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Zheng Huang, Ben Lowe, Andrea Seppi. Uniqueness and non-uniqueness for the asymptotic Plateau problem in hyperbolic space. 2023. hal-04235517

HAL Id: hal-04235517

<https://hal.science/hal-04235517v1>

Preprint submitted on 11 Oct 2023

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UNIQUENESS AND NON-UNIQUENESS FOR THE ASYMPTOTIC PLATEAU PROBLEM IN HYPERBOLIC SPACE

ZHENG HUANG, BEN LOWE, AND ANDREA SEPPI

ABSTRACT. We prove a number of results on the number of solutions to the asymptotic Plateau problem in \mathbb{H}^3 . In the direction of non-uniqueness, we construct an example of a quasicircle that is the asymptotic boundary of uncountably many pairwise distinct stable minimal disks.

Moreover, we discuss criteria that ensure uniqueness. Given a Jordan curve Λ in the asymptotic boundary of \mathbb{H}^3 , we show that uniqueness of the minimal surfaces with asymptotic boundary Λ is equivalent to uniqueness in the smaller class of stable minimal disks, and, when Λ is invariant by a Kleinian group, to uniqueness in the even smaller class of group invariant stable minimal disks.

Finally, we show that if a quasicircle (or more generally, a Jordan curve of finite width) Λ is the asymptotic boundary of a minimal surface Σ with principal curvatures less than or equal to 1 in absolute value, then uniqueness holds.

CONTENTS

1. Introduction	2
1.1. Characterizing uniqueness	3
1.2. Uniqueness criteria via curvature conditions	4
1.3. Strong non-uniqueness	6
1.4. Quasiconformal constant	7
1.5. Organization of the paper	8
1.6. Acknowledgements	8
2. Preliminaries	8
2.1. Hypersurfaces theory	8
2.2. Convex hull and width of a Jordan curve	9
2.3. Stable and area-minimizing minimal hypersurfaces	10

Date: September 4, 2023.

2020 Mathematics Subject Classification. Primary 53C42, 57K32.

2.4. Compactness results	11
3. Uniqueness among stable minimal disks	12
3.1. Outline	12
3.2. Adapting the arguments of Anderson	12
3.3. Conclusion of the proofs	15
4. Small curvatures and uniqueness	16
4.1. Small curvature conditions	16
4.2. Proof of Theorem 1.3 for $n = 2$	17
4.3. Alternative argument for Theorem 1.2	19
5. Higher dimension and codimension	21
5.1. Hypersurfaces in higher dimensions	21
5.2. Higher codimension submanifolds	22
6. Uncountably many minimal disks	23
6.1. Constructing the Jordan curve	23
6.2. Topological disks	28
6.3. Conclusion of the proof	31
Appendix A. Small curvatures	33
References	36

1. INTRODUCTION

The classical “asymptotic Plateau problem” asks, given a Jordan curve Λ on $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$, how to count the number of (properly embedded) minimal surfaces Σ in \mathbb{H}^3 , if any, that are asymptotic to Λ , in the sense that the closure of Σ in $\mathbb{S}_\infty^2 \cup \mathbb{H}^3$ is equal to $\Lambda \cup \Sigma$. The existence of minimal disk solutions to the asymptotic Plateau problem was obtained by Anderson ([[And83](#)]). Using geometric measure theory, Anderson also obtained existence results in higher dimensions.

The uniqueness does not hold in general: as shown in [[And83](#), [HW15](#)], taking advantage of group actions, one can construct a Jordan curve Λ in $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$ which is the limit set of some quasi-Fuchsian group such that Λ spans multiple minimal disks (even an arbitrarily large, but finite, number). Anderson ([[And83](#)]) even constructed a curve Λ which spans infinitely many minimal surfaces (the surfaces he constructs have positive genus). On the other hand, when Λ is a round circle,

the unique minimal surface it spans is a totally geodesic disk. To look for unique solutions, it is therefore natural to consider the class of minimal surfaces that are “close” to totally geodesic, for which Λ is “close” to a round circle. Related questions with conditions on natural invariants of Λ were studied in [Sep16] (for the quasi-conformal constant of Λ), and [HW13, San18] (for the Hausdorff dimension of Λ). Some properness questions for the asymptotic Plateau problem solutions for various classes of curves were addressed, for example, in [GS00, AM10]. For an overview of this active area of research see the survey paper [Cos14].

Motivated by the above results and by many natural questions arising from the study of the asymptotic Plateau problem, in this paper we address two basic questions, which, for the sake of simplicity, we state only in dimension three, but may naturally be extended to hypersurfaces in \mathbb{H}^{n+1} :

- under what conditions does a Jordan curve Λ on \mathbb{S}_∞^2 span exactly one minimal surface in \mathbb{H}^3 ?
- does there exist a Jordan curve Λ on \mathbb{S}_∞^2 that spans infinitely many minimal disks in \mathbb{H}^3 , and if yes, which cardinality may the set of solutions have?

1.1. Characterizing uniqueness. Recall that a (hyper)surface is *minimal* if it is a critical point of the area under compactly supported variations. It is *stable* if moreover the second variation of the area under any compactly supported variation is non-negative. In this introduction, we will always implicitly consider *properly embedded* (hyper)surfaces. Our first theorem shows that it suffices to check uniqueness in the class of *stable minimal disks*.

Theorem 1.1. *Let Λ be a Jordan curve on $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$. Then Λ spans a unique minimal surface if and only if it spans a unique stable minimal disk.*

A statement similar to Theorem 1.1, but in the context of the *finite* Plateau problem, was proved in [MY19]. When Λ is invariant under the action of a Kleinian group (i.e. a discrete subgroup of isometries of \mathbb{H}^3), which is for instance the case for the limit set of a quasi-Fuchsian group, we can prove a stronger statement.

Theorem 1.2. *Let Λ be a Jordan curve on $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$, and let Γ be any Kleinian group preserving Λ . Then Λ spans a unique minimal surface if and only if it spans a unique Γ -invariant stable minimal disk.*

The main idea in the proof of Theorem 1.1 is an adaptation of an argument in [And83, Theorem 3.1]: we show that if there is a minimal surface, which is not stable or is not topologically a disk, with asymptotic boundary Λ , then we can construct two distinct — actually, disjoint — stable minimal disks with the same

asymptotic boundary Λ (Theorem 3.1). The proof of Theorem 1.2 then relies on a further improvement of the arguments of [And83, Theorem 3.1], showing that if there is a non-invariant stable minimal disk, then we can construct two disjoint invariant stable minimal disks (Theorem 3.2).

1.2. Uniqueness criteria via curvature conditions. Next, we turn our attention to sufficient conditions for uniqueness. For an immersed hypersurface in \mathbb{H}^{n+1} , or more generally in a hyperbolic $(n+1)$ -manifold, we say it has *strongly small curvature* if its principal curvatures $\{\lambda_i\}$ satisfy that

$$(1.1) \quad |\lambda_i| \leq 1 - \epsilon, \quad i = 1, \dots, n, \quad \text{for some small } \epsilon > 0.$$

Similarly we say it has *small curvature* if $|\lambda_i| < 1$, and it has *weakly small curvature* if $|\lambda_i| \leq 1$. This definition has some immediate consequences: for instance, a complete immersion of weakly small curvature is in fact a properly embedded topological disk ([Eps84, Eps86], see also [EES22] for a generalization). See Section 4 and Appendix A for more details.

Surfaces of small curvature are very special in three-manifold theory: Thurston observed that a closed surface of small curvature in a complete hyperbolic three-manifold is incompressible ([Thu86, Lei06]); they are abundant in closed hyperbolic three-manifolds ([KM12]); many results have been extended to the study of complete noncompact hyperbolic three-manifold of finite volume ([Rub05, CF19, KW21]). It is often favorable to consider canonical representatives within a homotopy class of surfaces, and minimal surfaces are in many ways the most natural choice. Among hyperbolic three-manifolds, *almost-Fuchsian* manifolds are quasi-Fuchsian manifolds which admit a closed minimal surface of small curvature. This notion was introduced by Uhlenbeck and it played an important role in her study of parametrization of the moduli space of minimal surfaces in hyperbolic three-manifolds ([Uhl83]). Subsequently many different aspects of this subclass of quasi-Fuchsian manifold have been studied up to recent years (for instance [KS07, GHW10, HL21] and many others).

It is known ([Uhl83]) that any almost-Fuchsian manifold admits a unique closed minimal surface — in other words, identifying the almost-Fuchsian manifold with a quotient \mathbb{H}^3/Γ , the limit set Λ of the group Γ bounds a unique Γ -invariant minimal disk asymptotic to Λ . Inspired by this fact, we prove (Corollary 1.4 below) that if a Jordan curve Λ (not necessarily group equivariant) spans a minimal disk Σ of strongly small curvature in \mathbb{H}^3 , then Σ is the unique minimal surface asymptotic to Λ . Our results, however, are more general. The main result we prove in this direction is the following:

Theorem 1.3. *Let Λ be a topologically embedded n -sphere on $\mathbb{S}_\infty^n = \partial_\infty \mathbb{H}^{n+1}$ of finite width, and let Σ be a minimal hypersurface in \mathbb{H}^{n+1} of weakly small curvature asymptotic to Λ . Then Σ is the unique minimal hypersurface in \mathbb{H}^{n+1} asymptotic to Λ . Moreover, Σ is area-minimizing.*

Let us explain the terminology of the statement. First, recall that a hypersurface Σ is *area-minimizing* if any compact codimension-zero submanifold with boundary has smaller area than any rectifiable hypersurface with the same boundary in the ambient space. This implies that Σ is a stable minimal hypersurface. Second, the *width* of a Jordan curve Λ in $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$ has been introduced in [BDMS21] — and the definition is immediately extended to higher dimensions — as the supremum over all points in the convex hull of Λ of the sum of the distances from each boundary component of the convex hull.

We include two proofs of Theorem 1.3 as they offer different perspectives. The first proof only works for $n = 2$, but it is based on elementary geometric arguments, which we sketch here. The first observation is that the finite width condition implies that every minimal surface Σ' has, roughly speaking, finite normal distance from Σ . Moreover, by Theorem 1.1, it suffices to show uniqueness among *stable* minimal disks. Now, if the maximum normal distance between Σ and another stable minimal surface is realized in the interior, the conclusion follows from a maximum principle argument, taking advantage the properties of the normal flow from Σ . If the maximum normal distance is not realized in the interior, we use isometries to send the run-away sequence of points back to a fixed point and use the compactness theorems for stable minimal surfaces, so as to reduce essentially to the previous case. The second proof, which works in any dimension, rests on an application of a more general maximum principle proved in [Whi10] in the context of minimal varifolds (see also [JT03]).

We now derive several corollaries of Theorem 1.3. Firstly, observe that if a properly embedded hypersurface Σ has strongly small curvatures, then its asymptotic boundary has finite width (See Lemma A.4 in Appendix A). Hence we obtain:

Corollary 1.4. *Let Λ be a topologically embedded n -sphere on $\mathbb{S}_\infty^n = \partial_\infty \mathbb{H}^{n+1}$, and let Σ be a minimal hypersurface in \mathbb{H}^{n+1} of strongly small curvature asymptotic to Λ . Then Σ is the unique minimal hypersurface in \mathbb{H}^{n+1} asymptotic to Λ . Moreover, Σ is area-minimizing.*

Secondly, in dimension $n = 2$, quasicircles are an important class of Jordan curves, which are known to have finite width. We thus obtain immediately the following.

Corollary 1.5. *Let Λ be a quasicircle on $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$, and let Σ be a minimal surface in \mathbb{H}^3 of weakly small curvature asymptotic to Λ . Then Σ is the unique minimal surface in \mathbb{H}^3 asymptotic to Λ . Moreover, Σ is area-minimizing.*

We remark that the setting of Theorems 1.1 and 1.2 is more general than Theorem 1.3 and Corollary 1.5. Indeed, it follows from [HL21, Theorem 5.2] that there are examples of quasi-Fuchsian groups Γ whose limit set Λ bounds a unique Γ -invariant stable minimal disk Σ (hence, by Theorem 1.2, a unique minimal surface), but Σ does *not* have weakly small curvature.

1.3. Strong non-uniqueness. Now we turn to the other extreme case: we aim to construct a Jordan curve on \mathbb{S}_∞^2 which spans *a lot* of minimal disks. To give some context, Hass-Thurston conjectured that no closed hyperbolic 3-manifold admits a foliation by minimal surfaces. Anderson even conjectured ([And83]) that no hyperbolic 3-manifold admits a local 1-parameter family of closed minimal surfaces, and proved this statement for quasi-Fuchsian hyperbolic 3-manifolds. It has been a folklore conjecture that no Jordan curve in $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$ asymptotically bounds a 1-parameter family of minimal surfaces. The full extent of these conjectures remain as major open questions in the field.

What has been proven up to this point tends to support these conjectures. Huang-Wang [HW19] and Hass [Has15] made progress on the Hass-Thurston conjecture for certain fibered closed hyperbolic 3-manifolds containing short geodesics; Wolf-Wu ([WW20]) ruled out so-called geometric local 1-parameter families of closed minimal surfaces; it follows from the work of Alexakis-Mazzeo [AM10] that a generic $C^{3,\alpha}$ simple closed curve in the boundary at infinity of \mathbb{H}^3 bounds only finitely many surfaces of any given finite genus; Coskunuzer proved that generic simple closed curves in $\partial_\infty \mathbb{H}^3$ bound unique area-minimizing surfaces [Cos11].

On the other hand what we prove, while compatible with the folklore conjecture, is in the other direction. Based on the aforementioned results, one might be tempted to strengthen the folklore conjecture to the statement that any Jordan curve in $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$ bounds at most countably many minimal surfaces. We show that this stronger statement is false:

Theorem 1.6. *There exists a quasicircle in $\mathbb{S}_\infty^2 = \partial_\infty \mathbb{H}^3$ spanning uncountably many pairwise distinct stable minimal disks.*

Let us emphasize some important features of the construction of this extreme curve Λ . In ([And83]), Anderson constructed a Jordan curve which is the limit set of a quasi-Fuchsian group (hence continuous but almost nowhere differentiable) such

that it spans infinitely many minimal surfaces, one of which is a minimal disk. In [HW15], for each integer $N > 1$, also using the limit set of a quasi-Fuchsian group, an extreme curve spanning at least 2^N distinct minimal disks was constructed. However, Anderson ([And83]) has shown that any quasi-Fuchsian manifold only admits finitely many least area closed minimal surfaces diffeomorphic to the fiber, which poses a possible limitation on how much one can improve the aforementioned constructions to find infinitely many minimal disks if one insists on using the limit set of some quasi-Fuchsian group as the curve at infinity. The starting point of our construction is similar to the ideas in [HW15], but the Jordan curve is constructed in such a way to allow an improvement of the argument, leading to 2^N pairwise distinct minimal disk. Moreover, since this Jordan curve is not invariant under any quasi-Fuchsian group, we must adopt a different approach in order to produce the minimal disks, namely, in a spirit similar to the proofs of Theorems 1.1 and 1.2, we take the limit of a sequence of solutions of the finite Plateau problems inside \mathbb{H}^3 .

1.4. Quasiconformal constant. We conclude this introduction with an improvement of the curvature estimates obtained in [Sep16] in terms of quasiconformal constants, by a direct application of Corollary 1.5. More concretely, [Sep16, Theorem A] showed that there exist universal constants $C > 0$ and $K_0 > 1$ such that any stable minimal disk in \mathbb{H}^3 with asymptotic boundary a K -quasicircle, for $K < K_0$, has principal curvatures bounded in absolute value by $C \log K$. This result has been recently applied in several directions, see [Bis19, Low21, CMN22].

The proof, however, relies on the application of compactness for minimal surfaces, and therefore requires stability. However, when K is sufficiently small, the principal curvatures of the area-minimizing (hence stable) disk whose existence is guaranteed by [And83] are less than $1 - \epsilon$ in absolute value, and therefore, as a consequence of Corollary 1.5, Σ is the unique minimal surface. Up to taking a smaller constant K_0 , we can therefore remove the stability assumption:

Corollary 1.7. *There exist universal constants $C > 0$ and $K_0 > 1$ such that the principal curvatures λ_i of any minimal surface Σ in \mathbb{H}^3 with asymptotic boundary a K -quasicircle with $K \leq K_0$ satisfy*

$$|\lambda_i| \leq C \log K, \quad i = 1, 2.$$

In particular, we also improve [Sep16, Theorem B] (up to choosing a smaller constant) by removing the stability assumption.

Corollary 1.8. *There exists a universal constant $K'_0 > 1$ such that any K -quasicircle with $K \leq K'_0$ is the asymptotic boundary of a unique minimal surface, which is an area-minimizing disk of strongly small curvature.*

1.5. Organization of the paper. In the preliminary section §2, we collect and prove some facts in preparation to work towards proofs of our main results introduced above. In §3 we prove Theorems 1.1 and 1.2. In §4 we work in the case of small curvature conditions and prove Theorem 1.3 and, using similar methods, we provide an alternative argument for Theorem 1.2 in a special case; in §5 we address some generalizations of our results in higher dimensions and codimensions; in §6 we detail a construction of an extreme Jordan curve which spans uncountably many minimal disks in \mathbb{H}^3 and hence prove Theorem 1.6. In Appendix A we provide the details to extend some well-known arguments for hypersurfaces of small curvature to the setting of weakly small curvature that is of interest here.

1.6. Acknowledgements. The first-named author wishes to thank Bill Meeks for his insightful suggestions and Biao Wang for his generous help. Part of this work was done during a visit of the first-named author at the Institut Fourier (Université Grenoble Alpes) in the framework of the “Visiting Scientist Campaign 2023”, he wishes to thank the institute for excellent working environment. The second-named author was supported by NSF grant DMS-2202830. The third-named author was partially supported by the ANR JCJC grant GAPR (ANR-22-CE40-0001, Geometry and Analysis in the Pseudo-Riemannian setting).

2. PRELIMINARIES

In this article, we denote by \mathbb{H}^{n+1} the hyperbolic space of dimension $n + 1$, and by $\mathbb{S}_\infty^n = \partial_\infty \mathbb{H}^{n+1}$ its visual boundary.

2.1. Hypersurfaces theory. Given an immersed hypersurface Σ in \mathbb{H}^{n+1} , we recall that its first fundamental form is the restriction to $T\Sigma$ of the hyperbolic metric h of \mathbb{H}^{n+1} . The second fundamental form is defined as

$$A_\Sigma(v, w) = h(\nabla_v^h W, N_\Sigma) ,$$

where ∇^h is the Levi-Civita connection of h , W is a local smooth extension of w , and $N_\Sigma : \Sigma \rightarrow T\mathbb{H}^{n+1}$ is a continuous choice of a unit normal vector to the immersion. (Since N_Σ is uniquely determined only up to a sign, so too is A_Σ .)

The second fundamental form satisfies the identity

$$A_\Sigma(v, w) = h(B_\Sigma(v), w)$$

where B_Σ is the shape operator, namely the endomorphism of the tangent bundle of Σ given by $B_\Sigma(v) = -\nabla_v^h N_\Sigma$. The principal curvatures of Σ are the eigenvalues of B_Σ , denoted by $\lambda_1, \dots, \lambda_n$.

The mean curvature of the immersed hypersurface Σ is

$$H_\Sigma = \text{tr}(B_\Sigma) = \lambda_1 + \dots + \lambda_n ,$$

and Σ is minimal if and only if its mean curvature vanishes identically. Although H_Σ depends (but only up to a sign) on the choice of the normal vector N_Σ , the condition of being minimal does not. Also, the mean curvature vector, which is defined as $H_\Sigma \cdot N_\Sigma$, does not depend on such a choice.

Finally, the norm of the second fundamental form is

$$\|A_\Sigma\| = \sqrt{\text{tr}(B_\Sigma B_\Sigma^T)} = \sqrt{\lambda_1^2 + \dots + \lambda_n^2} .$$

2.2. Convex hull and width of a Jordan curve. Let Λ be a topologically embedded n -sphere in $\mathbb{S}_\infty^n = \partial\mathbb{H}^{n+1}$. We recall that the convex hull $\mathcal{C}(\Lambda) \subset \overline{\mathbb{H}^{n+1}} = \mathbb{H}^{n+1} \cup \mathbb{S}_\infty^n$ of Λ is the smallest geodesically convex subset that contains Λ . When Λ is not the boundary of a totally geodesic hyperplane in \mathbb{H}^{n+1} , $\mathcal{C}(\Lambda)$ is homeomorphic to a ball, and, by the Jordan-Brouwer separation theorem, its boundary is the union of Λ and two properly embedded disks, denoted by $\partial^+\mathcal{C}(\Lambda)$ and $\partial^-\mathcal{C}(\Lambda)$. When Λ is the boundary of a totally geodesic hyperplane P , $\mathcal{C}(\Lambda)$ equals $P \cup \Lambda$. In this case, by an abuse of notation, we will still use the symbols $\partial^+\mathcal{C}(\Lambda)$ and $\partial^-\mathcal{C}(\Lambda)$, meaning that $P = \partial^+\mathcal{C}(\Lambda) = \partial^-\mathcal{C}(\Lambda)$.

Following (and extending to all dimensions) the recent work [BDMS21], we now define the *width* of Λ .

Definition 2.1. Given a topologically embedded n -sphere Λ in \mathbb{S}_∞^n , the width of Λ is defined as:

$$w(\Lambda) = \sup_{x \in \mathcal{C}(\Lambda)} (d(x, \partial^+\mathcal{C}(\Lambda)) + d(x, \partial^-\mathcal{C}(\Lambda))) \in [0, +\infty] .$$

The following lemma, that relates minimal hypersurfaces and the convex hull of their asymptotic boundaries, is well-known.

Lemma 2.2. *Given a topologically embedded n -sphere Λ in \mathbb{S}_∞^n , let Σ be any properly embedded minimal hypersurface such that $\partial_\infty \Sigma = \Lambda$. Then Σ is contained in $\mathcal{C}(\Lambda)$. Moreover, if Λ is not the boundary of a totally geodesic hyperplane, then Σ is contained in the interior of $\mathcal{C}(\Lambda)$.*

Proof. Let P be any totally geodesic hyperplane disjoint from Λ . The signed distance function from P , defined in such a way that it goes to $-\infty$ as it approaches Λ , cannot have a positive maximum by a standard application of the geometric maximum principle (see also Corollary 4.4 for a more general statement). This implies that Σ

is contained in the half-space bounded by P whose closure contains Λ . Since $\mathcal{C}(\Lambda)$ is the intersection of all such half-spaces, this concludes the proof of the first assertion.

For the second assertion, suppose that Σ contains a point x in the boundary of $\mathcal{C}(\Lambda)$. Let P be any support hyperplane for $\mathcal{C}(\Lambda)$ containing x . This means that Σ is tangent to P and, by the first assertion, it is contained in a half-space bounded by P . By the strong maximum principle, $\Sigma = P$, and therefore $\Lambda = \partial_\infty P$. \square

2.3. Stable and area-minimizing minimal hypersurfaces. A minimal hypersurface Σ in a Riemannian manifold M^{n+1} is stable if and only if, for every $u \in C_0^\infty(\Sigma)$,

$$\int_{\Sigma} u L_{\Sigma}(u) d\text{Vol}_{\Sigma} \geq 0 ,$$

where L_{Σ} is the Jacobi operator:

$$(2.1) \quad L_{\Sigma}(u) = -\Delta_{\Sigma}u - (\|A_{\Sigma}\|^2 + \text{Ric}_M(N, N)) u .$$

Equivalently, integrating by parts, Σ is stable if and only if

$$(2.2) \quad \int_{\Sigma} (\|A_{\Sigma}\|^2 + \text{Ric}_M(N, N)) u^2 d\text{Vol}_{\Sigma} \leq \int_{\Sigma} \|\nabla u\|^2 d\text{Vol}_{\Sigma}$$

for every $u \in C_0^\infty(\Sigma)$.

A compact hypersurface Σ_c with boundary is a *least area hypersurface* if its area is less than or equal to that of any other compact hypersurface Σ'_c such that $\partial\Sigma_c = \partial\Sigma'_c$. For the non-compact case, we say a hypersurface Σ is area-minimizing if any compact codimension 0 submanifold with boundary of Σ has least area in the sense above.

An area-minimizing hypersurface is stable. Indeed, the area-minimizing condition implies that Σ is a critical point of the area functional among compactly supported variation (that is, Σ is minimal) and the second derivative of the area is non-negative (which is equivalent to the definition of stability given above).

When $n = 2$, the following theorem provides an important a priori bound on the curvature of a stable minimal surface. We state the theorem here in its version for embedded minimal surfaces in \mathbb{H}^3 , but it holds more generally for immersed CMC surfaces in a complete Riemannian manifold of bounded sectional curvature.

Theorem 2.3. [RST10, Main Theorem] *There exists a constant $c > 0$ such that, for any stable embedded two-sided minimal surface Σ in \mathbb{H}^3 ,*

$$\|A_{\Sigma}(p)\| \leq \frac{c}{\min\{d(p, \partial\Sigma), \pi/2\}} .$$

Applying Theorem 2.3 to a properly embedded minimal surface, which is automatically two-sided (see [Sam69]), we immediately get the following corollary. See also [Ros06, Corollary 11].

Corollary 2.4. *There exists a constant $c' > 0$ such that, for any stable properly embedded minimal surface Σ in \mathbb{H}^3 , $\|A_\Sigma\| \leq c'$.*

2.4. Compactness results. We conclude the preliminaries by stating an important compactness theorem for minimal surfaces. Again, we state the theorem here only for embedded minimal surfaces in \mathbb{H}^3 , but it holds more generally for immersed surfaces in a complete Riemannian manifold (see [And85, Section 2]). This result is well-known and is often stated under the assumption of bounded Gaussian curvature (in absolute value); however, by the Gauss' equation for minimal surfaces when $n = 2$, $K_\Sigma = -1 - \|A_\Sigma\|^2/2$, hence this is equivalent to a bound on the norm of the second fundamental form.

Theorem 2.5. *Let B be a bounded domain in \mathbb{H}^3 and let $\{\Sigma_n\}_{n \in \mathbb{N}}$ be a sequence of embedded minimal surfaces in B with $\partial\Sigma_n \subset \partial B$. Suppose there exists a constant C such that $\|A_{\Sigma_n}\| \leq C$ and that there exists a compact subset $K \subset \text{int}(B)$ such that $\Sigma_n \cap K \neq \emptyset$, for all n . Then, up to extracting a subsequence, Σ_n converges smoothly to an embedded minimal surface Σ_∞ .*

In particular, for properly embedded minimal surfaces, we obtain the following compactness result.

Corollary 2.6. *Let $\{\Sigma_n\}_{n \in \mathbb{N}}$ be a sequence of properly embedded minimal surfaces in \mathbb{H}^3 . Suppose there exists a constant C such that $\|A_{\Sigma_n}\| \leq C$ and that there exists a compact subset $K \subset \mathbb{H}^3$ such that $\Sigma_n \cap K \neq \emptyset$, for all n . Then, up to extracting a subsequence, Σ_n converges smoothly on compact subsets of \mathbb{H}^3 to a properly embedded minimal surface Σ_∞ .*

Proof. Fix a point $x_o \in \mathbb{H}^3$. Let $m_0 \in \mathbb{N}$ be such that $K \subset B(x_o, m_0)$. For any $m \geq m_0$, by applying Theorem 2.5 on the balls $B(x_o, m)$ of radius m centered at x_o , we find a subsequence converging smoothly to a minimal surface in $B(x_o, m)$. By a standard diagonal argument, we can then extract a subsequence converging smoothly on all compact subsets to an embedded minimal surface Σ_∞ .

It only remains to show that Σ_∞ is properly embedded. Suppose by contradiction that $x_i \in \Sigma_\infty$ is a diverging sequence in Σ_∞ , which accumulates to $x_\infty \in \mathbb{H}^3$. Then for m large, x_∞ would be contained in the interior of $B(x_o, m)$, and this would contradict the smooth convergence. \square

3. UNIQUENESS AMONG STABLE MINIMAL DISKS

In this section, we prove Theorems 1.1 and 1.2.

3.1. Outline. The key result to prove Theorem 1.1 is the following:

Theorem 3.1. *Let Λ be a Jordan curve on $\mathbb{S}_\infty^2 = \partial\mathbb{H}^3$. Suppose that there exists a properly embedded minimal surface in \mathbb{H}^3 , which is not a stable minimal disk, spanning Λ , then Λ spans at least two disjoint stable minimal disks in \mathbb{H}^3 .*

Before proving Theorem 3.1, let us explain how this implies Theorem 1.1. By the results of [And83], we know that Λ spans a stable (actually, area-minimizing) minimal disk. So uniqueness among minimal surfaces clearly implies uniqueness among stable minimal disks. For the converse direction, suppose by contradiction that Λ span a unique stable minimal disk, but it spans several minimal surfaces. Hence it spans a minimal surface U which is not a stable minimal disk (that is, either it is not stable, or it is not a disk, or both). By Theorem 3.1, Λ spans two distinct stable minimal disks, and this gives a contradiction.

When Λ is invariant by the action of a Kleinian group, in order to achieve Theorem 1.2, we will first prove the following result in the same spirit of Theorem 3.1.

Theorem 3.2. *Let Λ be a Jordan curve on $\mathbb{S}_\infty^2 = \partial\mathbb{H}^3$ and let Γ be a Kleinian group preserving Λ . Suppose that there exists a properly embedded minimal surface in \mathbb{H}^3 , which is not Γ -invariant, spanning Λ . Then Λ spans at least two disjoint Γ -invariant stable minimal disks in \mathbb{H}^3 .*

Theorem 3.2 implies Theorem 1.2 very similarly to above. Indeed, Anderson proved the existence of a Γ -invariant stable minimal disk, hence uniqueness among minimal surfaces implies uniqueness among Γ -invariant stable minimal disks. Conversely, suppose that Λ spans a minimal surface which is not a Γ -invariant stable minimal disk. If it is not a minimal disk, Theorem 3.1 applies as above. If it is instead not Γ -invariant, we apply Theorem 3.2 to produce two distinct Γ -invariant stable minimal disks, and this gives a contradiction.

3.2. Adapting the arguments of Anderson. To prove Theorems 3.1 and 3.2, we will adapt an argument in [And83]. We first need the following result.

Lemma 3.3. *Let Λ be a Jordan curve on $\mathbb{S}_\infty^2 = \partial\mathbb{H}^3$, let U be a properly embedded minimal surface in \mathbb{H}^3 spanning Λ , and let Ω be a connected component of $\mathbb{H}^3 \setminus U$. Then there exists a properly embedded stable minimal disk Σ in $\Omega \cup U$ spanning Λ .*

Moreover, if Λ is invariant under a Kleinian group Γ , then Σ can be taken such that

$$(3.1) \quad \Sigma \subset \bigcap_{g \in \Gamma} g(\Omega \cup U) .$$

Proof. We can clearly assume that Λ is not the boundary of a totally geodesic plane. Then, since U is contained in the interior of $\mathcal{C}(\Lambda)$ by Lemma 2.2, Ω contains either $\partial^+\mathcal{C}(\Lambda)$ or $\partial^-\mathcal{C}(\Lambda)$. Suppose it contains $\partial^+\mathcal{C}(\Lambda)$, the other case being analogous. Fix a homeomorphism $\phi : \mathbb{D} \rightarrow \Lambda \cup \partial^+\mathcal{C}(\Lambda)$ and let $\gamma_n = \phi(\{|z| = 1 - \frac{1}{n}\})$. Then $\{\gamma_n\}_{n \in \mathbb{N}}$ is a sequence of nullhomotopic simple closed curves in $\Omega \cap \mathcal{C}(\Lambda)$ that converge to Λ as $n \rightarrow +\infty$.

Now, fix a point $x_o \in U$. For any n , let R_n be a sufficiently large radius so that $\gamma_n \subset B(x_o, R_n)$. By genericity of the transversality of intersection of submanifolds, we can furthermore pick R_n so that U and $\partial B(x_o, R_n)$ intersect transversely. By [HS88, Theorem 6.3] applied to $M_n := B(x_o, R_n) \cap \Omega$, which (as a consequence of transversality) is sufficiently convex in the terminology of [HS88], we can solve the Plateau problem to find a minimal disk Σ_n , contained in M_n , with $\partial \Sigma_n = \gamma_n$, and of least area (in M_n), hence in particular stable.

Next we show that the limit of these Σ_n provides the desired stable minimal disk Σ . However, when Λ is Γ -invariant, we first need an additional adjustment which will ensure (3.1). Since Γ is a discrete subgroup of $\text{Isom}(\mathbb{H}^3)$, it is countable, and so too is the coset space $G/\text{Stab}(U)$. Let us enumerate its elements as

$$G/\text{Stab}(U) = \{g_0 = [\text{id}], g_1, g_2, \dots\}$$

and let us denote $U_n := g_n(U)$, which is a properly embedded minimal surface spanning Λ . Here, we construct Σ_n as the solution of the Plateau problem with $\partial \Sigma_n = \gamma_n$, but instead of M_n , we apply [HS88, Theorem 6.3] to the ambient manifold

$$M'_n := B(x_o, R_n) \cap \Omega \cap g_1(\Omega) \dots \cap g_n(\Omega) ,$$

where R_n is chosen so that $\partial B(x_o, R_n)$ intersects transversely all the minimal surfaces $\Omega, g_1(\Omega), \dots, g_n(\Omega)$. Since the points of tangency of two minimal surfaces are isolated (see [CM11, Section 5.3]), the boundary of M'_n is piecewise smooth, meaning that it is smooth (and minimal) in the complement of an embedded finite graph. Hence M'_n is sufficiently convex in the terminology of [HS88], which allows us to apply [HS88, Theorem 6.3].

We now claim that, up to extracting a subsequence, Σ_n converges smoothly on compact subsets of \mathbb{H}^3 to a properly embedded minimal disk Σ . For this, first observe that there exists $R_0 > 0$ such that $\Sigma_n \cap B(x_o, R_0) \neq \emptyset$ for all n . Indeed, since $\partial \Sigma_n = \gamma_n$ is contained in $\partial^+\mathcal{C}(\Lambda)$, by the maximum principle (as in the first

part of Lemma 2.2) Σ_n is entirely contained in $\mathcal{C}(\Lambda)$, and by the Jordan-Brouwer Theorem, Σ_n must disconnect $\mathcal{C}(\Lambda)$ in two components, one of which contains x_o , and the other contains $\phi(0) \in \partial^+\mathcal{C}(\Lambda)$. Fixing a path η connecting x_o to $\phi(0)$ which is contained in the interior of $\mathcal{C}(\Lambda)$ (except for its endpoints), Σ_n must therefore intersect η . Hence Σ_n intersects $B(x_o, R_0)$ for any choice of R_0 such that $B(x_o, R_0)$ contains η .

Now, fix any radius $R > 0$ and consider the ball $B(x_o, R)$. For $n = n(R)$ sufficiently large, γ_n is contained in $\mathbb{H}^3 \setminus B(x_o, R + 1)$. Hence the distance from any point of $\Sigma_n \cap B(x_o, R)$ and $\partial\Sigma_n = \gamma_n$ is bounded below by 1. Theorem 2.3 implies that there is a constant C , independent of n , such that $\|A_{\Sigma_n}(x)\| \leq C$ for every $x \in \Sigma_n \cap B(x_o, R)$. Applying Theorem 2.5, we can extract a subsequence n_k such that $\Sigma_{n_k} \cap B(x_o, R)$ converges smoothly to a minimal disk. By a diagonal argument, we can then find a subsequence of $\{\Sigma_n\}_{n \in \mathbb{N}}$ that converges smoothly on compact subsets to a minimal disk Σ_∞ . Moreover, the limit Σ_∞ is stable as a smooth limit of stable minimal surfaces, and is contained in $\Omega \cup U$ because every Σ_n is contained in Ω .

It remains to show that Σ_∞ is properly embedded and $\partial_\infty\Sigma = \Lambda$. To see that it is properly embedded, the same argument as in the proof of Corollary 2.6 applies. Observe that Σ_n is contained in $\mathcal{C}(\Lambda)$ for all n , and therefore so too is Σ_∞ . Since $\mathcal{C}(\Lambda) \cap \partial_\infty\mathbb{H}^3 = \Lambda$, we have that $\partial_\infty\Sigma = \Lambda$. This concludes the proof.

For the “moreover” part when Λ is Γ -invariant, recall that in that case we have constructed the Σ_n to be contained in M'_n . In particular, for fixed m , Σ_n is contained in $g_m(\Omega)$ for all $n \geq m$. Taking limits, Σ_∞ is contained in $g(\Omega \cup U)$ for all $g \in \Gamma$. This concludes the proof. \square

In the case where Λ is Γ -invariant, we improve Lemma 3.3 to the following lemma. We include a proof, which follows closely the construction done by Anderson in [And83, Theorem 3.1], for convenience of the reader.

Lemma 3.4. *Let Λ be a Jordan curve on $\mathbb{S}_\infty^2 = \partial\mathbb{H}^3$ and let Γ be a Kleinian group preserving Λ . Let U be a properly embedded minimal surface in \mathbb{H}^3 spanning Λ , and let Ω be a connected component of $\mathbb{H}^3 \setminus U$. Then there exists a Γ -invariant properly embedded stable minimal disk Σ in $\Omega \cup U$ spanning Λ .*

Proof. We will use the following notation. Choose a connected component D of $\partial_\infty\mathbb{H}^3 \setminus \Lambda$. Given a properly embedded (minimal) surface Σ with $\partial_\infty\Sigma = \Lambda$, recalling that Σ disconnects \mathbb{H}^3 by the Jordan-Brouwer separation theorem, we denote $\Omega^+(\Sigma)$ to be the connected component of $\mathbb{H}^3 \setminus \Sigma$ whose closure in $\mathbb{H}^3 \cup \partial_\infty\mathbb{H}^3$ contains D . Finally, given Σ_1 and Σ_2 , we write $\Sigma_1 \preceq \Sigma_2$ if $\Omega^+(\Sigma_1) \supseteq \Omega^+(\Sigma_2)$.

Now, given U , we define a sequence $\{\Sigma_n\}_{n \in \mathbb{N}}$ inductively as follows. Set $\Sigma_0 := U$ and let Σ_{n+1} , using Lemma 3.3, to be a properly embedded stable minimal surface such that

$$(3.2) \quad \Sigma_{n+1} \subset \bigcap_{g \in \Gamma} g(\Omega^+(\Sigma_n) \cup \Sigma_n) .$$

As in the proof of Lemma 3.3, there exists a fixed compact set (for example a compact set containing a path connecting the two boundary components of $\mathcal{C}(\Lambda)$) that intersects each Σ_n . Moreover, by Corollary 2.4, $\|A_{\Sigma_n}\| \leq C$ for some universal constant C . Hence by Corollary 2.6, Σ_n converges up to a subsequence to a properly embedded minimal surface Σ_∞ , which is moreover stable. Arguing as in the second last paragraph of Lemma 3.3, $\partial_\infty \Sigma_\infty = \Lambda$.

The main point of the construction is that Σ_∞ is Γ -invariant. To see this, fix $g \in \Gamma$. By the inductive step, $g(\Sigma_n) \preceq \Sigma_{n+1}$ for all n . Taking limits along the chosen subsequence, $g(\Sigma_\infty) \preceq \Sigma_\infty$. Repeating the argument for g^{-1} , this shows that $g(\Sigma_\infty) = \Sigma_\infty$ and concludes the proof. \square

3.3. Conclusion of the proofs.

We now prove Theorem 3.1.

Proof of Theorem 3.1. Suppose U is a properly embedded minimal surface in \mathbb{H}^3 spanned by Λ , which is not a stable minimal disk. By the Jordan-Brouwer separation theorem, U disconnects \mathbb{H}^3 . Let Ω^\pm be the two connected components of $\mathbb{H}^3 \setminus U$. By Lemma 3.3, there exists a properly embedded stable minimal disk Σ^\pm contained in $\Omega^\pm \cup U$ with $\partial_\infty \Sigma^\pm = \Lambda$. We claim that Σ^\pm is disjoint from U . Indeed, Σ^\pm is contained by construction in $\Omega^\pm \cup U$, so by the strong maximum principle, if Σ^\pm and U intersect, then they are equal. Since Σ^\pm is a stable minimal disk, and U is not, they must be disjoint. This shows that $\Sigma^+ \subset \Omega^+$ and $\Sigma^- \subset \Omega^-$, hence they are in particular disjoint. \square

The proof of Theorem 3.2 is completely analogous, relying on Lemma 3.4 instead of Lemma 3.3.

Proof of Theorem 3.2. If U is a properly embedded minimal surface in \mathbb{H}^3 spanned by Λ , which is not Γ -invariant, denoting by Ω_\pm the two connected components of the complement of U , then by Lemma 3.4 there exists a properly embedded Γ -invariant stable minimal disk Σ^\pm contained in $\Omega^\pm \cup U$ with $\partial_\infty \Sigma^\pm = \Lambda$. Since Σ^\pm is Γ -invariant, and U is not, they are distinct, and by the strong maximum principle they are disjoint. Hence Σ^+ and Σ^- are disjoint. \square

4. SMALL CURVATURES AND UNIQUENESS

In this section, we show that a small curvature condition can imply uniqueness for solutions to the asymptotic Plateau problem. This will prove Theorem 1.3 in the case of $n = 2$.

4.1. Small curvature conditions. We now introduce several conditions of having small curvature.

Definition 4.1. Let Σ be an immersed hypersurface in \mathbb{H}^{n+1} , and let λ_i denote its principal curvatures, for $i = 1, \dots, n$. We say that:

- Σ has *weakly small curvature* if $|\lambda_i(x)| \leq 1$ for every x in Σ and every $i = 1, \dots, n$;
- Σ has *small curvature* if $|\lambda_i(x)| < 1$ for every x in Σ and every $i = 1, \dots, n$;
- Σ has *strongly small curvature* if there exists $\epsilon > 0$ such that $|\lambda_i(x)| \leq 1 - \epsilon$ for every x in Σ and every $i = 1, \dots, n$.

In this article, we will need two results on hypersurfaces of weakly small curvatures, namely Theorem 4.2 and Theorem 4.3 below. These results are well known under the small curvatures assumptions, see [Eps84] and [EES22, Section 4]. In Appendix A we will explain how the arguments adapt under the weakly small curvatures assumption.

Theorem 4.2. *Let $\Sigma \subset \mathbb{H}^{n+1}$ be a properly immersed hypersurface of weakly small curvature. Then Σ is properly embedded and diffeomorphic to \mathbb{R}^n .*

By this result, we can hereafter assume that all the hypersurfaces of weakly small curvatures are properly embedded.

Theorem 4.3. *Let $\Sigma \subset \mathbb{H}^{n+1}$ be a properly embedded hypersurface of weakly small curvatures. Then the map $F : \Sigma \times \mathbb{R} \rightarrow \mathbb{H}^{n+1}$ defined by*

$$F_\Sigma(x, t) = \exp_x(tN(x))$$

is a diffeomorphism. For every $t \in \mathbb{R}$, $\Sigma_t := F_\Sigma(\Sigma, t)$ is a properly embedded hypersurface of weakly small curvature.

If moreover Σ is minimal, then the mean curvature of Σ_t , computed with respect to the unit normal vector field pointing towards the direction where t is increasing, has the same sign as $-t$.

A consequence of Theorem 4.3 is that, if Σ is a properly embedded hypersurface of weakly small curvature, the signed distance function $f_\Sigma : \mathbb{H}^{n+1} \rightarrow \mathbb{R}$ is smooth

and has no critical points. Indeed, f_Σ is simply the composition

$$(4.1) \quad f_\Sigma = \pi \circ F_\Sigma^{-1}$$

where $\pi : \Sigma \times \mathbb{R} \rightarrow \mathbb{R}$ is the projection to the second factor.

Let us focus again on minimal hypersurfaces. The following statement is an immediate consequence of great importance for this work.

Corollary 4.4. *Let $\Sigma \subset \mathbb{H}^{n+1}$ be a properly embedded minimal hypersurface of weakly small curvature. For any embedded minimal hypersurface $\Sigma' \subset \mathbb{H}^{n+1}$, the restriction of $d_\Sigma := d(\cdot, \Sigma)$ to Σ' has no positive local maximum.*

Proof. Assume by contradiction that $(d_\Sigma)|_{\Sigma'}$ has a positive local maximum. Let $x_{\max} \in \Sigma'$ the maximum point and $t_{\max} = d_\Sigma(x_{\max}) > 0$ be the corresponding maximum value. Up to switching the sign of f_Σ , d_Σ coincides with f_Σ in a neighbourhood of x_{\max} , where f_Σ is the signed distance function as in (4.1). Hence $(f_\Sigma)|_{\Sigma'}$ also has a positive local maximum at x_{\max} . This implies that Σ' is, in an embedded neighbourhood of x_{\max} , tangent to $\Sigma_{t_{\max}}$ and contained in the region $F_\Sigma(\Sigma \times \{t \leq t_{\max}\})$, which is strictly mean convex by the last part of Theorem 4.3. This contradicts the maximum principle for mean curvature. \square

Finally, it is immediate to check that minimal hypersurfaces of weakly small curvatures are stable.

Lemma 4.5. *Let Σ be a minimal hypersurface in \mathbb{H}^{n+1} of weakly small curvatures. Then Σ is stable.*

Proof. If Σ has weakly small curvatures, then

$$\|A_\Sigma\|^2 = \lambda_1^2 + \dots + \lambda_n^2 \leq n = -\text{Ric}_{\mathbb{H}^{n+1}}(N, N) .$$

Hence the left hand side of (2.2) is non-positive, and this concludes the proof. \square

4.2. Proof of Theorem 1.3 for $n = 2$. In this subsection, we prove Theorem 1.3 for surfaces in \mathbb{H}^3 . By Theorem 1.1, we know that it will be sufficient to prove the uniqueness in the class of stable minimal disks.

The fundamental result is therefore the following theorem.

Theorem 4.6. *Let Λ be a Jordan curve on $\mathbb{S}_\infty^2 = \partial\mathbb{H}^3$ of finite width, and let Σ be a properly embedded minimal surface in \mathbb{H}^3 of weakly small curvature asymptotic to Λ . Then Σ is the unique properly embedded stable minimal surface in \mathbb{H}^3 asymptotic to Λ .*

Combining Theorem 4.6 with Theorem 1.1, we immediately obtain the first part Theorem 1.3 in the case $n = 2$. The “moreover” part of Theorem 1.3 then follows from the results of [And83], which show that every Jordan curve bounds at least one area-minimizing minimal disk.

We begin by an easy lemma.

Lemma 4.7. *Let Λ be a Jordan curve on $\mathbb{S}_\infty^2 = \partial\mathbb{H}^3$ of width $w(\Lambda) < +\infty$, and let Σ be any properly embedded surface contained in $\mathcal{C}(\Lambda)$ with $\partial_\infty\Sigma = \Lambda$. Then for every $x \in \mathcal{C}(\Lambda)$, $d(x, \Sigma) \leq w(\Lambda)$.*

Proof. We may assume that Λ is not the boundary of a totally geodesic hyperplane, as in that case the conclusion holds trivially. Let $x \in \mathcal{C}(\Lambda)$. By definition of the width, there exist two geodesic segments joining x to $y_\pm \in \mathcal{C}^\pm(\Lambda)$, each of length at most $w(\Lambda) + \epsilon$. Since Σ is contained in the interior of $\mathcal{C}(\Lambda)$ by Lemma 2.2, and disconnects $\mathcal{C}(\Lambda)$ by the Jordan-Brouwer separation theorem, one of these two segments must meet Σ at a point y , which is on the segment between x and the other endpoint y_\pm . Hence

$$d(x, \Sigma) \leq d(x, y) \leq d(x, y_\pm) \leq w(\Lambda) + \epsilon .$$

Since ϵ was arbitrary, this concludes the proof. \square

Proof of Theorem 4.6. Let Σ' be a properly embedded stable minimal surface such that $\partial_\infty\Sigma' = \Lambda$. Consider the restriction to the surface Σ' of the distance function $d_\Sigma := d(\cdot, \Sigma)$. We want to show that d_Σ vanishes identically on Σ' . This will imply $\Sigma' \subseteq \Sigma$ and thus, since Σ' is also properly embedded, $\Sigma = \Sigma'$.

Suppose by contradiction that

$$(4.2) \quad \sup_{\Sigma'} d_\Sigma > 0 .$$

By Lemma 4.7, the supremum of d_Σ on Σ' is bounded above by $w(\Lambda)$, hence in particular it is finite. Let $\{x'_n\}_{n \in \mathbb{N}}$ be a maximizing sequence for $(d_\Sigma)|_{\Sigma'}$, namely

$$(4.3) \quad d(x'_n, \Sigma) \geq \sup_{\Sigma'} d_\Sigma - \frac{1}{n} .$$

Now, fix a point $x_o \in \mathbb{H}^3$. Let ϕ_n be isometries of \mathbb{H}^3 such that $\phi_n(x'_n) = x_o$. We denote $\Sigma_n = \phi_n(\Sigma)$ and $\Sigma'_n = \phi_n(\Sigma')$. Observe that Σ'_n contains x_o , and thus it intersects every ball centered at x_o . For Σ_n , we have

$$d(x_o, \Sigma_n) = d(\phi_n(x'_n), \phi_n(\Sigma)) = d(x'_n, \Sigma) \leq w(\Lambda) .$$

This shows that Σ_n intersects every ball centered at x_o of radius $R \geq w(\Lambda)$.

Recall that Σ has weakly small curvatures, hence $\|A_\Sigma\| \leq 2$, and since this condition is preserved by isometries, $\|A_{\Sigma_n}\| \leq 2$. By Corollary 2.6, Σ_n converges up to a subsequence, smoothly on a compact subsets in \mathbb{H}^3 , to a stable properly embedded minimal surface Σ_∞ . Moreover, since the weakly small curvatures condition is closed for the topology of smooth convergence on compact domains, Σ_∞ has weakly small curvatures.

Since Σ' is stable by assumption, and stability is preserved by isometries, Σ'_n is stable for every n . However, we do not have explicit curvature bounds as for Σ_n . In this case, we use Theorem 2.4 to find that each $\|A_{\Sigma'_n}\|$ is bounded by a universal constant. This enables us to apply Corollary 2.6 again to find that the sequence Σ'_n converges, up to a subsequence, smoothly on a compact domains in \mathbb{H}^3 to another properly embedded minimal surface Σ'_∞ .

Observe that, since by construction $x_o \in \Sigma'_n$ for all n , $x_o \in \Sigma'_\infty$. We now claim that

$$(4.4) \quad d(x_o, \Sigma_\infty) = \sup_{\Sigma'_\infty} d_{\Sigma_\infty}$$

where as usual $d_{\Sigma_\infty} = d(\cdot, \Sigma_\infty)$. Clearly we have $d(x_o, \Sigma_\infty) \leq \sup_{\Sigma'_\infty} d_{\Sigma_\infty}$. By contradiction, if the inequality were strict, then one would find $p \in \Sigma'_\infty$ and $\epsilon > 0$ such that $d(p, \Sigma_\infty) > d(x_o, \Sigma_\infty) + \epsilon$. By smooth convergence on compact sets, this would imply that, for large n ,

$$d(p_n, \Sigma_n) > d(x_o, \Sigma_n) + \frac{\epsilon}{2}$$

where $p_n \in \Sigma'_n$ and $p_n \rightarrow p$. But $d(p_n, \Sigma_n) = d(\phi^{-1}(p_n), \Sigma)$ and $d(x_o, \Sigma_n) = d(x'_n, \Sigma)$. Hence we would have

$$d(\phi^{-1}(p_n), \Sigma) > d(x'_n, \Sigma) + \frac{\epsilon}{2}$$

thus contradicting (4.3).

Having established the claim, (4.4) implies that x_o is a global maximum for the restriction to Σ' of the distance function d_{Σ_∞} . This contradicts Corollary 4.4 and thus concludes the proof. \square

4.3. Alternative argument for Theorem 1.2. We now explain an alternative proof of Theorem 1.2, which however only applies when the Kleinian group Γ is quasi-Fuchsian and isomorphic to the fundamental group of a closed surface. This is based on the work of Guaraco, Lima and Vargas-Pallete [GLVP21], which we summarize now. Let M be a quasi-Fuchsian manifold. One of their main results is that M admits a foliation by closed surfaces that are either minimal, or have non-vanishing mean curvature (i.e., are either strictly mean-convex, or strictly mean-concave.) The

sign of the mean curvature switches at the minimal leaves. Their work implies that if M contains a unique closed stable minimal surface S , then $M \setminus S$ has a foliation \mathcal{F} by surfaces that are isotopic to S and have mean-curvature vector everywhere pointing towards S .

Proof of Theorem 1.2 for Γ quasi-Fuchsian. Suppose Γ admits a unique invariant stable minimal disk Σ , which thus gives a unique closed stable minimal surface Σ/Γ in $M := \mathbb{H}^3/\Gamma$. Let $F : M \rightarrow \mathbb{R}$ be a function whose level sets $F^{-1}(t)$ are the leaves of the foliation of M provided by [GLVP21], and such that $F^{-1}(0) = \Sigma/\Gamma$. The foliation \mathcal{F} lifts to a foliation of \mathbb{H}^3 by properly embedded disks $\tilde{\Sigma}_t := \tilde{F}^{-1}(t)$ (where \tilde{F} is the lift of F to \mathbb{H}^3 , and therefore $\tilde{\Sigma}_0 = \Sigma$) with asymptotic boundary the limit set Λ of Γ .

By Theorem 1.1, it suffices to show that, if Σ' is a properly embedded stable minimal disk in \mathbb{H}^3 with $\partial_\infty \Sigma' = \Lambda$, then $\Sigma' = \Sigma$. Consider the function $f := \tilde{F}|_{\Sigma'} : \Sigma' \rightarrow \mathbb{R}$. We claim that $f \equiv 0$, which will imply $\Sigma' = \Sigma$ since both are properly embedded. First, observe that f is bounded. Indeed, since Γ is quasi-Fuchsian, the convex core $\mathcal{C}(\Lambda)/\Gamma$ of M is compact, and F is thus bounded on the convex core. Lifting to \mathbb{H}^3 , this shows that $\tilde{F}|_{\mathcal{C}(\Lambda)}$ is bounded, and so too is f by Lemma 2.2.

Now, the proof is similar to the proof of Theorem 4.6, using the foliation lifted from \mathcal{F} instead of the equidistant foliation. Suppose by contradiction that $\sup f > 0$, the other case being analogous. Let x'_n be a sequence such that $f(x'_n) \rightarrow \sup f$, let ϕ_n be a sequence of isometries sending x'_n to a fixed point x_o , and let $\Sigma'_n := \phi_n(\Sigma')$ and $\Sigma_n := \phi_n(\tilde{\Sigma}_{\sup f})$. Using Theorem 2.4 and Corollary 2.6, the sequence Σ'_n converges, up to a subsequence, smoothly on a compact domains in \mathbb{H}^3 to another properly embedded minimal surface Σ'_∞ . We claim that the sequence Σ_n also converges up to subsequences to a smooth surface in \mathbb{H}^3 . For this, first observe that the width $\mathcal{C}(\Lambda)$ is finite because Λ is a quasicircle, and, by cocompactness of the action of Γ on $\tilde{\Sigma}_{\sup f}$, the distance of $\tilde{\Sigma}_{\sup f}$ from $\mathcal{C}(\Lambda)$ is bounded by some constant c . It follows that Σ_n intersects every ball centered at x_o of radius $R \geq w(\Lambda) + c$. Second, by smoothness of $\tilde{\Sigma}_{\sup f}$ and cocompactness again, the norm of the second fundamental form of $\tilde{\Sigma}_{\sup f}$ and of all its covariant derivative is bounded. Since ϕ_n is an isometry, the norm of the second fundamental form of Σ_n and of all its covariant derivative is bounded too. This allows to extract a subsequence of $\{\Sigma_n\}_{n \in \mathbb{N}}$ converging smoothly on compact sets to a surface Σ_∞ (see for instance [Smi07]). As we are assuming $\sup f > 0$, $\tilde{\Sigma}_{\sup f}$ has negative mean curvature with respect to the unit normal pointing towards $\tilde{F} = +\infty$, hence the same holds for the limit Σ_∞ .

It follows from the construction that $x_o \in \Sigma'_\infty$. By cocompactness, the distance between different leaves $\tilde{\Sigma}_{\sup f}$ and $\tilde{\Sigma}_t$ goes to zero as $t \rightarrow \sup f$. Using this fact,

one can show similarly to the proof of Theorem 4.6 that $x_o \in \Sigma_\infty$. Moreover, since Σ' is contained in the region $\{x \mid \tilde{F}(x) \leq \sup f\}$ whose boundary is $\tilde{\Sigma}_{\sup f}$, after taking limits Σ'_∞ is contained in the side of Σ_∞ corresponding to decreasing values of \tilde{F} . This contradicts the geometric maximum principle at the tangency point x_o , and therefore shows that $\sup f \leq 0$. Repeating the argument for the infimum of f instead of the supremum shows that $f \equiv 0$ and concludes the proof. \square

5. HIGHER DIMENSION AND CODIMENSION

In this section, we outline some generalization of previous results to higher dimensions and higher codimensions.

5.1. Hypersurfaces in higher dimensions. We first provide a proof of Theorem 1.3 in higher dimensions. The argument is similar to the proof of Theorem 4.6. However, in that case it was sufficient to work with a *stable* minimal surface, by Theorem 1.1. The difference here is that instead of passing to a limit of minimal submanifolds (which we will be unable to do if they are unstable), we pass to a limit of metric balls in the ambient space. We are then able to apply a theorem of White [Whi10] in the setting of minimal varifolds to get a contradiction.

Proof of Theorem 1.3. Recall that, if $d_\Sigma = d(\cdot, \Sigma)$ denotes the distance function from Σ , by Theorem 4.3 the level hypersurfaces of d_Σ form a strictly mean-convex foliation of $\mathbb{H}^n \setminus \Sigma$, meaning that the mean curvature vector does not vanish and always points towards Σ . Let Σ' be another properly embedded minimal hypersurface with the same asymptotic boundary as Σ . Since Λ has finite width, we know that d_Σ is bounded on Σ' by some constant. (Indeed, the proof of Lemma 4.7 works identically in any dimension.) If $(d_\Sigma)|_{\Sigma'}$ attains its supremum at an interior point, then the strong maximum principle implies that $\Sigma' = \Sigma$. If not, we can take a sequence of points x_k on Σ' with $d_\Sigma(x_k)$ tending to the supremal distance $D := \sup_{\Sigma'} d_\Sigma$. Without loss of generality assume that all of the x_k are in the same connected component Ω_+ (say) of $\mathbb{H}^{n+1} \setminus \Sigma$.

Then we claim that the intersections of the open metric balls of radius $2D$ centered at the x_k intersected with $d_\Sigma^{-1}([0, D]) \cup \Omega_-$ (where Ω_- is the other connected component of $\mathbb{H}^{n+1} \setminus \Sigma$) are a sequence of smooth Riemannian manifolds with boundary, that we call M_k , that subsequentially converge to a smooth limit Riemannian manifold with strictly mean convex boundary M_∞ . Note that since we are intersecting with open balls, the M_k are incomplete and have boundary contained in $d_\Sigma^{-1}(D) \cap \Omega_+$. We just need to check smooth convergence at the boundary.

For every x_k there is a point p_k on Σ such that the ball of radius $4D$ centered at p_k contains M_k . Let ϕ_k be an isometry of \mathbb{H}^{n+1} sending p_k to a given point p , and fix any radius $R \geq 4D$. Using that $\|A_\Sigma\| \leq n$ by assumption, as in Corollary 2.6 (which holds in any dimensions) we can pass to a subsequential limit Σ_∞ of the minimal hypersurfaces $\phi_k(\Sigma)$ inside every ball $B(p, R)$.

Note that Σ_∞ has weakly small curvature. By Theorem 4.3, the set of points at signed distance D from Σ_∞ is a smooth strictly mean convex submanifold, and so is its intersection with the interior of $B(p, 4D)$. Since $\phi_k(d_\Sigma^{-1}(D) \cap B(p_k, 4D))$ converges smoothly to the set of points in $B(p, 4D)$ at signed distance D from Σ_∞ , it follows that the boundary of M_k (which is contained in $d_\Sigma^{-1}(D)$) subsequentially converges smoothly, after applying ϕ_k . In other words, we have shown that the M_k subconverge smoothly, in the sense of limits of Riemannian manifolds, to a manifold with strictly mean convex boundary M_∞ .

Now, since $d_\Sigma(x_n)$ tends to D , the set of limit points of $\Sigma \cap M_k$ in M_∞ contains a point in the boundary of M_∞ . But [Whi10, Theorem 3] says exactly that this is impossible.

For the “moreover” part, Anderson [And85] proved the existence of an area-minimizing integral current with asymptotic boundary Λ , which, however, is a smooth hypersurface only in the complement of a singular set of Hausdorff dimension at most $n-7$. Nevertheless, calling this integral current Σ' , the above argument based on the theorem of White [Whi10] applies to such a Σ' as well, showing that $\Sigma' = \Sigma$ is, in particular, a smooth hypersurface. Hence Σ is area-minimizing. \square

5.2. Higher codimension submanifolds. We briefly discuss some generalizations to higher codimension. By following Uhlenbeck’s approach [Uhl83], Jiang [Jia21, Lemma 2.2] defined a notion of almost Fuchsian for quotients M of \mathbb{H}^{n+1} by discrete faithful representations of surface groups in $\text{Isom}(\mathbb{H}^{n+1})$. Such a quotient M is called almost-Fuchsian if it admits a (two-dimensional) minimal surface Σ with principal curvatures smaller than 1 in magnitude. Jiang showed that an almost-Fuchsian M with Σ removed is foliated by equidistant surfaces to Σ that are mean 2-convex (the sum of the smallest two principal curvatures is positive), and thus by the maximum principle that Σ is the unique closed minimal surface in M . Bronstein recently showed that in four dimensions M could in fact be homeomorphic to a nontrivial disk bundle over a surface [Bro23]. The limit set of such an M would then be a self-similar everywhere-non-tame knot in $\partial_\infty \mathbb{H}^4 \cong S^3$ (see [GLT22].)

For any almost-Fuchsian M , the foliation by mean 2-convex equidistant surfaces to Σ lifts to a foliation of the complement of a lift $\tilde{\Sigma}$ of Σ to \mathbb{H}^{n+1} . We thus expect that the same argument by contradiction using [Whi10, Theorem 3] (which applies

in any codimension) can be used, to show that the limit set of M bounds a unique minimal disk, namely $\tilde{\Sigma}$. Jiang's work is for two dimension surfaces in arbitrary codimension, but we expect that similar arguments would work for k -dimensional minimal surfaces with small principal curvatures in any dimension. In that case, the same arguments would give a uniqueness result for asymptotic Plateau problems in that setting as well.

Finally here we mention that Fine has recently developed a theory for counting minimal surfaces in \mathbb{H}^4 with asymptotic boundary a given knot [Fin21]. In view of [Bro23] and the previous discussion, one should be able to give examples of (wild) knots in S^3 that bound a unique minimal surface in \mathbb{H}^4 .

6. UNCOUNTABLY MANY MINIMAL DISKS

The aim of this section is to prove Theorem 1.6.

6.1. Constructing the Jordan curve. We begin by constructing the Jordan curve Λ in $\mathbb{S}_\infty = \partial_\infty \mathbb{H}^3$, that will be the asymptotic boundary of uncountably many stable minimal disks. For this purpose, we will construct an explicit continuous injective map

$$f : \mathbb{R}P^1 = \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} .$$

We begin by fixing some notation. Let $\epsilon, \delta > 0$ be some constants which will be fixed later. For the moment, we only need to assume that $\epsilon < 1/3$, so that the circle of radius $2^n(1 + \epsilon)$ is smaller than the circle of radius $2^{n+1}(1 - \epsilon)$, and $\delta < \pi/8$. Let

$$\begin{aligned} p_{+,+}^n &:= 2^n (1 - \epsilon) e^{i(\frac{\pi}{2} - \delta)} & p_{+,-}^n &:= 2^n (1 - \epsilon) e^{i(-\frac{\pi}{2} + \delta)} \\ q_{+,+}^n &:= 2^n (1 + \epsilon) e^{i(\frac{\pi}{2} - \delta)} & q_{+,-}^n &:= 2^n (1 + \epsilon) e^{i(-\frac{\pi}{2} + \delta)} \\ p_{-,+}^n &:= 2^n (1 - \epsilon) e^{i(\frac{\pi}{2} + \delta)} & p_{-,-}^n &:= 2^n (1 - \epsilon) e^{i(-\frac{\pi}{2} - \delta)} \\ q_{-,+}^n &:= 2^n (1 + \epsilon) e^{i(\frac{\pi}{2} + \delta)} & q_{-,-}^n &:= 2^n (1 + \epsilon) e^{i(-\frac{\pi}{2} - \delta)} \end{aligned}$$

Observe that the points p_{σ_1, σ_2}^n and q_{σ_1, σ_2}^n lie in the quadrant determined by the two signs σ_1 and σ_2 , and that the Euclidean distance between p_{σ_1, σ_2}^n and q_{σ_1, σ_2}^n is $2^{n+1}\epsilon$. Moreover, denoting $\alpha(z) = 2z$, we see that $p_{\sigma_1, \sigma_2}^n = 2^n p_{\sigma_1, \sigma_2}^0$ and the same for q_{σ_1, σ_2}^n .

Now, we define the map f , which will be injective and piecewise smooth when restricted to $\mathbb{R} \setminus \{0\}$, with junctions exactly at the points p_{σ_1, σ_2}^n and q_{σ_1, σ_2}^n . The explicit formula for f is the following:

$$f(t) = \begin{cases} 0 & \text{if } t = 0 \\ 2^n (1 - \epsilon) e^{i((4\pi-8\delta)t - \frac{9\pi}{2} + 9\delta)} & \text{if } t \in [2^n, 2^n (1 + \frac{1}{4})] \\ 2^n (1 + \epsilon(-8t + 7)/3) e^{i(\frac{\pi}{2} - \delta)} & \text{if } t \in [2^n (1 + \frac{1}{4}), 2^n (1 + \frac{1}{2})] \\ 2^n (1 + \epsilon) e^{i((-4\pi+8\delta)t + \frac{5\pi}{2} - 5\delta)} & \text{if } t \in [2^n (1 + \frac{1}{2}), 2^n (1 + \frac{3}{4})] \\ 2^n ((-4/3 + 4\epsilon)t + (10/3 - 6\epsilon)) e^{i(-\frac{\pi}{2} + \delta)} & \text{if } t \in [2^n (1 + \frac{3}{4}), 2^{n+1}] \\ \infty & \text{if } t = \infty \\ -2^n (1 - \epsilon) e^{i((4\pi-8\delta)t + \frac{9\pi}{2} - 9\delta)} & \text{if } t \in [-2^n (1 + \frac{1}{4}), -2^n] \\ -2^n (1 + \epsilon(8t + 7)/3) e^{i(-\frac{\pi}{2} + \delta)} & \text{if } t \in [-2^n (1 + \frac{1}{2}), -2^n (1 + \frac{1}{4})] \\ -2^n (1 + \epsilon) e^{i((-4\pi+8\delta)t - \frac{5\pi}{2} + 5\delta)} & \text{if } t \in [-2^n (1 + \frac{3}{4}), -2^n (1 + \frac{1}{2})] \\ -2^n ((4/3 - 4\epsilon)t + (10/3 - 6\epsilon)) e^{i(\frac{\pi}{2} - \delta)} & \text{if } t \in [-2^{n+1}, -2^n (1 + \frac{3}{4})] \end{cases}$$

Let us explain this definition (see also Figure 1). First, observe that the image of $(0, +\infty)$ lies in the half-space $\{\Re(z) > 0\}$, while the image of $(-\infty, 0)$ lies in the half-space $\{\Re(z) < 0\}$. The intervals $[2^n, 2^n(1 + 1/4)]$ are mapped to the arcs of circles contained in $\{\Re(z) > 0\}$ connecting $p_{+,-}^n$ to $p_{+,+}^n$, and similarly the intervals $[2^n(1 + 1/2), 2^n(1 + 3/4)]$ are mapped to the arcs of circles contained in $\{\Re(z) > 0\}$ connecting $q_{+,+}^n$ to $q_{+,-}^n$, parameterized proportionally to arclength. The intervals $[2^n(1 + 1/4), 2^n(1 + 1/2)]$ are mapped to straight lines connecting $p_{+,+}^n$ to $q_{+,+}^n$. The intervals $[2^n(1 + 1/4), 2^n(1 + 1/2)]$ are mapped to straight lines connecting $q_{+,-}^n$ to $p_{+,-}^{n+1}$. For negative values of t , the situation is absolutely analogous up to composing with a reflection in the imaginary axis. Namely, f satisfies the symmetry:

$$(6.1) \quad f(-t) = -\overline{f(t)} .$$

Also, f satisfies the equivariance property

$$(6.2) \quad f \circ \alpha^n = \alpha^n \circ f ,$$

where we recall that $\alpha(z) = 2z$.

Lemma 6.1. *The image of f is a quasicircle in \mathbb{CP}^1 .*

Proof. We show that f is continuous and injective, which shows that the image of f is a Jordan curve. One immediately checks from the formula that f is well-defined and continuous (actually, piecewise smooth) when restricted to $\mathbb{R} \setminus \{0\}$. Observe moreover that, if $|t| \in [2^n, 2^{n+1}]$, then $|f(t)| \in [2^n(1 - \epsilon), 2^{n+1}(1 - \epsilon)]$. This immediately implies that f is continuous at $t = 0$ and $t = \infty$.

To check that f is injective, we have already observed that f maps $(0, +\infty)$ to the open right half-space and $(-\infty, 0)$ to the open left half-space. Hence, together

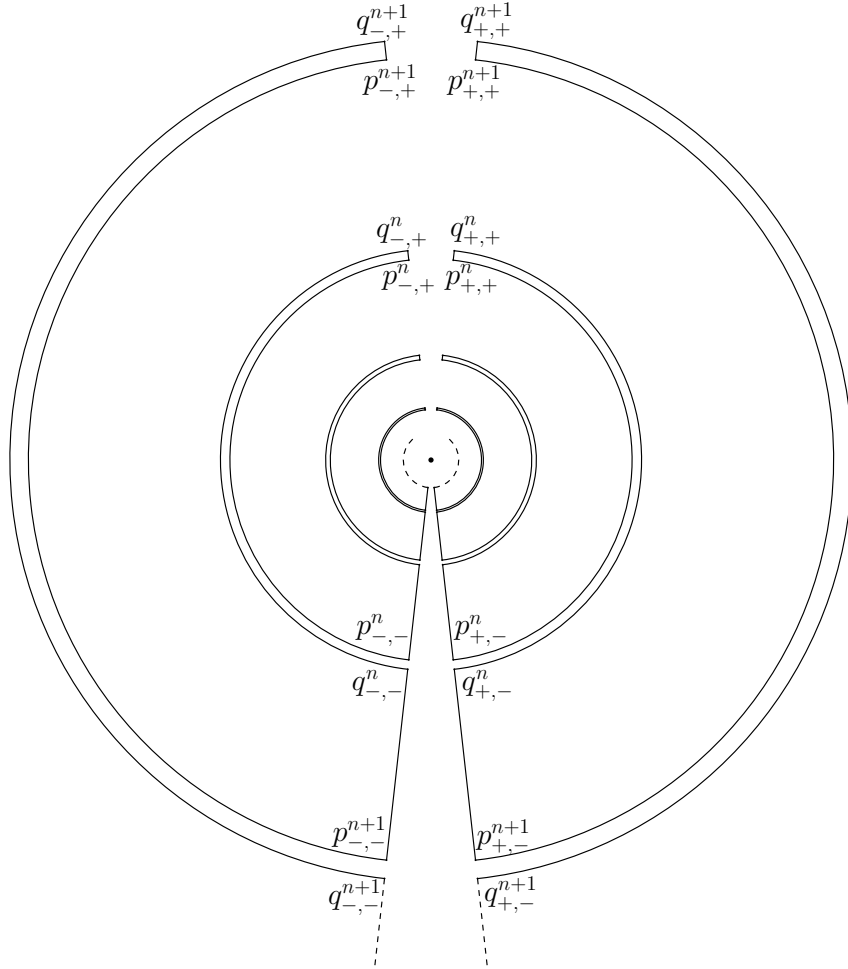


FIGURE 1. The construction of the Jordan curve Λ .

with (6.1), it suffices to check that f is injective on $(0, +\infty)$. By the condition on $|f(t)|$ in the previous paragraph and (6.2) it is actually sufficient to check that f is injective on $[1, 2]$, which is an immediate consequence of the construction.

Let us denote by Λ the image of f . We now show that Λ is a quasicircle, by using the following characterization of quasicircles given in [Ahl66, Chapter IV, Theorem 5]. Let Λ be a Jordan curve in $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ going through ∞ . Then Λ is a quasicircle if and only if there exists a constant $C > 0$ such that, for every $z_1, z_2 \in \Lambda \cap \mathbb{C}$ and every z_3 in the arc of $\Lambda \cap \mathbb{C}$ with endpoints z_1, z_2 ,

$$(6.3) \quad |z_3 - z_1| \leq C|z_2 - z_1| .$$

To check that (6.3) holds for the image of f , first observe that the condition (6.3) is scale-invariant. Hence, since Λ is invariant by the transformations $\alpha^n(z) = 2^n z$, we can assume that $\max\{|z_1|, |z_2|\} \in [1/2, 1]$. By definition of f , the arc of $\Lambda \cap \mathbb{C}$ with endpoints z_1, z_2 is then contained in the Euclidean ball $B(0, 1)$. It follows that it is enough to check (6.3) for $|z_2 - z_1| \leq \epsilon_0$, for any fixed $\epsilon_0 > 0$. Indeed, if $|z_2 - z_1| \geq \epsilon_0$, since by the triangle inequality $|z_3 - z_1| \leq 2$, we have $|z_3 - z_1| \leq (2/\epsilon_0)|z_2 - z_1|$. In conclusion, it is sufficient to check that (6.3) holds when either z_1 or z_2 has modulus in $[1/2, 1]$ and $|z_2 - z_1| \leq \epsilon_0$. This is immediately verified since $\Lambda \cap \{1/2 - \epsilon_0 \leq |z| \leq 1 + \epsilon_0\}$ is a union of finitely many straight line segments or arcs of circles meeting orthogonally at the adjacency points. \square

Remark 6.2. We remark that Λ is not a symmetric quasicircle. Indeed, as a consequence of the invariance under the scaling $\alpha(z) = 2z$, Λ cannot satisfy the so-called strong reverse triangle property in a neighbourhood of 0, see [GS92, Section 6].

Now, in order to fix the suitable values of ϵ and δ , we need to introduce minimal catenoids. A *minimal catenoid* is a minimal surface in \mathbb{H}^3 , homeomorphic to an open annulus, whose asymptotic boundary is the union of two disjoint circles C and C' in $\partial_\infty \mathbb{H}^3 = \mathbb{C}P^1$, and which is obtained as a surface of revolution with respect to the unique geodesic meeting orthogonally both totally geodesic planes P and P' with $\partial_\infty P = C$ and $\partial_\infty P' = C'$. See Figure 2.

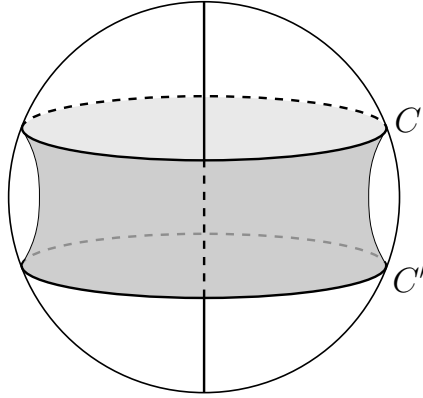


FIGURE 2. A minimal catenoid in \mathbb{H}^3 and its axis of revolution.

We will use the following existence result:

Theorem 6.3 ([dMG87]). *There exists a constant $d > 0$ such that, if C and C' are disjoint circles in $\partial_\infty \mathbb{H}^3 = \mathbb{C}P^1$ and the distance between the totally geodesic planes P and P' that they span is less than d , then there exists a minimal catenoid with asymptotic boundary equal to $C \cup C'$.*

Now, let us define the following circles in $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$:

$$\begin{aligned} C_{n,0} &= \left\{ z : \left| z - 2^n \left(1 - 2\epsilon - \frac{1+\epsilon}{100} \right) \right| = \frac{2^n}{100} \right\} \\ C'_{n,0} &= \left\{ z : \left| z - 2^n \left(1 + 2\epsilon + \frac{1+\epsilon}{100} \right) \right| = \frac{2^n}{100} \right\} \\ C_{n,1} &= \{ z : |z - 2^n e^{i(\frac{\pi}{2} - \delta - (1+\delta)\epsilon)}| = \epsilon \} \\ C'_{n,1} &= \{ z : |z - 2^n e^{i(\frac{\pi}{2} + \delta + (1+\delta)\epsilon)}| = \epsilon \} \end{aligned}$$

See Figure 3. Observe that

$$(6.4) \quad C_{n,0} = \alpha^n C_{0,0} \quad C'_{n,0} = \alpha^n C'_{0,0} \quad C_{n,1} = \alpha^n C_{0,1} \quad C'_{n,1} = \alpha^n C'_{0,1} .$$

and that $C_{n,0}$, $C'_{n,0}$, $C_{n,1}$ and $C'_{n,1}$ are all in the complement of the image of f .

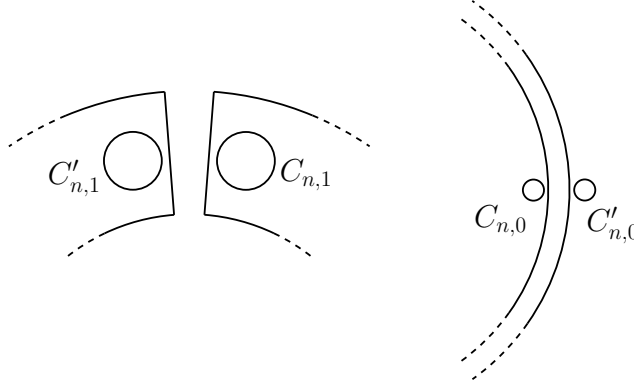


FIGURE 3. The circles $C_{n,0}$, $C'_{n,0}$, $C_{n,1}$ and $C'_{n,1}$.

Lemma 6.4. *There exists constants $\epsilon, \delta > 0$ such that for all $n \in \mathbb{Z}$ and all $j \in \{0, 1\}$, $C_{n,j} \cup C'_{n,j}$ is the asymptotic boundary of a minimal catenoid.*

Proof. By (6.4) and the fact that α is the boundary value of an isometry of \mathbb{H}^3 in the upper half-space model, it suffices to prove the statement for $n = 0$. We shall first choose ϵ in such a way that $C_{0,0} \cup C'_{0,0}$ bounds a minimal catenoid. For this, it suffices to observe that $C_{0,0}$ and $C'_{0,0}$ are circles of radii $1/100$, and the Euclidean distance between their centers is $2(2\epsilon + (1+\epsilon)/100) = 2/100 + 402\epsilon/100$. As $\epsilon \rightarrow 0$, the circles $C_{0,0}$ and $C'_{0,0}$ tend to a pair of tangent circles, each with radius $1/100$. Since the hyperbolic distance varies continuously, and the hyperbolic distance between

two totally geodesic planes tangent at infinity equals zero, the distance between the totally geodesic planes bounded by $C_{0,0}$ and $C'_{0,0}$ tends to zero. Hence by Theorem 6.3, when ϵ is sufficiently small, there exists a minimal catenoid with asymptotic boundary $C_{0,0} \cup C'_{0,0}$.

We shall now keep such ϵ fixed, and choose δ in such a way that $C_{0,1} \cup C'_{0,1}$ bounds a minimal catenoid. The reasoning is completely analogous to the previous case. Indeed, the circles $C_{0,1}$ and $C'_{0,1}$ have radius ϵ and the distance between their centers is bounded above by $2\epsilon + 2\delta(1 + \epsilon)$. Hence $C_{0,1}$ and $C'_{0,1}$ tend to be tangent when $\delta \rightarrow 0$, and we conclude again by Theorem 6.3. \square

In the rest of the section, we will denote by $\text{Cat}_{n,j}$ the minimal catenoid whose asymptotic boundary is $C_{n,j} \cup C'_{n,j}$, and by $\text{Cat}_{n,j}^{\text{solid}}$ the closure in \mathbb{H}^3 of the connected component of $\mathbb{H}^3 \setminus \text{Cat}_{n,j}$ that contains the totally geodesic planes spanned by $C_{n,j}$ and $C'_{n,j}$.

6.2. Topological disks. We will now need to construct topological disks with asymptotic boundary the Jordan curve $\Lambda = f(\mathbb{RP}^1)$, contained in the complement of suitably chosen solid catenoids. The choice of these solid catenoids will be encoded in a function $\iota : \mathbb{Z} \rightarrow \{0, 1\}$. Each such topological disk will be the starting point to construct pairwise distinct *minimal* disks.

Lemma 6.5. *For every $\iota : \mathbb{Z} \rightarrow \{0, 1\}$, there exists a continuous injective map F_ι from the upper half-plane \mathbb{H}^2 to \mathbb{H}^3 , whose image is contained in*

$$\mathbb{H}^3 \setminus \bigcup_{n \in \mathbb{Z}} \text{Cat}_{n, \iota(n)}^{\text{solid}}$$

and which extends continuously to $f : \mathbb{R} \cup \{\infty\} = \partial_\infty \mathbb{H}^2 \rightarrow \mathbb{C} \cup \{\infty\} = \partial_\infty \mathbb{H}^3$.

Proof. We will divide the construction in several steps. We fix $n \in \mathbb{Z}$, and we will first define F_ι on the strip $\{z \in \mathbb{H}^2 : 2^n \leq |z| \leq 2^n(1+3/4)\}$ in the upper half-plane. We will distinguish two cases here, according to whether $\iota(n) = 0$ or $\iota(n) = 1$. Then we will define F_ι on the strip $\{z \in \mathbb{H}^2 : 2^n(1+3/4) \leq |z| \leq 2^{n+1}\}$ in a way which is independent of $\iota(n)$.

Suppose first that $\iota(n) = 1$. We divide the strip $\{2^n \leq |z| \leq 2^n(1+3/4)\}$ in three portions, namely $\{2^n \leq |z| \leq 2^n(1+1/4)\}$, $\{2^n(1+1/4) \leq |z| \leq 2^n(1+1/2)\}$ and $\{2^n(1+1/2) \leq |z| \leq 2^n(1+3/4)\}$, and define F_ι on each of them in such a way that it extends to f_ι on the corresponding interval of the real axis. Moreover, the image of F_ι will have to avoid the solid catenoid $\text{Cat}_{n,1}^{\text{solid}}$. The idea is to map $\{2^n \leq |z| \leq 2^n(1+1/4)\}$ to the convex hulls of the arcs of circles $f_\iota([2^n, 2^n(1+1/4)])$ and $f_\iota([-2^n(1+1/4), -2^n])$, and similarly $\{2^n(1+1/2) \leq |z| \leq 2^n(1+3/4)\}$ to the convex

hull of $f_i([2^n(1 + 1/2), 2^n(1 + 3/4)])$ and $f_i([-2^n(1 + 3/4), -2^n(1 + 1/2)])$ — these convex hulls are portions of totally geodesic planes — and then connect them by a band connecting $f_i([2^n(1 + 1/4), 2^n(1 + 1/2)])$ and $f_i([-2^n(1 + 1/2), -2^n(1 + 1/4)])$. See Figure 4.

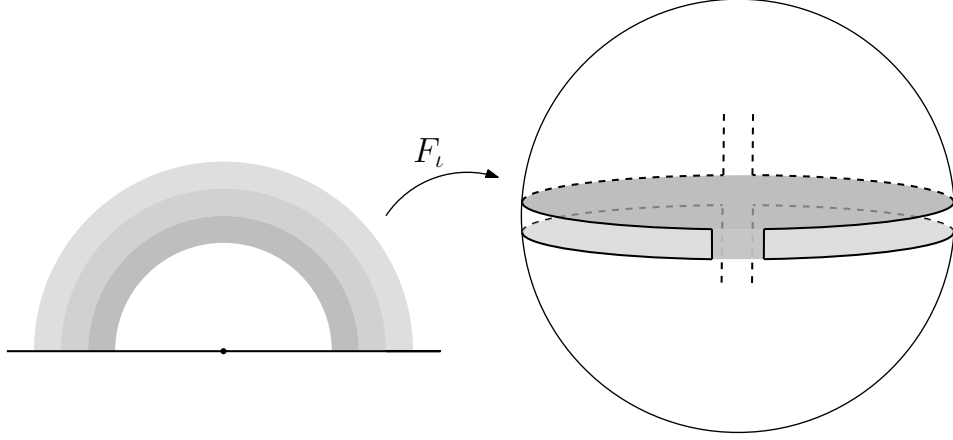


FIGURE 4. The construction of F_l if $\iota(n) = 1$, in the Klein model of \mathbb{H}^3 .

Formally, let us use the upper half-space model $\{(z, h) : z \in \mathbb{C}, h > 0\}$ of \mathbb{H}^3 . As t varies in $[2^n, 2^n(1 + 1/4)]$, let $\gamma(t)$ parameterize the curve in \mathbb{H}^3 (actually, a geodesic, although not parameterized by arclength) contained in the vertical plane $\Re(z) = 0$, in such a way that $\lim_{x \rightarrow \pm\infty} \exp_{\gamma(t)}(xv_{\gamma(t)}) = f_i(\pm t)$, where v_p is the vector based at p and orthogonal to the vertical plane $\Re(z) = 0$, pointing towards $\Re(z) > 0$. Then define

$$F_l(it e^{is}) = \exp_{\gamma(t)}(\chi(s)v_{\gamma(t)})$$

for a fixed decreasing diffeomorphism $\chi : [-\pi/2, \pi/2] \rightarrow \mathbb{R}$. By construction, F_l extends to f_i on $\pm[2^n, 2^n(1 + 1/4)]$. Starting from $t \in [2^n(1 + 1/2), 2^n(1 + 3/4)]$, we repeat exactly the same procedure to define F_l on $\{2^n(1 + 1/2) \leq |z| \leq 2^n(1 + 3/4)\}$, thus obtaining another portion of totally geodesic plane. We repeat the same again to define F_l on $\{2^n(1 + 1/4) \leq |z| \leq 2^n(1 + 1/2)\}$, simply by changing the interval of definition of t , with the sole irrelevant difference that in this case the image of γ is not a geodesic. Putting together the three pieces, F_l , defined on $\{2^n \leq |z| \leq 2^n(1 + 3/4)\}$, is continuous and piecewise smooth. Since the minimal catenoids are contained in the convex hull of their two asymptotic circles, it is immediate to see that the image of F_l is disjoint from the solid catenoid $\text{Cat}_{n,1}^{\text{solid}}$.

Having described this construction in detail when $\iota(n) = 1$, we now explain how to modify it for $\iota(n) = 0$. Here we divide again the strip $A := \{2^n \leq |z| \leq 2^n(1 + 3/4)\}$

in three parts, but differently from above. Let $A_{\pm} = A \cap \{|z \mp 2^n(1 + 3/8)| = 2^n(3/8)\}$, and let A_0 be the closure of $A \setminus (A_- \cup A_+)$. We can use again an auxiliary curve $\gamma : [-\pi/2, \pi/2]$ in \mathbb{H}^3 , having the property of being contained in the totally geodesic plane of symmetry between the two totally geodesic planes with asymptotic boundaries $\{|z| = 2^n(1 \pm \epsilon)\}$, such that $\gamma(\pm\pi/2) = f_{\iota}(\pm 2^n(1 + 3/8))$, and define $F_{\iota}(ie^s) = \gamma(s)$. Then we “fill” the strip A in such a way that $F_{\iota}(A_{\pm})$ is swept-out by the geodesics with endpoints $f_{\iota}(\pm 2^n(1 + 3/8 + t))$, for $t \in 2^n(-3/8, 3/8)$, and that $F_{\iota}(A_0)$ is swept-out by finite geodesic segments, in such a way that $F_{\iota}(2^n ie^s) = \exp_p(\chi(s)v)$ parameterizes, as in the previous case, a geodesic with endpoints $f_{\iota}(\pm 2^n)$, for p a point in the vertical plane $\Re(z) = 0$, and analogously for $F_{\iota}(2^n(1 + 3/4)ie^s)$, on the other boundary half-circle of A . The convex hull property will then ensure again that the image of F_{ι} is in the complement of $\text{Cat}_{n,0}^{\text{solid}}$. See Figure 5. We leave to the reader the details, which are only slightly more technical than in the case $\iota(n) = 1$.

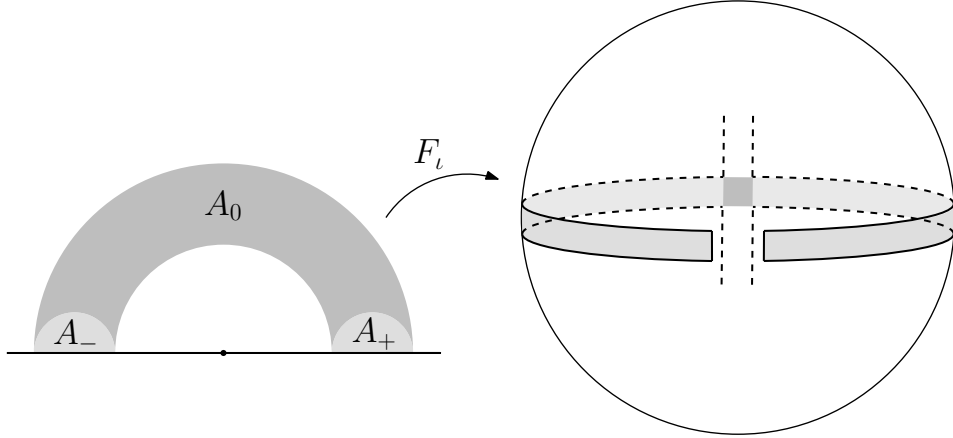


FIGURE 5. The construction of F_{ι} if $\iota(n) = 0$.

Finally, it remains to define F_{ι} on the strip $\{2^n(1 + 3/4) \leq |z| \leq 2^{n+1}\}$. Here we take again a curve $\gamma : [2^n(1 + 3/4), 2^{n+1}] \rightarrow \mathbb{H}^3$ with the property that its image is contained in $\Re(z) = 0$ and that $\lim_{x \rightarrow \pm\infty} \exp_{\gamma(t)}(xv_{\gamma(t)}) = f_{\iota}(\pm t)$. We can then define $F_{\iota}(ite^{is}) = \exp_{\gamma(t)}(\chi(s)v_{\gamma(t)})$. This definition will match the previous definitions of F_{ι} on the common boundaries $|z| = 2^n(1 + 3/4)$ or $|z| = 2^{n+1}$. See Figure 6.

Hence F_{ι} is globally well-defined, continuous and piecewise smooth. As in the proof of Lemma 6.1, one can observe that the image of $2^n \leq |z| \leq 2^{n+1}$ under the map F_{ι} is contained, in the upper half-space, in $B(0, 2^{n+1}(1 + \epsilon)) \setminus B(0, 2^n(1 + \epsilon))$. Using this property, following the same lines as the proof of Lemma 6.1 it is easy to

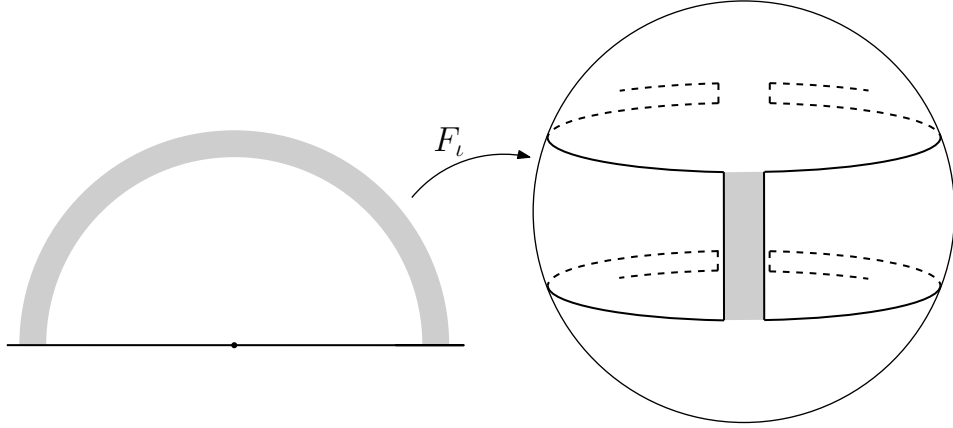


FIGURE 6. The construction of F_l on the remaining strip, which is in common for all choices of ι .

verify that F_l is injective and $\lim_{z \rightarrow 0} F_l(z) = 0$, $\lim_{z \rightarrow \infty} F_l(z) = \infty$. By construction F_l extends to f_l on \mathbb{R} , and therefore F_l is the desired continuous and injective extension of f_l . \square

6.3. Conclusion of the proof. We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let Λ be the image of the continuous injective map f , and fix $\iota : \mathbb{Z} \rightarrow \{0, 1\}$. We will construct a stable minimal disk Σ_ι contained in

$$\Omega_\iota := \mathbb{H}^3 \setminus \bigcup_{n \in \mathbb{Z}} \text{Cat}_{n, \iota(n)}^{\text{solid}}$$

with $\partial_\infty \Sigma_\iota = \Lambda$.

Let us denote by U_ι the topological disk constructed in Lemma 6.5, namely the image of the continuous injective map $F_l : \mathbb{D} \rightarrow \mathbb{H}^3$, where we identify here the upper half-plane model of \mathbb{H}^2 with the unit disc \mathbb{D} . We will denote by $\mathcal{C}(U_\iota)$ the convex hull of $\Lambda \cup U_\iota$ in $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$. As a preliminary step, observe that

$$(6.5) \quad \partial_\infty \mathcal{C}(U_\iota) = \Lambda .$$

Indeed, given any point p in $\partial_\infty \mathbb{H}^3 \setminus \Lambda$, since the map F_l extends to a continuous injective map from $\overline{\mathbb{D}}$ to $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$, $U_\iota \cup \Lambda$ is a closed subset of $\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3$, and one can find a neighbourhood \mathcal{U} of p in the topology of the closed ball which is disjoint from $U_\iota \cup \Lambda$. Then a totally geodesic plane P contained in \mathcal{U} disconnects p from the convex hull of $\Lambda \cup U_\iota$, thus showing $p \notin \mathcal{C}(U_\iota)$.

Now, let $\gamma_n = F_\iota(\{|z| = 1 - 1/n\})$. Then $\{\gamma_n\}_{n \in \mathbb{N}}$ is a sequence of nullhomotopic simple closed curves in $\mathcal{C}(U_\iota)$ that converge to Λ as $n \rightarrow +\infty$. The proof of the existence of Σ_ι now roughly follows the strategy of Lemma 3.3.

Fix the point $x_o = F_\iota(0) \in U_\iota$. For any n , let R_n be a sufficiently large radius so that $\gamma_n \subset B(x_o, R_n)$. Up to a small perturbation of R_n , assume moreover, as a consequence of genericity of transverse intersection, that the intersection of $\partial B(x_o, R_n)$ and $\partial\Omega_\iota$ is transverse. By [HS88, Theorem 6.3] applied to $M_n := \Omega_\iota \cap B(x_o, R_n)$, which is sufficiently convex in the terminology of [HS88], we can solve the Plateau problem to find a least area (hence stable) minimal disk Σ_ι^n contained in M_n with $\partial\Sigma_\iota^n = \gamma_n$.

We will show that Σ_ι^n converges, up to extracting a subsequence, smoothly on compact subsets of \mathbb{H}^3 to a properly embedded minimal disk Σ_ι contained in Ω_ι . We first observe that there exists $R_0 > 0$ such that $\Sigma_\iota^n \cap B(x_o, R_0) \neq \emptyset$ for all n . To see this, recall that $\mathcal{C}(U_\iota)$ is homeomorphic to a closed ball, hence Λ disconnects $\partial\mathcal{C}(U_\iota)$, which is homeomorphic to a sphere, into two topological disks which we denote $\partial^\pm\mathcal{C}(U_\iota)$. Pick a geodesic segment η connecting two points $x_\pm \in \partial^\pm\mathcal{C}(U_\iota)$ and containing x_o . Since γ_n and η are contained in $\mathcal{C}(U_\iota)$ and linked, and the disk Σ_ι^n is contained in $\mathcal{C}(U_\iota)$, it must intersect η . Hence Σ_ι^n intersects $B(x_o, R_0)$ for any choice of R_0 such that $B(x_o, R_0)$ contains η .

Now, fix any radius $R > 0$ and consider the ball $B(x_o, R)$. For $n = n(R)$ sufficiently large, γ_n is contained in $\mathbb{H}^3 \setminus B(x_o, R + 1)$. The distance from any point of $\Sigma_\iota^n \cap B(x_o, R)$ and $\partial\Sigma_\iota^n = \gamma_n$ is thus bounded below by 1. By Theorem 2.3 there is a constant C , independent of n , such that the Gaussian curvature of Σ_ι^n is bounded by C in absolute value. Applying Theorem 2.5, we can extract a subsequence n_k such that $\Sigma_\iota^{n_k} \cap B(x_o, R)$ converges smoothly to a minimal disc. By a diagonal argument, we can then find a subsequence of $\{\Sigma_\iota^n\}_{n \in \mathbb{N}}$ that converges smoothly on compact subsets to a minimal disk Σ_ι . Moreover the limit Σ_ι is stable since it is the smooth limit of the stable minimal surfaces Σ_ι^n .

An easy argument by contradiction as in the proof of Corollary 2.6 shows that Σ_ι is properly embedded. Moreover, since Σ_ι^n is contained in Ω_ι , then Σ_ι is contained in the closure of Ω_ι . This would be sufficient for our purpose; nonetheless the strong maximum principle actually implies that Σ_ι is contained in Ω_ι itself. Finally, since Σ_ι^n is contained in $\mathcal{C}(U_\iota)$ for all n , so too is Σ_ι . Then (6.5) implies that $\partial_\infty\Sigma_\iota = \Lambda$.

It only remains to show that, if $\iota_0 \neq \iota_1$, then $\Sigma_{\iota_0} \neq \Sigma_{\iota_1}$. By hypothesis, there exists $m \in \mathbb{Z}$ such that $\iota_0(m) \neq \iota_1(m)$. Up to switching the roles, suppose $\iota_0(m) = 0$ and $\iota_1(m) = 1$. Then, by construction, Σ_{ι_0} is contained in the complement of $\text{Cat}_{m,0}^{\text{solid}}$, whereas Σ_{ι_1} is contained in the complement of $\text{Cat}_{m,1}^{\text{solid}}$.

But the solid catenoids $\text{Cat}_{m,0}^{\text{solid}}$ and $\text{Cat}_{m,1}^{\text{solid}}$ are linked in the sense of [HW15, Section 3.4]. Then [HW15, Lemma 3.12] implies that Λ is not nullhomotopic in $(\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3) \setminus (\text{Cat}_{m,0}^{\text{solid}} \cup \text{Cat}_{m,1}^{\text{solid}})$. Now, if $\Sigma_{\iota_0} = \Sigma_{\iota_1} =: \Sigma$, then Σ would be a disk contained in the complement of $\text{Cat}_{m,0}^{\text{solid}} \cup \text{Cat}_{m,1}^{\text{solid}}$, hence contradicting that Λ is not nullhomotopic. This concludes the proof. \square

APPENDIX A. SMALL CURVATURES

In this appendix, we explain how to obtain Theorems 4.2 and 4.3. These results are well-known under the assumption that Σ has small curvatures (see for instance [Eps84]). However, we need to relax the hypothesis here to assume that Σ has only weakly small curvatures. We follow here the approach of [EES22, Section 4].

The fundamental observation is that, if Σ is a hypersurface in \mathbb{H}^{n+1} of weakly small curvatures and $\gamma : I \rightarrow \Sigma$ is a geodesic for the first fundamental form of Σ , then the acceleration of γ is

$$(A.1) \quad \nabla_{\gamma'(t)}^h \gamma'(t) = A(\gamma'(t), \gamma'(t))N(\gamma(t))$$

and therefore, by the weakly small curvature assumption,

$$(A.2) \quad \|\nabla_{\gamma'(t)}^h \gamma'(t)\| \leq 1 .$$

This leads us to study curves satisfying the bound (A.2) on the acceleration. Let S be a horosphere or a sphere in \mathbb{H}^{n+1} . (By sphere, here we always mean the set of points at distance $r > 0$ from a given points of \mathbb{H}^{n+1} .) The complement of S has two connected components, one of which is geodesically convex, and the other is not. We call the former the (open) convex side of S , and the latter the (open) concave side of S . The closed convex (resp. concave) side of S is the union of S with the open convex (resp. concave) side. The following lemma is an improvement of [EES22, Lemma 4.9], following a similar strategy of proof.

Lemma A.1. *Let $\gamma : [a, b] \rightarrow \mathbb{H}^{n+1}$ be a smooth curve satisfying (A.2). Then*

- *The image of γ lies in the open concave side of any sphere tangent to γ , except for the tangency point.*
- *The image of γ lies in the closed concave side of any horosphere tangent to γ .*

Proof. It suffices to prove the first item, as the second item follows by taking the limit of tangent spheres as the radius goes to infinity.

Assume that the tangency point is, in the upper half-space model $\gamma(0) = (0, \dots, 0, 1)$, that γ is parameterized by arclength, that $\gamma'(0) = (1, 0, \dots, 0)$ and that the tangent

horosphere is $\{x_n = 1\}$. Hence the tangent spheres S_r of radius r are, in a neighbourhood of $\gamma(0)$, graphs of functions $f_r : U_r \rightarrow \mathbb{R}$, where $U_r \subset \mathbb{R}^n$ is a sufficiently small ball, and by symmetry f_r only depends on the distance from the origin in \mathbb{R}^n .

First we show γ lies on the open concave side of any tangent sphere S_r for small t . We use a subscript $(\cdot)_{n+1}$ to denote the last coordinate in the upper half-space model. Thus $\gamma_{n+1}(0) = \gamma'_{n+1}(0) = 0$. By a direct computation of the Christoffel symbol $\Gamma_{11}^n = 1$, we see that

$$(\nabla_{\gamma'}^h \gamma')_{n+1}(0) = \gamma''_{n+1}(0) + 1.$$

Since γ has small acceleration, we obtain

$$\gamma''_{n+1}(0) \leq |(\nabla_{\gamma'}^h \gamma')_{n+1}(0)| - 1 \leq \|(\nabla_{\gamma'}^h \gamma')(0)\| - 1 \leq 0.$$

Now, denote $\hat{\gamma}$ the projection of γ to \mathbb{R}^n . By an easy computation following similar lines, $(f_r \circ \hat{\gamma})(0) = (f_r \circ \hat{\gamma})'(0) = 0$ and $(f_r \circ \hat{\gamma})''(0) = -1 + 1/\tanh(r) > 0$. Hence for $t \in (0, \epsilon)$, $\gamma_{n+1}(t) < f_r(t)$ and this shows that γ stays for small times in the open concave side of S_r .

It remains to show that $\gamma(t)$ stays in the open concave side of S_r for every $t > 0$. Suppose by contradiction that for some t_0 , $\gamma(t_0) \in S_r$. Let p be the center of S_r . Then the function $t \mapsto d(p, \gamma(t))$ is equal to r for $t = 0$ and $t = t_0$, and is larger than r for $t \in (0, \epsilon)$. Hence it admits a maximum point t_{\max} . This implies that γ is tangent to the geodesic sphere centered at p of radius $d(p, \gamma(t_{\max}))$ and thus contradicts the first part of the proof. \square

Having established Lemma A.1, the following lemma follows immediately.

Lemma A.2. *Let $\Sigma \subset \mathbb{H}^{n+1}$ be a properly immersed hypersurface of weakly small curvatures. Then Σ is contained in the closed concave side of every tangent horosphere.*

Proof. Let $p \in \Sigma$ and let S be a tangent horosphere. Given any other q , by completeness of Σ , we can find a geodesic segment γ joining p and q . As observed in (A.2), γ satisfies the hypothesis of Lemma A.1, hence q is contained in the closed concave side of S . \square

Now, the proof of Theorem 4.2 follows exactly the proof of [EES22, Proposition 4.15]. To prove Theorem 4.3, we need the following well-known result, which follows from a direct computation (see for instance [HW13, formula (2.2)] or [EES22, Lemma 4.5]).

Lemma A.3. *Let Σ be an embedded hypersurface in \mathbb{H}^{n+1} of weakly small curvatures, and let N be its unit normal vector. Then for every $t \in \mathbb{R}$, the map*

$\iota_t : \Sigma \rightarrow \mathbb{H}^{n+1}$ defined by $\iota_t(x) = \exp_x(tN(x))$ is an immersion, and its principal curvatures satisfy the identity

$$(A.3) \quad \lambda_i^t(\iota_t(x)) = \frac{\lambda_i(x) - \tanh(t)}{1 - \lambda_i(x) \tanh(t)},$$

where $\lambda_1, \dots, \lambda_n$ are the principal curvatures of Σ .

Proof of Theorem 4.3. It follows from Lemma A.3 that the map $F(x, t) = \iota_t(x)$ has injective differential, hence is a local diffeomorphism. We show that it is bijective. To show that F is surjective, let p be any point in \mathbb{H}^{n+1} , let $r = d(p, \Sigma)$ and let $x \in \Sigma$ be a point that realizes this distance. Then the geodesic sphere of center p and radius r is tangent to Σ at x , and $p = F(x, r)$. For injectivity, suppose that $p = F(x_1, t_1) = F(x_2, t_2)$. Let S_{t_i} be the geodesic spheres centered at p of radius t_i . As before, S_{t_i} is tangent to Σ at x_i . By Lemma A.1, Σ is contained, except for the tangency points, in the open concave side of both S_{t_1} and S_{t_2} . This implies that $t_1 = t_2$, because otherwise the open concave side of S_{t_1} would intersect the convex side of S_{t_2} (or vice versa), leading to a contradiction. By Lemma A.1 again, $\Sigma \setminus \{x_1\}$ is contained in the open concave side of $S_{t_1} = S_{t_2}$, and thus $x_2 = x_1$.

We have thus shown that each ι_t is a proper embedding, and it has weakly small curvatures by Lemma A.3. When Σ is minimal, Lemma A.3 again implies the last statement on the sign of the mean curvature of ι_t , because λ_i^t is a decreasing function of t . \square

We have already observed that, for $n = 2$, if the Jordan curve Λ spans a surface of strongly small curvatures, then Λ is a quasicircle, and this in turns implies that Λ has finite width. We conclude this appendix by a direct proof, that works in any dimensions.

Lemma A.4. *Let Λ be a topologically embedded n -sphere Λ in \mathbb{S}_∞^n and Σ be a properly embedded hypersurface of strongly small curvatures such that $\partial_\infty \Sigma = \Lambda$. Then $\mathcal{C}(\Lambda)$ has finite width.*

Proof. Assume that the principal curvatures $\lambda_1, \dots, \lambda_n$ of Σ are smaller than $1 - \epsilon$ in absolute value. By Lemma A.3, for $t_0 > \tanh^{-1}(1 - \epsilon)$ the hypersurface Σ_{t_0} (resp. Σ_{-t_0}) has negative (resp. positive) principal curvatures, that is, it is locally concave (resp. convex) with respect to the direction of the normal vector N . Since $\Sigma_{\pm t_0}$ are properly embedded, they bound a geodesically convex region U , which thus contains the $\mathcal{C}(\Lambda)$. By Theorem 4.3, every point in $\mathcal{C}(\Lambda)$ lies on a geodesic segment of the form $t \mapsto F(x, t)$, for $t \in [-t_0, t_0]$. Hence for every $x \in \mathcal{C}(\Lambda)$, $d(x, \partial^+ \mathcal{C}(\Lambda)) + d(x, \partial^- \mathcal{C}(\Lambda)) \leq 2t_0$. This shows that $w(\Lambda) \leq 2t_0$. \square

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