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Tobias Metzloff

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| Tobias Metzloff. On symmetry adapted bases in trigonometric optimization. 2023. hal-04236200

**HAL Id: hal-04236200**

**<https://hal.science/hal-04236200>**

Preprint submitted on 10 Oct 2023

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# On symmetry adapted bases in trigonometric optimization

Tobias Metzlaff\*

Monday 9<sup>th</sup> October, 2023

## Abstract

We consider the problem of computing the global minimum of a trigonometric polynomial  $f$ , when it is invariant under the exponential action of a Weyl group. We follow an established relaxation strategy in order to obtain a converging hierarchy of lower bounds. Those are obtained by numerically solving semi-definite programs (SDPs) on the cone of positive semi-definite Hermitian Toeplitz matrices, which is outlined in the book of Dumitrescu [Dum07]. Subsequently, we explain that the Weyl group has an induced action on the feasible region of each SDP and show that one may restrict to the invariant matrices. Then we construct a symmetry adapted basis tailored to this Weyl group action, which allows us to block-diagonalize the matrices and thus reduce the computational effort to solve the SDP.

The approach is novel for trigonometric optimization and complements the one that was proposed as a poster at the ISSAC 2022 conference [HMMR22] and later extended to [HMMR23]. In the previous work, we first used the invariance of  $f$  to obtain a classical polynomial optimization problem on the orbit space and subsequently relaxed the problem to an SDP. Now, we first make the relaxation and then exploit invariance.

Partial results of this article have been presented as a poster at the ISSAC 2023 conference [Met23].

**Keywords:** Trigonometric Optimization, Crystallographic Symmetry, Weyl Groups, Root Systems, Lattices

**MSC:** 05C15 17B22 33C52 52C07 90C23

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This work has been supported by the Deutsche Forschungsgemeinschaft transregional collaborative research centre (SFB–TRR) 195 “Symbolic Tools in Mathematics and their Application”.

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# 1 Introduction

Given an  $n$ -dimensional lattice  $\Omega \subseteq \mathbb{R}^n$ , a trigonometric polynomial is a function

$$f : \mathbb{R}^n \rightarrow \mathbb{R}, u \mapsto f(u) := \sum_{\mu \in \Omega} c_\mu \exp(-2\pi i \langle \mu, u \rangle),$$

where  $\langle \cdot, \cdot \rangle$  denotes the Euclidean scalar product and the finitely many nonzero coefficients  $c_\mu \in \mathbb{C}$  satisfy  $c_{-\mu} = \overline{c_\mu}$ . Such functions are periodic and assume their global minimum (and maximum) on the compact periodicity domain. We present a strategy to compute the minimum under symmetry assumptions.

In this article,  $\Omega$  is the weight lattice of a crystallographic root system in  $\mathbb{R}^n$ . The distinguishing feature of these lattices is their intrinsic symmetry. This latter is given by the so called Weyl group  $\mathcal{W}$ , a finite subgroup of the automorphism group  $\text{GL}(\mathbb{R}^n)$  generated by orthogonal reflections w.r.t.  $\langle \cdot, \cdot \rangle$ . It is this feature that we emphasize and offer to exploit in an optimization context. Given a group element  $s \in \mathcal{W}$ , it acts on a trigonometric polynomial  $f$  by

$$(s, f(u)) \mapsto f(s^{-1}(u)) = \sum_{\mu \in \Omega} c_\mu \exp(-2\pi i \langle \mu, s^{-1}(u) \rangle) = \sum_{\mu \in \Omega} c_\mu \exp(-2\pi i \langle s(\mu), u \rangle).$$

We consider the trigonometric optimization problem

$$f^* := \min_{u \in \mathbb{R}^n} f(u) \tag{1.1}$$

under the assumption of  $\mathcal{W}$ -invariance, that is, for all  $s \in \mathcal{W}$ , we have  $f(s^{-1}(u)) = f(u)$ , or equivalently  $c_{s(\mu)} = c_\mu$ .

The computation of the minimum is algorithmically hard and can be achieved numerically with an algorithm based on Hermitian sums of squares reinforcements. In the book of Dumitrescu [Dum07], this algorithm is explained for the univariate (Chapter 2) and the multivariate case (Chapters 3 and 4) with applications to filter design (Chapters 5, 6 and 8), stability and robustness (Chapter 7). The convergence rate of this approach was recently shown to be exponential [BR22]. Alternatively, one can apply Lasserre's hierarchy with complex variables [JM18], where one has to restrict to the compact torus.

The general idea can be summarized as follows (we give the details in Section 3). One restricts to suitable finite subsets  $\Omega_d \subseteq \Omega_{d+1} \subseteq \dots \subseteq \Omega$ . Then a lower bound for  $f^*$  is obtained by computing the maximal  $\lambda$ , such that  $f - \lambda$  is a Hermitian sum of squares, where the summands are supported by  $\Omega_d$ . This means, that  $f - \lambda$  can be represented by a positive semi-definite Hermitian Toeplitz matrix and this new problem is a semi-definite program. By increasing the parameter  $d$ , one improves the quality of the approximation, but the problem becomes more and more difficult to solve.

It is at this point where we exploit symmetry. Using symmetry adapted bases, the representing matrix can be transformed by a change of basis into block diagonal form, when  $f$  is  $\mathcal{W}$ -invariant. This means that instead of one large semi-definite program, one can solve several smaller ones, which amounts to a reduced computational effort.

Trigonometric polynomials are good  $L^2$ -approximations for real-valued  $\Lambda$ -periodic functions, where  $\Lambda$  is the dual lattice. They are the Fourier transformations of finite measures supported on  $\Omega$ . The lattices associated to crystallographic root systems provide optimal configurations for a variety of problems in geometry and information theory, with incidence in physics and chemistry. For example, the  $A_2$  lattice (the hexagonal lattice) is classically known to be optimal for sampling, packing, covering, and quantization in the plane [CS99, KAH05], but also proved, or conjectured, to be optimal for energy minimization problems [PS20, BF23]. More recently, the  $E_8$  lattice was proven to give an optimal solution for the sphere packing problem and a large class of energy minimization problems in dimension 8 [Via17, CKM<sup>+</sup>22]. From an approximation point of view, weight lattices of root systems describe Gaussian cubature [LX10, MP11], a rare occurrence on multidimensional domains. In a different direction, the triangulations associated with infinite families of root systems are relevant in graphics and computational geometry, see for instance [CKW20] and references therein.

The main tool in this article, the symmetry adapted basis, is obtained by decomposing a semi-simple representation of the Weyl group into isotypic components [Ser77]. The fact that one can do so is widely used in several areas of computational mathematics, for example combinatorics [Sta79, Ber09], optimization via polynomial sums of squares

[GP04, ALRT13], dynamical systems [Gat00, HL13], interpolation [RBH21, RBH22b], polynomial system solving [Gat90, FR09], invariant theory [HL16, RBH22a], as well as their fields of application [FS92].

So far, symmetry adapted bases were not used in the context of trigonometric optimization via Hermitian sums of squares. A different approach that exploits symmetry was explored by Hubert, Moustrou, Riener and the author in [HMMR23] and the thesis [Met22]. The idea there was to rewrite an invariant trigonometric polynomial to a classical one and then use relaxation tools from polynomial optimization [Las01, Par03]. The present paper complements this work in the sense that we first rewrite the problem to a semi-definite program and then exploit symmetry. One goal is to provide a framework for a fair comparison between polynomial and Hermitian sums of squares. This completes a recently presented poster at the ISSAC 2023 conference [Met23].

The article is structured as follows. In Section 2, we give the definitions of root systems, weight lattices and Weyl groups. Subsequently, we define trigonometric polynomials and introduce the concepts of degree and filtration. A trigonometric polynomial  $f$  can be written as a Hermitian Toeplitz matrix  $\mathbf{mat}(f)$ . The first novelty is to associate the action of  $\mathcal{W}$  on  $f$  with an action on  $\mathbf{mat}(f)$ , which is given by a semi-simple representation. Under the assumption that  $f$  is  $\mathcal{W}$ -invariant, we show that the matrix  $\mathbf{mat}(f)$  is an equivariant of the representation and can thus be block-diagonalized via symmetry adaptation.

In Section 3, we recall the SDP relaxation from [Dum07] and then show how to apply symmetry adapted bases in this context with a simple example. The feasible region of the SDP can be restricted to invariant Hermitian Toeplitz matrices, that is, the equivariants, and thus it suffices to consider block-diagonal matrices.

Finally, in Section 4, we show how to compute the symmetry adapted basis in the context of trigonometric polynomials and give a larger example.

## 2 Trigonometric polynomials with crystallographic symmetry

We require the lattice  $\Omega$  to be full-dimensional and stable under some finite reflection group  $\mathcal{W}$ , that is,  $\mathcal{W}\Omega = \Omega$ . Then  $\mathcal{W}$  must be the Weyl group of some crystallographic root system and  $\Omega$  is the associated weight lattice [Kan01, Chapter 9]. We need several facts from the theory of Lie algebras, root systems and lattices, which come from the books [Bou68, Hum72].

### 2.1 Root systems and Weyl groups

Let  $n \in \mathbb{N}$  and denote by  $\langle \cdot, \cdot \rangle$  the Euclidean scalar product on  $\mathbb{R}^n$ . A subset  $R \subseteq \mathbb{R}^n$  is called a **root system** in  $\mathbb{R}^n$ , if the following conditions<sup>1</sup> hold.

R1  $R$  is finite, spans  $\mathbb{R}^n$  and does not contain 0.

R2 If  $\rho, \tilde{\rho} \in R$ , then  $\langle \tilde{\rho}, \rho^\vee \rangle \in \mathbb{Z}$ , where  $\rho^\vee := \frac{2\rho}{\langle \rho, \rho \rangle}$ .

R3 If  $\rho, \tilde{\rho} \in R$ , then  $s_\rho(\tilde{\rho}) \in R$ , where  $s_\rho$  is the reflection defined by  $s_\rho(u) = u - \langle u, \rho^\vee \rangle \rho$  for  $u \in \mathbb{R}^n$ .

R4 For  $\rho \in R$  and  $c \in \mathbb{R}$ , we have  $c\rho \in R$  if and only if  $c = \pm 1$ .

The elements of  $R$  are called **roots** and the **rank** of  $R$  is the dimension of the space, that is,  $\text{Rank}(R) = n$ . The  $\rho^\vee$  are called the **coroots**.

Before we give illustrations in Example 2.1, we introduce the groups and lattices that arise from root systems.

<sup>1</sup>This is the definition given in [Bou68, Chapitre VI, §1, Définition 1] plus the “reduced” property R4.

### 2.1.1 Weyl group, weight lattice and coroot lattice

The **Weyl group**  $\mathcal{W}$  of  $\mathbb{R}$  is the group generated by the reflections  $s_\rho$  for  $\rho \in \mathbb{R}$ . We usually denote the elements by  $s \in \mathcal{W}$ . This is a finite subgroup of the orthogonal group  $O(\mathbb{R}^n)$  with respect to the inner product  $\langle \cdot, \cdot \rangle$ . The Weyl groups are the groups we consider in this article and now we introduce the lattices of interest.

A subset  $B = \{\rho_1, \dots, \rho_n\} \subseteq \mathbb{R}$  is called a **base**, if the following conditions hold.

B1  $B$  is a basis of  $\mathbb{R}^n$ .

B2 Every root  $\rho \in \mathbb{R}$  can be written as  $\rho = \alpha_1 \rho_1 + \dots + \alpha_n \rho_n$  or  $\rho = -\alpha_1 \rho_1 - \dots - \alpha_n \rho_n$  for some  $\alpha \in \mathbb{N}^n$ .

Every root system contains a base [Bou68, Chapitre VI, §1, Théorème 3]. A partial ordering  $\geq$  on  $\mathbb{R}^n$  is defined by  $u \geq v$  if and only if  $u - v$  is a sum of positive roots, that is,  $u - v = \alpha_1 \rho_1 + \dots + \alpha_n \rho_n$  for some  $\alpha \in \mathbb{N}^n$ .

A **weight** of  $\mathbb{R}$  is an element  $\mu \in \mathbb{R}^n$ , such that, for all  $\rho \in \mathbb{R}$ , we have  $\langle \mu, \rho^\vee \rangle \in \mathbb{Z}$ . The set of all weights forms a lattice  $\Omega$ , called the **weight lattice**. By the ‘‘crystallographic’’ property R2, every root is a weight. For a base  $B = \{\rho_1, \dots, \rho_n\}$ , the **fundamental weights** are the elements  $\{\omega_1, \dots, \omega_n\}$ , where, for  $1 \leq i, j \leq n$ , we have  $\langle \omega_i, \rho_j^\vee \rangle = \delta_{i,j}$ . The weight lattice is left invariant under the Weyl group, that is,  $\mathcal{W}\Omega = \Omega$ .

The **fundamental Weyl chamber** of  $\mathcal{W}$  relative to  $B$  is

$$\Lambda := \{u \in \mathbb{R}^n \mid \forall \rho \in B : \langle u, \rho_i \rangle > 0\}.$$

The closure  $\overline{\Lambda}$  is a fundamental domain of  $\mathcal{W}$  [Bou68, Chapitre V, §3, Théorème 2]. Hence,  $\overline{\Lambda}$  contains exactly one element per  $\mathcal{W}$ -orbit and the weights in  $\overline{\Lambda}$  are called **dominant**. We denote  $\Omega^+ := \Omega \cap \overline{\Lambda}$ . In particular, for every  $\mu \in \Omega^+$ , there exists a unique  $\hat{\mu} \in \Omega^+$  with  $-\mu \in \mathcal{W}\hat{\mu}$ .

The set of all coroots  $\rho^\vee$  spans a lattice  $\Lambda$  in  $\mathbb{R}^n$ , called the **coroot lattice**. As an Abelian group,  $\Lambda$  acts on  $\mathbb{R}^n$  by translation and is the dual lattice of the weight lattice, that is,  $\Lambda = \Omega^* = \{u \in \mathbb{R}^n \mid \forall \mu \in \Omega : \langle \mu, u \rangle \in \mathbb{Z}\}$ .

### 2.1.2 Irreducible root systems

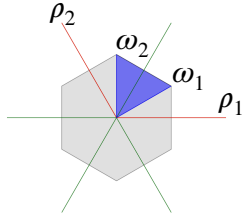
Assume that  $\mathbb{R}^n = V^{(1)} \oplus \dots \oplus V^{(k)}$  is the direct sum of proper orthogonal subspaces and that, for each  $1 \leq i \leq k$ , there is a root system  $\mathbb{R}^{(i)}$  in  $V^{(i)}$ . Then  $\mathbb{R} := \mathbb{R}^{(1)} \cup \dots \cup \mathbb{R}^{(k)}$  is a root system in  $\mathbb{R}^n$  called the **direct sum** of the  $\mathbb{R}^{(i)}$ . If a root system is not the direct sum of at least two root systems, then it is called **irreducible**, see [Bou68, Chapitre VI, §1.2].

The Weyl group  $\mathcal{W}$  is the product of the Weyl groups corresponding to the irreducible components, see the discussion before [Bou68, Chapitre VI, §1, Proposition 5]. If  $\mathbb{R}$  is irreducible with base  $B$ , then there is a unique positive root  $\rho_0 \in \mathbb{R}^+$ , which is maximal with respect to the partial ordering  $\geq$  induced by  $B$  [Bou68, Chapitre VI, §1, Proposition 25]. We call  $\rho_0$  the **highest root**.

Every root system can be uniquely decomposed into irreducible components [Bou68, Chapitre VI, §1, Proposition 6] and there are only finitely many cases [Bou68, Chapitre VI, §4, Théorème 3] denoted by  $A_{n-1}$ ,  $B_n$ ,  $C_n$  ( $n \geq 2$ ),  $D_n$  ( $n \geq 4$ ),  $E_{6,7,8}$ ,  $F_4$  and  $G_2$ . These are explicitly given in [Bou68, Planches I - IX].

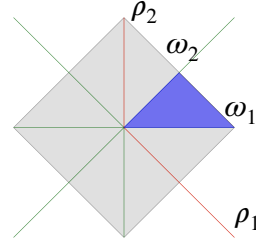
**Example 2.1.** For  $n = 1$ , a root system in  $\mathbb{R}$  must be of the form  $\mathbb{R} = \{\pm\rho\}$  for some  $\rho \in \mathbb{R}_{>0}$ , which is the only base element and the highest root. It admits the reflection at the origin  $s_\rho = -1$  and so the Weyl group is  $\mathcal{W} = \{\pm 1\}$ . The fundamental Weyl chamber is  $\Lambda = \mathbb{R}_{>0}$  and the coroot is  $\rho^\vee = 2\rho/\langle\rho, \rho\rangle = 2/\rho$ . For  $\omega \in \mathbb{R}$  to be the fundamental weight, we require  $1 = \langle\omega, \rho^\vee\rangle = 2\omega/\rho$ , that is,  $\omega = \rho/2$ . When we choose  $\rho = 2$ , then  $\mathbb{R}$  admits the self-dual lattice  $\Omega = \Lambda = \mathbb{Z}$ .

For  $n = 2$ , the irreducible root systems are depicted in Figures 1 to 4. The roots are depicted in green, the base in red and the fundamental weights in blue. The gray shaded region is a Voronoï cell of the coroot lattice  $\Lambda$  : we have two squares ( $C_2$  and  $B_2$ ) and two hexagons ( $A_2$  and  $G_2$ ). The blue shaded triangle is a fundamental domain of the semi-direct product  $\mathcal{W} \ltimes \Lambda$ .



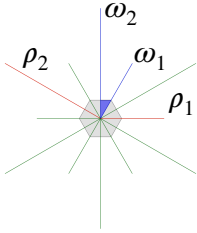
$$\begin{aligned} \mathcal{W}(A_2) &\cong \mathfrak{S}_3 \\ \omega_1 &= [2, -1, -1]^t/3 \\ \omega_2 &= [1, 1, -2]^t/3 \\ \rho_1 &= [1, -1, 0]^t = \rho_1^\vee \\ \rho_2 &= [0, 1, -1]^t = \rho_2^\vee \\ \rho_0 &= \rho_1^\vee + \rho_2^\vee \end{aligned}$$

Figure 1: The root system  $A_2$  in  $\mathbb{R}^3/\langle [1, 1, 1]^t \rangle$ .



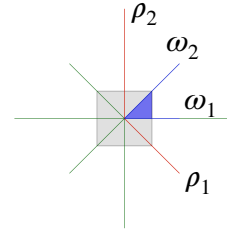
$$\begin{aligned} \mathcal{W}(B_2) &\cong \mathfrak{S}_2 \times \{\pm 1\}^2 \\ \omega_1 &= [1, 0]^t \\ \omega_2 &= [1, 1]^t/2 \\ \rho_1 &= [1, -1]^t = \rho_1^\vee \\ \rho_2 &= [0, 1]^t = \rho_2^\vee/2 \\ \rho_0 &= \rho_1^\vee + \rho_2^\vee \end{aligned}$$

Figure 2: The root system  $B_2$  in  $\mathbb{R}^2$ .



$$\begin{aligned} \mathcal{W}(G_2) &\cong \mathfrak{S}_3 \times \{\pm 1\} \\ \omega_1 &= [0, -1, 1]^t \\ \omega_2 &= [-1, -1, 2]^t \\ \rho_1 &= [1, -1, 0]^t = \rho_1^\vee \\ \rho_2 &= [-2, 1, 1]^t = 3\rho_1^\vee \\ \rho_0 &= 3\rho_1^\vee + 6\rho_2^\vee \end{aligned}$$

Figure 3: The root system  $G_2$  in  $\mathbb{R}^3/\langle [1, 1, 1]^t \rangle$ .



$$\begin{aligned} \mathcal{W}(C_2) &\cong \mathfrak{S}_2 \times \{\pm 1\}^2 \\ \omega_1 &= [1, 0]^t \\ \omega_2 &= [1, 1]^t \\ \rho_1 &= [1, -1]^t = \rho_1^\vee \\ \rho_2 &= [0, 2]^t = 2\rho_2^\vee \\ \rho_0 &= 2\rho_1^\vee + 2\rho_2^\vee \end{aligned}$$

Figure 4: The root system  $C_2$  in  $\mathbb{R}^2$ .

## 2.2 Trigonometric polynomials

From now on,  $R$  shall be a root system in  $\mathbb{R}^n$  with Weyl group  $\mathcal{W}$ , weight lattice  $\Omega = \mathbb{Z}\omega_1 \oplus \dots \oplus \mathbb{Z}\omega_n$  and coroot lattice  $\Lambda = \Omega^*$ . For  $\mu \in \Omega$ , we define the map

$$\begin{aligned} e^\mu : \mathbb{R}^n &\rightarrow \mathbb{C}, \\ u &\mapsto \exp(-2\pi i \langle \mu, u \rangle). \end{aligned}$$

The set  $\{e^\mu \mid \mu \in \Omega\}$  is closed under multiplication  $e^\mu e^{\tilde{\mu}} = e^{\mu+\tilde{\mu}}$  and thus a group with neutral element  $1 = e^0$  and inverse  $(e^\mu)^{-1} = e^{-\mu}$ . By abuse of notation, we denote the group algebra by  $\mathbb{C}[\Omega]$ . The elements of  $\mathbb{C}[\Omega]$  are  $\mathbb{C}$ -linear combinations of the form  $f = \sum_\mu f_\mu e^\mu$  with only finitely many nonzero coefficients  $f_\mu \in \mathbb{C}$  and called **trigonometric polynomials**.

### 2.2.1 Periodicity and crystallographic symmetry

As the coroot lattice  $\Lambda$  is the dual lattice of  $\Omega$ , any element  $f \in \mathbb{C}[\Omega]$  is  $\Lambda$ -periodic, that is, for all  $u \in \mathbb{R}^n$  and  $\lambda \in \Lambda$ , we have  $f(u + \lambda) = f(u)$ .

The Weyl group  $\mathcal{W}$  has a  $\mathbb{C}$ -linear action on the trigonometric polynomials

$$\begin{aligned} \cdot : \mathcal{W} \times \mathbb{C}[\Omega] &\rightarrow \mathbb{C}[\Omega], \\ (s, e^\mu) &\mapsto e^{s(\mu)}. \end{aligned} \tag{2.1}$$

A trigonometric polynomial  $f \in \mathbb{C}[\Omega]$  is called  **$\mathcal{W}$ -invariant**, if, for all  $u \in \mathbb{R}^n$  and  $s \in \mathcal{W}$ , we have  $s \cdot f(u) = f(s^{-1}(u)) = f(u)$ . The  $\mathbb{C}$ -algebra of all  $\mathcal{W}$ -invariant trigonometric polynomials is denoted by  $\mathbb{C}[\Omega]^\mathcal{W}$ .

### 2.2.2 Degree and filtration

We now introduce the notion of degree for trigonometric polynomials. Let us first assume that  $R$  is irreducible with highest root  $\rho_0$ . For  $d \in \mathbb{N}$ , the Weyl group  $\mathcal{W}$  leaves the finite set of weights

$$\Omega_d := \{\mu \in \Omega \mid \langle \hat{\mu}, \rho_0^\vee \rangle \leq d\} \tag{2.2}$$

invariant, where  $\hat{\mu} \in \Omega^+$  is the unique dominant weight in the  $\mathcal{W}$ -orbit of  $-\mu$ . We write  $\Omega_d^+ := \Omega_d \cap \Omega^+$  for the dominant weights in  $\Omega_d$ .

**Lemma 2.2.** *For  $d, d' \in \mathbb{N}$ , we have  $\Omega_0 = \{0\}$ ,  $\Omega_{d+1} \neq \Omega_d$  if and only if  $\Omega_1 \neq \{0\}$ , and  $\Omega_d + \Omega_{d'} \subseteq \Omega_{d+d'}$ .*

*Proof.* First, let  $\mu \in \Omega_0$  and  $\rho_0^\vee = \sum_i \alpha_i \rho_i$  for some  $\alpha \in \mathbb{R}^n$ . Then  $\alpha_i > 0$  and  $0 \geq \langle \hat{\mu}, \rho_0^\vee \rangle = \sum_i \alpha_i \langle \hat{\mu}, \rho_i \rangle \geq 0$ . Since  $\hat{\mu} \in \overline{\Lambda}$ , we have  $\langle \hat{\mu}, \rho_i \rangle = 0$ . As the  $\rho_i$  form a vector space basis, this means  $\hat{\mu} = 0$  and so  $\mu = 0$ .

For the second statement, note that the implication “ $\Rightarrow$ ” is clear. For “ $\Leftarrow$ ”, let  $\mu \in \Omega_1 \setminus \{0\}$ . We have  $0 < \langle \hat{\mu}, \rho_0^\vee \rangle \leq 1$  and  $\langle \hat{\mu}, \rho_0^\vee \rangle \in \mathbb{Z}$  implies  $\langle \hat{\mu}, \rho_0^\vee \rangle = 1$ . Thus,  $(d+1)\mu \in \Omega_{d+1} \setminus \Omega_d$ .

Finally, let  $\mu \in \Omega_d$ ,  $\nu \in \Omega_{d'}$  and  $s \in \mathcal{W}$ , such that  $s(\mu + \nu) = \widehat{\mu + \nu} \in \Omega^+$ . By [Bou68, Chapitre VI, §1, Proposition 18], the weights  $\widehat{\mu} - s(\mu)$  and  $\widehat{\nu} - s(\nu)$  are sums of positive roots, that is, there exist  $\alpha, \beta \in \mathbb{N}^n$ , such that

$$\langle s(\mu + \nu), \rho_0^\vee \rangle = \langle \widehat{\mu} - \sum_{i=1}^n \alpha_i \rho_i + \widehat{\nu} - \sum_{i=1}^n \beta_i \rho_i, \rho_0^\vee \rangle = \langle \widehat{\mu} + \widehat{\nu}, \rho_0^\vee \rangle - \underbrace{\sum_{i=1}^n (\alpha_i + \beta_i)}_{\geq 0} \underbrace{\langle \rho_i, \rho_0^\vee \rangle}_{\geq 0} \leq \langle \widehat{\mu} + \widehat{\nu}, \rho_0^\vee \rangle \leq d + d'.$$

Hence, we have  $\langle \widehat{\mu + \nu}, \rho_0^\vee \rangle \leq d + d'$  and the statement follows.  $\square$

**Example 2.3.** *The root system  $E_8$  from [Bou68, Planche VII] satisfies  $\langle \omega_i, \rho_0^\vee \rangle \geq 2$  for  $1 \leq i \leq 8$  and  $\langle \omega_1, \rho_0^\vee \rangle = 2$ . In this case, we have  $\Omega_1 = \{0\}$  and  $\Omega_1 + \Omega_1 = \{0\} \subsetneq \mathcal{W}\omega_1 \subseteq \Omega_2$ . For every other irreducible root system from [Bou68, Planche I - IX], we have  $\Omega_1 \neq \{0\}$  and  $\Omega_d + \Omega_{d'} = \Omega_{d+d'}$ .*

We define the **degree** of a trigonometric polynomial  $f = \sum_\mu f_\mu e^\mu \in \mathbb{C}[\Omega]$  as

$$\deg(f) := \max\{\langle \hat{\mu}, \rho_0^\vee \rangle \mid \mu \in \Omega, f_\mu \neq 0\} = \min\{d \in \mathbb{N} \mid \forall \mu \in \Omega : f_\mu \neq 0 \Rightarrow \mu \in \Omega_d\}.$$

The finite-dimensional subspace of  $\mathbb{C}[\Omega]$  consisting of all  $f \in \mathbb{C}[\Omega]$  with  $\deg(f) \leq d$  is denoted by  $\mathbb{C}[\Omega]_d$ .

**Proposition 2.4.**  *$(\mathbb{C}[\Omega]_d)_{d \in \mathbb{N}}$  is a filtration of  $\mathbb{C}[\Omega]$ , that is,*

1.  $\{0\} \subseteq \mathbb{C}[\Omega]_0 \subseteq \mathbb{C}[\Omega]_1 \subseteq \mathbb{C}[\Omega]_2 \subseteq \dots \subseteq \mathbb{C}[\Omega]$ ,
2.  $\mathbb{C}[\Omega] = \bigcup_{d \in \mathbb{N}} \mathbb{C}[\Omega]_d$ ,
3.  $\mathbb{C}[\Omega]_d \mathbb{C}[\Omega]_{d'} \subseteq \mathbb{C}[\Omega]_{d+d'}$  whenever  $d, d' \in \mathbb{N}$ .

*Proof.* The first two statements follow from the definition. A basis of  $\mathbb{C}[\Omega]_d$  is given by the set of all  $e^\mu$  with  $\mu \in \Omega_d$ . In order to show the third statement, let  $\mu \in \Omega_d$  and  $\nu \in \Omega_{d'}$ . Then we have  $e^\mu e^\nu \in \mathbb{C}[\Omega]_{d+d'}$  by Lemma 2.2 and the statement follows.  $\square$

### 2.2.3 Toeplitz matrix representations

Given a trigonometric polynomial  $f = \sum_\mu f_\mu e^\mu \in \mathbb{C}[\Omega]$ , let us assume that

- A1 the coefficients are conjugate sign symmetric, that is,  $f_{-\mu} = \overline{f_\mu}$ ,
- A2 we know the minimal  $d \in \mathbb{N}$ , such that  $\mu \in \Omega_d + \Omega_d$  whenever  $f_\mu \neq 0$ .

Then  $f$  assumes only real values on  $\mathbb{R}^n$  and, by Proposition 2.4, has degree at most  $2d$ . We want to rewrite  $f$  as

$$f(u) = \overline{\mathbf{E}_d(u)}^t \mathbf{mat}(f) \mathbf{E}_d(u), \quad (2.3)$$

where



1.  $\mathbf{E}_d(u)$  is the vector of all  $e^\mu(u)/\sqrt{|\Omega_d|}$  with  $\mu \in \Omega_d$ ,
2.  $\mathbf{mat}(f)$  is a matrix with rows and columns indexed by  $\Omega_d$  and independent of  $u$ .

By comparing coefficients in Equation (2.3) and taking assumption A2 into account, we see that

$$f_\eta = \frac{1}{|\Omega_d|} \sum_{\substack{(\mu, \nu) \in \Omega_d^2 \\ \mu - \nu = \eta}} \mathbf{mat}(f)_{\mu\nu}. \quad (2.4)$$

Therefore, we may assume without loss of generality that the entries  $\mathbf{mat}(f)_{\mu\nu}$  depend only on the weight  $\mu - \nu$ . In other words,  $\mathbf{mat}(f)$  can be chosen uniquely as a Toeplitz matrix. Then assumption A1 implies that  $\mathbf{mat}(f)$  is Hermitian, that is,  $\mathbf{mat}(f)_{\mu\nu} = \mathbf{mat}(f)_{\nu\mu}$ . The space of all Hermitian Toeplitz matrices of size  $|\Omega_d| \times |\Omega_d|$  is denoted by  $\text{Toep}_d$ . Since the Weyl group  $\mathcal{W}$  acts on the finite set  $\Omega_d$ , we have a left group action

$$\star : \mathcal{W} \times \text{Toep}_d \rightarrow \text{Toep}_d, (s, \mathbf{X} = (\mathbf{X}_{\mu\nu})) \mapsto s \star \mathbf{X} := (\mathbf{X}_{s^{-1}(\mu)s^{-1}(\nu)}). \quad (2.5)$$

The **fixed-point space** of this action is  $\text{Toep}_d^{\mathcal{W}} := \{\mathbf{X} \in \text{Toep}_d \mid \forall s \in \mathcal{W} : s \star \mathbf{X} = \mathbf{X}\}$ . We have  $\mathbf{mat}(f) \in \text{Toep}_d^{\mathcal{W}}$  if and only if  $f$  is  $\mathcal{W}$ -invariant under the action from Equation (2.1).

## 2.2.4 Symmetry adapted bases

The goal of this section is to find a basis that allows us to blockdiagonalize matrices in  $\text{Toep}_d^{\mathcal{W}}$  and will be of relevance for the optimization of trigonometric polynomials. First, we construct a representation  $\vartheta : \mathcal{W} \rightarrow \text{O}(\mathbb{R}^{\Omega_d})$ , which induces the action on  $\text{Toep}_d$  by Equation (2.5). Here, to induce means the following.

**Proposition 2.5.** *For  $s \in \mathcal{W}$ , let  $\vartheta(s) \in \text{O}(\mathbb{R}^{\Omega_d})$  be the permutation matrix with  $(\vartheta(s)\mathbf{x})_\mu = \mathbf{x}_{s^{-1}(\mu)}$  whenever  $\mathbf{x} \in \mathbb{R}^{\Omega_d}$ . Then  $\vartheta : \mathcal{W} \rightarrow \text{O}(\mathbb{R}^{\Omega_d})$  is a semi-simple representation and, for all  $\mathbf{X} \in \text{Toep}_d$ , we have*

$$s \star \mathbf{X} = \vartheta(s) \mathbf{X} \vartheta(s)^t.$$

*Proof.* We have  $\vartheta(st) = \vartheta(s)\vartheta(t)$ . Thus,  $\vartheta$  is an orthogonal representation of  $\mathcal{W}$  and as such semi-simple [Ser77, Chapter 1]. Now, let  $s \in \mathcal{W}$  and  $\mathbf{X} \in \text{Toep}_d$ . Then

$$(\vartheta(s) \mathbf{X} \vartheta(s)^t)_{\mu\nu} = \sum_{\kappa, \eta \in \Omega_d} \vartheta(s)_{\mu\kappa} \mathbf{X}_{\kappa\eta} \vartheta(s)_{\nu\eta} = \mathbf{X}_{s^{-1}\mu s^{-1}\nu} = (s \star \mathbf{X})_{\mu\nu},$$

because  $\vartheta(s)_{\mu\kappa} = 1$  when  $\kappa = s^{-1}\mu$  and 0 otherwise. □

Since  $\vartheta$  is semi-simple, the  $\mathcal{W}$ -module  $\mathbb{R}^{\Omega_d}$  has an **isotypic decomposition**

$$\mathbb{R}^{\Omega_d} = \bigoplus_{i=1}^h \left( \bigoplus_{j=1}^{m_i} V_{i,j} \right), \quad (2.6)$$

where, for all  $1 \leq i \leq h$ , the  $V_{i,1}, \dots, V_{i,m_i}$  are isomorphic irreducible  $\mathcal{W}$ -submodules with dimension  $d_i := \dim(V_{i,j})$  and multiplicity  $m_i \in \mathbb{N}$  so that  $\sum_i d_i m_i = |\Omega_d|$ , see [Ser77, Chapter 2].

There is an orthonormal change of basis  $\mathbf{T} \in \text{O}(\mathbb{R}^{\Omega_d})$  that transforms any  $\mathbf{X} \in \text{Toep}_d^{\mathcal{W}}$  into

$$\mathbf{T}^t \mathbf{X} \mathbf{T} = \begin{pmatrix} \boxed{\mathbf{X}_1} & & \\ & \ddots & \\ & & \boxed{\mathbf{X}_h} \end{pmatrix} \quad \text{with} \quad \mathbf{X}_i = \begin{pmatrix} \boxed{\tilde{\mathbf{X}}_i} & & \\ & \ddots & \\ & & \boxed{\tilde{\mathbf{X}}_i} \end{pmatrix} \in \mathbb{C}^{d_i m_i \times d_i m_i}, \quad (2.7)$$

where each  $\mathbf{X}_i$  consists of  $d_i$  identical blocks  $\tilde{\mathbf{X}}_i$  of size  $m_i \times m_i$ .

### 3 Optimization

We now minimize an invariant trigonometric polynomial  $f = \sum_{\mu} f_{\mu} e^{\mu} \in \mathbb{C}[\Omega]^{\mathcal{W}}$ , where  $\Omega$  is the weight lattice of an irreducible root system in  $\mathbb{R}^n$  with Weyl group  $\mathcal{W}$ . For this, we require  $f$  to assume only real values, which is the case under assumption A1: the coefficients are conjugate sign symmetric, that is,  $f_{-\mu} = \overline{f_{\mu}}$ . Then the minimum

$$f^* := \min_{u \in \mathbb{R}^n} f(u)$$

is well-defined and assumed in some minimizer  $u^* \in \mathbb{R}^n$ .

Furthermore, we assume A2, that is, we know the minimal  $d \in \mathbb{N}$ , such that  $\mu \in \Omega_d + \Omega_d$  whenever  $f_{\mu} \neq 0$ . Then  $f$  has degree at most  $2d$ .

Our strategy is to follow a known relaxation technique of the problem of computing  $f^*$  to a semi-definite program (SDP), whose solution will provide a lower bound for the minimum. This technique is well-explained in the book of Dumitrescu for the case  $\Omega = \mathbb{Z}^n$  [Dum07, Chapter 4]. Under assumption A2, it can be applied to any weight lattice. We then exploit the symmetry of  $f$  by rewriting the SDP in block-diagonal form according to the isotypic decomposition as in Equation (2.7).

The number  $d$  can be seen as the starting index of a hierarchy of lower bounds, which is non-decreasing and converges to  $f^*$ . Exploiting symmetry does not effect the quality of the bound, but greatly reduces the computational effort.

The fundamental difference to [HMMR23] is that we first relax to an SDP and then exploit symmetry.

#### 3.1 SDP relaxation

Recall that, for  $f \in \mathbb{C}[\Omega]$  satisfying assumptions A1 and A2, there is a unique Hermitian Toeplitz matrix  $\mathbf{mat}(f) \in \text{Toep}_d$ , such that

$$f(u) = \overline{\mathbf{E}_d(u)}^t \mathbf{mat}(f) \mathbf{E}_d(u).$$

On the  $\mathbb{R}$ -vector space of Hermitian Toeplitz matrices, the trace admits a scalar product. By [Dum07, Chapter 3], we have

$$f^* \geq f_d := \min_{\mathbf{X} \in \text{Toep}_d} \text{Trace}(\mathbf{mat}(f) \mathbf{X}) \quad \text{s.t.} \quad \mathbf{X} \geq 0, \text{Trace}(\mathbf{X}) = 1. \quad (3.1)$$

**Example 3.1.** Consider the case  $n = 1$  with  $\Omega = \Lambda = \mathbb{Z}$  and  $\mathcal{W} = \{\pm 1\}$  from Example 2.1. Then  $\Omega_1 = \{1, 0, -1\}$  and  $\mathbf{E}_1 = (e^1, 1, e^{-1})^t / \sqrt{3}$ . We have

$$\overline{\mathbf{E}_1}^t \mathbf{E}_1 = \frac{1}{3} (e^{-1} \quad 1 \quad e^1) \begin{pmatrix} e^1 \\ 1 \\ e^{-1} \end{pmatrix} = 1 \quad \text{and} \quad \mathbf{E}_1 \overline{\mathbf{E}_1}^t = \frac{1}{3} \begin{pmatrix} 1 & e^1 & e^2 \\ e^{-1} & 1 & e^1 \\ e^{-2} & e^{-1} & 1 \end{pmatrix}.$$

As a toy example, take  $f(u) := e^2(u) - 2e^1(u) + 3 - 2e^{-1}(u) + e^{-2}(u) = 2 \cos(4\pi u) - 4 \cos(2\pi u) + 3$  with degree 2 and global minimum  $f^* = f(\lambda \pm 1/6) = 0$  for  $\lambda \in \mathbb{Z}$ . Note that  $f$  can be rewritten as

$$f(u) = \overline{\mathbf{E}_1(u)}^t \mathbf{mat}(f) \mathbf{E}_1(u), \quad \text{where} \quad \mathbf{mat}(f) = \begin{pmatrix} 3 & -3 & 3 \\ -3 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix} \in \text{Toep}_1.$$

Thus, the SDP from Equation (3.1) is

$$f_1 = \min_{b, c \in \mathbb{C}} \text{Trace} \left( \begin{pmatrix} 3 & -3 & 3 \\ -3 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix} \begin{pmatrix} 1/3 & b & c \\ \bar{b} & 1/3 & b \\ \bar{c} & \bar{b} & 1/3 \end{pmatrix} \right) \quad \text{s.t.} \quad \begin{pmatrix} 1/3 & b & c \\ \bar{b} & 1/3 & b \\ \bar{c} & \bar{b} & 1/3 \end{pmatrix} \geq 0$$

and the optimal value  $f^* = f_1 = 0$  is obtained with  $b = -c = 1/6$ .

### 3.2 Symmetry reduction

We now assume  $f \in \mathbb{C}[\Omega]^{\mathcal{W}}$ , that is,  $f$  is invariant under the action of the Weyl group from Equation (2.1).

**Proposition 3.2.** *We have  $f_d = f_{d,\mathcal{W}}$ , where  $f_d$  is as in Equation (3.1) and*

$$f_{d,\mathcal{W}} := \min_{\mathbf{X} \in \text{Toep}_d^{\mathcal{W}}} \text{Trace}(\mathbf{mat}(f) \mathbf{X}) \quad \text{s.t.} \quad \mathbf{X} \geq 0, \text{Trace}(\mathbf{X}) = 1.$$

*Proof.* If  $\mathbf{X} \in \text{Toep}_d^{\mathcal{W}}$  is feasible for  $f_{d,\mathcal{W}}$ , then it is especially feasible for  $f_d$ . Hence, we have  $f_d \leq f_{d,\mathcal{W}}$ . For the converse, let  $\mathbf{X} \in \text{Toep}_d$  be feasible for  $f_d$ . We define

$$\tilde{\mathbf{X}} := \frac{1}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} s \star \mathbf{X} \in \text{Toep}_d^{\mathcal{W}}.$$

We claim that  $\tilde{\mathbf{X}}$  is feasible for  $f_{d,\mathcal{W}}$ . If  $\mathbf{X} \geq 0$  and  $s \in \mathcal{W}$ ,  $\mathbf{x} \in \mathbb{C}^{\Omega_d}$ , then

$$\bar{\mathbf{x}}^t (s \star \mathbf{X}) \mathbf{x} = \bar{\mathbf{x}}^t (\vartheta(s) \mathbf{X} \vartheta(s)^t) \mathbf{x} = \overline{\vartheta(s)^t \mathbf{x}}^t \mathbf{X} (\vartheta(s)^t \mathbf{x}) \geq 0.$$

Furthermore, if  $\text{Trace}(\mathbf{X}) = 1$ , then

$$\text{Trace}(s \star \mathbf{X}) = \sum_{\mu \in \Omega_d} \mathbf{X}_{s^{-1}(\mu) s^{-1}(\mu)} = \sum_{\nu \in \Omega_d} \mathbf{X}_{\nu\nu} = \text{Trace}(\mathbf{X}) = 1.$$

Thus,  $s \star \mathbf{X}$  is also feasible for  $f_d$ . Since the feasibility region of is convex, it also contains the convex combination  $\tilde{\mathbf{X}}$ . In particular,  $\tilde{\mathbf{X}}$  is feasible for  $f_{d,\mathcal{W}}$  with

$$\text{Trace}(\mathbf{mat}(f) \tilde{\mathbf{X}}) = \frac{1}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} \text{Trace}(\mathbf{mat}(f) (s \star \mathbf{X})) = \frac{1}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} \text{Trace}((s^{-1} \star \mathbf{mat}(f)) \mathbf{X}) = \text{Trace}(\mathbf{mat}(f) \mathbf{X}),$$

because  $\mathbf{mat}(f) \in \text{Toep}_d^{\mathcal{W}}$ . Hence, we also have  $f_d \geq f_{d,\mathcal{W}}$ . □

**Theorem 3.3.** *Assume that  $\mathbf{T}^t \mathbf{mat}(f) \mathbf{T}$  has blocks  $\tilde{\mathbf{F}}_i \in \mathbb{C}^{m_i \times m_i}$ . Then  $f_d = f_{d,\mathcal{W}} = f_{d,\mathcal{W}}^{\text{block}}$ , where*

$$f_{d,\mathcal{W}}^{\text{block}} := \min_{\mathbf{X} \in \text{Toep}_d^{\mathcal{W}}} \sum_{i=1}^h d_i \text{Trace}(\tilde{\mathbf{F}}_i \tilde{\mathbf{X}}_i) \quad \text{s.t.} \quad \text{Trace}(\mathbf{X}) = 1, \mathbf{T}^t \mathbf{X} \mathbf{T} \text{ has blocks } \tilde{\mathbf{X}}_i \geq 0.$$

*Proof.* Since the trace is basis invariant and linear, we have

$$\text{Trace}(\mathbf{mat}(f) \mathbf{X}) = \text{Trace}(\mathbf{T}^t \mathbf{mat}(f) \mathbf{T} \mathbf{T}^t \mathbf{X} \mathbf{T}) = \sum_{i=1}^h d_i \text{Trace}(\tilde{\mathbf{F}}_i \tilde{\mathbf{X}}_i).$$

Note that  $\mathbf{X} \geq 0$  if and only if  $\mathbf{T}^t \mathbf{X} \mathbf{T} \geq 0$ , that is, for all  $1 \leq i \leq h$ , we have  $\tilde{\mathbf{X}}_i \geq 0$ . Thus,  $f_d = f_{d,\mathcal{W}}^{\text{block}}$  and the statement follows from Proposition 3.2. □

**Example 3.4.** *We continue with Example 3.1 with  $n = 1$ ,  $d = 2$  and  $\mathcal{W} = \{\pm 1\}$ . An arbitrary  $\mathbf{X} \in \text{Toep}_1$  shall have rows and columns indexed by  $\Omega_1 = \{1, 0, -1\}$ . Then the left group action of  $-1 \in \mathcal{W}$  on  $\text{Toep}_1$  spells out as*

$$\begin{array}{c} 1 \\ 0 \\ -1 \end{array} \begin{pmatrix} 1 & 0 & -1 \\ \bar{a} & b & c \\ \bar{c} & \bar{b} & a \end{pmatrix} \xrightarrow{(-1) \star \dots} \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \begin{pmatrix} 1 & 0 & -1 \\ a & \bar{b} & \bar{c} \\ c & b & a \end{pmatrix}$$

for  $a \in \mathbb{R}$  and  $b, c \in \mathbb{C}$ . The fixed point space  $\text{Toep}_1^{\mathcal{W}}$  consists of those  $\mathbf{X} \in \text{Toep}_1$  with  $a, b, c \in \mathbb{R}$ . The above action is induced by the orthogonal representation  $\rho : \mathcal{W} \rightarrow \text{O}(\mathbb{R}^3)$  with

$$\rho(1) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(-1) := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have an isotypic decomposition

$$\mathbb{R}^3 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle_{\mathbb{R}} \oplus \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}} \oplus \left\langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle_{\mathbb{R}}$$

with  $h = 2, m_1 = 2, d_1 = 1, m_2 = 1, d_2 = 1$ . When  $\mathbf{T}$  denotes the orthogonal matrix with columns given by the above basis vectors and  $f$  is the  $\mathcal{W}$ -invariant trigonometric polynomial from [Example 3.1](#), then

$$\mathbf{T}^t \mathbf{X} \mathbf{T} = \begin{pmatrix} a & \sqrt{2}b & 0 \\ \sqrt{2}b & a+c & 0 \\ 0 & 0 & a-c \end{pmatrix} \quad \text{and} \quad \mathbf{T}^t \mathbf{mat}(f) \mathbf{T} = \begin{pmatrix} 3 & -3\sqrt{2} & 0 \\ -3\sqrt{2} & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now,  $\text{Trace}(\mathbf{X}) = 1$  implies  $a = 1/3$ . Thus,

$$f_{1, \mathcal{W}}^{\text{block}} = \min_{b, c \in \mathbb{R}} \text{Trace} \left( \begin{pmatrix} 3 & -3\sqrt{2} \\ -3\sqrt{2} & 6 \end{pmatrix} \begin{pmatrix} 1/3 & \sqrt{2}b \\ \sqrt{2}b & 1/3+c \end{pmatrix} \right) \quad \text{s.t.} \quad \begin{pmatrix} 1/3 & \sqrt{2}b \\ \sqrt{2}b & 1/3+c \end{pmatrix} \geq 0, \quad 1/3 \geq c.$$

Finally, the optimal value  $f^* = f_1 = f_{1, \mathcal{W}} = f_{1, \mathcal{W}}^{\text{block}} = 0$  is recovered with  $b = -c = 1/6$ .

## 4 Computing symmetry adapted bases

We have seen the immediate advantage that a symmetry adapted basis brings to the table when solving the trigonometric optimization problem via semi-definite relaxation techniques. In this section, we explain how to obtain said basis by giving an algorithm, which is based on [\[Ser77, Chapter 2\]](#).

### 4.1 Projection onto isotypic components

Recall that we have an orthogonal representation  $\vartheta : \mathcal{W} \rightarrow \text{O}(\mathbb{R}^{\Omega_d})$  of the Weyl group. Furthermore, we denote by  $W^{(1)}, \dots, W^{(h)}$  the irreducible representations of  $\mathcal{W}$  and by  $\chi_1, \dots, \chi_h$  their characters. Then the representation space  $\mathbb{R}^{\Omega_d}$  has an isotypic decomposition

$$\mathbb{R}^{\Omega_d} = V^{(1)} \oplus \dots \oplus V^{(h)}, \quad V^{(i)} = V_1^{(i)} \oplus \dots \oplus V_{m_i}^{(i)},$$

where  $V_j^{(i)}$  are the subrepresentations isomorphic to  $W^{(i)}$  with multiplicities  $m_i$  and dimensions  $d_i = \dim(W^{(i)})$ .

**Proposition 4.1.** [\[Ser77\]](#) A projection of  $V$  onto  $V^{(i)}$  is given by

$$\mathbf{P}^{(i)} := \frac{d_i}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} \chi_i(s^{-1}) \vartheta(s).$$

## 4.2 Computing the basis

We now recall how to compute a basis for  $\mathbb{R}^{\Omega_d}$  so that the  $\mathcal{W}$ -invariant matrices have the block-diagonal structure from Equation (2.7). Then we give some improvements to this algorithm.

We fix an index  $1 \leq i \leq h$  and follow the steps below.

1. Choose a basis for  $W_i$ , so that, for all  $s \in \mathcal{W}$ , the representation matrix is given by  $(\mathbf{D}_{k\ell}^{(i)}(s))_{1 \leq k, \ell \leq d_i}$ .
2. For  $1 \leq \ell \leq d_i$ , compute

$$\mathbf{P}_\ell^{(i)} := \frac{d_i}{|\mathcal{W}|} \sum_{s \in \mathcal{W}} \mathbf{D}_{1\ell}^{(i)}(s^{-1}) \vartheta(s).$$

The column dimension of this matrix is  $m_i$ .

3. Among the columns of  $\mathbf{P}_1^{(i)}$  choose  $m_i$  linearly independent ones and denote them by  $w_{11}^{(i)}, \dots, w_{1m_i}^{(i)}$ .
4. For  $2 \leq \ell \leq d_i$  and  $1 \leq j \leq m_i$ , compute

$$w_{\ell j}^{(i)} := \mathbf{P}_\ell^{(i)} w_{1j}^{(i)}.$$

**Proposition 4.2.** [Ser77, FS92] For  $1 \leq i \leq h$  and  $1 \leq j \leq m_i$ , the set  $\{w_{1j}^{(i)}, \dots, w_{d_i j}^{(i)}\}$  is a basis for  $V_j^{(i)}$ .

Furthermore, the matrix

$$\tilde{\mathbf{T}} := \left( \underbrace{w_{11}^{(1)}, \dots, w_{d_1 1}^{(1)}}_{V_1^{(1)}}, \dots, \underbrace{w_{1m_1}^{(1)}, \dots, w_{d_1 m_1}^{(1)}}_{V_{m_1}^{(1)}}, \dots, \underbrace{w_{11}^{(h)}, \dots, w_{d_h 1}^{(h)}}_{V_1^{(h)}}, \dots, \underbrace{w_{1m_h}^{(h)}, \dots, w_{d_h m_h}^{(h)}}_{V_{m_h}^{(h)}} \right)$$

has rank  $|\Omega_d|$  and, for  $s \in \mathcal{W}$ ,  $\tilde{\mathbf{T}}^{-1} \vartheta(s) \tilde{\mathbf{T}}$  has  $h$  blocks, where each block consists of  $m_i$  further identical blocks of size  $d_i \times d_i$ .

On the other hand, when the columns are reordered to

$$\mathbf{T} := (w_{11}^{(1)}, \dots, w_{1m_1}^{(1)}, \dots, w_{d_1 1}^{(1)}, \dots, w_{d_1 m_1}^{(1)}, \dots, w_{11}^{(h)}, \dots, w_{1m_h}^{(h)}, \dots, w_{d_h 1}^{(h)}, \dots, w_{d_h m_h}^{(h)}),$$

then, for  $\mathbf{X} \in \text{Toep}_d^{\mathcal{W}}$ ,  $\mathbf{T}^{-1} \mathbf{X} \mathbf{T}$  has  $h$  blocks, where each block consists of  $d_i$  further identical blocks of size  $m_i \times m_i$  (see Equation (2.7)).

## 4.3 Example

Since  $\vartheta$  is orthogonal, we may and do assume from now on that the change of basis matrix  $\mathbf{T}$  (and  $\tilde{\mathbf{T}}$ ) is also orthogonal, that is,  $\mathbf{T}^{-1} = \mathbf{T}^t$ . Thus, Hermitian matrices  $\mathbf{X}$  are transformed to Hermitian matrices  $\mathbf{T}^t \mathbf{X} \mathbf{T}$ .

**Example 4.3.** We consider the Weyl group of the  $A_2$  root system  $\mathcal{W} \cong \mathfrak{S}_3$ . It acts by permutation of coordinates on  $\mathbb{R}^3 / \langle [1, 1, 1]^t \rangle$  and thus on the hexagonal lattice  $\Omega = \mathbb{Z} \omega_1 \oplus \mathbb{Z} \omega_2$  from Figure 1. The group has order  $|\mathcal{W}| = 6$  and  $h = 3$  irreducible representations, which is encoded by the character table  $\chi \in \mathbb{Z}^{h \times \mathcal{W}}$ . This is a matrix with rows indexed by the irreducible representations  $1 \leq i \leq h$  and columns indexed by the group elements  $s \in \mathcal{W}$ , such that

$$\text{Trace}(\vartheta(s)) = m_1 \chi_1(s) + \dots + m_h \chi_h(s).$$

In particular, the multiplicities  $m_i$  are the solutions of a linear system and the dimensions are  $d_i = \chi_i(\text{id})$ , where  $\text{id} \in \mathcal{W}$  is the identity element. In the case of  $\mathcal{W} \cong \mathfrak{S}_3$ , we have

$$\chi = \begin{matrix} & \text{id} & s_1 & s_2 & s_1 s_2 & s_2 s_1 & s_1 s_2 s_1 \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 2 & 0 & 0 & -1 & -1 & 0 \\ 3 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{matrix},$$

where  $s_1$  and  $s_2$  are the two generating reflections. With the computer algebra system OSCAR [OSC23], this matrix can be computed as follows.

```
julia> using Oscar;
julia> W = symmetric_group(3);
julia> X = character_table(W);
julia> E = elements(W);
julia> [[X[i](E[s]) for s in 1:length(E)] for i in 1:length(X)]
```

Note that the first and third row of  $\chi$  correspond to the sign and trivial representation of dimension 1, respectively. By computing  $\text{Trace}(\vartheta(s))$  and solving for the  $m_i$ , we find the multiplicities in Table 1.

	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$
$m_1$	0	1	3	6	10	15
$m_2$	2	6	12	20	30	42
$m_3$	3	6	10	15	21	28

Table 1: The multiplicities  $m_i$  of the irreducible representations of  $\mathcal{W} \cong \mathfrak{S}_3$  appearing in  $\vartheta$  and depending on  $d$ .

To obtain the matrices of the irreducible representations, we use Proposition 4.1. For  $d = 1$ , we have  $\Omega_1 = \{0\} \cup \mathcal{W}\omega_2 \cup \mathcal{W}\omega_1 = \{0, -\omega_1, \omega_1 - \omega_2, \omega_2, -\omega_2, \omega_2 - \omega_1, \omega_1\}$  and the projections of  $\mathbb{R}^{\Omega_1} = \mathbb{R}^7$  onto  $V_1, V_2, V_3$  are

$$\mathbf{P}^{(1)} = 0, \quad \mathbf{P}^{(2)} = \frac{1}{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}, \quad \mathbf{P}^{(3)} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The spaces  $W_2 \cong \langle \mathbf{P}_{.2}^{(2)}, \mathbf{P}_{.3}^{(2)} \rangle_{\mathbb{R}}$  (second and third column) and  $W_3 \cong \langle \mathbf{P}_{.1}^{(3)} \rangle_{\mathbb{R}}$  (first column) are two irreducible, non-isomorphic, orthogonal  $\mathcal{W}$ -submodules and the representing matrices in this basis are

$$\mathbf{D}^{(1)}(s_1) = \mathbf{D}^{(1)}(s_2) = (-1), \quad \mathbf{D}^{(2)}(s_1) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{D}^{(2)}(s_2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{D}^{(3)}(s_1) = \mathbf{D}^{(3)}(s_2) = (1).$$

Following the steps preceding Proposition 4.2, we obtain a change of basis  $\tilde{\mathbf{T}}$ , so that, for  $s \in \mathcal{W}$ , the matrices  $\tilde{\mathbf{T}}^t \vartheta(s) \tilde{\mathbf{T}}$  follow the block diagonal structure

$$\begin{pmatrix} \begin{bmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix} & & & & & & \\ & \begin{bmatrix} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix} & & & & & \\ & & \blacksquare & & & & \\ & & & \blacksquare & & & \\ & & & & \blacksquare & & \\ & & & & & \blacksquare & \\ & & & & & & \blacksquare \end{pmatrix},$$

that is,  $m_2 = 2$  blocks of size  $d_2 = 2$  and  $m_3 = 3$  blocks of size  $d_3 = 1$ .

On the other hand, we obtain a change of basis  $\mathbf{T}$ , so that, for  $\mathbf{X} \in \text{Toep}_d^{\mathcal{W}}$ , the matrices  $\mathbf{T}^t \mathbf{X} \mathbf{T}$  follow the block

diagonal structure

$$\left( \begin{array}{c} \left[ \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array} \right] \\ \\ \left[ \begin{array}{cc} \blacksquare & \blacksquare \\ \blacksquare & \blacksquare \end{array} \right] \\ \\ \left[ \begin{array}{ccc} \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{array} \right] \end{array} \right),$$

that is,  $d_2 = 2$  blocks of size  $m_2 = 2$  and  $d_3 = 1$  block of size  $m_3 = 3$ .

Specifically, a matrix  $\mathbf{X} \in \text{Toep}_1^{\mathcal{W}}$  is of the form

$$\mathbf{X} = \begin{pmatrix} a & b & b & b & b & b & b \\ \bar{b} & a & c & c & b & b & d \\ \bar{b} & c & a & c & b & d & b \\ \bar{b} & c & c & a & d & b & b \\ \bar{b} & \bar{b} & \bar{b} & \bar{d} & a & c & c \\ \bar{b} & \bar{b} & \bar{d} & \bar{b} & c & a & c \\ \bar{b} & \bar{d} & \bar{b} & \bar{b} & c & c & a \end{pmatrix} \xrightarrow{\mathbf{T}^t \dots \mathbf{T}} \begin{pmatrix} a-c & b-d & 0 & 0 & 0 & 0 & 0 \\ \bar{b}-\bar{d} & a-c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a-c & d-b & 0 & 0 & 0 \\ 0 & 0 & \bar{d}-\bar{b} & a-c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & \sqrt{3}b & \sqrt{3}b \\ 0 & 0 & 0 & 0 & \sqrt{3}\bar{b} & a+2c & 2b+d \\ 0 & 0 & 0 & 0 & \sqrt{3}\bar{b} & 2\bar{b}+\bar{d} & a+2c \end{pmatrix}.$$

for some  $a, c \in \mathbb{R}$ ,  $b, d \in \mathbb{C}$ . We now consider the problem of minimizing the trigonometric polynomial

$$f := 6+4e^{\omega_1}+4e^{\omega_2}+4e^{\omega_2-\omega_1}+4e^{\omega_1-\omega_2}+4e^{-\omega_1}+4e^{-\omega_2}+2e^{2\omega_1}+2e^{2\omega_2}+2e^{2\omega_2-2\omega_1}+2e^{2\omega_1-2\omega_2}+2e^{-2\omega_1}+2e^{-2\omega_2}$$

with  $a = 6$ ,  $b = c = 7$ ,  $d = 14$ . Then we have

$$\begin{aligned} f^* &\geq f_{1,\mathcal{W}}^{\text{block}} = \min_{\mathbf{X} \in \text{Toep}_1^{\mathcal{W}}} \sum_{i=1}^h d_i \text{Trace}(\tilde{\mathbf{F}}_i \tilde{\mathbf{X}}_i) \quad \text{s.t.} \quad \text{Trace}(\mathbf{X}) = 1, \mathbf{T}^t \mathbf{X} \mathbf{T} \text{ has blocks } \tilde{\mathbf{X}}_i \geq 0. \\ &= \min_{a,c \in \mathbb{R}, b,d \in \mathbb{C}} 42(a+2b+2\bar{b}+d+\bar{d}) \\ &\quad \text{s.t.} \quad 7a = 1, \begin{pmatrix} a-c & b-d \\ \bar{b}-\bar{d} & a-c \end{pmatrix} \geq 0, \begin{pmatrix} a-c & d-b \\ \bar{d}-\bar{b} & a-c \end{pmatrix} \geq 0, \begin{pmatrix} a & \sqrt{3}b & \sqrt{3}b \\ \sqrt{3}b & a+2c & 2b+d \\ \sqrt{3}b & 2\bar{b}+\bar{d} & a+2c \end{pmatrix} \geq 0. \end{aligned}$$

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