

[6] M. Delgado, S. Eliahou, and J. Fromentin. Github - trim_tree_semigroups. `https://github.com/jfromentin/trim_tree_semigroups`, 2023.

[7] M. Delgado, P. A. Garcia-Sanchez, and J. Morais. NumericalSgps, a package for numerical semigroups, Version 1.3.1, July 2022. Refereed GAP package.

[8] Manuel Delgado. On a question of Eliahou and a conjecture of Wilf. *Math. Z.*, 288(1-2):595–627, 2018.

[9] Manuel Delgado. Trimming the numerical semigroups tree to probe Wilf's conjecture to higher genus, 2019. 40 pages.

[10] Manuel Delgado. Conjecture of Wilf: A survey. In V. Barucci, S. Chapman, M. D'Anna, and R. Fröberg, editors, *Numerical Semigroups*, volume 40 of *Springer INdAM Ser.*, pages 39–62. Springer, Cham, 2020.

[11] David E. Dobbs and Gretchen L. Matthews. On a question of Wilf concerning numerical semigroups. In *Focus on commutative rings research*, pages 193–202. Nova Sci. Publ., New York, 2006.

[12] Shalom Eliahou. Wilf's conjecture and Macaulay's theorem. *J. Eur. Math. Soc. (JEMS)*, 20(9):2105–2129, 2018.

[13] Shalom Eliahou. A graph-theoretic approach to Wilf 's conjecture. *Electron. J. Combin.*, 27 (2):Article No P2.15, 31 pages, 2020.

[14] Shalom Eliahou. Divsets, numerical semigroups and Wilf's conjecture. `https://hal.science/hal-04234167`, October 2023.

[15] Shalom Eliahou and Jean Fromentin. Near-misses in Wilf's conjecture. *Semigroup Forum*, 98(2):285–298, 2019.

[16] Shalom Eliahou and Daniel Marín-Aragón. On numerical semigroups with at most 12 left elements. *Comm. Algebra*, 49(6):2402–2422, 2021.

[17] R. Fröberg, C. Gottlieb, and R. Häggkvist. On numerical semigroups. *Semigroup Forum*, 35(1):63–83, 1987.

[18] Jean Fromentin and Florent Hivert. Exploring the tree of numerical semigroups. *Math. Comp.*, 85(301):2553–2568, 2016.

[19] Nathan Kaplan. Counting numerical semigroups by genus and some cases of a question of Wilf. *J. Pure Appl. Algebra*, 216(5):1016–1032, 2012.

[20] Jonathan Kliem and Christian Stump. A new face iterator for polyhedra and for more general finite locally branched lattices. *Discrete Comput. Geom.*, 67(4):1147–1173, 2022.

[21] Alessio Moscariello and Alessio Sammartano. On a conjecture by Wilf about the Frobenius number. *Math. Z.*, 280(1-2):47–53, 2015.

[22] J. L. Ramírez-Alfonsín. *The Diophantine Frobenius problem*, volume 30 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2005.

[23] J. C. Rosales and P. A. García Sánchez. *Numerical semigroups*, volume 20 of *Developments in Mathematics*. Springer, New York, 2009.

[24] Alessio Sammartano. Numerical semigroups with large embedding dimension satisfy Wilf's conjecture. *Semigroup Forum*, 85(3):439–447, 2012.

[25] Ernst S. Selmer. On the linear Diophantine problem of Frobenius. *J. Reine Angew. Math.*, 293(294):1–17, 1977.

[26] J. J. Sylvester. On Subinvariants, i.e. Semi-Invariants to Binary Quantics of an Unlimited Order. *Amer. J. Math.*, 5(1-4):79–136, 1882.

[27] Herbert S. Wilf. A circle-of-lights algorithm for the "money-changing problem". *Amer. Math. Monthly*, 85(7):562–565, 1978.

[28] Alex Zhai. Fibonacci-like growth of numerical semigroups of a given genus. *Semigroup Forum*, 86(3):634–662, 2013.

Authors addresses

• Manuel Delgado,
CMUP–Centro de Matemática da Universidade do Porto, Departamento de Matemática, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre s/n, 4169– 007 Porto, Portugal
mdelgado@fc.up.pt

• Shalom Eliahou[a,b],
[a]Univ. Littoral Côte d'Opale, UR 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France
[b]CNRS, FR2037, France
eliahou@univ-littoral.fr

• Jean Fromentin[a,b],
[a]Univ. Littoral Côte d'Opale, UR 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France
[b]CNRS, FR2037, France
fromentin@univ-littoral.fr