Generalized Hamiltonian drift-fluid and gyrofluid reductions

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Abstract.

We provide a procedure for deriving Hamiltonian reduced fluid models for plasmas, starting from a Hamiltonian gyrokinetic system in the δf approximation. The procedure generalizes, to a considerable extent, previous results. In particular, the evolution of moments with respect to the magnetic moment coordinate is also taken into account, together with background density and magnetic inhomogeneities. In the limit of vanishing Finite Larmor Radius (FLR) effects, an infinite family of reduced electron drift-fluid equations is derived, evolving all the electron moments g_{ij_e}, with $i = 0, \cdots, N$ and $j = 0, \cdots, M$, where N and M are arbitrary non-negative integers counting the maximum order of the moments taken with respect to the parallel velocity and to the magnetic moment coordinates, respectively. An analogous result is found in the gyrofluid case and applied to the ion species. The gyrofluid result holds for $M \leq 1$, finite FLR effects and a low ratio β_e between electron internal pressure and magnetic guide field pressure.

In both the drift and gyrofluid case, the key for the identification of the Hamiltonian structure resides in changes of variables based on orthogonal matrices that diagonalize the Jacobi matrices associated with Hermite and Laguerre polynomials. In terms of the transformed variables, the drift and gyrofluid equations are cast in a simple form, which reduces to advection equations for Lagrangian invariants in the two-dimensional case with homogeneous background.

Because the procedure requires to evolve, for a particle species s, all and only the moments g_{ij_s}, with $i = 0, \cdots, N$ and $j = 0, \cdots, M$, not every choice for the set of moments is admissible, for fixed N and M. This might also explain the scarcity of Hamiltonian reduced fluid models obtained so far which account for anisotropic temperature fluctuations.

1. Introduction

Reduced fluid models proved to be an effective tool to describe the dynamics of plasmas in the presence of a strong guide field. These evolution equations are characterized by quadratic nonlinear terms describing the dynamics perpendicular to the guide field and by linear terms expressing weak variations along the guide field, as well as further effects associated with additional weak inhomogeneities (e.g. weak density and temperature equilibrium gradients or moderate spatial variations of the guide field). The paradigm of such reduced fluid models is reduced magnetohydrodynamics [1, 2]. Several other reduced fluid models have been derived in the last decades and applied to different problems, such as, for instance, nonlinear tokamak dynamics [3], drift-wave turbulence [4], magnetic reconnection [5] and astrophysical plasma turbulence [6, 7]. We refer to such models as to "reduced", meaning that they have been derived from a parent model

which, loosely speaking, consists of a set of partial differential equations with a larger number of unknowns, with respect to the reduced model. For instance, in the case of low-β reduced magnetohydrodynamics, the parent model used in Ref. [2] is the set of magnetohydrodynamics equations evolving the three components of the magnetic field plus the three components of the plasma bulk velocity field, whereas the reduced model only evolves two scalar potentials. In this case the reduction is carried out by first identifying a small parameter suggested by the physical problem of interest (the ratio between the poloidal and the strong guide field component of the magnetic field) and by applying to the physical quantities in the parent model an ordering based on the small parameter. With such an ordering, the use of scalar potentials instead of vector field components, becomes natural. Then, in the parent model equations, only terms up to a certain order are retained. The advantage of the reduced model is that, in its regime of validity, it is generally easier to treat analytically, with respect the parent model. Not only the number of unknowns is lesser but also some nonlinearities are of smaller order or eliminated (e.g., in the case of reduced magnetohydrodynamics, those associated with terms expressing variations along the guide field).

While in the case of reduced magnetohydrodynamics the parent model is already a fluid model, in the present paper we are concerned with reductions starting from a gyrokinetic system, which has, as unknown variables, perturbations of distribution functions, defined on domains described by spatial but also velocity coordinates. Such functions can be written as an infinite series (see Eq. (27)) where the coefficients are fluid moments. The reduction, in our case, consists of deriving, from the parent gyrokinetic system, fluid systems evolving only a finite number of moments. From this perspective, one can easily see how reduced fluid models can also provide faster numerical computations, with respect to the parent gyrokinetic models.

In this article we will consider a drift-fluid reduction, valid for scales much larger than the thermal Larmor radius, as well as a gyrofluid reduction, for which Finite Larmor Radius (FLR) effects are retained. We recall that gyrokinetic systems, unlike

kinetic systems such as Vlasov-Maxwell, evolve the distribution functions of gyrocenters, rather than of actual particles. For phenomena occurring at frequencies much lower than the ion cyclotron frequency based on the guide field amplitude, a convenient change of coordinates (see, e.g. Ref. [8]) can transform the system describing the particle dynamics, into a system in which the fast dynamics of the particle gyration motion around the magnetic field, gets decoupled. When electromagnetic perturbations are taken into account, the new system of coordinates is referred to as the gyrocenter coordinates. Gyrofluid models evolve then quantities such as the densities, velocities and temperatures of the gyrocenters. One of the advantages of gyrofluid models is that they are valid on scales as small as the Larmor radius scales, but their derivation bypasses the complications related to the so-called diamagnetic cancellation [9] which arise in the derivation of FLR fluid models evolving ordinary particle moments. An accurate modelling of FLR effects is important for describing, for instance, turbulent transport in tokamaks [8] and turbulent cascades in astrophysical plasmas [6].

From the point of view of the analysis of both the linear and nonlinear dynamics of such reduced models, a relevant question concerns the existence of a Hamiltonian structure, when these models are taken in their supposedly non-dissipative limit. Besides providing the expression for the total conserved energy of the system, the knowledge of the Hamiltonian structure helps to find further conserved quantities (Casimir invariants, in the noncanonical case) and to carry out stability analyses [10]. In this respect, a first breakthrough came with the identification of the noncanonical Hamiltonian structure of reduced magnetohydrodynamics [11]. Subsequently, the Hamiltonian structures of a number of other reduced fluid models were identified (see, for instance Ref. [12] for a review). In recent years, systematic procedures for deriving infinite families of Hamiltonian reduced fluid models were developed [13, 14]. In these approaches, reduced models are derived by taking moments of drift [13] and gyro-kinetic [14] equations and truncating the resulting hierarchy of drift and gyrofluid equations with an appropriate closure that guarantees the existence of a Hamiltonian structure. The

parent drift and gyrokinetic systems, adopted within this approach, assume the so-called δf approximation, in which the gyrocenter distribution function is decomposed as the sum of an equilibrium distribution function with a small perturbation which evolves in time. These procedures work for an arbitrary number of moments (of the perturbation of the distribution function) retained in the hierarchy of drift and gyrofluid equations, which leads to an infinite family of Hamiltonian reduced fluid models. Previously known Hamiltonian reduced fluid models [5, 15, 16, 17, 18, 19], when taken in the appropriate limits, were shown to be particular cases of such infinite class of models. Although the procedures described in Refs. [13, 14] are rather general, a strong limitation of theirs is that they only deal with moments with respect to the velocity along the direction of the guide field. The evolution of moments involving the perpendicular velocity, as, for instance, perpendicular temperature fluctuations and heat fluxes, is then excluded by this approach. This limitation prevents the description of all the anisotropies, for instance due to parallel and perpendicular temperature and heat flux fluctuations, developing in collisionless plasmas in particular at finite β, with β indicating the ratio between equilibrium kinetic pressure and magnetic pressure exerted by the guide field. With regard to this, we also point out that, more in general, to the best of our knowledge, no Hamiltonian reduced fluid model derived so far, accounts for the evolution of both parallel and perpendicular moments, apart from the simple electrostatic case for the perpendicular temperature [20] or from the electromagnetic case [21, 22], but with "ad hoc" closures, without providing a general procedure or a physical insight. Moreover, the procedures described in Refs. [13, 14], assumed a background state with spatially uniform density and guide field. Although this might be appropriate for some astrophysical applications, where the length scale of variations are so large that background gradients can be neglected, in laboratory plasmas background gradients can be important.

In this article, we considerably remedy these deficiencies, by providing a procedure for obtaining Hamiltonian drift-fluid and gyrofluid reductions of gyrokinetic equations,

including the evolution of both parallel and perpendicular moments, as well as density and magnetic background gradients. For the drift-fluid case, valid in the limit of vanishing FLR, the proposed Hamiltonian reduction permits to obtain, for a given species, a set of equations evolving $(N+1)(M+1)$ moments, where N and M are arbitrary non-negative integers, referring to the highest order of the moments retained in the parallel and perpendicular direction, respectively. We apply the Hamiltonian reduction to the electron species, for which the small FLR approximation is more appropriate, and leave the ions be described by a gyrokinetic equation, although other choices are admissible. For the gyrofluid case, accounting for finite FLR effects, we obtain a result valid in the low-β_e regime and with $M \leq 1$, i.e. retaining at most the first two moments in the perpendicular direction. In this case, we apply the Hamiltonian reduction to the ion species, and assume that the electrons are governed by a drift-kinetic equation.

In both the drift-fluid and gyrofluid case, the presence of the Hamiltonian structure is unveiled thanks to the identification of a change of variables (consisting of two consecutive orthogonal transformations) which casts the drift-fluid and gyrofluid equations in a particularly simple form. From such form, it becomes easy to see that the Poisson bracket for the drift-fluid and gyrofluid equations can be written as the direct sum of $(N+1)(M+1)$ independent Poisson brackets, which, in particular, enormously simplifies the verification of the Jacobi identity. In the two-dimensional (2D) limit without background inhomogeneities, the transformed variables also reveal that the drift-fluid and gyrofluid equations can be cast in the form of advection equations for Lagrangian invariants transported by generalized velocity fields combining the $\mathbf{E} \times \mathbf{B}$ drift, the free streaming along the perpendicular magnetic field and the nonlinear grad B drift. A physical meaning to such transformed variables can be ascribed, by extending the argument provided in Ref. [23].

The paper is organized as follows. In Sec. 2 we introduce the parent gyrokinetic model from which we start our reduction procedure, present its Hamiltonian structure

and introduce some notions about the moments we will use. In Sec. 3 we describe the Hamiltonian drift-fluid reduction, applied to the electron species, which leads to a Hamiltonian hybrid drift-fluid/gyrokinetic model. A physical interpretation of the electron equations in terms of the transformed variables is also provided. Such general procedure is illustrated with an example in Sec. 4, where a Hamiltonian four-moment reduction of the electron equations is carried out in detail. The Hamiltonian gyrofluid reduction is treated in Sec. 5 and applied to the ion species, yielding, in this case, a hybrid drift-kinetic/gyrofluid model. We conclude in Sec. 6 and in Appendix A we show that the Poisson bracket for the parent gyrokinetic model satisfies the Jacobi identity.

2. Parent gyrokinetic model and its Hamiltonian structure

The Hamiltonian gyrokinetic model that we adopt as starting point for drift-fluid and gyrofluid Hamiltonian reductions is given by

$$\begin{aligned}
&\frac{\partial g_s}{\partial t} + \rho_s \left[\mathcal{J}_{0s}\phi - \frac{\tau_s}{\rho_{th_s}} v_s \mathcal{J}_{0s} A_\parallel + \mathrm{sgn}(q_s)\tau_s\mu_s\mathcal{J}_{1s}B_\parallel, g_s \right]_s \\
&+ \frac{\rho_s}{\rho_{th_s}} v_s \frac{\partial}{\partial z_s}\left(\tau_s g_s + \mathrm{sgn}(q_s)\mathcal{F}_{0_s}\left(\mathcal{J}_{0s}\phi - \frac{\tau_s}{\rho_{th_s}} v_s \mathcal{J}_{0s}A_\parallel + \mathrm{sgn}(q_s)\tau_s\mu_s\mathcal{J}_{1s}B_\parallel \right)\right) \\
&- \mathrm{sgn}(q_s)\rho_s\tau_s\mu_s\zeta_B\frac{\partial g_s}{\partial y_s} - \rho_s\mathcal{F}_{0_s}(\mu_s\zeta_B - \zeta_n)\frac{\partial}{\partial y_s}\left(\mathcal{J}_{0s}\phi - \frac{\tau_s}{\rho_{th_s}} v_s \mathcal{J}_{0s}A_\parallel + \mathrm{sgn}(q_s)\tau_s\mu_s\mathcal{J}_{1s}B_\parallel \right) = 0, \qquad (1)
\end{aligned}$$

$$\sum_s \mathrm{sgn}(q_s)\int d\mathcal{W}_s\, \mathcal{J}_{0s} g_s = \sum_s \int d\mathcal{W}_s\, \mathcal{F}_{0_s}\left((1-\mathcal{J}_{0s}^2)\frac{\phi}{\tau_s} - \mathrm{sgn}(q_s)\mu_s\mathcal{J}_{0s}\mathcal{J}_{1s}B_\parallel \right), \qquad (2)$$

$$\Delta_\perp A_\parallel = -\frac{\beta_e}{2}\sum_s \mathrm{sgn}(q_s)\frac{\tau_s}{\rho_{th_s}}\int d\mathcal{W}_s\, v_s\mathcal{J}_{0s}\left(g_s - \mathrm{sgn}(q_s)\frac{v_s}{\rho_{th_s}}\mathcal{F}_{0_s}\mathcal{J}_{0s}A_\parallel \right), \qquad (3)$$

$$\begin{aligned}
&\sum_s \beta_s \int d\mathcal{W}_s\, \mu_s\mathcal{J}_{1s} g_s = -\sum_s \mathrm{sgn}(q_s)\beta_s \int d\mathcal{W}_s\, \mu_s\mathcal{F}_{0_s}\mathcal{J}_{0s}\mathcal{J}_{1s}\frac{\phi}{\tau_s} \\
&- \left(2 + \sum_s \beta_s \int d\mathcal{W}_s\, \mathcal{F}_{0_s}\mu_s^2\mathcal{J}_{1s}^2 \right) B_\parallel, \qquad (4)
\end{aligned}$$

where Eq. (1) indicates the gyrokinetic evolution equation for a given particle species s, whereas Eqs. (2)-(4) are three static relations that permit to close the system and which correspond to quasi-neutrality and to the parallel and perpendicular components of Ampère's law, respectively. The quantities appearing in system (1)-(4) will be defined

below.

The model is formulated in a slab geometry and assumes the presence of a weakly inhomogeneous, strong guide field directed along $\hat{z}$, which is the unit vector of a Cartesian coordinate system $\{x, y, z\}$. The guide field is assumed to have a weak spatial variation along the x coordinate. If we denote with B_0 the guide field amplitude at $x = 0$, the expression for the adopted total magnetic field, normalized with respect to B_0, reads

$$\mathbf{B}(x, y, z, t) = \epsilon \nabla A_{\parallel}(x, y, z, t) \times \hat{z} + ((1 - \epsilon \zeta_B x) + \epsilon B_{\parallel}(x, y, z, t))\hat{z} + \mathcal{O}(\epsilon^2), \qquad (5)$$

where $A_{\parallel}$ and $B_{\parallel}$ are perturbations of the magnetic vector potential and of the magnetic field parallel to the guide field, respectively, depending on space coordinates and on the time coordinate t. The parameters ζ_B and ζ_n are defined by

$$\zeta_B = \frac{L_{\parallel}}{L_B}, \qquad \zeta_n = \frac{L_{\parallel}}{L_n}, \qquad (6)$$

where $L_{\parallel}$ is the characteristic scale of variation of the perturbations along the guide field, while L_B and L_n are the spatial scales of variation of the guide field and of the equilibrium density (for both electrons and ions), respectively. While we account for the presence of an equilibrium density gradient, we also assume no equilibrium temperature gradients. This is mainly a simplifying assumption, although we believe that the extension of the present theory to the case with inhomogeneous temperature profiles should not be a major challenge.

The parameter ϵ is given by

$$\epsilon = \frac{L}{L_{\parallel}}, \qquad (7)$$

with L indicating the characteristic scale of variations along directions perpendicular to the guide field. Consistently with the gyrokinetic ordering, one has

$$\epsilon \ll 1. \qquad (8)$$

The function $\phi = \phi(x, y, z, t)$ represents the electrostatic potential perturbation, whereas $g_s = g_s(x_s, y_s, z_s, v_s, \mu_s, t)$ is referred to as generalized perturbed distribution function.

We indicated with x_s, y_s, z_s the spatial coordinates of the guiding centers of species s, with v_s and μ_s the parallel and lowest order magnetic moment, respectively, referring to the species s. The subscript s labels the particle species. We consider a plasma composed by electrons and one ion species, assuming the latter to be singly ionized. The index s can then take values $s = e$ or $s = i$ when referring to the electron or ion species, respectively.

The generalized perturbed distribution function g_s is related to the corresponding perturbation of the gyrocenter distribution function f_s by

$$g_s(x_s, y_s, z_s, v_s, \mu_s, t) = f_s(x_s, y_s, z_s, v_s, \mu_s, t) + \mathrm{sgn}(q_s)\mathcal{F}_{0_s}\frac{v_s}{\rho_{th_s}}(\mathcal{J}_{0s}A)(x_s, y_s, z_s, \mu_s, t). \tag{9}$$

In Eq. (9) q_s and ρ_{th_s} are the charge and the thermal Larmor radius of the particle of species s, respectively, so that $q_e = -e$ and $q_i = e$, and $\rho_{th_s} = v_{th_s}/\omega_{cs}$, with $v_{th_s} = \sqrt{T_{0s}/m_s}$ and $\omega_{cs} = |q_s|B_0/(m_s c)$. In the latter expression we indicated with T_{0s} and m_s the equilibrium temperature and the mass of the particles of species s, whereas c is the speed of light. The symbol ρ_s, on the other hand, represents the sonic Larmor radius, which can be defined as $\rho_s = \rho_{th_e}/(m_e/m_i)^{1/2}$ (note that, unlike in the symbol ρ_{th_s} for the thermal Larmor radii, in the symbol ρ_s the subscript s, following a standard notation, is not an index for the particle species).

The lowest order equilibrium distribution functions $\mathcal{F}_{0s}$ are homogeneous Maxwellian distribution functions defined as:

$$\mathcal{F}_{0s}(v_s, \mu_s) = \frac{\mathrm{e}^{-\frac{v_s^2}{2}-\mu_s}}{(2\pi)^{3/2}}. \tag{10}$$

The weak spatial variations of the equilibrium density, associated with the parameter ζ_n, in Eq. (1), come from the gradient of contributions of order ϵ to the equilibrium distribution functions given by $\mathcal{F}_{1_s}(x_s, v_s, \mu_s) = -\epsilon\zeta_n x_s \mathcal{F}_{0s}(v_s, \mu_s)$.

Next, we define the parameters

$$\beta_s = 8\pi\frac{n_0 T_{0s}}{B_0^2}, \qquad \tau_s = \frac{T_{0s}}{T_{0e}} \tag{11}$$

and the operators

$$[f,g]_s = \frac{\partial f}{\partial x_s}\frac{\partial g}{\partial y_s} - \frac{\partial f}{\partial y_s}\frac{\partial g}{\partial x_s}, \qquad \Delta_\perp f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}. \tag{12}$$

We also recall [24] that, from Ampère's law, the balance among equilibrium contributions at order ϵ implies the relation

$$\zeta_B + \zeta_n\left(1+\tau_i\right)\frac{\beta_e}{2} = 0, \tag{13}$$

which corresponds to perpendicular pressure balance among contributions due to spatial variations of the equilibrium magnetic and density fields.

The normalization adopted for the system (1)-(4) reads

$$t = \sqrt{\frac{\beta_e}{2}}\frac{v_A}{L_\parallel}\hat{t}, \qquad x = \frac{\hat{x}}{L}, \qquad y = \frac{\hat{y}}{L}, \qquad z = \frac{\hat{z}}{L_\parallel}, \qquad v_s = \frac{\hat{v}}{v_{th_s}}, \qquad \mu_s = \frac{\hat{\mu}_s B_0}{T_{0_s}},$$
$$\mathcal{F}_{0s} = \frac{v_{th_s}^3}{n_0}\hat{\mathcal{F}}_{0s}, \qquad f_s = \frac{L_\parallel}{L}\frac{v_{th_s}^3}{n_0}\hat{f}_s, \qquad \phi = \frac{L_\parallel}{L}\frac{e\hat{\phi}}{T_{0e}}, \qquad A_\parallel = \frac{L_\parallel}{L}\frac{\hat{A}}{B_0 L}, \qquad B_\parallel = \frac{L_\parallel}{L}\frac{\hat{B}_\parallel}{B_0}, \tag{14}$$

where $v_A = B_0/\sqrt{4\pi m_i n_0}$ is the Alfvèn speed based on the guide field, and where we denoted with a hat the dimensional variables. This normalization is consistent with the ordering adopted in Refs. [6, 25] which is suitable, in particular, for the expansion in the limit $b_e \to 0$ that will be carried out in Sec. 3.1.‡

The coordinates of the perturbed distribution functions belong to the domain $\mathcal{D} = \{|x_s| \le L_x\,, |y_s| \le L_y\,, |z_s| \le L_z\,, -\infty < v_s < +\infty\,, 0 \le \mu_s < +\infty\}$, for both species s, with L_x, L_y and L_z positive constants. Periodic conditions are imposed at the boundaries of the spatial coordinates. The condition $g_s = 0$ is assumed at $\mu_s = 0$, whereas $g_s \to 0$ is the condition for $\mu_s \to +\infty$ and $v_s \to 0$. The velocity volume element $d\mathcal{W}_s$ corresponds to $d\mathcal{W}_s = 2\pi dv_s d\mu_s$. The electromagnetic quantities ϕ, $A_\parallel$ and $B_\parallel$ also have to satisfy periodic boundary conditions on the spatial domain given by $\mathbb{D} = \{|x| \le L_x\,, |y| \le L_y\,, |z| \le L_z\}$.

‡ Actually, in Refs. [6, 25], frequencies ω are supposed to be of order $\omega \sim v_A/L_\parallel$, whereas, according to our normalization of the time coordinate in Eq. (14), we have $\omega \sim \sqrt{\beta_e/2}\,v_A/L_\parallel$. Therefore, our normalization remains compatible with that of Refs. [6, 25] as long as we do not take the $\beta_e \to 0$ limit, which occurs in Sec. 5. On the other hand, our normalization simplifies the expressions of the gyrokinetic evolution equations.

We complete the required definitions by providing the expressions for the gyroaverage operators $\mathcal{J}_{0s}$ and $\mathcal{J}_{1s}$. We begin with the former by expressing it, as customary in gyrokinetic theory, in Fourier space. Given a function $f(x, y, z, t)$ periodic on the above defined spatial domain $\mathbb{D}$, we write its Fourier series as

$$f(x, y, z, t) = \sum_{\mathbf{k}\in\mathscr{D}} f_{\mathbf{k}}(t)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{15}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector, the components of which belong to the lattice $\mathscr{D} = \{(2\pi l/(2L_x), 2\pi m/(2L_y), 2\pi n/(2L_z)) : (l, m, n) \in \mathbb{Z}^3\}$. The Fourier series representation of the operator $\mathcal{J}_{0s}$ acting on f is then defined as

$$(\mathcal{J}_{0s}f)(x_s, y_s, z_s, \mu_s, t) = \sum_{\mathbf{k}\in\mathscr{D}} (\mathcal{J}_{0s}f)_{\mathbf{k}}(\mu_s, t)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}_s}, \tag{16}$$

where, for every $\mathbf{k} \in \mathscr{D}$, one has

$$(\mathcal{J}_{0s}f)_{\mathbf{k}}(\mu_s, t) = J_0(\sqrt{2b_s\mu_s})f_{\mathbf{k}}(t), \qquad \text{where } b_s = k_\perp^2\rho_{th_s}^2, \tag{17}$$

with J_0 indicating the zeroth order Bessel function of the first kind and $k_\perp = \sqrt{k_x^2 + k_y^2}$ indicating the perpendicular wave number. In an analogous way, the operator $\mathcal{J}_{1s}$ is defined in terms of the Fourier coefficients of $(\mathcal{J}_{1s}f)$, which are given by

$$(\mathcal{J}_{1s}f)_{\mathbf{k}}(\mu_s, t) = 2\frac{J_1(\sqrt{2b_s\mu_s})}{\sqrt{2b_s\mu_s}}f_{\mathbf{k}}(t), \tag{18}$$

with J_1 representing the first order Bessel function of the first kind §. We recall that, according to the gyrokinetic approach, in the evolution equation (1), the gyroaveraged fields are evaluated at the guiding center position $\mathbf{x}_s$, whereas in the static relations (2)-(4), the gyroaveraged generalized perturbed distribution functions $\mathcal{J}_{0s}g_s$ and $\mathcal{J}_{1s}g_s$ are evaluated at a given position $\mathbf{x}$.

The model (1)-(4) can be obtained as a particular limit of the gyrokinetic equations derived, for instance, in Refs. [26, 9, 24]. With respect to such more general models,

§ We remark that a different notation is used, with respect for instance to Ref. [14], for indicating the dependence of $\mathcal{J}_{0s}f$ and $\mathcal{J}_{1s}f$ on the independent variables. In the present paper, when the dependence on such variables is written explicitly, as in Eq. (16), the expressions are written between parentheses, as for instance $(\mathcal{J}_{0s}f)$ in Eq. (16). When the dependence is not explicit, as for instance in Eq. (1), the parentheses are omitted.

we point out in particular that the model we are considering assumes no curvature of the guide field and no equilibrium temperature gradients. Similar equations are also presented as starting point from the derivation of gyrofluid models in Refs. [27, 28] and also account for more general background states, although they refer to low-β plasmas and, as a consequence, parallel magnetic fluctuations are neglected.

2.1. Hamiltonian formulation of the parent gyrokinetic model

We show that the system (1)-(4) admits a Hamiltonian formulation, i.e., it can be cast in the form

$$\frac{\partial \Xi_\alpha}{\partial t} = \{\Xi_\alpha, H\}, \qquad \alpha = 1, \cdots, \mathcal{N}, \tag{19}$$

where $\Xi_1, \cdots, \Xi_\mathcal{N}$ is a set of dynamical field variables, $\{\,,\,\}$ is a Poisson bracket and $H = H(\Xi_1, \cdots, \Xi_\mathcal{N})$ is the Hamiltonian functional. In our case one has $\mathcal{N} = 2$ and a natural choice for the field variables is $\Xi_1 = g_e$, $\Xi_2 = g_i$. The electromagnetic quantities $\phi, A_\parallel$ and $B_\parallel$ can be expressed in terms of $g_{e,i}$ by means of the static relations (2)-(4). We could then write

$$\phi = \mathcal{L}_{\phi_e} g_e + \mathcal{L}_{\phi_i} g_i, \tag{20}$$

$$A_\parallel = \mathcal{L}_{A_e} g_e + \mathcal{L}_{A_i} g_i, \tag{21}$$

$$B_\parallel = \mathcal{L}_{B_e} g_e + \mathcal{L}_{B_i} g_i, \tag{22}$$

where $\mathcal{L}_{\phi_s}, \mathcal{L}_{A_s}$ and $\mathcal{L}_{B_s}$ are linear operators, the explicit expressions of which can be obtained in Fourier space from Eqs. (2)-(4).

As is customary with infinite-dimensional Hamiltonian systems describing continuous media from the Eulerian point of view, the Hamiltonian structure of our gyrokinetic model is of noncanonical type [10]. Such structure is given by the Hamiltonian

$$H(g_e, g_i) = \frac{1}{2} \sum_{s=e,i} \int d^3x_s d\mathcal{W}_s \left(\tau_s \frac{g_s^2}{\mathcal{F}_{0s}} + \mathrm{sgn}(q_s) g_s \chi_s - 2\tau_s \zeta_n x_s g_s \right), \tag{23}$$

and by the noncanonical Poisson bracket

$$\{F,G\} = -\rho_s \sum_{s=e,i} \mathrm{sgn}(q_s) \int d^3x_s d\mathcal{W}_s \left(g_s[F_{g_s}, G_{g_s}]_s + \frac{\mathrm{sgn}(q_s)}{\rho_{th_s}} v_s \mathcal{F}_{0s} F_{g_s} \frac{\partial G_{g_s}}{\partial z_s} \right.$$
$$\left. -(\mu_s \zeta_B - \zeta_n) \mathcal{F}_{0s} F_{g_s} \frac{\partial G_{g_s}}{\partial y_s} \right), \tag{24}$$

where F and G are functionals of g_e and g_i. In Eq. (24) the subscripts on functionals denote functional derivatives (so that, for instance, $F_{g_s} = \delta F/\delta g_s$), while in Eq. (23) we introduced the following short-hand notation for the gyroaveraged potential

$$\chi_s = \mathcal{J}_{0s}\phi - \frac{\tau_s}{\rho_{th_s}} v_s \mathcal{J}_{0s} A_\parallel + \mathrm{sgn}(q_s)\tau_s \mu_s \mathcal{J}_{1s} B_\parallel. \tag{25}$$

One can verify by direct calculations that, from the expression (19), one obtains Eq. (1) when one chooses as Hamiltonian and Poisson bracket those given by Eq. (23) and (24), respectively. With regard to this, it is useful to note that

$$H_{g_s} = \tau_s \frac{g_s}{\mathcal{F}_{0s}} + \mathrm{sgn}(q_s)\chi_s - \tau_s \zeta_n x_s. \tag{26}$$

We find it appropriate to point out that, in order to derive the second term on the right-hand side of Eq. (26), one has to recall that, for each s, χ_s in Eq. (23) depends on g_e and g_i through $\phi, A_\parallel$ and $B_\parallel$ (see Eqs. (20)-(22)). The second term on the right-hand side of Eq. (23) is therefore a quadratic form applied to the vector (g_e, g_i). Also, one needs to use in the Hamiltonian the explicit form, in Fourier space, of the operators $\mathcal{L}_{\phi_s}, \mathcal{L}_{A_s}, \mathcal{L}_{B_s}$ obtained from Eqs. (2)-(4). Finally, the identities $\int d^3x_s d\mathcal{W}_s\, g(\mathbf{x}_s, v_s, \mu_s, t)(\mathcal{J}_{0s} f)(\mathbf{x}_s, \mu_s, t) = \int d^3x d\mathcal{W}_s\, f(\mathbf{x}, t)(\mathcal{J}_{0s} g_s)(\mathbf{x}, v_s, \mu_s, t)$ and $\int d^3x_s d\mathcal{W}_s\, g(\mathbf{x}_s, v_s, \mu_s, t)(\mathcal{J}_{1s} f)(\mathbf{x}_s, \mu_s, t) = \int d^3x d\mathcal{W}_s\, f(\mathbf{x}, t)(\mathcal{J}_{0s} g_s)(\mathbf{x}, v_s, \mu_s, t)$, valid for a function f of the form (15), must also be used. Note also that the Poisson bracket (24) consists of the direct sum of two independent noncanonical Poisson brackets, one depending only on g_e and involving functional derivatives with respect to g_e, and the other one depending only on g_i and possessing only functional derivatives with respect to g_i. The coupling between the species, in the dynamics, comes indeed from the Hamiltonian (23) through the electromagnetic fields, in particular from the term involving the gyroaveraged potentials χ_s.

The Hamiltonian structure (23)-(24) can actually be obtained as a simple extension of the Hamiltonian structure of the gyrokinetic system treated in Ref. [14], in the limit of isotropic equilibrium temperature. The extension concerns the inclusion of the terms associated with the weak background inhomogeneities. Indeed, for $\zeta_B = \zeta_n = 0$, one retrieves, up to the normalization, the system of Ref. [14] with isotropic equilibrium temperatures. Note that the presence of the equilibrium magnetic inhomogeneity only affects the Poisson bracket, whereas the equilibrium density inhomogeneity modifies both the bracket and the Hamiltonian.

The Poisson bracket (24) can be shown to satisfy the properties defining a Poisson bracket, i.e. bilinearity, antisymmetry, the Leibniz identity and the Jacobi identity. The first three of these properties are rather straightforward to show, whereas the proof of the Jacobi identity is slightly more involved. However, one can take advantage from the fact that, for $\zeta_B = \zeta_n = 0$, one has a valid Poisson bracket [29, 30, 23], as can be shown by using a procedure analogous to the one adopted in Ref. [31] for three-dimensional reduced fluid models. Taking advantage from the results (in particular Theorem 1) of Ref. [32], one can show that the Jacobi identity holds also for the expression (24). The explicit calculation is provided in Appendix A.

2.2. Moments

We introduce in this Section the dynamical variables (moments) that will be used for the fluid models. Following a customary procedure in gyrokinetic theory (see, for instance, Refs.[28, 33, 6]), it is convenient to perform an expansion of the perturbed distribution functions f_s, in terms of Hermite and Laguerre polynomials:

$$f_s(x_s, y_s, z_s, v_s, \mu_s, t) = \mathcal{F}_{0s}(v_s, \mu_s) \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} \frac{H_i(v_s)}{\sqrt{i!}} L_j(\mu_s) f_{ijs}(x_s, y_s, z_s, t), \qquad (27)$$

where H_i and L_j are the Hermite and Laguerre polynomial of order i and j, respectively.

Analogously, by virtue of the relation (9), it is possible to expand the perturbed

generalized distribution function:

$$g_s(x_s, y_s, z_s, v_s, \mu_s, t) = \mathcal{F}_{0s}(v_s, \mu_s) \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} \frac{H_i(v_s)}{\sqrt{i!}} L_j(\mu_s) g_{ij_s}(x_s, y_s, z_s, t), \tag{28}$$

We refer to the quantities g_{ij_s} as to moments (possibly with a slight abuse of language, as these are perturbations of the actual moments of the distribution functions) of order ij for the species s. In particular, from the orthogonality relation

$$\int d\mathcal{W}_s \frac{H_i(v_s)}{\sqrt{i!}} \frac{H_h(v_s)}{\sqrt{h!}} L_j(\mu_s) L_k(\mu_s) \mathcal{F}_{0s}(v_s, \mu_s) = \delta_{ih}\delta_{jk}, \tag{29}$$

where δ is the Kronecker delta, it follows that

$$g_{ij_s} = \int d\mathcal{W}_s \frac{H_i(v_s)}{\sqrt{i!}} L_j(\mu_s) g_s. \tag{30}$$

The moments g_{ij_s} are proportional to normalized fluctuations of physically relevant macroscopic quantities. Indeed, for instance, given that $H_0(v_s) = 1, H_1(v_s) = v_s, H_2(v_s) = v_s^2 - 1, H_3(v_s) = v_s^3 - 3v_s, L_0(\mu_s) = 1, L_1(\mu_s) = 1 - \mu_s$, the following relations hold:

$$g_{00_s} = N_s, \qquad g_{10_s} = M_s, \tag{31}$$

$$g_{20_s} = \frac{T_{\parallel s}}{\sqrt{2}}, \qquad g_{30_s} = \sqrt{\frac{2}{3}} q_{\parallel s}, \tag{32}$$

$$g_{01_s} = -T_{\perp s}, \qquad g_{11_s} = -Q_s, \tag{33}$$

where $N_s, M_s, T_{\parallel s}, q_{\parallel s}, T_{\perp s}, Q_s$ indicate the gyrocenter fluctuations of density , parallel canonical momentum, parallel temperature , parallel heat flux, perpendicular temperature and what we refer to as canonical perpendicular heat flux, respectively.

Through Eq. (9), the moments g_{ij_s} are related to the quantities f_{ij_s}, which are obviously interpreted as moments of the gyrocenter perturbed distribution function. For the lowest order moments we thus have the following relations:

$$g_{00_s} = f_{00_s} = N_s, \qquad g_{10_s} = f_{10_s} + \mathrm{sgn}(q_s) \frac{\mathcal{G}_{10_s} A_\parallel}{\rho_{th_s}} = U_{\parallel s} + \mathrm{sgn}(q_s) \frac{\mathcal{G}_{10_s} A_\parallel}{\rho_{th_s}}, \tag{34}$$

$$g_{20_s} = f_{20_s} = \frac{T_{\parallel s}}{\sqrt{2}}, \qquad g_{30_s} = f_{30_s} = \sqrt{\frac{2}{3}} q_{\parallel s}, \tag{35}$$

$$g_{01_s} = f_{01_s} = -T_{\perp s}, \qquad g_{11_s} = f_{11_s} + \mathrm{sgn}(q_s) \frac{\mathcal{G}_{11_s} A_\parallel}{\rho_{th_s}} = -q_{\perp s} + \mathrm{sgn}(q_s) \frac{\mathcal{G}_{11_s} A_\parallel}{\rho_{th_s}}, \tag{36}$$

where $U_{\parallel s}$ and $q_{\perp s}$ indicate the gyrocenter parallel velocity and the perpendicular heat flux, respectively. In order to obtain the relations (34)-(36), we also made use of the orthogonality relation (29), as well as of the following decomposition of the Bessel function J_0 in terms of Laguerre polynomials:

$$J_0(\sqrt{2b_s\mu_s}) = \sum_{n=0}^{+\infty} \mathcal{G}_{1n_s}(b_s) L_n(\mu_s), \tag{37}$$

where

$$\mathcal{G}_{1n_s}(b_s) = \frac{\mathrm{e}^{-\frac{b_s}{2}}}{n!} \left(\frac{b_s}{2}\right)^n \tag{38}$$

are gyroaverage operators commonly appearing in gyrofluid and FLR models [9, 34, 35]. The definitions of the operators in Eq. (38) has to be intended as those of multiplication operators in Fourier space, analogously to the definitions (17) and (18) for the gyroaverge operators $\mathcal{J}_{0s}$ and $\mathcal{J}_{1s}$.

3. Hamiltonian drift-fluid reduction

In this Section we show how a Hamiltonian fluid reduction can be obtained from the parent Hamiltonian gyrokinetic model (1)-(4), in the limit of small FLR effects. We refer to this, as to a Hamiltonian *drift-fluid* reduction, as opposed to the Hamiltonian *gyrofluid* reduction that will be considered in Sec. 5, and for which finite FLR effects will be retained.

Taking a small FLR limit for a species s, in the parent gyrokinetic system leads to a *drift-kinetic* equation for that species. This is obtained in principle by expanding in the limit $b_s \to 0$. The most straightforward expansion is the zero-th order expansion, formally leading to

$$\mathcal{J}_{0s} \to 1, \qquad \mathcal{J}_{1s} \to 1, \tag{39}$$

so that the gyroaverage operators reduce to the identity operator.

Recently [35], a more refined expansion was adopted, corresponding to

$$\mathcal{J}_{0s} \to \mathrm{e}^{-\frac{b_s}{2}}, \qquad \mathcal{J}_{1s} \to (1 - \mathrm{e}^{-\frac{b_s}{2}}) \frac{2}{b_s}. \tag{40}$$

Such approximation consists of retaining, for both gyroaverage operators, only the first term in the corresponding series expansions of the operators in terms of Laguerre polynomials. For $b_s \to 0$, these terms dominate over all the remaining terms in the corresponding series.

Other expansions can be chosen, in principle. In general, the drift-fluid reduction that will be presented in this Section, requires the gyrokinetic gyroaverage operators $\mathcal{J}_{0s}$ and $\mathcal{J}_{1s}$ to be replaced by drift-kinetic operators $\bar{\mathcal{G}}_{0s}$ and $\bar{\mathcal{G}}_{1s}$, so that

$$\mathcal{J}_{0s}(\sqrt{2b_s\mu_s}) \to \bar{\mathcal{G}}_{0s}(b_s), \qquad \mathcal{J}_{1s}(\sqrt{2b_s\mu_s}) \to \bar{\mathcal{G}}_{1s}(b_s), \tag{41}$$

where $\bar{\mathcal{G}}_{0s}$ and $\bar{\mathcal{G}}_{1s}$ no longer depend on μ_s and are multiplication operators in Fourier space, symmetric with respect to integration over space, i.e. $\int d^3x_s f\bar{\mathcal{G}}_{0s}g = \int d^3x_s g\bar{\mathcal{G}}_{0s}f$ and $\int d^3x_s f\bar{\mathcal{G}}_{1s}g = \int d^3x_s g\bar{\mathcal{G}}_{1s}f$, for two functions f and g.

In the following, we will leave the expression for $\bar{\mathcal{G}}_{0s}$ and $\bar{\mathcal{G}}_{1s}$ unspecified, but the Reader can make use, for instance, of the options (39) or (40) for having concrete examples.

We will also apply the drift-kinetic limit (and consequently the drift-fluid reduction) only to the electron species, leaving the ion species be described by gyrokinetics. On one hand, this is justified by physical relevance, given that the condition $b_e \ll b_i$ is often met, due to the small electron-to-ion mass ratio. Moreover, one also sees that, when applying the zero-th order drift-kinetic approximation (39) to both species in Eq. (2), the gyrokinetic equations get constrained by the relation $\int d\mathcal{W}_i\, g_i - \int d\mathcal{W}_e g_e = 0$ and the three static equations (2)-(4) no longer provide relations of the form (20)-(22). In particular, ϕ is not uniquely determined in terms of g_e and g_i. This indicates that, at least the zero-th order approximation (39) does not yield a well-posed system when applied to both species. Consequently, we aim at deriving a Hamiltonian hybrid gyrokinetic/drift-fluid model. However, the procedure is flexible and can be generalized to other cases, for instance carrying out a Hamiltonian gyrofluid reduction of the ion species, to obtain an actual drift-fluid/gyrofluid model, or focusing on sub-ion scales, which greatly simplifies the ion dynamics in such a way that the resulting system still

allows for a Hamiltonian structure.

The derivation of the Hamiltonian hybrid model proceeds by first obtaining the model equations and then by identifying the Hamiltonian structure. The model equations are obtained by neglecting electron FLR effects (see Sec. 3.1) and by taking moments of the electron gyrokinetic equation, after having replaced, in the system, the series expansion of the generalized electron perturbed distribution with a truncated version (see Sec. 3.2). Two change of variables are then carried out in Sec. 3.3. Such transformations permit to cast the resulting electron drift-fluid equations in a very simple form, from which the Poisson bracket for the electron drift-fluid equations can be "guessed". The corresponding Hamiltonian, on the other hand, is obtained by replacing the truncated generalized distribution function for the electrons, in the Hamiltonian of the parent model.

3.1. Hybrid drift-fluid/gyro-kinetic parent model

We begin the Hamiltonian electron drift-fluid reduction by first applying the replacements

$$\mathcal{J}_{0e} \to \bar{\mathcal{G}}_{0e}, \qquad \mathcal{J}_{1e} \to \bar{\mathcal{G}}_{1e} \tag{42}$$

in the parent system (1)-(4). Upon solving also well known integrals (see, for instance, Ref. [9]) in the right-hand sides of Eqs. (2)-(4), the parent system becomes

$$\frac{\partial g_e}{\partial t} + \rho_s \left[\bar{\chi}_e, g_e\right]_e + \frac{\rho_s}{\rho_{th_e}} v_e \frac{\partial}{\partial z_e}\left(g_e - \bar{\chi}_e \mathcal{F}_{0e}\right) + \rho_s \mu_e \zeta_B \frac{\partial g_e}{\partial y_e} - \rho_s \mathcal{F}_{0e}(\mu_e \zeta_B - \zeta_n)\frac{\partial \bar{\chi}_e}{\partial y_e} = 0, \tag{43}$$

$$\frac{\partial g_i}{\partial t} + \rho_s \left[\chi_i, g_i\right]_i + \frac{\rho_s}{\rho_{th_i}} v_i \frac{\partial}{\partial z_i}\left(\tau_i g_i + \chi_i \mathcal{F}_{0i}\right) - \rho_s \tau_i \mu_i \zeta_B \frac{\partial g_i}{\partial y_i} - \rho_s \mathcal{F}_{0i}(\mu_i \zeta_B - \zeta_n)\frac{\partial \chi_i}{\partial y_i} = 0, \tag{44}$$

$$\int d\mathcal{W}_i \, \mathcal{J}_{0i} g_i - \bar{\mathcal{G}}_{0e} \int d\mathcal{W}_e \, g_e = (1 - \Gamma_{0i})\frac{\phi}{\tau_i} + (1 - \bar{\mathcal{G}}_{0e}^2)\phi - (\Gamma_{0i} - \Gamma_{1i})B_\parallel + \bar{\mathcal{G}}_{0e}\bar{\mathcal{G}}_{1e} B_\parallel, \tag{45}$$

$$\Delta_\perp A_\parallel - \frac{\beta_e}{2}\frac{\tau_i}{\rho_{th_i}^2}\Gamma_{0i} A_\parallel - \frac{\beta_e}{2}\frac{\bar{\mathcal{G}}_{0e}^2 A_\parallel}{\rho_{th_e}^2} = -\frac{\beta_e}{2}\frac{\tau_i}{\rho_{th_i}} \int d\mathcal{W}_i \, v_i \mathcal{J}_{0i} g_i + \frac{\beta_e}{2}\frac{\bar{\mathcal{G}}_{0e}}{\rho_{th_e}} \int d\mathcal{W}_e \, v_e g_e, \tag{46}$$

$$\beta_i \int d\mathcal{W}_i \, \mu_i \mathcal{J}_{1i} g_i + \beta_e \bar{\mathcal{G}}_{1e} \int d\mathcal{W}_e \, \mu_e g_e = -\beta_e(\Gamma_{0i} - \Gamma_{1i})\phi + \beta_e \bar{\mathcal{G}}_{0e}\bar{\mathcal{G}}_{1e}\phi$$
$$- (2 + 2\beta_i(\Gamma_{0i} - \Gamma_{1i}) + 2\beta_e \bar{\mathcal{G}}_{1e}^2) B_\parallel. \tag{47}$$

In Eqs. (45)-(47) we introduced the operators

$$\Gamma_{0s}(b_s) = I_0(b_s)\mathrm{e}^{-b_s}, \qquad \Gamma_{1s}(b_s) = I_1(b_s)\mathrm{e}^{-b_s}, \tag{48}$$

where I_0 and I_1 are modified Bessel functions of the first kind. The operators (48) are also to be intended as multiplication operators in Fourier space.

We also found it convenient to introduce an electron drift-fluid counterpart of the gyroaveraged potentials (25), defined by

$$\bar{\chi}_e = \bar{\mathcal{G}}_{0e}\phi - v_e \frac{\bar{\mathcal{G}}_{0e} A_\parallel}{\rho_{th_e}} - \mu_e \bar{\mathcal{G}}_{1e} B_\parallel. \tag{49}$$

Equations (45)-(47) provide relations of the form (20)-(22), with $\mathcal{L}_{\phi_s}, \mathcal{L}_{A_s}$ and $\mathcal{L}_{B_s}$ integro-differential operators, defined in Fourier space.

The Hamiltonian functional of the system (43)-(47) descends from Eq. (23) and we find it convenient to decompose it as

$$H(g_e, g_i) = H_{dk_e}(g_e) + H_{gk_i}(g_i) + H_{dk\,gk}(g_e, g_i), \tag{50}$$

where

$$H_{dk_e}(g_e) = \frac{1}{2}\int d^3x_e d\mathcal{W}_e \left(\frac{g_e^2}{\mathcal{F}_{0e}} - 2\zeta_n x_e g_e\right), \tag{51}$$

$$H_{gk_i}(g_i) = \frac{1}{2}\int d^3x_i d\mathcal{W}_i \left(\tau_i\frac{g_i^2}{\mathcal{F}_{0i}} - 2\tau_i\zeta_n x_i g_i\right), \tag{52}$$

$$H_{dk\,gk}(g_e,g_i) = \frac{1}{2}\int d^3x_i d\mathcal{W}_i\, g_i\left(\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}}v_i\mathcal{J}_{0i}A_\parallel + \tau_i\mu_i\mathcal{J}_{1i}B_\parallel\right)$$
$$-\frac{1}{2}\int d^3x_e d\mathcal{W}_e\, g_e\left(\bar{\mathcal{G}}_{0e}\phi - v_e\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - \mu_e\bar{\mathcal{G}}_{1e}B_\parallel\right). \tag{53}$$

The modifications introduced by neglecting electron FLR only affect the Hamiltonian. The Poisson bracket remains identical to (24), which we rewrite in the form

$$\{F,G\}(g_e,g_i) = \{F,G\}_{gk_e}(g_e) + \{F,G\}_{gk_i}(g_i), \tag{54}$$

where

$$\{F,G\}_{gk_s}(g_s) = -\operatorname{sgn}(q_s)\rho_s\int d^3x_s d\mathcal{W}_s\left(g_s[F_{g_s},G_{g_s}]_s + \frac{\operatorname{sgn}(q_s)}{\rho_{th_s}}v_s\mathcal{F}_{0s}F_{g_s}\frac{\partial G_{g_s}}{\partial z_s}\right.$$
$$\left.-(\mu_s\zeta_B - \zeta_n)\mathcal{F}_{0s}F_{g_s}\frac{\partial G_{g_s}}{\partial y_s}\right). \tag{55}$$

Note that the coupling between g_e and g_i in the system, occurs only through the term $H_{dk\,gk}(g_e,g_i)$ in the Hamiltonian. In particular, as above mentioned, the Poisson bracket writes as the direct sum (54) of two Poisson brackets depending only on g_e and g_i, respectively.

3.1.1. Remarks on the ordering The system (43)-(47) represents a hybrid model combining a drift-kinetic description for the electrons with a gyrokinetic description for the ions. As mentioned earlier in Sec. 3, the electron drift-kinetic limit is obtained letting $b_e = k_\perp^2\rho_{th_e}^2 \to 0$. If the scales of the derivatives do not change, this corresponds to $\rho_{th_e} \to 0$ or, equivalently, assuming that ρ_s remains finite, to $(m_e/m_i)^{1/2} \to 0$. In Refs. [6, 25] it was shown that considering an expansion $f_e = f_{e0} + (m_e/m_i)^{1/2}f_{e1}$ for the perturbed electron distribution function and solving the system order by order in powers of $(m_e/m_i)^{1/2}$, leads to a condition compatible with isothermal electrons. This indicates

that temporal evolution of temperatures, heat fluxes and higher order moments come from higher order terms in the mass ratio (or electron thermal Larmor radius) expansion. Assuming that all the quantities $k_\perp, \partial_z, \partial_t, f_i, \mathcal{F}_{0e}, \mathcal{F}_{0i}, A_\parallel, \phi, B_\parallel, \rho_{th_i}, \beta_e, \tau_i, \zeta_B, \zeta_n$ remain finite in the limit $\rho_{th_e} \to 0$, the drift-kinetic equation (43) contains all the dominant contributions of order $\mathcal{O}(\rho_{th_e}^{-1})$, all the terms of order $\mathcal{O}(1)$ and some sub-dominant contributions of order $\mathcal{O}(\rho_{th_e})$. Indeed, for instance, the inspection of the last term on the right-hand side of the parallel Ampère's law (46), reveals that $U_{\parallel e}$ should be of order $\mathcal{O}(\rho_{th_e})$. Therefore, the expansion (27) of f_e, contains terms of order $\mathcal{O}(\rho_{th_e})$. Also, if one chooses approximations for $\bar{\mathcal{G}}_{0e}$ and $\bar{\mathcal{G}}_{1e}$ depending on b_e, this also introduces higher order contributions of order at least $\mathcal{O}(\rho_{th_e})$. The resulting moment equations will also contain the contributions of the first two leading orders in ρ_{th_e} and some subdominant contributions of higher order, but not all those that would be present if all the fields and gyroaverage operators had been expanded in powers of ρ_{th_e} up to a certain order. So, in this respect, the drift-kinetic equation (43), and the resulting moment equations, do not strictly respect the ordering in ρ_{th_e}. However, the concomitance of a non-trivial evolution of temperatures, heat fluxes and higher order moments, with the Hamiltonian structure, appears to occur in the presence of such higher order corrections. Although this could be unsatisfactory from the point of view of asymptotic analysis, from the point of view of physical relevance of the system, one could argue that the fact that such higher-order corrections are not retained consistently, should have little impact on the dynamics, as the the first two leading orders are, on the other hand, treated consistently.

3.2. Electron drift-fluid equations

The Hamiltonian electron drift-fluid reduction is carried out considering a truncation of the expansion (28) for the electron species. More precisely, we consider the truncated expansion

$$\bar{g}_e(x_e, y_e, z_e, v_e, \mu_e, t) = \mathcal{F}_{0e}(v_e, \mu_e) \sum_{i=0}^{N} \sum_{j=0}^{M} \frac{H_i(v_s)}{\sqrt{i!}} L_j(\mu_e) g_{ij_e}(x_e, y_e, z_e, t), \qquad (56)$$

with arbitrary integers $M, N \geq 0$. The numbers M and N will of course determine the highest order of the moment that one wants retain, along the perpendicular and parallel direction, respectively, in the electron drift-fluid description.

Inserting the truncated expansion (56) into the static relations (45)-(47) leads to

$$\int d\mathcal{W}_i \, \mathcal{J}_{0i} g_i - \bar{\mathcal{G}}_{0e} g_{00_e} = (1 - \Gamma_{0i}) \frac{\phi}{\tau_i} + (1 - \bar{\mathcal{G}}_{0e}^2)\phi - (\Gamma_{0i} - \Gamma_{1i}) B_\parallel + \bar{\mathcal{G}}_{0e} \bar{\mathcal{G}}_{1e} B_\parallel, \quad (57)$$

$$\Delta_\perp A_\parallel - \frac{\beta_e}{2} \frac{\tau_i}{\rho_{th_i}^2} \Gamma_{0i} A_\parallel - \frac{\beta_e}{2} \frac{\bar{\mathcal{G}}_{0e}^2 A_\parallel}{\rho_{th_e}^2} = -\frac{\beta_e}{2} \frac{\tau_i}{\rho_{th_i}} \int d\mathcal{W}_i \, v_i \mathcal{J}_{0i} g_i + \frac{\beta_e}{2} \frac{\bar{\mathcal{G}}_{0e} g_{10_e}}{\rho_{th_e}}, \quad (58)$$

$$\beta_i \int d\mathcal{W}_i \, \mu_i \mathcal{J}_{1i} g_i + \beta_e \bar{\mathcal{G}}_{1e} (g_{00_e} - g_{01_e}) = -\beta_e (\Gamma_{0i} - \Gamma_{1i}) \phi + \beta_e \bar{\mathcal{G}}_{0e} \bar{\mathcal{G}}_{1e} \phi$$
$$- (2 + 2\beta_i (\Gamma_{0i} - \Gamma_{1i}) + 2\beta_e \bar{\mathcal{G}}_{1e}^2) B_\parallel. \quad (59)$$

Therefore, in the electron drift-fluid reduction, electromagnetic perturbations are expressed in terms of the dynamical variables via relations of the form

$$\phi = \mathcal{L}_{\phi_{e00}} g_{00_e} + \mathcal{L}_{\phi_{e01}} g_{01_e} + \mathcal{L}_{\phi_i} g_i, \quad (60)$$

$$A_\parallel = \mathcal{L}_{A_{e10}} g_{10_e} + \mathcal{L}_{A_i} g_i, \quad (61)$$

$$B_\parallel = \mathcal{L}_{B_{e00}} g_{00_e} + \mathcal{L}_{B_{e01}} g_{01_e} + \mathcal{L}_{B_i} g_i, \quad (62)$$

where $\mathcal{L}_{\phi_{e00}}, \mathcal{L}_{\phi_{e01}}, \mathcal{L}_{A_{e10}}, \mathcal{L}_{B_{e00}}, \mathcal{L}_{B_{e01}}$ are linear operators.

We can then define the matrix Γ_e, with $N+1$ rows and $M+1$ columns, the elements of which correspond to the moments g_{ij_e}, so that

$$\Gamma_{e_{ij}} = g_{ij_e}, \qquad i = 0, \cdots N, \quad j = 0, \cdots M. \quad (63)$$

The evolution equations for the moments g_{ij_e} can be obtained by inserting the truncated expansion (56) into the drift-kinetic equation (43), multiplying every term of the resulting equation by $(H_i(v_e)/\sqrt{i!}) L_j(\mu_e)$ and integrating over $d\mathcal{W}_e$. Using the matrix notation introduced in Eq. (63), the evolution equations for the moments can be written

as

$$\frac{\partial \Gamma^T_{e_{ji}}}{\partial t} + \rho_s \left([\bar{\mathcal{G}}_{0e}\phi, \Gamma^T_{e_{ji}}]_e - \left[\frac{\bar{\mathcal{G}}_{0e} A_\parallel}{\rho_{th_e}}, \Gamma^T_{e_{jl}} W^T_{li} \right]_e - [\bar{\mathcal{G}}_{1e} B_\parallel, V_{jl} \Gamma^T_{e_{li}}]_e \right)$$
$$+ \frac{\rho_s}{\rho_{th_e}} \frac{\partial}{\partial z_e} \Gamma^T_{e_{jl}} W^T_{li} - \frac{\rho_s}{\rho_{th_e}} \frac{\partial}{\partial z_e} \left(\delta_{j0}\delta_{1i} \bar{\mathcal{G}}_{0e}\phi - \delta_{j0}(\sqrt{2}\delta_{2i} + \delta_{0i}) \frac{\bar{\mathcal{G}}_{0e} A_\parallel}{\rho_{th_e}} - (\delta_{j0} - \delta_{j1})\delta_{1i} \bar{\mathcal{G}}_{1e} B_\parallel \right) \tag{64}$$
$$+ \rho_s \zeta_B \frac{\partial}{\partial y_e} V_{jl} \Gamma^T_{e_{li}} - \rho_s \frac{\partial}{\partial y_e} \left((\delta_{j0} - \delta_{j1})\delta_{0i}\zeta_B \bar{\mathcal{G}}_{0e}\phi - (\delta_{j0} - \delta_{j1})\delta_{1i}\zeta_B \frac{\bar{\mathcal{G}}_{0e} A_\parallel}{\rho_{th_e}} \right.$$
$$-((2j+1)(\delta_{j0} - \delta_{j1}) - (j+1)(\delta_{j+1,0} - \delta_{j+1,1}) - j(\delta_{j-1,0} - \delta_{j-1,1}))\delta_{0i}\zeta_B \bar{\mathcal{G}}_{1e} B_\parallel$$
$$\left. -\delta_{j0}\delta_{0i}\zeta_n \bar{\mathcal{G}}_{0e}\phi + \delta_{j0}\delta_{1i}\zeta_n \frac{\bar{\mathcal{G}}_{0e} A_\parallel}{\rho_{th_e}} + (\delta_{j0} - \delta_{j1})\delta_{0i}\zeta_n \bar{\mathcal{G}}_{1e} B_\parallel \right) = 0,$$

where sum over repeated indices is understood, while $j = 0, \cdots, M$ and $i = 0, \cdots, N$. For a matrix A, we also indicated with A^T the transpose of A.

In order to derive Eq. (64) we made use of the recurrence relations

$$v_e H_n(v_e) = H_{n+1}(v_e) + n H_{n-1}(v_e), \tag{65}$$

$$\mu_e L_n(\mu_e) = (2n+1) L_n(\mu_e) - (n+1) L_{n+1}(\mu_e) - n L_{n-1}(\mu_e), \tag{66}$$

for Hermite and Laguerre polynomials, respectively. Also, we introduced the Jacobi matrices W and V, the elements of which are given by

$$W_{mn} = \sqrt{m}\delta_{m,n+1} + \sqrt{m+1}\delta_{m,n-1}, \qquad 0 \le m, n \le N, \tag{67}$$

$$V_{mn} = -m\delta_{m,n+1} + (2m+1)\delta_{mn} - (m+1)\delta_{m,n-1}, \qquad 0 \le m, n \le M. \tag{68}$$

The matrices W and V are $(N+1)\times(N+1)$ and $(M+1)\times(M+1)$, respectively, and their expressions are of the form

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 1 & 0 & \sqrt{2} & 0 & ... & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & ... & 0 \\ 0 & 0 & \sqrt{3} & 0 & ... & 0 \\ & ... & & & ... & \\ & ... & & & ... & \\ 0 & 0 & 0 & ... & 0 & \sqrt{N} \\ 0 & 0 & 0 & ... & \sqrt{N} & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & -1 & 0 & ... & 0 \\ -1 & 3 & -2 & ... & 0 \\ 0 & -2 & 5 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ & ... & & ... & \\ 0 & 0 & 0 & 2M-1 & -M \\ 0 & 0 & 0 & -M & 2M+1 \end{pmatrix}. \tag{69}$$

These matrices are symmetric and can be diagonalized, so that

$$U^T W U = \mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_N), \qquad Z^T V Z = \mathrm{diag}(\eta_0, \eta_1, \cdots, \eta_M), \tag{70}$$

where $U \in O(N+1)$ and $Z \in O(M+1)$ are $(N+1)\times(N+1)$ and $(M+1)\times(M+1)$ orthogonal matrices, respectively. We indicated with $\lambda_0, \lambda_1, \cdots, \lambda_N$ and $\eta_0, \eta_1, \cdots, \eta_M$ the eigenvalues of W and V, respectively. It is well known [36] that $\lambda_0, \lambda_1, \cdots, \lambda_N$ correspond to the zeros of the Hermite polynomial of order $N+1$, whereas $\eta_0, \eta_1, \cdots, \eta_M$ correspond to the zeros of the Laguerre polynomial of order $M+1$. Also, it is known that, for $0 \le i \le N$, the eigenvector associated with the eigenvalue λ_i is proportional to $(H_0(\lambda_i), H_1(\lambda_i), \cdots, H_N(\lambda_i))^T$. Likewise, for $0 \le i \le M$ the eigenvector associated with η_i is proportional to $(L_0(\eta_i), L_1(\eta_i), \cdots, L_M(\eta_i))^T$. With the help of the Christoffel-Darboux identity [36], it is possible to write in a compact form the proportionality constant that makes such eigenvectors orthonormal. For the case of the matrix W this was carried out in Refs. [37, 38]. As a result, we obtain that the elements of the matrices U and Z are given explicitly by

$$U_{mn} = \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_m(\lambda_n)}{|H_N(\lambda_n)|}, \qquad m, n = 0, \cdots, N, \tag{71}$$

$$Z_{mn} = \frac{\sqrt{\eta_n}}{M+1} \frac{L_m(\eta_n)}{|L_M(\eta_n)|}, \qquad m, n = 0, \cdots, M, \tag{72}$$

where

$$\hat{H}_n(v_e) = \frac{H_n(v_e)}{\sqrt{n!}}, \qquad n \ge 0. \tag{73}$$

Consequently, the matrices U and Z have the forms

$$U = \begin{pmatrix} \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_0(\lambda_0)}{|H_N(\lambda_0)|} & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_0(\lambda_1)}{|H_N(\lambda_1)|} & \cdots & \cdots & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_0(\lambda_N)}{|H_N(\lambda_N)|} \\ \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_1(\lambda_0)}{|H_N(\lambda_0)|} & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_1(\lambda_1)}{|H_N(\lambda_1)|} & \cdots & \cdots & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_1(\lambda_N)}{|H_N(\lambda_N)|} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_N(\lambda_0)}{|H_N(\lambda_0)|} & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_N(\lambda_1)}{|H_N(\lambda_1)|} & & & \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_N(\lambda_N)}{|H_N(\lambda_N)|} \end{pmatrix}, \tag{74}$$

$$Z = \begin{pmatrix} \frac{\sqrt{\eta_0}}{M+1}\frac{L_0(\eta_0)}{|L_M(\eta_0)|} & \frac{\sqrt{\eta_1}}{M+1}\frac{L_0(\eta_1)}{|L_M(\eta_1)|} & \cdots & \cdots & \frac{\sqrt{\eta_M}}{M+1}\frac{L_0(\eta_M)}{|L_M(\eta_M)|} \\ \frac{\sqrt{\eta_0}}{M+1}\frac{L_1(\eta_0)}{|L_M(\eta_0)|} & \frac{\sqrt{\eta_1}}{M+1}\frac{L_1(\eta_1)}{|L_M(\eta_1)|} & \cdots & \cdots & \frac{\sqrt{\eta_M}}{M+1}\frac{L_1(\eta_M)}{|L_M(\eta_M)|} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \\ \frac{\sqrt{\eta_0}}{M+1}\frac{L_M(\eta_0)}{|L_M(\eta_0)|} & \frac{\sqrt{\eta_1}}{M+1}\frac{L_M(\eta_1)}{|L_M(\eta_1)|} & & & \frac{\sqrt{\eta_M}}{M+1}\frac{L_M(\eta_M)}{|L_M(\eta_M)|} \end{pmatrix}. \tag{75}$$

3.3. Changes of variables

We carry out two consecutive changes of variables that cast the electron drift-fluid equations (64) in a remarkably simple form. The first change of variables (but the order can be inverted) is given, in matrix form, by

$$J_e = Z^T \Gamma_e^T. \tag{76}$$

The new variables are the elements of the $(M+1)\times(N+1)$ matrix J_e and their explicit expression is given by

$$J_{e_{mi}} = Z^T_{ml}\Gamma^T_{e_{li}} = \frac{\sqrt{\eta_m}}{M+1}\frac{L_l(\eta_m)}{|L_M(\eta_m)|}g_{il_e}, \qquad m = 0,\cdots,M, \quad i = 0,\cdots,N. \tag{77}$$

Note also that, due to the orthogonality of Z, one has $ZZ^T = I$, so that the relation

$$\Gamma_e^T = Z J_e \tag{78}$$

holds.

By multiplying Eqs. (64) by the matrix Z^T one can obtain the following moment equations, formulated in terms of the variables $J_{e_{mi}}$:

$$\begin{aligned} &\frac{\partial J_{e_{mi}}}{\partial t} + \rho_s\left([\bar{\mathcal{G}}_{0e}\phi, J_{e_{mi}}]_e - \left[\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}, J_{e_{ml}}W^T_{li}\right]_e - \eta_m[\bar{\mathcal{G}}_{1e}B_\parallel, J_{e_{mi}}]_e\right) \\ &+ \frac{\rho_s}{\rho_{th_e}}\frac{\partial}{\partial z_e}J_{e_{ml}}W^T_{li} - \frac{\rho_s}{\rho_{th_e}}\frac{\partial}{\partial z_e}\left(Z^T_{m0}\delta_{1i}\bar{\mathcal{G}}_{0e}\phi - Z^T_{m0}(\sqrt{2}\delta_{2i}+\delta_{0i})\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - (Z^T_{m0}-Z^T_{m1})\delta_{1i}\bar{\mathcal{G}}_{1e}B_\parallel\right) \\ &+ \rho_s\zeta_B\eta_m\frac{\partial}{\partial y_e}J_{e_{mi}} - \rho_s\frac{\partial}{\partial y_e}\left((Z^T_{m0}-Z^T_{m1})\delta_{0i}\zeta_B\bar{\mathcal{G}}_{0e}\phi - (Z^T_{m0}-Z^T_{m1})\delta_{1i}\zeta_B\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right. \\ &-(Z^T_{m0}-3Z^T_{m1}+Z^T_{m0}-Z^T_{m1}+2Z^T_{m2})\delta_{0i}\zeta_B\bar{\mathcal{G}}_{1e}B_\parallel \\ &\left.-Z^T_{m0}\delta_{0i}\zeta_n\bar{\mathcal{G}}_{0e}\phi + Z^T_{m0}\delta_{1i}\zeta_n\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} + (Z^T_{m0}-Z^T_{m1})\delta_{0i}\zeta_n\bar{\mathcal{G}}_{1e}B_\parallel\right) = 0, \end{aligned} \tag{79}$$

for $m = 0, \cdots, M$ and $i = 0, \cdots, N$. To derive Eq. (79) we also made use of the relation (78). We point out that, when an eigenvalue multiplies a matrix element, as in the fourth term of Eq. (79), the repeated indices do not imply a summation (the indices m and i are indeed fixed in Eq. (79)). When carrying out this change of variables (and the following one) we also assume it is understood that the relations (60)-(62) are replaced by the corresponding relations expressing ϕ, $A_\parallel$ and $B_\parallel$ in terms of the new variables.

The second orthogonal transformation that we perform is given by

$$G_e = U^T J_e^T. \tag{80}$$

The matrix G_e is therefore a $(N+1) \times (M+1)$ matrix with elements

$$\begin{aligned} &G_{e_{nm}} = U^T_{nl} \Gamma_{e_{lk}} Z_{km} = \sqrt{\frac{N!}{N+1}} \frac{\hat{H}_l(\lambda_n)}{|H_N(\lambda_n)|} g_{lk_e} \frac{\sqrt{\eta_m}}{M+1} \frac{L_k(\eta_m)}{|L_M(\eta_m)|}, \\ &n = 0, \cdots, N, \quad m = 0, \cdots, M. \end{aligned} \tag{81}$$

Analogously to Eq. (78), due to the orthogonality of U, one has also the relation

$$J_e^T = U G_e. \tag{82}$$

Note that the transformation (80) is analogous to the transformation already adopted in Refs. [13, 14, 23] for the case of moments only with respect to the parallel velocity. Equation (79) possesses indeed terms of form analogous to those appearing in the simpler case treated in such References.

Taking the transpose of Eq. (79) and multiplying it by the matrix U^T , one obtains

the following set of equations in terms of the variables $G_{e_{nm}}$:

$$\begin{aligned}
&\frac{\partial G_{e_{nm}}}{\partial t} + \rho_s \left[\bar{\mathcal{G}}_{0e}\phi - \lambda_n \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - \eta_m \bar{\mathcal{G}}_{1e} B_\parallel, G_{e_{nm}} \right]_e \\
&+ \frac{\rho_s}{\rho_{th_e}} \lambda_n \frac{\partial}{\partial z_e} G_{e_{nm}} - \frac{\rho_s}{\rho_{th_e}} \frac{\partial}{\partial z_e} \left(U^T_{n1} Z_{0m} \bar{\mathcal{G}}_{0e}\phi - (\sqrt{2}U^T_{n2} + U^T_{n0}) Z_{0m} \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - U^T_{n1}(Z_{0m} - Z_{1m}) \bar{\mathcal{G}}_{1e} B_\parallel \right) \\
&+ \rho_s \zeta_B \eta_m \frac{\partial}{\partial y_e} G_{e_{nm}} - \rho_s \frac{\partial}{\partial y_e} \left(U^T_{n0}(Z_{0m} - Z_{1m}) \zeta_B \bar{\mathcal{G}}_{0e}\phi - U^T_{n1}(Z_{0m} - Z_{1m}) \zeta_B \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} \right. \\
&- U^T_{n0}(2Z_{0m} - 4Z_{1m} + 2Z_{2m}) \zeta_B \bar{\mathcal{G}}_{1e} B_\parallel \\
&\left. - U^T_{n0} Z_{0m} \zeta_n \bar{\mathcal{G}}_{0e}\phi + U^T_{n1} Z_{0m} \zeta_n \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} + U^T_{n0}(Z_{0m} - Z_{1m}) \zeta_n \bar{\mathcal{G}}_{1e} B_\parallel \right) = 0.
\end{aligned} \tag{83}$$

Making use of the expressions (71) and (72), we can provide a more explicit formulation of Eq. (83) in terms of known quantities, such as the zeros of Hermite and Laguerre polynomials λ_n and η_m, and the parameters M and N. In particular, we can make use of the relations

$$U^T_{n0} = \frac{1}{u_{(n)}}, \qquad U^T_{n1} = \frac{\lambda_n}{u_{(n)}}, \qquad U^T_{n2} = \frac{\lambda_n^2 - 1}{\sqrt{2}u_{(n)}}, \qquad n = 0, \cdots, N, \tag{84}$$

$$Z_{0m} = \frac{1}{z_{(m)}}, \qquad Z_{1m} = \frac{1 - \eta_m}{z_{(m)}}, \qquad Z_{2m} = \frac{\eta_m^2 - 4\eta_m + 2}{2z_{(m)}}, \qquad m = 0, \cdots, M, \tag{85}$$

where we defined the normalization constants

$$u_{(n)} = \sqrt{\frac{N+1}{N!}} |H_N(\lambda_n)|, \qquad z_{(m)} = \frac{M+1}{\sqrt{\eta_m}} |L_M(\eta_m)|. \tag{86}$$

Inserting the relations (84) and (85) into Eq. (83) we obtain

$$\begin{aligned}
&\frac{\partial G_{e_{nm}}}{\partial t} + \rho_s \left[\bar{\mathcal{G}}_{0e}\phi - \lambda_n \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - \eta_m \bar{\mathcal{G}}_{1e} B_\parallel, G_{e_{nm}} \right]_e \\
&+ \frac{\rho_s}{\rho_{th_e}} \lambda_n \frac{\partial}{\partial z_e} G_{e_{nm}} - \frac{\rho_s}{\rho_{th_e}} \frac{\lambda_n}{u_{(n)} z_{(m)}} \frac{\partial}{\partial z_e} \left(\bar{\mathcal{G}}_{0e}\phi - \lambda_n \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - \eta_m \bar{\mathcal{G}}_{1e} B_\parallel \right) \\
&+ \rho_s \zeta_B \eta_m \frac{\partial}{\partial y_e} G_{e_{nm}} - \rho_s \frac{\eta_m}{u_{(n)} z_{(m)}} \frac{\partial}{\partial y_e} \left(\zeta_B \bar{\mathcal{G}}_{0e}\phi - \zeta_B \lambda_n \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - \zeta_B \eta_m \bar{\mathcal{G}}_{1e} B_\parallel \right) \\
&+ \rho_s \frac{\zeta_n}{u_{(n)} z_{(m)}} \frac{\partial}{\partial y_e} \left(\bar{\mathcal{G}}_{0e}\phi - \lambda_n \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - \eta_m \bar{\mathcal{G}}_{1e} B_\parallel \right) = 0,
\end{aligned} \tag{87}$$

or, in an even more compact form,

$$\begin{aligned}
&\frac{\partial G_{e_{nm}}}{\partial t} + \rho_s \left[\chi_{e_{nm}}, G_{e_{nm}} \right]_e + \rho_s \left(\frac{\lambda_n}{\rho_{th_e}} \frac{\partial}{\partial z_e} + \zeta_B \eta_m \frac{\partial}{\partial y_e} \right) \left(G_{e_{nm}} - \frac{\chi_{e_{nm}}}{u_{(n)} z_{(m)}} \right) \\
&+ \rho_s \frac{\zeta_n}{u_{(n)} z_{(m)}} \frac{\partial \chi_{e_{nm}}}{\partial y_e} = 0,
\end{aligned} \tag{88}$$

where we introduced the potentials

$$\chi_{e_{nm}} = \bar{\mathcal{G}}_{0e}\phi - \lambda_n \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} - \eta_m \bar{\mathcal{G}}_{1e}B_\parallel. \tag{89}$$

The formulation (88) shows how, thanks to the transformations (76) and (80), the set of drift-fluid equations (64) can be cast in a much simpler form. In particular, we note that in each evolution equation (88) with fixed m and n, only quantities indexed by m and n appear explicitly. Thus, the mixing with moments of different order in m and n, which occurred in the original formulation (64) has been removed. This "diagonalization" of the system will turn out to be useful, in particular, for identifying the Hamiltonian structure. The dependence on variables $G_{e_{n'm'}}$ with $m' \neq m$ and $n' \neq n$ remains of course present, in each equation, in implicit form, through the electromagnetic quantities $\phi, A_\parallel$ and $B_\parallel$.

3.4. Remarks on the interpretation of the system in terms of the new variables

The formulation (88) generalizes to an arbitrary number of moments in both parallel and perpendicular velocities, as well as to the case of inhomogeneous background magnetic field and density profiles, the formulations found for the drift-fluid electron dynamics in simpler cases [5, 16, 18, 13]. We remark that the potentials $\chi_{e_{nm}}$ can be interpreted as stream functions for generalized perpendicular velocity fields

$$\mathbf{v}_{e_{nm}} = \hat{z} \times \nabla \chi_{e_{nm}} = \mathbf{v}_{\mathbf{E}\times\mathbf{B}} + \frac{\lambda_n}{\rho_{th_e}} \mathbf{v}_{\mathbf{B}_\perp} + \eta_m \mathbf{v}_{nl\,\nabla B} \tag{90}$$

where

$$\mathbf{v}_{\mathbf{E}\times\mathbf{B}} = \hat{z} \times \nabla \bar{\mathcal{G}}_{0e}\phi, \qquad \mathbf{v}_{\mathbf{B}_\perp} = \nabla \bar{\mathcal{G}}_{0e}A_\parallel \times \hat{z}, \qquad \mathbf{v}_{nl\,\nabla B} = -\hat{z} \times \nabla \bar{\mathcal{G}}_{1e}B_\parallel. \tag{91}$$

The vector fields $\mathbf{v}_{\mathbf{E}\times\mathbf{B}}$, $\mathbf{v}_{\mathbf{B}_\perp}$ and $\mathbf{v}_{nl\,\nabla B}$ corresponds to normalized versions of the $\mathbf{E}\times\mathbf{B}$ velocity, of the free streaming velocity along the perpendicular perturbation of the magnetic field and of the nonlinear grad B drift, respectively. These velocity fields account also for small FLR corrections possibly induced by the operators $\bar{\mathcal{G}}_{0e}$ and $\bar{\mathcal{G}}_{1e}$.

In the 2D limit (where $\partial_z = \partial_{z_e} = 0$) in the absence of background inhomogeneities (i.e. $\zeta_B = \zeta_n = 0$), the electron fluid equations (88) reduce to

$$\frac{\partial G_{e_{nm}}}{\partial t} + \rho_s \mathbf{v}_{e_{nm}} \cdot \nabla G_{e_{nm}} = 0. \tag{92}$$

The fields G_{nm} become then Lagrangian invariants advected by the velocity fields $\mathbf{v}_{e_{nm}}$. This generalizes, with the inclusion of the nonlinear grad B drift, the Lagrangian invariant formulation found in Refs. [13, 23]. Considering Eq. (90), it follows that the advecting fields $\mathbf{v}_{e_{nm}}$ consist of linear combination of the $\mathbf{E}\times\mathbf{B}$ drift with the magnetic drift $\mathbf{v}_{\mathbf{B}_\perp}$ and the nonlinear grad B drift $\mathbf{v}_{nl\,\nabla B}$. The relative amplitudes, with respect to the $\mathbf{E}\times\mathbf{B}$ drift, of the magnetic and nonlinear grad B drift, are given by the eigenvalues λ_n and η_m, corresponding to zeros of Hermite and Laguerre polynomials. From known properties of such polynomials, it is possible to infer some features of how the magnetic and nonlinear grad B drift add up to the $\mathbf{E}\times\mathbf{B}$ drift, as the number of retained moments N and M increases. As already recalled in Ref. [23], the zeros of the Hermite polynomial $H_{N+1}(v_e)$, corresponding to λ_n, are real, distinct and symmetrically distributed around $v_e = 0$. Therefore, when N is even, so that the number of zeros is odd, one of the λ_n will be equal to zero, implying no contribution from the perpendicular magnetic drift for that particular value of n. Also, the zeros of Hermite polynomials interlace. Therefore, if λ_i and λ_{i+1} are two consecutive eigenvalues for a given N, with $\lambda_i \leq \lambda_{i+1}$, then, for $N+1$, there will be an eigenvalue $\lambda_{i'}$ such that $\lambda_i \leq \lambda_{i'} \leq \lambda_{i+1}$. Moreover, for any interval $v_1 \leq v_e \leq v_2$, it is always possible to find a zero λ_i such that $v_1 \leq \lambda_i \leq v_2$, provided N is large enough. This indicates how the relative weight between $\mathbf{E}\times\mathbf{B}$ and magnetic drifts will fill the set of roots of the Hermite polynomials as the number of retained parallel moments N increases. Most of these properties are shared also by zeros of Laguerre polynomials, and therefore by the relative weights η_m between $\mathbf{E}\times\mathbf{B}$ and nonlinear grad B drift. Also in this case, the zeros are distinct, real, they interlace and it is always possible, for any interval $\mu_1 \leq \mu_e \leq \mu_2$, to find a zero η_i such that $\mu_1 \leq \eta_i \leq \mu_2$, provided M is large enough. However, zeros of Laguerre polynomials are positive, unlike zeros of Hermite polynomials, for which we noticed, in particular, that $\lambda_i = 0$ is always

an eigenvalue if N is even. Therefore, for every $G_{e_{nm}}$ we expect a finite contribution of the nonlinear grad B drift in the advecting velocity field $\mathbf{v}_{e_{nm}}$. Also, because $\eta_m > 0$, for every m and every M, such contribution always adds up with the same sign to the $\mathbf{E} \times \mathbf{B}$ drift. When the nonlinear grad B drift is negligible, which we expect to occur for low-β plasmas, and N is even, one of the Lagrangian invariants is purely advected, also in 3D, by the $\mathbf{E} \times \mathbf{B}$ drift, in the absence of background inhomogeneities [23]. We also remark that, from the third term on the left-hand side of Eq. (88), it follows that the above mentioned properties of λ_n and η_m concern of course also the amplitudes of the transport of the quantities $G_{e_{nm}} - \chi_{e_{nm}}/(u_{(n)} z_{(m)})$ along the guide field and due to the linear grad B drift, respectively. The last term on the left-hand side of Eq. (88), which originates from the electron diamagnetic drift, on the other hand, is influenced by the values of λ_n and η_m in a more implicit way, through the normalization constants $u_{(n)}$ and $z_{(m)}$. Further properties about the location of the eigenvalues λ_n and η_m can be found, for instance in Refs. [36, 39].

With regard to the fields $G_{e_{nm}}$, it is possible to generalize the physical interpretation provided in Ref. [23]. Indeed, we can consider the truncated expansion (56):

$$\bar{g}_e(x_e, y_e, z_e, v_e, \mu_e, t) = \mathcal{F}_{0e}(v_e, \mu_e) \sum_{i=0}^{N} \sum_{j=0}^{M} \frac{H_i(v_e)}{\sqrt{i!}} L_j(\mu_e) g_{ij_e}(x_e, y_e, z_e, t). \tag{93}$$

Combining the relations (63), (76) and (82), one obtains

$$g_{ij_e} = U_{il} G_{e_{lk}} Z^T_{kj}, \tag{94}$$

which, when inserted into Eq. (93), yields

$$\bar{g}_e(x_e, y_e, z_e, v_e, \mu_e, t) = \mathcal{F}_{0e}(v_e, \mu_e) \sum_{i,l=0}^{N} \sum_{j,k=0}^{M} \frac{H_i(v_s)}{\sqrt{i!}} L_j(\mu_e) U_{il} G_{e_{lk}}(x_e, y_e, z_e, t) Z^T_{kj}. \tag{95}$$

Then one evaluates Eq. (95) at $v_e = \lambda_n$ and $\mu_e = \eta_m$, for $n = 0, \cdots, N$ and $m = 0, \cdots, M$. In the resulting expression one can make use of the relations

$$\frac{N!}{N+1} \sum_{i=0}^{N} \frac{\hat{H}_i(\lambda_n)}{|H_N(\lambda_n)|} \frac{\hat{H}_i(\lambda_l)}{|H_N(\lambda_l)|} = \delta_{nl}, \qquad \frac{\sqrt{\eta_m}\sqrt{\eta_k}}{(M+1)^2} \sum_{j=0}^{M} \frac{L_j(\eta_m)}{|L_M(\eta_m)|} \frac{L_j(\eta_k)}{|L_M(\eta_k)|} = \delta_{mk}, \tag{96}$$

which descend from $U^TU = I$ and $Z^TZ = I$, respectively. As a result, one obtains

$$\bar{g}_e(x_e, y_e, z_e, \lambda_n, \eta_m, t) = \mathcal{F}_{0e}(\lambda_n, \eta_m)\sqrt{\frac{N+1}{N!}}(M+1)|H_N(\lambda_n)|\frac{|L_M(\eta_m)|}{\sqrt{\eta_m}}G_{e_{nm}}(x_e, y_e, z_e, t), \tag{97}$$

from which it follows that

$$G_{e_{nm}}(x_e, y_e, z_e, t) = \sqrt{\frac{N!}{N+1}}\frac{1}{|H_N(\lambda_n)|}\frac{\sqrt{\eta_m}}{(M+1)|L_M(\eta_m)|}\frac{\bar{g}_e(x_e, y_e, z_e, \lambda_n, \eta_m, t)}{\mathcal{F}_{0e}(\lambda_n, \eta_m)}. \tag{98}$$

Therefore, the values of the new field variables $G_{e_{nm}}$ are proportional to those of the truncated perturbed generalized distribution function $\bar{g}_e$, evaluated at $v_e = \lambda_n$ and at $\mu_e = \eta_m$. Studying the dynamics of $G_{e_{nm}}$ permits then to follow directly the evolution (up to the proportionality coefficient) of the truncated version of g_e, at specific values of the parallel velocity and of the magnetic moment. Of course, increasing the number of variables $G_{e_{nm}}$ by increasing M and N, provides a more and more accurate reconstruction of $\bar{g}_e$. We notice that a similarity between the structures formed by the drift-kinetic distribution functions at a fixed value of the parallel velocity, and those formed by Lagrangian invariants of drift-fluid models was pointed out, by means of numerical simulations, in Refs. [40, 41, 42].

3.5. Hamiltonian structure of the hybrid drift-fluid/gyrokinetic model

We finally proceed with providing the Hamiltonian structure of the model we derived combining a drift-fluid description for the electrons with the gyrokinetic description for the ions. We first write the complete system, in terms of the dynamical variables g_i and

$G_{e_{nm}}$, for $n = 0, \cdots, N$ and $m = 0, \cdots, M$:

$$\frac{\partial G_{e_{nm}}}{\partial t} + \rho_s \left[\chi_{e_{nm}}, G_{e_{nm}}\right]_e + \rho_s \left(\frac{\lambda_n}{\rho_{th_e}}\frac{\partial}{\partial z_e} + \zeta_B \eta_m \frac{\partial}{\partial y_e}\right)\left(G_{e_{nm}} - \frac{\chi_{e_{nm}}}{u_{(n)} z_{(m)}}\right)$$
$$+ \rho_s \frac{\zeta_n}{u_{(n)} z_{(m)}} \frac{\partial \chi_{e_{nm}}}{\partial y_e} = 0, \tag{99}$$

$$\frac{\partial g_i}{\partial t} + \rho_s \left[\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}} v_i \mathcal{J}_{0i} A_\parallel + \tau_i \mu_i \mathcal{J}_{1i} B_\parallel, g_i\right]_i$$
$$+ \frac{\rho_s}{\rho_{th_i}} v_i \frac{\partial}{\partial z_i}\left(\tau_i g_i + \mathcal{F}_{0i}\left(\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}} v_i \mathcal{J}_{0i} A_\parallel + \tau_i \mu_i \mathcal{J}_{1i} B_\parallel\right)\right)$$
$$- \rho_s \tau_i \mu_i \zeta_B \frac{\partial g_i}{\partial y_i} - \rho_s \mathcal{F}_{0i}(\mu_i \zeta_B - \zeta_n)\frac{\partial}{\partial y_i}\left(\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}} v_i \mathcal{J}_{0i} A_\parallel + \tau_i \mu_i \mathcal{J}_{1i} B_\parallel\right) = 0, \tag{100}$$

$$\int d\mathcal{W}_i \, \mathcal{J}_{0i} g_i - \sum_{l=0}^{N}\sum_{k=0}^{M} \frac{\bar{\mathcal{G}}_{0e} G_{e_{lk}}}{u_{(l)} z_{(k)}} = (1 - \Gamma_{0i})\frac{\phi}{\tau_i} + (1 - \bar{\mathcal{G}}_{0e}^2)\phi - (\Gamma_{0i} - \Gamma_{1i}) B_\parallel + \bar{\mathcal{G}}_{0e}\bar{\mathcal{G}}_{1e} B_\parallel, \tag{101}$$

$$\Delta_\perp A_\parallel - \frac{\beta_e}{2}\frac{\tau_i}{\rho_{th_i}^2}\Gamma_{0i} A_\parallel - \frac{\beta_e}{2}\frac{\bar{\mathcal{G}}_{0e}^2 A_\parallel}{\rho_{th_e}^2} = -\frac{\beta_e}{2}\frac{\tau_i}{\rho_{th_i}}\int d\mathcal{W}_i \, v_i \mathcal{J}_{0i} g_i + \frac{\beta_e}{2\rho_{th_e}} \sum_{l=0}^{N}\sum_{k=0}^{M} \lambda_l \frac{\bar{\mathcal{G}}_{0e} G_{e_{lk}}}{u_{(l)} z_{(k)}}, \tag{102}$$

$$\beta_i \int d\mathcal{W}_i \, \mu_i \mathcal{J}_{0i} g_i + \beta_e \sum_{l=0}^{N}\sum_{k=0}^{M} \eta_k \frac{\bar{\mathcal{G}}_{1e} G_{e_{lk}}}{u_{(l)} z_{(k)}} = -\beta_e(\Gamma_{0i} - \Gamma_{1i})\phi + \beta_e \bar{\mathcal{G}}_{0e}\bar{\mathcal{G}}_{1e}\phi$$
$$- (2 + 2\beta_i(\Gamma_{0i} - \Gamma_{1i}) + 2\beta_e \bar{\mathcal{G}}_{1e}^2) B_\parallel. \tag{103}$$

The Hamiltonian of the system (99)-(103) is obtained by analogy with the procedure adopted in Refs. [28, 14, 23] for finding a conserved energy. The procedure consists of replacing in the Hamiltonian of the parent gyrokinetic model, the truncated expansion of the perturbed generalized distribution function. Because in our hybrid model the fluid reduction concerns only the electron species, we will only perform the replacement

$$g_e \to \bar{g}_e \tag{104}$$

into the parent Hamiltonian (50). We denote the resulting functional as

$$\bar{H}(g_{e_{00}}, \cdots, g_{e_{NM}}, g_i) = \bar{H}_{df_e}(g_{e_{00}}, \cdots, g_{e_{NM}}) + H_{gk_i}(g_i) + \bar{H}_{df\,gk}(g_{e_{00}}, g_{e_{10}}, g_{e_{01}}, g_i), \tag{105}$$

where $\bar{H}_{df_e}$ and $\bar{H}_{df\,gk}$ correspond to the functionals that one obtains by inserting the replacement (104) into H_{dk_e} and $H_{dk\,gk}$, respectively. Note in particular that, inserting

the truncated expansion $\bar{g}_e$ into $H_{dk\,gk}$, the resulting coupling Hamiltonian $\bar{H}_{df\,gk}$ depends on only three electron moments.

However, because we are expressing the system using $G_{e_{nm}}$ as fluid variables, rather than $g_{e_{nm}}$, we consider, as candidate Hamiltonian, the functional $H(G_{e_{00}}, \cdots, G_{e_{NM}}, g_i)$ given by

$$H(G_{e_{00}}, \cdots, G_{e_{NM}}, g_i) = H_{df_e}(G_{e_{00}}, \cdots, G_{e_{NM}}) + H_{gk_i}(g_i) + H_{df\,gk}(G_{e_{00}}, \cdots, G_{e_{NM}}, g_i), \tag{106}$$

where

$$\begin{aligned} H_{df_e}(G_{e_{00}}, \cdots, G_{e_{NM}}) &= \bar{H}_{df_e}(g_{e_{00}}, \cdots, g_{e_{NM}}), \\ H_{df\,gk}(G_{e_{00}}, \cdots, G_{e_{NM}}, g_i) &= \bar{H}_{df\,gk}(g_{e_{00}}, g_{e_{10}}, g_{e_{01}}, g_i). \end{aligned} \tag{107}$$

In explicit form we have

$$H_{df_e}(G_{e_{00}}, \cdots, G_{e_{NM}}) = \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{M}\int d^3x_e \left(G_{e_{ij}}^2 - 2\zeta_n x_e \frac{G_{e_{ij}}}{u_{(i)}z_{(j)}}\right), \tag{108}$$

$$H_{gk_i}(g_i) = \frac{1}{2}\int d^3x_i d\mathcal{W}_i \left(\tau_i \frac{g_i^2}{\mathcal{F}_{0i}} - 2\tau_i\zeta_n x_i g_i\right), \tag{109}$$

$$\begin{aligned} H_{df\,gk}(G_{e_{00}}, \cdots, G_{e_{NM}}, g_i) &= \frac{1}{2}\int d^3x_i d\mathcal{W}_i\, g_i \left(\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}} v_i \mathcal{J}_{0i} A_\parallel + \tau_i \mu_i \mathcal{J}_{1i} B_\parallel\right) \\ &- \frac{1}{2}\sum_{i=0}^{N}\sum_{j=0}^{M}\int d^3x_e\, G_{e_{ij}} \frac{\chi_{e_{ij}}}{u_{(i)}z_{(j)}}. \end{aligned} \tag{110}$$

Therefore, the functional derivatives read

$$\frac{\delta H}{\delta g_i} = \tau_i \frac{g_i}{\mathcal{F}_{0i}} + \chi_i - \tau_i\zeta_n x_i, \tag{111}$$

$$\frac{\delta H}{\delta G_{e_{nm}}} = G_{e_{nm}} - \frac{\chi_{e_{nm}}}{u_{(n)}z_{(m)}} - \frac{\zeta_n}{u_{(n)}z_{(m)}} x_e. \tag{112}$$

Note that, thanks to the orthogonality of the matrices U and Z, the first term in the expression of H_{df_e} remains a sum of squares when moving from the moments $g_{e_{kl}}$ to the variables $G_{e_{ij}}$.

With regard to the Poisson bracket, we first notice that the ion gyrokinetic equation (100) can be obtained from $\partial_t g_i = \{g_i, H\}_{gk_i}$, where H is the candidate Hamiltonian

(106) and $\{\,,\,\}_{gk_i}$, given in Eq. (55), is the Poisson bracket, referring to ions, of the parent drift/gyrokinetic model. Because the ion gyrokinetic equation is left untouched by replacing the parent Hamiltonian (50) with the candidate Hamiltonian (106), we assume the Poisson bracket of the hybrid drift fluid/gyrokinetic model still has a direct sum structure of the form $\{\,,\,\} = \{\,,\,\}_{gk_i}(g_i) + \{\,,\,\}_{df_e}(G_{e_{00}}, \cdots, G_{e_{NM}})$, where the Poisson bracket $\{\,,\,\}_{df_e}$ generates the electron dynamics, when combined with H given by Eq. (106). Moreover, the form of the equations (99) and the functional derivatives (112) suggest the ansatz

$$\begin{aligned}
&\{F,G\}_{df_e}(G_{e_{00}}, \cdots, G_{e_{NM}}) = \sum_{n=0}^{N}\sum_{m=0}^{M}\{F,G\}_{e_{nm}}(G_{e_{nm}}) \\
&= \sum_{n=0}^{N}\sum_{m=0}^{M}\int d^3x_e \left(a_{nm} G_{e_{nm}} \left[\frac{\delta F}{\delta G_{e_{nm}}}, \frac{\delta G}{\delta G_{e_{nm}}}\right]_e + b_{nm}\frac{\delta F}{\delta G_{e_{nm}}}\frac{\partial}{\partial z_e}\frac{\delta G}{\delta G_{e_{nm}}} \right. \\
&\left. + c_{nm}\frac{\delta F}{\delta G_{e_{nm}}}\frac{\partial}{\partial y_e}\frac{\delta G}{\delta G_{e_{nm}}} \right),
\end{aligned} \tag{113}$$

where a_{nm}, b_{nm} and c_{nm} are coefficients to be determined. The formulation in terms of the $G_{e_{nm}}$ variables, as mentioned in Sec. 3.3, has the advantage of casting the electron moments equations in a diagonal form, in which the evolution equation for each $G_{e_{nm}}$ explicitly depends only on quantities with the same indices n and m. Together with the expression for $\delta H/\delta G_{e_{nm}}$ in Eq. (112), which also only depends on quantities indexed by n and m, this suggests that, in terms of these variables, the Poisson bracket $\{\,,\,\}_{df_e}$ has the direct sum structure (113), given by the sum of $(N+1)(M+1)$ independent Poisson brackets, each of which only depends on $G_{e_{nm}}$. In particular, this greatly simplifies the proof of the Jacobi identity, which follows as a straightforward modification of the proof provided in Appendix A for the gyrokinetic Poisson bracket. The specific ansatz for the individual Poisson brackets $\{F,G\}_{e_{mn}}$ also comes from a rather intuitive extension of the ansatz adopted in Ref. [13] for the case with no perpendicular moments and no background inhomogeneities.

By comparing the expressions $\{G_{e_{nm}}, H\}_{e_{mn}}$, where H is the candidate Hamiltonian (106), with the equations of motion (99), it is possible to see that $\partial_t G_{e_{nm}} = \{G_{e_{nm}}, H\}_{e_{mn}}$

if

$$a_{nm} = \rho_s u_{(n)} z_{(m)}, \qquad b_{nm} = -\frac{\rho_s}{\rho_{th_e}} \lambda_n, \qquad c_{nm} = -\rho_s (\zeta_B \eta_m - \zeta_n). \tag{114}$$

This permits to completely identify the Poisson bracket of the system and confirms that the candidate Hamiltonian is the Hamiltonian of the system.

To summarize, for any choice of the numbers of moments N and M, the hybrid drift fluid/gyrokinetic model (100)-(103) is a Hamiltonian system with Hamiltonian given by the functional (106) and with a noncanonical Poisson bracket given by

$$\{F, G\} = \{F, G\}_{gk_i} + \sum_{n=0}^{N} \sum_{m=0}^{M} \{F, G\}_{e_{nm}}, \tag{115}$$

where $\{F, G\}_{gk_i}$ can be found in Eq. (55) and

$$\{F, G\}_{e_{nm}} = \int d^3 x_e \left(\rho_s u_{(n)} z_{(m)} G_{e_{nm}} \left[\frac{\delta F}{\delta G_{e_{nm}}}, \frac{\delta G}{\delta G_{e_{nm}}} \right]_e - \frac{\rho_s}{\rho_{th_e}} \lambda_n \frac{\delta F}{\delta G_{e_{nm}}} \frac{\partial}{\partial z_e} \frac{\delta G}{\delta G_{e_{nm}}} \right.$$
$$\left. - \rho_s (\zeta_B \eta_m - \zeta_n) \frac{\delta F}{\delta G_{e_{nm}}} \frac{\partial}{\partial y_e} \frac{\delta G}{\delta G_{e_{nm}}} \right). \tag{116}$$

By means of the relation (98) and of its inverse, it is possible to express the Hamiltonian (106) and the Poisson bracket (115) in terms of the original moment variables, if one so wishes.

We finally remark that, in the 2D limit with homogeneous background, each Poisson bracket $\{\,,\,\}_{e_{nm}}$ admits the infinite families of Casimir invariants given by

$$C_{e_{nm}} = \int d^2 x_e \, \mathcal{C}_{nm}(G_{e_{nm}}), \tag{117}$$

where $\mathcal{C}_{nm}$ are arbitrary functions. This property reflects in the fact that $G_{e_{nm}}$ are Lagrangian invariants with respect to the velocity fields $\mathbf{v}_{e_{nm}}$, and generalizes the results obtained in special cases [5, 43, 15, 16, 17, 18].

4. Example: a four-moment drift-fluid reduction

We illustrate the general theory developed in Sec. 3 by considering the simple case $N = 1$, $M = 1$. This amounts to consider the evolution of four moments, corresponding to combinations of the first two moments with respect to the v_e coordinate with the

first two moments with respect to the μ_e coordinate. Because the ion evolution remains unchanged, we focus on the electron dynamics.

The matrix of the moments, for this case, is given by

$$\Gamma_e = \begin{pmatrix} g_{00_e} & g_{01_e} \\ g_{10_e} & g_{11_e} \end{pmatrix}, \tag{118}$$

and the equations of motion (64) read

$$\frac{\partial g_{00_e}}{\partial t} + \rho_s \left([\bar{\mathcal{G}}_{0e}\phi, g_{00_e}]_e - \left[\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}, g_{10_e} \right]_e - [\bar{\mathcal{G}}_{1e}B_\parallel, g_{00_e} - g_{01_e}]_e \right)$$
$$+ \frac{\rho_s}{\rho_{th_e}} \frac{\partial}{\partial z_e} \left(g_{10_e} + \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} \right) + \rho_s \zeta_B \frac{\partial}{\partial y_e} \left(g_{00_e} - g_{01_e} - \bar{\mathcal{G}}_{0e}\phi + 2\bar{\mathcal{G}}_{1e}B_\parallel \right) + \rho_s \zeta_n \frac{\partial}{\partial y_e} \left(\bar{\mathcal{G}}_{0e}\phi - \bar{\mathcal{G}}_{1e}B_\parallel \right) = 0, \tag{119}$$

$$\frac{\partial g_{10_e}}{\partial t} + \rho_s \left([\bar{\mathcal{G}}_{0e}\phi, g_{10_e}]_e - \left[\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}, g_{00_e} \right]_e - [\bar{\mathcal{G}}_{1e}B_\parallel, g_{10_e} - g_{11_e}]_e \right)$$
$$+ \frac{\rho_s}{\rho_{th_e}} \frac{\partial}{\partial z_e} \left(g_{00_e} - \bar{\mathcal{G}}_{0e}\phi + \bar{\mathcal{G}}_{1e}B_\parallel \right) + \rho_s \zeta_B \frac{\partial}{\partial y_e} \left(g_{10_e} - g_{11_e} + \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} \right) - \rho_s \zeta_n \frac{\partial}{\partial y_e} \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} = 0, \tag{120}$$

$$\frac{\partial g_{01_e}}{\partial t} + \rho_s \left([\bar{\mathcal{G}}_{0e}\phi, g_{01_e}]_e - \left[\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}, g_{11_e} \right]_e - [\bar{\mathcal{G}}_{1e}B_\parallel, 3g_{01_e} - g_{00_e}]_e \right)$$
$$+ \frac{\rho_s}{\rho_{th_e}} \frac{\partial}{\partial z_e} g_{11_e} + \rho_s \zeta_B \frac{\partial}{\partial y_e} \left(3g_{01_e} - g_{00_e} + \bar{\mathcal{G}}_{0e}\phi - 4\bar{\mathcal{G}}_{1e}B_\parallel \right) + \rho_s \zeta_n \frac{\partial}{\partial y_e} \bar{\mathcal{G}}_{1e}B_\parallel = 0, \tag{121}$$

$$\frac{\partial g_{11_e}}{\partial t} + \rho_s \left([\bar{\mathcal{G}}_{0e}\phi, g_{11_e}]_e - \left[\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}, g_{01_e} \right]_e - [\bar{\mathcal{G}}_{1e}B_\parallel, 3g_{11_e} - g_{10_e}]_e \right)$$
$$+ \frac{\rho_s}{\rho_{th_e}} \frac{\partial}{\partial z_e} \left(g_{01_e} - \bar{\mathcal{G}}_{1e}B_\parallel \right) + \rho_s \zeta_B \frac{\partial}{\partial y_e} \left(3g_{11_e} - g_{10_e} - \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} \right) = 0, \tag{122}$$

or, in a physically more perspicuous form,

$$\frac{d_e N_e}{dt} + \nabla_{\parallel e} U_{\parallel e} + \nabla_{B_e}(N_e + T_{\perp e}) - \rho_s \zeta_B \frac{\partial}{\partial y_e}(\bar{\mathcal{G}}_{0e}\phi - 2\bar{\mathcal{G}}_{1e}B_\parallel) + \rho_s \zeta_n \frac{\partial}{\partial y_e}(\bar{\mathcal{G}}_{0e}\phi - \bar{\mathcal{G}}_{1e}B_\parallel) = 0, \tag{123}$$

$$\frac{d_e}{dt} \left(U_{\parallel e} - \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} \right) + \nabla_{\parallel e} N_e + \nabla_{B_e} \left(U_{\parallel e} - \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} + q_{\perp e} \right) - \frac{\rho_s}{\rho_{th_e}} \frac{\partial}{\partial z_e} (\bar{\mathcal{G}}_{0e}\phi - \bar{\mathcal{G}}_{1e}B_\parallel)$$
$$+ \rho_s (\zeta_B - \zeta_n) \frac{\partial}{\partial y_e} \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} = 0, \tag{124}$$

$$\frac{d_e T_{\perp e}}{dt} + \nabla_{\parallel e} q_{\perp e} + \nabla_{B_e}(N_e + 3T_{\perp e}) - \rho_s \zeta_B \frac{\partial}{\partial y_e}(\bar{\mathcal{G}}_{0e}\phi - 4\bar{\mathcal{G}}_{1e}B_\parallel) - \rho_s \zeta_n \frac{\partial \bar{\mathcal{G}}_{1e}B_\parallel}{\partial y_e} = 0, \tag{125}$$

$$\frac{d_e q_{\perp e}}{dt} + \nabla_{\parallel e} T_{\perp e} + \nabla_{B_e} \left(U_{\parallel e} - \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} + 3q_{\perp e} \right) + \frac{\rho_s}{\rho_{th_e}} \frac{\partial \bar{\mathcal{G}}_{1e}B_\parallel}{\partial z_e} + \rho_s \zeta_B \frac{\partial}{\partial y_e} \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}} = 0. \tag{126}$$

In Eqs. (123)-(126) we expressed the moments in terms of the physical quantities introduced in Eqs. (34)-(36) and we also introduced the operators

$$\begin{aligned}
\frac{d_e}{dt} &= \frac{\partial}{\partial t} + \rho_s[\bar{\mathcal{G}}_{0e}\phi, \cdot]_e, \\
\nabla_{\parallel e} &= -\rho_s \left[\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}, \cdot\right]_e + \frac{\rho_s}{\rho_{th_e}}\frac{\partial}{\partial z_e}, \\
\nabla_{B_e} &= -\rho_s[\bar{\mathcal{G}}_{1e}B_\parallel, \cdot]_e + \rho_s\zeta_B\frac{\partial}{\partial y_e}
\end{aligned} \tag{127}$$

representing the advective derivative with respect to the $\mathbf{v_{E\times B}}$, the leading order gradient along the magnetic field and the advection with respect to nonlinear and linear grad B velocities, respectively.

The Jacobi matrices are

$$W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}, \tag{128}$$

and the corresponding eigenvalues are

$$\lambda_0 = -1, \qquad \lambda_1 = 1, \tag{129}$$

$$\eta_0 = 2 + \sqrt{2}, \qquad \eta_1 = 2 - \sqrt{2}. \tag{130}$$

The orthogonal matrices read

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad Z = \begin{pmatrix} \frac{\sqrt{2}-1}{2}\sqrt{2+\sqrt{2}} & \frac{\sqrt{2}+1}{2}\sqrt{2-\sqrt{2}} \\ -\frac{\sqrt{2+\sqrt{2}}}{2} & \frac{\sqrt{2-\sqrt{2}}}{2} \end{pmatrix}, \tag{131}$$

and the normalization constants are

$$u_{(0)} = \sqrt{2}, \qquad u_{(1)} = \sqrt{2}, \tag{132}$$

$$z_{(0)} = \frac{2}{\sqrt{2+\sqrt{2}}(\sqrt{2}-1)}, \qquad z_{(1)} = \frac{2}{\sqrt{2-\sqrt{2}}(\sqrt{2}+1)}. \tag{133}$$

From Eq. (81) one obtains the expression for the new variables

$$G_{e_{00}} = \frac{\sqrt{2+\sqrt{2}}}{2\sqrt{2}}\left((\sqrt{2}-1)N_e - (\sqrt{2}-1)\left(U_{\parallel e} - \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right) + T_{\perp e} - q_{\perp e}\right), \tag{134}$$

$$G_{e_{10}} = \frac{\sqrt{2+\sqrt{2}}}{2\sqrt{2}}\left((\sqrt{2}-1)N_e + (\sqrt{2}-1)\left(U_{\parallel e} - \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right) + T_{\perp e} + q_{\perp e}\right), \tag{135}$$

$$G_{e_{01}} = \frac{\sqrt{2-\sqrt{2}}}{2\sqrt{2}}\left((\sqrt{2}+1)N_e - (\sqrt{2}+1)\left(U_{\parallel e} - \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right) - T_{\perp e} + q_{\perp e}\right), \tag{136}$$

$$G_{e_{11}} = \frac{\sqrt{2-\sqrt{2}}}{2\sqrt{2}}\left((\sqrt{2}+1)N_e + (\sqrt{2}+1)\left(U_{\parallel e} - \frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right) - T_{\perp e} - q_{\perp e}\right). \tag{137}$$

In terms of such variables, as described in Sec. 3.3, Eqs. (123)-(126) can be cast in the simple form

$$\frac{\partial G_{e_{nm}}}{\partial t} + \rho_s \left[\chi_{e_{nm}}, G_{e_{nm}}\right]_e + \rho_s \left(\frac{\lambda_n}{\rho_{th_e}}\frac{\partial}{\partial z_e} + \zeta_B \eta_m \frac{\partial}{\partial y_e}\right)\left(G_{e_{nm}} - \frac{\chi_{e_{nm}}}{u_{(n)} z_{(m)}}\right) + \rho_s \frac{\zeta_n}{u_{(n)} z_{(m)}}\frac{\partial \chi_{e_{nm}}}{\partial y_e} = 0, \tag{138}$$

with $n = 0, 1$ and $m = 0, 1$. The expression for the generalized stream functions $\chi_{e_{nm}}$ is provided in Eq. (89).

The Hamiltonian structure of the hybrid system consisting of Eqs. (138), (100) and (101)-(103) (with $N = 1, M = 1$), is then given by the Hamiltonian

$$H(G_{e_{00}}, G_{e_{10}}, G_{e_{01}}, G_{e_{11}}, g_i) = \frac{1}{2}\sum_{i=0}^{1}\sum_{j=0}^{1}\int d^3x_e \left(G_{e_{ij}}^2 - 2\zeta_n x_e \frac{G_{e_{ij}}}{u_{(i)} z_{(j)}}\right) + \frac{1}{2}\int d^3x_i d\mathcal{W}_i \left(\tau_i \frac{g_i^2}{\mathcal{F}_{0i}} - 2\tau_i \zeta_n x_i g_i\right) + \frac{1}{2}\int d^3x_i d\mathcal{W}_i \, g_i \left(\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}} v_i \mathcal{J}_{0i} A_\parallel + \tau_i \mu_i \mathcal{J}_{1i} B_\parallel\right) - \frac{1}{2}\sum_{i=0}^{1}\sum_{j=0}^{1}\int d^3x_e \, G_{e_{ij}} \frac{\chi_{e_{ij}}}{u_{(i)} z_{(j)}} \tag{139}$$

and the Poisson bracket

$$\{F, G\} = -\rho_s \int d^3x_i d\mathcal{W}_i \left(g_i \left[\frac{\delta F}{\delta g_i}, \frac{\delta G}{\delta g_i}\right]_i + \frac{1}{\rho_{th_i}} v_i \mathcal{F}_{0i} \frac{\delta F}{\delta g_i}\frac{\partial}{\partial z_i}\frac{\delta G}{\delta g_i} - (\mu_i \zeta_B - \zeta_n)\mathcal{F}_{0i}\frac{\delta F}{\delta g_i}\frac{\partial}{\partial y_i}\frac{\delta G}{\delta g_i}\right) + \sum_{n=0}^{1}\sum_{m=0}^{1}\int d^3x_e \left(\rho_s u_{(n)} z_{(m)} G_{e_{nm}} \left[\frac{\delta F}{\delta G_{e_{nm}}}, \frac{\delta G}{\delta G_{e_{nm}}}\right]_e - \frac{\rho_s}{\rho_{th_e}}\lambda_n \frac{\delta F}{\delta G_{e_{nm}}}\frac{\partial}{\partial z_e}\frac{\delta G}{\delta G_{e_{nm}}} - \rho_s(\zeta_B \eta_m - \zeta_n)\frac{\delta F}{\delta G_{e_{nm}}}\frac{\partial}{\partial y_e}\frac{\delta G}{\delta G_{e_{nm}}}\right). \tag{140}$$

Using Eqs. (139) and (140) in the general expression (19) one indeed retrieves the evolution equations of the hybrid model.

All the considerations of Sec. 3.4 can of course be applied to this case. So, for instance, the fields $G_{e_{mn}}$ are proportional to the truncation of g_e, evaluated at $(v_e, \mu_e) = (\lambda_n, \eta_m)$. The generalized stream functions $\chi_{e_{nm}}$ correspond to linear combinations of stream functions for $\mathbf{E} \times \mathbf{B}$ drift, velocity along the magnetic field and grad B drift, with relative amplitudes depending on the coefficients λ_n and η_m. In the 2D limit without background gradients, the electron dynamics reduced to the advection of $G_{e_{mn}}$ by the corresponding generalized velocity fields $\mathbf{v}_{e_{nm}}$.

This example shows how one can build a Hamiltonian drift-fluid model for the electrons (coupled with gyrokinetic ions) describing the evolution of gyrocenter density, parallel momentum, perpendicular temperature and heat flux fluctuations, for a finite plasma β. The resulting electron drift-fluid dynamics generalizes that described by the Hamiltonian two-field model of Refs. [5, 43], by adding perpendicular temperature and heat flux dynamics. To the best of our knowledge, this model was not present in the literature before. By considering the case $N = 2$, $M = 1$, one can easily add parallel temperature fluctuations, thus providing a Hamiltonian model describing anisotropic electron temperature fluctuations by a fluid approach. However, if parallel temperature fluctuations are added, our procedure requires the inclusion also of another moment, namely g_{21_e}. Indeed, in order to be able to initiate the procedure with the matrix Γ_e, one is forced to consider a total number of $(N + 1)(M + 1)$ moments, according to Eq. (63). This could explain why, as mentioned in Sec. 1, although so far some Hamiltonian reduced fluid models evolving temperature fluctuations were derived [18, 15, 20, 22], none of such models accounted also for perpendicular temperature fluctuations, if not for the case of trivial or not systematic closures. The present procedure indeed reveals that, in order to preserve in a systematic way the Hamiltonian structure, while adding moments to evolve, the condition of considering the $(N + 1)(M + 1)$ moments dictated by Eq. (63) has to be respected.

We also stress that the presence of a Hamiltonian structure automatically implies the existence of a total conserved energy, but the converse is not true. Requiring a Hamiltonian structure is thus more demanding than requiring energy conservation, which makes Hamiltonian models more difficult to build.

5. Hamiltonian gyrofluid reduction

In the presence of FLR effects, the general procedure described in Sec. 3 does not work. Indeed, the dependence on μ_s of the gyroaverage operators $\mathcal{J}_{0s}$ and $\mathcal{J}_{1s}$, prevents the presence of the two commutative transformations that allow to diagonalize the system.

Nevertheless, in a simplified setting, an analogous diagonalization procedure is still possible, also in the presence of FLR effects. When applied to ions, the simplified setting consists of considering, at most, the first two ion moments along the perpendicular direction (i.e. imposing $M \leq 1$), taking the $\beta_e \to 0$ limit and approximating the gyroaverage operators arising when taking moments of the gyrokinetic equations. Electrons are still treated in the limit $\rho_{th_e} \to 0$. In particular, we assume

$$\rho_{th_e} \sim \left(\frac{\beta_e}{2}\right)^{1/2}, \qquad \text{for } \beta_e \to 0. \tag{141}$$

In this $\beta_e \to 0$ limit, we also rescale the z variable in the following way

$$z = \sqrt{\frac{\beta_e}{2}}\frac{\hat{z}}{L_\parallel}. \tag{142}$$

Moreover, although not necessary, we retain background magnetic inhomogeneity, even though the equilibrium balance relation (13) would give $\zeta_B \to 0$ as $\beta_e \to 0$.

Inspection of the perpendicular Ampère's law (4) in the parent model, yields that

$$B_\parallel = \mathcal{O}(\beta_e). \tag{143}$$

Taking into account the assumption (141) and neglecting terms of order β_e, while retaining contributions of order $\beta_e^{1/2}$, the parent gyrokinetic model (1)-(4) reduces to

$$\frac{\partial g_e}{\partial t} + \rho_s \left[\bar{\mathcal{G}}_{0e}\phi - v_e\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}, g_e\right]_e + \sqrt{\frac{\beta_e}{2}}\frac{\rho_s}{\rho_{th_e}}v_e\frac{\partial}{\partial z_e}\left(g_e - \mathcal{F}_{0e}\left(\bar{\mathcal{G}}_{0e}\phi - v_e\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right)\right)$$
$$+ \rho_s\mu_e\zeta_B\frac{\partial g_e}{\partial y_e} - \rho_s\mathcal{F}_{0e}(\mu_e\zeta_B - \zeta_n)\frac{\partial}{\partial y_e}\left(\bar{\mathcal{G}}_{0e}\phi - v_e\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right) = 0, \tag{144}$$

$$\frac{\partial g_i}{\partial t} + \rho_s \left[\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}}v_i\mathcal{J}_{0i}A_\parallel, g_i\right]_i + \sqrt{\frac{\beta_e}{2}}\frac{\rho_s}{\rho_{th_i}}v_i\frac{\partial}{\partial z_i}\left(\tau_i g_i + \mathcal{F}_{0i}\left(\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}}v_i\mathcal{J}_{0i}A_\parallel\right)\right)$$
$$- \rho_s\tau_i\mu_i\zeta_B\frac{\partial g_i}{\partial y_i} - \rho_s\mathcal{F}_{0i}(\mu_i\zeta_B - \zeta_n)\frac{\partial}{\partial y_i}\left(\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}}v_i\mathcal{J}_{0i}A_\parallel\right) = 0, \tag{145}$$

$$\int d\mathcal{W}_i\, \mathcal{J}_{0i}g_i - \bar{\mathcal{G}}_{0e}\int d\mathcal{W}_e\, g_e = (1-\Gamma_{0i})\frac{\phi}{\tau_i} + (1-\bar{\mathcal{G}}_{0e}^2)\phi, \tag{146}$$

$$\Delta_\perp A_\parallel - \frac{\beta_e}{2}\frac{\tau_i}{\rho_{th_i}^2}\Gamma_{0i}A_\parallel - \frac{\beta_e}{2}\frac{\bar{\mathcal{G}}_{0e}^2 A_\parallel}{\rho_{th_e}^2} = -\frac{\beta_e}{2}\frac{\tau_i}{\rho_{th_i}}\int d\mathcal{W}_i\, v_i\mathcal{J}_{0i}g_i + \frac{\beta_e}{2}\frac{\bar{\mathcal{G}}_{0e}}{\rho_{th_e}}\int d\mathcal{W}_e\, v_e g_e. \tag{147}$$

The contributions of $B_\parallel$ are no longer present in the system and parallel Ampère's law (147) implies, due to assumption (141), that

$$A_\parallel = \mathcal{O}(\beta_e^{1/2}). \tag{148}$$

We also notice that the parallel current is mainly provided by the electron parallel velocity, as the parallel ion velocity is smaller by a factor of order $\beta_e^{1/2}$. In the electron drift-kinetic equation (144) all terms are of order 1, whereas in the ion gyrokinetic equation (145) higher order contributions of order $\beta_e^{1/2}$, associated with gradients along the magnetic field, are also present. Neglecting such higher order contributions and background inhomogeneities leads to the electrostatic limit considered in Ref. [44].

The Hamiltonian of the model is given by

$$H(g_e, g_i) = H_{dk_e}(g_e) + H_{gk_i}(g_i) + H_{dk\,gk}(g_e, g_i), \tag{149}$$

where

$$\begin{aligned} H_{dk\,gk}(g_e, g_i) = &\frac{1}{2}\int d^3x_i d\mathcal{W}_i\, g_i \left(\mathcal{J}_{0i}\phi - \frac{\tau_i}{\rho_{th_i}} v_i \mathcal{J}_{0i} A_\parallel\right) \\ &- \frac{1}{2}\int d^3x_e d\mathcal{W}_e\, g_e \left(\bar{\mathcal{G}}_{0e}\phi - v_e \frac{\bar{\mathcal{G}}_{0e} A_\parallel}{\rho_{th_e}}\right), \end{aligned} \tag{150}$$

no longer contains the contributions of $B_\parallel$. The Poisson bracket is still given by Eq. (54).

5.1. Hamiltonian gyrofluid reduction

As above anticipated, we carry out the Hamiltonian gyrofluid reduction on the ion species. We assume that the evolution of the electron species remains governed by the drift-kinetic equation (144), so that the final result of the reduction will be a Hamiltonian hybrid drift-kinetic/gyrofluid model. However, one could also obtain a Hamiltonian drift-fluid/gyrofluid model by applying to the electron species, the Hamiltonian reduction described in Sec. 3, although in the low-β_e limit, if one wants to maintain consistency with the hypotheses for the ion gyrofluid equations.

Focusing on the ion species, we consider the truncated expansion

$$\bar{g}_i(x_i, y_i, z_i, v_i, \mu_i, t) = \mathcal{F}_{0i}(v_i, \mu_i) \sum_{h=0}^{N} \sum_{j=0}^{1} \frac{H_i(v_i)}{\sqrt{i!}} L_j(\mu_i) g_{hj_i}(x_i, y_i, z_i, t). \tag{151}$$

As above anticipated, we only consider the first two orders in the moments with respect to μ_i. The number of moments with respect to v_i, on the other hand, remains arbitrary and is given by $N+1$.

We introduce the matrix Γ_i with elements

$$\Gamma_{i_{hj}} = g_{hj_i}, \qquad h = 0, \cdots N, \quad j = 0, 1, \tag{152}$$

and consider the evolution equations for the ion moments

$$\begin{aligned}
&\frac{\partial \Gamma^T_{i_{jh}}}{\partial t} + \rho_s \left([\mathcal{G}_{10_i}\phi, \Gamma^T_{i_{jh}}]_i + [\mathcal{G}_{11_i}\phi, S_{jl}\Gamma^T_{i_{lh}}]_i - \tau_i \left[\frac{\mathcal{G}_{10_i} A_\parallel}{\rho_{th_i}}, \Gamma^T_{i_{jl}} W^T_{lh} \right]_i - \tau_i \left[\frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}}, S_{jk}\Gamma^T_{i_{kl}} W^T_{lh} \right]_i \right) \\
&+ \sqrt{\frac{\beta_e}{2}} \tau_i \frac{\rho_s}{\rho_{th_i}} \frac{\partial}{\partial z_i} \Gamma^T_{i_{jl}} W^T_{lh} + \sqrt{\frac{\beta_e}{2}} \frac{\rho_s}{\rho_{th_i}} \frac{\partial}{\partial z_i} \left(\mathcal{G}_{1j_i}\phi\delta_{1h} - \tau_i \frac{\mathcal{G}_{1j_i} A_\parallel}{\rho_{th_i}} (\sqrt{2}\delta_{2h} + \delta_{0h}) \right) \\
&- \rho_s \tau_i \zeta_B \frac{\partial}{\partial y_i} V_{jl} \Gamma^T_{i_{lh}} - \rho_s \frac{\partial}{\partial y_i} \left((\zeta_B V_{jl} - \zeta_n \delta_{jl}) \left(\mathcal{G}_{1l_i}\phi\delta_{0h} - \frac{\tau_i}{\rho_{th_i}} \mathcal{G}_{1l_i} A_\parallel \delta_{1h} \right) \right).
\end{aligned} \tag{153}$$

In Eqs. (153), FLR terms have been treated in an approximated way. In particular, the FLR contributions arising from the electromagnetic nonlinearity $[\mathcal{J}_{0i}\phi, \mathcal{J}_{0i}A_\parallel]_i$ in Eq. (1), when calculated exactly [9], yield additional contributions that we omitted, because we approximated the exact expansion (37) with $J_0(\sqrt{2b_s\mu_s}) \approx \sum_{n=0}^{1} \mathcal{G}_{1n_s}(b_s) L_n(\mu_s)$. The adopted approximation results in the gyrofluid model depending on only two gyroaverage operators, namely $\mathcal{G}_{10_i}$ and $\mathcal{G}_{11_i}$, defined in Eq. (38).

In Eq. (153) we also introduced the matrix

$$S = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}. \tag{154}$$

Insertion of the expression (151) into Eqs. (146) and (147), on the other hand, leads to the following quasi-neutrality relation and parallel Ampère's law:

$$\mathcal{G}_{10_i} g_{00_i} + \mathcal{G}_{11_i} g_{01_i} - \bar{\mathcal{G}}_{0e} \int d\mathcal{W}_e \, g_e = (1 - \Gamma_{0i}) \frac{\phi}{\tau_i} + (1 - \bar{\mathcal{G}}^2_{0e})\phi, \tag{155}$$

$$\Delta_\perp A_\parallel - \frac{\beta_e}{2} \frac{\tau_i}{\rho^2_{th_i}} \Gamma_{0i} A_\parallel - \frac{\beta_e}{2} \frac{\bar{\mathcal{G}}^2_{0e} A_\parallel}{\rho^2_{th_e}} = -\frac{\beta_e}{2} \frac{\tau_i}{\rho_{th_i}} (\mathcal{G}_{10_i} g_{10_i} + \mathcal{G}_{11_i} g_{11_i}) + \frac{\beta_e}{2} \frac{\bar{\mathcal{G}}_{0e}}{\rho_{th_e}} \int d\mathcal{W}_e \, v_e g_e. \tag{156}$$

Consequently, the electromagnetic potentials are determined from the dynamical variables by relations of the form

$$\phi = \mathcal{L}_{\phi_{i00}} g_{00_i} + \mathcal{L}_{\phi_{i01}} g_{01_i} + \mathcal{L}_{\phi_e} g_e, \tag{157}$$

$$A_\parallel = \mathcal{L}_{A_{i10}} g_{10_i} + \mathcal{L}_{A_{i11}} g_{11_i} + \mathcal{L}_{A_e} g_e. \tag{158}$$

Note that, compared to Eqs. (60)-(61), in the gyrofluid case, FLR effects make the field g_{01_i} (or, equivalently, ion gyrocenter perpendicular temperature fluctuations) intervene in the quasi-neutrality relation (157). Similarly, gyrocenter perpendicular heat flux fluctuations (embedded in g_{11_i}) affect parallel Ampère's law (158) in the gyrofluid case.

5.2. Changes of variables

It is possible to cast Eq. (153) in a simpler form by a procedure totally analogous to the one described in Sec. 3.3. In particular, we notice that, for $M = 1$, the relation

$$Z^T S Z = \mathrm{diag}(1-\eta_0, 1-\eta_1) = \begin{pmatrix} -1-\sqrt{2} & 0 \\ 0 & -1+\sqrt{2} \end{pmatrix} \tag{159}$$

holds. Therefore, the orthogonal matrix Z diagonalizes also S (which commutes with V).

Analogously to the procedure followed for the drift-fluid case, we introduce

$$J_i = Z^T \Gamma_i^T, \tag{160}$$

$$G_i = U^T J_i^T. \tag{161}$$

In terms of the variables $G_{i_{nm}}$, the gyrofluid equations (153) read

$$\begin{aligned} &\frac{\partial G_{i_{nm}}}{\partial t} + \rho_s \left([(\mathcal{G}_{10_i} + (1-\eta_m)\mathcal{G}_{11_i})\phi, G_{i_{nm}}]_i - \frac{\tau_i}{\rho_{th_i}} \lambda_n [(\mathcal{G}_{10_i} + (1-\eta_m)\mathcal{G}_{11_i}) A_\parallel, G_{i_{nm}}]_i \right) \\ &+ \sqrt{\frac{\beta_e}{2}} \tau_i \frac{\rho_s}{\rho_{th_i}} \lambda_n \frac{\partial G_{i_{nm}}}{\partial z_i} + \sqrt{\frac{\beta_e}{2}} \frac{\rho_s}{\rho_{th_i}} \frac{\lambda_n}{u_{(n)} z_{(m)}} \frac{\partial}{\partial z_i} \left((\mathcal{G}_{10_i} + (1-\eta_m)\mathcal{G}_{11_i})\phi \right. \\ &\left. - \frac{\tau_i}{\rho_{th_i}} \lambda_n (\mathcal{G}_{10_i} + (1-\eta_m)\mathcal{G}_{11_i}) A_\parallel \right) - \rho_s \tau_i \zeta_B \eta_m \frac{\partial G_{i_{nm}}}{\partial y_i} \\ &- \rho_s \frac{\eta_m \zeta_B - \zeta_n}{u_{(n)} z_{(m)}} \frac{\partial}{\partial y_i} \left((\mathcal{G}_{10_i} + (1-\eta_m)\mathcal{G}_{11_i})\phi - \frac{\tau_i}{\rho_{th_i}} \lambda_n (\mathcal{G}_{10_i} + (1-\eta_m)\mathcal{G}_{11_i}) A_\parallel \right), \end{aligned} \tag{162}$$

for $n = 0, \cdots, N$ and $m = 0, 1$.

In a compact form, Eq. (162) reads

$$\frac{\partial G_{i_{nm}}}{\partial t} + \rho_s \left[\chi_{i_{nm}}, G_{i_{nm}}\right]_i + \rho_s \left(\frac{\lambda_n}{\rho_{th_i}}\sqrt{\frac{\beta_e}{2}}\frac{\partial}{\partial z_i} - \zeta_B \eta_m \frac{\partial}{\partial y_i}\right)\left(\tau_i G_{i_{nm}} + \frac{\chi_{i_{nm}}}{u_{(n)} z_{(m)}}\right) + \rho_s \frac{\zeta_n}{u_{(n)} z_{(m)}}\frac{\partial \chi_{i_{nm}}}{\partial y_i} = 0, \tag{163}$$

where the ion gyrofluid potentials $\chi_{i_{nm}}$ are defined by

$$\chi_{i_{nm}} = (\mathcal{G}_{10_i} + (1-\eta_m)\mathcal{G}_{11_i})\phi - \frac{\tau_i}{\rho_{th_i}}\lambda_n(\mathcal{G}_{10_i} + (1-\eta_m)\mathcal{G}_{11_i})A_\parallel. \tag{164}$$

Equation (163) has a form analogous to that of its electron drift-fluid counterpart (88). The ion gyrofluid potentials do not contain the contribution due to the nonlinear grad B drift but, on the other hand, they include the gyroaverage operators $\mathcal{G}_{10_i}$ and $\mathcal{G}_{11_i}$, valid for finite b_i. We also point out that alternative expressions for the gyroaverage operators (38) can be used. For instance, in Refs. [45, 27, 34], approximated forms for such operators are given, which provide a better agreement with linear theory for a finite number of moments in the perpendicular direction is retained.

Because of the formal analogy between the two equations, the considerations of Sec. 3.4 on Eq. (88) can be transferred, with the appropriate replacements, to Eq. (163). We only remark that the relative weights, $1-\eta_0$ and $1-\eta_1$, between the zero and first order gyroaveraged electromagnetic potentials, have opposite signs. Therefore we expect such higher order gyroaveraged potentials to modify in opposite ways the advection of $G_{i_{nm}}$ due to the zeroth order potentials $\mathcal{G}_{10_i}\phi - (\tau_i/\rho_{th_i})\lambda_n \mathcal{G}_{10_i} A_\parallel$.

5.3. Hamiltonian structure of the hybrid drift-kinetic/gyrofluid model

The Hamiltonian structure of the hybrid model under consideration follows in a straightforward way from the procedure adopted in Sec. 3.5. In particular, the Hamiltonian functional is obtained upon inserting the truncated expansion (151) into the parent Hamiltonian (149) and expressing the result using the variables $G_{i_{nm}}$ instead of the moments g_{nm_i}. The Poisson bracket is the direct sum of the parent Poisson bracket $\{\,,\,\}_{gk_e}$ for the electrons, with the Poisson bracket for the ions which is totally

analogous to bracket found for the drift-fluid electrons in Sec. 3.5, as both are intuited from analogous equations for the transformed variables. Therefore, we present the final result stating that, for arbitrary N, the hybrid drift-kinetic/gyrofluid model given by

$$\frac{\partial g_e}{\partial t} + \rho_s \left[\bar{\mathcal{G}}_{0e}\phi - v_e\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}, g_e\right]_e + \sqrt{\frac{\beta_e}{2}}\frac{\rho_s}{\rho_{th_e}}v_e\frac{\partial}{\partial z_e}\left(g_e - \mathcal{F}_{0e}\left(\bar{\mathcal{G}}_{0e}\phi - v_e\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right)\right)$$
$$+ \rho_s\mu_e\zeta_B\frac{\partial g_e}{\partial y_e} - \rho_s\mathcal{F}_{0e}(\mu_e\zeta_B - \zeta_n)\frac{\partial}{\partial y_e}\left(\bar{\mathcal{G}}_{0e}\phi - v_e\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right) = 0, \tag{165}$$

$$\frac{\partial G_{i_{nm}}}{\partial t} + \rho_s\left[\chi_{i_{nm}}, G_{i_{nm}}\right]_i + \rho_s\left(\frac{\lambda_n}{\rho_{th_i}}\sqrt{\frac{\beta_e}{2}}\frac{\partial}{\partial z_i} - \zeta_B\eta_m\frac{\partial}{\partial y_i}\right)\left(\tau_i G_{i_{nm}} + \frac{\chi_{i_{nm}}}{u_{(n)}z_{(m)}}\right)$$
$$+ \rho_s\frac{\zeta_n}{u_{(n)}z_{(m)}}\frac{\partial\chi_{i_{nm}}}{\partial y_i} = 0, \tag{166}$$

$$\sum_{l=0}^{N}\sum_{k=0}^{1}\frac{\mathcal{G}_{10_i}G_{i_{lk}} + (1-\eta_k)\mathcal{G}_{11_i}G_{i_{lk}}}{u_{(l)}z_{(k)}} - \bar{\mathcal{G}}_{0e}\int d\mathcal{W}_e\, g_e = (1-\Gamma_{0i})\frac{\phi}{\tau_i} + (1-\bar{\mathcal{G}}_{0e}^2)\phi, \tag{167}$$

$$\Delta_\perp A_\parallel - \frac{\beta_e}{2}\frac{\tau_i}{\rho_{th_i}^2}\Gamma_{0i}A_\parallel - \frac{\beta_e}{2}\frac{\bar{\mathcal{G}}_{0e}^2 A_\parallel}{\rho_{th_e}^2}$$
$$= -\frac{\beta_e}{2}\frac{\tau_i}{\rho_{th_i}}\sum_{l=0}^{N}\sum_{k=0}^{1}\lambda_l\frac{\mathcal{G}_{10_i}G_{i_{lk}} + (1-\eta_k)\mathcal{G}_{11_i}G_{i_{lk}}}{u_{(l)}z_{(k)}} + \frac{\beta_e}{2}\frac{\bar{\mathcal{G}}_{0e}}{\rho_{th_e}}\int d\mathcal{W}_e\, v_e g_e, \tag{168}$$

with $n = 0, \cdots, N$ and $m = 0, 1$, is a Hamiltonian system with Hamiltonian given by

$$H(G_{i_{00}}, \cdots, G_{i_{N1}}, g_e) = H_{gf_i}(G_{i_{00}}, \cdots, G_{i_{N1}}) + H_{dk_e}(g_e) + H_{dk\,gf}(G_{i_{00}}, \cdots, G_{i_{N1}}, g_e), \tag{169}$$

where

$$H_{gf_i}(G_{i_{00}}, \cdots, G_{i_{N1}}) = \frac{1}{2}\sum_{h=0}^{N}\sum_{j=0}^{1}\int d^3x_i\,\left(\tau_i G_{i_{hj}}^2 - 2\zeta_n x_i\frac{G_{i_{hj}}}{u_{(h)}z_{(j)}}\right), \tag{170}$$

$$H_{dk_e}(g_e) = \frac{1}{2}\int d^3x_e d\mathcal{W}_e\,\left(\frac{g_e^2}{\mathcal{F}_{0e}} - 2\zeta_n x_e g_e\right), \tag{171}$$

$$H_{dk\,gf}(G_{i_{00}}, \cdots, G_{i_{N1}}, g_e) = \frac{1}{2}\sum_{h=0}^{N}\sum_{j=0}^{1}\int d^3x_i\, G_{i_{hj}}\frac{\chi_{i_{hj}}}{u_{(h)}z_{(j)}} - \frac{1}{2}\int d^3x_e d\mathcal{W}_e g_e\left(\bar{\mathcal{G}}_{0e}\phi - v_e\frac{\bar{\mathcal{G}}_{0e}A_\parallel}{\rho_{th_e}}\right)$$

and Poisson bracket corresponding to

$$\{F, G\} = \{F, G\}_{gk_e} + \sum_{n=0}^{N}\sum_{m=0}^{1}\{F, G\}_{i_{nm}}, \tag{172}$$

where $\{F,G\}_{gk_e}$ is given in Eq. (55) and

$$\{F,G\}_{i_{nm}} = -\int d^3x_i \left(\rho_s u_{(n)} z_{(m)} G_{i_{nm}} \left[\frac{\delta F}{\delta G_{i_{nm}}}, \frac{\delta G}{\delta G_{i_{nm}}} \right]_i + \frac{\rho_s}{\rho_{th_i}} \lambda_n \frac{\delta F}{\delta G_{i_{nm}}} \frac{\partial}{\partial z_i} \frac{\delta G}{\delta G_{i_{nm}}} \right.$$
$$\left. -\rho_s (\zeta_B \eta_m - \zeta_n) \frac{\delta F}{\delta G_{i_{nm}}} \frac{\partial}{\partial y_i} \frac{\delta G}{\delta G_{i_{nm}}} \right). \tag{173}$$

The procedure adopted to identify the Hamiltonian structure of the ion gyrofluid equations can be used to generalize low-β Hamiltonian gyrofluid models present in the literature [16, 17, 18] with the inclusion of moments also in the perpendicular direction. For instance, in the case $N = 1$, the ion gyrofluid evolution equations read

$$\frac{d_i N_i}{dt} + \tau_i \nabla_{\parallel i} U_{\parallel i} - \rho_s [\mathcal{G}_{11_i}\phi, T_{\perp i}]_i + \rho_s \tau_i \left[\frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}}, q_{\perp i} \right]_i$$
$$- \rho_s \tau_i \zeta_B \frac{\partial}{\partial y_i} (N_i + T_{\perp i} + \mathcal{G}_{10_i}\phi - \mathcal{G}_{11_i}\phi) + \rho_s \zeta_n \frac{\partial \mathcal{G}_{10_i}\phi}{\partial y_i} = 0, \tag{174}$$
$$\frac{d_i}{dt} \left(U_{\parallel i} + \frac{\mathcal{G}_{10_i} A_\parallel}{\rho_{th_i}} \right) + \tau_i \nabla_{\parallel i} N_i + \rho_s \tau_i \left[\frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}}, T_{\perp i} \right]_i - \rho_s \left[\mathcal{G}_{11_i}\phi, q_{\perp i} - \frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}} \right]_i$$
$$+ \sqrt{\frac{\beta_e}{2}} \frac{\rho_s}{\rho_{th_i}} \frac{\partial \mathcal{G}_{10_i}\phi}{\partial z_i} - \rho_s \tau_i \zeta_B \frac{\partial}{\partial y_i} \left(U_{\parallel i} + q_{\perp i} \right) - \rho_s \zeta_n \tau_i \frac{\partial}{\partial y_i} \frac{\mathcal{G}_{10_i} A_\parallel}{\rho_{th_i}} = 0, \tag{175}$$
$$\frac{d_i T_{\perp i}}{dt} + \tau_i \nabla_{\parallel i} q_{\perp i} - \rho_s [\mathcal{G}_{11_i}\phi, N_i + 2T_{\perp i}]_i + \rho_s \tau_i \left[\frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}}, U_{\parallel i} + 2q_{\perp i} \right]_i$$
$$- \rho_s \zeta_B \frac{\partial}{\partial y_i} \left(\tau_i (N_i + 3T_{\perp i}) + \mathcal{G}_{10_i}\phi - 3\mathcal{G}_{11_i}\phi \right) - \rho_s \zeta_n \frac{\partial \mathcal{G}_{11_i}\phi}{\partial y_i} = 0, \tag{176}$$
$$\frac{d_i}{dt} \left(q_{\perp i} - \frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}} \right) + \tau_i \nabla_{\parallel i} T_{\perp i} - \rho_s \left[\mathcal{G}_{11_i}\phi, U_{\parallel i} + 2q_{\perp i} + \frac{\mathcal{G}_{10_i} A_\parallel}{\rho_{th_i}} - 2\frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}} \right]_i$$
$$+ \rho_s \tau_i \left[\frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}}, N_i + 2T_{\perp i} \right]_i - \sqrt{\frac{\beta_e}{2}} \frac{\rho_s}{\rho_{th_i}} \frac{\partial}{\partial z_i} \mathcal{G}_{11_i}\phi - \rho_s \tau_i \zeta_B \frac{\partial}{\partial y_i} (U_{\parallel i} + 3q_{\perp i}) \tag{177}$$
$$+ \rho_s \zeta_n \tau_i \frac{\partial}{\partial y_i} \frac{\mathcal{G}_{11_i} A_\parallel}{\rho_{th_i}} = 0,$$

where we introduced the operators

$$\frac{d_i}{dt} = \frac{\partial}{\partial t} + \rho_s [\mathcal{G}_{10_i}\phi, \cdot\,]_i,$$
$$\nabla_{\parallel i} = -\rho_s \left[\frac{\mathcal{G}_{10_i} A_\parallel}{\rho_{th_i}}, \cdot \right]_i + \sqrt{\frac{\beta_e}{2}} \frac{\rho_s}{\rho_{th_i}} \frac{\partial}{\partial z_i}. \tag{178}$$

$$\tag{179}$$

Equations (174)-(177) generalize the two ion equations, for density and parallel momentum, of the gyrofluid model of Ref. [17]. The four equations (174)-(177) can

be cast in the simple form (163) making use of the transformed variables $G_{i_{mn}}$ defined in Eq. (161). Note that, because the matrices U and Z do not depend on the species, but only on the number of retained moments, i.e. N and M, the expressions for the variables $G_{i_{mn}}$, in the case of the system (174)-(177), are totally analogous to Eqs. (134)-(137) of the example for the electron drift-fluid equations. The generalized stream functions $\chi_{i_{nm}}$ are obtained from Eq. (164) and the Hamiltonian structure of the total hybrid drift-kinetic/gyrofluid system follows immediately from Eqs. (169) and (172).

Also in this case, we point out that Hamiltonian structure, associated with the possibility of casting the evolution equations in the form (163), requires evolving the $2(N+1)$ moments necessary to construct the matrix Γ_i. Therefore, for instance, the six-field (for each species) gyrofluid model of Ref. [28], although energy-conserving, is not amenable to our Hamiltonian formulation. In fact, it evolves, for the ions (in our notation) the moments $g_{00_i}, g_{10_i}, g_{01_i}, g_{11_i}, g_{20_i}$ and g_{30_i}. Our Hamiltonian gyrofluid reduction for a six-field model, on the other hand, would require evolving g_{21_i} instead of g_{30_i}.

6. Conclusions

We presented a systematic framework for building Hamiltonian drift-fluid and gyrofluid reductions from Hamiltonian gyrokinetic equations. This procedure considerably extends previous results [13, 14] by including the evolution of an arbitrary number of moments in the perpendicular direction for the drift-fluid case, and of the first two perpendicular moments in the gyrofluid case. The systematic inclusion of background density and magnetic gradients is a further new feature of the present procedure. Although the final result of the derivations are hybrid models, the procedure can easily be extended to obtain fluid reductions for both species, provided that the evolutions of the species are governed by independent Poisson brackets.

The potential for deriving Hamiltonian reduced fluid models is thus considerably increased and, as a consequence, also the number of possible applications. The

inclusion of perpendicular temperature fluctuations would allow to study secondary instabilities due to temperature anisotropy (e.g. secondary firehose instability). Investigations of the distribution of the total energy (given by the Hamiltonian), or of other conserved quantities, among energy contributions due to fluctuations also of perpendicular moments becomes now possible in the Hamiltonian framework. This can have relevance for turbulence and reconnection studies. Concerning 2D Hamiltonian magnetic reconnection, in particular, the approach based on the Lagrangian invariants [43, 46, 47, 15, 48] can now be extended in a substantial way using the generalized Lagrangian invariants $G_{e,i_{mn}}$ introduced in this paper. The inclusion of the background density gradient should also allow to study, for instance, drift-waves and magnetic island rotations, in a Hamiltonian context, accounting for a considerable number of moments, as an alternative to full gyrokinetic or kinetic models. Of course, dissipative terms can be added to all models obtained with our procedure, in order to account for effects that are ruled out in the Hamiltonian core of the model.

As already pointed out, the Hamiltonian character of our models is based on the diagonalization procedure obtained with the change of variables, and this imposes constraints on the number of moments to be retained.

As mentioned in Sec. 3.1.1, a price to pay for the Hamiltonian structure in the drift-fluid, finite β_e case, is the presence of subdominant terms in the small parameter ρ_{th_e}. To the best of our knowledge, few examples of non-dissipative reduced fluid models possess a Hamiltonian structure and, at the same time, can be derived from a consistent asymptotic expansion. This appears to be a main obstacle in the derivation of such models, where the Hamiltonian structure is of noncanonical type and orderings have to be applied both to the Hamiltonian and to the noncanonical Poisson bracket. On the other hand, asymptotic limits taken directly on the equations of motion, do not necessarily preserve structures (such as a Hamiltonian structure) of the equations.

Although progress has been made in the derivation of Hamiltonian reduced fluid models for plasmas, some problems remain open. In the context of reduced fluid

models, the general scheme presented in this paper still misses the inclusion of magnetic curvature, which is of course relevant for local models of tokamak dynamics. The systematic inclusion of perpendicular moments with $m > 1$ in gyrofluid models, appears to remain a much more complex problem, in particular at finite β_e. Background temperature anisotropy, on the other hand, could be incorporated adopting, as parent gyrokinetic model, the one presented in Ref. [25], as it was done in Ref. [14].

From a more theoretical point of view, an aspect of the present approach which remains unsatisfactory is that the Poisson bracket of the fluid models is not derived from the Poisson bracket of the parent gyrokinetic model, but rather "guessed" from the equations of motion. Unlike what happens with the Hamiltonian, the replacement of the functions g_s, in the gyrokinetic Poisson bracket, with their truncated expansions, was shown not to lead to the fluid Poisson brackets [23], already in the case with only parallel moments. Therefore, the connection between the gyrokinetic and the fluid Poisson brackets still remains to be understood.

Finally, beyond the domain of reduced fluid models, which are derived from so-called δf gyrokinetic equations, the derivation of Hamiltonian full-f drift- and gyrofluid models is a subject that still remains largely unexplored, although some of such models (e.g. those in Refs. [49, 50, 33]) have already been shown to be energy-conserving.

7. Acknowledgments

This work was carried out within the framework of the activities of the GNFM.

Appendix A. Jacobi identity for the Poisson bracket (24)

To show that the bracket (24) satisfies the Jacobi identity, we need to show that it satisfies

$$\mathbb{J}(F,G,K) = \{\{F,G\},K\} + \{\{G,K\},F\} + \{\{K,F\},G\} = 0, \tag{A.1}$$

for arbitrary functionals F, G, K of the dynamical variables g_e and g_i.

The bracket (24) is the direct sum of two independent brackets depending on g_e

and g_i, respectively. Therefore, in order to prove the Jacobi identity, it is sufficient to prove it for such independent brackets. To this purpose, it is convenient to decompose the bracket for the species s in the following way:

$$\{F, G\}_{gk_s} = \{F, G\}_{0s} + \{F, G\}_{1s}, \tag{A.2}$$

where

$$\{F, G\}_{0s} = -\operatorname{sgn}(q_s)\rho_s \int d^3x_s d\mathcal{W}_s \left(g_s[F_{g_s}, G_{g_s}]_s + \operatorname{sgn}(q_s)\frac{v_s}{\rho_{th_s}}\mathcal{F}_{0s}F_{g_s}\frac{\partial G_{g_s}}{\partial z_s}\right), \tag{A.3}$$

$$\{F, G\}_{1s} = \operatorname{sgn}(q_s)\rho_s \int d^3x_s d\mathcal{W}_s \left(\mu_s\zeta_B - \zeta_n\right)\mathcal{F}_{0s}F_{g_s}\frac{\partial G_{g_s}}{\partial y_s}. \tag{A.4}$$

We can define, for each species s, the expression

$$\mathbb{J}_s = \{\{F, G\}_{0s}, K\}_{0s} + \{\{F, G\}_{0s}, K\}_{1s} + \{\{F, G\}_{1s}, K\}_{0s} + \{\{F, G\}_{1s}, K\}_{1s} + \circlearrowleft, \tag{A.5}$$

where the symbol $\circlearrowleft$ indicates all the terms obtained by cyclic permutation of F, G and K. Then, we take advantage from the fact that $\{\,,\,\}_{0s}$ and $\{\,,\,\}_{1s}$ are both Poisson brackets on their own and thus satisfy the Jacobi identity. Indeed, $\{\,,\,\}_{0s}$ is the Poisson bracket appearing in Refs. [29, 30, 23], whereas $\{\,,\,\}_{1s}$ satisfies the Jacobi identity because it is constant, i.e. independent on the dynamical variables.

Moreover, Theorem 1 of Ref. [32] implies that the contributions of second functional derivatives of F, G and K do not contribute to the expression of $\mathbb{J}_s$, as they cancel by virtue of antisymmetry. Because only the functional derivative of the cosymplectic form contributes, we can conclude that the contribution to $\mathbb{J}_s$ due to the third term in Eq. (A.5) (together with the corresponding terms obtained by cyclic permutation) vanishes. As above said, the cosymplectic operator of $\{\,,\,\}_{1s}$ is indeed constant, and thus its functional derivatives vanish.

The expression (A.5) then reduces to

$$\mathbb{J}_s = \{\{F, G\}_{0s}, K\}_{1s} + \circlearrowleft . \tag{A.6}$$

Making use of the above mentioned Theorem 1 of Ref. [32], we obtain that

$$\begin{aligned}
\mathbb{J}_s &= -\rho_s^2 \int d^3x_s d\mathcal{W}_s \, (\mu_s \zeta_B - \zeta_n) \mathcal{F}_{0s} [F_{g_s}, G_{g_s}]_s \frac{\partial K_{g_s}}{\partial y_s} + \circlearrowleft, \\
&= -\rho_s^2 \int d^3x_s d\mathcal{W}_s \, (\mu_s \zeta_B - \zeta_n) \mathcal{F}_{0s} \left([F_{g_s}, G_{g_s}]_s \frac{\partial K_{g_s}}{\partial y_s} + [G_{g_s}, K_{g_s}]_s \frac{\partial F_{g_s}}{\partial y_s} + [K_{g_s}, F_{g_s}]_s \frac{\partial G_{g_s}}{\partial y_s} \right) \\
&= -\rho_s^2 \int d^3x_s d\mathcal{W}_s \, (\mu_s \zeta_B - \zeta_n) \mathcal{F}_{0s} \left(- \left[\frac{\partial F_{g_s}}{\partial y_s}, G_{g_s} \right]_s K_{g_s} - \left[F_{g_s}, \frac{\partial G_{g_s}}{\partial y_s} \right]_s K_{g_s} \right. \\
&\left. + [G_{g_s}, K_{g_s}]_s \frac{\partial F_{g_s}}{\partial y_s} + [K_{g_s}, F_{g_s}]_s \frac{\partial G_{g_s}}{\partial y_s} \right) = 0, \qquad \text{(A.7)}
\end{aligned}$$

where integration by parts was carried out in the last two steps.

This shows that the bracket $\{\ ,\ \}_{gk_s}$ satisfies the Jacobi identity. Consequently, the gyrokinetic Poisson bracket $\{\ ,\ \} = \sum_s \{\ ,\ \}_{gk_s}$ satisfies the Jacobi identity.

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