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Uniform attachment with freezing: Scaling limits

Étienne Bellin [◆] Arthur Blanc-Renaudie [♥] Emmanuel Kammerer [◆] Igor Kortchemski [♣]

Abstract

We investigate scaling limits of trees built by uniform attachment with freezing, which is a variant of the classical model of random recursive trees introduced in a companion paper. Here vertices are allowed to freeze, and arriving vertices cannot be attached to already frozen ones. We identify a phase transition when the number of non-frozen vertices roughly evolves as the total number of vertices to a given power. In particular, we observe a critical regime where the scaling limit is a random compact real tree, closely related to a time non-homogenous Kingman coalescent process identified by Aldous. Interestingly, in this critical regime, a condensation phenomenon can occur.

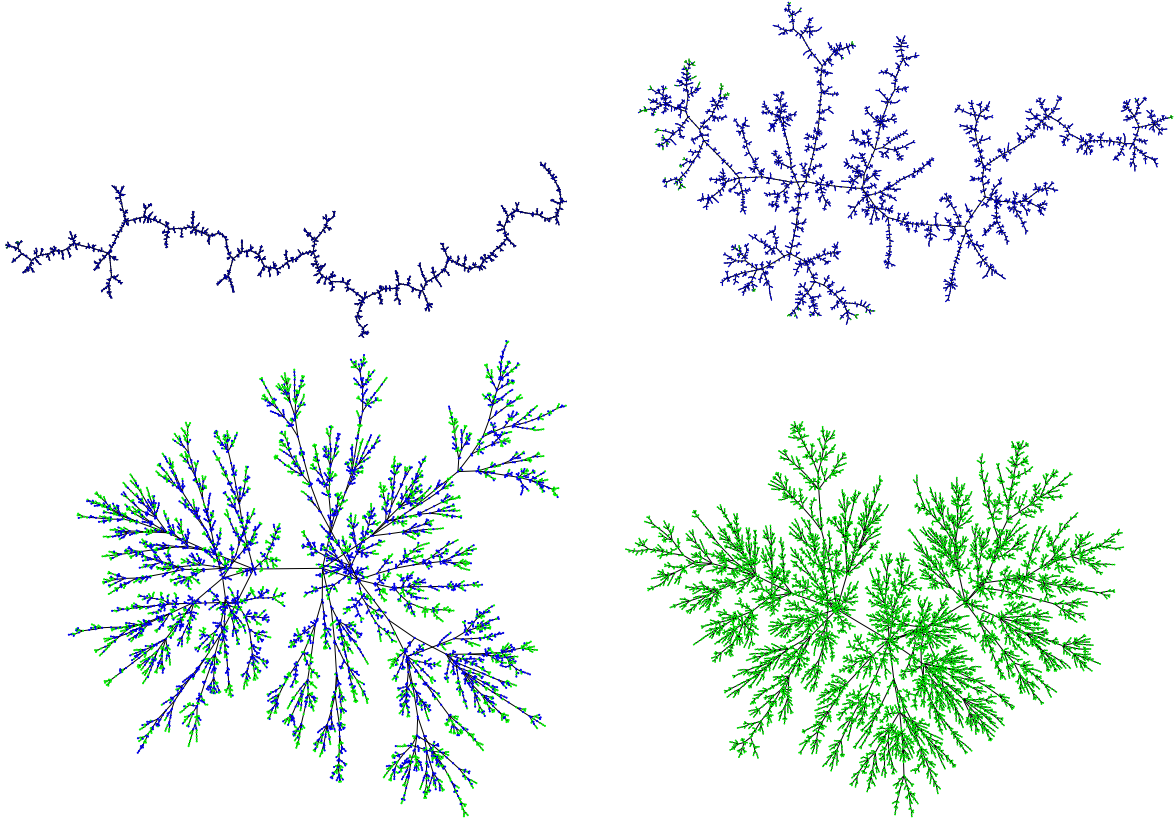


Figure 1: Simulations of the model of uniform attachment with freezing, when the number of active (i.e. non-frozen) vertices roughly evolves as the total number of vertices to the power α ; top: $\alpha = 0.2, 0.5$, bottom: $\alpha = 0.8, 1$. Frozen vertices are blue; active vertices are green.

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1 Introduction

We are interested in the geometry of uniform attachment trees with freezing, which we introduced in [8]. In the classical model of uniform recursive trees (sometimes also called uniform attachment trees), trees are constructed recursively starting with one single vertex, and successively attaching new vertices to a previous existing vertex, chosen uniformly at random. In the model of uniform attachment trees with freezing, existing vertices can freeze and new vertices cannot be attached to frozen vertices.

The motivations in introducing this model were multiple. In the context of real-world networks such mechanisms naturally appear: for instance, on social media (such as Twitter) a user can choose to set

their account to “private” which prevents strangers from “following” them; also performing an infection-tracing of an SIR epidemics falls within this framework (see [8]). Second, this model presents interesting mathematical features: for instance, it extends both uniform recursive trees and uniform plane trees, and in [8] universal bounds on the height have been obtained.

The main purpose of this paper is to study scaling limits of uniform attachment trees with freezing, in the specific regime where the number of non-frozen vertices roughly evolves as the total number of vertices to a given power.

Uniform attachment with freezing. Let us recall the model, which is parametrized by a deterministic sequence $\mathbf{x} = (x_i)_{i \geq 1}$ of elements of $\{-1, +1\}$. Starting from a unique active vertex, we recursively build random trees by reading the elements of the sequence one after the other, by applying a “freezing” step when reading -1 (which amounts to freezing an active vertex chosen uniformly at random) and a “uniform attachment” step when reading $+1$ (which amounts to attaching a new vertex to an active vertex chosen uniformly at random). For every $n \geq 1$ we denote by $\mathcal{T}_n(\mathbf{x})$ the random tree recursively built in this fashion after reading the first n elements of \mathbf{x} (see Sec. 3.1 for a precise definition and Fig. 2 for an example).

Let us comment on our choice of parametrization. It would have been possible to define the model starting with a random sequence \mathbf{x} , but the choice of a deterministic sequence defines a more general model. Our results can then be applied in the context of a random sequence.

It is interesting to note that this model encompasses the two classical models of random recursive trees (when $x_i = 1$ for every $i \geq 1$) and random uniform plane trees (when the sequence $\mathbf{X} = (X_i)_{i \geq 1}$ is a sequence of non constant i.i.d. uniform random variables on $\{-1, +1\}$, see [8, Proposition 12]).

Scaling limits. In order to make explicit the regime we are interested in, we need to introduce some notation. Given a sequence $\mathbf{x} = (x_i)_{i \geq 1}$ of elements of $\{-1, +1\}$, we set $S_0(\mathbf{x}) := 1$ and for every $n \geq 1$

$$S_n(\mathbf{x}) := 1 + \sum_{i=1}^n x_i; \quad \tau(\mathbf{x}) := \inf\{n \geq 1 : S_n(\mathbf{x}) = 0\}. \quad (1)$$

Observe that $S_n(\mathbf{x})$ represents the number of active vertices of $\mathcal{T}_n(\mathbf{x})$. In particular, $\tau(\mathbf{x})$, if it is finite, is the first time when all the vertices are frozen, so that the tree does not evolve anymore. In order to consider large trees, we thus need this time to be large. For this reason, we consider a sequence $(\mathbf{x}^n)_{n \geq 1}$ of sequences of elements of $\{-1, 1\}$ such that $\tau(\mathbf{x}^n) > n$ for every $n \geq 1$ (if $\tau(\mathbf{x}) = \infty$ we can of course take $\mathbf{x}^n = \mathbf{x}$).

We investigate scaling limits for a particular class of sequences $(\mathbf{x}^n)_{n \geq 1}$ where the number of active vertices, that is $S_n(\mathbf{x}^n)$, roughly evolves as the total number of vertices to the power α , for fixed $\alpha \in (0, 1]$. More precisely, we require $(S_{nt}(\mathbf{x}^n)/n^\alpha)_{0 \leq t \leq 1}$ to converge towards some function f (see Sec. 2 for precise conditions). It turns out that the geometry of $\mathcal{T}_n(\mathbf{x}^n)$ can be understood within this class, through scaling limits for which we identify different regimes according to the value of α . In order to keep this introduction at reasonable length, we postpone the precise statements to Sec. 2, and summarize here the essence of our results.

- *Subcritical regime* $\alpha \in (0, 1/2)$: $\mathcal{T}_n(\mathbf{x}^n)/n^{1-\alpha}$ converges in distribution to a deterministic line-segment for the Gromov–Hausdorff–Prokhorov topology (Theorem 2.2);
- *critical regime* $\alpha = 1/2$: $\mathcal{T}_n(\mathbf{x}^n)/\sqrt{n}$ converges in distribution to a random compact measured real tree for the Gromov–Hausdorff–Prokhorov topology, with a condensation phenomenon when $1/f^2$ is integrable near 0 (Theorem 2.3);
- *supercritical regime* $\alpha \in (1/2, 1)$: $\mathcal{T}_n(\mathbf{x}^n)$ looks like a “star” with branches of random lengths of order $n^{1-\alpha}$ (Theorem 2.6).

Here the terminology “ X looks like Y ” means that the joint laws of the graph-distances of k points chosen in an i.i.d. way from X and from Y are “close”.

The boundary case $\alpha = 1$ has been analyzed in our companion paper [8]: $\mathcal{T}_n(\mathbf{x}^n)$ looks like a “tentacular bush” in the sense that at the first order two typical vertices are always at the same deterministic distance of order $\ln(n)$, smaller than the height of the tree.

Our proof is based on an alternative construction of uniform attachment trees with freezing, which we introduced in [8]: by reversing time, uniform attachment trees with freezing can be built using a growth-coalescent process of rooted forests. This extends the well-known connection between recursive trees and the Kingman coalescent. This construction enables us to control distances using classical concentration tools combined with the chaining method. This method is an important technique in concentration theory [25] whose goal is to estimate the maximum of a given function on a given space using a sequence of increasing subspaces. It has found many applications in the study of random metric spaces, see e.g. [4, 7, 14, 24, 11, 10]. Moreover, in the case $\alpha = 1/2$, the continuous analogue of the alternative construction can be viewed as a time-change of a time non-homogeneous Kingman coalescent process introduced by Aldous.

Plan of the paper. We start in Sec. 2 by stating our main results concerning scaling limits in various regimes. Sec. 3 then recalls an alternative construction of \mathcal{T}_n , based on time-reversal, through a growth-coalescent process of rooted forests, which we have introduced in [8]. We then obtain in Sec. 4 general estimates for distances in \mathcal{T}_n , and introduce in particular another distance based on coalescence times. This enables us to establish our scaling limit results, first in the non-critical cases $\alpha \neq 1/2$ (Section 5) and then in the critical case $\alpha = 1/2$ (Section 6). Finally, Appendix A contains background on different topologies that we use and Appendix B contains a tightness criterion.

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2 Scaling limits: main results

We consider a sequence $(\mathbf{x}^n)_{n \geq 1}$ of sequences of elements of $\{-1, 1\}$ such that $\tau(\mathbf{x}^n) > n$ for every $n \geq 1$ (if $\tau(\mathbf{x}) = \infty$ we can of course take $\mathbf{x}^n = \mathbf{x}$ for every $n \geq 1$), and set $\mathcal{T}_n = \mathcal{T}_n(\mathbf{x}^n)$. We keep the notation introduced in Sec. 1, recall in particular (1). To simplify the notation, we set $S_k^n = S_k(\mathbf{x}^n)$ for every $0 \leq k \leq n$ and define S_t^n for $t \in [0, n]$ by linear interpolation. We also set

$$h_n^+ = \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}}.$$

We study the geometry of \mathcal{T}_n under certain assumptions, which we now state. Given $\alpha \in (0, 1)$ and a function $f : [0, 1] \rightarrow \mathbb{R}_+$, we consider the following set of conditions:

- (H_f) f is continuous, $f(0) = 0$ and $f(x) > 0$ for every $x \in (0, 1)$;
- (Lim_f^α) $(S_{nt}^n/n^\alpha)_{0 \leq t \leq 1}$ converges uniformly towards f ;
- (Tight_α) $\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n^{1-\alpha}} \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{S_i^n \leq \delta n^\alpha} = 0$.

We say that condition (Tight_α) is α -tightness. We will later see that conditions (H_f) , (Lim_f^α) and (Tight_α) imply that $1/f$ is integrable on $[0, 1]$ (see Lemma 4.2). Observe that condition (Tight_α) can be alternatively formulated by saying that if $(U_n)_{n \geq 1}$ are independent random variables with U_n uniform on $\{1, 2, \dots, n\}$, then the sequence of random variables $((S_{U_n}^n/n^\alpha)^{-1})_{n \geq 1}$ is uniformly integrable.

We first identify the height of \mathcal{T}_n under these assumptions:

Theorem 2.1. *Assume that $\alpha \in (0, 1)$ and that $f : [0, 1] \mapsto \mathbb{R}^+$ satisfies (H_f) and (Lim_f^α) . Also assume that $(\mathbf{x}^n)_{n \geq 1}$ satisfies (Tight_α) . Then for all $p \geq 1$:*

$$\frac{\text{Height}(\mathcal{T}_n)}{n^{1-\alpha}} \xrightarrow[n \rightarrow \infty]{\mathbb{L}^p} \int_0^1 \frac{dt}{2f(t)}.$$

This follows from [8, Theorem 2], in virtue of which the convergence $\text{Height}(\mathcal{T}_n)/h_n^+ \rightarrow 1$ holds in \mathbb{L}^p , combined with estimating the asymptotic behavior of h_n^+ .

Let us mention that condition (Tight_α) allows to discard undesirable behavior at the first steps of the process: for example, for $\varepsilon \in (0, 1)$, if $x_{2i+1}^n = 1$ and $x_{2i}^n = -1$ for $1 \leq i \leq n^{1-\varepsilon}$, then with high probability the tree \mathcal{T}_n will have height at least of order $n^{1-\varepsilon}$.

2.1 Subcritical regime $\alpha \in (0, 1/2)$

Theorem 2.2. *Assume that $\alpha \in (0, 1/2)$ and that $f : [0, 1] \mapsto \mathbb{R}^+$ satisfies (H_f) and (Lim_f^α) . Also assume that $(\mathbf{x}^n)_{n \geq 1}$ satisfies (Tight_α) . Then the convergence*

$$\frac{1}{n^{1-\alpha}} \cdot \mathcal{T}_n \xrightarrow[n \rightarrow \infty]{(d)} \left[0, \int_0^1 \frac{dt}{2f(t)} \right]$$

holds in distribution for the Gromov–Hausdorff–Prokhorov topology, where the segment $\left[0, \int_0^1 1/(2f(t))dt \right]$ is equipped with the image measure of the uniform measure on $[0, 1]$ by the mapping $x \mapsto \int_0^x \frac{dt}{2f(t)}$.

We refer to Sec. A for details concerning the Gromov–Hausdorff–Prokhorov topology. See Theorem 5.2 for the case where $\|S^n\|_\infty$ is bounded (which roughly speaking may be viewed as the case $\alpha = 0$).

2.2 Critical regime $\alpha = 1/2$

Theorem 2.3. *Assume that $\alpha = 1/2$ and that $f : [0, 1] \mapsto \mathbb{R}^+$ satisfies (H_f) and (Lim_f^α) . Also assume that $(\mathbf{x}^n)_{n \geq 1}$ satisfies (Tight_α) . Then the convergence*

$$\frac{1}{\sqrt{n}} \cdot \mathcal{T}_n \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{T}(f)$$

holds in distribution for the Gromov–Hausdorff–Prokhorov topology, where $\mathcal{T}(f)$ is a random measured compact real tree.

The proof is divided in two steps: first the identification of the limit through finite-dimensional distributions by constructing a continuous analog of Algorithm 2, second establishing tightness for the Gromov–Hausdorff–Prokhorov topology by relying on the chaining method.

We show that this tree is similar in many aspects to the Brownian tree introduced by Aldous [4] (see Sec. 6 for background concerning real trees). Specifically, we establish the following properties concerning the geometry of $\mathcal{T}(f)$:

Theorem 2.4. *The following assertions hold almost surely.*

- (1) *The mass measure on $\mathcal{T}(f)$ has full support and gives full measure to the set of its leaves.*
- (2) *$\mathcal{T}(f)$ is binary if and only if $\int_0^{1/2} \frac{1}{f^2} = \infty$. When $\int_0^{1/2} \frac{1}{f^2} < \infty$, the root has infinite degree and all the other branchpoints are binary.*

Observe that, quite surprisingly, a condensation phenomenon may occur; in particular the topological class of $\mathcal{T}(f)$ depends on the integrability of $1/f^2$ near 0. Let us mention that it is not too hard to check that almost surely the Minkowski, packing and Hausdorff dimensions of $\mathcal{T}(f)$ are at least equal to 2. However, the upper dimensions depend on the the fine behavior of f near 0, and will be investigated in future work.

It turns out that by appropriately choosing f one can recover the Brownian CRT.

Corollary 2.5. *If e is a normalized Brownian excursion then $\mathcal{T}(e)$ has the same law as the Brownian CRT.*

This comes from Theorem 2.3 applied with a carefully chosen random sequence $(\mathbf{x}^n)_{n \geq 1}$ (see Sec. 6.7 for details). This result can also be seen as a consequence of [6, Theorem 2].

2.3 Supercritical regime $\alpha \in (1/2, 1)$

We denote by d^n the graph distance on the vertices of \mathcal{T}_n .

Theorem 2.6. *Assume $\alpha \in (1/2, 1)$ and $f : [0, 1] \mapsto \mathbb{R}^+$ satisfies (H_f) and (Lim_f^α) . Also assume $(\mathbf{x}^n)_{n \geq 1}$ satisfies (Tight_α) . For every $k \geq 2$, conditionally given \mathcal{T}_n , let $V_1^n, V_2^n, \dots, V_k^n$ be independent uniform vertices in \mathcal{T}_n and let U_1, \dots, U_n be i.i.d. uniform random variables in $[0, 1]$. Also let V_0^n be the root of \mathcal{T}_n and let $U_0 = 0$. Then the following convergence holds in distribution.*

$$\left(\frac{1}{n^{1-\alpha}} \cdot d^n(V_i^n, V_j^n) \right)_{0 \leq i, j \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_0^{U_i} \frac{dt}{2f(t)} + \int_0^{U_j} \frac{dt}{2f(t)} \right)_{0 \leq i, j \leq k}$$

where d^n denotes the graph distance in the tree \mathcal{T}_n .

Intuitively speaking, this tells us that the tree \mathcal{T}_n roughly looks like a “star” with branches of length $n^{1-\alpha} \int_0^1 1/(2f(t))dt$. It is worth noting that this case gives a natural example of trees growing polynomially in their size, without scaling limits. Indeed, $(\mathcal{T}_n/n^{1-\alpha})_{n \geq 1}$ is not tight for the GHP topology (since for $\varepsilon > 0$ the number of balls of radius ε needed to cover this space tends to infinity in probability). However, for another topology closely related to the Gromov–Prokhorov topology (see [17, 19]), $\mathcal{T}_n/n^{1-\alpha}$ converges to the so-called long dendron Υ_ν (we use the notation of [19, Example 3.12]) with ν being the image measure of the uniform measure on $[0, 1]$ by the mapping $x \mapsto \int_0^x 1/(2f(t))dt$.

3 Trees constructed by uniform attachment with freezing

Here we give a precise definition of trees constructed by uniform attachment with freezing, and recall from [8] an alternative growth-coalescence algorithm to generate them. We also provide Table 1 which contains the notation that will be used in the future sections.

3.1 Uniform attachment with freezing: recursive construction

Given a sequence $\mathbf{x} = (x_n)_{n \geq 1}$ of elements of $\{-1, 1\}$, for every $n \geq 1$ the tree $\mathcal{T}_n(\mathbf{x})$ is built by reading the first n elements of the sequence \mathbf{x} , namely x_1, \dots, x_n . Here the trees will be rooted, vertex-labelled and edge-labelled; edges have the label corresponding to their time of appearance and vertices have the label

Table 1: Table of the main notation and symbols introduced in Section 3 and used later.

\mathbb{N}	$= \{1, 2, 3, \dots\}$ positive integers
$[n]$	$= \{1, 2, \dots, n\}$ integers between 1 and n
$\#A$	cardinality of a finite set A
$\mathbf{x} = (x_n)_{n \in \mathbb{N}}$	a sequence of elements of $\{-1, 1\}$
$S_n(\mathbf{x})$	$= 1 + \sum_{i=1}^n x_i$
$\tau(\mathbf{x})$	$= \inf\{n \geq 1 : S_n(\mathbf{x}) = 0\}$
$\mathcal{T}_n(\mathbf{x})$	tree built at time n by Algorithm 1; $S_n(\mathbf{x})$ is its number of active vertices
$N_n(\mathbf{x})$	total number of vertices in $\mathcal{T}_n(\mathbf{x})$; $N_n(\mathbf{x}) = (S_n(\mathbf{x}) + n + 1)/2$ when $n \leq \tau(\mathbf{x})$
$\mathcal{F}_n^n(\mathbf{x}), \mathcal{F}_{n-1}^n(\mathbf{x}), \dots, \mathcal{F}_0^n(\mathbf{x})$	the forest of trees built by Algorithm 2
$\mathcal{F}^n(\mathbf{x})$	$= \mathcal{F}_0^n(\mathbf{x})$ the output of Algorithm 2
$\mathbb{F}_n(\mathbf{x})$	$= \{i \in \llbracket 1, n \rrbracket : x_i = -1\}$ the labels of frozen vertices of $\mathcal{T}^n(\mathbf{x})$
$\mathbb{A}_n(\mathbf{x})$	$= \{a_1, \dots, a_{S_n(\mathbf{x})}\}$ the labels of active vertices of $\mathcal{T}^n(\mathbf{x})$
$\mathbb{V}_n(\mathbf{x})$	$= \mathbb{F}_n(\mathbf{x}) \cup \mathbb{A}_n(\mathbf{x})$ the labels of all vertices of $\mathcal{T}^n(\mathbf{x})$
$b_n(u)$	the birth time of $u \in \mathbb{V}_n(\mathbf{x})$ in the construction of $\mathcal{T}^n(\mathbf{x})$ by Algorithm 2
$c_n(u, v)$	the coalescence time between $u, v \in \mathbb{V}_n(\mathbf{x})$ in the construction of $\mathcal{T}^n(\mathbf{x})$ by Algorithm 2
$H_i^n(u)$	the height of vertex u in $\mathcal{F}_i^n(\mathbf{x})$

corresponding to their time of freezing or the label “ a ” if they are still active (meaning they have not frozen yet). We set $S_0(\mathbf{x}) := 1$ and for every $n \geq 1$

$$S_n(\mathbf{x}) := 1 + \sum_{i=1}^n x_i; \quad \tau(\mathbf{x}) := \inf\{n \geq 1 : S_n(\mathbf{x}) = 0\}. \quad (2)$$

Algorithm 1.

- Start with the tree $\mathcal{T}_0(\mathbf{x})$ made of a single root vertex labelled a (vertices labelled a are called active vertices, and the others frozen vertices).
- For every $n \geq 1$, if $\mathcal{T}_{n-1}(\mathbf{x})$ has no vertices labelled a , then set $\mathcal{T}_n(\mathbf{x}) = \mathcal{T}_{n-1}(\mathbf{x})$. Otherwise let V_n be a random uniform active vertex of $\mathcal{T}_{n-1}(\mathbf{x})$, chosen independently from the previous ones. Then:
 - if $x_n = -1$, build $\mathcal{T}_n(\mathbf{x})$ from $\mathcal{T}_{n-1}(\mathbf{x})$ simply by replacing the label a of V_n with the label n ;
 - if $x_n = 1$; build $\mathcal{T}_n(\mathbf{x})$ from $\mathcal{T}_{n-1}(\mathbf{x})$ by adding an edge labelled n between V_n and a new vertex labelled a .

For $n \geq 0$, we view $\mathcal{T}_n(\mathbf{x})$ as a rooted, double-labelled tree (that is edge-labelled and vertex-labelled). If $N_n(\mathbf{x})$ represents the total number of vertices of $\mathcal{T}_n(\mathbf{x})$, observe that by construction $N_n(\mathbf{x}) = (S_n(\mathbf{x}) + n + 1)/2$ for $0 \leq n \leq \tau(\mathbf{x})$.

The law of the random tree $\mathcal{T}_n(\mathbf{x})$ obviously depends on the sequence \mathbf{x} (actually it only depends on the first n elements x_1, \dots, x_n). Notice that, by construction, the sequence of trees $(\mathcal{T}_n(\mathbf{x}))_{n \geq 0}$ is non-decreasing and reaches a stationary state if and only if $\tau(\mathbf{x}) < \infty$, in which case it reaches a stationary state for the first time at $\tau(\mathbf{x})$. In this case, $\mathcal{T}_n(\mathbf{x})$ has no more active vertices for the first time at $n = \tau(\mathbf{x})$.

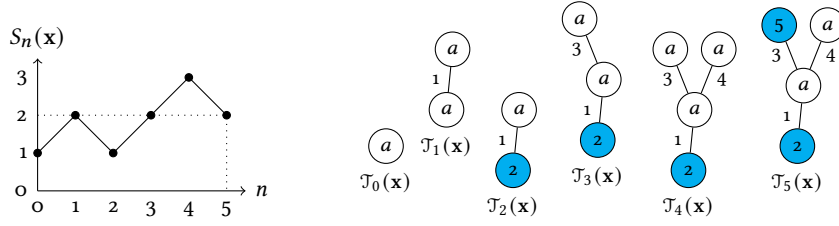


Figure 2: On the left is represented the walk $(S_n(\mathbf{x}))_{n \geq 0}$ up to time $n = 5$ associated with the sequence $\mathbf{x} = +1, -1, +1, +1, -1, \dots$. On the right, a possible realisation of the trees $\mathcal{T}_0(\mathbf{x})$ to $\mathcal{T}_5(\mathbf{x})$ given this sequence. Frozen vertices have been colored in blue.

3.2 Uniform attachment with freezing: growth-coalescent construction

In order to analyse the geometry of uniform attachment trees with freezing, an alternative time-reversed construction based on a growth-coalescence process of forests, that we introduced in [8], is particularly useful. This algorithm is a generalization of the connection between random recursive trees and Kingman's coalescent which first appeared in [16] (see [15, Sec. 6], [23, Sec. 3], [2, Sec. 2.2], [3] for applications), in the context of union-find data structures (which are data structures that store a collection of disjoint sets where merging sets and finding a representative member of a set), see e.g. [20].

Algorithm 2. Fix $0 \leq n \leq \tau(\mathbf{x})$. We construct a sequence $(\mathcal{F}_n^n(\mathbf{x}), \mathcal{F}_{n-1}^n(\mathbf{x}), \dots, \mathcal{F}_0^n(\mathbf{x}))$ of forests of rooted, edge-labelled, vertex-labelled, unoriented trees by induction as follows.

- Let $\mathcal{F}_n^n(\mathbf{x})$ be a forest made of $S_n(\mathbf{x})$ one-vertex trees labelled $a_1, \dots, a_{S_n(\mathbf{x})}$.
- For every $1 \leq i \leq n$, if $\mathcal{F}_i^n(\mathbf{x})$ has been constructed, define $\mathcal{F}_{i-1}^n(\mathbf{x})$ as follows:
 - if $x_i = -1$, $\mathcal{F}_{i-1}^n(\mathbf{x})$ is obtained by adding to $\mathcal{F}_i^n(\mathbf{x})$ a new one-vertex tree labelled i ;
 - if $x_i = 1$, let (T_1, T_2) be a pair of different random trees in $\mathcal{F}_i^n(\mathbf{x})$ chosen uniformly at random, independently of the previous choices; then $\mathcal{F}_{i-1}^n(\mathbf{x})$ is obtained from $\mathcal{F}_i^n(\mathbf{x})$ by adding an edge labelled i between the roots $r(T_1)$ and $r(T_2)$ of respectively T_1 and T_2 , and rooting the tree thus obtained at $r(T_1)$;
- Let $\mathcal{T}^n(\mathbf{x})$ be the only tree of $\mathcal{F}_0^n(\mathbf{x})$.

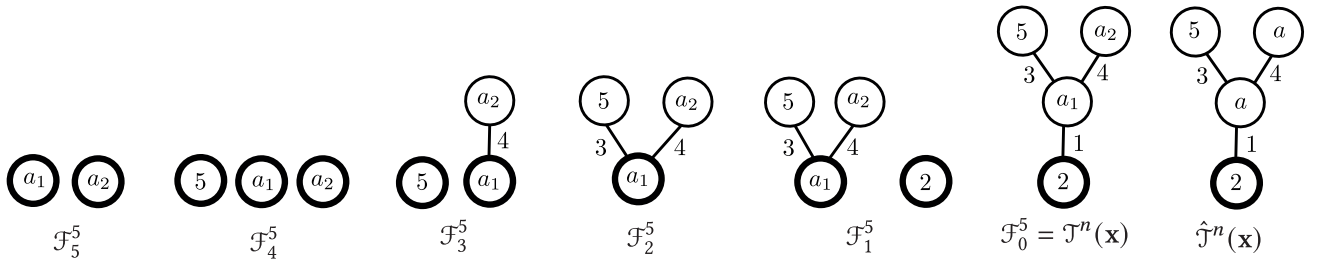


Figure 3: An illustration of Algorithm 2 with $n = 5$ and $(x_5, x_4, x_3, x_2, x_1) = (-1, 1, 1, -1, 1)$ (this is the same sequence as in Fig. 2). For example, since $x_2 = -1$, \mathcal{F}_1^5 is obtained from \mathcal{F}_2^5 by adding a new tree made of a vertex labeled 2. Since $x_4 = 1$, to build \mathcal{F}_3^5 from \mathcal{F}_4^5 we have chosen in \mathcal{F}_4^5 two trees (T_1, T_2) with T_1 being the vertex a_1 and T_2 being the vertex a_2 , and we have added an edge labelled 4 between the roots $r(T_1) = a_1$ and $r(T_2) = a_2$ of respectively T_1 and T_2 , and rooting the tree thus obtained at $r(T_1) = a_1$.

Observe that the active vertices of $\mathcal{T}_n(\mathbf{x})$ are all labelled a while the active vertices of $\mathcal{T}^n(\mathbf{x})$ are labelled $a_1, \dots, a_{S_n(\mathbf{x})}$ (see Fig. 3). It turns out that $\mathcal{T}_n(\mathbf{x})$ is equal in law to $\mathcal{T}^n(\mathbf{x})$ when its active vertices are all relabelled a . More precisely, denote by $\hat{\mathcal{T}}^n(\mathbf{x})$ the tree obtained from $\mathcal{T}^n(\mathbf{x})$ by relabelling its S_n active vertices by a (see the right-most part of Fig. 3 for an illustration).

Theorem 3.1 (Theorem 7 from [8]). *The two trees $\mathcal{T}_n(\mathbf{x})$ and $\hat{\mathcal{T}}^n(\mathbf{x})$ have the same distribution.*

3.3 Laws of the birth and coalescence times

We now introduce some notation related to birth and coalescence times, and identify their laws in view of the analysis of scaling limits of analysis of trees built by uniform attachment with freezing.

In the rest of the paper we shall use the notation $[n] = \{1, 2, \dots, n\}$. First, define

$$\mathbb{F}_n(\mathbf{x}) = \{i \in [n] : x_i = -1\}, \quad \mathbb{A}_n(\mathbf{x}) = \{a_1, \dots, a_{S_n(\mathbf{x})}\}, \quad \mathbb{V}_n(\mathbf{x}) = \mathbb{F}_n(\mathbf{x}) \cup \mathbb{A}_n(\mathbf{x}). \quad (3)$$

Note that while $\mathcal{T}^n(\mathbf{x})$ is a random tree, the labels of its vertices are deterministic as they depend only on \mathbf{x} : $\mathbb{A}_n(\mathbf{x})$ are the labels of the active vertices of $\mathcal{T}^n(\mathbf{x})$ and $\mathbb{F}_n(\mathbf{x})$ are the labels of the frozen vertices of $\mathcal{T}^n(\mathbf{x})$. In particular, the elements of $\mathbb{V}_n(\mathbf{x})$ will be called vertices of $\mathcal{T}^n(\mathbf{x})$.

Next, for every $u \in \mathbb{V}_n(\mathbf{x})$, we denote by $b_n(u)$ the largest $i \in \{0, 1, \dots, n\}$ such that u belongs to the forest $\mathcal{F}_i^n(\mathbf{x})$. Explicitly, if $u \in \mathbb{A}_n(\mathbf{x})$ is an active vertex then $b_n(u) = n$, and if $u \in \mathbb{F}_n(\mathbf{x})$ (note that u is then an integer) then $b_n(u) = u - 1$ (see Fig. 3 for an example). We say that $b_n(u)$ is the *birth time* of u , since it encodes the first time when vertex u appears in Algorithm 2.

For $0 \leq i \leq b_n(u)$, denote by $H_i^n(u)$ the height of vertex u in \mathcal{F}_i^n (that is, the graph distance between u and the root of \mathcal{F}_i^n).

Finally, for $u, v \in \mathbb{V}_n(\mathbf{x})$, we denote by $c_n(u, v)$ the largest $i \in \{0, 1, \dots, n\}$ such that u and v belong to the same tree in the forest $\mathcal{F}_i^n(\mathbf{x})$ obtained when building $\mathcal{T}^n(\mathbf{x})$ in Algorithm 2. We say that $c_n(u, v)$ is the *coalescence time* between u and v , since it encodes the first time u and v belong to the same tree in Algorithm 2 (observe that while $b_n(u)$ is deterministic, $c_n(u, v)$ is random).

We now state several simple consequences, which will be useful in the analysis of the geometry of $\mathcal{T}^n(\mathbf{x})$ (see Section 2.4 in [8] for proofs).

Lemma 3.2. *Fix $1 \leq n \leq \tau(\mathbf{x})$ and $u \in \mathbb{V}_n(\mathbf{x})$. For every $1 \leq i \leq b_n(u)$:*

$$\mathbb{P}(H_{i-1}^n(u) - H_i^n(u) = 1 | \mathcal{F}_n^n, \dots, \mathcal{F}_i^n) = \frac{1}{S_i} \mathbb{1}_{\{x_i=1\}}.$$

In particular, if $(Y_i^n)_{1 \leq i \leq n}$ are independent Bernoulli random variables of parameter $1/S_i$, we have for every $u \in \mathbb{V}^n$

$$H_0^n(u) \stackrel{(d)}{=} \sum_{i=1}^{b_n(u)} Y_i^n \mathbb{1}_{\{x_i=1\}}. \quad (4)$$

Lemma 3.3. *Fix $1 \leq n \leq \tau(\mathbf{x})$. Let V be an element of $\mathbb{V}_n(\mathbf{x})$ chosen uniformly at random. For every $1 \leq m \leq n$ we have*

$$\mathbb{P}(b_n(V) < m) = \frac{m+1 - S_m(\mathbf{x})}{n+1 + S_n(\mathbf{x})}.$$

The last useful result identifies the law of the coalescence times between two vertices.

Lemma 3.4. *Fix $1 \leq n \leq \tau(\mathbf{x})$ and consider $u, v \in \mathbb{V}_n(\mathbf{x})$. Then for every $0 \leq c < b_n(u) \wedge b_n(v)$ such that $x_{c+1} = 1$:*

$$\mathbb{P}(c_n(u, v) = c) = \frac{1}{\binom{S_{c+1}}{2}} \prod_{\substack{i=c+2 \\ \text{s.t. } x_i=1}}^{b_n(u) \wedge b_n(v)} \left(1 - \frac{1}{\binom{S_i}{2}}\right).$$

4 Distances in uniform attachment trees with freezing

We shall now study the geometry of uniform attachment trees with freezing. It will be convenient to work with $\mathcal{T}^n(\mathbf{x})$ as built using Algorithm 2. Recall Theorem 3.1: the only difference between $\mathcal{T}_n(\mathbf{x})$ and $\mathcal{T}^n(\mathbf{x})$ is that all the active vertices of $\mathcal{T}^n(\mathbf{x})$ are labelled $a_1, \dots, a_{S_n(\mathbf{x})}$, while all the active vertices of $\mathcal{T}_n(\mathbf{x})$ are labelled a . In particular the graph structure of both trees is the same in law, so it is equivalent to establish our main results with $\mathcal{T}_n(\mathbf{x})$ replaced with $\mathcal{T}^n(\mathbf{x})$.

Also recall from (3) the definition of $\mathbb{V}_n(\mathbf{x})$, which is a deterministic set representing the labels of the vertices of $\mathcal{T}^n(\mathbf{x})$, and that by a slight abuse of notation we view elements of $\mathbb{V}_n(\mathbf{x})$ as vertices of $\mathcal{T}^n(\mathbf{x})$.

As in [8], we will use Bennett's inequality many times, under the following variant tailored for our purpose:

Proposition 4.1 (Bennett's inequality). *Let $(Y_i)_{1 \leq i \leq n}$ be independent random variables, such that Y_i follows the Bernoulli distribution of parameter $p_i \in [0, 1]$. Set $m_n = \sum_{i=1}^n p_i$. Assume $m_n > 0$. Then for every $t > 0$:*

$$\mathbb{P}\left(\sum_{i=1}^n Y_i > m_n + t\right) \leq \exp\left(-m_n g\left(\frac{t}{m_n}\right)\right) \quad \text{and} \quad \mathbb{P}\left(\sum_{i=1}^n Y_i < m_n - t\right) \leq \exp\left(-m_n g\left(\frac{t}{m_n}\right)\right),$$

where $g(u) = (1+u) \ln(1+u) - u$ for $u > 0$.

4.1 Deterministic estimates

From now on, we consider a sequence $(\mathbf{x}^n)_{n \geq 1}$ of sequences of elements of $\{-1, 1\}$ such that $\tau(\mathbf{x}^n) > n$ for every $n \geq 1$.

To simplify notation, in the sequel, we drop the dependence in \mathbf{x}^n , so for instance we will write \mathcal{T}^n for the tree $\mathcal{T}^n(\mathbf{x}^n)$ built from \mathbf{x}^n using Algorithm 2. We also set $S_k^n := S_k(\mathbf{x}^n)$ for all $k \leq n$, and define for every $1 \leq a \leq b < \tau(\mathbf{x}^n)$:

$$h_n(a, b) := \sum_{i=a}^b \frac{1}{S_i^n}, \quad h_n := h_n(1, n), \quad h_n^+(a, b) := \sum_{i=a}^b \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}}, \quad h_n^+ := h_n^+(1, n). \quad (5)$$

Our goal is now to study the geometry of \mathcal{T}^n . To this end, it is useful to estimate h_n^+ .

Lemma 4.2. *Let $\alpha \in (0, 1)$ and $f : [0, 1] \rightarrow \mathbb{R}_+$. Assume that (H_f) , (Lim_f^α) and (Tight_α) are in force. Then the following assertions hold:*

(i) *The following convergence*

$$\frac{2}{n} \sum_{i=1}^n \mathbb{1}_{x_i^n=1} \delta_{i/n} \xrightarrow{n \rightarrow \infty} \text{Leb}_{[0,1]},$$

holds weakly, where $\text{Leb}_{[0,1]}$ is the Lebesgue measure on $[0, 1]$.

(ii)

$$\int_0^1 \frac{1}{f(t)} dt < +\infty.$$

(iii)

$$2n^{\alpha-1} h_n^+ \sim n^{\alpha-1} h_n \sim \int_0^1 \frac{1}{f(t)} dt.$$

(iv)

$$\frac{1}{n^{1-\alpha}} \max_{1 \leq a \leq b \leq n} \left| h_n^+(a, b) - \frac{1}{2} h_n(a, b) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. We first introduce some notation. For $0 \leq a < b \leq 1$, set

$$p_n(a, b) = \#\{\lfloor an \rfloor + 1 \leq i \leq \lfloor bn \rfloor : x_i = 1\} = \frac{\lfloor bn \rfloor - \lfloor an \rfloor + S_{\lfloor bn \rfloor}^n - S_{\lfloor an \rfloor}^n}{2} \underset{n \rightarrow \infty}{\sim} \frac{(b-a)n}{2} \quad (6)$$

since $S_{\lfloor an \rfloor}^n = o(n)$ and $S_{\lfloor bn \rfloor}^n = o(n)$.

By the Portmanteau theorem, (i) directly follows from (6). Let us show (ii) and (iii) simultaneously. Set $\varepsilon_n = \max_{1 \leq i \leq n} |n^{-\alpha} S_i^n - f(i/n)|$. Let $0 < a < b < 1$, then

$$h_n^+ \geq n^{-\alpha} \sum_{i=\lfloor an \rfloor}^{\lfloor bn \rfloor} \frac{1}{f(\frac{i}{n}) + \varepsilon_n} \mathbb{1}_{\{x_i^n=1\}} = n^{-\alpha} \sum_{i=\lfloor an \rfloor}^{\lfloor bn \rfloor} \frac{1}{f(\frac{i}{n})} \mathbb{1}_{\{x_i^n=1\}} - \varepsilon_n n^{-\alpha} \sum_{i=\lfloor an \rfloor}^{\lfloor bn \rfloor} \frac{1}{(f(\frac{i}{n}) + \varepsilon_n)f(\frac{i}{n})} \mathbb{1}_{\{x_i^n=1\}}.$$

Observe that $1/f$ and $1/f^2$ are continuous and bounded on $[a, b]$ by (H_f), so that by (i):

$$\frac{2}{n} \sum_{i=\lfloor an \rfloor}^{\lfloor bn \rfloor} \frac{1}{f^2(\frac{i}{n})} \mathbb{1}_{\{x_i^n=1\}} \xrightarrow{n \rightarrow \infty} \int_a^b \frac{1}{f(t)^2} dt, \quad \frac{2}{n} \sum_{i=\lfloor an \rfloor}^{\lfloor bn \rfloor} \frac{1}{f(\frac{i}{n})} \mathbb{1}_{\{x_i^n=1\}} \xrightarrow{n \rightarrow \infty} \int_a^b \frac{1}{f(t)} dt.$$

Therefore, for all $0 < a < b < 1$:

$$\liminf_{n \rightarrow \infty} (n^{\alpha-1} h_n^+) \geq \frac{1}{2} \int_a^b \frac{1}{f(t)} dt.$$

Making $a \rightarrow 0$ and $b \rightarrow 1$ we obtain that $\liminf (n^{\alpha-1} h_n^+)$ is at least equal to half of the integral of $1/f$ between 0 and 1 (which, at this stage, might be infinite). Now, for the upper bound, fix $\varepsilon > 0$. By (Tight_α) for every $\delta > 0$ small enough, for every n large enough,

$$\sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{\{S_i^n \leq \delta n^\alpha\}} \leq \varepsilon n^{1-\alpha}.$$

Then, for all δ small enough, and n large enough with $\varepsilon_n \leq \delta/2$

$$h_n^+ \leq \varepsilon n^{1-\alpha} + \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{\{S_i^n > \delta n^\alpha\}} \mathbb{1}_{\{x_i^n=1\}} \leq \varepsilon n^{1-\alpha} + n^{-\alpha} \sum_{i=1}^n \frac{1}{f(\frac{i}{n}) - \varepsilon_n} \mathbb{1}_{\{f(\frac{i}{n}) > \delta/2\}} \mathbb{1}_{\{x_i^n=1\}}.$$

Then by (i), the last sum, divided by $n^{1-\alpha}$, converges towards $\frac{1}{2} \int_0^1 dt/f \times \mathbb{1}_{\{f > \delta/2\}}$. Therefore,

$$\limsup_{n \rightarrow \infty} (n^{\alpha-1} h_n^+) \leq 2\varepsilon + \frac{1}{2} \int_0^1 \frac{1}{f(t)} \mathbb{1}_{\{f(t) > \delta/2\}} dt \xrightarrow{\delta \rightarrow \infty} 2\varepsilon + \frac{1}{2} \int_0^1 \frac{1}{f(t)} dt \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{2} \int_0^1 \frac{1}{f(t)} dt.$$

Finally we show that this integral is finite. Let $\delta > 0$ such that for all $n \geq 1$

$$\sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{\{S_i^n \leq \delta n^\alpha\}} \leq n^{1-\alpha}.$$

Then

$$h_n^+ = \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{\{x_i^n=1\}} \leq n^{1-\alpha} + \sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{\{S_i^n > \delta n^\alpha\}} \leq n^{1-\alpha} \left(1 + \frac{1}{\delta}\right).$$

Therefore $n^{\alpha-1} h_n^+$ is bounded and by what precedes converges towards $\frac{1}{2} \int_0^1 1/f$, which hence is finite. Following the exact same steps one can similarly show that $n^{\alpha-1} h_n \rightarrow \int_0^1 1/f$, which concludes for (iii).

Finally, one can readily adapt the proof of (iii) to get for every $a, b \in [0, 1]$

$$2n^{\alpha-1} h_n^+(\lfloor an \rfloor, \lfloor bn \rfloor) \xrightarrow{n \rightarrow \infty} \int_a^b \frac{1}{f(t)} dt \quad \text{and} \quad n^{\alpha-1} h_n(\lfloor an \rfloor, \lfloor bn \rfloor) \xrightarrow{n \rightarrow \infty} \int_a^b \frac{1}{f(t)} dt.$$

By an extension of Dini's theorem to \mathbb{R}^2 , by monotonicity the last convergences hold uniformly on $[0, 1]^2$, and (iv) directly follows. \square

4.2 An alternative distance using coalescence times

In this section, we introduce a new distance on the vertices of \mathcal{T}^n which will be more convenient to study, based on the coalescence time between vertices. For every $n \in \mathbb{N}$, and $u, v \in \mathcal{T}^n$, set

$$D^n(u, v) := \frac{1}{2} \sum_{c_n(u, v) \leq i \leq b_n(u)} \frac{1}{S_i^n} + \frac{1}{2} \sum_{c_n(u, v) \leq i \leq b_n(v)} \frac{1}{S_i^n} = \frac{1}{2} h_n(c_n(u, v), b_n(u)) + \frac{1}{2} h_n(c_n(u, v), b_n(v)),$$

where we use the notation h introduced in (5).

Proposition 4.3. *Let $\alpha \in (0, 1)$ and $f : [0, 1] \rightarrow \mathbb{R}_+$. Assume that (H_f) , (Lim_f^α) and (Tight_α) are in force. Then for every $\varepsilon > 0$,*

$$\mathbb{P} \left(\max_{u, v \in \mathbb{V}_n} |d^n(u, v) - D^n(u, v)| > \varepsilon n^{1-\alpha} \right) \xrightarrow{n \rightarrow \infty} 0.$$

Proof. Recall that, for $v \in \mathbb{V}_n$ and $0 \leq i \leq b_n(v)$, $H_i^n(v)$ is the height of vertex v in \mathcal{F}_i^n . First note that by the definition of Algorithm 2, for every $u, v \in \mathbb{V}_n$, the nearest common ancestor of u and v in \mathcal{T}^n is the root of the tree in the forest $\mathcal{F}_{c_n(u, v)}^n$ which contains u and v . As a result, $d^n(u, v) = H_{c_n(u, v)}^n(u) + H_{c_n(u, v)}^n(v)$. So

$$|d^n(u, v) - D^n(u, v)| = \left| H_{c_n(u, v)}^n(u) + H_{c_n(u, v)}^n(v) - \frac{1}{2} h_n(c_n(u, v), b_n(u)) - \frac{1}{2} h_n(c_n(u, v), b_n(v)) \right|.$$

Hence

$$\max_{u, v \in \mathbb{V}_n} |d^n(u, v) - D^n(u, v)| \leq 2 \max_{\substack{v \in \mathbb{V}_n \\ 0 \leq a < b_n(v)}} \left| H_a^n(v) - \frac{1}{2} h_n(a, b_n(v)) \right|. \quad (7)$$

We next estimate the right hand side of (7). It follows from Lemma 3.2 that for every $v \in \mathbb{V}_n$ and $0 \leq a \leq b_n(v)$,

$$H_a^n(v) \stackrel{(d)}{=} \sum_{i=a+1}^{b_n(v)} Y_i^n \mathbb{1}_{x_i^n=1},$$

where $(Y_i^n)_{1 \leq i \leq n}$ are independent Bernoulli random variables of respective parameters $(1/S_i^n)_{1 \leq i \leq n}$. As before, set $g(u) = (1+u) \log(1+u) - u$ for $u > 0$. Then, by Bennett's inequality, recalled in Proposition 4.1, and using the fact that $x \mapsto xg(t/x)$ is decreasing, for every $t > 0$, we see that $\mathbb{P}(|H_a^n(v) - h_n^+(a+1, b_n(v))| > t)$ is smaller than $2 \exp(-h_n^+ g(t/h_n^+))$. Therefore, by a union bound,

$$\mathbb{P} \left(\max_{v \in \mathbb{V}_n, 0 \leq a < b_n(v)} |H_a^n(v) - h_n^+(a+1, b_n(v))| > t \right) \leq 2n^2 \exp(-h_n^+ g(t/h_n^+)). \quad (8)$$

By Lemma 4.2 (iii) it follows that for every $\varepsilon > 0$,

$$\mathbb{P} \left(\max_{v \in \mathbb{V}_n, 0 \leq a < b_n(v)} |H_a^n(v) - h_n^+(a+1, b_n(v))| > \varepsilon n^{1-\alpha} \right) \leq 2n^2 \exp(-h_n^+ g(\varepsilon n^{1-\alpha}/h_n^+)) \rightarrow 0.$$

In addition, by Lemma 4.2 (iv), and since $1/S_i \leq 1$ for every $0 \leq i \leq n$, we have

$$\frac{1}{n^{1-\alpha}} \max_{1 \leq a \leq b \leq n} \left| h_n^+(a+1, b) - \frac{1}{2} h_n(a, b) \right| \xrightarrow{n \rightarrow \infty} 0,$$

which completes the proof. \square

5 Scaling limits: non-critical regimes

As before, we consider a sequence $(\mathbf{x}^n)_{n \geq 1}$ of sequences of elements of $\{-1, 1\}$ such that $\tau(\mathbf{x}^n) > n$ for every $n \geq 1$.

5.1 Subcritical regime $\alpha \in (0, 1/2)$

We first show that, roughly speaking, vertices coalesce quickly.

Lemma 5.1. *Let $\alpha \in (0, 1/2)$. Let $\alpha \in (0, 1)$ and $f : [0, 1] \rightarrow \mathbb{R}_+$. Assume that (H_f) , (Lim_f^α) and (Tight_α) are in force. Then,*

$$\frac{1}{n} \max_{u, v \in \mathbb{V}_n} |b_n(u) \wedge b_n(v) - c_n(u, v)| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Proof. Recall that the total number of vertices of \mathcal{T}^n is $(n + S_n^n + 1)/2$. By a union bound and then by the uniform convergence of $x \mapsto S_{nx}^n/n^\alpha$ towards f , for n large enough

$$\begin{aligned} \mathbb{P}(\exists u, v \in \mathbb{V}_n : |b_n(u) \wedge b_n(v) - c_n(u, v)| \geq \varepsilon n) &\leq \left(\frac{n + S_n^n + 1}{2} \right)^2 \max_{\lfloor \varepsilon n \rfloor + 1 \leq b \leq n} \prod_{\substack{i=b-\lfloor \varepsilon n \rfloor + 1 \\ \text{s.t. } x_i^n = 1}}^b \left(1 - \frac{1}{\binom{S_i^n}{2}} \right) \\ &\leq (n+1)^2 \left(1 - \frac{2}{n^{2\alpha}(1 + \sup_{[0,1]} f^2)} \right)^{\varepsilon n/3} \xrightarrow[n \rightarrow \infty]{} 0. \end{aligned}$$

For the second inequality, we also used the fact that for n large enough, for all $\lfloor \varepsilon n \rfloor + 1 \leq b \leq n$,

$$\#\{i \in [b - \lfloor \varepsilon n \rfloor + 1, b], x_i^n = 1\} = (\lfloor \varepsilon n \rfloor + S_b^n - S_{b-\lfloor \varepsilon n \rfloor + 1}^n)/2 \geq \varepsilon n/3.$$

This completes the proof. \square

Proof of Theorem 2.2. Let $k \geq 1$. For all $n \geq 1$, let V_1^n, \dots, V_k^n be uniform random vertices of \mathcal{T}^n . Let U_1, \dots, U_k be i.i.d. uniform random variables in $[0, 1]$. Let us first show that

$$\left(\frac{d^n(V_j^n, V_\ell^n)}{n^{1-\alpha}} \right)_{1 \leq j, \ell \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_{U_j \wedge U_\ell}^{U_j \vee U_\ell} \frac{dt}{2f(t)} \right)_{1 \leq j, \ell \leq k}.$$

By Proposition 4.3, it suffices to prove that

$$\left(\frac{1}{n^{1-\alpha}} \left(\sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_j^n)} \frac{1}{2S_i^n} + \sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_\ell^n)} \frac{1}{2S_i^n} \right) \right)_{1 \leq j, \ell \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_{U_j \wedge U_\ell}^{U_j \vee U_\ell} \frac{dt}{2f(t)} \right)_{1 \leq j, \ell \leq k}. \quad (9)$$

However, one can write

$$\sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_j^n)} \frac{1}{2S_i^n} + \sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_\ell^n)} \frac{1}{2S_i^n} = \sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_j^n) \wedge b_n(V_\ell^n)} \frac{1}{S_i^n} + \sum_{(b_n(V_j^n) \wedge b_n(V_\ell^n)) + 1}^{b_n(V_j^n) \vee b_n(V_\ell^n)} \frac{1}{2S_i^n}.$$

Besides, $S_n^n = o(n)$, so that with high probability the vertices V_j^n 's are frozen. Moreover, conditionally on being frozen, the $b_n(V_j^n)$'s are independent and uniform in $\{i \in [n]; x_i^n = -1\}$. Therefore, by Lemma 4.2 (i), we know that the $b_n(V_j^n)/n$'s converge in distribution towards the U_j 's. Finally, combining this remark with Lemma 5.1, (Lim_f^α) and a Riemann sum, we obtain (9).

So as to end the proof of the GHP convergence, by Lemma A.2, it suffices to establish the convergence in the sense Gromov-Hausdorff since we already have the Gromov-Prokhorov convergence. Let us define the correspondence

$$\mathcal{C}_n = \left\{ \left(u, \int_0^t \frac{1}{2f(s)} ds \right); u \in \mathbb{V}_n \text{ and } \left| t - \frac{b_n(u)}{n} \right| = \min_{v \in \mathbb{V}_n} \left| t - \frac{b_n(v)}{n} \right| \right\}.$$

By Theorem 7.3.25 in [12] it then suffices to show that

$$\text{dis}(\mathcal{C}_n) := \sup_{(u, \int_0^t \frac{1}{2f(s)} ds), (v, \int_0^r \frac{1}{2f(s)} ds) \in \mathcal{C}_n} \left| \frac{d^n(u, v)}{n^{1-\alpha}} - \int_{t \wedge r}^{t \vee r} \frac{1}{2f(s)} ds \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (10)$$

We upper bound

$$\text{dis}(\mathcal{C}_n) \leq \sup_{u, v \in \mathbb{V}_n} \frac{|d^n(u, v) - D^n(u, v)|}{n^{1-\alpha}} \quad (11)$$

$$+ \sup_{u, v \in \mathbb{V}_n} \left| \frac{1}{n^{1-\alpha}} \sum_{i=c_n(u, v)}^{b_n(u) \wedge b_n(v)} \frac{1}{S_i^n} \right| \quad (12)$$

$$+ \sup_{(u, \int_0^t \frac{1}{2f(s)} ds), (v, \int_0^r \frac{1}{2f(s)} ds) \in \mathcal{C}_n} \left| \frac{1}{n^{1-\alpha}} \sum_{i=b_n(u) \wedge b_n(v)+1}^{b_n(u) \vee b_n(v)} \frac{1}{2S_i^n} - \int_{t \wedge r}^{t \vee r} \frac{1}{2f(s)} ds \right| \quad (13)$$

The first line (11) converges to zero in probability as $n \rightarrow \infty$ thanks to Proposition 4.3. Let $\varepsilon > 0$. Let δ given by (Tight $_\alpha$) such that $(1/n^{1-\alpha}) \sum_{i=1}^n (1/S_i^n) \mathbb{1}_{S_i^n \leq \delta n^\alpha} \leq \varepsilon$ for all $n \geq 1$. Then,

$$\sup_{u, v \in \mathbb{V}_n} \left| \frac{1}{n^{1-\alpha}} \sum_{i=c_n(u, v)}^{b_n(u) \wedge b_n(v)} \frac{1}{S_i^n} \right| \leq \varepsilon + \sup_{u, v \in \mathbb{V}_n} \frac{1}{\delta n} |b_n(u) \wedge b_n(v) - c_n(u, v)|,$$

and the right-hand side supremum goes to zero in probability due to Lemma 5.1. Thus the second line (12) converges to zero in probability as $n \rightarrow \infty$.

For the last line (13), we again set $\varepsilon > 0$ and take $\delta > 0$ given by (Tight $_\alpha$). We can upper-bound the quantity appearing in (13) by

$$\begin{aligned} \varepsilon + \int_0^1 \frac{1}{2f(s)} \mathbb{1}_{\{S_{[ns]}^n \leq \delta n^\alpha\}} ds + \int_0^1 \left| \frac{n^\alpha}{2S_{[ns]}^n} - \frac{1}{2f(s)} \right| \mathbb{1}_{\{S_{[ns]}^n \geq \delta n^\alpha\}} ds \\ + \sup_{(u, \int_0^t \frac{1}{2f(s)} ds), (v, \int_0^r \frac{1}{2f(s)} ds) \in \mathcal{C}_n} \left| \int_{\frac{b_n(u) \wedge b_n(v)+1}{n}}^{\frac{b_n(u) \vee b_n(v)}{n}} \frac{ds}{2f(s)} - \int_{t \wedge r}^{t \vee r} \frac{ds}{2f(s)} \right|. \end{aligned}$$

Observe that

$$\sum_{i=1}^n \mathbb{1}_{S_i^n \leq \delta n^\alpha} \leq \sum_{i=1}^n \frac{\delta n^\alpha}{S_i^n} \mathbb{1}_{S_i^n \leq \delta n^\alpha} \leq \varepsilon n,$$

so that by the integrability of $1/f$ the first integral in the above expression is a $o(1)$ as $\varepsilon \rightarrow 0$ uniformly in n . Moreover, by dominated convergence,

$$\int_0^1 \left| \frac{n^\alpha}{2S_{[ns]}^n} - \frac{1}{2f(s)} \right| \mathbb{1}_{\{S_{[ns]}^n \geq \delta n^\alpha\}} ds \xrightarrow[n \rightarrow \infty]{} 0.$$

Also, by definition of \mathcal{C}_n and thanks to Lemma 4.2 (i) we have

$$\sup_{(u, \int_0^t \frac{1}{2f(s)} ds), (v, \int_0^r \frac{1}{2f(s)} ds) \in \mathcal{C}_n} \left(\left| t \wedge r - \frac{b_n(u) \wedge b_n(v)}{n} \right| + \left| t \vee r - \frac{b_n(u) \vee b_n(v)}{n} \right| \right) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

and combined with the integrability of $1/f$ this completes the proof. \square

We complete the study of the subcritical regime by establishing a scaling limit result when the sequence $\|S^n\|_\infty$ is bounded (which roughly speaking may be thought of as the case $\alpha = 0$). The main difference with Theorem 2.2 is that now the limit is for the Gromov–Hausdorff topology, and not in the Gromov–Hausdorff–Prokhorov sense (in general there is no universal behavior for the latter topology).

Theorem 5.2. *Let $(\mathbf{x}^n)_n$ be such that $\tau(\mathbf{x}^n) > n$ for all n and $S_k^n \leq M$ for all $k \leq n$ where M is some positive constant. Then*

$$\frac{1}{h_n^+} \cdot \mathcal{T}_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} [0, 1]$$

holds in probability for the Gromov-Hausdorff topology.

Proof. For all n , let us call a *side sub-tree* of \mathcal{T}_n , a sub-tree rooted at a vertex of the longest branch of \mathcal{T}_n and containing only vertices that are not on the longest branch (except the root). The height of the longest branch of \mathcal{T}_n is of order h_n^+ . Indeed, it is easy to check that $h_n^+ = \Theta(n)$. Therefore, by [?, Theorem 3 (iii)] we deduce that $\text{Height}(\mathcal{T}_n)/h_n^+$ converges to 1 in \mathbb{L}^p for all $p \geq 1$. To show the proposition, it is enough to show that, for all $\varepsilon > 0$, the probability to see a side sub-tree of height larger than εn in \mathcal{T}_n goes to 0 when n goes to infinity.

Suppose that there is a side sub-tree t^n in \mathcal{T}_n of height at least εn . Denote by t_k^n the sub-tree of t^n observed when reading only the first k steps of \mathbf{x}^n . Since t^n has size at least εn , there must be an interval of integers I_n of length at least εn such that t_k^n and $\mathcal{T}_n \setminus t_k^n$ have both, at least, one active vertex for all $k \in I_n$. Consider an interval $J \subset I_n$ where there are at least M steps -1 . Then there is a positive probability $p > 0$ that t_k^n has no active vertices for some $k \in J$. Indeed there is a positive probability that, every time we see a step -1 , we freeze a vertex of t_k^n (if there are still active vertices), and every time we see a step $+1$, we add a vertex outside of t_k^n . This event would imply that, eventually, t_k^n has no active vertices since there are at most M active vertices in the whole tree \mathcal{T}_n at each time. Notice that any interval of size at least $3M$ contains at least M steps -1 . Thus we can decompose the interval I_n into $N \approx \varepsilon n/(3M)$ disjoint intervals all containing at least M steps -1 . Therefore the probability that t_k^n has an active vertex for all $k \in I_n$ is bounded by $(1-p)^N$ which goes to 0 when n goes to infinity. We conclude using a union bound. \square

5.2 Supercritical regime $\alpha \in (1/2, 1)$

We first show that, roughly speaking, vertices coalesce near the origin.

Lemma 5.3. *Let $\alpha \in (1/2, 1)$ and $f : [0, 1] \rightarrow \mathbb{R}_+$. Assume that (H_f) , (Lim_f^α) and (Tight_α) are in force. For all $n \geq 1$, let V_1^n, V_2^n be two independent uniform vertices of \mathbb{V}_n . Then*

$$\frac{c_n(V_1^n, V_2^n)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Proof. Let $\varepsilon, \delta \in (0, 1)$ with $\varepsilon < 1 - \delta$. By Lemma 3.4,

$$\mathbb{P}(c_n(V_1^n, V_2^n) \geq \varepsilon n) \leq \mathbb{P}(b_n(V_1^n) \wedge b_n(V_2^n) \geq (1 - \delta)n) + \sum_{\substack{\varepsilon n \leq i \leq (1 - \delta)n \\ x_i^n = 1}} \frac{1}{\binom{S_i^n}{2}}.$$

By Lemma 3.3 and the fact that $S_n^n = o(n)$,

$$\mathbb{P}(b_n(V_1^n) \wedge b_n(V_2^n) \geq (1 - \delta)n) \xrightarrow[n \rightarrow \infty]{} \delta^2.$$

For the remaining sum, we use Lemma 4.2 (i). Since (Lim_f^α) holds and $1/f^2$ is continuous and bounded on $[\varepsilon, 1 - \delta]$, when $n \rightarrow \infty$ we have

$$\sum_{\substack{\varepsilon n \leq i \leq (1-\delta)n \\ x_i^n = 1}} \frac{1}{\binom{S_i^n}{2}} \sim n^{1-2\alpha} \int_\varepsilon^{1-\delta} \frac{2}{f(t)^2} dt = o(1).$$

Making $\delta \rightarrow 0$ we conclude that $\mathbb{P}(c_n(V_1^n, V_2^n) \geq \varepsilon n) \rightarrow 0$ when $n \rightarrow \infty$. \square

Proof of Theorem 2.6. By Proposition 4.3 it suffices to show that

$$\frac{1}{n^{1-\alpha}} \sum_{i=1}^{b_n(V_1^n)} \frac{1}{2S_i^n} \xrightarrow[n \rightarrow \infty]{(d)} \int_0^{U_1} \frac{dt}{2f(t)} \quad (14)$$

and that

$$\left(\frac{1}{n^{1-\alpha}} \left(\sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_j^n)} \frac{1}{2S_i^n} + \sum_{i=c_n(V_j^n, V_\ell^n)}^{b_n(V_\ell^n)} \frac{1}{2S_i^n} \right) \right)_{1 \leq j, \ell \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_0^{U_j} \frac{dt}{2f(t)} + \int_0^{U_\ell} \frac{dt}{2f(t)} \right)_{1 \leq j, \ell \leq k}. \quad (15)$$

We know that $S_n^n = o(n)$ and that conditionally on being frozen, $b_n(V_1^n)$ is uniform in $\{i \in [n]; x_i^n = -1\}$. Therefore, by Lemma 4.2 (i), we know that $b_n(V_1^n)/n$ converges in distribution towards U_1 . The convergence (14) then comes from a Riemann sum using also (Lim_f^α) and (Tight_α) . For (15), we again identify Riemann sums, using also that $c_n(V_j^n, V_\ell^n)/n$ converges in probability towards zero by Lemma 5.3. \square

6 Scaling limits: critical regime

The goal of Section 6 is to establish Theorem 2.3. We first explain in Section 6.1 the strategy of the proof of Theorem 2.3. Roughly speaking, the identification of the “finite dimensional marginal distributions” (Gromov–Prokhorov convergence) is based on a continuous-time coalescent process, defined in Section 6.2, and which is closely related to a non-homogenous analogue of Kingman’s coalescent introduced by Aldous in [6]. The law of the genealogy of clusters in this continuous-time coalescent process can be explicitly described (Section 6.3), which allows us to establish the Gromov–Prokhorov convergence of Theorem 2.3. Tightness is then established in Section 6.5.

6.1 Definition of the limit

The way we define the limiting compact measured tree $\mathcal{T}(f)$ is rather indirect. For every $k \geq 1$, let V_1^n, \dots, V_k^n be k i.i.d. vertices in \mathbb{V}_n chosen uniformly at random (conditionally given \mathcal{T}^n). Our strategy is to establish the following two facts:

- Gromov–Prokhorov convergence:

$$\left(\frac{d^n(V_j^n, V_\ell^n)}{\sqrt{n}} \right)_{1 \leq j, \ell \leq k} \quad (16)$$

converges in distribution as $n \rightarrow \infty$, where we recall that d^n denotes the graph distance on \mathcal{T}^n . This is established in Sec. 6.4

- “Leaf-tightness”: for every $\varepsilon > 0$

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\max_{v \in \mathbb{V}_n} \min_{1 \leq i \leq k} \frac{d^n(v, V_i^n)}{\sqrt{n}} > \varepsilon \right) = 0. \quad (17)$$

This is established in Sec. 6.5.

Indeed, by Lemma B.1, (16) and (17) entail the existence of a limiting compact metric space $\mathcal{T}(f)$ equipped with a probability measure μ with full support such that the convergence of Theorem 2.3 holds. The compact metric space $\mathcal{T}(f)$ is a real tree, being the Gromov–Hausdorff limit of real trees.

6.2 A continuous-time coalescent

Here we introduce a continuous-time coalescent process, which appears in the limit of the quantity (16). Let $f : [0, 1] \rightarrow \mathbb{R}_+$ be a measurable function satisfying the following conditions:

$$\int_0^1 \frac{1}{f(t)} dt < \infty, \quad ; \quad \forall a, b \in (0, 1), \int_a^b \frac{1}{f(t)^2} dt < \infty.$$

The continuous coalescent construction proceeds as follows (see Figure 4): let each of k particles be born at independent uniform random times B_1, \dots, B_k in $[0, 1]$. Particles coalesce into clusters according to the following rule: in time $[t, t - dt]$, each pair of clusters merges with rate $\frac{1}{f(t)^2}$. If we furthermore assume

$$\forall a \in (0, 1), \int_0^a \frac{1}{f(t)^2} dt = \infty \tag{18}$$

this ensures that all the particles will eventually merge into one cluster (Aldous assumes this condition in [6] but it will not be necessary for our purpose). Since we will not assume this condition, two clusters may not merge with positive probability. If there are r clusters which have not coalesced at time 0, we denote by $0 < C_{r+1} < \dots < C_{k-1}$ the times of coalescence into clusters in the increasing order (they are a.s. distinct) and by convention, we also set $C_1 = \dots = C_r = 0$.

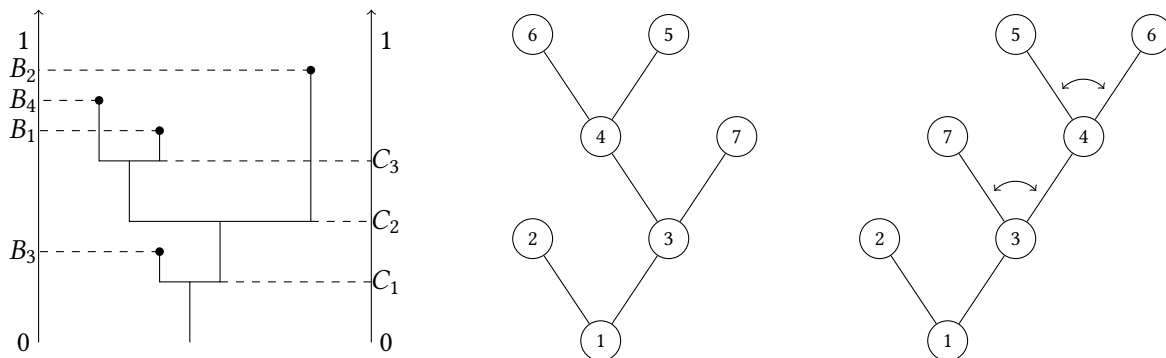


Figure 4: Illustration of the continuous coalescence construction on the left for $k = 4$. In the middle is represented the binary forest \mathcal{F} composed here of only one tree associated with this construction. On the right is the binary forest where the vertices of each internal node have been exchanged uniformly at random. In this case, the child of the vertices 3 and 4 have been exchanged but those of 1 have not.

This process is actually a time-change of a non-homogenous analogue of Kingman’s coalescent introduced by Aldous in [6], see Remark 6.3 below.

6.3 The genealogy of clusters in the continuous-time coalescent

The genealogy of the clusters is described by an increasing binary forest \mathcal{F} . The trees forming the forest are associated with the clusters which have not merged at time 0. Each internal vertex in one of these trees corresponds to a coalescence of two clusters and the leafs correspond to the particles’ births. We also give a uniform random order to the children of each internal node, so that the trees of \mathcal{F} can be viewed

as plane trees. We equip \mathcal{F} with a labelling from $r + 1$ to $2k - 1$ of its vertices according to the order of the times of coalescence or birth. If $k > r \geq 0$, we denote by $\mathfrak{F}_{r,k}^+$ the set of ordered forests of $r + 1$ plane binary trees with $2k - 1 - r$ vertices with a vertex-labelling from $r + 1$ to $2k + 1$ which increases along the branches. If there are $r + 1$ clusters at time 0, then the forest \mathcal{F} belongs to $\mathfrak{F}_{r,k}^+$. To express the law of $(\mathcal{F}, (C_j)_{j \in [k-1]}, (B_j)_{j \in [k]})$, we first describe what will be its support.

Definition 6.1. Let $k > r \geq 0$. Let $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$. Let $b_1, \dots, b_k \in \mathbb{R}_+$ and $0 \leq c_{r+1} < \dots < c_{k-1}$ be distinct real numbers. The triplet $(\mathcal{F}_0, (c_j)_{j \in \{r+1, \dots, k-1\}}, (b_j)_{j \in [k]})$ is called admissible if there is an increasing bijection $\varphi : \{c_j; j \in \{r+1, \dots, k-1\}\} \cup \{b_j; j \in [k]\} \rightarrow \{r+1, \dots, 2k-1\}$ such that for all $j \in [k]$, the integer $\varphi(b_j)$ is the label of a leaf in \mathcal{F}_0 and for all $j \in \{r+1, \dots, k-1\}$, the integer $\varphi(c_j)$ is the label of an internal vertex of \mathcal{F}_0 .

Roughly speaking, $(\mathcal{F}_0, (c_j)_{j \in \{r+1, \dots, k-1\}}, (b_j)_{j \in [k]})$ is admissible if it is possible to associate the times (c_j) with cluster coalescence times and $(b_j)_{j \in [k]}$ with particle birth times in such a way that their relative order is compatible with \mathcal{F}_0 . In the example of Fig. 4 we have $r = 0$, $B_1 = 7/10$, $B_2 = 9/10$, $B_3 = 3/10$, $B_4 = 8/10$, $C_1 = 2/10$, $C_2 = 4/10$, $C_3 = 6/10$ and φ is defined by $\varphi(2/10) = 1$, $\varphi(3/10) = 2$, $\varphi(4/10) = 3$, $\varphi(6/10) = 4$, $\varphi(7/10) = 5$, $\varphi(8/10) = 6$, $\varphi(9/10) = 7$.

Proposition 6.2. For all $k > r \geq 0$, for all $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$, for all $b_1, \dots, b_k \in [0, 1]$ and $0 < c_{r+1} < \dots < c_{k-1} < 1$ distinct real numbers such that the triplet $(\mathcal{F}_0, (c_j)_{j \in \{r+1, \dots, k-1\}}, (b_j)_{j \in [k]})$ is admissible, we set

$$g_{r,k}(\mathcal{F}_0, (b_j)_{1 \leq j \leq k}, (c_j)_{j \in \{r+1, \dots, k-1\}}) = \frac{1}{2^{k-1-r}} \exp\left(-\int_0^1 \frac{\binom{a(t)}{2}}{f(t)^2} dt\right) \prod_{j=r+1}^{k-1} \frac{1}{f(c_j)^2}$$

where $a(t) := \#\{j \in [k]; b_j > t\} - \#\{j \in \{r+1, \dots, k-1\}; c_j > t\}$ corresponds to the number of clusters at time t . Then for all $k \geq 1$, the joint law of the increasing binary forest \mathcal{F} associated with the genealogy (where the order of the children of each inner vertex is chosen uniformly at random), of the coalescing times $C_1 \leq \dots \leq C_{k-1}$ and birth times B_1, \dots, B_k is given by: for all $r \in \llbracket 0, k-1 \rrbracket$, for all $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$,

$$\begin{aligned} \mathbb{P}(\mathcal{F} = \mathcal{F}_0, B_1 \in db_1, \dots, B_k \in db_k, C_1 = 0, \dots, C_r = 0, C_{r+1} \in dc_{r+1}, \dots, C_{k-1} \in dc_{k-1}) \\ = g_{r,k}(\mathcal{F}_0, (b_j)_{j \in [k]}, (c_j)_{j \in \{r+1, \dots, k-1\}}) db_1 \cdots db_k dc_{r+1} \cdots dc_{k-1}. \end{aligned}$$

Proof. The proof is the same as the proof of the density formula (2) in [6] except that we use that the probability that none of the $r + 1$ clusters coalesce between time c_{r+1} and 0 is

$$\exp\left(-\int_0^{c_{r+1}} \frac{\binom{r+1}{2}}{f(t)^2} dt\right).$$

The factor $1/2^{k-1-r}$ comes from the uniform order assigned to the children of each internal node of \mathcal{F} . \square

Remark 6.3. The change of time relating the continuous coalescent in [6] and the continuous coalescent presented here is constructed as follows: for all $s \in [0, 1]$, let $G(s) = \int_0^s \frac{1}{2f(t)} dt$, which is an increasing continuous function from $[0, 1]$ to \mathbb{R}_+ . We define F as the inverse of G , extended to \mathbb{R}_+ by setting $F(t) = 1$ if $t > G(1)$. Let $\ell(t) = 2f(F(t))$. Then one can show that if we change the time by G , the continuous coalescent becomes the one introduced in [6]. Besides, if we assume condition (18), then the law of the coalescent is fully characterized by the density $g_{0,k}$ by Proposition 6.2, which is nothing else but the time-change of the density (2) in [6].

6.4 Gromov–Prokhorov convergence

Our goal is now to establish (16). Recall from (3) the notation $\mathbb{F}_n(\mathbf{x}^n) = \{i \in [n] : x_i^n = -1\}$, which denotes the labels of the frozen vertices of \mathcal{T}^n . Let V_1^n, \dots, V_k^n be k vertices in \mathcal{T}^n chosen uniformly (conditionally to the construction of \mathcal{T}^n). Let \mathcal{B}^n be the (rooted non-plane) binary tree associated with the genealogy of those k vertices in the coalescent construction: its leaves correspond to the birth-times $B_1^n := b_n(V_1^n), \dots, B_k^n := b_n(V_k^n)$, of V_1^n, \dots, V_k^n and its internal vertices correspond to a coalescence of two trees which contain each one a V_i^n for $i \in [k]$.

Note that $B_1^n + 1, \dots, B_k^n + 1 \in \mathbb{F}_n(\mathbf{x}^n)$ with large probability when $n \rightarrow \infty$, since the number of non frozen vertices at time n is of order \sqrt{n} . Moreover, conditionally on the fact that they belong to $\mathbb{F}_n(\mathbf{x}^n)$, the integers $B_1^n + 1, \dots, B_k^n + 1$ are taken uniformly in $\{i \in [n]; x_i^n = -1\}$. In the sequel, for the sake of simplicity, we will thus assume implicitly that they belong to $\mathbb{F}_n(\mathbf{x}^n)$.

Each coalescence between two trees both containing vertices among V_1^n, \dots, V_k^n in the coalescent construction is associated with an internal node v of \mathcal{B}^n and we write those coalescence times in the increasing order $C_1^n < \dots < C_{k-1}^n$. We also label the vertices of \mathcal{B}^n from 1 to $2k - 1$ in the increasing order of the associated coalescence times or birth times. We then equip \mathcal{B}^n with an ordering of the children of each internal nodes, taking into account which one of the two roots remains the root of the tree after the coalescence, so that \mathcal{B}^n is a plane tree. With these additional structures, $\mathcal{B}^n \in \mathfrak{B}_{0,k}^+$. If $r \in \{0, \dots, k - 1\}$, we denote by \mathcal{F}_r^n the increasing binary forest in $\mathfrak{F}_{r,k}^+$ which describes the genealogy but forgetting the r last coalescences.

Let us state a local limit estimate for $\left(\mathcal{F}^n, (B_j^n)_{j \in [k]}, (C_j^n)_{j \in [k-1]}\right)$.

Lemma 6.4. *let $k > r \geq 0$. Let $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$. For all $n \geq 1$, let b_1^n, \dots, b_k^n be a (not necessarily increasing) sequence in $\{0 \leq i < n, x_{i+1}^n = -1\}$ and let $c_{r+1}^n < \dots < c_{k-1}^n$ be an increasing sequence in $\{0 \leq i < n, x_{i+1}^n = 1\}$ such that the triplet $\left(\mathcal{F}_0, (b_j^n)_{j \in [k]}, (c_j^n)_{j \in \{r+1, \dots, k-1\}}\right)$ is admissible. Let $b_1, \dots, b_k \in [0, 1]$ and $0 < c_{r+1} < \dots < c_{k-1} < 1$ be distinct real numbers such that the triplet $(\mathcal{F}_0, (c_j)_{j \in \{r+1, \dots, k-1\}}, (b_j)_{j \in [k]})$ is admissible. We also assume that*

$$\forall j \in [k], \frac{b_j^n}{n} \xrightarrow[n \rightarrow \infty]{} b_j \text{ and } \forall j \in \{r+1, \dots, k-1\}, \frac{c_j^n}{n} \xrightarrow[n \rightarrow \infty]{} c_j.$$

Then we have the local limit convergence

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left(\frac{n}{2}\right)^{2k-1-r} \mathbb{P}\left(\mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = b_j^n, \forall j \in [r], C_j^n \leq \varepsilon n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = c_j^n\right) \\ = g_{r,k}(\mathcal{F}_0, (b_j)_{j \in [k]}, (c_j)_{j \in \{r+1, \dots, k-1\}}). \end{aligned} \quad (19)$$

In particular, if, for all $i, j \in [k]$ we denote by $C_{i,j}$ the coalescing time between the cluster containing the i -th particle with the cluster containing the j -th particle, then we have

$$\left(\left(B_j^n/n\right)_{j \in [k]}, \left(c_n(V_i^n, V_j^n)/n\right)_{i,j \in [k]}\right) \xrightarrow[n \rightarrow \infty]{(d)} \left((B_j)_{j \in [k]}, (C_{i,j})_{i,j \in [k]}\right). \quad (20)$$

Proof. For all $i \in [n]$, we set

$$a_i^n = \#\{j \in [k]; b_j^n \geq i\} - \#\{j \in \{r+1, \dots, k-1\}; c_j^n \geq i\}.$$

Notice that if we replace the c_j^n by C_j^n and b_j^n by B_j^n , then for $i \geq C_r^n + 1$ we have that a_i^n is the number of trees in $\mathcal{F}_i^n(\mathbf{x}^n)$ containing vertices among V_1^n, \dots, V_k^n .

Then, by the definition of Algorithm 2,

$$\begin{aligned}
& \mathbb{P}\left(C_1^n \leq \varepsilon n, \dots, C_r^n \leq \varepsilon n, \mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = b_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = c_j^n\right) \\
&= \mathbb{P}\left(\mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = b_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = c_j^n\right) \\
&\quad - \mathbb{P}\left(C_r^n > \varepsilon n, \mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = b_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = c_j^n\right) \\
&= \frac{1}{k! \binom{(n-S_r^n+1)/2+S_r^n}{k}} \left(\prod_{j=r+2}^k \prod_{\substack{i=c_{j-1}^n+2 \\ \text{s.t. } x_i^n=1}}^{c_j^n} \left(1 - \frac{\binom{a_i^n}{2}}{\binom{S_i^n}{2}}\right) \right) \prod_{j=r+1}^{k-1} \frac{1}{2 \binom{S_{j+1}^n}{2}} \\
&\quad - \sum_{\varepsilon n < c_r^n \leq n} \binom{r+1}{2} \frac{1}{k! \binom{(n-S_r^n+1)/2+S_r^n}{k}} \left(\prod_{j=r+1}^k \prod_{\substack{i=c_{j-1}^n+2 \\ \text{s.t. } x_i^n=1}}^{c_j^n} \left(1 - \frac{\binom{a_i^n}{2}}{\binom{S_i^n}{2}}\right) \right) \prod_{j=r}^{k-1} \frac{1}{2 \binom{S_{j+1}^n}{2}},
\end{aligned}$$

where by convention $c_k^n = n$. Indeed, the factor $1/(k! \binom{(n-S_r^n+1)/2}{k})$ comes from the choice of the uniform (distinct) birth-times B_j^n , the factor $1/(2 \binom{S_{j+1}^n}{2})$ corresponds to the coalescence at time c_j^n of a pair of trees containing some V_j^n but the choice of the trees is prescribed by \mathcal{F}_0 and its planar ordering. The factor $1 - \frac{\binom{a_i^n}{2}}{\binom{S_i^n}{2}}$ encodes the fact that there is no coalescence between the trees containing V_1^n, \dots, V_k^n at times different from c_1^n, \dots, c_{k-1}^n . Finally, the binomial coefficient $\binom{r+1}{2}$ comes from the different possibilities for the coalescence at time c_r^n (since the forest \mathcal{F}_0 does not impose the genealogical structure for this coalescence).

By dominated convergence and using Lemma 4.2 (i), it is easy to derive the local limit convergence

$$\begin{aligned}
& \mathbb{P}\left(C_1^n \leq \varepsilon n, \dots, C_r^n \leq \varepsilon n, \mathcal{F}_r^n = \mathcal{F}_0, \forall j \in [k], B_j^n = b_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = c_j^n\right) \\
&\underset{n \rightarrow \infty}{\sim} \left(\frac{2}{n}\right)^k \left(\prod_{j=r+2}^k \exp\left(-\int_{c_{j-1}}^{c_j} \frac{\binom{a(t)}{2}}{f(t)^2} dt\right) \right) \left(\prod_{j=r+1}^{k-1} \frac{1}{nf(c_j)^2} \right) \left(1 - \binom{r+1}{2} \int_{\varepsilon}^{c_{r+1}} \exp\left(-\int_c^{c_{r+1}} \frac{\binom{r+1}{2}}{f(t)^2} dt\right) \frac{1}{f(c)^2} dc\right) \\
&= \left(\frac{2}{n}\right)^k \exp\left(-\int_{c_{r+1}}^1 \frac{\binom{a(t)}{2}}{f(t)^2} dt\right) \left(\prod_{j=r+1}^{k-1} \frac{1}{nf(c_j)^2} \right) \left(1 - \binom{r+1}{2} \int_{\varepsilon}^{c_{r+1}} \exp\left(-\int_c^{c_{r+1}} \frac{\binom{r+1}{2}}{f(t)^2} dt\right) \frac{1}{f(c)^2} dc\right),
\end{aligned}$$

where by convention $c_k = 1$ and where $a(t)$ is defined in Proposition 6.2. Besides,

$$1 - \binom{r+1}{2} \int_{\varepsilon}^{c_{r+1}} \exp\left(-\int_c^{c_{r+1}} \frac{\binom{r+1}{2}}{f(t)^2} dt\right) \frac{1}{f(c)^2} dc \xrightarrow{\varepsilon \rightarrow 0} \exp\left(-\int_0^{c_{r+1}} \frac{\binom{r+1}{2}}{f(t)^2} dt\right),$$

hence the desired local limit (19).

For all $n \geq 1, \varepsilon > 0$, let $R^n(\varepsilon)$ be the smallest $r' \geq 0$ such that $C_{r'}^n > \varepsilon n$. Let

$$\begin{aligned}
& g_{\varepsilon, n, r, k} \left(\mathcal{F}_0, (b_j^n)_{j \in [k]}, (c_j^n)_{j \in \{r+1, \dots, k-1\}} \right) \\
&= \left(\frac{n}{2}\right)^{2k-1-r} \mathbb{P}\left(\mathcal{F}_{R^n(\varepsilon)}^n = \mathcal{F}_0, \forall j \in [k], B_j^n = b_j^n \text{ and } \forall j \in \{r+1, \dots, k-1\}, C_j^n = c_j^n\right).
\end{aligned}$$

Then (19) can be rewritten

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} g_{\varepsilon, n, r, k} \left(\mathcal{F}_0, (b_j^n)_{j \in [k]}, (c_j^n)_{j \in \{r+1, \dots, k-1\}} \right) = g_{r, k} \left(\mathcal{F}_0, (b_j)_{j \in [k]}, (c_j)_{j \in \{r+1, \dots, k-1\}} \right). \quad (21)$$

Let us check that this implies the convergence in law

$$\left(\mathcal{F}_{R^n(\varepsilon)}^n, \left(\frac{B_j^n}{n}\right)_{j \in [k]}, \left(\frac{C_j^n}{n}\right)_{j \in [k-1]} \right) \xrightarrow{n \rightarrow \infty, \varepsilon \rightarrow 0} (\mathcal{F}, (B_j)_{j \in [k]}, (C_j)_{j \in [k-1]}). \quad (22)$$

As before, let $r \in \{0, \dots, k-1\}$ and let $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$. For all $j \in \{r+1, \dots, k-1\}$, let \tilde{C}_j^n be independent uniform random variables in $\{0 \leq i < n; x_{i+1}^n = 1\}$. By Lemma 4.2 (i) and the analogous statement for +1 steps, we know that

$$\left(\left(\frac{B_j^n}{n} \right)_{j \in [k]}, \left(\frac{\tilde{C}_j^n}{n} \right)_{j \in \{r+1, \dots, k-1\}} \right) \xrightarrow[n \rightarrow \infty]{(d)} \left((B_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \{r+1, \dots, k-1\}} \right), \quad (23)$$

where the \tilde{C}_j 's are i.i.d. uniform random variables in $[0, 1]$ which are independent of the B_j 's. By Skorokhod's representation theorem, we may assume that the above convergence holds almost surely. As a consequence, by (21), we have a.s.

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} g_{\varepsilon, n, r, k} \left(\mathcal{F}_0, (B_j^n)_{j \in [k]}, (\tilde{C}_j^n)_{j \in \{r+1, \dots, k-1\}} \right) = g_{r, k} \left(\mathcal{F}_0, (B_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \{r+1, \dots, k-1\}} \right), \quad (24)$$

where we set $g_{r, k}(\mathcal{F}_0, (b_j)_{1 \leq j \leq k}, (c_j)_{j \in \{r+1, \dots, k-1\}}) = 0$ if the triplet $(\mathcal{F}_0, (c_j)_{j \in \{r+1, \dots, k-1\}}, (b_j)_{j \in [k]})$ is not admissible. But by definition of $g_{\varepsilon, n, r, k}$ we have

$$\begin{aligned} \mathbb{P} \left(\mathcal{F}_{R^n(\varepsilon)}^n = \mathcal{F}_0 \right) &= \frac{(n - \#\mathbb{F}_n(\mathbf{x}^n))^{k-1-r} k! \binom{\#\mathbb{F}_n(\mathbf{x}^n)}{k}}{\left(\frac{n}{2}\right)^{2k-1-r}} \cdot \mathbb{E} \left[g_{\varepsilon, n, r, k} \left(\mathcal{F}_0, \left(\frac{B_j^n}{n} \right)_{j \in [k]}, \left(\frac{\tilde{C}_j^n}{n} \right)_{j \in \llbracket r+1, k-1 \rrbracket} \right) \right] \\ &\sim_{n \rightarrow \infty} \mathbb{E} \left[g_{\varepsilon, n, r, k} \left(\mathcal{F}_0, (B_j^n)_{j \in [k]}, (\tilde{C}_j^n)_{j \in \{r+1, \dots, k-1\}} \right) \right] \end{aligned}$$

when the latter quantity is positive, since $\#\mathbb{F}_n(\mathbf{x}^n) \sim n/2$ as $n \rightarrow \infty$. In addition,

$$\mathbb{E} \left[g_{r, k} \left(\mathcal{F}_0, (B_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \{r+1, \dots, k-1\}} \right) \right] = \mathbb{P}(\mathcal{F} = \mathcal{F}_0).$$

Therefore, using (24) with Fatou's lemma and then using the fact that if we sum over all possible $\mathcal{F}_0 \in \bigcup_{0 \leq r \leq k-1} \mathfrak{F}_{r,k}^+$ then we get one, one can see that

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P} \left(\mathcal{F}_{R^n(\varepsilon)}^n = \mathcal{F}_0 \right) = \mathbb{P}(\mathcal{F} = \mathcal{F}_0).$$

So

$$\liminf_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{E} \left[g_{\varepsilon, n, r, k} \left(\mathcal{F}_0, (B_j^n)_{j \in [k]}, (\tilde{C}_j^n)_{j \in \{r+1, \dots, k-1\}} \right) \right] = \mathbb{E} \left[g_{r, k} \left(\mathcal{F}_0, (B_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \{r+1, \dots, k-1\}} \right) \right].$$

Here we use the following result: if $(X_n^k)_{n \geq 1, k \in I}$, X are nonnegative random variables, if $X_n^k \rightarrow X^k$ almost surely as $n \rightarrow \infty$ for every $k \in I$, if for every $n \geq 1$ we have $\sum_{k \in I} \mathbb{E}[X_n^k] = \sum_{k \in I} \mathbb{E}[X^k] = 1$, then $\mathbb{E}[X_n^k] \rightarrow \mathbb{E}[X^k]$ as $n \rightarrow \infty$ for every $k \in I$. Thus, by Scheffé's lemma, the convergence in (24) holds also in \mathbb{L}^1 . Let $\varepsilon_0 > 0$. Let $F : \bigcup_{0 \leq r < k} \mathfrak{F}_{r,k} \times [0, 1]^k \times [0, 1]^{k-1} \rightarrow \mathbb{R}$ be a continuous bounded function which is constant on the elements which have one coordinate in $[0, \varepsilon_0]$. Then for all $\varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} &\mathbb{E} \left[F \left(\mathcal{F}_{R^n(\varepsilon)}^n, \left(\frac{B_j^n}{n} \right)_{j \in [k]}, \left(\frac{C_j^n}{n} \right)_{j \in [k-1]} \right) \right] \\ &= \sum_{0 \leq r < k, \mathcal{F}_0 \in \mathfrak{F}_{r,k}} \frac{(n - \#\mathbb{F}_n(\mathbf{x}^n))^{k-1-r} k! \binom{\#\mathbb{F}_n(\mathbf{x}^n)}{k}}{\left(\frac{n}{2}\right)^{2k-1-r}} \\ &\quad \cdot \mathbb{E} \left[F \left(\mathcal{F}_0, \left(\frac{B_j^n}{n} \right)_{j \in [k]}, \left(\frac{\tilde{C}_j^n}{n} \right)_{j \in [k-1]} \right) g_{\varepsilon, n, r, k} \left(\mathcal{F}_0, \left(\frac{B_j^n}{n} \right)_{j \in [k]}, \left(\frac{\tilde{C}_j^n}{n} \right)_{j \in \llbracket r+1, k-1 \rrbracket} \right) \right] \\ &\xrightarrow[n \rightarrow \infty]{\varepsilon \rightarrow 0} \sum_{0 \leq r < k, \mathcal{F}_0 \in \mathfrak{F}_{r,k}} \mathbb{E} \left[F \left(\mathcal{F}_0, (B_j)_{j \in [k]}, (\tilde{C}_j)_{j \in [k-1]} \right) g_{r, k} \left(\mathcal{F}_0, (B_j)_{j \in [k]}, (\tilde{C}_j)_{j \in \llbracket r+1, k-1 \rrbracket} \right) \right] \\ &= \mathbb{E} \left[F(\mathcal{F}, (B_j)_{j \in [k]}, (C_j)_{j \in [k-1]}) \right], \end{aligned}$$

where by convention for all $j \in [r]$ we have set $\widetilde{C}_j^n = 0$ and $\widetilde{C}_j = 0$; here we use the following generalized version of the dominated convergence theorem: if X_n converges almost surely to X , if $|X_n| \leq Y_n$ almost surely, if Y_n converges in \mathbb{L}^1 to Y , then X_n converges in \mathbb{L}^1 to X . The convergence (22) follows.

Now, one can see that given the forest $\mathcal{F}_{R^n(\varepsilon)}^n$ and given the order of the births, one can recover the order of the coalescences occurring at times greater than εn , hence identifying to which C_j^n 's the $c_n(V_i^n, V_\ell^n)$'s which are greater than εn are equal. More precisely, for all $r \in \{0, \dots, k-1\}$, for all $\mathcal{F}_0 \in \mathfrak{F}_{r,k}^+$, for all permutation σ of $\{1, \dots, k\}$ encoding a labelling by $1, \dots, k$ of the leafs of \mathcal{F}_0 , we associate the subset $I \subseteq [k]^2$ of couples of labelled leaves which have a common ancestor in \mathcal{F}_0 and the unique continuous function $\varphi_{\mathcal{F}_0, \sigma}$ from $\{(c_{r+1}, \dots, c_{k-1}) \in [0, 1]^{k-r-1}, \varepsilon < c_{r+1} < \dots < c_{k-1}\}$ to $[0, 1]^I$ which sends $(c_{r+1}, \dots, c_{k-1})$ to the unique family $(c_{i,j})_{(i,j) \in I}$ such that for all $(i, j) \in I$, the entry $c_{i,j}$ equals c_ℓ where the nearest common ancestor of the leaves labelled i, j by σ has the ℓ -th smallest label among the internal nodes of the labelled forest \mathcal{F}_0 .

Let $\varepsilon > 0$ and $n \geq 1$. Let π^n (resp. π) be the random uniform permutation associated with the ordering of B_1^n, \dots, B_k^n (resp. B_1, \dots, B_k), which defines a labelling of the leaves of $\mathcal{F}_{R^n(\varepsilon)}^n$ (resp. \mathcal{F}). Let $I_\varepsilon^n = \{(i, j) \in [k]^2, c_n(V_i^n, V_j^n) > \varepsilon n\}$. Then

$$\left(\frac{c_n(V_i^n, V_j^n)}{n} \right)_{(i,j) \in I_\varepsilon^n} = \varphi_{\mathcal{F}_{R^n(\varepsilon)}^n, \pi^n} \left(\left(\frac{C_j^n}{n} \right)_{j \in \{R^n(\varepsilon)+1, \dots, k-1\}} \right),$$

hence the convergence in law (20) by (22). \square

Proof of (16). Now, let us prove the convergence in law of $(d^n(V_j^n, V_\ell^n)/\sqrt{n})_{1 \leq j, \ell \leq k}$. For all $j, \ell \in [k]$,

$$D^n(V_j^n, V_\ell^n) = \sum_{i=c_n(V_j^n, V_\ell^n)}^{B_j^n} \frac{1}{2S_i^n} + \sum_{i=c_n(V_j^n, V_\ell^n)}^{B_\ell^n} \frac{1}{2S_i^n}.$$

Using Lemma 4.2 (i) and the convergence (20), it is henceforth straightforward to conclude that

$$\left(\frac{d^n(V_j^n, V_\ell^n)}{\sqrt{n}} \right)_{1 \leq j, \ell \leq k} \xrightarrow[n \rightarrow \infty]{(d)} \left(\int_{C_{j,\ell}}^{B_j} \frac{1}{2f(t)} dt + \int_{C_{j,\ell}}^{B_\ell} \frac{1}{2f(t)} dt \right)_{1 \leq j, \ell \leq k}. \quad (25)$$

\square

6.5 Gromov–Hausdorff tightness

We now establish (17). We first introduce some notation: for $n \geq 1$, let V^n and $(V_i^n)_{i \geq 1}$ be independent uniform vertices in \mathbb{V}_n . For simplicity, we drop the superscript and write $V, (V_i)_{i \geq 1}$. For $u, v \in \mathbb{V}_n$ and $B \subset \mathbb{V}_n$ we write

$$\Delta^n(u, v) := \frac{1}{2}(b_n(u) + b_n(v) - 2c_n(u, v)), \quad \Delta^n(u, B) := \min_{v \in B} \Delta^n(u, v).$$

Finally, we set

$$M_n := \max_{0 \leq i \leq n} S_i^n.$$

Our main technical input is the following uniform estimate.

Proposition 6.5. *For every $k, n \geq 1$ large enough with $k \leq n^{1/3}$,*

$$\mathbb{P} \left(\Delta^n(V, \{V_1, \dots, V_k\}) > \frac{\ln(k)^2}{\sqrt{k}} n \right) \leq \frac{1}{k^4}.$$

The idea underlining the proof of Proposition 6.5 is the following: on every time-interval of length n/\sqrt{k} , we can find roughly \sqrt{k} vertices among V_1, \dots, V_k and the probability that \sqrt{k} vertices do not coalesce with a given vertex during n/\sqrt{k} time steps is strictly positive. Let us first explain how tightness follows from this estimate.

Proof of Theorem 2.3 (tightness) using Proposition 6.5. Step 1. We first check that for every $\varepsilon > 0$ we have

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\max_{v \in \mathbb{V}_n} \Delta^n(v, \{V_1, \dots, V_k\})/n > \varepsilon \right) = 0. \quad (26)$$

To this end, we apply the so-called chaining method. Specifically, for $k, n \geq 1$ we set

$$\varepsilon_k^n := \max_{2^k < i \leq 2^{k+1}} \Delta^n(V_i, \{V_1, \dots, V_{2^k}\}),$$

and roughly speaking, we show that $\varepsilon_k^n/n \lesssim 2^{-k/2}$, where \lesssim denotes an informal upper bound. Indeed, this entails

$$\frac{1}{n} \max_{v \in \mathbb{V}_n} \Delta^n(v, \{V_1, \dots, V_{2^k}\}) \lesssim \frac{1}{n} \sum_{i \geq k} \varepsilon_i^n \lesssim 2^{-k/2}$$

and (26) will follow. Let us now be more precise. For $n \geq 1$, let N_n be the largest integer with $2^{N_n} \leq n^{1/3}$ and set $\varepsilon_\infty^n := \max_{v \in \mathbb{V}_n} \Delta^n(v, \{V_1, \dots, V_{2^{N_n}}\})$. By Proposition 6.5, for every k, n large enough with $k < N_n$,

$$\mathbb{P} \left(\varepsilon_k^n > k^2 2^{-k/2} n \right) \leq 2^k \mathbb{P} \left(\Delta^n(V, \{V_1, \dots, V_{2^k}\}) > k^2 2^{-k/2} n \right) \leq 2^{-3k}.$$

Similarly, since there are at most n vertices in \mathcal{T}^n ,

$$\mathbb{P} \left(\varepsilon_\infty^n > (N_n)^2 2^{-N_n/2} n \right) \leq n \mathbb{P} \left(\Delta^n(V, \{V_1, \dots, V_{2^{N_n}}\}) > (N_n)^2 2^{-N_n/2} n \right) \leq n 2^{-4N_n} = o(1).$$

Hence, for every k large enough, for every n large enough, with probability at least $1 - 2 \sum_{i=k}^\infty 2^{-3i} \geq 1 - 2^{-k}$,

$$\max_{v \in \mathbb{V}_n} \Delta^n(v, \{V_1, \dots, V_{2^k}\}) \leq \sum_{i=k}^{N_n-1} \varepsilon_i^n + \varepsilon_\infty^n \leq \sum_{i=k}^{N_n} i^2 2^{-i/2} n \leq 2^{-k/3} n.$$

The convergence (26) readily follows.

Step 2. We now show (17) using (26). By Proposition 4.3 (applied with $\alpha = 1/2$), it suffices to show that for every $\varepsilon > 0$,

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\max_{v \in \mathbb{V}_n} \min_{1 \leq i \leq k} D^n(v, V_i)/\sqrt{n} > \varepsilon \right) = 0. \quad (27)$$

Now, since $(\mathbf{x}^n)_{n \in \mathbb{N}}$ is $1/2$ -tight (recall (Tight $_\alpha$)), we may fix $\delta > 0$ such that

$$\sum_{i=1}^n \frac{1}{S_i^n} \mathbb{1}_{S_i^n \leq \delta \sqrt{n}} \leq \varepsilon \sqrt{n}.$$

Then, by the definition of D^n , write for every $\delta > 0$ and $u, v \in \mathbb{V}_n$:

$$\frac{D^n(u, v)}{\sqrt{n}} \leq \sum_{i=1}^n \frac{1}{\sqrt{n} S_i^n} \mathbb{1}_{S_i^n \leq \delta \sqrt{n}} + \frac{\Delta^n(u, v)}{n\delta} \leq \varepsilon + \frac{\Delta^n(u, v)}{n\delta}.$$

The estimate (27) then follows from (26). \square

We now turn to the proof of Proposition 6.5. Recall that for $n \in \mathbb{N}$, $M_n := \max_{0 \leq i \leq n} S_i^n$. Let $M > 0$ be such that $M_n \leq M\sqrt{n}$ for all $n \geq 1$. (This is possible by (Lim $_\alpha^c$)). For $k, n \in \mathbb{N}$ let $\delta_{k,n} := 4 \lceil Mn/\sqrt{k} \rceil$. Observe that $\delta_{k,n} > M_n$ for k, n sufficiently large with $k \leq n^{1/3}$.

Lemma 6.6. For every $k, n \geq 1$ large enough with $k \leq n^{1/3}$, for every $0 \leq i \leq n/\delta_{k,n}$, for every $v_0 \in \mathbb{V}_n$ with $b_n(v_0) > (i+1)\delta_{k,n}$, for every $K \geq 1$ and $v_1 \neq \dots \neq v_K \in \mathbb{V}_n$ such that $i\delta_{k,n} < b_n(v_j) \leq (i+1)\delta_{k,n}$, we have

$$\mathbb{P}\left(\text{the tree of } \mathcal{F}_{(i-1)\delta_{k,n}}^n \text{ containing } v_0 \text{ does not contain any vertex among } v_1, \dots, v_K\right) \leq \frac{3M\sqrt{k}}{4K}.$$

Proof. For every $0 \leq t \leq n$, we define N_t^n by induction as follows: $N_n^n = K + 1$, and then for $1 \leq t \leq n$ we set $N_{t-1}^n = N_t^n - 1$ if $x_t^n = 1$ and two trees of \mathcal{F}_t^n which contain a vertex in $\{v_0, \dots, v_K\}$ coalesce at time $t - 1$ and $N_{t-1}^n = N_t^n$ otherwise. In words, N_t^n represents the number of trees in \mathcal{F}_t^n which contain a vertex in $\{v_0, \dots, v_K\}$, taking also into account vertices which are not born yet.

Step 1. We first show that

$$\begin{aligned} \mathbb{P}\left(\text{the tree of } \mathcal{F}_{(i-1)\delta_{k,n}}^n \text{ containing } v_0 \text{ does not contain any vertex among } v_1, \dots, v_K\right) \\ \leq \frac{1}{K+1} \mathbb{E}\left[N_{(i-1)\delta_{k,n}}^n\right]. \end{aligned} \quad (28)$$

To this end, we say that $u \in \{v_0, \dots, v_K\}$ stays alone at time t if either $t > b_n(u)$, or if the tree of \mathcal{F}_t^n which contains u does not contain any other vertex of $\{v_0, \dots, v_K\} \setminus \{u\}$. Plainly, by definition N_t^n is at least equal to the number of vertices which stay alone at time t . Moreover, since $b_n(v_0) > b_n(v_i)$ for $1 \leq i \leq K$, v_0 is the vertex with the less chance to stay alone at time t among $\{v_0, \dots, v_K\}$. Hence

$$\frac{1}{K+1} \mathbb{E}\left[N_t^n\right] \geq \frac{1}{K+1} \sum_{i=0}^{K+1} \mathbb{P}(v_i \text{ stays alone at time } t) \geq \mathbb{P}(v_0 \text{ stays alone at time } t).$$

The bound (28) follows by noting that if v_0 stays alone at time t then the tree of \mathcal{F}_t^n containing v_0 does not contain any vertex among v_1, \dots, v_K , with $t = (i-1)\delta_{k,n}$.

Step 2. We next show that

$$\mathbb{E}\left[N_{(i-1)\delta_{k,n}}^n\right] \leq 1 + \frac{2(M_n)^2}{\delta_{k,n} - M_n - 1} \quad (29)$$

for $k, n \geq 1$ sufficiently large with $k \leq n^{1/3}$. For every $(i-1)\delta_{k,n} < t \leq i\delta_{k,n}$, depending on whether two trees which contain a vertex in $\{v_0, \dots, v_K\}$ coalesce or not at time $t - 1$ in Algorithm 2, we get

$$\mathbb{E}\left[N_{t-1}^n\right] = \mathbb{E}\left[N_t^n\right] - \mathbb{1}_{x_t^n=1} \frac{\mathbb{E}\left[N_t^n(N_t^n - 1)\right]}{S_t^n(S_t^n - 1)} \leq \mathbb{E}\left[N_t^n\right] - \mathbb{1}_{x_t^n=1} \frac{\mathbb{E}\left[N_t^n\right]\mathbb{E}\left[N_t^n - 1\right]}{(M_n)^2}.$$

Then, for every $(i-1)\delta_{k,n} < t \leq i\delta_{k,n}$, write

$$\frac{1}{\mathbb{E}\left[N_{t-1}^n\right] - 1} - \frac{1}{\mathbb{E}\left[N_t^n\right] - 1} = \frac{\mathbb{E}\left[N_t^n\right] - \mathbb{E}\left[N_{t-1}^n\right]}{(\mathbb{E}\left[N_{t-1}^n\right] - 1)(\mathbb{E}\left[N_t^n\right] - 1)} \geq \frac{\mathbb{1}_{x_t^n=1}\mathbb{E}\left[N_t^n\right]\mathbb{E}\left[N_t^n - 1\right]/(M_n)^2}{(\mathbb{E}\left[N_{t-1}^n\right] - 1)(\mathbb{E}\left[N_t^n\right] - 1)} \geq \frac{\mathbb{1}_{x_t^n=1}}{(M_n)^2}.$$

As a consequence, for $k, n \geq 1$,

$$\frac{1}{\mathbb{E}\left[N_{(i-1)\delta_{k,n}}^n\right] - 1} \geq \frac{1}{\mathbb{E}\left[N_{i\delta_{k,n}}^n\right] - 1} + \sum_{j=(i-1)\delta_{k,n}+1}^{i\delta_{k,n}} \frac{\mathbb{1}_{x_j^n=1}}{(M_n)^2} \geq \frac{S_{i\delta_{k,n}}^n - S_{(i-1)\delta_{k,n}}^n + i\delta_{k,n} - (i-1)\delta_{k,n}}{2(M_n)^2}$$

and (29) follows $k, n \geq 1$ sufficiently large with $k \leq n^{1/3}$ (so that $\delta_{k,n} > M_n$).

Step 3. To complete the proof of Lemma 6.6, it just remains to note that for $k, n \geq 1$ sufficiently large with $k \leq n^{1/3}$ we have $\delta_{k,n} - M_n - 1 \geq 4Mn/\sqrt{k} - M_n - 1 \geq 3Mn/\sqrt{k}$ so that

$$\frac{1}{K+1} \left(1 + \frac{2(M_n)^2}{\delta_{k,n} - M_n - 1}\right) \leq \frac{1}{K+1} \left(1 + \frac{2M^2n}{3Mn/\sqrt{k}}\right) \leq \frac{M\sqrt{k}}{K+1} \left(\frac{1}{M\sqrt{k}} + \frac{2}{3}\right) \leq \frac{3M\sqrt{k}}{4K}.$$

This completes the proof. \square

Proof of Proposition 6.5. Recall that $M_n \leq M\sqrt{n}$ for every $n \geq 1$ and that $\delta_{k,n} = 4\lceil Mn/\sqrt{k} \rceil$.

To simplify notation, set $b_n(V_j) = b_n^j$ for $1 \leq j \leq n$ and $b_n = b_n(V)$ and define the event E by

$$E = \left\{ \forall 0 \leq i \leq \lfloor n/\delta_{k,n} \rfloor - 1, \#\{b_n^j : 1 \leq j \leq k \text{ and } i\delta_{k,n} < b_n^j \leq (i+1)\delta_{k,n}\} > M\sqrt{k} \right\}.$$

Step 1. We first check that

$$\mathbb{P}(E^c) = o\left(\frac{1}{k^4}\right). \quad (30)$$

To this end, first note that for every $0 \leq i \leq \lfloor n/\delta_{k,n} \rfloor - 1$,

$$\sum_{i\delta_{k,n} < t \leq (i+1)\delta_{k,n}} \mathbb{1}_{x_t^n = -1} = (\delta_{k,n} - S_{(i+1)\delta_{k,n}}^n + S_{i\delta_{k,n}}^n)/2 \geq \delta_{k,n}/2 - M\sqrt{n}/2 \geq \frac{3Mn}{2\sqrt{k}}$$

for k, n sufficiently large with $k \leq n^{1/3}$. Since there are at most n vertices in \mathcal{T}^n , it follows that we have $\mathbb{P}(i\delta_{k,n} < b_n \leq (i+1)\delta_{k,n}) \geq 3M/(2\sqrt{k}) - O(1/\sqrt{n})$. Next, if $\text{Bin}(k, p)$ denotes a Binomial random variable with parameter (k, p) , by Bennett's inequality (Proposition 4.1) we have for $0 < u < pk$:

$$\mathbb{P}(\text{Bin}(k, p) < u) \leq \exp\left(-pk g\left(1 - \frac{u}{pk}\right)\right).$$

Since $\text{Bin}(k, p)$ is stochastically increasing in p , it follows that

$$\mathbb{P}\left(\#\{1 \leq j \leq k : b_n^j \in (i\delta_{k,n}, (i+1)\delta_{k,n})\} \leq 4M\sqrt{k}\right) \leq \exp\left(-6M\sqrt{k}g(1/3)\right) \leq \exp\left(-\frac{\sqrt{k}}{100}\right),$$

since M can be chosen large. So, by a union bound,

$$\mathbb{P}\left(\exists 0 \leq i \leq \lfloor n/\delta_{k,n} \rfloor - 1, \#\{1 \leq j \leq k : b_n^j \in (i\delta_{k,n}, (i+1)\delta_{k,n})\} \leq 4M\sqrt{k}\right) \leq \frac{n}{\delta_{k,n}} e^{-\sqrt{k}/100} \leq \frac{\sqrt{k}}{4M} e^{-\sqrt{k}/100},$$

which is $o(k^{-4})$. To establish (30) it thus remains to check that

$$\mathbb{P}(\exists 1 \leq i_1 < i_2 < i_3 < i_4 \leq k : b_n^{i_1} = b_n^{i_2} = b_n^{i_3} = b_n^{i_4}) = o\left(\frac{1}{k^4}\right).$$

To this end, using the fact that there are at least $n/3$ vertices in \mathcal{T}^n for n sufficiently large, a union bound yields $\mathbb{P}(\exists 1 \leq i_1 < i_2 < i_3 < i_4 \leq k : b_n^{i_1} = b_n^{i_2} = b_n^{i_3} = b_n^{i_4}) \leq 27k^4/n^3 = o(1/k^4)$ since $k \leq n^{1/3}$.

Step 2. Next we need some more notation. Set

$$\mathbb{V}_n(i) = \{V_j : 1 \leq j \leq k \text{ and } i\delta_{k,n} < b_n^j \leq (i+1)\delta_{k,n}\}.$$

Observe that under E we have $\#\mathbb{V}_n(i) \geq M\sqrt{k}$ for every $0 \leq i \leq \lfloor n/\delta_{k,n} \rfloor - 1$. Setting $p_k = \lfloor 100 \ln(k) \rfloor$, we show that

$$\mathbb{P}(\Delta^n(V, \{V_1, \dots, V_k\}) > 2p_k \delta_{k,n} \mid E) = o(k^{-4}), \quad (31)$$

and the desired result will follow

First case: b_n is small. First, if $b_n \leq 2p_k \delta_{k,n}$, under E there exists $1 \leq j \leq k$ such that $b_n^j \leq 2\delta_{k,n}$, and then $\Delta^n(V, \{V_1, \dots, V_k\}) \leq \Delta^n(V, V_j) \leq 2p_k \delta_{k,n}$. Thus

$$\mathbb{P}(\Delta^n(V, \{V_1, \dots, V_k\}) > 2p_k \delta_{k,n} \mid E, b_n \leq 2p_k \delta_{k,n}) = 0$$

Second case. b_n is large. For $i \geq 1$, let A_i be the event:

“the tree of $\mathcal{T}_{(i-1)\delta_{k,n}}^n$ containing V does not contain any vertex of $\mathbb{V}_n(i)$.”

By definition of Algorithm 2, for $i_0 \geq 2p_k$, conditionally given E and $i_0\delta_{k,n} < b_n \leq (i_0 + 1)\delta_{k,n}$, the events $(A_i)_{2 \leq i \leq i_0-1, i \text{ even}}$ are independent and by Lemma 6.6 (applied with $K = \lceil M\sqrt{k} \rceil$) their probability is at most $3/4$. As a consequence, for every $i_0 \geq 2p_k$:

$$\mathbb{P}\left(\bigcap_{i=i_0-p_k-1}^{i_0-1} A_i \mid E, i_0\delta_{k,n} < b_n \leq (i_0 + 1)\delta_{k,n}\right) \leq (3/4)^{(p_k-1)/2}.$$

But then observe that when $i_0\delta_{k,n} < b_n \leq (i_0 + 1)\delta_{k,n}$, if for $i_0 - p_k - 1 \leq i \leq i_0 - 1$, if the tree of $\mathcal{F}_{i\delta_{k,n}}^n$ containing V contains a vertex of $\mathbb{V}_n(i)$, say V_j , then

$$\Delta^n(V, V_j) \leq \frac{1}{2}((i_0 + 1)\delta_{k,n} + (i + 1)\delta_{k,n} - 2(i - 1)\delta_{k,n}) \leq \left(\frac{i_0 - i}{2} + 3\right)\delta_{k,n} + 1 \leq 2p_k\delta_{k,n}$$

for k, n sufficiently large. We conclude from the previous discussion that

$$\mathbb{P}(\Delta^n(V, \{V_1, \dots, V_k\}) > 2p_k\delta_{k,n} \mid E, b_n > 2p_k\delta_{k,n}) = o\left((3/4)^{(p_k-1)/2}\right) = o(k^{-4}),$$

where the last equality comes from our choice of p_k . This completes the proof. \square

6.6 Properties of $\mathcal{T}(f)$

Here we study some properties of $\mathcal{T}(f)$ by establishing Theorem 2.4.

Proof of Theorem 2.4 (1). The fact that the mass measure on $\mathcal{T}(f)$ has full support was already mentioned in Section 6.1 as a consequence of Lemma B.1, taking into account (16) and (17). The fact that it gives full support to the leaves follows from the property the coalescence time of two particles of the continuous-time coalescent is strictly smaller than the birth times, combined with (20) and (25): this implies that for two vertices sampled uniformly at random in $\mathcal{T}(f)$, one of them is not a descendant of the other. \square

Proof of Theorem 2.4 (3). Let us first assume that $\int_0 \frac{1}{f^2} < \infty$. To show that the degree of the root of $\mathcal{T}(f)$ is a.s. infinite, it suffices to show that a.s. there exist infinitely many clusters which do not coalesce with each other. First, observe that since $\int_0 \frac{1}{f^2} < \infty$, we can build a decreasing sequence $(\varepsilon_k)_{k \geq 0}$ of positive numbers such that

$$\int_{\varepsilon_{k+1}}^{\varepsilon_k} \frac{1}{f^2} \leq \frac{2^{-k}}{\binom{k}{2}}.$$

Next, since $(B_i)_{i \geq 1}$ are i.i.d. uniform random variables, we know that a.s. there exists a subsequence $(B_{i_k})_{k \geq 1}$ such that for all $k \geq 1$, we have $B_{i_k} \in [\varepsilon_{k+1}, \varepsilon_k]$. But then, conditionally on the B_{i_k} 's, for all $k_0 \geq 1$, the probability that none of the clusters containing a B_{i_k} for $k \geq k_0$ coalesce with each other is

$$\prod_{k \geq k_0} \exp\left(-\int_{B_{i_{k+1}}}^{B_{i_k}} \frac{\binom{k}{2}}{f(t)^2} dt\right) \geq \prod_{k \geq k_0} \exp\left(-\int_{\varepsilon_{k+1}}^{\varepsilon_k} \frac{\binom{k}{2}}{f(t)^2} dt\right) \geq \exp\left(-\sum_{k \geq k_0} 2^{-k}\right) \xrightarrow[k_0 \rightarrow \infty]{} 1.$$

Thus, with probability one there exist infinitely many clusters which do not coalesce with each other. Conversely, when $\int_0 \frac{1}{f^2} = \infty$, the probability that two clusters which have not merged yet at time $t > 0$ do not coalesce until time zero is $\exp\left(-\int_0^t \frac{1}{f(t)^2} dt\right) = 0$, which proves that the root has a.s. degree 1. For the other branchpoints, the fact that they have degree two is due to the fact that a.s. the coalescences happen at distinct times and thus at distinct heights. \square

6.7 Application: the Brownian CRT

Here we establish Corollary 2.5.

Proof of Corollary 2.5. The idea is to apply Theorem 2.3 with a random sequence $(\mathbf{x}^{2n+1})_{n \geq 1}$ chosen in such a way that $(S_k^{2n+1})_{0 \leq k \leq 2n+1}$ has the law of a simple random walk started at 1 and conditioned to hit 0 for the first time at time $2n+1$. Indeed, by [8, Theorem 2], as a random compact metric space, \mathcal{T}_{2n+1} has the same law as a uniform plane tree with $n+1$ vertices, so that $\frac{1}{\sqrt{2n+1}} \cdot \mathcal{T}_{2n+1}$ converges in distribution to the Brownian CRT. As a consequence, it suffices to check that the conditions of application of Theorem 2.3 are satisfied almost surely with the sequence $(\mathbf{x}^{2n+1})_{n \geq 1}$ and $f = \mathfrak{e}$.

First, it is standard that the convergence

$$\left(\frac{S_{(2n+1)t}^{2n+1}}{\sqrt{2n+1}} : 0 \leq t \leq 1 \right) \xrightarrow[n \rightarrow \infty]{(d)} \mathfrak{e} \quad (32)$$

holds in distribution in the space of all real-valued continuous functions defined on $[0, 1]$ equipped with the topology of uniform convergence, where \mathfrak{e} is the Brownian excursion. Thus (H_f) and (Lim_f^α) hold with $f = \mathfrak{e}$.

It remains to check (Tight_α) . To simplify notation, for $n \geq 1$ and $0 \leq \eta < \delta \leq 1$ set

$$X_n(\delta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{S_i^{2n+1}} \mathbb{1}_{S_i^{2n+1} \leq \delta \sqrt{n}}, \quad X(\eta, \delta) = \int_0^1 \frac{1}{\mathfrak{e}(t)} \mathbb{1}_{\eta \leq \mathfrak{e}(t) \leq \delta} dt.$$

It suffices to check that for every $\varepsilon > 0$, for a well chosen coupling, almost surely there exists $\delta > 0$ such that every $n \geq 1$ sufficiently large we have $X_n(\delta) \leq \varepsilon$.

Step 1. We show that

$$\mathbb{P}(X_n(\delta) \geq \varepsilon) \leq \frac{C}{\varepsilon} \delta^2 \quad (33)$$

for all $n \geq 1$ for a certain constant $C > 0$. This is essentially [13, Lemma 3.6], but let us give a different proof based on the local limit theorem which gives a slightly better upper bound. To this end, denote by $(S_i)_{i \geq 0}$ a simple random walk started from 1. By Kemperman's formula (see e.g. [22, Sec. 6.1]), we have for $1 \leq i \leq 2n+1$

$$\mathbb{P}(S_i^{2n} = k) = \frac{2n+1}{\mathbb{P}(S_{2n+1} = 0)} \cdot \frac{k-1}{i} \mathbb{P}(S_i = k) \cdot \frac{k}{2n-i+1} \mathbb{P}(S_{2n-i+1} = -k+1). \quad (34)$$

By the local limit theorem (see e.g. [18, Theorem 4.2.1]), we get for all $n \geq 1$, for $1 \leq i \leq n$,

$$\mathbb{P}(S_i^{2n} = k) \leq C n^{3/2} \cdot \frac{k}{i} \cdot \frac{1}{\sqrt{i}} \cdot \frac{k}{n} \cdot \frac{1}{\sqrt{n}} = C \frac{k^2}{i^{3/2}},$$

where C is a constant that may change from line to line. It follows that

$$\mathbb{E} \left[\frac{1}{S_i^{2n}} \mathbb{1}_{S_i^{2n} \leq \delta \sqrt{n}} \right] \leq \sum_{k=1}^{\delta \sqrt{n}} C \frac{k}{i^{3/2}} \leq C \frac{\delta^2 n}{i^{3/2}}.$$

Thus

$$\mathbb{P} \left(\sum_{i=1}^n \frac{1}{S_i^{2n}} \mathbb{1}_{S_i^{2n} \leq \delta \sqrt{n}} \geq \varepsilon \sqrt{n} \right) \leq \frac{1}{\varepsilon \sqrt{n}} \sum_{i=1}^n \mathbb{E} \left[\frac{1}{S_i^{2n}} \mathbb{1}_{S_i^{2n} \leq \delta \sqrt{n}} \right] \leq \frac{C}{\varepsilon} \delta^2 \sqrt{n} \sum_{i=1}^n \frac{1}{i^{3/2}} \leq \frac{C}{\varepsilon} \delta^2$$

and (33) follows.

Step 2. We show that

$$X_n(\delta) \xrightarrow[n \rightarrow \infty]{(d)} X(0, \delta) \quad (35)$$

jointly with (32). Observe that by continuity, for $\eta \in (0, \delta)$,

$$X_n(\delta) - X_n(\eta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{S_i^{2n}} \mathbb{1}_{\eta\sqrt{n} \leq S_i^{2n} \leq \delta\sqrt{n}} \xrightarrow[n \rightarrow \infty]{(d)} X(\eta, \delta)$$

jointly with (32). Then we have:

- $X_n(\eta) \rightarrow 0$ in probability uniformly in n as $\eta \rightarrow 0$ by (33);
- $X(\eta, \delta) \rightarrow X(0, \delta)$ as $\eta \rightarrow 0$;
- for every $\eta \in (0, \delta)$ we have $X_n(\delta) - X_n(\eta) \rightarrow X(\eta, \delta)$ in distribution as $n \rightarrow \infty$.

This entails (35).

Step 3. By Skorokhod's representation theorem, we may assume that the convergences (32) and (35) both hold almost surely. Choose $\delta > 0$ such that $X(0, \delta) \leq \varepsilon$. Then for every n sufficiently large we have $X_n(\delta) \leq 2\varepsilon$. This completes the proof. \square

Remark 6.7. By combining Theorem 2.1 and Corollary 2.5 we get the following well-known distributional identity (see e.g. [13]):

$$\frac{1}{2} \int_0^1 \frac{t}{\mathbb{E}(t)} \stackrel{(d)}{=} \sup \mathbb{E}.$$

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A Background on the GP, GH and GHP topologies

We give background on various topologies that we use, based on [9, 11].

A.1 The Gromov–Prokhorov (GP) topology

A measured metric space is a triple (X, d, μ) such that (X, d) is a Polish space and μ is a Borel probability measure on X . Two such spaces $(X, d, \mu), (X', d', \mu')$ are called GP-isometry-equivalent if and only if there exists an isometry $f : \text{supp}(X) \rightarrow \text{supp}(X')$ such that if $f_\star \mu$ is the image of μ by f then $f_\star \mu = \mu'$. Let \mathbb{K}_{GP} be the set of GP-equivalent classes of measured metric space. Given a measured metric space (X, d, μ) , we write $[X, d, \mu]$ for the GP-isometry-equivalence class of (X, d, μ) and frequently use the notation X for either (X, d, μ) or $[X, d, \mu]$.

We now recall the definition of the Prokhorov distance. Consider a metric space (X, d) . For every $A \subset X$ and $\varepsilon > 0$ let $A^\varepsilon := \{x \in X, d(x, A) < \varepsilon\}$ be the open ε -neighborhood of A . Then given two (Borel) probability measures μ, ν on X , the Prokhorov distance between μ and ν is defined by

$$d_P(\mu, \nu) := \inf \{ \varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon, \text{ for all Borel set } A \subset X \}.$$

The Gromov–Prokhorov (for short GP) distance is an extension of the Prokhorov’s distance: For every $(X, d, \mu), (X', d', \mu') \in \mathbb{K}_{\text{GP}}$ the Gromov–Prokhorov distance between X and X' is defined by

$$d_{\text{GP}}((X, d, \mu), (X', d', \mu')) := \inf_{S, \phi, \phi'} d_P(\phi_{\star}\mu, \phi'_{\star}\mu'),$$

where the infimum is taken over all metric spaces S and isometric embeddings $\phi : X \rightarrow S, \phi' : X' \rightarrow S$. d_{GP} is indeed a distance on \mathbb{K}_{GP} and $(\mathbb{K}_{\text{GP}}, d_{\text{GP}})$ is a Polish space (see e.g. [1]).

We use another convenient characterization of the GP topology using the convergence of distance matrices: For every measured metric space (X, d^X, μ^X) let $(x_i^X)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables of common distribution μ^X and let $M^X := (d^X(x_i^X, x_j^X))_{i, j \in \mathbb{N}}$. We have the following result from [21],

Lemma A.1. *Let $(X^n)_{n \in \mathbb{N}} \in \mathbb{K}_{\text{GP}}^{\mathbb{N}}$ and let $X \in \mathbb{K}_{\text{GP}}$ then $X^n \xrightarrow{\text{GP}} X$ as $n \rightarrow \infty$ if and only if M^{X^n} converges in distribution toward M^X .*

A.2 The Gromov–Hausdorff (GH) topology

Let \mathbb{K}_{GH} be the set of isometry-equivalent classes of compact metric space. For every metric space (X, d) , we write $[X, d]$ for the isometry-equivalent class of (X, d) , and frequently use the notation X for either (X, d) or $[X, d]$.

For every metric space (X, d) , the Hausdorff distance between $A, B \subset X$ is given by

$$d_H(A, B) := \inf\{\varepsilon > 0, A \subset B^\varepsilon, B \subset A^\varepsilon\}.$$

The Gromov–Hausdorff distance between $(X, d), (X', d') \in \mathbb{K}_{\text{GH}}$ is given by

$$d_{\text{GH}}((X, d), (X', d')) := \inf_{S, \phi, \phi'} (d_H(\phi(X), \phi'(X'))),$$

where the infimum is taken over all metric spaces S and isometric embeddings $\phi : X \rightarrow S, \phi' : X' \rightarrow S$. d_{GH} is indeed a distance on \mathbb{K}_{GH} and $(\mathbb{K}_{\text{GH}}, d_{\text{GH}})$ is a Polish space (see e.g. [1]).

A.3 The Gromov–Hausdorff–Prokhorov (GHP) topology

Two measured metric spaces $(X, d, \mu), (X', d', \mu')$ are called GHP-isometry-equivalent if and only if there exists an isometry $f : X \rightarrow X'$ such that if $f_{\star}\mu$ is the image of μ by f then $f_{\star}\mu = \mu'$. Let $\mathbb{K}_{\text{GHP}} \subset \mathbb{K}_{\text{GP}}$ be the set of isometry-equivalent classes of compact measured metric space.

The Gromov–Hausdorff–Prokhorov distance between $(X, d, \mu), (X', d', \mu') \in \mathbb{K}_{\text{GHP}}$ is given by

$$d_{\text{GHP}}((X, d, \mu), (X', d', \mu')) := \inf_{S, \phi, \phi'} (d_P(\phi_{\star}\mu, \phi'_{\star}\mu') + d_H(\phi(X), \phi'(X'))),$$

where the infimum is taken over all metric spaces S and isometric embeddings $\phi : X \rightarrow S, \phi' : X' \rightarrow S$. d_{GHP} is indeed a distance on \mathbb{K}_{GHP} and $(\mathbb{K}_{\text{GHP}}, d_{\text{GHP}})$ is a Polish space (see [1]).

Note that random variables which are GHP measurable are also GH measurable. For every $[X, d, p] \in \mathbb{K}_{\text{GHP}}$, let $[X, d]_{\text{GH}}$ denote its natural projection on \mathbb{K}_{GH} . Note that GHP convergence implies GH convergence of the projections on \mathbb{K}_{GH} , then that the projection on \mathbb{K}_{GH} is a measurable function. The same statements hold for the GP topology. We will need the following statement.

Lemma A.2. *[Lemma 4 in [9]] Let $([X^n, d^n, p^n])_{n \in \mathbb{N}}$ and $[X, d, p]$ be GHP measurable random variables in \mathbb{K}_{GHP} . Assume that almost surely $[X, d, p]$ has full support. Assume that $([X^n, d^n, p^n])_{n \in \mathbb{N}}$ converges weakly toward $[X, d, p]$ in a GP sense, and that $([X^n, d^n])_{n \in \mathbb{N}}$ converges weakly toward $[X, d]$ in a GH sense. Then $([X^n, d^n, p^n])_{n \in \mathbb{N}}$ converges weakly toward $[X, d, p]$ in a GHP sense.*

B The leaf-tightness criterion

In this section, $\mathbf{X} = ((X^n, d^n, p^n))_{n \in \mathbb{N}}$ denotes a sequence of random compact measured metric spaces GHP-measurable. For every $n \in \mathbb{N}$ let $(x_i^n)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables of common distribution p^n , then let $M^n := (d^n(x_i^n, x_j^n))_{i, j \in \mathbb{N}}$. We say that \mathbf{X} is leaf-tight if and only if

$$\lim_{\delta \rightarrow 0} \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}(d_H(X^n, \{x_1^n, x_2^n, \dots, x_k^n\}) > \delta) = 0. \quad (36)$$

Although the criterion is unrelated with leaves, its name ‘‘leaf-tight’’ stayed as the original name from Aldous [5] which first introduced this criterion to study random trees with random leaves.

Proposition B.1. *If $(M^n)_{n \in \mathbb{N}}$ converges weakly toward a random matrix M , and \mathbf{X} is leaf-tight, then \mathbf{X} converges weakly for the GHP topology toward a random compact measured metric space (X, d, p) . Furthermore, if $(x_i)_{i \in \mathbb{N}}$ are i.i.d. random variables of law p then $M^X := (d(x_i, x_j))_{i, j \in \mathbb{N}} \stackrel{(d)}{=} M$. In addition, a.s. p has full support.*

Proof. We focus on the first statement, as the other two will naturally follow alongside the proof. First, it directly follows from the convergence of $(M_n)_{n \in \mathbb{N}}$ and from (36) that:

- $(\text{Diam}(X^n))_{n \in \mathbb{N}}$ is tight.
- For every $\delta > 0$, $(N_\delta(X^n))_{n \in \mathbb{N}}$ is tight, where N_δ stands for the usual δ -covering number (see [1])

As a result, by [1, Theorem 2.4], \mathbf{X} is tight for the weak GHP-topology.

Let $[X, d, p], [X', d', p']$ be two GHP subsequential limits of \mathbf{X} . Let us show $[X, d, p] \stackrel{(d)}{=} [X', d', p']$. Let $(x_i)_{i \in \mathbb{N}}$ be i.i.d. random variables of law p and let $M^X := (d(x_i, x_j))_{i, j \in \mathbb{N}}$. Define similarly $(x'_i)_{i \in \mathbb{N}}, M^{X'}$. By Lemma A.1, jointly with the above convergence, along the proper subsequence, we have weakly $M^n \rightarrow M^X$ and $M^n \rightarrow M^{X'}$. Note that this implies $M^X \stackrel{(d)}{=} M$. By the Skorokhod representation theorem, and up to further extract other subsequences, we may assume that those two convergences hold almost surely, and that almost surely $M^X = M^{X'}$. This implies that almost surely,

$$d_{GH}((\{x_i\}_{i \in \mathbb{N}}, d), (\{x'_i\}_{i \in \mathbb{N}}, d)) = 0.$$

And by the leaf-tightness criterion (36), almost surely $\{x_i\}_{i \in \mathbb{N}}$ and $\{x'_i\}_{i \in \mathbb{N}}$ are dense on X, X' so

$$d_{GH}((X, d), (X', d')) = 0.$$

Hence $[X, d]_{GH} \stackrel{(d)}{=} [X', d']_{GH}$. Also by Lemma A.1, since $(M^n)_{n \in \mathbb{N}}$ converges weakly, we have the equality in distribution $[X, d, p]_{GP} \stackrel{(d)}{=} [X', d', p']_{GP}$. Moreover, since almost surely $\{x_i\}_{i \in \mathbb{N}}$ and $\{x'_i\}_{i \in \mathbb{N}}$ are dense on X and X' , almost surely p and p' have full support on X and X' . Therefore, by Lemma A.2 we have $[X, d, p] \stackrel{(d)}{=} [X', d', p']$. This shows that there is only one subsequential possible limit for \mathbf{X} , and thus concludes the proof. \square