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# DYNC: a Cast3M module for the bifurcation analysis of nonlinear mechanical vibrations

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**Abstract.** This work presents a recently developed implementation of numerical methods for vibration problems involving nonlinear mechanical systems in the finite element software Cast3M. The strategy consists in solving the modal equations of motion for generic structures through a continuation method. Hence, the evolution of steady-state periodic solutions over given parameter ranges can be assessed. Of particular interest are stability computations, which are performed at each continuation step and through which any potential bifurcation is detected. Examples showing forced and free responses of beam structures are presented.

## 1 Introduction

The need for numerical methods capable of tackling nonlinear differential equations has motivated a great deal of research over the course of the past few decades, leading to the creation of efficient and robust programs dealing with problems such as the computation and continuation of (quasi-)periodic solutions, stability analyses, and bifurcation tracking for generic dynamical systems. The most widely-known example is probably the software AUTO [1], which employs orthogonal collocation techniques coupled with arc-length continuation to compute branches of solutions with respect to the system's parameters. With MANLAB [2], the same types of problems are solved through the use of a frequency-domain (harmonic balance, HMB) approach and asymptotic numerical continuation. While the generic nature of these and other freely-available codes offers great flexibility, they are seldom used for the analysis of large-scale finite element models, as this functionality is not an intrinsic part of their current versions. This fact suggests that a numerical toolbox which similar capabilities, but streamlined for the analysis of structures and integrated in a finite-element software, would be very convenient for engineering applications. The objective of this paper is to present such a toolbox, which has been implemented in the open-source multi-physics code Cast3M [3]. Section 2 reviews the concepts underlying the solution of vibration problems by the coupled continuation/harmonic balance method. An overview of the DYNC operator, which implements these

algorithms, is presented in Section 3, followed by example applications in Section 4.

## 2 Scope

The problem considered herein is to find periodic solutions of the discrete equations of motion:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + a\mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}_{\text{NL}}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + a\mathbf{f}_e(t) \quad (1)$$

obtained by projecting a full finite-element model onto a basis consisting of the first  $n$  linear eigenmodes. Hence,  $\mathbf{q} \in \mathbb{R}^n$  is the vector of modal displacements, and the physical displacements  $\mathbf{d}$  of any given point  $\mathbf{p} = (x, y, z)$  on the structure are obtained by linear combination:

$$\mathbf{d}(\mathbf{p}, t) = \boldsymbol{\Phi}(\mathbf{p})\mathbf{q}(t) \quad (2)$$

with  $\boldsymbol{\Phi}(\mathbf{p}) \in \mathbb{R}^{s \times n}$  a matrix containing the eigenvectors evaluated at  $\mathbf{p}$ .  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  are, respectively, the (modal) mass, damping, and stiffness matrices, while the vectors  $\mathbf{f}_e$ ,  $\mathbf{f}_{\text{NL}}$  contain the modal contributions of applied external loads (which must be periodic) and nonlinear forces. The parameter  $a$  in Eq. (1) is introduced to distinguish between the two following cases:

(a) Forced response,  $a = 1$  :

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}_{\text{NL}}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{f}_e(t)$$

(b) Free response,  $a = 0$  :

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}_{\text{NL}}(\mathbf{q}(t), \dot{\mathbf{q}}(t))$$

In the latter case, the system is Hamiltonian and admits at least  $n$  families of periodic solutions with fundamental frequency<sup>3</sup>  $\omega$ , parametrized by the total cycle energy  $E$ , i.e.  $\omega = \omega(E)$ . This corresponds to the definition of Nonlinear Normal Modes (NNM), as initially defined by Rosenberg [4]. In the former case,  $\omega$  is fixed and equal to the driving frequency. Resonances may occur if, at the energy level  $E$  imposed by damping and forcing,  $\omega$  is close to (a multiple of) the associated frequency of the NNM solution. This situation is clearly analogous to the linear case.

### 2.1 Continuation with the Harmonic Balance Method

We adopt the real HBM formalism, i.e. the modal displacement vector is assumed to be periodic and therefore expanded as a real Fourier series truncated at the  $H$ -th harmonic:

$$\mathbf{q}(t) = (\mathbb{F}_H(\omega t) \otimes \mathbf{I}_n) \mathbf{Q} \quad (3)$$

$$\mathbb{F}_H(\omega t) = [1, \cos(\omega t), \sin(\omega t), \dots, \cos(H\omega t), \sin(H\omega t)]$$

<sup>3</sup> Throughout this paper, the term *frequency* will be used to mean *angular frequency*, with units of rad/s.

where  $\otimes$  represents the Kronecker tensor product and  $\mathbf{Q} \in \mathbb{R}^{n(2H+1)}$  is the vector of Fourier coefficients. Once expressions similar to Eq.(3) are introduced for every time-dependent term, a Galerkin projection of the problem onto the frequency domain is performed by taking the scalar product of Eq. (1) with  $(\mathbb{F}_H^T(\omega t) \otimes \mathbf{I}_n)$ . This leads to the following set of  $n(2H+1)$  nonlinear algebraic equations to be solved for the Fourier coefficients  $\mathbf{Q}$  and (potentially) the frequency  $\omega$ :

$$\mathbf{R}(\mathbf{Q}, \omega) = [\omega^2(\nabla^2 \otimes \mathbf{M}) + a\omega(\nabla \otimes \mathbf{C}) + \mathbf{I}_L \otimes \mathbf{K}] \mathbf{Q} - \mathbf{F}_{\text{NL}}(\mathbf{Q}, \omega) - a\mathbf{F}_e(\omega) = \mathbf{0} \quad (4)$$

where uppercase terms contain the Fourier coefficients of the time-domain variables appearing in Eq. (1). Solving Eq. (4) is equivalent to finding an approximate periodic solution of the (free or forced) equations of motion which is correct up to the  $H$ -th harmonic. To this end, a typical root-finding algorithm is used, such as the Newton-Raphson method. Two cases may arise, depending on the type of problem:

- (a) Forced response: the residue  $\mathbf{R}$  is a function of  $\mathbf{Q}$  only and the problem  $\mathbf{Y}_F(\mathbf{Q}) = \mathbf{R}(\mathbf{Q}) = \mathbf{0}$  is well-posed.
- (b) Free response: the frequency is an additional unknown. Since the system is autonomous, solutions with a given total energy are invariant with respect to translations in time. In other words, an infinite family of solutions exist, parametrised by phase, and so a particular solution must be selected through the inclusion of a *phase equation*. Without loss of generality, we here choose:  $\dot{q}_1(0) = 0 \implies g(\mathbf{Q}) = 0$ . In addition, since solutions exist for any  $E = \mathbf{Q}^T \mathbf{Q}$ , the equation  $h(\mathbf{Q}) = E(\mathbf{Q}) - E_0$  is appended to fix the norm of  $\mathbf{Q}$ . In this way we may choose  $E_0$  to be small, so that an initialisation from a linear mode is close to the nonlinear solution. This greatly facilitates convergence but over-constrains the problem, so:  $\mathbf{F}_b = b(\nabla \otimes \mathbf{I}_n)\mathbf{Q}$  is added to  $\mathbf{R}$ . By treating the relaxation parameter  $b$  as an additional unknown, a closed system is obtained, i.e.: the problem  $\mathbf{Y}_{\text{NNM}}(\mathbf{Q}, \omega, b) = [\mathbf{R}(\mathbf{Q}, \omega, b), g(\mathbf{Q}), h(\mathbf{Q})] = \mathbf{0}$  is well-posed. Please note that, as  $\mathbf{F}_b$  is dissipative and there is no forcing, the only allowable value is  $b = 0$ , thus it is actually the free response which is been computed.

Since no analytical expression is available for the Fourier coefficients of generic nonlinear functions, the Alternating Frequency-Time (AFT) approach proposed by Cameron and Griffin [5] is used instead. Briefly stated, this consists in evaluating the forces in the time domain by first applying an inverse Fourier transform on  $\mathbf{Q}$ , and then projecting them back on the frequency domain by a direct transform. Hence, the iterative solution of Eq. (4) can be summarized by diagram shown in Fig. 1. It should be noted that the Jacobian matrix  $\mathbf{J}$  of  $\mathbf{Y}_F$  or  $\mathbf{Y}_{\text{NNM}}$ , which contains derivatives with respect to  $\mathbf{Q}$  and  $\omega$ , can be obtained in the same way, by means of a finite-difference scheme or by direct computation if a closed-form definition of the corresponding derivative in time is known.

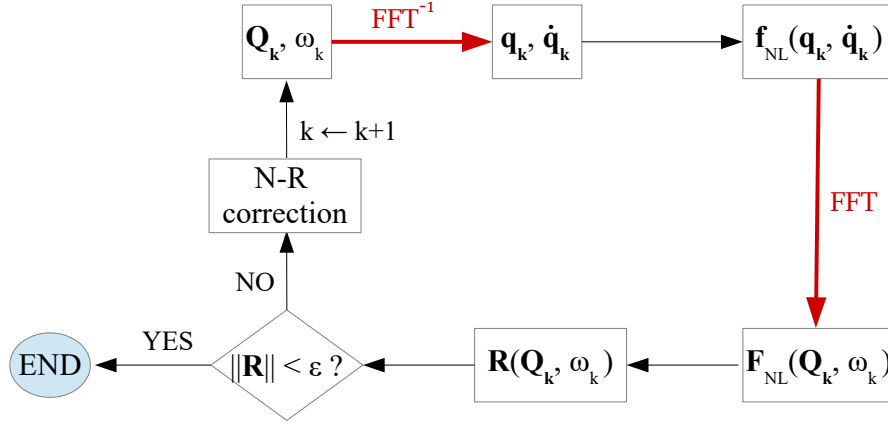


Fig. 1. AFT algorithm.

Starting from a converged solution  $(\mathbf{Q}_0, \omega_0)$  to the equations of motion, whole branches in the  $(\mathbf{Q}, \gamma)$ -space are calculated step-by-step through the use of *pseudo arc-length continuation* (see for example [6]), where  $\gamma$  is usually equal to  $\omega$  in both cases, although a different parameter -e.g., forcing amplitude- can be used for the forced case. Assuming  $\gamma = \omega$  for simplicity, the algorithm operates in the following way:

1. Lower and upper continuation bounds are specified:  $\omega \in [\omega_{\min}, \omega_{\max}]$  for the forced response, or  $E \in [E_{\min}, E_{\max}]$  for NNMs.
2. From the initial solution, a step of size  $\Delta s$  is taken along the direction  $\mathbf{t}_1 = [\Delta \mathbf{Q}^T, \Delta \omega]$ , tangent to the implicit curve  $\mathfrak{C}(\mathbf{Q}_0, \omega_0) = 0$ . This implies that an *arc-length* equation of the form:  $\|\Delta \mathbf{Q}\|^2 + (\Delta \omega)^2 = (\Delta s)^2$  has to be included in the system. For the forced response, the equation is simply appended and the frequency is freed. For the free response, it is the total energy which is now free to vary, so the arc-length equation replaces  $h(\mathbf{Q})$  and the artificial parameter  $b$  is kept.
3. In the likely case that the new point  $(\mathbf{Q}_1, \omega_1) = (\mathbf{Q}_0 + \Delta \mathbf{Q}, \omega_0 + \Delta \omega)$  does not satisfy dynamical equilibrium, Newton-Raphson corrections are performed in a direction perpendicular to  $\mathbf{t}_1$  until convergence.
4. Step-size  $\Delta s$  is automatically adapted for the next step as a function of the number of iterations required for convergence in the present step.
5. The process is repeated while the condition stated in the first step is verified.

## 2.2 Stability & bifurcations

Whereas time-integration methods can only find stable solutions of Eq. (1), unstable ones are also within the scope of the HBM. A local stability evaluation is performed at each continuation step by considering a small perturbation  $\boldsymbol{\eta}(t)$  applied to a converged cycle. This leads to a linear, autonomous, periodic-coefficient

differential equation of second order for the perturbation, which admits  $2n$  solutions of the form:  $\boldsymbol{\eta}_j(t) = e^{\lambda_j t} [(\mathbb{F}_H(\omega t) \otimes \mathbf{I}_n) \boldsymbol{\phi}_j]$  according to Floquet theory, with  $\boldsymbol{\phi}_j \in \mathbb{R}^{n(2H+1)}$ . The *Floquet exponents*  $\lambda_j, j = 1, \dots, 2n$  determine the system's stability: if  $\Re(\lambda_j) < 0 \forall j$ , the perturbation decays with time and the underlying cycle is stable. Following [7], the Floquet exponents can be directly computed in the frequency domain within the real HBM framework through Hill's method, which consists in solving the quadratic eigenvalue problem:

$$[\mathbf{R}_Q + \lambda_j \mathbf{D}_1 + \lambda_j^2 \mathbf{D}_2] \boldsymbol{\phi}_j = \mathbf{0} \quad (5)$$

$$\mathbf{R}_Q = \frac{\partial \mathbf{R}}{\partial \mathbf{Q}}; \mathbf{D}_1 = 2\omega(\nabla \otimes \mathbf{M}) + a(\mathbf{I}_{2H+1} \otimes \mathbf{C}); \mathbf{D}_2 = \mathbf{I}_{2H+1} \otimes \mathbf{M}$$

Eq. (5) holds for both forced and free systems, although the latter includes a singular value of multiplicity 2 owing from its Hamiltonian nature, which must therefore be shifted away from zero in order to have a meaningful stability analysis, as explained in [8].

Codimension-1 bifurcations are detected whenever a stability change occurs between two consecutive steps. In that case, the way in which the imaginary axis is crossed characterises the type of bifurcation; letting  $\rho$  equal  $\Im(\lambda_j)$  for  $j$  such that  $\Re(\lambda_j) = 0$ :

1.  $\rho = 0 \implies$  Limit Point (generally) or Branch Point (for symmetric systems).
2.  $\rho = \pm\kappa, 0 < |\kappa| < \omega/2 \implies$  Neimark-Sacker (torus) bifurcation.
3.  $\rho = \pm\omega/2 \implies$  period doubling.

A precise localisation of bifurcation points is achieved through Newton-Raphson iterations applied on particular extended systems, which include constraints characterising each type along with dynamical equilibrium. More details can be found in references [9] and [10].

### 3 The DYNC operator

The methods described above have been implemented in the form of an operator, called **DYNC** (for DYNAMIC Continuation), in the most recent version of Cast3M. In this way, not only can systems of arbitrarily complicated geometries be studied, but one also benefits from the wide library of non-linear forces (*linkages*) which is already present in the software.

#### 3.1 Linkages

Nonlinear forces may be defined in terms of either modal or physical displacements and velocities. The former, which we will refer to as *base-A linkages* in Cast3M jargon, are straightforwardly computed by AFT. The latter, *base-B linkages*, require two additional steps. Firstly, the physical displacements at the  $N_c$  contact points must be computed through Eq. (2), thus yielding  $\mathbf{d}_i = \mathbf{d}(\mathbf{p}_i, t), \forall i = 1, \dots, N_c$ , so that the forces in physical space,  $\hat{\mathbf{f}}_{\text{NL}}(\mathbf{d}_i, \dot{\mathbf{d}}_i)$ , can

be evaluated. Secondly, these forces must be projected to modal basis. Looping over all linkages, the total force vector is given by:

$$\begin{aligned} \mathbf{f}_{\text{NL}}(\mathbf{q}, \dot{\mathbf{q}}) &= \sum_{i=1}^{N_c} \boldsymbol{\Phi}^T(\mathbf{p}_i) \left[ \hat{\mathbf{f}}_{\text{NL}}(\mathbf{d}_i, \dot{\mathbf{d}}_i) \right] + \mathbf{f}_{\text{NL}}^A(\mathbf{q}, \dot{\mathbf{q}}) \\ &= \mathbf{f}_{\text{NL}}^B(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{f}_{\text{NL}}^A(\mathbf{q}, \dot{\mathbf{q}}) \end{aligned} \quad (6)$$

where the second terms groups all base-A linkages. As the contact points are specified beforehand in most situations, the matrices  $\boldsymbol{\Phi}(\mathbf{p}_i)$  are computed and stocked during the pre-processing stage.

### 3.2 Problem setup

In Cast3M, a structural dynamics problem is defined by writing a text file with *.dgibi* extension, containing a series of statements in the OOP-language GIBIANE<sup>4</sup>. This leads to the definition of several *Table* objects, which operators can read and exploit. In our case, the call to DYNC has the syntax:

```
TAB1 = DYNC TMOD TCHR TLIA TAMOR TINI TNUM NHBM NFFT;
```

where NHBM and NFFT are two integers indicating, respectively, the desired number of harmonics and FFT samples, while the remaining inputs are the following Table objects:

1. TMOD: modal basis.
2. TCHR: external loadings (if forced response).
3. TLIA: linkages.
4. TAMOR: damping (if forced response).
5. TINI: approximation to the first solution (optional).
6. TNUM: setting for the numerical continuation, including: problem type, max./min. step-size, among others.

**Remarks:** The modal basis table TMOD is simply computed by a call to the Cast3M operator VIBR. A detailed description of all currently-supported linkages is included in the *Manual Pages* page of the website, under the tag DYNE<sup>5</sup>. The output table TAB1 consists of two sub-tables: the first one stocks the Fourier coefficients, frequencies, Floquet exponents, and stabilities at each continuation step, while the second one stocks the bifurcations found along the response curves.

<sup>4</sup> Detailed explanations, tutorials and examples can be found on the Cast3M website. While most commands in the GIBIANE language use French words, documentation is available both in French and English.

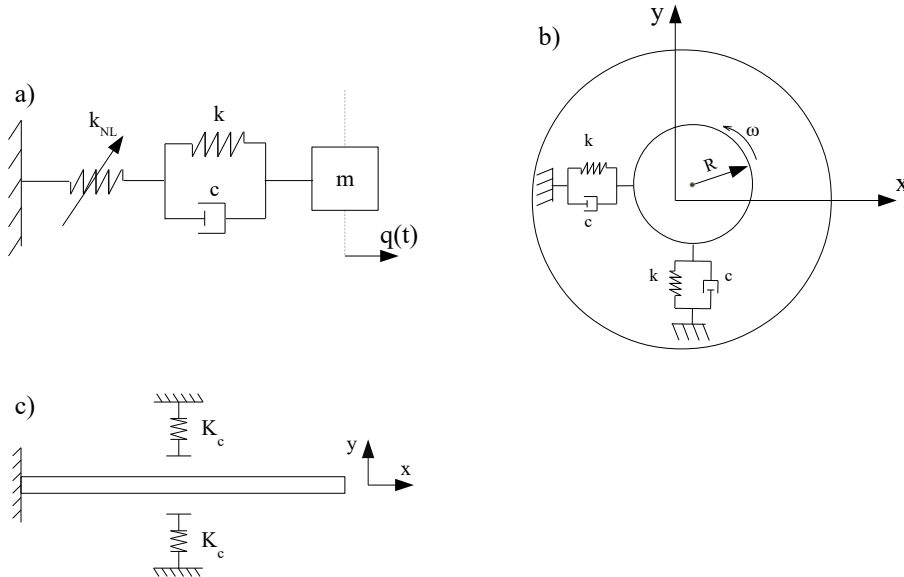
<sup>5</sup> DYNE is the explicit time-integration operator in Cast3M, which uses the same library of nonlinear forces.

### 4 Examples

This section briefly presents example calculations performed with DYNC. The *.dgibi* files corresponding to the test cases herein are available under the *Examples* page of the Cast3M website, and described succinctly in Fig. 2 and Tab. 1. A summary of the main parameters is as follows:

1. Duffing oscillator:  $NHBM = 5$ ,  $NFFT = 2^8$ ,  $n = 1$ , point mass, base-A linkage.
2. Jeffcot rotor:  $NHBM = 7$ ,  $NFFT = 2^8$ ,  $n = 2$ , point mass, base-B linkage.
3. Cantilever beam:  $NHBM = 15$ ,  $NFFT = 2^{10}$ , system modelled with 50 beam elements, base-B linkage. The case  $n = 3$  is shown for NNM computation.

**Note:** The inherently non-smooth impact forces involved in the last examples have been replaced by a smooth approximation in order to enhance the performance of pseudo arc-length continuation, following the method proposed in [11]. This is not strictly necessary if contact stiffness is moderate (which is not the case of the *dync03.dgibi* test case). The option to smooth contact forces is supported by the current version of the program.



**Fig. 2.** Schematic representations of the three example systems: a) Duffing oscillator, b) Jeffcott rotor, c) cantilever beam.



*.dgibi	System	Cast3M Linkage	Force
dync01	Duffing oscillator	COUPLAGE_DEPLACEMENT	Geometric nonlinearity
dync02	Jeffcott rotor	POINT_CERCLE_FROTTEMENT	Frictional (annular) contact
dync03	Cantilever beam	POINT_PLAN	Bilateral elastic impacts

Table 1. Test cases.

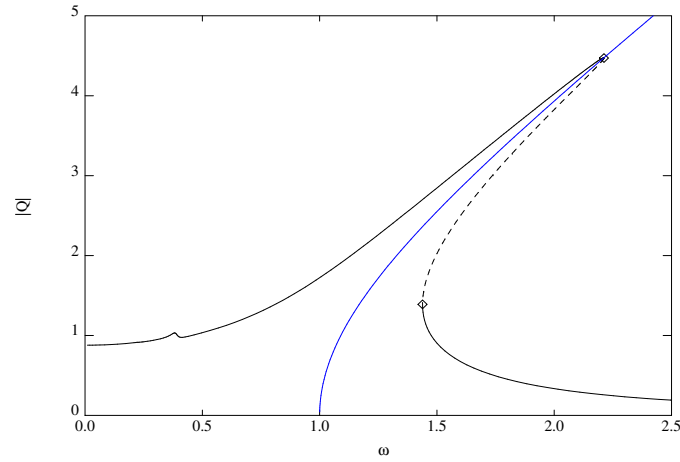


Fig. 3. Duffing oscillator: frequency response (black) and backbone curve (blue). Notice the limit points (diamonds).

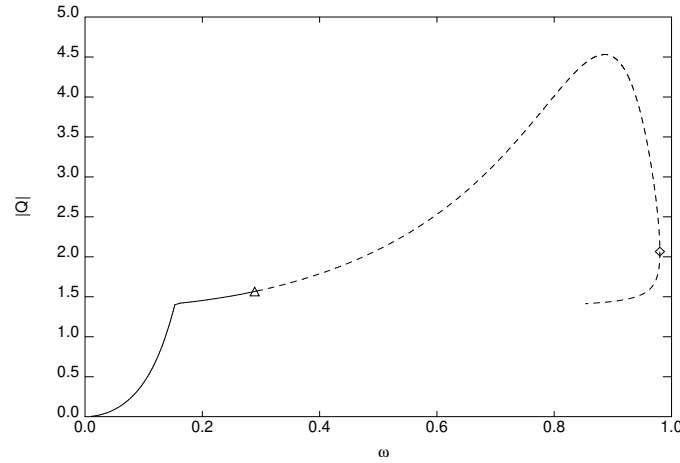
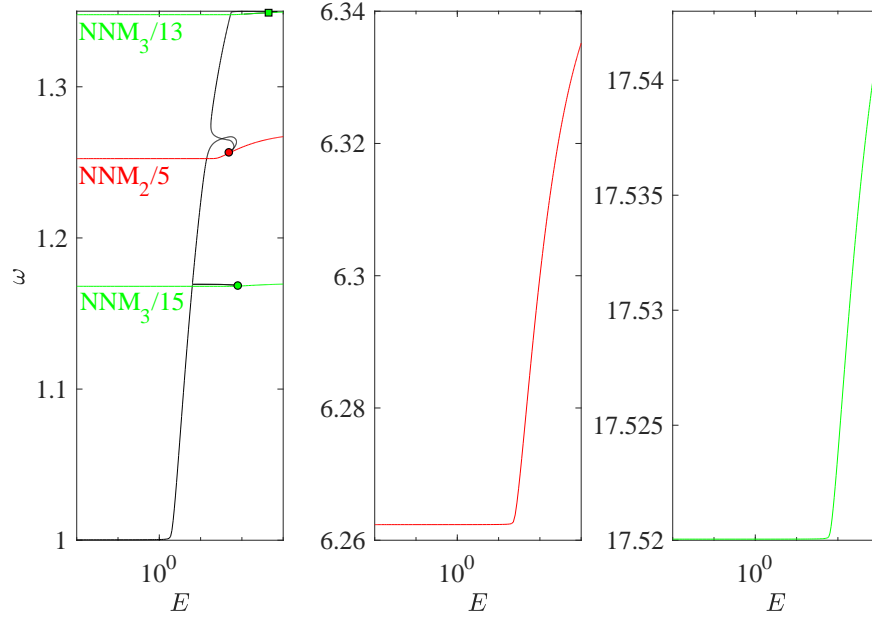


Fig. 4. Jeffcott rotor: frequency response; notice the Neimark-Sacker (triangle) and limit point (diamond) bifurcations.

## 5 Conclusions

The objective of the present contribution was to introduce the novel Cast3M operator DYNC. While other excellent resources exist for the bifurcation anal-



**Fig. 5.** Frequency-energy plot of cantilever beam with bilateral elastic stops. Left to right: first NNM (internal resonances labelled by markers), second NNM, third NNM.

ysis of general nonlinear systems, DYNC is tailored for the study of mechanical engineering problems and is thus a convenient tool when geometrically-complex structures with strong nonlinearities are considered. The current version of the algorithm support continuation of frequency responses as well as Nonlinear Normal Modes, and will be freely available as a part of the forthcoming 2021 edition of Cast3M. Further developments, under implementation at the time of writing and scheduled to be released within Cast3M 2021, are bifurcation tracking and continuation of self-excited systems.

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