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The Jacobian criterion of smoothness; a course at the Hausdorff Institute Spring/Summer 2023

Laurent Fargues

CHAPTER 1

1st lecture: Mai 16 2023

- The Jacobian criterion of smoothness is a key tool in our joint work with Scholze on the geometrization of the local Langlands correspondence.
- This allows us to construct "nice charts" on Bun_G the stack of G-bundles on the curve. Here charts=charts for the "smooth topology" (we will precise later what this means).

1. Algebraic classical analog

X is a smooth projective curve over the field k. The datum is the following:

$$\begin{tikzcd} Z \arrow[d, "\text{quasi-projective smooth morphism}"] \\ X \end{tikzcd}$$

Definition 1.1. *$\mathcal{M}_Z \longrightarrow Spec(k)$ represents the functor*

$$S \longmapsto \left\{ \text{sections } s \quad \begin{tikzcd} & Z \arrow[d] \\ X \times_k S \arrow[ur, bend left, "s"] \arrow[r] & X \end{tikzcd} \right\}$$

Then: $\mathcal{M}_Z$ is representable by a quasi-projective k-scheme.

Example 1.2.
- *Let $\mathscr{E}$ be a vector bundle on X and $Z = \mathbb{V}(\mathscr{E})$ be its geometric realization. Then, $\mathcal{M}_Z$ is representable by the affine space $\mathbb{V}(H^0(X, \mathscr{E}))$.*
- *$\mathscr{E}$ = v.b. on X and $d \geq 1$ an integer. Let $Z = \mathrm{Gr}_d(\mathscr{E})$ be the Grassmianian of quotients of $\mathscr{E}$ that are locally free of rank d. Then*

$$\mathcal{M}_Z(S) = \Big\{ \textit{quotients of } \underbrace{\mathscr{E}_{|X \times_k S}}_{\textit{pullback of } \mathscr{E} \textit{ via } X \times_k S \to X} \textit{ that are locally free of rank } d \Big\}.$$

One has an open immersion

$$\mathcal{M}_Z \underset{open}{\subset} Quot_{\mathscr{E}/X/k}$$

where

$$Quot_{\mathscr{E}/X/k}(S) = \Big\{ \textit{coherent quotients of } \mathscr{E}_{|X \times_k S} \textit{ that are flat over } S \Big\}.$$

Definition 1.3. *We note $\mathcal{M}_Z^{sm} \subset \mathcal{M}_Z$ the open sub-functor defined by*

$$\mathcal{M}_Z^{sm}(S) = \Big\{ \textit{sections } s \textit{ satisfying: } \forall t \in S,\ H^1\big(X \otimes_k k(t), (s^*T_{Z/X})_{|X\otimes_k k(t)}\big) = 0 \Big\}$$

where

- *$T_{Z/X}$ is the relative tangent bundle (that is well defined as a vector bundle on Z since $Z \to X$ is smooth)*
- *$s^*T_{Z/X}$ is a vector bundle on $X \times_k S$*
- *$(s^*T_{Z/X})_{|X\otimes_k k(t)}$ is the pullback of this vector bundle via $X \otimes_k k(t) \to X \times S$ via $Spec(k(t)) \to S$.*

Proposition 1.4 (Jacobian criterion of smoothness, classical case). *The morphism*

$$\mathcal{M}_Z^{sm} \longrightarrow Spec(k)$$

is smooth.

Proof. Let

$$\begin{tikzcd} & Z \arrow[d] \\ X \times_k S \arrow[ur, bend left, "s"] \arrow[r] & X \end{tikzcd}$$

lie in $\mathcal{M}_Z^{sm}(S)$. Note

$$\pi : X \times_k S \longrightarrow S$$

the projection. Using

- $R\pi_*(s^*T_{Z/X}) \in \mathrm{Perf}^{[0,1]}(\mathcal{O}_S)$ (perfect complex with amplitude in $[0,1]$) since π is proper and flat
- $+$ the vanishing condition fiberwise
- $+$ proper base change in coherent cohomology (aka Zariski formal function theorem, see EGA III: if S is noetherian then for $t \in S$, $R\pi_*(s^*T_{Z/X})_t \otimes^{\mathbb{L}}_{\mathcal{O}_{S,t}} \widehat{\mathcal{O}}_{S,t} = R\widehat{\pi}_{t*}\widehat{s^*T_{Z/X}}$ where $\widehat{\pi}_t : X \hat{\otimes}_k \widehat{\mathcal{O}}_{S,t} \to \mathrm{Spf}(\widehat{\mathcal{O}}_{S,t})$)

one deduces that $R^1\pi_*(s^*T_{Z/X}) = 0$ and thus if S is affine then

$$H^1(X \times_k S, s^*T_{Z/X}) = 0.$$

Now, if $S \hookrightarrow S'$ is a nilpotent immersion of affine schemes defined by $\mathcal{I}$ with $\mathcal{I}^2 = (0)$ and we are looking for s' as in the following diagram

$$\begin{tikzcd} X \times_k S' \arrow[r, dotted, red, "s'"] & Z \arrow[d] \\ X \times_k S \arrow[u, hook] \arrow[ur, "s"] \arrow[r] & X \end{tikzcd}$$

the sheaf on $(X \times_k S)_{Zar}$ of liftings of s to s' is a $s^*T_{Z/X} \otimes_{\mathcal{O}_{X\times_k S}} \pi^*\mathcal{I}$-torsor. Using the projection formula, $R\pi_*(s^*T_{Z/X} \otimes_{\mathcal{O}_{X\times_k S}} \pi^*\mathcal{I}) = R\pi_*(s^*T_{Z/X}) \otimes^{\mathbb{L}} \mathcal{I}$, one has

$$H^1(X \times_k S, s^*T_{Z/X} \otimes_{\mathcal{O}_{X\times_k S}} \pi^*\mathcal{I}) = 0$$

and we conclude. □

Remark 1.5. *We used the infinitesimal criterion for formal smoothness of Grothendieck → not available in the perfectoid world since there are no infinitesimals. This is why this is much more difficult in the perfectoid world.*

2. Example of application of the classical Jacobian criterion

X/k smooth projective curve as before, $n \geq 1$,

$$\mathrm{Bun}_n$$

the stack of rank n vector bundles on X,

$$\mathrm{Bun}_n(S) = \big\{\text{rank } n \text{ vector bundles on } X \times_k S\big\}$$

where the notation $\{\dots\}$ here means "the groupoid of" and not the set. $\mathcal{O}(1)$ is an ample line bundle on X. $r \geq 1, N \geq 0$ integers,

$$U_{r,N} \longrightarrow \mathrm{Spec}(k)$$

is the moduli of quotients of $\mathcal{O}_X(-N)^r$, on S this means a morphism

$$u : \mathcal{O}_{X\times_k S}(-N)^r \twoheadrightarrow \mathscr{E},$$

with $\mathscr{E}$ locally free of rank n s.t. fiberwise (on S with the preceding notations) $\underbrace{\mathscr{H}om(\ker u, \mathscr{E})}_{\text{vector bundle}}$ has no H^1. Then,

$$\big(U_{r,N} \longrightarrow \mathrm{Bun}_n\big)_{r,N}$$

is a set of smooth charts of Bun_n.

3. The Jacobian criterion of smoothness

3.1. Background on the curve. E local field, $\mathbb{F}_q = \mathcal{O}_E/\pi$:

(1) either $|E : \mathbb{Q}_p] < +\infty$,
(2) or $E = \mathbb{F}_q(\!(\pi)\!)$.

(R, R^+) affinoid perfectoid over $\mathbb{F}_q$

$$\underbrace{X_{R,R^+}}_{E\text{-analytic adic space}} = \text{adic curve over } \mathrm{Spa}(E) \text{ attached to } (R, R^+)$$

Recall:

$$\boxed{X_{R,R^+} = \underbrace{Y_{R,R^+}}_{\substack{\text{Stein } E\text{-analytic}\\ \text{adic space}}} /\varphi^{\mathbb{Z}}}$$

where

- the quotient $Y_{R,R^+} \longrightarrow Y_{R,R^+}/\varphi^{\mathbb{Z}}$ is for the analytic topology i.e. this is a local isomorphism,
- the group of deck transformations is $\varphi^{\mathbb{Z}}$.

One has

$$\boxed{Y_{R,R^+} = \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \smallsetminus V(\pi[\underbrace{\varpi}_{\substack{\text{pseudo}\\ \text{-uniformizer}}}])}$$

$\to$ remove two divisors stable under φ, (π) and $([\varpi])$, after removing those two fixed divisors the action of $\varphi^{\mathbb{Z}}$ is without fixed points totally discontinuous

- (π) = étale divisor,
- $([\varpi])$ = crystalline divisor

$\to$ see prismatic cohomology where one adds another divisor to the picture: the de Rham divisor associated to an untilt.

Here

$$W_{\mathcal{O}_E}(R^+) = \text{ramified Witt vectors}$$

that is $R^+\hat{\otimes}_{\mathbb{F}_q}\mathcal{O}_E = R^+[\![\pi]\!]$ if $E = \mathbb{F}_q(\!(\pi)\!)$ (in equal characteristic the Teichmüller is additive, $[-] : R^+ \hookrightarrow W_{\mathcal{O}_E}(R^+)$). Moreover,

$$\varphi\big(\sum_{n\geq 0}[a_n]\pi^n\big) = \sum_{n\geq 0}[a_n^q]\pi^n.$$

There is an "ample" line bundle $\mathcal{O}_{X_{R,R^+}}(1)$ on X_{R,R^+} that is trivial when pulled back to Y_{R,R^+}. It corresponds to the φ-equivariant line bundle

$$\big(\mathcal{O}_{Y_{R,R^+}}, \pi^{-1}\varphi\big)$$

on Y_{R,R^+}. Set

$$\underbrace{\mathbb{B}(R,R^+)}_{\substack{\text{Fontaine's}\\ \text{period ring}}} = \mathcal{O}\big(\underbrace{Y_{R,R^+}}_{\substack{\text{Stein}\\ \Rightarrow\text{lot of hol. fcts}}}\big)$$

<u>Ample</u>: Kedlaya-Liu; $\forall\mathscr{E}$ v.b. on X_{R,R^+} for $n \gg 0$, $\mathscr{E}(n)$ is generated by its global sections and $H^1(X_{R,R^+}, \mathscr{E}(n)) = 0$. See the article where we give a full proof.

If

$$\underbrace{P}_{\substack{\text{graded alg.}\\ \text{of periods}}} = \bigoplus_{d\geq 0} \underbrace{H^0\big(X_{R,R^+}, \mathcal{O}(d)\big)}_{\mathbb{B}(R,R^+)^{\varphi=\pi^d}=\{f|\varphi(f)=\pi^d f\}}$$

set

$$\mathfrak{X}_{R,R^+} = \mathrm{Proj}(P) = \text{“algebraic curve”}$$

<u>Fact</u>: there is a canonical morphism of ringed spaces

$$X_{R,R^+} \longrightarrow \mathfrak{X}_{R,R^+}$$

that induces a GAGA equivalence by pullback

$$\boxed{\big\{\text{vector bundles on } \mathfrak{X}_{R,R^+}\big\} \xrightarrow{\sim} \big\{\text{vector bundles on } X_{R,R^+}\big\}}$$

This morphism of ringed spaces is constructed in the following way: for $t \in \mathbb{B}(R,R^+)^{\varphi=\pi}$

$$Y_{R,R^+} \smallsetminus V(t) \longrightarrow \operatorname{Spec}\big(\mathbb{B}(R,R^+)[\tfrac{1}{t}]^{\varphi=\mathrm{Id}}\big) \underbrace{\hookrightarrow}_{\substack{\text{affine}\\ \text{open}}} \mathfrak{X}_{R,R^+}$$

is induced by

$$\mathbb{B}(R,R^+)[\tfrac{1}{t}]^{\varphi=\mathrm{Id}} \hookrightarrow \mathbb{B}(R,R^+)[\tfrac{1}{t}] \longrightarrow \mathcal{O}\big(Y_{R,R^+} \smallsetminus V(t)\big).$$

When t varies

$$Y_{R,R^+} = \bigcup_t Y_{R,R^+} \smallsetminus V(t)$$

and this glues to a morphism of ringed spaces

$$Y_{R,R^+} \longrightarrow \mathfrak{X}_{R,R^+}$$

that is φ-invariant. This defines our morphism of ringed spaces.

Remark 3.1. *When we say that $\mathbb{B}(R,R^+)$ is a Fontaine's period ring we can give a more precise content to this sentence. Suppose $E = \mathbb{Q}_p$. Let $(R^\sharp, R^{\sharp,+})$ be an untilt of (R,R^+). There is then a natural morphism*

$$\mathbb{B}(R,R^+) \longrightarrow B^+_{cris}(R^{\sharp,+}) := H^0(Spec(R^{\sharp,+}/p)/Spec(\mathbb{Z}_p), \mathcal{O}_{cris})\big[\tfrac{1}{p}\big].$$

inducing an isomorphism $\mathbb{B}(R,R^+)^{\varphi=\pi^d} \xrightarrow{\sim} B^+_{cris}(R^{\sharp,+})^{\varphi=\pi^d}$ for all $d \geq 0$.

4. The Jacobian criterion

Let

$$\begin{tikzcd}
Z \arrow[r, hook, "\exists"] \arrow[d, "\text{smooth morphism of sous-perfectoid spaces}"'] & \mathbb{P}^n_{X_{R,R^+}} \arrow[dl] & \text{analytically open inside Zariski closed} \arrow[d] \\
X_{R,R^+} & & \text{quasi-projectivity } \textit{assumption}
\end{tikzcd}$$

In fact there is a good notion of smooth morphisms of sous-perfectoid spaces. Recall in fact; X_{R,R^+} is not perfectoid but

$$X_{R,R^+} \hat{\otimes}_E \widehat{\overline{E}}$$

is perfectoid. More generally, by definition, (A,A^+) is sous-perfectoid if there exists a morphism $(A,A^+) \to (B,B^+)$ with (B,B^+) affinoid perfectoid and an A-linear continuous section $\begin{tikzcd} A \arrow[r, hook] & B \arrow[l, bend right] \end{tikzcd}$. Sous-perfectoid implies sheafy.

Remark 4.1. *The sous-perfectoid space X_{R,R^+} has a nice formula for its diamond. Namely,*

$$X_S^\diamond = (X_S \hat{\otimes}_E \widehat{\overline{E}})^\flat / \underline{Gal(\overline{E}|E)}$$

is canonically identified with

$$\underbrace{S \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spd}(E)}_{Y_S^\diamond} / \varphi_S^{\mathbb{Z}} \times \mathrm{Id}$$

Definition 4.2. *Define*

$$\begin{array}{rcl} \mathcal{M}_Z : \mathrm{Perf}_{R,R^+} & \longrightarrow & \mathrm{Sets} \\ S & \longmapsto & \left\{ \textit{sections } s \quad \begin{tikzcd} & Z \arrow[d] \\ X_S \arrow[ur, bend left, "s"] \arrow[r] & X_{R,R^+} \end{tikzcd} \right\} \end{array}$$

Remark 4.3. *If Z is Zariski closed inside some $\mathbb{P}^n_{X_{R,R^+}}$ it is of the form $\mathfrak{Z}^{ad}$ for some Zariski closed $\mathfrak{Z} \to \mathfrak{X}_{R,R^+}$ (GAGA). Then, GAGA applies to give for $S = \mathrm{Spa}(A, A^+)$ aff. perf. with a morphism $S \to \mathrm{Spa}(R, R^+)$*

$$\left\{ \textit{sections } s \quad \begin{tikzcd} & \mathfrak{Z} \arrow[d] \\ \mathfrak{X}_{A,A^+} \arrow[ur, bend left, "s"] \arrow[r] & \mathfrak{X}_{R,R^+} \end{tikzcd} \right\} \xrightarrow{\sim} \left\{ \textit{sections } s \quad \begin{tikzcd} & Z \arrow[d] \\ X_S \arrow[ur, bend left, "s"] \arrow[r] & X_{R,R^+} \end{tikzcd} \right\}$$

$\to$ *can compute $\mathcal{M}_Z$ using the adic or algebraic curve.*

The first basic result is the following.

Proposition 4.4. *$\mathcal{M}_Z$ is representable by a locally spatial diamond with*

$$\mathcal{M}_Z \longrightarrow \mathrm{Spa}(R, R^+)$$

compactifiable of finite dim trg.

$\to$ compactifiable of finite dim trg.: to be explained later what this means.

Definition 4.5. *$\mathcal{M}_Z^{sm} \subset \mathcal{M}_Z$ is the open sub-diamond such that*

$$\mathcal{M}_Z^{sm}(S) = \Big\{ \textit{sections } s \textit{ s.t. } \forall t \in S,\ \big(\underbrace{s^* \overbrace{T_{Z/X_{R,R^+}}}^{\textit{v.b. on } Z}}_{\textit{v.b. on } X_S} \big)_{|X_{K(t),K(t)^+}} \textit{ has } > 0 \textit{ H.N. slopes} \Big\}.$$

$\to$ we use the fact that for a smooth morphism of sous-perfectoid spaces one can define its relative tangent bundle as a vector bundle.

$\rightarrow$ the fact that $\mathcal{M}_Z^{sm}$ is open inside $\mathcal{M}_Z$ is a consequence of the semi-continuity of the H.N. polygon of a vector bundle on the curve, see the article where we give a simple proof of this using diamonds.

Remark 4.6 (link with the classical Jacobian criterion of smoothness). *For (K, K^+) an $\mathbb{F}_q$-affinoid perfectoid field and $\mathscr{E}$ a v.b. on X_{K,K^+},*

$$\mathscr{E} \text{ has } > 0 \text{ H.N. slopes } \Leftrightarrow \forall (L, L^+)|(K, K^+),\ H^1(X_{L,L^+}, \mathscr{E}_{|X_{L,L^+}}) = 0$$

where (L, L^+) is an affinoid perfectoid field.

The purpose of those lectures is then to prove the following.

Theorem 4.7 (Jacobian criterion of smoothness). *The morphism*

$$\mathcal{M}_Z^{sm} \longrightarrow \mathrm{Spa}(R, R^+)$$

is ℓ-cohomologically smooth of dimension $\deg(s^* T_{Z/X_{R,R^+}})$ at a section s.

$\rightarrow$ We will explain later what ℓ-cohomologically smooth means.

Example 4.8. *(1) $\mathscr{E}$ = a v.b. on X_{R,R^+}, $Z = \mathbb{V}(\mathscr{E})$. Then*

$$\mathcal{M}_Z = BC(\mathscr{E})$$

and

$$\mathcal{M}_Z^{sm} = BC(\mathscr{E}) \times_{\mathrm{Spa}(R,R^+)} U$$

where $U \subset \mathrm{Spa}(R, R^+)$ is the open subset where $\mathscr{E}$ has > 0 H.N. slopes.

(2) $\mathscr{E}$ = v.b. on X_{R,R^+}, $Z = \mathbb{P}(\mathscr{E})$. One has a decomposition

$$\mathcal{M}_Z = \coprod_{d \in \mathbb{Z}} \mathcal{M}_Z^d$$

where $\mathcal{M}_Z^d$ is the open/closed subset where $s^* \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$ has degree d (s =section). One has moreover

$$\begin{array}{ccc} \overbrace{\mathcal{P}ic^d}^{\substack{\textit{Picard stack} \\ \textit{of deg. } d \textit{ line bundles}}} & \xrightarrow{\sim} & \overbrace{[*/\underline{E}^\times]}^{\substack{\textit{classifying stack} \\ \textit{of pro-étale } \underline{E}^\times\textit{-torsors}}} \\ \mathscr{L} & \longmapsto & \mathrm{Isom}(\mathcal{O}(d), \mathscr{L}). \end{array}$$

From this we deduce that

$$\mathcal{M}_Z^d \simeq U/\underline{E}^\times \subset \underbrace{\overbrace{BC\big(\mathscr{E}^\vee(d)\big) \smallsetminus \{0\}}^{\substack{\textit{punctured} \\ BC\textit{- space}}} / \underline{E}^\times}_{\substack{\textit{"projective space"} \\ \textit{associated to a BC space}}}$$

where $U \subset BC\big(\mathscr{E}^\vee(d)\big) \smallsetminus \{0\}$ is the open sub-diamond defined by

$$\begin{array}{c} U(S) \\ \| \\ \Big\{u : \mathscr{E}_{|X_S} \to \mathcal{O}_{X_S}(d) \mid \forall t \in S,\ u_{|X_{K(t),K(t)^+}} : \mathscr{E}_{|X_{K(t),K(t)^+}} \to \mathcal{O}_{X_{K(t),K(t)^+}}(d) \text{ is surjective}\Big\} \\ \big\downarrow \\ \Big\{u : \mathscr{E}_{|X_S} \to \mathcal{O}_{X_S}(d) \mid \forall t \in S,\ u_{|X_{K(t),K(t)^+}} : \mathscr{E}_{|X_{K(t),K(t)^+}} \to \mathcal{O}_{X_{K(t),K(t)^+}}(d) \text{ is non-zero}\Big\} \\ \| \\ \big(BC\big(\mathscr{E}^\vee(d)\big) \smallsetminus \{0\}\big)(S) \end{array}$$

(3) *$Z = X_{R,R^+} \times_{\mathrm{Spa}(E)} W$ where $W \subset \mathbb{P}^n_E$ is smooth equal to $V(f_i)_{i\in I}$, f_i = homogeneous polynomial in $n+1$-variables with coefficients in E. Then,*

$$\mathcal{M}_Z = \coprod_{d\in\mathbb{N}} \mathcal{M}_Z^d$$

where

$$\mathcal{M}_Z^d = \Big\{(x_0,\dots,x_n) \in U \subset \big(\mathbb{B}^{\varphi=\pi^d}\big)^{n+1} \smallsetminus \{(0,\dots,0)\} \mid \underbrace{\forall i \in I,\ f_i(x_0,\dots,x_n) = 0}_{\text{alg. equation in a BC space}}\Big\}/\underline{E}^\times$$

where U is the open subset of $\big(\mathbb{B}^{\varphi=\pi^d}\big)^{n+1} \smallsetminus \{(0,\dots,0)\}$ defined as before: $U(S)$ is the set of

$$(x_0,\dots,x_n) \in \big(\mathbb{B}(S)^{\varphi=\pi^d}\big)^{n+1}$$

satisfying

$$\forall \textit{aff. perf. field } (K,K^+)|E,\ \forall \mathrm{Spa}(K^\flat,K^{\flat,+}) \to S,$$
$$(\theta_K(x_0),\dots,\theta_K(x_n)) \in K^{n+1} \smallsetminus \{(0,\dots,0)\}$$

where $\theta_K : \mathbb{B}(K^\flat,K^{\flat,+}) \to K$ is Fontaine's θ map.

→Banach-Colmez spaces = linear objects of the category of diamonds, analogs of affine spaces. For $Z = \mathbb{V}(\mathscr{E})$ the Jacobian criterion of smoothness is "easy", established first much before the Jacobian criterion

→ here we look more generally at solutions of algebraic equations inside BC spaces

→ this is already needed if we take $Z = \mathrm{Gr}_d(\mathscr{E})$ with $d > 1$ since the Plücker embedding of $\mathrm{Gr}_d(\mathscr{E})$ inside $\mathbb{P}(\wedge^d\mathscr{E})$ is defined by quadratic equations.

5. The application to Bun_G

Here is how we use the Jacobian criterion in our work. Let G be a reductive group over E and P a parabolic subgroup of G.

$\mathscr{E}$ a G-bundle on X_{R,R^+} (one can give a meaning to this as an étale G^{ad}-torsor over X_{R,R^+}, at the end this is the same as an étale G-torsor over the scheme $\mathfrak{X}_{R,R^+}$, the "algebraic curve"). Take

$$Z = P\backslash\mathscr{E} \longrightarrow X_{R,R^+}.$$

Then,

$$\mathcal{M}_Z = \text{moduli of reductions of } \mathscr{E} \text{ to } P$$

i.e.

$$\mathcal{M}_Z(S) = \left\{ \mathscr{E}_P \text{ a } P\text{-torsor} + \text{iso. } \mathscr{E}_P \overset{P}{\times} G \xrightarrow{\sim} \mathscr{E} \right\}.$$

Then,

$$\mathcal{M}_Z^{sm}(S) = \left\{ \mathscr{E}_P \mid \underbrace{\mathscr{E}_P \overset{P}{\times} \mathfrak{g}/\mathfrak{p}}_{\text{v.b.}} \text{ has } > 0 \text{ H.N. slopes fiberwise}/S \right\}$$

where $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{p} = \mathrm{Lie(P)}$ and $P \to \mathrm{GL}(\mathfrak{g}/\mathfrak{p})$ is given by the adjoint representation.

Jacobian criterion $\Rightarrow$

$$\begin{array}{ccc} \mathrm{Bun}_P^{sm} & \longrightarrow & \mathrm{Bun}_G \\ \mathscr{E}_P & \longmapsto & \mathscr{E}_P \overset{P}{\times} G \end{array}$$

is ℓ-coho. smooth where Bun_P^{sm} is the open substack of Bun_P formed by P-bundles $\mathscr{E}_P$ such that $\mathscr{E}_P \overset{P}{\times} \mathfrak{g}/\mathfrak{p}$ has > 0 H.N. slopes fiberwise.

Remark 5.1 (Final remark). *"$X_S = X \times S$" although this has no meaning: the curve exists only after pullback to any $\mathbb{F}_q$-perfectoid space S but not absolutely over* $\mathrm{Spa}(\mathbb{F}_q)$.

CHAPTER 2

2nd lecture: Mai 23 2023

1. We lied to you

X a (Grothendieck) topos, Λ a ring

$$\underbrace{D^+(X,\Lambda)}_{\text{good object}} \subset \underbrace{\overbrace{D(X,\Lambda)}^{\substack{\text{usual derived category}\\ \text{of complexes of } \Lambda\text{-modules on } X}}}_{\substack{\text{not good object in general, does}\\ \text{not satisfy hyperdescent in general, not left complete}}}$$

Let

$$X^{\mathbb{N}}$$

be the topos of projective systems of objects of X i.e. functors $(\mathbb{N},\leq) \to X$.

For each $n \geq 0$ there is a "stage n" morphism of topos $i_n : X \to X^{\mathbb{N}}$ with $i_n^{-1}(U_k)_{k\geq 0} = U_n$ where $\cdots \to U_k \to \cdots U_1 \to U_0$ is a projective system.

Definition 1.1. *We note*

$$\widehat{D}(X,\Lambda) \subset D(X^{\mathbb{N}},\Lambda)$$

for the sub-category of $A \in D(X^{\mathbb{N}},\Lambda)$ satisfying

(1) $\forall n \geq 0,\ \underbrace{A_n}_{i_n^{-1}A} \in D^{\geq -n}(X,\Lambda)$,

(2) $\forall n \geq 0,\ \tau_{\geq -n}A_{n+1} \underbrace{\xrightarrow{\sim}}_{\substack{\text{in } D^+(X,\Lambda)\\ \text{i.e. quasi-iso.}}} A_n$

Two adjoint functors

$$\widehat{D}(X,\Lambda) \underset{\tau}{\overset{R\varprojlim}{\rightleftarrows}} D(X,\Lambda)$$

where

- $R\varprojlim$ is the derived functor of projective limits from Λ-modules in $X^{\mathbb{N}}$ to Λ-modules in X,
- τ sends A to the projective system of truncations $(\tau_{\geq -n}A)_{n\geq 0}$.

Definition 1.2. *(1) $D(X,\Lambda)$ is left complete if those two adjoint functors are equivalences i.e.*

$$\forall A \in D(X,\Lambda), \quad A \xrightarrow{\sim} R\varprojlim_{n\geq 0} \tau_{\geq -n} A$$

(2) $\widehat{D}(X,\Lambda)$ = left completion of $D(X,\Lambda)$.

Remark 1.3. *Although canonically defined as a composite of two functors applied to A, $R\varprojlim_{n\geq 0} \tau_{\geq -n} A$ can be though of as a homotopy limit*

$$h\varprojlim_{n\geq 0} \tau_{\geq -n} A$$

where by definition a homotopy limit of a projective system $(B_n)_{n\geq 0}$ in a triangulated category admitting countable products is $C(f)[-1]$ where $C(f)$ is a cone of $f : \prod_{n\geq 0} B_n \to \prod_{n\geq 0} B_n$ sending $(x_n)_{n\geq 0}$ to $(u_{n+1}(x_{n+1}) - x_n)_{n\geq 0}$, $u_{n+1} : B_{n+1} \to B_n$. One can thus give a meaning to the notion of a left complete triangulated category equipped with a t-structure and admitting countable products.

For the next proposition recall that the topos X is replete if for any projective system $(\mathscr{F}_n)_{n\geq 0} \in X^{\mathbb{N}}$ such that for all n, $\mathscr{F}_{n+1} \to \mathscr{F}_n$ is surjective,

$$\varprojlim_{n\geq 0} \mathscr{F}_n \longrightarrow \mathscr{F}_0$$

is surjective (where we use the word surjective in the topos X as an abuse of terminology for epimorphism). For example, if k is a field with $[k^s : k] = +\infty$ then $\widetilde{\mathrm{Spec}(k)}_{\text{ét}}$ is not replete. In fact, if $(k_i)_{i\geq 0}$ is a sequence of finite degree separable extensions of k with $k_{i+1}|k_i$ then

$$\emptyset = \varprojlim_{i\geq 0} \mathrm{Spec}(k_i) \longrightarrow \mathrm{Spec}(k)$$

is not surjective in $\widetilde{\mathrm{Spec}(k)}_{\text{ét}}$. The topos of sets is replete.

Proposition 1.4. *(1) (finite coho. dim. $\Rightarrow$ left complete) If $X = \widetilde{S}$ with S a site and $\forall \mathscr{F} \in X$ a Λ-module, $\exists d \geq 0$ s.t. $\forall U \in S$ $\exists (V_i \to U)_{i\in I}$ a cover in S s.t.*

$$\forall i \in I, \ H^q(V_i, \mathscr{F}) = 0 \text{ for } q > d$$

then $D(X,\Lambda)$ is complete.

(2) (replete implies left complete) If X is replete then $D(X,\Lambda)$ is left complete.

Example 1.5. *If X is a finite type k-scheme, $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, ℓ any prime number, and $\mathrm{cd}_\ell(k) < +\infty$ then $D(X_{\text{ét}},\Lambda)$ is left complete. This is for example the case when k is algebraically closed or when k is a finite field.*

Finally, left completeness has the following advantage. Let $f : X \to Y$ be a morphism of topoi. Then the usual derived functor $Rf_* : D^+(X, \Lambda) \to D^+(Y, \Lambda)$ extends to a functor

$$\widehat{D}(X, \Lambda) \xrightarrow{Rf_*} \widehat{D}(Y, \Lambda) \xrightarrow{R\varprojlim} D(Y, \Lambda)$$

and thus if $D(X, \Lambda)$ is left complete we obtain a functor

$$Rf_* : D(X, \Lambda) \to D(Y, \Lambda)$$

that is right adjoint to f^*.

2. Étale/pro-étale/v-topology for perfectoid spaces

2.1. The étale topology on perfectoid spaces.

2.1.1. *Finite étale morphisms of perfectoid spaces.* Recall: there is a good notion of vector bundles on perfectoid spaces. Let X be a perfectoid space. By definition a vector bundle on X is a locally free of finite rank $\mathcal{O}_X$-module. Then, if $X = \mathrm{Spa}(A, A^+)$ is affinoid perfectoid

$$\Gamma(X, -) : \{\text{v.b. on } X\} \xrightarrow{\sim} \{A\text{-modules that are projective of finite type}\}$$

with inverse given by $P \mapsto P \otimes_A \mathcal{O}_X$. This property is what we call "a good notion".

From this and the purity theorem we deduce that there is a good notion of finite étale morphisms of perfectoid spaces such that for X perfectoid,

$$\{\text{finite étale perf. spaces}/X\} \xrightarrow{\sim} \left\{\begin{matrix}\text{finite locally free } \mathcal{O}_X\text{-algebras } \mathcal{A} \text{ s.t.} \\ \text{the quad. form } \mathrm{tr}_{\mathcal{A}/\mathcal{O}_X} : \mathcal{A} \times \mathcal{A} \to \mathcal{O}_X \text{ is perfect}\end{matrix}\right\}$$

where here by perfect we mean $\mathcal{A} \xrightarrow{\sim} \mathcal{A}^\vee$. Moreover, with this definition GAGA applies: for (A, A^+) affinoid perfectoid

$$\{\text{finite étale}/\mathrm{Spec}(A)\} \xrightarrow{\sim} \{\text{finite étale}/\mathrm{Spa}(A, A^+)\}.$$

2.1.2. *A result by Huber.* The following result by Huber about étale morphisms of noetherian analytic adic spaces is a key point for the definition of an étale morphism of perfectoid spaces. For morphisms of analytic noetherian adic spaces there is a "good" notion of étale and smooth morphisms analogous to the one for morphisms of schemes using the infinitesimal criterion for formal smoothness couples with some locally (topologically) of finite type hypothesis.

PROPOSITION 2.1 (Huber). *Let $f : X \to Y$ be a morphism between adic spectra of strongly noetherian Tate rings. Then, for any $y \in Y$ there exists a nbd. V of y and a factorization in the category of Noetherian analytic adic spaces*

$$\begin{tikzcd} f^{-1}(V) \arrow[r, hook, "\textit{open immersion}"] \arrow[d, "f"'] & W \arrow[dl, "\textit{finite étale}"] \\ V & \end{tikzcd}$$

SKETCH OF PROOF. Write $X = \mathrm{Spa}(B, B^+)$ and $Y = \mathrm{Spa}(A, A^+)$. General results about étale morphisms of analytic noetherian adic spaces show that one can write

$$B = A\langle X_1, \dots, X_n\rangle/(f_1, \dots, f_n)$$

with

$$\det\left(\frac{\partial f_i}{\partial X_j}\right)_{1\le i,j\le n} \mod (f_1, \dots, f_n) \in B^\times$$

(and B^+ is the integral closure of the image of $A^+\langle X_1, \dots, X_n\rangle$). An approximation result "à la Elkik" shows that if we "modify slightly" the equations $f_1, \dots, f_n$ then the obtained topological A-algebra B' is isomorphic to B. We can thus suppose that

$$f_1, \dots, f_n \in A[X_1, \dots, X_n]$$

(this is a typical algebraization result "à la Elkik"). Up to replacing $A[X_1, \dots, X_n]/(f_1, \dots, f_n)$ by its localization with respect to the image of an element of $1 + A^{\circ\circ}[X_1, \dots, X_n]$ we can suppose that the Jacobian becomes invertible and thus that we have a finite type étale A-algebra C with a finite set of elements $g_1, \dots, g_n \in C$ such that

$$X = \big\{x \in \mathrm{Spec}(C)^{ad} \mid |g_1(x)| \le 1, \dots, |g_n(x)| \le 1\big\}.$$

where $(-)^{ad}$ is the analytification functor

$$(-)^{ad} : \{\text{finite type } \mathrm{Spec}(A)\text{-schemes}\} \longrightarrow \{\text{loc. of finite type } \mathrm{Spa}(A, A^+)\text{-adic spaces}\}$$

that sends $\mathbb{A}^n_{\mathrm{Spec}(A)}$ to $\mathbb{A}^{n,ad}_{\mathrm{Spa}(A,A^+)}$ (affine schematical space to affine adic space). Since $\mathcal{O}_{Y,y}$ is Henselian, the étale $\mathrm{Spec}(\mathcal{O}_{Y,y})$-scheme

$$\mathrm{Spec}(C) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathcal{O}_{Y,y})$$

splits as a disjoint union of a finite étale étale scheme together with an étale scheme over $\mathrm{Spec}(\mathcal{O}_{Y,y})$ with image in $\mathrm{Spec}(\mathcal{O}_{Y,y})$ not containing the closed point. Since $\mathcal{O}_{Y,y} = \varinjlim_{V \ni y} \mathcal{O}(V)$ with V a rational neighborhood of y, a finite presentation argument shows that up to replacing (A, A^+) by a rational localization (A', A'^+) such that $y \in \mathrm{Spa}(A', A'^+)$, and C by $C \otimes_A A'$, we can suppose that

$$\mathrm{Spec}(C) = \mathrm{Spec}(C_1) \coprod \mathrm{Spec}(C_2)$$

with

(1) $\mathrm{Spec}(C_2) \to \mathrm{Spec}(A)$ finite étale,
(2) and $\mathrm{supp}(y) \notin \mathrm{Im}\big(\mathrm{Spec}(C_2) \to \mathrm{Spec}(A)\big)$.

Now, the image of $\mathrm{Spec}(C_2)^{ad} \cap \{|g_1| \leq 1, \ldots, |g_n| \leq 1\}$ in $\mathrm{Spa}(A, A^+)$ is a quasi-compact open subset and thus, since pro-constructible, its closure is its set of specializations in the analytic adic space $\mathrm{Spa}(A, A^+)$. Since in an analytic affinoid adic space $z_1 \geq z_2 \Rightarrow \mathrm{supp}(z_1) = \mathrm{supp}(z_2)$, y does not ly in this closure. Up to another rational localization we can thus suppose that

$$\mathrm{Spec}(C_2)^{ad} \cap \{|g_1| \leq 1, \ldots, |g_n| \leq 1\} = \emptyset.$$

The morphism f then factorizes as

$$\mathrm{Spec}(C_1)^{ad} \cap \{|g_1| \leq 1, \ldots, |g_n| \leq 1\} \xhookrightarrow{\text{open immersion}} \mathrm{Spec}(C_1)^{ad} \xrightarrow{\text{finite étale}} \mathrm{Spa}(A, A^+). \qquad \square$$

$\rightarrow$ thus, the fact that such a result is true for noetherian analytic adic spaces but not for schemes is, at the end, a consequence of the fact that the local rings are Henselian. For schemes, Zariski's main theorem says that we can only find a compactification of a spearated étale morphism that is finite but no étale in general.

2.1.3. *Étale morphisms of perfectoid spaces.* Motivated by Huber's definition we take the following.

Definition 2.2. *A morphism of perfectoid spaces $X \to Y$ is étale if locally on X and Y it can be written as a composite*

$$\begin{tikzcd} X \arrow[r, hook, "\textit{open immersion}"] \arrow[d] & \overline{X} \arrow[dl, "\textit{finite étale morphism of perf. spaces}"] \\ X & \end{tikzcd}$$

$\rightarrow$ good definition. Typically:

- étale morphisms are open
- a morphism between étale X-perfectoid spaces, X perfectoid, is étale.

This type of result is reduced via the tilting equivalence to characteristic p. For X affinoid perfectoid space of char.p, if $X = \varprojlim_i X_i$ with X_i affinoid top. of finite type over $\mathrm{Spa}(\mathbb{F}_p(\!(\varpi)\!))$, and the limit is cofiltered, then

$$2 - \varinjlim_i \{\text{qc qs, étale}/X_i\} \xrightarrow{\sim} \{\text{qc qs, étale}/X\}.$$

2.1.4. *The étale site of a perfectoid space.* The following definition is now evident.

Definition 2.3. *X perfectoid space*

$$X_{\text{ét}} := \{\textit{small étale site of étale perf. spaces over } X\}.$$

Coverings: families $(U_i \to V)_i$ such that $V = \cup_i \mathrm{Im}(U_i \to V)$.

We now set the following. $\Lambda =$ torsion ring.

Definition 2.4.

$$\begin{aligned} D^+_{\text{ét}}(X,\Lambda) &:= D^+(X_{\text{ét}},\Lambda) \\ D_{\text{ét}}(X,\Lambda) &:= \widehat{D}(X_{\text{ét}},\Lambda) \quad (\textit{left completion of } D(X_{\text{ét}},\Lambda)) \end{aligned}$$

$\to$ we will give later a "geometric incarnation of $D_{\text{ét}}(X,\Lambda)$", i.e. not using the abstract left completion process, using the pro-étale topology on X.

2.2. The pro-étale topology on perfectoid spaces.

2.2.1. *Pro-étale morphisms.*

<u>Recall:</u> $\underbrace{\text{Aff. Perf.}}_{\text{affinoid perfetoid spaces}}$ admits cofiltered limits:

$$\varprojlim_i \operatorname{Spa}(A_i, A_i^+) = \operatorname{Spa}(A_\infty, A_\infty^+)$$

where if ϖ = image of a p.u. of some A_i,

$$A_\infty^+ = \widehat{\varinjlim_i A_i^+}, \quad A_\infty = A_\infty^+[\tfrac{1}{\varpi}]$$

(ϖ-adic completion).

$\to$ not true for "classical" Noetherian analytic adic spaces; perfectoid spaces are much more "flexible"

Definition 2.5. *(1) A morphism of affinoid perfectoid spaces is affinoid pro-étale if it can be written as*

$$\varprojlim_{i \geq i_0} X_i \longrightarrow X_{i_0}$$

where

(a) the limit is cofiltered with smallest index i_0,
(b) each X_i is affinoid perfectoid,
(c) the transition morphisms in the projective system are étale.

(2) A morphism $X \to Y$ is pro-étale if locally on X and Y it is affinoid pro-étale.

Example 2.6. *For X perfectoid and $x \in X$, the loalization of X at x,*

$$\operatorname{Spa}(K(x), K(x)^+) = \varprojlim_{U \ni x} U \hookrightarrow X$$

is pro-étale. In particular in general pro-étale morphisms, even the surjective one, are not open in general. For example,

$$\coprod_{x \in X} \operatorname{Spa}(K(x), K(x)^+) \longrightarrow X$$

is pro-étale surjective but not open in general.

EXAMPLE 2.7. *For $X = \mathrm{Spa}(A, A^+)$ aff. perf. and $I \subset A$ an ideal, $V(I) \subset |X|$ is represented by an aff. perf. space pro-étale inside X,*

$$V(I) = \varprojlim_{\substack{n\geq 1 \\ f_1,\dots,f_n\in I}} \{|f_1| \leq 1, \dots, |f_n| \leq 1\}.$$

2.2.2. *The pro-étale site of a perfectoid space.* Here X is a perfectoid space.

DEFINITION 2.8. *$X_{\text{pro-ét}}$ = the small site whose objects are perfectoid spaces that are pro-étale over X.*
Coverings=families $(U_i \to V)_{i\in I}$ of X-morphisms satisfying the strong surjectivity property

$$\forall W \underset{qc\ open}{\subset} V,\ \exists \underbrace{J}_{finite} \subset I \text{ and } \forall j \in J,\ Z_j \underset{qc\ open}{\subset} U_j \quad s.t. \quad W = \bigcup_{j\in J} \mathrm{Im}(Z_j \to V).$$

REMARK 2.9. *The definition of a covering is subtle, like for the fpqc topology for schemes, since pro-étale morphisms are not open in general. In fact, if $(U_i \to V)_{i\in I}$ is a family of morphisms of perfectoid spaces such that $\forall i,\ U_i \to V$ is open then*

$$\forall W \underset{qc\ open}{\subset} V,\ \exists \underbrace{J}_{finite} \subset I \text{ and } \forall j \in J,\ Z_j \underset{qc\ open}{\subset} U_j \quad s.t. \quad W = \textstyle\bigcup_{j\in J} \mathrm{Im}(Z_j \to V)$$

$$\Updownarrow$$

$$V = \textstyle\bigcup_{i\in I} \mathrm{Im}(U_i \longrightarrow V) \text{ i.e. } \coprod_{i\in I} U_i \longrightarrow V \text{ is surjective}$$

i.e. for families of open morphisms strongly surjective $\Leftrightarrow$ surjective in the usual "naive" sense.

For example, if $\mathbb{B}^{1/1/p^\infty}_{K,K^+}$ is the one dimensional perfectoid closed ball over the affinoid perfectoid field (K, K^+) then

$$\mathrm{Spa}(K, K^+) \coprod \mathbb{B}^{1,1/p^\infty}_{K,K^+} \smallsetminus \{0\} \longrightarrow \mathbb{B}^{1,1/p^\infty}_{K,K^+},$$

where $\mathrm{Spa}(K, K^+) \to \mathbb{B}^{1,1/p^\infty}_{K,K^+}$ is given by the inclusion of the origin of the ball, is pro-étale surjective but not a pro-étale cover.

DEFINITION 2.10.

$$\begin{aligned} D^+_{\text{pro-ét}}(X, \Lambda) &:= D^+(X_{\text{pro-ét}}, \Lambda) \\ D_{\text{pro-ét}}(X, \Lambda) &:= D(X_{\text{pro-ét}}, \Lambda) \end{aligned}$$

$\to$ since $\widetilde{X}_{\text{pro-ét}}$ is replete, $D(X_{\text{pro-ét}}, \Lambda)$ is left complete.

Here is the verification that $\widetilde{X}_{\text{pro-ét}}$ is replete. Let $(\mathscr{F}_n)_{n\geq 0}$ be a projective system of pro-étale sheaves on X with surjective transition morphisms. Let U be an object of $X_{\text{pro-ét}}$ and $s \in \mathscr{F}_0(U)$. Up to replacing U by an affinoid perfectoid cover and s by its restriction to this cover we can suppose that U is affinoid perfectoid. Set $U_0 = 0$ and $s_0 = s$. Suppose

by induction that we have constructed $U_n \to U$ an affinoid pro-étale cover and $s_n \in \mathscr{F}_n(U_n)$ such that $s_n \mapsto s_{|U_n}$. Any pro-étale cover of U_n is dominated by an affinoid pro-étale cover formed by one element. We can thus find $U_{n+1} \to U_n$ an affinoid pro-étale cover and $s_{n+1} \in \mathscr{F}_{n+1}(U_{n+1})$ such that $s_{n+1} \mapsto s_{n|U_{n+1}}$. Let now $U_\infty = \varprojlim_{n\geq 0} U_n$ that is an affinoid pro-étale cover of U. The collection $(s_{n|U_\infty})_{n\geq 0}$ lies in $\varprojlim_{n\geq 0} \mathscr{F}_n(U_\infty)$ and is mapped to $s_{|U_\infty}$. This proves the result.

2.3. The v-topology on perfectoid spaces. Analog of fpqc topology for schemes.

Definition 2.11. *X a perfectoid space. $X_v =$ big site whose objects are Perf_X and coverings are families of morphisms $(U_i \to V)_{i\in I}$ of X-perfectoid spaces that are strongly surjective as in Definition 2.8.*

As for the pro-étale site, $\widetilde{X}_v$ is replete and we set.

Definition 2.12.

$$\begin{aligned} D_v^+(X,\Lambda) &= D^+(X_v,\Lambda) \\ D_v(X,\Lambda) &= D(X_v,\Lambda) \end{aligned}$$

Remark 2.13. *$X_{\text{ét}}$ and $X_{\text{pro-ét}}$ are equivalent to small sites. This is not the case for X_v and to do cohomology one needs to fix some set-theoretical bounds by fixing a "sufficiently large cardinal" κ and consider only κ-small perfectoid spaces in the sense that the cardinal of $|X|$ and A for any $\mathrm{Spa}(A,A^+) \subset X$ affinoid perfectoid is less than κ. Doing this the v-site of κ-small perf. spaces is equivalent to a small site, and if κ is taken sufficiently large then all results and constructions do not depend on κ.*

2.4. Comparison of étale/pro-étale/v. X =perfectoid space. Evident continuous morphisms of sites

$$X_v \xrightarrow{\lambda} X_{\text{pro-ét}} \xrightarrow{\nu} X_{\text{ét}}$$

Proposition 2.14. *The functors*

$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\nu^*} D_{\text{pro-ét}}(X,\Lambda) \xrightarrow{\lambda^*} D_v(X,\Lambda)$$

satisfy!

(1) ν^* is fully faithful and $\mathrm{Id} \xrightarrow{\sim} R\nu_*\nu^*$
(2) $\nu\circ\lambda$ is fully faithful and $\mathrm{Id} \xrightarrow{\sim} R(\nu\circ\lambda)_*(\nu\circ\lambda)^*$.

$\to$ using left completness this is reduced to prove that for $\mathscr{F}$ an étale sheaf of Λ-modules on X and $\mathscr{G}$ a pro-étale sheaf of Λ-modules on X, $\forall q \in \mathbb{N}$,

$$\begin{aligned} H^q(X_{\text{ét}},\mathscr{F}) &\xrightarrow{\sim} H^q(X_{\text{pro-ét}},\nu^*\mathscr{F}) \\ H^q(X_{\text{pro-ét}},\nu^*\mathscr{F}) &\xrightarrow{\sim} H^q(X_v,\lambda^*\nu^*\mathscr{F}) \end{aligned}$$

$\to$ analogous to the following "classical" result for schemes: X a scheme,

$$X_{fpqc} \xrightarrow{\lambda} \underbrace{X_{\text{pro-ét}}}_{\text{Bhatt-Scholze pro-étale site}} \xrightarrow{\nu} X_{\text{ét}}$$

then one can compute the étale cohomology of an étale torsion sheaf using the pro-étale or even the fpqc site: for $\mathscr{F}$ an étale sheaf of Λ-modules on X and $\mathscr{G}$ a pro-étale sheaf of Λ-modules on X,

$$H^q(X_{\text{ét}}, \mathscr{F}) \xrightarrow{\sim} H^q(X_{\text{pro-ét}}, \nu^*\mathscr{F}) \xrightarrow{\sim} H^q(X_{fpqc}, \lambda^*\nu^*\mathscr{F})$$

This type of results itself is an abelian generalization of well known results about the non-abelian H^1. Typically, if G is a smooth X-group scheme then any fpqc G-torsor on X is representable by a smooth X-scheme (smooth morphisms satisfy fpqc descent). It has thus a section over an étale cover of X and is thus an étale G-torsor. This gives

$$H^1_{\text{ét}}(X, G) \xrightarrow{\sim} H^1_{fpqc}(X, G).$$

Here is the key point in the proof of $H^\bullet_{\text{ét}}(X, \mathscr{F}) \xrightarrow{\sim} H^\bullet_{\text{pro-ét}}(X, \nu^*\mathscr{F})$. This is the following result.

Proposition 2.15. *$X =$affinoid perfectoid.*

(1) The functor $\varprojlim$ induces an equivalence

$$\textit{Pro category (affinoid perfectoid, étale}/X) \xrightarrow{\sim} \{\textit{affinoid pro-étale}/X\}$$

(2) If $U = \varprojlim_i U_i$ with U_i affinoid perfectoid étale over X then

$$\nu^*\mathscr{F}(U) = \varinjlim_i \mathscr{F}(U_i).$$

Point (2) is an easy consequence of point (1) that reduces the problem to an "algebraic statement" like in Bhatt-Scholze. Point (1) is a "decompletion argument" that is obtained by devissage from the following using a "graph of a morphism" argument.

Proposition 2.16 (Elkik). *If $(A_i, A_i^+)_{i\in I}$ is a filtered inductive system of affinoid complete Tate rings and $A_\infty^+ = \widehat{\varinjlim_{i\in I} A_i^+}$, $A_\infty = A_\infty^+[\frac{1}{\varpi}]$ then*

$$2-\varinjlim_{i\in I}\{\textit{finite étale } A_i\textit{-alg.}\} \xrightarrow{\sim} \{\textit{finite étale } \varinjlim_{i\in I} A_i\textit{-alg.}\} \xrightarrow{\sim} \{\textit{finite étale } A_\infty\textit{-alg.}\}$$

2.5. Description of the essential image of $\nu^* : D_{\text{ét}} \hookrightarrow D_{\textbf{pro-ét}}$.

2.5.1. *Strictly totally disconnected perfectoid spaces.* Let us recall the following.

Definition 2.17. *A perfectoid space is strictly totally disconnected if it is quasi-compact quasi-separated and any étale cover has a section.*

It is immediately checked that any s.t.d. perfectoid space is affinoid perfectoid (a section of $\coprod_i U_i \to X$ with $(U_i)_i$ an aff. perf. covering gives a decomposition $X = \coprod_i V_i$ with V_i open/closed in U_i and thus aff. perf.).

Proposition 2.18. *X = qc qs perfectoid space. The following are equivalent:*

(1) X is strictly totally disconnected
(2) $\forall \mathscr{F}$ étale sheaf of abelian groups on X and $q > 0$, $H^q_{\text{ét}}(X, \mathscr{F}) = 0$
(3) for all $x \in X$, $K(x)$ is algebraically closed and the connected components of $|X|$ have a unique closed point i.e. are of the form $\mathrm{Spa}(K(x), K(x)^+)$ for some $x \in X$.

$\to$ If X is a qc qs perfectoid space there is a continuous surjective map

$$\underbrace{|X|}_{\text{spectral space}} \longrightarrow \underbrace{\pi_0(|X|)}_{\text{profinite space}}$$

and thus a surjective morphism of v-sheaves

$$X \longrightarrow \underline{\pi_0(X)}$$

where for P a profinite set $\underline{P}$ = v-sheaf on Perf s.t.

$$\underline{P}(S) = \mathscr{C}(|S|, P),$$

if $P = \varprojlim_i \underbrace{P_i}_{\text{finite set}}$ then $\underline{P} = \varprojlim_i \underbrace{\underline{P_i}}_{\text{constant sheaf wt. value } P_i}$.

$\Rightarrow$ any qc qs perf. space X is fibered naturally over a profinite set with connected fibers.

Then,

$$X \text{ strictly totally disconnected}$$
$$\Updownarrow$$
$$\text{the fibers of } X \to \underline{\pi_0(X)} \text{ are of the form } \mathrm{Spa}(C, C^+) \text{ wt. } C \text{ alg.closed}$$

Remark 2.19. *There is a stronger notion than s.t.d. perf. spaces: strictly w-local perf. spaces. For this one adds the condition that $X \to \underline{\pi_0(X)}$ has a section*

$$X \rightleftarrows \underline{\pi_0(X)}.$$

In this case one can really think about strictly w-local perf. spaces as "amalgmations of $\mathrm{Spa}(C(x), C(x)^+)$, $x \in P$, along a profinite set P with $C(x)$ alg. closed.

PROPOSITION 2.20. *Any perfectoid space X admits a pro-étale cover $(U_i \to X)_i$ such that for all i, $U_i \to X$ is open and U_i is strictly totally disconnected.*

$\to$ the proof consist in giving a meaning to

$$\text{"} \varprojlim_{\substack{U \to X \\ \text{étale cover} \\ U \text{ aff. perf.}}} U \text{"} \to \text{ has no sense from the set theoretical point of view}$$

This is done via an induction process that stops at some point; we don't even need any transfinite induction!

REMARK 2.21. *There is no explicit formula for such a s.t.d. pro-étale cover in general, typically for the ball $\mathbb{B}^{1,1/p^\infty}_{K,K^+}$ over the affinoid perfectoid field (K, K^+). The only "most general case" where one can give such an explicit forumla is for $\mathrm{Spa}(K, K^+) \times \underline{P} = \mathrm{Spa}(\mathscr{C}(P, K), \mathscr{C}(P, K^+))$ where P is a profinite set. For such a space one can take $\mathrm{Spa}(\widehat{\overline{K}}, \widehat{\overline{K}}^+) \times \underline{P} \to \mathrm{Spa}(K, K^+) \times \underline{P}$ as a s.t.d. pro-étale cover.*

2.5.2. *Description of the image.*

$\to$ $\nu^* : D_{\text{ét}}(X, \Lambda) \hookrightarrow D_{\text{pro-ét}}(X, \Lambda)$ is far from being essentially surjective in general.

For example, if $X = \mathrm{Spa}(C, \mathcal{O}_C)$ with C alg. closed this is the embedding

$$\nu^* : D(\Lambda) \hookrightarrow D(\text{condensed } \Lambda_{disc}\text{-modules}).$$

More generally, if $X = \mathrm{Spa}(K, K^+)$ is the spectrum of an affinoid perfectoid field this is the embedding

$$\begin{array}{c} D(\Lambda\text{-modules} + \text{discrete } \mathrm{Gal}(\overline{K}|K) \text{ linear action}) \\ \Big\downarrow \nu^* \\ D(\text{condensed } \Lambda_{disc}\text{-modules} + \text{linear action of } \underbrace{\mathrm{Gal}(\overline{K}|K)}_{\text{condensed group}}) \end{array}$$

The starting point is the following remark.

PROPOSITION 2.22. *If X is a s.t.d. perf. space then*

(1) any qc open subset is strictly totally disconnected
(2) the projection $X_{\text{ét}} \to |X|$ induces an equivalence of topoi $\widetilde{X}_{\text{ét}} \xrightarrow{\sim} |X|^\sim$
(3) $\nu^* : D_{\text{ét}}(X, \Lambda) \xrightarrow{\sim} D(|X|, \Lambda)$
(4) for any étale sheaf of ab. gp. $\mathscr{F}$ on X and $U \to X$ étale with U perfectoid qc qs, $H^q_{\text{ét}}(U, \mathscr{F}) = 0$ for $q > 0$.

$\rightarrow$ proof is easy using Prop. 2.18. Vanishing of cohomlogy (point (4)) implies that $D(X_{\text{ét}}, \Lambda)$ is left complete, see point (1) of Prop. 1.4.

Remark 2.23. *In the scheme context, see Bhatt-Scholze's article, any scheme X admits a pro-étale cover $(U_i \to X)_{i\in I}$ with U_i strictly totally disconnected schemes (which means in this context that U_i is affine with connected components spectra of strictly Henselian local rings). In this context Prop. 2.22 is false: a qc open subset of a s.t.d. scheme may not be s.t.d.. For example, if K is a complete non-archimedean field for a rank 1 valuation with residue field k_K separably closed then $Spec(\mathcal{O}_K)$ is strictly totally disconnected but the open subset $Spec(K) \hookrightarrow Spec(\mathcal{O}_K)$ may not be s.t.d. i.e. K may not be separably closed. One of the points is that spectral spaces associated to analytic adic spaces satisfy: for any $x \in$ this space, the set of generalizations of x is a chain. This is not true for schemes in general. What is true for X a s.t.d. scheme is that if $\mathscr{F}$ an étale sheaf of abelian groups on X then $H^q_{\text{ét}}(X, \mathscr{F}) = 0$ when $q > 0$.*

Proposition 2.24. *For X a perfectoid space*

(1) There are equivalences

$$\begin{tikzcd} D_{\text{ét}}(X,\Lambda) \arrow[d, "\nu^*", "\wr"'] \\ \left\{A \in D_{pro\text{-}ét}(X,\Lambda) \mid \forall S \to X,\ S \text{ s.t.d. }, A_{|S} \in D_{\text{ét}}(S,\Lambda) = D(|S|,\Lambda)\right\} \arrow[d, "\lambda^*", "\wr"'] \\ \left\{A \in D_v(X,\Lambda) \mid \forall S \to X,\ S \text{ s.t.d. }, A_{|S} \in D_{\text{ét}}(S,\Lambda) = D(|S|,\Lambda)\right\} \end{tikzcd}$$

(2) If $(U_i \to X)_{i\in I}$ is a pro-étale cover with $\forall i,\ U_i$ is s.t.d. then this reduces to

$$\begin{tikzcd} D_{\text{ét}}(X,\Lambda) \arrow[d, "\nu^*", "\wr"'] \\ \left\{A \in D_{pro\text{-}ét}(X,\Lambda) \mid \forall i \in I,\ A_{|U_i} \in D_{\text{ét}}(U_i,\Lambda) = D(|U_i|,\Lambda)\right\} \arrow[d, "\lambda^*", "\wr"'] \\ \left\{A \in D_v(X,\Lambda) \mid \forall i \in I, A_{|U_i} \in D_{\text{ét}}(U_i,\Lambda) = D(|U_i|,\Lambda)\right\} \end{tikzcd}$$

$\rightarrow$ the fact that ν^* is fully faithful on $D^+_{\text{ét}} + \widetilde{X}_{\text{pro-ét}}$ is replete and thus $D(X_{\text{pro-ét}}, \Lambda)$ is left complete + for all S perfectoid s.t.d. $D(S_{\text{ét}}, \Lambda))$ is complete $\Rightarrow$ we get a "geometric concrete incarnation" of the abstractly defined left completion $\widehat{D}(X_{\text{ét}}, \Lambda)$.

3. Étale/quasi-pro-étale/v-topology for locally spatial diamonds

3.1. A Key descent result.

PROPOSITION 3.1. *Separated étale morphisms of perfectoid spaces satisfy descent for the v-topology.*

$\to$ if $(U_i \to X)_i$ is a v-cover of perf. spaces then

$$\begin{array}{c} \{\text{separated étale perf. spaces } /X\} \\ \downarrow \wr \\ \{\text{separated étale perf. spaces } / \coprod_i U_i \ + \ \text{descent datum w.r.t.}\} \end{array}$$

By a descent datum we mean cartesian separated étale perfectoid spaces over the diagram

$$\coprod_{i,j,k} U_i \times_X U_j \times_X U_k \rightrightarrows \coprod_{i,j} U_i \times_X U_j \longrightarrow \coprod_i U_i$$

$\to$ For finite étale morphism of perf. spaces this is "easy" because vector bundles on perfectoid spaces satisfy v-descent:

PROPOSITION 3.2. *If $(U_i \to X)_{i\in I}$ is a v-cover of perfectoid spaces,*

$$\begin{array}{c} v.b. \ on \ X \\ \downarrow \wr \\ v.b. \ on \ \coprod_i U_i \ + \ descent \ datum \end{array}$$

$\to$ One of the key tools of the descent results for the v-topology on perf. spaces is that after a pro-étale covering one can suppose that our perfectoid spaces are s.t.d. and the use of the following key remark:

Let $X = \mathrm{Spa}(A, A^+)$ be s.t.d. and let $Y \to X$ be a morphism with $Y = \mathrm{Spa}(B, B^+)$ affinoid perfectoid then:

(1) $\mathrm{Spec}(B^+/\varpi) \longrightarrow \mathrm{Spec}(A^+/\varpi)$ is flat
(2) if $Y \to X$ is surjective i.e. a v-cover then

$$\mathrm{Spec}(B^+/\varpi) \longrightarrow \mathrm{Spec}(A^+/\varpi) \text{ is faithfully flat}$$

EXAMPLE 3.3. *Here is an application of Proposition 3.1 that we use all the time. Let X be a perf. space.*

(1) Let G be a finite group and $T \to X$ be a $\underline{G}$-torsor for the v-topology. Then T is represented by an étale separated perf. space over X and is thus an étale $\underline{G}$-torsor;

$$\boxed{H^1_{\text{ét}}(X,G) \xrightarrow{\sim} H^1_v(X,G).}$$

(2) *Suppose now that G is locally profinite. If $T \to X$ is a $\underline{G}$-torsor for the v-topology then $T \xrightarrow{\sim} \varprojlim_{\substack{K\subset G \\ \text{compact open subgp.}}} \underline{K}\backslash T$ and $\underline{K}\backslash T$ is represented by a separated étale X-perfectoid space. Thus, $T \to X$ is represented by a pro-étale perfectoid space and is thus a pro-étale $\underline{G}$-torsor,*

$$\boxed{H^1_{\text{pro-ét}}(X,\underline{G}) \xrightarrow{\sim} H^1_v(X,\underline{G}).}$$

3.2. Locally spatial diamonds.

Recall: A diamond is a pro-étale sheaf X on $\mathrm{Perf}_{\mathbb{F}_p}$ (i.e. on the big pro-étale site) satisfying:

$\exists \widetilde{X}$ perf. space and $R \subset \widetilde{X} \times \widetilde{X}$ an eq. relation representable by a perf. space s.t.

(1) $\widetilde{X} \rightrightarrows X$ are pro-étale
(2) $X \simeq \widetilde{X}/R$ as pro-étale sheaves on the big pro-étale site $(\mathrm{Perf}_{\mathbb{F}_p})_{\text{pro-ét}}$.

$\to$ thus a diamond is an "algebraic space for the pro-étale topology" on $\mathrm{Perf}_{\mathbb{F}_p}$. One can verify that a diamond is in fact a v-sheaf (analogous result to : any algebraic space in the sense of Artin is in fact an fpqc sheaf (Gabber)).

$\to$ the category of diamonds is "too large" to work with. For Artin "classical" algebraic spaces it is for example usual to assume that our algebraic spaces are quasi-separated to remove pathological objects like $\mathbb{G}_{a,k}/\mathbb{Z}$, k a field of char. 0.

Definition 3.4. *A spatial diamond is a diamond X satisfying*
(1) X is qc qs as a v-sheaf
(2) each point of $|X|$ has a basis of qc open nbd.

For X a spatial diamond the first basic result is that in fact

$$|X| \text{ is spectral.}$$

The typical example of a locally spatial diamond is $X^\diamond$ for X a locally Noetherian analytic adic space where

$$|X| = |X^\diamond|.$$

But those are not the only one locally spatial diamonds we deal with: we deal with "more exotic ones" like punctured absolute BC's.

Spatial diamonds share a lot of properties with analytic adic spaces, typically:

- in the spectral space $|X|$ the set of generalizations of a point form a chain,
- for $f : X \to Y$ a morphism of spatial diamonds, $|f| : |X| \to |Y|$ is generalizing.
- Any morphism between spatial diamonds is qc qs.

One has to be careful still: any morphism of perfectoid spaces is locally separated (since any morphism of aff. perf. spaces is separated) but this is not the case for morphisms of locally spatial diamonds: they are only locally quasi-separated in general.

Finally let us cite the following.

Proposition 3.5. *X =spatial diamond, $Z \subset |X|$ pro-constructible generalizing subset then the v-sheaf*

$$\mathrm{Perf}_{\mathbb{F}_p} \ni S \longmapsto \{S \to X \mid \mathrm{Im}(|S| \to |X|) \subset Z\}$$

is represented by a spatial diamond Y with $Y \hookrightarrow X$ quasi-compact quasi-pro-étale and $|Y| = Z$.

$\to$ if X is a qs finite type adic space over $\mathrm{Spa}(K)$, K non-archi. field, any $Z \subset |X| = |X^\diamond|$ defines a sub spatial diamond of $X^\diamond$ $\to$ can speak about the étale cohomology of such a subset even if this is not a rigid analytic space this has a nice geometric structure in the world of diamonds

3.3. The étale site of a locally spatial diamond.

Let X be a loc. spatial diamond. Since separated étale morphisms of perf. spaces descend for the v-topology and thus the pro-étale one, there is a good notion of a locally separated étale morphism of locally spatial diamond. As for perfectoid spaces, those are open morphisms.

Definition 3.6. *For X a loc. spatial diamond we note*

$$X_{\text{ét}}$$

for the small site of locally separated étale loc. spatial diamonds over X. A family of morphisms $(U_i \to V)_i$ in $X_{\text{ét}}$ is a cover if $\coprod_i |U_i| \to |V|$ is surjective.

$\to$ if X is a perfectoid space one recovers the preceding étale site of a perfectoid space; the definition is thus coherent.

Definition 3.7. *For X a loc. spatial diamond we note*

$$\begin{aligned} D^+_{\text{ét}}(X, \Lambda) &= D^+(X_{\text{ét}}, \Lambda) \\ D_{\text{ét}}(X, \Lambda) &= \widehat{D}(X_{\text{ét}}, \Lambda) \quad (\textit{left completion}) \end{aligned}$$

Finally let us cite.

PROPOSITION 3.8. *If X is an adic space locally of finite type over* $\mathrm{Spa}(K, K^+)$*, K a complete non-archimedean field, the continuous morphism of sites*

$$(X^\diamond)_{\text{ét}} \longrightarrow \underbrace{X_{\text{ét}}}_{\substack{\text{étale site} \\ \text{as defined by Huber}}}$$

induces an equivalence of topoi

$$(X^\diamond)^{\widetilde{}}_{\text{ét}} \xrightarrow{\sim} \widetilde{X}_{\text{ét}}.$$

$\to$ in particular one can compute étale cohomology of rigid analytic spaces using diamonds

3.4. Quasi-pro-étale morphisms.

Contrary to (separated) étale morphisms that satisfy descent wrt v-covers, pro-étale morphisms do not even descend along pro-étale morphisms and we need to take care of this. This is for example the case for the Kummer morphism $z \mapsto z^2$ from the perfectoid closed ball to itself that is not pro-étale but becomes pro-étale after a pro-étale localization of the target.

DEFINITION 3.9. *A morphism $X \to Y$ of perfectoid spaces is quasi-pro-étale if there exists a pro-étale cover $\widetilde{Y} \to Y$ s.t. $X \times_Y \widetilde{Y} \to \widetilde{Y}$ is pro-étale.*

This definition is a little bit abstract and hopefully we have this geometric caracterization.

PROPOSITION 3.10. *For $f : X \to Y$ a morphism of perfectoid spaces the following are equivalent:*

(1) f is quasi-pro-étale
(2) if $\widetilde{Y} \to Y$ is a pro-étale cover with $\widetilde{Y}$ a disjoint union of s.t.d. perf. spaces then $X \times_Y \widetilde{Y} \to \widetilde{Y}$ is pro-étale
(3) if $(U_i)_i$ is a cover of X by qc qs open subsets s.t. $f_{|U_i} : U_i \to Y$ is separated,

$$f_{|U_i} : U_i \to Y \text{ has profinite geometric fibers}$$

$\to \forall \mathrm{Spa}(C, C^+) \to Y$, $\forall i$, $U_i \times_Y \mathrm{Spa}(C, C^+)$ is isomorphic to $\mathrm{Spa}(C, C^+) \times \underline{P}$ with P a profinite set.

$\to$ thus, quasi-pro-étale morphisms=morphisms with locally on the source profinite geometric fibers.

3.5. The quasi-pro-étale site.

Definition 3.11. *(1) A morphism $X \to Y$ of locally spatial diamonds is quasi-pro-étale if $\forall S$ s.t.d. perf. space and $S \to Y$, $X \times_Y S \to S$ is pro-étale*

(2) For X a loc. spatial diamond

$$X_{\textit{q-pro-ét}} = \textit{small site of quasi-pro-étale loc. spatial diamonds } /X$$

where the covers are defined as for the pro-étale site of a perf. space using the "strong surjectivity condition".

(3) For X a loc. spatial diamond

$$\begin{aligned} D^+_{\textit{pro-ét}}(X,\Lambda) &= D^+(X_{\textit{q-pro-ét}},\Lambda) \\ D_{\textit{pro-ét}}(X,\Lambda) &= D(X_{\textit{q-pro-ét}},\Lambda) \end{aligned}$$

$\to$ If X is a perf. space then the continuous morphism of sites $X_{\text{q-pro-ét}} \to X_{\text{pro-ét}}$ induces an equivalence of topoi

$$\widetilde{X}_{\text{q-pro-ét}} \xrightarrow{\sim} \widetilde{X}_{\text{pro-ét}}$$

and thus our definition of $D_{\text{pro-ét}}$ is coherent.

Proposition 3.12. *Propositions 2.14 and 2.24 remain valid by replacing the perfectoid space X by a locally spatial diamond:*

$$\begin{tikzcd} D_{\text{ét}}(X,\Lambda) \arrow[r, hook, "\nu^*"] \arrow[dr, hook] & D_{\textit{pro-ét}}(X,\Lambda) \arrow[d, "\lambda^*"] \\ & D_v(X,\Lambda) \end{tikzcd}$$

where here $D_v(X,\Lambda) = D(X_v,\Lambda)$ where X_v is the big site Perf$/X$ *equipped with the localized v-topology on* Perf*. And the essential image is given by*

$$\big\{A \in D_v(X,\Lambda) \mid \forall S \textit{ s.t.d. perf. space }, \ \forall S \to X, \ A_{|S} \in D(|S|,\Lambda)\big\}.$$

Remark 3.13. *The functor $\lambda^* : D_{\textit{pro-ét}}(X,\Lambda) \to D_v(X,\Lambda)$ is in general not fully-faithful and does not satisfy* $\mathrm{Id} \xrightarrow{\sim} R\lambda_*\lambda^*$*. That being said, this is the case when restricted to*

$$D_{\textit{pro-ét},\blacksquare}(X,\Lambda) \subset D_{\textit{pro-ét}}(X,\Lambda)$$

and we have a diagram

$$\begin{tikzcd} D_{\text{ét}}(X,\Lambda) \arrow[r, hook, "\nu^*"] \arrow[dr, hook] & D_{\textit{pro-ét},\blacksquare}(X,\Lambda) \arrow[d, hook, "\lambda^*"] \\ & D_v(X,\Lambda) \end{tikzcd}$$

CHAPTER 3

3rd lecture: Mai 30 2023

1. $D_{\text{ét}}(X, \Lambda)$ for X a small v-sheaf

1.1. Small v-sheaves. $X = v$-sheaf of sets on the big site $\mathrm{Perf}_{\mathbb{F}_p}$.

Definition 1.1. *X is small if there exists a perfectoid space U and an epimorphism of v-sheaves $U \to X$.*

$\to$ one has to be careful that there exists non-small v-sheaves. For example, $\underline{\{s, \eta\}}$ is non-small where $\underline{\{s, \eta\}}(S) = \{\text{open subsets of } S\}$

$\to$ for example, diamonds are small v-sheaves since if X is a diamond, $X \simeq \widetilde{X}/R$ with $\widetilde{X}$ perfectoid and $R \subset \widetilde{X} \times \widetilde{X}$ a pro-étale equivalence relation and thus $\widetilde{X} \to X$ is a v-cover.

$\to$ need to work with more general objects than loc. spatial diamonds.

The good category of geometric objects we work with is the category of small v-sheaves equipped with morphisms that are relatively representable in loc. spatial diamonds

Example 1.2. *(1) If k is a char. p discrete field then $\mathrm{Spa}(k)$ is a small v-sheaf, not representable by a loc. spatial diamond*

(2) (Loc. profinite sets) If P is a locally profinite set then $\underline{P}$ is a small v-sheaf not representable by a loc. spatial diamond although

$$\begin{tikzcd} \underline{P} \arrow[d, "\text{rel. rep. in perf. spaces}"'] & \\ \mathrm{Spa}(\mathbb{F}_p) & \end{tikzcd} \qquad \text{not a loc. spatial diamond}$$

(3) (Formal schemes) If $\mathfrak{X}$ = more generally an $\mathbb{F}_p$-formal scheme $\mapsto$ a small v-sheaf $\mathfrak{X}^\diamond$ by taking the analytic sheaf associated to the presheaf $(R, R^+) \mapsto \mathfrak{X}(Spf(R^+))$

$\to$ small v-sheaf not representable by a loc. spatial diamond. For example, $\mathfrak{X} = Spf(\mathbb{F}_p[\![x_1, \ldots, x_d]\!])$, $\mathfrak{X}^\diamond(S) = (\Gamma(S, \mathcal{O}_S)^{\circ\circ})^d$,

$$\mathfrak{X}^\diamond \times_{\mathrm{Spa}(\mathbb{F}_p)} S \simeq \underbrace{\mathring{\mathbb{B}}_S^{d, 1/p^\infty}}_{\text{open perf. ball}/S}$$

but $\mathfrak{X}^\diamond$ is not a loc. spatial diamond. The small v-sheaf

$$\mathfrak{X}^\diamond \smallsetminus (\mathfrak{X}_{red})^\diamond \underbrace{\subset}_{\substack{\text{represented by}\\ \text{an open immersion}}} \mathfrak{X}^\diamond$$

is always representable by a perf. space for any $\mathfrak{X}$, for example

$$\begin{aligned} & \mathrm{Spf}(\mathbb{F}_p[\![x_1,\dots,x_d]\!])^\diamond \smallsetminus Spec(\mathbb{F}_p) \\ = \; & \underbrace{\mathrm{Spa}\big(\mathbb{F}_p[\![x_1^{1/p^\infty},\dots,x_d^{1/p^\infty},\mathbb{F}_p[\![x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]\!]\big) \smallsetminus V(x_1,\dots,x_d)}_{\text{qc qs perfectoid space}} \end{aligned}$$

$$\begin{tikzcd} \text{qc qs perf. space} \arrow[r, bend right] & \mathfrak{X}^\diamond \smallsetminus (\mathfrak{X}_{red})^\diamond \arrow[r, hook] & \mathfrak{X}^\diamond \arrow[d, "\text{rel. rep. in perf. spaces}"'] & \text{not a loc. spatial diamond} \arrow[l, bend right] \\ & & \mathrm{Spa}(\mathbb{F}_p) & \end{tikzcd}$$

(4) *(Absolute positive BC spaces). For (D,φ) an $\overline{\mathbb{F}}_q$-isocrystal relative to E, i.e. $D = \underbrace{\breve{E}}_{\widehat{E^{un}}}$-v.s. and φ is a $\underbrace{\sigma}_{\text{Frob of } E^{un}|E}$-linear auto, with ≤ 0 slopes the functor*

$$\begin{aligned} BC(D,\varphi) : \mathrm{Perf}_{\overline{\mathbb{F}}_q} &\longrightarrow \mathrm{Sets} \\ S &\longmapsto H^0(X_S, \mathscr{E}(D,\varphi)) \end{aligned}$$

is

(a) *representable by a formal scheme ($\mathcal{G}^\diamond$, $\mathcal{G}$ a formal p-divisible gp./$\overline{\mathbb{F}}_q$) for slopes $\in \big[-[E:\mathbb{Q}_p],0\big[$.*
(b) *representable by a formal scheme $\times$ a locally profinite set for slopes $\in [-[E:\mathbb{Q}_p],0]$,*
(c) *only a small v-sheaf for any slopes.*

If $* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$ there is a "zero section" $* \hookrightarrow BC(D,\varphi)$. In general, for any slope, one has the picture for any (D,φ)

$$\begin{tikzcd} \text{spatial diamond} \arrow[r, bend right] & BC(D,\varphi) \smallsetminus \{0\} \arrow[r, hook] & BC(D,\varphi) \arrow[d, "\text{rel. rep. in loc. spatial diamonds}"'] & \text{not a loc. spatial diamond} \arrow[l, bend right] \\ & & * & \end{tikzcd}$$

(5) *(Absolute negative BC spaces) (D,φ) has < 0 slopes,*

$$\begin{aligned} BC((D,\varphi)[1]) : \mathrm{Perf}_{\overline{\mathbb{F}}_q} &\longrightarrow \mathrm{Sets} \\ S &\longmapsto H^1(X_S, \mathscr{E}(D,\varphi)) \end{aligned}$$

is a small v-sheaf and we have the same picture

$$\begin{tikzcd}
\textcolor{red}{\textit{spatial diamond}} & BC((D,\varphi)[1]) \smallsetminus \{0\} \arrow[r, hook] & BC((D,\varphi)[1]) \arrow[d, "\textit{rel. rep. in loc. spatial diamonds}"'] & \textcolor{red}{\textit{not a loc. spatial diamond}} \arrow[ul, bend right] \\
& & * &
\end{tikzcd}$$

For example,

$$\underbrace{BC(\mathcal{O}(-1)[1])}_{\substack{\textit{small } v\textit{-sheaf} \\ \textit{not a diamond}}} \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(E)^\diamond \simeq \underbrace{(\mathbb{G}_{a/E})^\diamond/\underline{E}}_{\textit{loc. spatial diamond}}$$

and the spatial diamond $BC(\mathcal{O}(-1)[1]) \smallsetminus \{0\}$ is an absolute version of

$$\Omega^\diamond \Big/ \begin{pmatrix} 1 & \underline{E} \\ 0 & 1 \end{pmatrix}$$

PROPOSITION 1.3. *If X is a small v-sheaf there exists a v-hypercover*

$$U_\bullet \longrightarrow X$$

such that for all $n \geq 0$, U_n is a $\coprod$ of strictly totally disconnected perfectoid spaces.

$\to$ any U as in the definition of a small v-sheaf can be replaced, up to a pro-étale cover, by a $\coprod$ of s.t.d. perfectoid spaces. Now,

$$U \times_X U \underbrace{\subset}_{\text{sub } v\text{-sheaf}} \underbrace{U \times U}_{\text{perf. space}}$$

Use the following:

Any sub-v-sheaf of a diamond is a diamond and in particular a small v-sheaf.

That is deduced from the following:

If X is a s.t.d. perf. space then any pro-constructible generalizing subset of $|X|$ is representable by a perfectoid space pro-étale inside X.

Here is how to use the preceding. Let $\mathscr{F} \subset X$ be a sub-v-sheaf of X s.t.d. perfectoid. For each S affinoid perfectoid and each element of $\mathscr{F}(S)$ there is an associated morphism $S \to X$ to which is associated $\mathrm{Im}(|S| \to |X|)$ that is pro-constructible generalizing. Thus, applying the preceding result, for each element of $\mathscr{F}(S)$, S aff. perf., is associated an affinoid perfectoid $Z \subset X$ that is pro-étale inside X. When S and the element of $\mathscr{F}(S)$ vary this forms a subset of the set Σ of such $Z \subset X$. Then, using the v-sheaf property, there is a quasi-pro-étale surjection $\coprod_{Z \in \Sigma} Z \to \mathscr{F}$.

1.2. $D_{\text{ét}}(X,\Lambda)$**.** Let X be a small v-sheaf. Let $X_v = (\mathrm{Perf})_v/X$ be the v-site of X whose underlying category is the one of perfectoid spaces over X.

Definition 1.4. *Set*

$$D_{\text{ét}}(X,\Lambda) = \big\{A \in D(X_v,\Lambda) \mid \forall S \to X \ \ S \textit{ perf. s.t.d.}, \quad A_{|S} \in D_{\text{ét}}(S,\Lambda) = D(|S|,\Lambda)\big\}.$$

$\to$ one recovers $D_{\text{ét}}(X,\Lambda)$ for X a locally spatial diamond.

$\to$ since $D(X_v,\Lambda)$ and $D_{\text{ét}}(S,\Lambda)$ for S perf. s.t.d. are left complete, $D_{\text{ét}}(X,\Lambda)$ is left complete.

The main remark is now the following. Let

$$S_\bullet \longrightarrow X$$

be a v-hypercover s.t. for all $n \geq 0$, S_n is a $\coprod$ of s.t.d. perf. spaces. Then, pull back from the topos $\widetilde{X}_v$ to the topos of cartesian sheaves on $S_{\bullet,v}$ induces

$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} \underbrace{D_{cart}(|S_\bullet|,\Lambda)}_{\substack{\text{derived cat. of cartesian}\\ \text{sheaves of } \Lambda\text{-mod. on}\\ \text{the simplicial top. space } |S_\bullet|}}$$

Moreover if $A \in D_{\text{ét}}(X,\Lambda)$ corresponds to $\mathscr{F}_\bullet$ then

$$R\Gamma(X,A) \xrightarrow{\sim} \Gamma(|S_\bullet|,\mathscr{F}_\bullet).$$

$\to$ étale cohomology of perfectoid spaces / locally spatial diamonds / small v-sheaves is simpler than étale cohomology of schemes: everything is reduced to simplicial cartesian sheaves on top. spaces !!

We have in fact the ore general formula for $S_\bullet \to X$ a v-hypercover by perfectoid space

$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} \underbrace{\widehat{D}_{cart}(S_{\bullet,\text{ét}},\Lambda)}_{\substack{\text{left completion of}\\ \text{derived cat. of cartesian étale sheaves}\\ \text{on } S_\bullet}}$$

We will need the following Lemma.

Lemma 1.5. *The inclusion* $D_{\text{ét}}(X,\Lambda) \subset D_v(X,\Lambda)$ *admits a right adjoint* $R_{X_{\text{ét}}}$: $D_v(X,\Lambda) \to D_{\text{ét}}(X,\Lambda)$.

Proof. One can apply Freyd's adjunction theorem (or its upgrade to ∞-categories by Lurie) and the result is then a consequence of the fact that $D_{\text{ét}}(X,\Lambda)$ is stable under colimits. A slightly more constructive proof consists in replacing X by a v-hypercover $S_\bullet$ with S_n a $\coprod$ of s.t.d. perf. spaces for all $n \geq 0$. Then,

$$R_{X_{\text{ét}}} = R\,\mathrm{Cart}\,R(\nu_{S_\bullet} \circ \lambda_{S_\bullet})_*$$

where

$$\big((\nu_{S_\bullet}\circ\lambda_{S_\bullet})^*,(\nu_{S_\bullet}\circ\lambda_{S_\bullet})_*\big):\widetilde{S}_{\bullet,v}\longrightarrow|S_\bullet|^{\sim}$$

is a morphism of simplicial topoi and Cart is the cartesianification functor that is the right adjoint of the inclusion of cartesian sheaves on $|S_\bullet|$ inside all sheaves on $|S_\bullet|$. □

→*No explicit formula in general for the Cartesianification functor*. This cartesianification functor exists by an application of Freyd's adjunction theorem since the cat. of cart. sheaves is stable under colimits.

EXAMPLE 1.6. *If* $*=\mathrm{Spa}(\overline{\mathbb{F}}_q)$ *as a small* v*-sheaf then one has* $\underbrace{D(\Lambda)}_{\substack{\text{usual derived cat.}\\ \text{of }\Lambda\text{-modules}}}\xrightarrow{\sim}D_{\text{ét}}(*,\Lambda)$.

This is a consequence of the fact that if $C=\widehat{\overline{\overline{\mathbb{F}}_p((T))}}$ *then* $\mathrm{Spa}(C)\times_{\mathrm{Spa}(\overline{\mathbb{F}}_p)}\mathrm{Spa}(C)$ *is a connected perfectoid space isomorphic to a projective limit with finite étale transition morphisms of open punctured disks over* C *(write* $C=\widehat{\cup_{r\geq 0}K_r}$ *with* $K_r\subset K_{r+1}$ *and* $K_r|\overline{\mathbb{F}}_p((T))$ *separable of finite degree,* $K_r\simeq\overline{\mathbb{F}}_p((T))$*).*

2. $D_{\text{ét}}(\mathfrak{X},\Lambda)$ for $\mathfrak{X}$ a small v-stack

2.1. Small v-stacks. → I lied:

The good category of geometric objects we work with is the category of small v-stacks equipped with morphisms that are 0-truncated representable in loc. spatial diamonds (compactifiable loc. of finite dim trg.)

DEFINITION 2.1. *A stack* $\mathfrak{X}$ *on* $\mathrm{Perf}_{\mathbb{F}_p}$ *equipped with the* v*-top. is small if* $\exists S\to\mathfrak{X}$ *and* $T\to S\times_{\mathfrak{X}}S$ *that are* v*-surjective with* S *and* T *perf. spaces.*

→ $\mathfrak{X}$ is a rule that sends S an $\mathbb{F}_p$-perf. space to a groupoid $\mathfrak{X}(S)$ s.t. $S\mapsto\mathfrak{X}(S)$ is a fibered category over $\mathrm{Perf}_{\mathbb{F}_p}$ s.t. if $T\to S$ is a v-cover of aff. perf. spaces then

$$\mathfrak{X}(S)\longrightarrow\underbrace{2-\varprojlim}_{\substack{\text{objects of }\mathfrak{X}(T)\\ +\text{ descent datum}}}\big[\,\mathfrak{X}(T)\rightrightarrows\mathfrak{X}(T\times_S T)\rrrightarrow\mathfrak{X}(T\times_S T\times_S T)\big]$$

is an equivalence of categories.

→ $S\to\mathfrak{X}$, S perf. space, is v-surjective if $\forall T$ perf. space, $\forall T\to\mathfrak{X}$, $\exists\widetilde{T}\to T$ a v-cover and a morphism $\widetilde{T}\to S$ such that $\widetilde{T}\to T\to\mathfrak{X}$ and $\widetilde{T}\to S\to\mathfrak{X}$ are isomorphic as elements

of the groupoid $\mathfrak{X}(\widetilde{T})$

$$\begin{tikzcd}
 & \widetilde{T} \arrow[d, two heads, "\exists"', "v\text{-cover}"] \arrow[dd, bend right, "\exists"'] & & \\
 & T \arrow[r, squiggly] \arrow[d] & \widetilde{T} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\exists\Downarrow"] & \mathfrak{X} \\
S \arrow[r] & \mathfrak{X} & &
\end{tikzcd}$$

EXAMPLE 2.2. *(1) If S is a small v-sheaf and $H \to S$ is v-sheaf in groups that is small we can consider the classifying stack*

$$\mathfrak{X} = \big[S/H\big] \longrightarrow S.$$

This is the small v-stack over S such that for a perf. space T over S, $\mathfrak{X}(T)$ is the groupoid of $H \times_S T$-v-torsors over T.

(2) If $\mathfrak{X}$ is a small v-stack, S an $\mathbb{F}_p$-perfectoid space and $x \in \mathfrak{X}(S)$ we can consider the small v-sheaf in groups $\underline{Aut}(x) \to S$. The morphism $x : S \to \mathfrak{X}$ then factorizes canonically as a morphism of small v-stacks

$$\big[S/\underline{Aut}(x)\big] \longrightarrow \mathfrak{X}.$$

(3) The stack Bun_G of G-bundles on the curve is small.

As for small v-sheaves:

PROPOSITION 2.3. *If $\mathfrak{X}$ is a small v-stack then $\exists$ a v-hypercover*

$$S_\bullet \longrightarrow \mathfrak{X}$$

such that for all $n \geq 0$, S_n is a $\coprod$ of s.t.d. perfectoid spaces.

2.2. $D_{\text{ét}}(\mathfrak{X}, \Lambda)$**.** $\mathfrak{X} =$ small v-stack. We note

$$\widetilde{\mathfrak{X}}_v = 2 - \varprojlim_{\substack{S \to \mathfrak{X} \\ \text{perf. space}}} \widetilde{S}_v$$

for the topos of cartesian v-sheaves on $\mathfrak{X}$.

DEFINITION 2.4. *We note for $\mathfrak{X}$ a small v-stack*
$D_{\text{ét}}(\mathfrak{X}, \Lambda) = \{A \in D(\widetilde{\mathfrak{X}}_v, \Lambda) \mid \forall S \to \mathfrak{X},\ S$ *s.t.d. perf. space,* $A_{|S} \in D_{\text{ét}}(S, \Lambda) = D(|S|, \Lambda)\}$

$\to$ left complete by construction since $\widetilde{S}_v$ is replete and $D_{\text{ét}}(S, \Lambda)$ is left complete for S s.t.d. perf. space.

As before for small v-sheaves, if

$$S_\bullet \longrightarrow \mathfrak{X}$$

is a v-hypercover by s.t.d. perf. spaces then

$$D_{\text{ét}}(\mathfrak{X},\Lambda) \xrightarrow{\sim} D_{cart}(|S_\bullet|,\Lambda).$$

More generally for a v-hypercover by locally spatial diamonds $S_\bullet$ one has

$$D_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} \underbrace{\widehat{D}_{cart}(S_{\bullet,\text{ét}},\Lambda)}_{\substack{\text{left completion of}\\ \text{derived cat. of cartesian étale sheaves}\\ \text{on } S_\bullet}}$$

Let now H be a locally pro-p topological group, typically $G(E)$ where G is an affine algebraic group over E. We consider the small v-stack

$$[\,*\,/\underline{H}]$$

where $* = \mathrm{Spa}(\overline{\mathbb{F}}_p)$ is the final object of the v-topos. If M is a Λ-module it defines a v-sheaf by setting for $S \in \mathrm{Perf}_{\mathbb{F}_p}$

$$\underline{M}(S) = \{f : |S| \to M \mid f \text{ is locally constant}\}.$$

Recall that we set

$$\underline{H}(S) = \mathscr{C}(|S|, H).$$

Suppose now that M is equipped with a smooth action of H. Then, $\underline{M}$ is equipped with a smooth action of $\underline{H}$. In fact, if S is qc, $f : |S| \to M$ is loc. constant, $g : |S| \to H$ is continuous: there exists $K \subset H$ compact open s.t. $f : |S[\to M^K$. Then, the composite $|S| \xrightarrow{g} H \to H/K$ is loc. constant and thus

$$\begin{array}{rcl} |S| & \longrightarrow & M \\ s & \longmapsto & g(s).f(s) \end{array}$$

is locally constant. This defines an action of the v-sheaf $\underline{H}$ on the v-sheaf $\underline{M}$ and thus a v-sheaf on $[*/\underline{H}]$. This v-sheaf is étale since isomorphic to $\underline{M}$ after pull-back to $*$ via the v-cover $* \to [*/\underline{H}]$. This defines an exact functor

$$\{\text{smooth rep. of } H \text{ on } \Lambda\text{-modules}\} \longrightarrow \{\text{étales sheaves of } \Lambda\text{-modules on } [*/\underline{H}]\}$$

and thus an exact functor

$$\underbrace{D(H,\Lambda)}_{\substack{\text{derived cat. of}\\ \text{smooth rep. of}\\ H \text{ wt. coeff. in } \Lambda}} \longrightarrow D_{\text{ét}}([*/\underline{H}],\Lambda).$$

We prove the following theorem.

THEOREM 2.5. *If Λ is killed by a power of prime number different from p then the preceding functor*

$$D(H,\Lambda) \longrightarrow D_{\text{ét}}([*/\underline{H}],\Lambda)$$

is an equivalence.

Proof. One uses the v-hypercover

$$S_\bullet \longrightarrow [*/\underline{H}]$$

where if $C = \widehat{\overline{\mathbb{F}}_p(\!(T)\!)}$ for $n \geq 0$

$$S_n = \underbrace{\mathrm{Spa}(C) \times_{\mathrm{Spd}(\overline{\mathbb{F}}_p)} \cdots \times_{\mathrm{Spd}(\overline{\mathbb{F}}_p)} \mathrm{Spa}(C)}_{(n+1)\text{-times, connected}} \times \underline{H}^n$$

One obtains an identification

$$D_{\text{ét}}([*/\underline{H}], \Lambda) = \widehat{D}(H, \Lambda).$$

Now, the category of Λ-modules with a linear smooth H action is the category of Λ-modules in the topos of discrete H-sets (i.e. sets + action of H s.t. the stabilizer of a point is open). Any object in this topos has a cover formed of discrete H-sets of the form H/K for K compact open. Now, the cohoology of H/K with values in the smooth module M is $H^\bullet(K, M) := \varinjlim_{K' \subset K} H^\bullet(K/K', M^{K'})$. $\square$

Remark 2.6. *The proof gives that we always have an equivalence $\widehat{D}(H, \Lambda) \xrightarrow{\sim} D_{\text{ét}}([*/\underline{H}], \Lambda)$ and that if H has a basis of compact open subgroups K such that $cd_\Lambda(K) < +\infty$ (cohomological dimension of the category Λ-modules equipped with a smooth action of K) then $D(H, \Lambda)$ is left complete.*

2.3. ∞-categorical point of view. At the end we can apply the ∞-categorical point of view in the preceding although this is not strictly necessary.

Proposition 2.7. *There exists a unique v-hypersheaf of presentable stable ∞-categories on* $\mathrm{Perf}_{\mathbb{F}_p}$,

$$S \mapsto \mathcal{D}_{\text{ét}}(S, \Lambda)$$

such that if S is a s.t.d. perfectoid space then $\mathcal{D}_{\text{ét}}(S, \Lambda) = \mathcal{D}(|S|, \Lambda)$. One has for $\mathfrak{X}$ a small v-stack

$$\mathcal{D}_{\text{ét}}(\mathfrak{X}, \Lambda) = \varprojlim_{S \to \mathfrak{X}} \mathcal{D}_{\text{ét}}(S, \Lambda)$$

with $\pi_0 \mathcal{D}_{\text{ét}}(\mathfrak{X}, \Lambda) = D_{\text{ét}}(\mathfrak{X}, \Lambda)$.

CHAPTER 4

4th lecture: June 6 2023

1. The four operations $(f^*, Rf_*, R\mathscr{H}om, \otimes^{\mathbb{L}}_{\Lambda})$

1.1. (Rf_*, f^*) in general.

1.1.1. *Morphisms of small v-sheaves.* Let

$$f : X \longrightarrow Y$$

be a morphism of small v-stacks. There is an evident continuous morphism of topoi

$$(f_v^*, f_{v*}) : \widetilde{X}_v \longrightarrow \widetilde{Y}_v$$

that is a particular case of the following: if $\mathcal{T}$ is a topos and $g : U \to V$ is a morphism in $\mathcal{T}$ there is a morphism of localized topos

$$(g^*, g_*) : \mathcal{T}/U \longrightarrow \mathcal{T}/V.$$

This induces a couples of adjoint functors (use the left complete property to see that Rf_{v*} extends to the whole derived category and not only the D^+)

$$D(X_v, \Lambda) \underset{Rf_{v*}}{\overset{f_v^*}{\leftrightarrows}} D(Y_v, \Lambda)$$

Now the point is the following.

Proposition 1.1. *(1) f_v^* sends $D_{\text{ét}}(X, \Lambda)$ to $D_{\text{ét}}(Y, \Lambda)$ and induces a functor*

$$f^* : D_{\text{ét}}(Y, \Lambda) \longrightarrow D_{\text{ét}}(X, \Lambda)$$

(2) f^* admits a right adjoint Rf_*

$$D_{\text{ét}}(X, \Lambda) \underset{f^*}{\overset{Rf_*}{\rightleftarrows}} D_{\text{ét}}(Y, \Lambda)$$

Proof. Point (1) is evident since we work "in a big topos" and f_v^* is just a restriction functor. More precisely, if S is a s.t.d. perfectoid space with a morphism $S \to X$, and $B \in D(Y_v, \Lambda)$ then $(f_v^* B)_{|S} = B_{|S}$ via the composite $S \to X \to Y$. For point (2) one can take

$$Rf_* = R_{Y_{\text{ét}}} \circ (Rf_{v*})_{|D_{\text{ét}}(X,\Lambda)}.$$

□

$\to$ no explicit formula in general for $Rf_* : D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda)$.

There is an evident case when there is an explicit formula for Rf_*.

If X and Y are represented by locally spatial diamonds, via the identifications $D_{\text{ét}}(X, \Lambda) = \widehat{D}(X_{\text{ét}}, \Lambda)$, $D_{\text{ét}}(Y, \Lambda) = \widehat{D}(Y_{\text{ét}}, \Lambda)$ (left completion), one has

$$Rf_* = Rf_{\text{ét}*} \text{ and } f^* = f^*_{\text{ét}}$$

where $f_{\text{ét}} : X_{\text{ét}} \to Y_{\text{ét}}$ is the continuous morphism of étale sites.

$\to$ it suffices to verify that $f^* = f^*_{\text{ét}}$ that is evident, the equality $Rf_* = Rf_{\text{ét}*}$ follows by adjunction.

Example 1.2. *If $f : X \to Y$ is a morphism of locally of finite type K-adic spaces, K =non-archi. field, then via the identifications $D_{\text{ét}}(X^\diamond, \Lambda) = D(X_{\text{ét}}, \Lambda)$, $(f^{\diamond *}, Rf^\diamond_*)$ is the usual couple of adjoint functors $(f^*_{\text{ét}}, Rf_{\text{ét}*})$ defined by Huber.*

1.1.2. 0-*truncated morphisms of small v-stacks.* Let

$$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$$

be a morphism of small v-stacks. Suppose it is 0-truncated; this means that if $\mathscr{F}$ is a v-sheaf together with a morphism $\mathscr{F} \to \mathfrak{Y}$ then the v-stack

$$\mathfrak{X} \times_{\mathfrak{Y}} \mathscr{F}$$

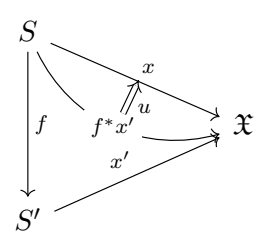
is a v-sheaf in the sense that it is a fibered category in " discrete groupoids" i.e. groupoids where objects have no automorphisms, that is to say a set. Another way to say it is that it is relatively representable in v-sheaves. This is for example the case if $\mathfrak{X}$ is itself is a small v-sheaf.

There is associated a morphism of topoi

$$(f^*_v, f_{v*}) : \widetilde{\mathfrak{X}}_v \longrightarrow \widetilde{\mathfrak{Y}}_v$$

of cartesian sheaves. This morphism of topoi exists even when f is not 0-truncated but there is a simpler expression for f_{v*} when f is 0-truncated (and this is the only case we use in our work). More precisely, the category of perfectoid spaces over $\mathfrak{X}$, $\mathrm{Perf}/\mathfrak{X}$, whose objects are perfectoid spaces $S \to \mathfrak{X}$ together with morphisms given by

$$\mathrm{Hom}\big(S \xrightarrow{x} \mathfrak{X}, S' \xrightarrow{x'} \mathfrak{X}\big) = \{(f, u) \mid f : S \to S',\ u : f^*x' \xrightarrow{\sim} x\}$$



is equiped with an evident functor by composing with f

$$\mathrm{Perf}/\mathfrak{X} \longrightarrow \mathrm{Perf}/\mathfrak{Y}$$

that induces

$$f_v^* : \underbrace{2-\varprojlim_{T\to\mathfrak{Y}} \widetilde{T}_v}_{\widetilde{\mathfrak{X}}_v} \longrightarrow \underbrace{2-\varprojlim_{S\to\mathfrak{X}} \widetilde{S}_v}_{\widetilde{\mathfrak{Y}}_v}$$

i.e. the value (as an element of $\widetilde{S}_v$) of the cartesian sheaf $f_v^*\mathscr{F}$ on $S \to \mathfrak{X}$ is given by the value of $\mathscr{F}$ on $S \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ i.e.

$$(f_v^*\mathscr{F})_{|S} = \mathscr{F}_{|S}$$

via $S \to \mathfrak{X} \to \mathfrak{Y}$.

The functor f_{v*} sends the cartesian sheaf $\mathscr{F}$ to the cartesian sheaf whose value on $T \to \mathfrak{Y}$ is the pushforward via $\widetilde{\mathrm{Perf}}_v/\mathfrak{X} \times_{\mathfrak{Y}} S \to S$ of the value (as a v-sheaf sitting over the v-sheaf $\mathfrak{X} \times_{\mathfrak{Y}} S$) of $\mathscr{F}$ restricted to $\mathfrak{X} \times_{\mathfrak{Y}} S$ (this defines a cartesian sheaf since we are working with big topoi and pullback is nothing else than restriction) i.e.

$$(f_{v*}\mathscr{F})_{|S} = f_{S,v*}(\mathscr{F}_{|\mathfrak{X}\times_{\mathfrak{Y}} S})$$

where $f_S : \mathfrak{X} \times_{\mathfrak{Y}} S \longrightarrow S$.

We thus obtain a couple of adjoint functors

$$D_v(\mathfrak{X}, \Lambda) \underset{Rf_{v*}}{\overset{f_v^*}{\leftrightarrows}} D_v(\mathfrak{Y}, \Lambda)$$

It is immediately checked that f_v^* sends $D_{\text{ét}}(\mathfrak{Y}, \Lambda)$ to $D_{\text{ét}}(\mathfrak{X}, \Lambda)$. This is not the case for Rf_{v*} in general. We set

$$Rf_* = R_{\mathfrak{Y}_{\text{ét}}} \circ Rf_{v*|D_{\text{ét}}(\mathfrak{X},\Lambda)}$$

This defines a couple of adjoint functors

$$D_{\text{ét}}(\mathfrak{X}, \Lambda) \underset{Rf_*}{\overset{f^*}{\leftrightarrows}} D_{\text{ét}}(\mathfrak{Y}, \Lambda)$$

As before, there is in general no explicit formula for Rf_*.

Example 1.3. *$H' \subset H$ closed subgroup $\to f : [*/\underline{H'}] \to [*/\underline{H}]$. Then, $f^* = \mathrm{Res}^H_{H'}$ (exact functor, extends immediately to the derived category) and $Rf_* = \mathrm{Ind}^H_{H'}$ (smooth induction, exact functor).*

1.2. The case of a qc qs morphism representable in locally spatial diamonds.

There is a particular case when on can compute Rf_*. This is the following.

PROPOSITION 1.4 (Quasi-compact base change). *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a qc qs morphism of small v-stacks representable in locally spatial diamonds i.e. $\forall S \to Y$ with S a loc. spatial diamonds $X \times_Y S$ is a loc. spatial diamond. Suppose that Λ is killed by a power of a prime to p integer. Let $A \in D^+_{\text{ét}}(\mathfrak{X}, \Lambda)$:*

(1) $Rf_{v*}A \in D_{\text{ét}}(\mathfrak{Y}, \Lambda)$ and is equal to Rf_*A.

(2) If S is a loc. spatial diamond, $S \to \mathfrak{Y}$ and

$$f_S : \mathfrak{X} \times_{\mathfrak{Y}} S \longrightarrow S,$$

inducing

$$(f_S)_{\text{ét}} : (\mathfrak{X} \times_{\mathfrak{Y}} S)_{\text{ét}} \longrightarrow S_{\text{ét}}$$

one has for $A \in D^+_{\text{ét}}(\mathfrak{X}, \Lambda)$

$$R(f_S)_{\text{ét}*}\big(A_{|\mathfrak{X}\times_{\mathfrak{Y}} S}\big) \xrightarrow{\sim} (Rf_*A)_{|S}$$

via the identifications $D((\mathfrak{X} \times_{\mathfrak{Y}} S)_{\text{ét}}, \Lambda) = D_{\text{ét}}(\mathfrak{X} \times_{\mathfrak{Y}} S, \Lambda)$ and $D(S_{\text{ét}}, \Lambda) = D_{\text{ét}}(S, \Lambda)$.

$\to$ uses Huber's quasi-compact base change.

$\to$ The hypothesis Λ killed by a power of ℓ with $\ell \neq p$ is essential. In fact, already the étale cohomology of the one dimensional ball over $C|\mathbb{Q}_p$ algebraically closed, $H^\bullet_{\text{ét}}(\mathbb{B}^1_C, \mathbb{F}_p)$, depends on C and thus qc base change does not hold in this situation.

Here is a strinking application.

COROLLARY 1.5. *Let $j : \mathfrak{U} \hookrightarrow \mathfrak{X}$ be an open immersion of small v-stacks. Suppose that j is qc qs. Then for A an $\underbrace{\text{étale } v\text{-sheaf of } \Lambda\text{-modules}}_{\substack{\text{an object of } D_{\text{ét}}(\mathfrak{U},\Lambda) \\ \text{concentrated in deg. } 0}}$ on $\mathfrak{U}$ with Λ killed by a power of $\ell \neq p$,*

$$R^i j_* A = 0 \text{ for } i > 0.$$

PROOF. Compute the pullback of $R^i j_* A$ via pullback via $\mathrm{Spa}(C, C^+) \to \mathfrak{X}$ using the qc base change theorem. But now if U is a qc open subset of $\mathrm{Spa}(C, C^+)$, $j' : U \hookrightarrow \mathrm{Spa}(C, C^+)$, since any qc open subset of $\mathrm{Spa}(C, C^+)$ is strictly totally disconnected one has $R^i j'_* = 0$ for $i > 0$. $\square$

$\to$ the quasi-compactness assertion is essential. For example, if $j : \mathbb{B}^{1,\diamond}_{K,K^+} \smallsetminus \{0\} \hookrightarrow \mathbb{B}^{1,\diamond}_{K,K^+}$ is the inclusion of the punctured closed ball over the affioid field (K, K^+) inside the ball then $R^1 j_* \mathbb{F}_\ell \neq 0$ if ℓ is invertible in K.

REMARK 1.6. *The stack* Bun_G *is not quasi-separated and thus Corollary 1.5 does not apply to open subs-tacks of* Bun_G. *For example,* $j : [*/\underline{G(E)}] \hookrightarrow \mathrm{Bun}_G$ *and to any* π *a smooth rep. of* $G(E)$ *one can associate* $\mathscr{F}_\pi$ *an étale v-sheaf on* $[*/\underline{G(E)}]$. *Then in general* $R^i j_* \mathscr{F}_\pi \neq 0$ *for* $i > 0$.

1.3. $\otimes^{\mathbb{L}}_\Lambda$ **and** $R\mathscr{H}om(-,-)$**.**

$\mathfrak{X}$ =small v-stack

It is easy to verify that $- \otimes^{\mathbb{L}}_\Lambda -$ on $D_v(\mathfrak{X}, \Lambda) \times D_v(\mathfrak{X}, \Lambda)$ sends $D_{\text{ét}}(\mathfrak{X}, \Lambda) \times D_{\text{ét}}(\mathfrak{X}, \Lambda)$ to $D_{\text{ét}}(\mathfrak{X}, \Lambda)$.

Now, for $A \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$ we can look at the functor

$$\begin{array}{ccc} D_{\text{ét}}(\mathfrak{X}, \Lambda) & \longrightarrow & D_{\text{ét}}(\mathfrak{X}, \Lambda) \\ B & \longmapsto & A \otimes^{\mathbb{L}}_\Lambda B \end{array}$$

This commutes with colimits and thus has a right adjoint (Freyd's adjunction theorem)

$$C \longmapsto R\mathscr{H}om_\Lambda(A, C).$$

Once again there is no explicit formula in general for this functor, like Rf_* in general.

If $S_\bullet \to \mathfrak{X}$ is a v-hypercover by $\coprod$ of s.t.d. perfectoid spaces then via

$$D_{\text{ét}}(\mathfrak{X}, \Lambda) \xrightarrow{\sim} D_{cart}(|S_\bullet|, \Lambda),$$

One has

$$R\mathscr{H}om_\Lambda(A, B) = R \underbrace{\mathrm{Cart}}_{\substack{\text{Cartesianification} \\ \text{functor}}} \underbrace{R\mathscr{H}om_\Lambda(A, B)}_{\substack{\text{usual } R\mathscr{H}om \\ \text{in } D(|S_\bullet|, \Lambda)}}$$

$\to$ as for Rf_*, if $\mathfrak{X} = X$ is a locally spatial diamond then $R\mathscr{H}om_\Lambda(A, B)$ is the usual derived functor computed in $\widehat{D}(X_{\text{ét}}, \Lambda)$.

2. The two operations $(Rf_!, Rf^!)$

2.1. Huber's canonical compactification.

2.1.1. *classical context.* Recall: Let

$$f : X \to Y$$

be a morphism of adic spaces locally of finite type over $\mathrm{Spa}(K, K^+)$ an affinoid field. We say that f is proper if it is qc separated and universally closed. This is equivalent to f qc qs and $\forall (R, R^+)$ top. of finite type over (K, K^+),

$$\begin{tikzcd} \mathrm{Spa}(R, R^\circ) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spa}(R, R^+) \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

This last property is called partially proper. Thus,

$$\text{proper} \iff \text{quasi-compact quasi-separated and partially proper.}$$

Separated partially proper morphisms are exactly the good one for which the derived functor of $f_{\text{ét}!}$ is the good notion for relatic cohomology with proper support. More precisely, if $\mathscr{F}$ is an étale sheaf on X and $U \to Y$ is étale then

$$\Gamma(U, f_{\text{ét}!}\mathscr{F}) = \{s \in \Gamma(X \times_Y U, \mathscr{F}) \mid \operatorname{supp}(s) \xrightarrow{f_{|\operatorname{supp}(s)}} U \text{ is proper}\}.$$

Then,

$$Rf_{\text{ét}!} : D(X_{\text{ét}}, \Lambda) \longrightarrow D(Y_{\text{ét}}, \Lambda)$$

is "the good relative cohomology with proper support" functor.

Example 2.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes locally formally of finite type over Spf($\mathcal{O}_K$). Let $\mathfrak{X}_\eta$, resp. $\mathfrak{Y}_\eta$, be their generic fiber as adic spaces locally of finite type over* $\operatorname{Spa}(K, \mathcal{O}_K)$. *Then if $f_\eta : \mathfrak{X}_\eta \longrightarrow \mathfrak{Y}_\eta$,*

$$\forall Z \text{ irred. comp. of } \mathfrak{X}_{red},\ f_{red|Z} \text{ proper} \iff f_\eta \text{ partially proper.}$$

For $f : X \to Y$ as before Huber says that f is taut if $\forall V \subset Y$ open qc qs and $U \subset f^{-1}(V)$ qc then $\overline{U}$ is qc. Let f be separated and taut. Then Huber defines a canonical compactification

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "f"'] & \overline{X}^{/Y} \arrow[dl, "\bar{f}"] \\ Y & \end{tikzcd}$$

where j is an open immersion and $\bar{f}$ is separated partially proper. Then he defines

$$Rf_{\text{ét}!} = R\bar{f}_{\text{ét}!} \circ j_!$$

and proves Poincaré duality in this context when f is moreover smooth $\to$ good definition for relative cohomology with proper support.

When $X = \operatorname{Spa}(B, B^+)$ and $Y = \operatorname{Spa}(A, A^+)$ define $(B^+)' =$ integrale closure of $f^*(A^+) + B^{\circ\circ}$. Then,

$$X = \operatorname{Spa}(B, B^+) \xhookrightarrow[\text{open immersion}]{} \overline{X}^{/Y} = \operatorname{Spa}(B, (B^+)')$$

is an open immersion.

$\to$ In fact, since f is of finite type, by definition, there exists an open surjective morphism $A\langle T_1, \dots, T_n\rangle \to B$ such that B^+ is the integral closure of the image of $A^+\langle T_1, \dots, T_n\rangle$. Thus $A^+/A^{\circ\circ} \to B^+/B^{\circ\circ}$ is integral over a finite type $A^+/A^{\circ\circ}$-algebra. Now, if $g_1, \dots, g_n \in B^+$ is a lift of a set of elements $\bar{g}_1, \dots, \bar{g}_n \in B^+/B^{\circ\circ}$ such that $B^+/B^{\circ\circ}$ is integral over $A^+/A^{\circ\circ}[\bar{g}_1, \dots, \bar{g}_n]$ then

$$X = \{|g_1| \le 1, \dots, |g_n| \le 1\} \subset \overline{X}^{/Y}$$

$|\overline{X}^{/Y}| \smallsetminus |X|$ is made of rank > 1 valuations only.

EXAMPLE 2.2. *Take* $X = \mathbb{B}^1_K = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle) \to \mathrm{Spa}(K, \mathcal{O}_K) = Y$. *Then,*

$$\overline{X}^{/Y} = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K + K^{\circ\circ}\langle T\rangle).$$

One has $|\overline{X}^{/Y}| = |X| \cup \{x\}$ *where*

$$v\Big(\sum_{n\geq 0} a_n T^n(x)\Big) = \inf\Big\{(v(a_n), -n) \in \underbrace{\Gamma_K \times \mathbb{Z}}_{\substack{+\textit{lexicographic}\\ \textit{order}}} \mid n \geq 0\Big\}$$

with x *a specialization of the Gauss norm.*

X = *adic space associated to a classical rigid space i.e.* $\mathrm{Sp}(R) \mapsto \mathrm{Spa}(R, R^\circ)$
$\overline{X}^{/Y}$ = *adic space not associated to a classical rigid space i.e.* $\mathrm{Spa}(R, R^+)$ *with* $R^+ \neq R^\circ$.

$\to$ for Berkovich spaces one considers only rank 1 valuations and $|\mathbb{B}^1_K|$ is compact $\to$ cohomology with compact support=cohomology $\to$ not the good definition of coho. with compact support $\partial\mathbb{B}^{1,an}_K \neq \emptyset \Leftrightarrow \mathbb{B}^{1,ad}_K \to \mathrm{Spa}(K, \mathcal{O}_K)$ not partially proper. We need to consider non-overconvergent étale sheaves like $i_{x*}\Lambda$ to define coho. with compact support. This is why we can not define coho. with compact support in general for K-Berkovich spaces X such that $\partial(X/K) \neq \emptyset$, typically for affinoid Berkovixh spaces X where $|X|$ is compact and thus $\Gamma_c(X, -) = \Gamma(X, -)$. In fact if $X^{Berk} = \mathcal{M}(A)$ and $X^{ad} = \mathrm{Spa}(A, A^\circ)$ then there is an equivalence of topoi

$$\widetilde{(X^{Berk})_{\text{ét}}} \xrightarrow{\sim} \{\text{overconvergent étale sheaves on } X^{ad}\}$$

where overconvergent means that for $x \in X$, if $x : \mathrm{Spa}(C, C^+) \to X$, C alg. closed,

$$x^*\mathscr{F}$$

is a constant sheaf on $|\mathrm{Spa}(C, C^+)|$ with value its stalk at the generic point $\mathrm{Spa}(C, \mathcal{O}_C)$.

$\to$ morale of the story: even to define compactly supported cohomology for overconvergent sheaves like $\underline{\Lambda}$ we need to go through non-overconvergent sheaves when f is not partially proper.

2.2. Compactifiable morphisms of small v-stacks. Let

$$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$$

be a 0-trunncated morphism of small v-stacks. Define

$$\underbrace{\overline{\mathfrak{X}}}_{\substack{\text{absolute compactification}\\ \text{of } \mathfrak{X}\to *}} = v\text{-stack}$$

such that $\overline{\mathfrak{X}}(R, R^+) = \mathfrak{X}(R, R^\circ)$. This is the "absolute compactification of $\mathfrak{X}$ over $*$". Then, one verifies that

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook] \arrow[d, "f"] & \overbrace{\overline{\mathfrak{X}}}^{\text{small}v\text{-stack}} \arrow[d, "\bar{f}=0\text{-truncated}"] \\ \mathfrak{Y} \arrow[r, hook] & \underbrace{\overline{\mathfrak{Y}}}_{\text{small}v\text{-stack}} \end{tikzcd}$$

And define

$$\underbrace{\overline{\mathfrak{X}}^{/\mathfrak{Y}}}_{\substack{\text{relative}\\ \text{compactification}}} = \overline{\mathfrak{X}} \times_{\overline{\mathfrak{Y}}} \mathfrak{Y}$$

There is a diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, hook, "j"] \arrow[d, "f"'] & \overline{\mathfrak{X}}^{/\mathfrak{Y}} \arrow[dl, "\overline{f}^{/\mathfrak{Y}}"] \\ \mathfrak{Y} & \end{tikzcd}$$

We now take the result that says that for separated taut morphisms of adic spaces locally of finite type over $\mathrm{Spa}(K, K^+)$, j is an open immersion as definition.

Definition 2.3. *The morphism f is compactifiable if it is separated and $j; \mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}^{/\mathfrak{Y}}$ is representable by an open immersion.*

$\to$ f qc compactifiable $\Rightarrow$ $\overline{f}^{/\mathfrak{Y}}$ is proper $\Rightarrow$ canonical compactification. In general, if f is not qc then $\overline{f}^{/\mathfrak{Y}}$ is only partially proper.

Remark 2.4. *One has to be careful that f representable in loc. spatial diamonds compactifiable does only imply that $\overline{f}^{/\mathfrak{Y}}$ representable in diamonds but à priori non locally spatial one..although in all cases when we apply this compactification construction this is the case.*

Example 2.5. *If $f : X \to Y$ is a separated taut morphism of adic spaces locally of finite type over $\mathrm{Spa}(K, K^+)$ then $f^\diamond : X^\diamond \to Y^\diamond$ is compactifiable.*

2.3. Geometric transcendance degree. $\to$ need to bound some cohomological dimension to have a "good" $Rf_!$.

Let $C'|C$ be an extension of complete algebraically closed non-archimedean fields. Define the topological transcendance degree

$$\mathrm{tr.c}(C'/C) \in \mathbb{N} \cup \{+\infty\}$$

as the minimum of the cardinal of I where there exists $(x_i)_{i\in I} \in C'^I$ such that the sub-field $\overline{C(x_i)_{i\in I}}$ of C' is dense in C'.

$\to$ well behaved when finite (Temkin): if $C \subset C' \subset C''$ and $\mathrm{tr.c}(C'/C) < +\infty$ then $\mathrm{tr.c}(C'/C) \leq \mathrm{tr.c}(C''/C)$. But it may happen (the answer to this is not known) that $\mathrm{tr.c}(C'/C) = +\infty$ and $\mathrm{tr.c}(C''/C) < +\infty$. Since we don't know we set

$$\underbrace{\widetilde{\mathrm{tr.c}}(C'/C)}_{\substack{\text{topological}\\ \text{transcendance degree}}} = \inf_{C''/C'} \mathrm{tr.c}(C''/C).$$

Then $C'|C \mapsto \widetilde{\mathrm{tr.c}}(C'/C)$ is monotonic sub-additive: for $C''|C'|C$

$$\boxed{\begin{aligned} \widetilde{\mathrm{tr.c}}(C'/C) &\leq \widetilde{\mathrm{tr.c}}(C''/C) \\ \widetilde{\mathrm{tr.c}}(C''/C) &\leq \widetilde{\mathrm{tr.c}}(C''/C') + \widetilde{\mathrm{tr.c}}(C'/C). \end{aligned}}$$

Definition 2.6. *(1) $f : X \to Y$ morphism of diamonds. Set*

$$\underbrace{\mathrm{dim.trg}}_{\substack{\textit{geometric}\\ \textit{transcendance degree}}} f = \sup_{x\in X} \widetilde{\mathrm{tr.c}}(C(x)|C(f(x))$$

where $\mathrm{Spa}(C(x), C(x)^+) \to X$ and $\mathrm{Spa}(C(f(x), C(f(x))^+) \to Y$ are quasi-pro-étale with x and $f(x)$ in their image and we have an extension

$$\begin{tikzcd} \mathrm{Spa}(C(x), C(x)^+) \arrow[r] \arrow[d, dashed] & X \arrow[d, "f"] \\ \mathrm{Spa}(C(f(x), C(f(x)^+) \arrow[r] & Y \end{tikzcd}$$

(2) if $f : \mathfrak{X} \to \mathfrak{Y}$ is representable in diamonds,

$$\mathrm{dim.trg}(f) = \sup_{\substack{S\to Y\\ S \text{ diamond}}} \mathrm{dim}.trg(X \times_{\mathfrak{Y}} S \to S)$$

Example 2.7. *Let (D, φ) be an isocrystal with ≤ 0 slopes. Then*

$$BC(D, \varphi) \longrightarrow *$$

is of finite dim. trg $\to$ There exists $d \geq 0$, V a finite dim. E-v.s., such that for any S there exists a pro-étale surjection

$$\mathring{\mathbb{B}}_S^{d,1/p^\infty} \times \underline{V} \longrightarrow BC(D, \varphi) \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} S$$

and $\mathrm{dim}.trg(f) \leq d$.

PROPOSITION 2.8 (Key cohomological bound). *Let $f : X \to \mathrm{Spa}(C, C^+)$ be a spatial diamond*

(1) for all maximal point $x \in X$, $X_x \simeq \mathrm{Spa}(C', \mathcal{O}_{C'})/\underline{G_x}$ for a profinite group $G_x \subset \mathrm{Aut}(C')$ satisfying $cd_\ell(G_x) \leq$ dim. trg.(f)

(2) $\dim |X| \leq$ dim. trg(f)

(3) for all $\mathscr{F}$ étale sheaf on Λ-modules on X with Λ killed by a power of ℓ one has $H^i_{\text{ét}}(X, \mathscr{F}) = 0$ for $i > 2$dim. trg.(f).

(4) if f is compactifiable then $H^i_{\text{ét,c}}(X, \mathscr{F}) = 0$ for $i > 3 \dim .trg.(f)$ ($2 \dim .trg.(f)$ if $\overline{X}^{/\mathrm{Spa}(C,C^+}$, that is à priori only a diamond, is moreover spatial).

$\to$ here $\dim |X|$ is the usual dimension of a spectral space: the maximal length of a chain of specializations

$\to$ the proof is like the one for schemes: if X is a finite type k-scheme, k alg. closed, and $\mathscr{F}$ an étale torsion sheaf on X then $H^i_{\text{ét}}(X, \mathscr{F}) = 0$ for $i > 2 \dim X$:

(1) One has first that (Tsen theorem: k alg. closed implies $k(T)$ is (C1) and thus $cd(G_{k(T)}) \leq 1$; and thus if $K|k$ is finite type then $cd(G_K) \leq tr.deg(K/k)$) for all $x \in X$, $cd(\mathrm{Spec}(k(x))) \leq \dim(X)$

(2) We use the projection $\mu : X_{\text{ét}} \to X_{Zar}$

(3) We use (Grothendieck: Noetherian induction on open subsets of $|X|$) that if S is a Noetherian topological space then $cd(S) \leq \dim(S)$.

CHAPTER 5

5th lecture: June 21 2023

1. The two operations $(Rf_!, Rf^!)$

1.1. $Rf_!$ for f representable in spatial diamonds. We are seeking to define $(Rf_!, Rf^!)$ for f represntable in locally spatial diamonds compactifiable of finite dim. trg. Let us begin first with the case when f is qc i.e. f is representable in diamonds.

from now on Λ is killed by a power of $\ell \neq p$

$\to$ used to bound coh. dimensions of $Rf_!$ by 3dim. trg.(f) (and even 2dim. trg.(f) if $\overline{f}^{/\mathfrak{Y}}$ is representable in loc. spatial diamonds).

Definition 1.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of small v-sheaves* $\underbrace{\textit{representable in spatial diamonds}}_{\textit{and thus qc}}$ *compactifiable of finite* dim. trg. *We define*

$$Rf_! = R(\underbrace{\overline{f}^{/\mathfrak{Y}}}_{\substack{\textit{proper morphism}\\ \textit{representable in qc qs diamonds}}})_* \circ j_! : D_{\text{ét}}(\mathfrak{X}, \Lambda) \longrightarrow D_{\text{ét}}(\mathfrak{Y}, \Lambda).$$

where $j : \mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}^{/\mathfrak{Y}}$.

$\to$ finite dim. trg $\Rightarrow$ $Rf_!$ commutes with direct sums + finite cohomological dimension.

1.2. $Rf_!$ for f a morphism of loc. spatial diamonds. $f : X \to Y$ a morphism of loc. spatial diamonds, compactifiable of finite dim. trg.

Definition 1.2. *We define*

$$Rf_! : D^+_{\text{ét}}(X, \Lambda) \longrightarrow D^+_{\text{ét}}(Y, \Lambda)$$

as $R(\overline{f}^{/Y})_{\text{ét}!} \circ j_!$ where $(\overline{f}^{/Y})_{\text{ét}!}$ is the functor from sheaves of Λ-modules on $\underbrace{(\overline{X}^{/Y})_{\text{ét}}}_{\substack{\textit{small site of}\\ \textit{loc. separated}\\ \textit{étale morphisms}}}$ *to sheaves of Λ-modules on $Y_{\text{ét}}$ that is the subfunctor $f_{\text{ét}*}$ of section with relative proper support.*

The extension to the full derived category $D_{\text{ét}}(X, \Lambda)$ is delicate. For Rf_*, the extension from $D^+_{\text{ét}}$ to $D_{\text{ét}}$ is straightforward since Rf_* commutes with limits and we can use the formula

$Rf_*A = \varprojlim_{n\geq 0} Rf_*\tau_{\geq -n}A$ that makes sense since our target category is left complete. This is not the case of $Rf_!$ that commutes with colimits but not with any limit.

We have to use a process of left Kan extension to solve this, this can only be done in the ∞-categorical setting $\to$ since $Rf_!$ has to commute with colimits this has to commute with left Kan extensions.

Definition 1.3. *Let $\mathcal{D}_{\text{ét,prop/Y}}(X,\Lambda)$ be the presentable stable ∞-category of $A \in \mathcal{D}_{\text{ét}}(X,\Lambda)$ such that there exists $U \subset X$ such that $U \to Y$ is qc. and $j_!j^*A \xrightarrow{\sim} A$. The funtor $Rf_!$ is the left Kan extension of*

$$R(\overline{f}^{/Y})_* \circ j_! : \mathcal{D}_{\text{ét,prop/Y}}(X,\Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(Y,\Lambda)$$

to $\mathcal{D}_{\text{ét}}(X,\Lambda)$.

$\to$ for $A \in \mathcal{D}_{\text{ét}}(X,\Lambda)$, since $X \to Y$ is taut

can write $A = \varinjlim_i A_i$ (filtered colimit as a complex of v-sheaves) with $k_{i!}k_i^*A_i \xrightarrow{\sim} A_i$ with $k_i : U_i \hookrightarrow X$ and $U_i \to Y$ qc.. Take simply $A_i = k_{i!}k_i^*A$.

Then,

$$Rf_!A = \underbrace{\varinjlim_i R(\overline{f_{|U_i}}^{/Y})_* j_{i!}A_i}_{\text{homotopy colimit}}$$

where $j_i : U_i \hookrightarrow \overline{U_i}^{/Y}$ and $\overline{f_{|U_i}}^{/Y} : \overline{U_i}^{/Y} \to Y$ proper. Here the homotopy limit is only defined up to a non-canonical isomorphism in the usual triangulated category $D_{\text{ét}}(Y,\Lambda) \to$ this defines $Rf_!A$ only up to a non-canonical isomorphism as $\underbrace{\text{a cone}}_{\text{not defined canonically}}$, if $B_i = R(\overline{f_{|U_i}}^{/Y})_* j_{i!}A_i$

$$\bigoplus_{i\leq j} B_i \longrightarrow \bigoplus_i B_i \longrightarrow \varinjlim_i B_i \xrightarrow{+1}$$

To have a definition of $Rf_!$ as a functor we need to upgrade to ∞-categories where this limit, the so-called process of Kan extension, is defined canonically.

1.3. $Rf_!$ for f representable in locally spatial diamonds. Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a morphism of small v-stacks representable in locally spatial diamonds compactifiable of finite dim. trg..

We use the preceding with the proper base change theorem to construct $Rf_!$. Let

$$T_\bullet \longrightarrow \mathfrak{Y}$$

be a v-hypercover with T_n a locally spatial diamond for all n. We note

$$S_\bullet = \mathfrak{X} \times_{\mathfrak{Y}} T_\bullet.$$

One has ($S \mapsto \mathcal{D}_{\text{ét}}(S, \Lambda)$ is a v-hypersheaf on loc. spatial diamonds)

$$\begin{aligned} \mathcal{D}_{\text{ét}}(\mathfrak{X}, \Lambda) &\xrightarrow{\sim} \varprojlim_{[n]\in\Delta} \mathcal{D}_{\text{ét}}(S_n, \Lambda) \\ \mathcal{D}_{\text{ét}}(\mathfrak{Y}, \Lambda) &\xrightarrow{\sim} \varprojlim_{[n]\in\Delta} \mathcal{D}_{\text{ét}}(T_n, \Lambda). \end{aligned}$$

For each $n \geq 0$, we have the ∞-functor

$$Rf_{n!} : \mathcal{D}_{\text{ét}}(S_n, \Lambda) \longrightarrow \mathcal{D}_{\text{ét}}(T_n, \Lambda).$$

Proper base change (that is an immediate application of quasi-compact base change) implies this extends to an ∞-functor

$$\boxed{Rf_! : \varprojlim_{[n]\in\Delta} \mathcal{D}_{\text{ét}}(S_n, \Lambda) \longrightarrow \varprojlim_{[n]\in\Delta} \mathcal{D}_{\text{ét}}(T_n, \Lambda).}$$

All of this being done, proper base change applies

THEOREM 1.4 (proper base change). *$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ morphism of small v-stacks representable in loc. spatial diamonds compactifiable of finite dim. trg., Λ killed by a power of $\ell \neq p$. Cartesian diagram of small v-stacks*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \mathfrak{Y}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

For all $A \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$ one has

$$g^* Rf_! A \xrightarrow{\sim} Rf'_! g'^* A.$$

1.4. $Rf^!$**.**

$Rf_!$ commutes with direct sums $\Rightarrow$ there exists a right adjoint $Rf^!$. Let us be more precise since this is a delicate point. Let $f : X \to Y$ is a compactifiable morphism of locally spatial diamonds. Let $(A_i)_{i\in I}$ be a collection of objects in $D^+(X_{\text{ét}}, \Lambda)$ satisfying

$$\exists N \in \mathbb{Z}, \quad \forall i \in I, \ A_i \in D^{\geq N}(X_{\text{ét}}, \Lambda).$$

Then it is easily checked

$$\bigoplus_{i\in I} Rf_!(A_i) \xrightarrow{\sim} Rf_!\Big(\bigoplus_{i\in I} A_i\Big).$$

Suppose now that f is of finite dim. trg. and thus $Rf_!$ has cohomological degree bounded by some integer $c \in \mathbb{N}$. Then, for all $i \in \mathbb{Z}$,

$$\mathcal{H}^i(Rf_! A) = \mathcal{H}^i(Rf_! \tau_{\geq i-c} A)$$

and one deduces from the preceding case that $Rf_!$ commutes with arbitrary direct sums.

Definition 1.5. *For $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ a morphism of small v-stacks representable in locally spatial diamonds, compactifiable of finite dim. trg.*

$$Rf^! : D_{\text{ét}}(\mathfrak{Y}, \Lambda) \longrightarrow D_{\text{ét}}(\mathfrak{X}, \Lambda)$$

is the right adjoint of $Rf_!$.

2. Annexe: the catalog of operations

Cartesian diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \mathfrak{Y}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

where all morphisms are compactifiable representable in loc. spatial diamonds of finite dim. trg.. Λ is killed by a power of $\ell \neq p$.

Relative tautological adjunction	$R\mathscr{H}om(A, Rf_*B) \xrightarrow{\sim} Rf_*R\mathscr{H}om(f^*A, B)$
Tautological base change map (iso. if g qc qs: qc base change)	$f^*Rg_* \longrightarrow Rg'_*f'^*$
Relative proper adjunction	$R\mathscr{H}om(Rf_!A, B) \xrightarrow{\sim} Rf_*R\mathscr{H}om(A, Rf^!B)$
Proper base change	$f^*Rg_! \xrightarrow{\sim} Rg'_!f'^*$
Dual proper base change	$Rg'_*Rf'^! \xrightarrow{\sim} Rf^!Rg_*$
Projection formula	$Rf_!(f^*A \otimes^{\mathbb{L}}_{\Lambda} B) \xrightarrow{\sim} A \otimes^{\mathbb{L}}_{\Lambda} Rf_!B$
Expectational pull-back of Hom	$R\mathscr{H}om(f^*A, Rf^!B) \xrightarrow{\sim} Rf^!R\mathscr{H}om(A, B)$

$\to$ formulas are deduced from the proper base change and the projection formula. For example, for the "dual proper base change",

$$\begin{aligned}\mathrm{Hom}(A, Rf^! Rg_* B) &\underbrace{=}_{\text{adjunction}} \mathrm{Hom}(g^* Rf_! A, B) \underbrace{=}_{\text{proper BC}} \mathrm{Hom}(Rf'_! g'^* A, B) \\ &\underbrace{=}_{\text{adjunction}} \mathrm{Hom}(A, Rg'_* Rf'_! B)\end{aligned}$$

and thus Yoneda $\Rightarrow$ the result. The same goes on for the exceptional pullback of Hom.

3. Cohomologically smooth morphisms

3.1. Definition. $\to$ The definition of coho. smooth morphisms is more subtle than what one may think: we have to force the property to be stable under base change. Here $\ell \neq p$.

Definition 3.1. *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a separated morphism of small v-stacks representable in locally spatial diamonds. Then, f is ℓ-coho. smooth if*

(1) it is compactifiable of finite dim. trg.,
(2) for any $S \to \mathfrak{Y}$ with S strictly totally disconnected, if

$$f_S : \mathfrak{X} \times_{\mathfrak{Y}} S \longrightarrow S$$

then there exists $D \in D_{\text{ét}}(\mathfrak{X} \times_{\mathfrak{Y}} S, \mathbb{F}_\ell)$ invertible and an isomorphism of functors from $D_{\text{ét}}(S, \mathbb{F}_\ell)$ to $D_{\text{ét}}(\mathfrak{X} \times_{\mathfrak{Y}} S, , \mathbb{F}_\ell)$,

$$Rf_S^!(-) \simeq D \otimes_{\mathbb{F}_\ell} f_S^*(-).$$

$\to$ For any f as before there is a natural morphism obtained by playing with the different adjunctions

$$Rf^!(\Lambda) \otimes^{\mathbb{L}}_{\Lambda} f^*(-) \longrightarrow Rf^!(-)$$

Then f is ℓ-coho. smooth iff for all $S \to \mathfrak{Y}$ with S a s.t.d. perf. space,

(1) $Rf_S^!(\mathbb{F}_\ell) \otimes f_S^*(-) \longrightarrow Rf_S^!(-)$ is an iso.
(2) $Rf_S^! \mathbb{F}_\ell$ is invertible.

$\to$ here invertible with respect to the monoidal structure $\otimes^{\mathbb{L}}_{\Lambda}$ is in fact equivalent to étale locally iso. to $\mathbb{F}_\ell[2d]$ for some $d \in \frac{1}{2}\mathbb{Z}$ that we call the dimension of f as a locally constant function on $|S|$.

Can descend the preceding and prove:

Theorem 3.2. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be separated ℓ-coho. smooth. Then, if Λ is killed by a power of ℓ,*

$$Rf^!(\Lambda) \otimes^{\mathbf{L}}_{\Lambda} f^*(-) \xrightarrow{\sim} Rf^!(-)$$

as functors from $D_{\text{ét}}(\mathfrak{Y}, \Lambda)$ to $D_{\text{ét}}(\mathfrak{X}, \Lambda)$, and $Rf^!\Lambda$ is invertible in $D_{\text{ét}}(\mathfrak{X}, \Lambda)$. Moreover, the formation of $Rf^!(\Lambda)$ is compatible with base change.

$\to$ now the function "dimension of f" is a locally constant function $|\mathfrak{X}| \to \frac{1}{2}\mathbb{Z}$.

3.2. Examples.

3.2.1. *First easy examples.*

EXAMPLE 3.3. *(1) Any loc. separated étale morphism of loc. spatial diamonds is ℓ-coho. smooth*
(2) Any perfectoid ball $\mathbb{B}^d \to *$ is ℓ-coho. smooth
(3) $f : X \to Y$ smooth morphism of (K, K^+)-adic spaces loc. of finite type implies $f^\diamond : X^\diamond \to Y^\diamond$ is ℓ-coho. smooth (Huber)
(4) k discrete field, $\mathrm{Spd}(k(T)) \to \mathrm{Spd}(k)$ is ℓ-coho. sm. since representable in annuli $\mathrm{Spa}(R\langle T^{\pm p^{-\infty}}\rangle, R^+\langle T^{\pm p^{-\infty}}\rangle) \to \mathrm{Spa}(R, R^+)$.

3.2.2. *Open B_{dR}-Schubert cells.*

EXAMPLE 3.4. *G split reductive gp. over E. For a dominant coweight μ let*

$$\mathrm{Gr}^{\mu}_{B_{dR}} \longrightarrow \mathrm{Spa}(E)^\diamond$$

be the open Schubert cell of the B_{dR}-affine Grassmanian. Then this is an ℓ-coho. smooth morphism.

$\to$ use Bialynicki-Birula morphism

$$\mathrm{Gr}^{\mu}_{B_{dR}} \underbrace{\longrightarrow}_{\substack{\text{iterated loc. trivial}\\ \text{fibration in } \mathbb{A}^{1,\diamond}}} (\underbrace{G/P_\mu}_{\substack{\text{flag variety}\\ \text{smooth}/E}})^\diamond \longrightarrow \mathrm{Spa}(E)^\diamond$$

3.2.3. *Quotient by a pro-p group.*

Here is a new example where we leave the "usual world" of rigid spaces even further.

PROPOSITION 3.5. *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ representable in loc. spatial diamonds. $K =$ pro-p group such that $\underline{K}$ acts on $\mathfrak{X}$ over $\mathfrak{Y}$ and $\underline{K} \times \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is qc. 0-truncated. Then,*

$$\mathfrak{X} \to \mathfrak{Y} \ \ell\text{-coho. smooth} \Longrightarrow \mathfrak{X}/\underline{K} \to \mathfrak{Y} \ \ell\text{-coho. smooth.}$$

One has to be careful that $\mathfrak{X} \to \mathfrak{X}/\underline{K}$ is not ℓ-coho. smooth in general (unless K is finite in which case this is finite étale)

Let P be a profinite set. Then $\underline{P} \longrightarrow * = \mathrm{Spa}(\overline{\mathbb{F}}_p)$ is not ℓ-coho. smooth unless P is finite.

$\to$ for S a perfectoid space

$$D_{\text{ét}}(\underline{P} \times S, \Lambda) = D_{\text{ét}}(S, \underbrace{\mathscr{C}(P, \Lambda)}_{\substack{\text{loc. constant} \\ \text{functions}}})$$

and

$$D_{\text{ét}}(\underline{P}, \Lambda) = D(\mathscr{C}(P, \Lambda)).$$

Let $f : \underline{P} \longrightarrow *$.

- The functor $Rf_* : D(\mathscr{C}(P, \Lambda)) \to D(\Lambda)$ is the evident one given by the morphism of rings $\Lambda \to \mathscr{C}(P, \Lambda)$.
- The functor f^* is $- \otimes^{\mathbb{L}}_{\Lambda} \mathcal{C}(P, \Lambda)$.
- f is proper and thus $Rf_* = Rf_!$
- One has $Rf^!(-) = R\mathrm{Hom}_{\Lambda}(\mathscr{C}(P, \Lambda), -)$.

In particular, $Rf^!\Lambda = \mathscr{D}(P, \Lambda) =$ distributions on P with values in Λ as a $\mathscr{C}(P, \Lambda)$-module $\to$ projective of finite type iff P is finite.

EXAMPLE 3.6. $* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$. *Then* $\mathrm{Spa}(\breve{E})^\diamond \to *$ *is* ℓ*-coho. smooth and*

$$\underbrace{\mathrm{Div}^1}_{\substack{\textit{moduli of deg. 1} \\ \textit{eff. divisors} \\ \textit{on the curve}}} = \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}} \longrightarrow *$$

is proper ℓ*-coho. smooth.*

$\to$ $\breve{E}_\infty$ completion of the extension generated by torsion points of a LT group = perfectoid field with $\breve{E}^\flat_\infty = \overline{\mathbb{F}}_q((T^{1/p^\infty}))$. Then, $\mathrm{Spa}(E^\flat_\infty) \to *$ is ℓ-coho. smooth since representable in perfectoid open punctured disks. Thus,

$$\begin{tikzcd}
\mathrm{Spa}(\breve{E}^\flat_\infty) \arrow[r, "\substack{\text{not } \ell\text{-coho.} \\ \text{sm.}}"] \arrow[rrrr, bend right, "\ell\text{-coho. sm.}"'] & \mathrm{Spa}(\breve{E})^\diamond \arrow[r, equal] \arrow[rrr, bend left, "\ell\text{-coho. sm.}"] & \mathrm{Spa}(E^\flat_\infty)/\mathcal{O}_E^\times \arrow[r, "\text{finite étale}"] & \mathrm{Spa}(\breve{E}^\flat_\infty)/1 + \pi\mathcal{O}_E \arrow[r, "\substack{\ell\text{-coho. sm.} \\ \text{by Prop. 3.5}}"] & * \\
& & & &
\end{tikzcd}$$

$\to$ as an application finds back Tate-Nakayama duality for finite $\mathrm{Gal}(\overline{E}|E)$-modules killed by a power of ℓ.

EXAMPLE 3.7 (Ivanov, Weinstein). *Let* $X \simeq \mathring{\mathbb{B}}^{d-1}_{\breve{E}}$ *be the generic fiber of Lubin-Tate space associated to* GL_d *and* $(X_K)_{K \subset GL_d(\mathcal{O}_E)} \to X$ *be Lubin-Tate tower. Let* $X_\infty = \varprojlim_K X_K$*, a perfectoid space over* $\mathrm{Spa}(\breve{E}_\infty)$*. Then, for each* K*,* $X_K^\diamond \to \mathrm{Spa}(\breve{E})^\diamond$ *is* ℓ*-coho. smooth as the*

diamond of a smooth morphism of rigid spaces. But going to the limit, $X_\infty^\flat \to \mathrm{Spa}(\breve{E}_\infty^\flat)$ is not ℓ-coho. smooth. In fact, as an application of the Jacobian criterin of smoothness Ivanov and Weinstein prove that the (partially proper) open subset $U \subset X_\infty$ where there is no complex multiplication is such that $U^\flat \to \mathrm{Spa}(\breve{E}_\infty^\flat)$ is ℓ-coho. sm..

3.2.4. *BC spaces.*

$\to$ linear case of the Jacobian criterion of smoothness

THEOREM 3.8 (Linear case of the Jacobian criterion).

(1) $\mathscr{E}$ a v.b. on X_S such that $\forall s \in S$, $\mathscr{E}_{|X_{K(s),K(s)^+}}$ has > 0 H.N. slopes. Then

$$BC(\mathscr{E}) \longrightarrow S$$

is ℓ-coho. smooth of dim. $\deg(\mathscr{E})$

(2) $\mathscr{E}$ a v.b. on X_S such that $\forall s \in S$, $\mathscr{E}_{|X_{K(s),K(s)^+}}$ has < 0 H.N. slopes. Then

$$BC(\mathscr{E}[1]) \longrightarrow S$$

is ℓ-coho. smooth of dim. $-\deg(\mathscr{E})$

$\to$ For point (1), we prove that up to replacing S by an étale cover, one can find an exact sequence $0 \to \mathscr{E}' \to \mathscr{E}'' \to \mathscr{E} \to 0$ where $\mathscr{E}''$ is $\simeq$ to $\oplus_i \mathcal{O}(\lambda_i)$ with $0 < \lambda_i \leq 1$ and $\mathscr{E}'$ is fiberwise s.s. of slope 0. $\Rightarrow$ after replacing S by a s.t.d. perf. space $BC(\mathscr{E})$ is a quotient of an open perfectoid ball by the action of $\underline{E^n}$ for some $n \geq 0$,

$$\begin{tikzcd}
\mathring{\mathbb{B}}_S^{1/p^\infty,d} \arrow[r] \arrow[dr, "\ell\text{-coho.sm.}"'] & \mathring{\mathbb{B}}_S^{1/p^\infty,d}/\underline{\mathcal{O}_E^n} \arrow[r, "\text{étale}"] \arrow[d, "\ell\text{-coho.sm.}", "\text{by Prop. 3.5}"'] & \mathring{\mathbb{B}}_S^{1/p^\infty,d}/\underline{E^n} \arrow[r, equal] & BC(\mathscr{E}) \arrow[dl, bend left, "\ell\text{-coho.sm.}"] \\
& S & &
\end{tikzcd}$$

3.3. Openness of smooth morphisms.

The following is quite important.

If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of small v-sheaves representable in locally spatial diamonds, separated ℓ-coho. smooth morphism then $\mathrm{Im}(f) \subset \mathfrak{Y}$ is represented by an open substack.

$\to$ We will give the proof later using constructible sheaves.

This result is quite important for the following reason.

Corollary 3.9. *Let $\mathfrak{X}$ be a small v-stack and consider a family $(U_i \to \mathfrak{X})_{i\in I}$ of morphisms toward $\mathfrak{X}$ where each U_i is a locally spatial diamond, $U_i \to \mathfrak{X}$ is representable in locally spatial diamonds separated ℓ-coho. smooth. Then, the family $(U_i \to \mathfrak{X})_{i\in I}$ is a v-cover if and only if and only if it is "in the naïve sense" that is to say $\coprod_{i\in I} |U_i| \to |\mathfrak{X}|$ is surjective !*

4. Smooth base change

Start with

$$f : \mathfrak{X} \longrightarrow \mathfrak{Y}$$

a 0-truncated morphism of small v-stacks and let $A \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$. As we said before, it is difficult to compute

$$Rf_* A \in D_{\text{ét}}(\mathfrak{Y}, \Lambda)$$

in general unless f is qc qs (quasi-compact base change). There is another case that is very useful and allows us to compute this in terms of "smooth charts".

Proposition 4.1 (Smooth base change). *Consider a cartesian diagram*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \mathfrak{Y}' \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

where

- *f is a 0-truncated morphism of small v-stacks,*
- *g is representable in locally spatial diamonds separated ℓ-coho. sm.*

Then, for any $A \in D_{\text{ét}}(X, \Lambda)$,

$$g^* Rf_* A \xrightarrow{\sim} Rf'_* g'^* A.$$

Proof. Use the "dual proper base change formula"

$$Rf'_* Rg'^! A \xrightarrow{\sim} Rg^! Rf_* A$$

coupled with g and g' separated ℓ-coho. smooth and $D_{g'} \xrightarrow{\sim} f'^* D_g$. □

In the same vein, we have the smooth base change of Hom' using the exceptional pull-back of Hom's

Proposition 4.2 (Smooth base change of Hom). *For $f : \mathfrak{X} \to \mathfrak{Y}$ a separated ℓ-coho. smooth of small v-stacks and $A, B \in D_{\text{ét}}(\mathfrak{Y}, \Lambda)$ one has*

$$f^* R\mathscr{H}om_\Lambda(A, B) \xrightarrow{\sim} R\mathscr{H}om_\Lambda(f^* A, f^* B).$$

5. Artin v-stacks

5.1. The example of classifying stacks.

Recall: $* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$.

PROPOSITION 5.1. *Let G be an affine algebraic group over E and $\mathfrak{X} = [*/\underline{G(E)}]$ seen as a small v-stack. Its diagonal is representable in loc. spatial diamonds and there exists $f : U \to \mathfrak{X}$ v-surjective with*

(1) U a locally spatial diamonds
(2) f v-surjective
(3) f separated ℓ-coho. smooth.
(4) $U \to *$ separated ℓ-coho. smooth.

$\to$ Take $U = G^{ad,\diamond}/\underline{K} \to [*/\underline{G(E)}]$ for $K \subset G(E)$ open pro-p:

$$\begin{tikzcd}
G^{ad,\diamond}/\underline{K} \arrow[r, "\substack{\ell\text{-coho. sm.} \\ \text{since } G^{ad}\to\mathrm{Spa}(E) \\ \text{sm.}\Rightarrow G^{ad,\diamond}\to\mathrm{Spa}(E)^\diamond \\ \ell\text{-coho. sm.}}"] \arrow[d, "\ell\text{-coho. sm.}", "\substack{\text{since } G^{ad,\diamond}\to * \\ \text{is } \ell\text{-coho. sm.} \\ +\text{Prop. 3.5}}"'] & [\mathrm{Spa}(E)/\underline{K}] \arrow[r, "\substack{\ell\text{-coho. sm. since} \\ \mathrm{Spa}(E)^\diamond\to * \text{ is} \\ \ell\text{-coho. sm.}}"] & [*/\underline{K}] \arrow[r, "\text{separated étale}"] & [\,*\,/\underline{G(E)}] \\
* & & &
\end{tikzcd}$$

5.2. Artin v-stacks.

5.2.1. *Definition.*

$\to$ Leads to the following definition.

DEFINITION 5.2. *An artin v-stack is a small v-stack $\mathfrak{X}$ s.t.*

(1) Its diagonal is representable in loc. spatial diamonds
(2) The exists a locally spatial diamond U and a separated surjective ℓ-coho. sm. morphism $U \to \mathfrak{X}$.

$\to$ if one can take $U \to *$ ℓ-coho. sm. we say that $\mathfrak{X}$ is ℓ-coho. smooth. If this is the case then this is true for any $U \to \mathfrak{X}$ that is ℓ-coho. sm.. One can then define its dualizing complex $D_{\mathfrak{X}}$ as an invertible object in $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ canonically.

THEOREM 5.3. *The small v-stack Bun_G of G-bundles on the curve is an Artin v-stack ℓ-coho. sm. with $D_{\mathrm{Bun}_G} \simeq \Lambda$.*

PROOF. Suppose G is split to simplify. Fix μ a dominant coweight. Beauville-Laszlo morphism

$$BL_\mu : \mathrm{Gr}^\mu_{B_{dR}} \longrightarrow \mathrm{Bun}_G \times \mathrm{Spa}(E)^\diamond$$

induces an ℓ-coho. smooth morphism

$$[\underline{G(E)} \backslash \mathrm{Gr}^\mu_{B_{dR}}] \longrightarrow \mathrm{Bun}_G \times \mathrm{Spa}(E)^\diamond.$$

□

5.2.2. *Cohomological operations on Artin v-stacks using smooth charts.*

For a 0-truncated morphism of Artin v-stacks

$$f : \mathfrak{X} \to \mathfrak{Y}$$

and $A \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$, Rf_*A is computable using the smooth base change theorem and smooth charts. More precisely, if V is a locally spatial diamond and $V \to \mathfrak{Y}$ is separated ℓ-coho. smooth then if

$$f_V : U := \mathfrak{X} \times_{\mathfrak{Y}} V \longrightarrow V$$

one has

$$(Rf_*A)_{|V} = R(f_V)_{\text{ét}*}(\underbrace{A_{|U}}_{\in \widehat{D}(U_{\text{ét}}, \Lambda)})$$

and

$$(f_V)_{\text{ét}} : U_{\text{ét}} \longrightarrow V_{\text{ét}}$$

is the morphism of small étale sites induced by f_V. One can even go further:

In the same vein, if $\mathfrak{X}$ is an Artin v-stack one can compute smooth locally $R\mathscr{H}om(A, B)$ for $A, B \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$. In fact, if $U \to \mathfrak{X}$ is separated ℓ-coho. smooth with U a locally spatial diamonds then

$$\underbrace{R\mathscr{H}om_\Lambda}_{\substack{\text{abstract } R\mathscr{H}om_\Lambda \\ \text{defined for any small} \\ v\text{-stack}}} (A, B)_{|U} = \underbrace{R\mathscr{H}om_\Lambda}_{\substack{\text{usual concrete} \\ R\mathscr{H}om_\Lambda \text{ in } D(U_{\text{ét}}, \Lambda)}} (A_{|U}, B_{|U})$$

CHAPTER 6

6th Lecture: June 23 2023

1. Constructible sheaves and complexes

1.1. Constructible sheaves on spectral spaces. $X =$ spectral space. $\Lambda =$ Noetherian

<u>Recall</u>:

$$\begin{array}{c} \text{constructible sets} \\ \| \\ \text{Boolean algebra generated by quasi-compact open subsets} \\ \| \\ \coprod_{\text{finite}} \underbrace{\text{locally closed constructible sets}}_{U \smallsetminus V \text{ with } U \text{ and } V \text{ open qc.}} \end{array}$$

$\to X_{cons} =$ topology generated by constructible subsets $=$ topology whose closed subsets are the pro-constructible subsets.

$=$ compact totally disconnected space i.e. profinite space whose closed/open subsets are exactly the constructible subsets of X.

Definition 1.1. *A sheaf of Λ-module $\mathscr{F}$ on the spectral space X is constructible if there exists a finite partition of X, $X = \bigcup_i Z_i$, in locally closed constructible subsets such that for all i, $\mathscr{F}_{|Z_i}$ is a constant sheaf with value a Λ-module of finite type.*

$\to$ sub-abelian category of the category of sheaves of Λ-modules on X

$\to \mathscr{F}$ is constructible iff it is a successive extension of $j_!\underline{M}$ where $j : Z \hookrightarrow X$ with Z loc. closed constructible and M of finite type. Thus, $\mathrm{Shv}_\Lambda(X)_{cons} =$ thick sub-category generated by the $j_!\underline{M}$ as before.

$\to \mathscr{F}$ is constructible iff its pullback to X_{cons} is locally constant, locally isomorphism to a finite type Λ-module

Remark 1.2. *If X is a spectral space and $Z \subset X$ constructible then Z is a nbd. of any maximal point of X lying in Z. In fact, if x is such a point, $\{x\} = X_x$. And thus $\cap_{U \ni x}(U \cap X \smallsetminus Z) = \emptyset$ where U is a qc open nbd. of x. Since $U \cap X \smallsetminus Z$ is constructible, the compacity of X_{cons} then implies that a finite sub-intersection is empty.*

For example, if $X = |\mathrm{Spa}(A, A^+)|$ where A is topologically of finite type over a non-archimedean field K then any Tate classical point of X that is contained in Z has a neighborhood contained in Z. For example, if $Z \subset |\mathbb{B}^d_K|$ containing the origin 0 then $\mathbb{B}^d(0, \varepsilon) \subset Z$ for some $\varepsilon > 0$.

Proposition 1.3.

(1) *The constructible sheaves are exactly the compact objects of the category* $\mathrm{Shv}_\Lambda(X)$

(2) *Any sheaf of Λ-modules is a filtered colimit of constructible sheaves and*

$$\varinjlim : \underbrace{\mathrm{Ind}\left(\mathrm{Shv}_\Lambda(X)_{cons}\right)}_{\textit{Ind-category}} \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X).$$

One of the great properties of constructible sheaves is the following.

Proposition 1.4. *If $X = \varprojlim_i X_i$ with X_i spectral and the transition morphisms are qc qs then*

$$2 - \varinjlim_i \mathrm{Shv}_\Lambda(X_i)_{cons} \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X)_{cons}.$$

At the end for any X spectral one can write $X = \varprojlim_i X_i$ with X_i a finite (T0) space. Then,

$$\mathrm{Ind}\left(2 - \varinjlim_i \underbrace{\mathrm{Shv}_\Lambda(X_i)}_{\text{sheaves on a finite ordered set}}\right) \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X)$$

$\to$ combinatorial description of sheaves of Λ-modules on X.

1.2. Background on overconvergent étale sheaves. X=spatial diamond.

Definition 1.5. *$\mathscr{F}$ an étale sheaf on X is overconvergent if $\forall \bar{x}$ a geometric point of X and $\forall \bar{y}$ a generalization of $\bar{x}$,*

$$\mathscr{F}_{\bar{x}} \xrightarrow{\sim} \mathscr{F}_{\bar{y}}.$$

This is equivalent to saying that

$$\forall \bar{x} : \mathrm{Spa}(C, C^+) \longrightarrow X,$$

the sheaf

$$x^* \mathscr{F}$$

on $|\mathrm{Spa}(C, C^+)|$ is constant.

Recall that if S is a spectral space such that for all $s \in S$, S_s is a chain then S has a biggest Hausdorff quotient

$$\boxed{\underbrace{S^B}_{\substack{\text{Berkovich quotient}\\ \text{=compact}\\ \text{Hausdorff}}} = S/\sim}$$

where $\sim$ is the equivalence relation generated by the specialization order. Equivalently,

$$s \sim t \Longleftrightarrow s^{max} = t^{max}$$

where s^{max} is the maximal generalization of s (if S is the top. space of an analytic adic space and $S_s = |\mathrm{Spa}(K(s), K(s)^+)|$ then s^{max} is the maximal point of $\mathrm{Spa}(K(s), K(s)^+)$ given by the rank 1 valuation $\mathrm{Spa}(K(s), \mathcal{O}_{K(s)})$. The quotient map

$$\beta : S \longrightarrow S^B$$

induces

$$\begin{tikzcd}
\{\text{sheaves on } S^B\} \arrow[r, "\beta^*", "\sim"'] & \{\text{overconvergent sheaves on } S\} \arrow[d, equal] \\
& \left\{\text{sheaves } \mathscr{F} \text{ on } S \text{ s.t. } \forall U \underbrace{\subset}_{\substack{\text{q.c.}\\ \text{open}}} X, \varinjlim_{\substack{V \subset X \text{ open}\\ \overline{U} \subset V}} \mathscr{F}(V) \xrightarrow{\sim} \mathscr{F}(U)\right\}
\end{tikzcd}$$

Example 1.6. *If $S = |\mathrm{Spa}(A, A^+)|$ with (A, A^+) an affinoid Tate ring. Then a sheaf $\mathscr{F}$ on S is overconvergent iff $\forall f_1, \ldots, f_n \in A$ that generate the unit ideal, $\forall g \in A$,*

$$\boxed{\varinjlim_{k \geq 1} \Gamma\Big(\underbrace{S\Big\langle \frac{\varpi f_1^k, \ldots, \varpi f_n^k}{g^k} \Big\rangle}_{\substack{\textit{basis of nbd. of}\\ \overline{S\langle \frac{f_1, \ldots, f_n}{g} \rangle}^S\\ \textit{when } k \textit{ varies}}}, \mathscr{F} \Big) \xrightarrow{\sim} \Gamma\Big(\underbrace{S\Big\langle \frac{f_1, \ldots, f_n}{g} \Big\rangle}_{\textit{rational open subset}}, \mathscr{F} \Big)}$$

If

$$\underbrace{S}_{\substack{\text{s.t.d.}\\ \text{perf. space}}} \xrightarrow[\text{surjective}]{\text{quasi-pro-étale}} X$$

then the overconvergent étale sheaves on X are identified with the sheaves $\mathscr{F}$ on $X_{\text{ét}}$ such that $\mathscr{F}_{|S}$ comes from a sheaf on $|S|^B$.

There is a partially proper étale site together with a continuous morphism of sites

$$\pi : X_{\text{ét}} \longrightarrow X_{\text{p.p.ét}}$$

that induces an equivalence of topoi

$$\widetilde{X}_{\text{p.p.ét}} \xrightarrow{\sim} \{\text{overconvergent sheaves on } X_{\text{ét}}\}$$

Thus, we have the following equivalent description of overconvergent étale sheaves.

Étale sheaves $\mathscr{F}$ on X satisfying $\mathscr{F}_{\bar{x}} \xrightarrow{\sim} \mathscr{F}_{\bar{y}}$ if $\bar{x} \leq \bar{y}$
Étale sheaves $\mathscr{F}$ on X satisfying: $\mathscr{F}_{\mid S}$ comes from a sheaf on the Berkovich spectrum $\lvert\mathcal{M}(R)\rvert = S^B$ if $\mathrm{Spa}(R, R^+) = S \longrightarrow X$ is a quasi-pro-étale cover with S a strictly totally disconnected perf. spaces
Sheaves on the partially proper étale site of X

1.3. Étale constructible sheaves on spatial diamonds.

$X =$ spatial diamond. Λ Noetherian.

Definition 1.7. *$\mathscr{F}$ a sheaf of Λ-modules on $X_{\text{ét}}$ is constructible if $\forall S \to X$ with S a s.t.d. perf. space, $\mathscr{F}_{\mid S}$ is constructible as a sheaf on $\lvert S\rvert$.*

Proposition 1.8.

(1) $\mathscr{F}$ constructible $\Leftrightarrow$ $\exists$ finite partition $\lvert X\rvert = \bigcup_i Z_i$ with Z_i loc. closed constructible and $\forall f : S \to X$ with S s.t.d. perf. space, $\forall i$, $\mathscr{F}_{\mid f^{-1}(Z_i)}$ is isomorphic to $\underline{M}$ with M a finite type Λ-module

(2) $\mathscr{F}$ constructible $\Leftrightarrow$ $\mathscr{F}$ is a compact object of $\mathrm{Shv}_\Lambda(X_{\text{ét}})$

(3) $\mathrm{Shv}_\Lambda(X_{\text{ét}})$ is compactly generated and

$$\varinjlim : \mathrm{Ind}\left(\mathrm{Shv}_\Lambda(X_{\text{ét}})_{cons}\right) \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X_{\text{ét}})$$

(4) If $X = \varprojlim_i X_i$, a cofiltered limit with qc transition morphisms of spatial diamonds,

$$2 - \varinjlim_i \mathrm{Shv}_\Lambda(X_{i,\text{ét}})_{cons} \xrightarrow{\sim} \mathrm{Shv}_\Lambda(X_{\text{ét}})_{cons}$$

EXAMPLE 1.9. *Consider $X = \mathbb{B}^{1,\diamond}_K$ the 1-dim. closed ball over the non-archi. field K and $j : (\mathbb{B}^1_K \smallsetminus \{0\})^\diamond \hookrightarrow \mathbb{B}^{1,\diamond}_K$ the inclusion of the punctured ball. For a radius $\rho \in]0,1[\cap|K|$ let $j_\rho : \{\rho \leq |z| \leq 1\}^\diamond \hookrightarrow \mathbb{B}^{1,\diamond}_K$ be the inclusion of the qc annulus with radii $\{\rho, 1\}$. Then*

$$j_!\Lambda = \varinjlim_{\substack{\rho \to 0 \\ >}} j_{\rho!}\Lambda$$

is a writing of $j_!\Lambda$ as an ind-constructible étale sheaf on $\mathbb{B}^{1,\diamond}_K$.

Here is a key remark/application.

LEMMA 1.10 (loc. systems=overconvergent constructible sheaves). *Let $\mathscr{F} \in \mathrm{Shv}_\Lambda(X_{\text{ét}})$. The following are equivalent:*

(1) $\mathscr{F}$ is constructible and overconvergent
(2) $\mathscr{F}$ is étale locally isomorphic to $\underline{M}$ with M a finitely generated Λ-module

PROOF. For $x \in X$ one has

$$\mathrm{Spa}(K(x), K(x)^+) = \varprojlim_{\substack{U \ni x \\ \text{open aff. perf.} \\ \text{nbd. of } x}} U$$

and thus

$$2 - \varinjlim_{\substack{U \ni x \\ \text{open aff. perf.} \\ \text{nbd. of } x}} \mathrm{Shv}_\Lambda(U_{\text{ét}})_{cons} \xrightarrow{\sim} \mathrm{Shv}_\Lambda(\mathrm{Spa}(K(x), K(x)^+)_{\text{ét}})_{cons}.$$

Now, $\mathscr{F}$ overconvergent exactly means that for all $x \in X$, if $x : \mathrm{Spa}(K(x), K(x)^+) \to X$, then $x^*\mathscr{F}$ is étale locally constant. □

Here is another example.

EXAMPLE 1.11. *Let $\mathscr{F}$ be constructible on $X_{\text{ét}}$. Then, $\mathscr{F}$ is locally constant in any nbd. of a maximal point of $|X|$. For example, if $X = Y^\diamond$ where Y is a qc qs rigid analytic analytic space then any étale constructible sheaf on Y is locally constant on an open subset U of Y containing all maximal points and in particular all classical Tate points of Y.*

→ This is where Berkovich theory breaks down: does not see the difference between constructible sheaves and local systems

1.4. Étale perfect-constructible complexes on spatial diamonds. X =spatial diamond. Any Λ.

Definition 1.12. *An object $A \in D_{\text{ét}}(X, \Lambda)$ is perfect constructible if $\forall S \to X$ with S a s.t.d. perf. space there exists a finite partition $|S| = \bigcup_i Z_i$ in loc. closed cons. subsets such that for all i, $(\underbrace{A_{|S}}_{\in D(|S|,\Lambda)})_{|Z_i}$ is constant with value a perfect complex of Λ-modules via $D(\Lambda) \to D(Z_i, \Lambda)$.*

One can characterize them in sheafy/fiberwise terms and descend the stratification along which they are loc. cst..

Proposition 1.13.

(1) *The object $A \in D_{\text{ét}}(X, \Lambda)$ is perfect constructible iff*
   (a) *it is bounded,*
   (b) *$\forall i \in \mathbb{Z}$, $\mathcal{H}^i(A)$ is a constructible étale sheaf on X,*
   (c) *$\forall x : \mathrm{Spa}(C, C^+) \to X$, $x^* A \in D(\Lambda)$ is a perfect complex.*

(2) *The object $A \in D_{\text{ét}}(X, \Lambda)$ is perfect constructible iff there exists a finite stratification $|X| = \bigcup_i Z_i$ by loc. closed constructible subsets s.t. $\forall f : S \to X$ with S s.t.d. perf. space s.t. $\forall i$, $(\underbrace{A_{|S}}_{\in D(|S|,\Lambda)})_{|f^{-1}(Z_i)}$ is constant with value a perfect complex of Λ-modules.*

Remark 1.14. *One has to be careful that for $Z \subset |X|$ locally closed constructible, Z does not define a "sub-spatial diamond" of X (unless it is open) and the expression "$A_{|Z}$"does not make any sense. The only sub-spaces of $|X|$ defining sub-spatial diamonds are the pro-constructible <u>generalizing</u> subsets.*

Let us come to the main point why we are interested in perfect constructible complexes and the way they will be related to our cohomological operations.

Proposition 1.15. *Let X be a spatial diamond satisfying: $\exists N \in \mathbb{N}$ s.t. $\forall U \to X$ separated étale qc, $\forall \mathscr{F} \in \mathrm{Shv}_\Lambda(X_{\text{ét}})$, $H^i_{\text{ét}}(U, \mathscr{F}) = 0$ for $i > N$.*

Then,

$$D_{\text{ét}}(X, \Lambda) = D(X_{\text{ét}}, \Lambda)$$

i.e. $D(X_{\text{ét}}, \Lambda)$ is left complete and $D_{\text{ét}}(X, \Lambda)$ is compactly generated with compact objects the perfect constructible étale complexes.

$\to$ the finite coh. dimension hypothesis is satisfied if for example there exists $f : X \to S$ qc. of finite dim. trg. with S a s.t.d. perf. space.

Finally this has a description as successive extensions of some "simple" perfect constructible objects.

PROPOSITION 1.16. *For X a spatial diamond, the triangulated category*

$$D_{\text{ét}}(X, \Lambda)_{p.c.}$$

is the thick triangulated sub-category generated by the objects

$$j_! \mathscr{L}$$

where j is the inclusion of a locally closed constructible subset Z of $|X|$ and $\mathscr{L}$ an object of $D_{\text{ét}}(U, \Lambda)$, U an open nbd. of Z, étale locally isomorphic to a perfect complex of Λ-modules.

$\to$ One has to be careful in the preceding statement: such a Z has no structure of a spatial diamond, unless it is open, and $j_! \mathscr{L}$ is a notation for $j'_! i_* i^* \mathscr{L}$ where $Z \subset U$, $j' : U \hookrightarrow X$, $i : Z \hookrightarrow X$ and $i_* i^* \mathscr{L}$ is a notation for an object sitting in an exact triangle

$$i_* i^* \mathscr{L} \to \mathscr{L} \to k_! k^* \mathscr{L} \xrightarrow{+1}$$

where $k : U \setminus Z \hookrightarrow U$.

CHAPTER 7

7th Lecture: June 30 2023

1. First applications of étale perfect constructible complexes

1.1. Toward f-ULA complexes.

PROPOSITION 1.1. *Let $f : X \to Y$ be a separated ℓ-coho. smooth morphism of spatial diamonds. Then, for $A \in D_{\text{ét}}(X, \Lambda)$ perfect constructible, $Rf_! A$ is perfect constructible.*

PROOF. Using the proper base change theorem we can base change and suppose that Y is a s.t.d. perf. space. It then suffices to prove that $Rf_! A$ is compact. But this follows from the fact that since f is ℓ-coho. sm. $Rf_!$ has a right adjoint that commutes with direct sums and the fact that A is compact and thus $\mathrm{Hom}(A, -)$ commutes with direct sums. $\square$

$\to$ Key lemma

LEMMA 1.2 (Neeman). *Let F be an additive functor between additive categories admitting arbitrary direct sums. If F as a right adjoint that commutes with arbitrary direct sums then F sends compact objects to compact objects.*

PROPOSITION 1.3. *Let $f : X \longrightarrow Y$ be a proper ℓ-coho. smooth morphism of spatial diamonds and $\mathscr{F} \in D_{\text{ét}}(X, \Lambda)$ be étale locally isomorphic to $\underline{M}$ where M is a perfect complex of Λ-modules. Then, $Rf_* \mathscr{F}$ is étale locally isomorphic to the constant complex associated to a perfect complex of Λ-modules.*

REMARK 1.4. *Proposition 1.3 remains true under the weaker assumption that $\mathscr{F}$ is perfect constructible and "overconvergent along f" in the sense that $\forall y, y' \in Y$ with $y \geq y'$, if $j : X_y \hookrightarrow X_{y'}$ then $\mathscr{F}_{|X_{y'}} \xrightarrow{\sim} j_*(\mathscr{F}_{|X_y})$. Using qc. base change this is equivalent to $\forall \bar{x}$, $\forall \bar{s} \geq f(\bar{x})$,*

$$\mathscr{F}_{\bar{x}} \xrightarrow{\sim} \Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} S_{\bar{s}}, \mathscr{F}).$$

1.2. Openness of cohomologically smooth morphisms.

COROLLARY 1.5. *Separated ℓ-coho. smooth morphisms are open.*

PROOF. Let $f : X \to S$ be a separated ℓ-coho. smooth morphism of spatial diamonds. Since the image of f is generalizing it suffices to prove it is constructible. since f is qc separated ℓ-coho. smooth,

$$A = Rf_! Rf^! \mathbb{F}_\ell$$

is perfect constructible. Let us prove that

$$\mathrm{Im}(|f|) = \{s \in S \mid A_{\bar{s}} \neq 0\}$$

where $\bar{s}$ is a geo. point over s. The formation of A is compatible with base change on S and thus we can suppose that $S = \mathrm{Spa}(C, C^+)$. Then, $A_{\bar{s}}$ is $R\Gamma(S, Rf_! Rf^! \mathbb{F}_\ell)$. Suppose that the closed point s of S is not in the image of f. Then, f factorizes as $j \circ g$ where $j : S \setminus \{s\} \hookrightarrow S$. Thus, $Rf_! = j_! Rg_!$. But since any non-empty closed subset of S contains s, $\Gamma(S, -) \circ j_! = 0$. We thus have $A_{\bar{s}} = 0$.

Suppose now that s is in the image of f. One computes, using the cohomological smoothness of f,

$$Rf_! Rf^! j_! \mathbb{F}_\ell = j_! j^* Rf_! Rf^! \mathbb{F}_\ell.$$

In particular, if one has a distinguished triangle

$$j_! j^* A \to A \to B \xrightarrow{+1}$$

then B is concentrated on $\{s\}$ with stalk $A_{\bar{s}}$ at s. But now,

$$\mathrm{Hom}(B, \mathbb{F}_\ell) = \mathrm{Hom}(Rf^! i_* \mathbb{F}_\ell, Rf^! i_* \mathbb{F}_\ell)$$

where $i_* \mathbb{F}_\ell$ is a notation for a cone of $j_! \mathbb{F}_\ell \to \mathbb{F}_\ell$. This is identified with endomorphisms of the étale local system $Rf^! \mathbb{F}_\ell$ restricted to $f^{-1}(s)$. Since $f^{-1}(s) \neq \emptyset$ the identity is such a non-zero endomorphism and $B \neq 0$. □

2. f-ULA complexes: the classical scheme case

Motivation: If S is a base scheme and $f : X \to S$ is a finite presentation morphism of schemes, the question is "What is a family of coherent sheaves parametrized by X?" The answer given by Grothendieck is: this is a coherent sheaf on X that is flat over S. Another question is "What is an étale complex of Λ-modules parametrized by X?" The answser is "this an f-ULA étale complex on X".

Here Λ is a Noehterian ring killed by a power of ℓ invertible on our schemes. All our schemes are qc qs.

If X is a scheme we note

$$D_{\text{ét}}(X, \Lambda)$$

for the left completion of $D(X_{\text{ét}}, \Lambda)$, that is equal to $D(X_{\text{ét}}, \Lambda)$ if for example X is of finite type over a field k satisfying $\mathrm{cd}_\ell(k) < +\infty$.

Recall the following definition. For a scheme S and a geometric point $\bar{s}$ of S we note $S_s = \mathrm{Spec}\big(\underbrace{\mathcal{O}_{S,\bar{s}}}_{\substack{\text{strict henselization}\\ \text{at } \bar{s}}}\big)$. If $\bar{s} : \mathrm{Spec}(K) \to S$ with K separably closed,

$$S_{\bar{s}} = \varprojlim_{\substack{\mathrm{Spec}(K) \xrightarrow{\bar{s}} S \\ \nearrow \\ U \\ \downarrow \text{étale}}} U$$

the étale localization of S at $\bar{s}$. This is the spectrum of the strict Henselization of S at $\bar{s}$.

Recall that a specialization $\bar{s'}$ of $\bar{s}$ is the datum of a geometric point $\bar{s'}$ of S together with a factorization of $\bar{s} : \mathrm{Spec}(K) \to S$ via $S_{\bar{s'}} \to S$. This is possible if and only if $s \geq s'$ as point of S. There is then induced a canonical pro-étale morphism

$$S_{\bar{s}} \longrightarrow S_{\bar{s'}}$$

For any étale sheaf $\mathscr{F}$ on S, its stalk at $\bar{s}$ is

$$\mathscr{F}_{\bar{s}} = \Gamma(S_{\bar{s}}, \mathscr{F}_{|S_{\bar{s}}}).$$

Since $S_{\bar{s}}$ is the spectrum of a strictly Henselian ring, $\Gamma(S_{\bar{s}}, -)$ is exact and for $A \in D_{\text{ét}}(X, \Lambda)$, $\Gamma(S_{\bar{s}}, A_{S_{\bar{s}}}) = R\Gamma(S_{\bar{s}}, A_{|S_{\bar{s}}})$ is the complex of stalks of A at $\bar{s}$, $A_{\bar{s}}$. There is a specialization map

$$\mathscr{F}_{\bar{s'}} \longrightarrow \mathscr{F}_{\bar{s}}.$$

Recall that a complex $A \in D^b_{\text{ét}}(X, \Lambda)$ with constructible cohomology is étale locally constant on X if and only if for all geometric point $\bar{x}$ of X together with a specialization $\bar{x'}$,

$$A_{\bar{x'}} \xrightarrow{\sim} A_{\bar{x}}$$

i.e. all specialization maps are isomorphisms.

We now use the following notation

$$\underbrace{D_{\text{ét}}(X, \Lambda)_{p.c.}}_{\substack{\text{perfect constructible}\\ \text{complexes}}}$$

is the subcategory of $D_{\text{ét}}(X, \Lambda)$ formed by complexes that are bounded with constructible cohomology sheaves and whose stalks at geometric points are perfect complexes of Λ-modules. When $D_{\text{ét}}(X, \Lambda) = D(X_{\text{ét}}, \Lambda)$ those are exactly the compact objects of $D_{\text{ét}}(X, \Lambda)$.

Definition 2.1 (Classical defintion of ULA complexes). *Let $f : X \to S$ be a finite presentation morphism of schemes.*

(1) *A complex $A \in D_{\text{ét}}(X, \Lambda)$ is f-locally acyclic if it is perfect constructible and $\forall \bar{x}$ a geometric point of X, $\forall \bar{s}$ a generalization of $f(\bar{x})$ there is an isomorphism*

$$A_{\bar{x}} = R\Gamma(X_{\bar{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{s}, A).$$

(2) *A is f-universally locally acyclic if it is locally acyclic after any base change $S' \to S$.*

Remark 2.2. *We wrote $X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{s}$ for the pullback via $\bar{s} : Spec(L) \to S_{f(\bar{x})}$ of $X_{\bar{x}} \to S_{f(\bar{x})}$. Thus, $R\Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{s}, A)$ is the étale cohomology of the L-scheme $X_{\bar{x}} \times_{S_{f(\bar{x})}} \bar{s}$ with coefficients in (the pullback of) A.*

Remark 2.3. *When S is a curve this is equivalent to $\forall \bar{s}$ a geometric closed point of S, if $X_{\bar{s}} = X \times_S S_{\bar{s}}$, for a choice $\bar{\eta}$ of a geometric point over the generic point of $S_{\bar{s}}$*

$$R\Phi_{\bar{\eta}}(A_{|X_{\bar{s}}}) = 0 \quad \text{in } D_{\text{ét}}(X \times_S Spec(k(\bar{s})))$$

where $R\Phi$ is the vanishing cycle functor

$$R\Phi_{\bar{\eta}} : D_{\text{ét}}(X_{\bar{s}}, \Lambda) \longrightarrow D_{\text{ét}}(X \times_S Spec(k(\bar{s})), \Lambda).$$

$\to$ locally acyclic complexes are a generalization of complexes without vanishing cycles when the base is a curve. Terminology locally acyclic="without any vanishing cycles"

In fact one can verify the following.

Proposition 2.4. *A perfect constructible is f-ULA if and only if for every morphism $g : Spec(V) \to S$ with V a Henselian rank 1 valuation ring and a choice of a separable closure of* $\mathrm{Frac}(V)$,

$$R\Phi_{\bar{\eta}}\big(A_{|X \times_S Spec(V)}\big) = 0.$$

Example 2.5. *A complex $A \in D^b_c(X_{\text{ét}}, \Lambda)$ is f-ULA for $f = \mathrm{Id}$ if and only if A is étale locally constant.*

$\to$ ULA complexes are some kind of relative notion for locally constant.

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r, "\bar{x}", "\text{closed géà. point}"'] & X \times_S \bar{s} \arrow[r, hook, "\bar{i}"] \arrow[d] & X \times_S \mathrm{Spec}(\overline{V}) \arrow[d] & X_{\bar{\eta}} \arrow[l, hook', "\bar{j}"'] \arrow[d] \\
& \bar{s} \arrow[r, hook] & \mathrm{Spec}(\overline{V}) & \bar{\eta} \arrow[l, hook']
\end{tikzcd}$$

$$\begin{aligned}
A_{\bar{x}} \xrightarrow{\sim} \left(\bar{i}^* R\bar{j}_*(A_{|X_{\bar{\eta}}})\right)_{\bar{x}} &= \underbrace{R\Gamma\Big(\big(X \times_S \mathrm{Spec}(\overline{V})\big)_{\bar{x}}[\tfrac{1}{t}], A\Big)}_{\substack{\text{cohomology of the schematical} \\ \text{Milnor fiber over } \bar{x}}} \\
&= R\Gamma\Big(\underbrace{sp^{-1}(\bar{x})}_{\substack{\text{rigid analytic} \\ \text{Milnor fiber} \\ \text{as a tube}}}, \big(A_{|X_{\hat{\bar{\eta}}}}\big)^{ad}\Big)
\end{aligned}$$

Recall now the following classical theorem. The first part implies the smooth base change theorem.

THEOREM 2.6. *Let $f : X \to S$ be a smooth morphism of schemes.*

(1) If A is an étale local system i.e. is étale locally isomorphic to $\underline{M}$ with M a free Λ-module of finite type then A is f-ULA.

(2) Being ULA is smooth local on the source and on the target.

REMARK 2.7. *The fact that being ULA is smooth local implies ther is a "good notion" of f-ULA complexes in $D_{\text{ét}}(\mathfrak{X}, \Lambda)$ where $f : \mathfrak{X} \to \mathfrak{Y}$ is a finite type morphism of Artin stacks.*

Let us note the following now. Proper base change implies that if we have a sequence

$$X \xrightarrow{g} Y \xrightarrow{f} S$$

and $A \in D_{\text{ét}}(X, \Lambda)$ that is $f \circ g$-ULA then g proper $\Rightarrow Rg_*A$ is f-ULA.

EXAMPLE 2.8. *If A is f-ULA and f is proper then Rf_*A is étale locally constant, étale locally isomorphic to $\underline{M^\bullet}$ where $M^\bullet$ is a perfect complex of Λ-modules.*

THEOREM 2.9 (Gabber). *LA implies ULA for finite type morphisms of Noetherian schemes.*
In particular, if $A \in D_{\text{ét,pc}}(X, \Lambda)$ where X is a finite type k-schemes, k a field, A is ULA with respect to $X \to Spec(k)$ and thus for any k-scheme S, $A_{|X \times_{Spec(k)} S}$ is locally acyclic relatively to $X \times_{Spec(k)} S \to S$.

Gaitsgory realized that ULA complexes behave well wrt to Verdier duality.

THEOREM 2.10 (ULA complexes behave well with respect to Verdier duality). *Let A be f-ULA where $f : X \to S$. Then,*

(1) The formation of the Verdier dual $\mathbb{D}_{X/S}(A)$ behaves well with respect to base change: for $S' \to S$,

$$\mathbb{D}_{X/S}(A)_{|X \times_S S'} = \mathbb{D}_{X \times_S S'/S'}(A_{|X \times_S S'})$$

and $A_{|X \times_S S'}$ is ULA relatively to $X \times_S S' \to S'$

(2) For $B \in D^b_c(X_{\text{ét}}, \Lambda)$ one has

$$\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} f^* B \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(f^* A, Rf^! B)$$

.

→ this has lead at the end to a new "Verdier duality point of view" on ULA complexes

THEOREM 2.11 (Verdier duality characterization of ULA complexes, Lu-Zheng). *For $A \in D_{\text{ét}}(X, \Lambda)$ one has if p_1 and p_2 are the two projections from $X \times_S X$ to X*

$$\begin{aligned} A \text{ is } f\text{-ULA} \iff & \left[\ \mathbb{D}_{X/S}(A) \boxtimes^{\mathbb{L}}_{\Lambda} A \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(p_1^* A, Rp_2^! A)\ \right] \\ \iff & X \xrightarrow{A} S \text{ is a left adjoint in the 2-cat.of correspondences} \end{aligned}$$

→ second condition to be explained later
→ "Compact" definition, does not involve something like "for any $S' \to S$..." or "$\forall \text{Spec}(V) \to S$", just one thing to test.
→ Let us cite as an example an immediate corollary of this point of view that is difficult to obtain without it.

COROLLARY 2.12. *If A is f-ULA then it is dualizable, $A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$.*

The following 3 definitions are then equivalent:

Classical definition (Grothendieck): Complexes without vanishing cycles	$\forall\ \mathrm{Spec}(\underbrace{V}_{\substack{\text{rank one}\\ \text{valuation ring}}}) \to S,$ $R\Phi_{\bar\eta}\big(A_{\mid X\times_S \mathrm{Spec}(V)}\big) = 0$
Modern definition (Gaitsgory): Complexes that behave nicely with respect to Verdier duality	$\mathbb{D}_{X/S}(A)_{\mid X\times_S S'} = \mathbb{D}_{X\times_S S'/S'}(A_{\mid X\times_S S'})$ $\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} f^*B \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(f^*A, Rf^!B)$ universally $/S$
Ultra modern 2-categorical super chic definition (Lu-Zheng)	The 1-morphism $X \xrightarrow{A} S$ is a left adjoint in the 2-category of correspondences

Let us cite another example of application of this ULA formalism.

COROLLARY 2.13. *Let $f : X \to S$ and $g : Y \to S$ be two morphisms of finite presentation and $A \in D_{\text{ét}}(X, \Lambda), B \in D_{\text{ét}}(Y, \Lambda)$ be ULA over S. Then,*

(1) $A \boxtimes^{\mathbb{L}}_{\Lambda} B$ is ULA over S and there is an isomorphism

$$\mathbb{D}_{X/S}(A) \boxtimes^{\mathbb{L}}_{\Lambda} \mathbb{D}_{Y/S}(B) \xrightarrow{\sim} \mathbb{D}_{X\times_S Y/S}(A \boxtimes^{\mathbb{L}}_{\Lambda} B).$$

(2) One has the Künneth formula: if $h : X \times_S Y \to S$,

$$Rf_*A \otimes^{\mathbb{L}}_{\Lambda} Rg_*B \xrightarrow{\sim} Rh_*\big(A \boxtimes^{\mathbb{L}}_{\Lambda} B\big).$$

Point (2) is immediately deduced from point (1) by applying Künneth formula with compact support (that is a formal consequence of the projection formula and is always true without any ULA hypothesis) that implies

$$Rf_!\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} Rg_!\mathbb{D}_{Y/S}(B) \xrightarrow{\sim} Rh_!\big(\mathbb{D}_{X/S}(A) \boxtimes^{\mathbb{L}}_{\Lambda} \mathbb{D}_{Y/S}(B)\big)$$

and an application of $R\mathscr{H}om(-, \Lambda)$ + Point (1) + the biduality of ULA complexes.

CHAPTER 8

8th Lecture: July 4 2023

1. ULA complexes on locally spatial diamonds: the classical definition

1.1. First step of finding the classical definition: mimicking the classical definition. X a locally spatial diamond.

For $\bar{x} : \mathrm{Spa}(C, C^+) \to X$ a geometric point we define

$$X_{\bar{x}}$$

as for a scheme,

$$X_{\bar{x}} = \varprojlim_{\substack{\mathrm{Spa}(C,C^+) \xrightarrow{\bar{x}} S \\ \nearrow \\ U \\ \downarrow \text{separated étale} \\ S}} U$$

where U is a spatial diamond. If $Y \to X$ is quasi-pro-étale where Y is a perfectoid space then we can lift $\bar{x}$ to a geometric point $\bar{y} : \mathrm{Spa}(C, C^+) \to Y$. Then, $Y_{\bar{y}} \xrightarrow{\sim} X_{\bar{x}}$ where $Y_{\bar{y}} = \mathrm{Spa}\left(\widehat{\overline{K(y)}}, \widehat{\overline{K(y)}}^+\right)$ with $y \in |Y|$ the image of the closed point of $\mathrm{Spa}(C, C^+)$ and $\overline{K(y)}$ the algebraic closure of $K(y)$ inside C.

Suppose

$$f : X \to S$$

is a morphism of locally spatial diamonds and

$$A \in D_{\text{ét}}(X, \Lambda).$$

For such a $\bar{x}$ let $\bar{s}$ be a generalization of $f(\bar{x})$ i.e. $\bar{s}$ is a geometric point of $S_{f(\bar{s})}$. There is a diagram of strictly local perfectoid spaces

$$\begin{tikzcd}
 & & X_{\bar{x}} \arrow[d] \\
\mathrm{Spa}(C, C^+) \arrow[r] \arrow[rr, bend right, "\bar{s}"'] & S_{\bar{s}} \arrow[r, hook, "\substack{\text{qc open} \\ \text{subset}}"] & S_{f(\bar{x})}
\end{tikzcd}$$

Since Λ is killed by a power of $\ell \neq p$, qc base chnage applies and there are maps

$$A_{\bar{x}} \longrightarrow R\Gamma(\underbrace{X_{\bar{x}} \times_{S_{f(\bar{x})}} S_{\bar{s}}}_{\substack{\text{qc open} \\ \text{subset of } X_{\bar{x}} \\ \text{thus s.t.d.}}}, A) = \Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} S_{\bar{s}}, A) \xrightarrow{\sim} R\Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} \mathrm{Spa}(C, C^+), A)$$

$\rightarrow$ the classical definition in terms of the cohomology of Milnor fibers translates here in an overconvergence like statement:

$$\boxed{A_{\bar{x}} \xrightarrow{\sim} \Gamma(X_{\bar{x}} \times_{S_{f(\bar{x})}} S_{\bar{s}}, A)}$$

$\rightarrow$ this involves no "higher degree cohomology classes" since, contrary to schemes, any qc open subset of a strictly local perf. spaces is strictly local

$\rightarrow$ if R is a strictly Henselian local ring then, $\mathscr{F}$ an étale sheaf of abelian groups on $\mathrm{Spec}(R)$, in general, there exists qc open subsets U of $\mathrm{Spec}(R)$ such that $H^i_{\text{ét}}(U, \mathscr{F}) \neq 0$ for some $i > 0$.

$\rightarrow$ the set of open subsets of $X_{\bar{x}}$ is totally ordered $\Rightarrow$ a sheaf on $X_{\bar{x}}$ is the same as a contravariant functor

$$\mathscr{F} : \{\text{qc non-empty open subsets of } X_{\bar{x}}\} \rightarrow \mathrm{Sets}$$

Then the preceding condition is that if $U \subset V \subset X_{\bar{x}}$ are pullback of qc non-empty open subsets of $S_{f(\bar{x})}$ then

$$\mathscr{F}(V) \xrightarrow{\sim} \mathscr{F}(U)$$

that is thus identified with $\mathscr{F}(X_{\bar{x}})$.

We now have the following Lemma.

Lemma 1.1. *A is overconvergent along f after any base change/S iff A is overconvergent.*

Proof. It suffices to take the base change $X_{\bar{x}} \rightarrow S$. $\square$

1.2. Second step of finding the classical definition: putting Verdier duality in the machine.

$\rightarrow$ contrary to schemes the preceding condition is not enough to define ULA: involves only degree 0 cohomology classes

1.2.1. *A first remark.* Before going further let us remark the following: let

$$f : X \rightarrow S$$

be as before and

$$A \in D_{\text{ét}}(X, \Lambda)$$

satisfy the "overconvergence along f" condition of the preceding section. Let $j : U \rightarrow X$ be a separated étale morphism of loc. spatial diamonds such that $f \circ j : U \rightarrow X$ is qc qs. Moreover, $A_{|U}$ is "overconvergent along $f \circ j$".

Lemma 1.2. *For such $j : U \rightarrow X$, $R(f \circ j)_*(A_{|U}) \in D_{\text{ét}}(S, \Lambda)$ is overconvergent.*

PROOF. We can use the qc base change Theorem to compute $\big(R(f\circ j)_*(A_{|U})\big)_{\bar{s}}$. □

We can now come to the main point.

LEMMA 1.3. *Suppose that after any base change over S to a strictly totally disconnected perfectoid space, for any $B\in D_{\text{ét}}(S,\Lambda)$ one has*

$$\mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda} f^*B \xrightarrow{\sim} R\mathscr{H}om(A, Rf^!B).$$

Then for any $j:U\to X$ separated étale such that $f\circ j$ is qc,

$$R(f\circ j)_!(A_{|U})\in D_{\text{ét}}(S,\Lambda)$$

is perfect constructible.

PROOF. Using the proper base change Theorem we can suppose that S is a s.t.d. perf. space. Replacing f by $f\circ j$ and A by $A_{|U}$ we can suppose that $j=\mathrm{Id}$ and f is qc and thus X is qc. We then have for any B,

$$\mathrm{Hom}(Rf_!A,B)=H^0(X,\mathbb{D}_{X/S}(A)\otimes^{\mathbb{L}}_{\Lambda} f^*B).$$

Since X is qc we deduce that $Rf_!A$ is a compact object of $D_{\text{ét}}(S,\Lambda)$ and thus perfect constructible. □

Let us now remark that this "perfect constructible" property is compatible with the following Lemma that will play an important role later.

LEMMA 1.4 (Key ULA Lemma). *Let X be a spatial diamond and $A\in D_{\text{ét}}(X,\Lambda)$ perfect constructible. Then $R\mathscr{H}om_\Lambda(A,\Lambda)$ is overconvergent and its formation is compatible with base change: for $f:X'\to X$ a morphism of spatial diamonds,*

$$f^*R\mathscr{H}om_\Lambda(A,\Lambda)\xrightarrow{\sim} R\mathscr{H}om_\Lambda(f^*A,\Lambda).$$

PROOF. The complex A has a finite filtration whose graded pieces are of the form

$$j_!\mathscr{L}$$

where $j:Z\hookrightarrow X$ is locally closed constructible and $\mathscr{L}$, defined in a nbd. of Z, is étale locally constant associated to a perfect complex of Λ-modules. Then,

$$R\mathscr{H}om(j_!\mathscr{L},\Lambda)=Rj_*\mathscr{L}^\vee$$

where $\mathscr{L}^\vee$ is étale locally constant associated to a perfect complex of Λ-modules. The result is then a consequence of the qc base change Theorem that says this is $j_*\mathscr{L}^\vee$. □

1.3. The classical definition of ULA.

$f : X \longrightarrow S$ is a morphism of locally spatial diamonds, compactifiable locally of finite dim.trg..

Definition 1.5 (Classical definition of ULA). *A complex $A \in D_{\text{ét}}(X, \Lambda)$ is ULA if*

(1) It is overconvergent

(2) after any base change over S, i.e. up to replacing S by S', X by $X \times_S S'$ and A by $A_{|X \times_S S'}$ for $S' \to S$ a morphism of loc. spatial diamonds, for any $j : U \to X$ a separated étale morphism of loc. spatial diamonds such that $f \circ j$ is qc,

$$R(f \circ j)_!(A_{|U}) \in D_{\text{ét}}(S, \Lambda)_{p.c.}$$

in the sense that is is perfect constructible when restricted to any spatial open subsets.

→ contrary to the scheme case we don't ask that A itself is p.c. and as a matter of fact, in the cases we will consider this will almost never be the case. In fact since we ask that A is overconvergent this would imply that A is étale locally constant and we don't want to put such a restriction.

→ there is a definition of LA complexes where we ask that A is only o.c. along f and property (2) is satisfied only over S, not necessarily after any basce change, but we don't need it.

→ it is enough to check property (2) after base change to any s.t.d. perf. space.

1.4. Basic properties. The following is evident and shows that ULA complexes are a natural generalization of local systems.

Proposition 1.6. *The following is statisfied.*

(1) If f is separated ℓ-coho. smooth then any complex étale locally constant with value a perfect complex of Λ-modules is ULA.

(2) Being ULA is ℓ-coho. smooth local on the source and the target.

Proposition 1.7. *Let*

$$X \xrightarrow{g} Y \xrightarrow{f} S.$$

If $A \in D_{\text{ét}}(X, \Lambda)$ is $f \circ g$-ULA and g is proper then Rg_*A is f-ULA.

Finally let us remark the following.

PROPOSITION 1.8. *Let A be f-ULA, $f : X \to S$, and $B \in D_{\text{ét}}(X, \Lambda)_{pc}$. Then for any $j : U \to X$ separated étale such that $f \circ j$ is quasicompact then*

$$R(f \circ j)_!((A \otimes^{\mathbb{L}}_{\Lambda} B)_{|U})$$

is perfect constructible.

$\to$ sufficient to do it for $j_!\mathscr{L}$, easy reduction.

DEFINITION 1.9. *Let $\mathfrak{X}$ ba an Artin v-stack and $A \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$. We say that A is ULA (relatively to $\mathfrak{X} \to *$) if for one (and thus all) morphism*

$$f : U \longrightarrow \mathfrak{X}$$

separated ℓ-coho. smooth surjective with U a locally spatial diamond, for all S a locally spatial diamond,

$$(f^*A)_{|U \times S} \in D_{\text{ét}}(U \times S, \Lambda)$$

if ULA relatively to $U \times S \to S$.

1.5. An example. Let us prove the following as an exercise.

THEOREM 1.10 (ULA $\Leftrightarrow$ admissible for classifying stacks of locally pro-p groups). *Let G be an affine algebraic group over E. Let $\mathfrak{X} = [*/\underline{G(E)}]$. Then, via the equivalence*

$$D(G(E), \Lambda) \xrightarrow{\sim} D_{\text{ét}}(\mathfrak{X}, \Lambda),$$

a complex $\pi^\bullet$ is ULA iff for all compact open pro-p subgroup K of $G(E)$, $(\pi^\bullet)^K$ is a perfect complex of Λ-modules.

PROOF. Let $\mathscr{F}_{\pi^\bullet} \in D_{\text{ét}}(\mathfrak{X}, \Lambda)$ be associated to $\pi^\bullet$. Chose $K \subset G(E)$ compact open pro-p. Since $[*/\underline{K}] \to [*/\underline{G(E)}]$ is étale surjective, $\mathscr{F}_{\pi^\bullet}$ is ULA iff $\mathscr{F}_{\pi^\bullet_{|K}} \in D_{\text{ét}}([*/\underline{K}], \Lambda)$ is ULA.

Let $\mathcal{G}$ be a flat model of G that is of finite type over $\mathcal{O}_E$, $\widehat{\mathcal{G}}$ be its π-adic completion and $\widehat{\mathcal{G}}_\eta$ be its generic fiber as a finite type adic space over $\mathrm{Spa}(E)$. We can suppose that $K \subset \mathcal{G}(\mathcal{O}_E)$. Let $X = \left(\widehat{\mathcal{G}}_\eta\right)^\diamond$, a spatial diamond separated ℓ-coho. smooth over $*$. Then,

$$X/\underline{K} \longrightarrow [*/\underline{K}]$$

is a surjective ℓ-coho. smooth morphism. Let $\mathscr{G} \in D_{\text{ét}}(X/\underline{K}, \Lambda)$ be the pull-back of $\mathscr{F}_{\pi^\bullet_{|K}}$.

Let S be a s.t.d perfectoid space and

$$j : U \longrightarrow X/\underline{K} \times S$$

be a separated étale morphism such that the composite

$$U \longrightarrow X/\underline{K} \times S \longrightarrow S$$

is quasi-compact. We can form the cartesian diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "\underline{K}"'] \arrow[dd, bend right=60, "\text{qc qs smooth}"'] & X \times S \arrow[d] \\
U \arrow[r] & X/\underline{K} \times S \arrow[d] \\
 & S
\end{tikzcd}$$

Let $f : X/\underline{K} \times S \to S$ be the projection. We have to compute

$$R(f \circ j)_!(\mathscr{G}_{|U}) \in D_{\text{ét}}(S, \Lambda).$$

Let us note $g : T \longrightarrow S$ the is a qc. ℓ-coho. smooth morphism invariant under the action of $\underline{K}$ on T. Let us note

$$\widetilde{f} : X/\underline{K} \times S \longrightarrow [S/\underline{K}].$$

We have

$$R(f \circ j)_!(\mathscr{G}_{|U}) = \left[(R\widetilde{f}_!\Lambda) \otimes^{\mathbb{L}}_{\Lambda} \pi^\bullet\right]^K$$

After forgetting the action of $\underline{K}$, the image of $R\widetilde{f}_!\Lambda$ under $D_{\text{ét}}([S/\underline{K}], \Lambda) \longrightarrow D_{\text{ét}}(S, \Lambda)$ is $Rg_!\Lambda$ equipped with its action of K. Since g is qc smooth, $Rg_!\Lambda$ is perfect constructible and thus the action of K on $Rg_!\Lambda$ is smooth. Let $K' \subset K$ be compact open such that K' acts trivially on $Rg_!\Lambda$. Then,

$$\left[(R\widetilde{f}_!\Lambda) \otimes^{\mathbb{L}}_{\Lambda} \pi^\bullet\right]^K \subset Rg_!\Lambda \otimes^{\mathbb{L}}_{\Lambda} (\pi^\bullet)^{K'}$$

that is a direct factor. Thus, $(\pi^\bullet)^{K'}$ is a perfect complex of Λ-modules $\Rightarrow R(f \circ j)_!(\mathscr{G}_{|U})$ is perfect constructible.

The reciprocal is obtained in the following way. If $h : X/\underline{K} \to *$ then

$$Rh_!\Lambda = \left[R\Gamma_c(X, \Lambda) \otimes^{\mathbb{L}}_{\Lambda} \pi^\bullet\right]^K$$

with $R\Gamma_c(X, \Lambda) \in D(\Lambda)$ a perfect complex equipped with a smoth action of K. The result then follows, after taking K smaller so that it acts trivially on $R\Gamma_c(X, \Lambda)$, from the fact that if M is a non-zero perfect complex of Λ-modules with $H^i(M) \simeq \Lambda$ for some $i \in \mathbb{Z}$ and $N \in D(\Lambda)$ then N is perfect iff $M \otimes^{\mathbb{L}}_{\Lambda} N$ is perfect. $\square$

Remark 1.11. *We will give another more conceptual proof later.*

2. Behavior with respect to Verdier duality

Let

$$f : X \to S$$

be a morphism of locally spatial diamonds, compactifiable locally of finite dim.trg..

THEOREM 2.1. *Let A be f-ULA.*

(1) The formation of the dualizing complex $\mathbb{D}_{X/S}(A)$ is compatible with base change: for any $S' \to S$,

$$\mathbb{D}_{X/S}(A)_{|X\times_S S'} = \mathbb{D}_{X\times_S S'/S'}\big(A_{|X_{X\times_S S'}}\big).$$

(2) For any $B \in D_{\text{ét}}(S,\Lambda)$

$$\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} f^*B \xrightarrow{\sim} R\mathscr{H}om(A, Rf^!B).$$

Let us give a full proof of point (1). Let us start with a Lemma. Here by $\mathbb{B}^d_S$ we mean the spatial diamond that sends (R,R^+) to morphisms $\mathrm{Spa}(R,R^+) \to S$ together with an element of $(R^+)^d$. Let us moreover recall that cofiltered limits of spatial diamonds as v-sheaves are again spatial diamonds.

LEMMA 2.2. *Let $X \to S$ be a morphism where X is affinoid perfectoid and S is a spatial diamond. Then one caw write*

$$X = \varprojlim_i U_i,$$

a colfiltered limit where U_i is a quasi-compact open subset inside a ball $\mathbb{B}^{d_i}_S$ for some integer d_i.

PROOF. For I a set we note

$$\mathbb{B}^I_S$$

for the spatial diamond over S such that $\mathbb{B}^I_S(R,R^+)$ is the set of morphisms $\mathrm{Spa}(R,R^+) \to S$ together with an element of $(R^+)^I$. When S is perfectoid this is representable by a perfectoid space, if $S = \mathrm{Spa}(A,A^+)$ then $\mathbb{B}^I_S = \mathrm{Spa}(A\langle X_i\rangle_{i\in I}, A^+\langle X_i\rangle)$ where $A^+\langle X_i\rangle_{i\in I}$ is the ϖ-adic completion of $A^+[X_i]_{i\in I}$ and $A\langle X_i\rangle_{i\in I}$ is $A^+\langle X_i\rangle_{i\in I}[\frac{1}{\varpi}]$.

Take $I = \mathcal{O}(X)$. Then there is an evident monomorphism of v-sheaves over S

$$X \hookrightarrow \mathbb{B}^I_S.$$

We have

$$\mathbb{B}^I_S = \varprojlim_{\substack{J\subset I\\ \text{finite}}} \mathbb{B}^J_S.$$

For any $J \subset I$ finite, the image $Z_J \subset |\mathbb{B}^J_S|$ of $X \hookrightarrow \mathbb{B}^I_S \twoheadrightarrow \mathbb{B}^J_S$ is pro-constructible generalizing and thus an intersection of quasi-compact open subsets of $\mathbb{B}^J_S$. We thus have

$$X = \varprojlim_{\substack{J\subset I\\ \text{finite}}} \varprojlim_{\substack{U\supset Z_J\\ \text{qc open}}} U.$$

This proves the result. □

Let us go with another Lemma.

LEMMA 2.3. *Let X be a spatial diamond satisfying: $\exists N$ such that for any $U \to X$ separated étale qc and any $\mathscr{F}$ an étale sheaf of Λ-modules on X, $H^i_{\text{ét}}(U,\mathscr{F}) = 0$ for $i > N$. A morphism $A \to B$ in $D_{\text{ét}}(X,\Lambda)$ is an isomorphism if and only if for any $U \to X$ separated étale qc, $R\Gamma(U,A) \xrightarrow{\sim} R\Gamma(U,B)$.*

Proof. We have $D_{\text{ét}}(X,\Lambda) = D(X_{\text{ét}},\Lambda)$. Now, for any geometric point $\bar{x} : \mathrm{Spa}(C,C^+) \to X$ and any $D \in D(X_{\text{ét}},\Lambda)$,

$$\begin{tikzcd} & \varinjlim U \arrow[d, "\text{sep. étale. qc.}"] \\ \mathrm{Spa}(C,C^+) \arrow[ur] \arrow[r] & X \end{tikzcd} \qquad R\Gamma(U,D) = D_{\bar{x}}$$

and the result follows. □

Proof of point (1) of Theorem 2.1. The assertion is local on X and S and we may assume that X and S are spatial diamonds. There exists a v-cover $S'' \to S'$ that is a strictly totally disconnected perfectoid space. Now, a morphism $D_1 \to D_2$ in $D_{\text{ét}}(X \times_S S',\Lambda)$ is an isomorphism if and only if it is an isomorphism after restriction to $X \times_S S''$. This is a consequence of the fact that $D_{\text{ét}} \subset D_v$. The result for $S'' \to S$ and $S'' \to S'$ thus implies the result for $S' \to S$. We can thus suppose that S' is a strictly totally disconnected perfectoid space.

Since S' is strictly totally disconnected and $X \times_S S' \to S'$ is quasicompact of finite dim.trg., $X \times_S S'$ satisfies the hypothesis of Lemma 2.3 and thus, if $g : X \times_S S' \to X$, we have to prove that for any $U \to X \times_S S'$ separated étale qc,

$$R\Gamma(U, g^*\mathbb{D}_{X/S}(A)) \xrightarrow{\sim} R\Gamma(U, \mathbb{D}_{X\times_S S'/S'}(g^*A)).$$

We can now apply Lemma 2.2 to write

$$S' = \varprojlim_i S'_i$$

(cofiltered limit) where S'_i is quasi-compact open inside a finite dimension ball over S. Since

$$2 - \varinjlim_i \{\text{separated étale qc}\}/S'_i \xrightarrow{\sim} \{\text{separated étale qc}\}/S',$$

we can find an index i and some separated étale qc morphism $V \to S'_i$ such that $U \to S'$ is a pullback of $V \to S'_i$. We have a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \arrow[dr] & \\ X \times_S S' \arrow[r] \arrow[d] & X \times_S S'_i \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S'_i \arrow[r] & S \end{tikzcd}$$

The result we want to prove is immediate when $S' \to S$ is ℓ-cohomologically smooth. This is in particular the case for $S'_i \to S$. We are thus reduce, up to replacing S by S'_i and X by V, to proving that for any cartesian diagram of spatial diamonds

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

with $X \to S$ compactifiable of finite dim.trg., and any $A \in D_{\text{ét}}(X, \Lambda)$ that is ULA over S,

$$R\Gamma(X', g'^*\mathbb{D}_{X/S}(A)) \xrightarrow{\sim} R\Gamma(X', \mathbb{D}_{X'/S'}(g'^*A)).$$

For this is suffices to prove the result after applying Rf'_*. We have

$$\begin{aligned} Rf'_* g'^* R\mathscr{H}om_\Lambda(A, Rf^!\Lambda) &\underbrace{=}_{\text{qc BC}} g^* Rf_* R\mathscr{H}om_\Lambda(A, Rf^!\Lambda) \\ &= g^* R\mathscr{H}om_\Lambda(Rf_! A, \Lambda). \end{aligned}$$

We have moreover,

$$\begin{aligned} Rf'_* R\mathscr{H}om_\Lambda(g'^*A, Rf'^!\Lambda) &= R\mathscr{H}om_\Lambda(Rf'_! g'^*A, \Lambda) \\ &= R\mathscr{H}om_\Lambda(g^* Rf_! A, \Lambda). \end{aligned}$$

The result is thus a consequence of the fact that $Rf_! A$ is perfect constructible and Lemma 1.4. $\square$

CHAPTER 9

9th Lecture: July 28 2023

1. The 2-category of correspondences

Let S be a locally spatial diamond.

Definition 1.1. *Define $\mathcal{C}_S$ for the 2-category whose objects are morphisms $X \to S$ that are compacitfiable of finite dim. trg. and*

(1) for $X \to S$ and $Y \to S$, the 1-morphisms between X/S and Y/S are given by an object of $D_{\text{ét}}(X \times_S Y, \Lambda)$,

(2) the 2-morphisms from A to B elements of $D_{\text{ét}}(X \times_S Y, \Lambda)$

$$\begin{tikzcd} X \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

are given by usual morphisms from A to B in $D_{\text{ét}}(X \times_S Y, \Lambda)$

The composition of one morphisms is given by "convolution" in the sense that for a sequence $X \xrightarrow{A} Y \xrightarrow{B} Z$, in the diagram

$$\begin{tikzcd} & & X \times_S Z \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\ & & X \times_S Y \times_S Z \arrow[u, "\pi_{13}"] \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{23}"] & & \\ & X \times_S Y \arrow[dl] \arrow[dr] & & Y \times_S Z \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

we set

$$B \circ A := \pi_{13!}\big(\pi_{12}^* A \otimes_\Lambda^{\mathbb{L}} \pi_{23}^* B\big) : X \longrightarrow Z.$$

Here the identity 1-morphism from X to X is given by $\Delta_! \Lambda$ where $\Delta : X \hookrightarrow X \times_S X$.

Remark 1.2. *Any correspondence of locally spatial diamonds*

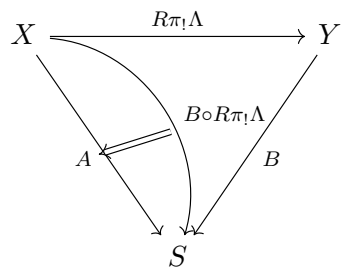
$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

over S (where both morphisms are compactifiable of finite dim.trg.) together with $A \in D_{\text{ét}}(C, \Lambda)$ give rise to a morphism from X to Y in $\mathcal{C}_S$: if $\pi : C \to X \times_S Y$ this is

$$X \xrightarrow{R\pi_! A} Y.$$

We could have defined the 2-category $\mathcal{C}_S$ by fixing a support like C for our correspondences but this is unnecessary and up to replacing (C, A) by $(X \times_S Y, R\pi_! A)$ this does not change anything.

Remark 1.3. *Given $A \in D_{\text{ét}}(X, \Lambda)$, $B \in D_{\text{ét}}(Y, \Lambda)$ and $\pi : C \to X \times_S Y$ as before, a cohomological correspondence from A to B with support in C is nothing else than a 2-morphism*



$$\underbrace{B \circ R\pi_! \Lambda \Longrightarrow A}_{\substack{\textit{coho. corr. from } B \textit{ to } A \\ \textit{supported on } C}}$$

There is a morphism of 2-categories

$$\begin{array}{lrcl} (1) & \mathcal{C}_S & \longrightarrow & \{\text{2-category of triangulated categories}\} \\ (2) & X/S & \longmapsto & D_{\text{ét}}(X, \Lambda) \end{array}$$

that sends X/S to $D_{\text{ét}}(X, \Lambda)$, the 1-morphism $X \xrightarrow{A} Y$ to the 1-morphism, i.e. the functor, given by the kernel A

$$\begin{array}{rcl} D_{\text{ét}}(X, \Lambda) & \longrightarrow & D_{\text{ét}}(Y, \Lambda) \\ \mathscr{F} & \longmapsto & p_{2!}(A \otimes^{\mathbb{L}}_{\Lambda} p_1^* \mathscr{F}) \end{array}$$

where $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$.

In a 2-category like $\mathcal{C}_S$ there is a notion for a 1-morphism to be a left adjoint (or if you want to have a right adjoint). More precisely, a 1-morphism $X \xrightarrow{A} Y$ is a left adjoint if there exists $Y \xrightarrow{B} X$ and 2-morphisms

$$\begin{cases} \eta : \mathrm{Id} \Rightarrow BA \\ \varepsilon : AB \Rightarrow \mathrm{Id} \end{cases}$$

satisfying

$$\begin{cases} B\varepsilon \circ \eta B = \mathrm{Id}_B \\ \varepsilon A \circ A\eta = \mathrm{Id}_A \,. \end{cases}$$

When this is the case such a B is unique up to a 2-isomorphism. In fact, suppose that $Y \xrightarrow{B'} X$ is equipped with 2-morphisms

$$\begin{cases} \eta' : \mathrm{Id} \Rightarrow B'A \\ \varepsilon' : AB' \Rightarrow \mathrm{Id} \end{cases}$$

satisfying

$$\begin{cases} B'\varepsilon' \circ \eta' B' = \mathrm{Id}_{B'} \\ \varepsilon' A \circ A\eta' = \mathrm{Id}_A \,. \end{cases}$$

Consider

$$u = B'\varepsilon \circ \eta' B : B \Rightarrow B'$$

and

$$v = B\varepsilon' \circ \eta B' : B' \Rightarrow B.$$

One then has

$$\begin{aligned} v \circ u &= B\varepsilon' \circ \underbrace{\eta B' \circ B'\varepsilon}_{BAB'\varepsilon \circ \eta B'AB} \circ \eta' B \\ &= \underbrace{B\varepsilon' \circ BAB'\varepsilon}_{B\varepsilon \circ B\varepsilon' AB} \circ \underbrace{\eta B'AB \circ \eta' B}_{BA\eta' B \circ \eta B} \\ &= B\varepsilon \circ B\varepsilon' AB \circ BA\eta' B \circ \eta B \\ &= B\varepsilon' AB \circ BA\eta' B \\ &= \mathrm{Id} \end{aligned}$$

2. Relation to the ULA condition

THEOREM 2.1. *The following are equivalent:*

(1) A is f-ULA,

(2) The natural morphism

$$\mathbb{D}_{X/S}(A) \boxtimes^{\mathbb{L}}_{\Lambda} A \longrightarrow R\mathscr{H}om_{\Lambda}(p_1^* A, Rp_2^! A)$$

is an isomorphism in $D_{\text{ét}}(X \times_S X, \Lambda)$,

(3) The 1-morphism $X \xrightarrow{A} S$ is a left adjoint in $\mathcal{C}_S$, in which case its right adjoint is given by $S \xrightarrow{\mathbb{D}_{X/S}(A)} X$.

PROOF. (1)⇒(2) is a consequence of Theorem 2.1.

(2)⇒(3): One computes

$$\begin{aligned} \mathbb{D}_{X/S}(A) \circ A &= A \boxtimes^{\mathbb{L}}_{\Lambda} \mathbb{D}_{X/S}(A) \\ A \circ \mathbb{D}_{X/S}(A) &= Rf_!(\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_{\Lambda} A) \end{aligned}$$

Then to give oneself η is the same a morphism

$$\eta : \Delta_! \Lambda \longrightarrow A \boxtimes^{\mathbb{L}}_{\Lambda} \mathbb{D}_{X/S}(A) \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(p_1^* A, Rp_2^! A)$$

By adjunction this is the same a morphism

$$\Lambda \longrightarrow R\Delta^! R\mathscr{H}om_\Lambda(p_1^* A, Rp_2^! A) = R\mathscr{H}om_\Lambda(A, A)$$

and we can take the Identity of A.

For ε, this is a morphism

$$\varepsilon : Rf_!(\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda A) \longrightarrow \Lambda$$

that is to say by adjunction a morphism

$$\mathbb{D}_{X/S}(A) \otimes^{\mathbb{L}}_\Lambda A \longrightarrow Rf^!\Lambda$$

and we can take the evident morphism.

One verifies easily those satisfy the adjunction properties.

(3)⇒(1) We use the morphism of 2-categories (1). If $S \xrightarrow{B} X$ is a right adjoint of $X \xrightarrow{A} S$ then by a application of our morphism of 2-categories, then the functor

$$Rf_!(A \otimes^{\mathbb{L}}_\Lambda -) : D_{\text{ét}}(X, \Lambda) \longrightarrow D_{\text{ét}}(S, \Lambda)$$

has

$$f^*(-) \otimes^{\mathbb{L}}_\Lambda B : D_{\text{ét}}(S, \Lambda) \longrightarrow D_{\text{ét}}(X, \Lambda)$$

as a right adjoint. To prove that A is f-ULA we can suppose that S is a strictly totally disconnected perfectoid space. But since the right adjoint commutes with all colimits, it sends compact objects to compact objects and we can conclude that for any $A' \in D_{\text{ét}}(X, \Lambda)_{p.c.}$, $Rf_!(A \otimes^{\mathbb{L}}_\Lambda A')$ is perfect constructible.

It remains to prove the overconvergence of A, see the article. □

Let us look at two applications of this that may be very difficult without this 2-categorical point of view.

Proposition 2.2. *(1) Any $A \in D_{\text{ét}}(X, \Lambda)$ that is f-ULA is bidual with respect to the Verdier duality,*

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A)).$$

(2) For $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$, and $A \in D_{\text{ét}}(X, \Lambda)$ that is f-ULA, and $B \in D_{\text{ét}}(Y, \Lambda)$ that is g-ULA, $A \boxtimes^{\mathbb{L}}_\Lambda B$ is ULA relatively to $X \times_S Y \to S$ and

$$\mathbb{D}_{X/S}(A) \boxtimes^{\mathbb{L}}_\Lambda \mathbb{D}_{Y/S}(B) \xrightarrow{\sim} \mathbb{D}_{X \times_S Y/S}(A \boxtimes^{\mathbb{L}}_\Lambda B).$$

Proof. Point (1) is a consequence of the isomorphism of 2-categories

$$\mathcal{C}_S \simeq \mathcal{C}_S^{op}.$$

In fact, the 1-morphisms between X and Y are exactly the same as the 1-morphisms between Y and X. Thus, if A is f-ULA then $\mathbb{D}_{X/S}(A)$ is a right adjoint of A and thus A is a right adjoint of $\mathbb{D}_{X/S}(A)$ that is thus f-ULA. From this we deduce point (1) via the identification of the right adjoint of $\mathbb{D}_{X/S}(A)$ with $\mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$.

Point (2) is a consequence of the fact that $B \circ A$ is a left adjoint and its right adjoint is the composite of the right adjoint of A with the one of B and is thus $\mathbb{D}_{X/S}(A) \circ \mathbb{D}_{X/S}(B)$. □

3. An application

THEOREM 3.1. *Let $\mathfrak{X} = [*/\underline{H}]$ with H a locally pro-p group. Suppose that $\mathfrak{X}$ is ℓ-cohomologically smooth of dimension 0, for example H is a closed sub-group of $G(E)$ where G is an affine algebraic group over E. Then, via the identification*

$$D(H, \Lambda) \xrightarrow{\sim} D(\mathfrak{X})$$

we have π corresponds to an ULA complex iff for all $K \subset H$ compact open pro-p, π^K is a perfect complex of Λ-modules.

PROOF. We use the identification

$$D(H \times H, \Lambda) \xrightarrow{\sim} D(\mathfrak{X} \times \mathfrak{X}).$$

If π corresponds to an ULA complex then for any π',

$$\widetilde{\pi} \boxtimes^{\mathbb{L}}_{\Lambda} \pi' \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(\pi \boxtimes 1, 1 \boxtimes \pi')$$

in $D(H \times H, \Lambda)$ (derived smooth dual and derived smooth Hom's). In particular, taking for π' a complex of Λ-modules with trivial H-action, M

$$\widetilde{\pi} \otimes^{\mathbb{L}}_{\Lambda} M \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(\pi, M).$$

(smooth duals and hom's) Taking the K-invariants with K open pro-p we obtain

$$R\mathrm{Hom}_{\Lambda}(\pi^K, \Lambda) \otimes^{\mathbb{L}}_{\Lambda} M \xrightarrow{\sim} R\mathrm{Hom}_{\Lambda}(\pi^K, M).$$

It is then immediate that π^K is a perfect complex of Λ-modules.

In the other direction, let π be such that for all K open pro-p, π^K is perfect. We have to verify that

$$\widetilde{\pi} \boxtimes^{\mathbb{L}}_{\Lambda} \pi \xrightarrow{\sim} R\mathscr{H}om_{\Lambda}(\pi, \pi).$$

It suffices to prove this is the case after applying $D(H \times H, \Lambda) \to D(\Lambda)$. Then the morphism is written as

$$\varinjlim_{K} R\mathrm{Hom}_{\Lambda}(\pi^K, \Lambda) \otimes^{\mathbb{L}}_{\Lambda} \pi^K \longrightarrow \varinjlim_{K} R\mathrm{Hom}_{\Lambda}(\pi^K, \pi^K)$$

and the result follows. □

4. A criterion of smoothness

PROPOSITION 4.1. *Let $f : X \to S$ be a compactifiable of finite dim.trg. morphism of locally spatial diamonds. Then, f is ℓ-cohomologically smooth if and only if $\mathbb{F}_\ell$ is f-ULA and $Rf^!\mathbb{F}_\ell$ is invertible.*

CHAPTER 10

10th Lecture: August 1st 2023

$\rightarrow$ the ULA property is used at two places in the article:

- for the geometric Satake correspondence, in fact the B_{dR}-affine Grassmanian that shows up in the geometric Satake correspondence should be though of as being a Beilinson-Drinfeld type affine Grassmanian i.e. something relative "sitting over the curve" (more precisely over $\mathrm{Div}^1 = \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}}$), and we thus need to speak about "families of perverse sheaves" i.e. ULA perverse sheaves,
- for the proof of the Jacobian criterion via Proposition 4.1.

We deal here with the proof of the Jacobian criterion.

1. A key remark: stability under retracts of the ULA property

Let us begin with a lemma whose proof is a simple computation.

LEMMA 1.1. *(1) Let*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

be a diagram of compactifiable of finite dim. trg. morphisms of locally spatial diamonds. Let $\Gamma_h : X \to X \times_S Y$ be the graph of h. Let $A \in D_{\text{ét}}(Y, \Lambda)$. The following diagram in $\mathcal{C}_S$

$$\begin{tikzcd} X \arrow[rr, "(\Gamma_h)_!\Lambda"] \arrow[dr, "h^*A"'] & & Y \arrow[dl, "A"] \\ & S & \end{tikzcd}$$

commutes i.e. there is a canonical (in A) isomorphism

$$A \circ (\Gamma_h)_!\Lambda \overset{\sim}{\Longrightarrow} h^*A.$$

(2) Let $\mathcal{D}_S$ be the category of locally spatial diamonds compactifiable of finite dim. trg. over S. We upgrade it to a 2-category by setting $2 - Hom(f, g) = \{\mathrm{Id}\}$ if $f = g$ and $\emptyset$ if $f \neq g$. Then the correspondence

$$\mathcal{D}_S \longrightarrow \mathcal{C}_S$$

that sends X/S to X/S and $f : X \to Y$ an S-morphism to $X \xrightarrow{R(\Gamma_f)_!\Lambda} Y$ is a morphism of 2-categories.

PROPOSITION 1.2. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of locally spatial diamonds that are compactifiable of finite dim. trg.. Suppose that f is a retract of g i.e. there exists morphisms*

$$\begin{tikzcd} X \arrow[rr, shift left, "i"] \arrow[dr, "f"'] & & Y \arrow[ll, shift left, "r"] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying

$$r \circ i = \mathrm{Id}_X .$$

Then, if $A \in D_{\text{ét}}(Y, \Lambda)$ is g-ULA, i^*A is f-ULA.

PROOF. Point (1) of the preceding lemma says that is suffices to verify that $X \xrightarrow{\Gamma_{i,!}\Gamma} Y$ is a left adjoint. Point (2) shows that the composite $X \xrightarrow{\Gamma_{i,!}\Gamma} Y \xrightarrow{R(\Gamma_r)_!\Lambda} X$ is canonically isomorphic to Id_X. This gives us a unit

$$\eta : \mathrm{Id}_X \overset{\sim}{\Longrightarrow} R(\Gamma_r)_!\Lambda \circ \Gamma_{i,!}\Lambda.$$

We moreover have

$$R(\Gamma_r)_!\Lambda \circ \Gamma_{i,!}\Lambda = R(\Gamma_{r\circ i})_!\Lambda.$$

Since f is separated, $\mathrm{Id}_X = \Delta_{X/S!}\Lambda = \Delta_{X/S*}\Lambda$. To give a 2-morphism

$$R(\Gamma_{r\circ i})_!\Lambda \Longrightarrow \mathrm{Id}_X$$

is thus the same as a morphism in $D_{\text{ét}}(X, \Lambda)$

$$\Delta^*_{X/S} R(\Gamma_{r\circ i})_! \longrightarrow \Lambda.$$

Proper base change says that the left term is Λ. We thus have a counit

$$\varepsilon : R(\Gamma_r)_!\Lambda \circ \Gamma_{i,!}\Lambda \Longrightarrow \mathrm{Id}_X .$$

One verifies that η and ε define an adjunction. □

Here is the corollary that we will use.

COROLLARY 1.3. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of locally spatial diamonds that are compactifiable of finite dim. trg.. Suppose that f is a retract of g i.e. there exists morphisms*

$$\begin{tikzcd} X \arrow[rr, shift left, "i"] \arrow[dr, "f"'] & & Y \arrow[ll, shift left, "r"] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying

$$r \circ i = \mathrm{Id}_X .$$

Then if g is ℓ-cohomologically smooth, $\mathbb{F}_\ell$ is f-ULA.

$\to$ contrary to the cohomological smoothness condition, the ULA property is stable under retracts.

2. Formal smoothness

The notion of formal smoothness we introduce is a tool we use to prove that $\mathbb{F}_\ell$ is ULA for $\mathcal{M}_Z^{sm} \to Z$ in the proof of the Jacobian criterion of smoothness. It is complementary to the notion of ℓ-coho. sm.. All "natural" morphisms that show up that are ℓ-coho. sm. are formally smooth too. If there exists a "good" notion of smooth morphisms in our context it has to imply ℓ-coho. sm. for all $\ell \neq p$ <u>and</u> formally smooth.

2.1. Background on Zariski closed immersions.

Recall the following basic result.

Proposition 2.1. *Let $S = \mathrm{Spa}(A, A^+)$ be affinoid perfectoid and I an ideal of A. The closed subset $V(I) = \{s \in S \mid \forall f \in I,\ |f(s)| = 0\} \subset |S|$ is representable by an affinoid perfectoid space pro-étale inside S.*

$\to$ $V(I) = \varprojlim_{\substack{n \geq 1 \\ f_1, \ldots, f_n \in I}} \{|f_1| \leq 1, \ldots, |f_n| \leq 1\}$ that is thus a cofiltered limit of affinoid perfectoid spaces.

Definition 2.2. *For S affinoid perfectoid the immersion $S_0 \hookrightarrow S$ defined by an ideal I of $\mathcal{O}(S)$ as before is called a Zariski closed immersion.*

$\to$ one has to be careful that, contrary to the case of schemes, this is not a local condition: if $S = \cup_i U_i$ is a finite rational cover of S and a closed subset $Z \subset |S|$ is such that for all i, $Z \cap U_i$ is Zariski closed in U_i then Z may not be Zariski closed in S in general).

Recall moreover the following basic result that is easy for $\mathbb{F}_p$-perfectoid spaces but more difficult in general.

Theorem 2.3. *Let $S_0 \subset S$ be a Zariksi closed immersion of affinoid perfectoid spaces.*

(1) $S_0 \subset S$ is strongly Zariski closed in the sense that the morphism
$$\mathcal{O}(S)^+ \to \mathcal{O}(S_0)^+$$
is almost surjective.

(2) $S_0^\flat \to S^\flat$ is a Zariksi closed immersion and thus if $Z \subset |S|$ is Zariski closed then it is Zariski closed in $|S^\flat|$ via the equality $|S| = |S^\flat|$.

2.2. Formally smooth morphisms.

We can now come to our definition. This is related to the topological notion of retracts in the sense of Borsuk.

Definition 2.4. *A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of v-stacks is formally smooth if for all diagrams*

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, "f"] \\ S \arrow[r] \arrow[ur, dashed] & \mathfrak{Y} \end{tikzcd}$$

where $S_0 \hookrightarrow S$ is a Zariski closed immersion of affinoid perfectoid spaces, up to replacing $S_0 \hookrightarrow S$ by $S' \times_S S_0 \hookrightarrow S'$ where $S' \to S$ is an étale neighborhood of S_0, we can complete the diagram with the dashed arrow:

$$\begin{tikzcd} S' \times_S S_0 \arrow[r] \arrow[d, hook] & S_0 \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, "f"] \\ S' \arrow[r, "\text{étale nbd. of } S_0"'] \arrow[urr] & S \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

commutes up to a 2-isomorphism i.e. the composite $S' \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ is isomorphic to $S' \to S \to \mathfrak{Y}$ in the groupoid $\mathfrak{Y}(S')$.

$\to$ some evident examples

Example 2.5. *(1) Any separated étale morphism of v-stacks is formally smooth.*
(2) The composite of two f.s. morphisms is f.s..
(3) The base change of a f.s. morphism is f.s..
(4) The f.s. property is étale local on the source

Let us give more non-trivial examples.

Lemma 2.6. *Let $\mathbb{B} \to \mathrm{Spd}(\mathbb{Z}_p)$ be the diamond of the morphism sending a $\mathbb{Z}_p$-perfectoid space S to $\mathcal{O}(S)^+$. This is formally smooth.*

Proof. Let $S_0 \hookrightarrow S$ be a Zariski closed immersion of affinoid perfectoid spaces and $f \in \mathcal{O}(S_0)^+$. Since $\mathcal{O}(S) \to \mathcal{O}(S_0)$ is surjective we can lift it to some $\tilde{f} \in \mathcal{O}(S)$. Now, if $U = \{|f| \leq 1\}$ then $f_{|U} \in \mathcal{O}(U)^+$ and U is a nbd. of S_0. $\square$

Lemma 2.7. *The morphism $\mathrm{Spd}(\mathbb{Z}_p) \to \mathrm{Spd}(\mathbb{F}_p)$ is formally smooth.*

Proof. Let $\xi = \sum_{n \geq 0} [a_n] p^n$ be a degree one primitive element in $W(R^+)$, (R, R^+) a $\mathbb{F}_p$-affinoid perfectoid algebra. Up to multiplying ξ by a unit in $W(R^+)^\times$ we can suppose that $\xi \in p + [\varpi] W(R^+)$ where ϖ is a p.u. in R. Now, if $(A, A^+) \to (R, R^+)$ is a morphism of affinoid rings such that $A^+ \to R^+$ is almost surjective then $A^{\circ\circ} \to R^{\circ\circ}$ is surjective. We deduce that, up to multiplying ξ by a unit it lifts to a degree one primitive element in $W(A^+)$. $\square$

COROLLARY 2.8. *If $f : X \to Y$ is a smooth morphism of Noetherian analytic adic spaces over $\mathbb{Z}_p$ then $f^\diamond : X^\diamond \to Y^\diamond$ is formally smooth.*

PROOF. Since the notion of f.s. is local for the analytic topology on the source we are reduce to proving that if f is the composite of an étale morphism toward $\mathbb{B}^d_Y$, a ball$/Y$, with the projection to Y then $f^\diamond$ is f.s.. Since the diamond of an étale morphism is étale we are reduced to proving that $(\mathbb{B}^d_Y \to Y)^\diamond$ is f.s.. But $(\mathbb{B}^d_Y)^\diamond = \mathbb{B}^d \times_{\mathrm{Spd}(\mathbb{Z}_p)} Y^\diamond$ and the result is deduced from lemma 2.6. □

3. Formal smoothness and the ULA property

→ the following says that, up to some technical "finiteness property of f",

$$f \text{ formally smooth } \implies \mathbb{F}_\ell \text{ is } f\text{-ULA}.$$

PROPOSITION 3.1. *Let $f : X \to S$ be a (compactifiable of finite dim. trg.) morphism of locally spatial diamonds satisfying: there exists a v-cover $(T_i \to X)_i$ such that for all i*

(1) T_i is affinoid perfectoid Zariski closed in an ℓ-cohomologically smooth affinoid perfectoid S-space
(2) $T_i \to X$ is formally smooth and ℓ-cohomologically smooth

Then, if f is formally smooth, $\mathbb{F}_\ell$ is f-ULA.

PROOF. Since $T_i \to X$ is ℓ-cohomologically smooth, the ULA notion being ℓ-coho. smooth local, it is enough to prove that $\mathbb{F}_\ell$ is ULA relatively to $T_i \to S$. This morphism is formally smooth as a composite of two formally smooth morphisms and thus, up to replacing T_i by an étale cover, $T_i \to S$ is a retract of an ℓ-cohomologically smooth affinoid perfectoid space. We can now apply Corollary 1.3. □

4. The main result about formal smoothness

THEOREM 4.1. *The morphism $\mathcal{M}^{sm}_Z \to S$ is formally smooth.*

→ we refer to the article; the proof is technical but natural.

5. Application of the formal smoothness: first step in the proof of the Jacobian criterion

PROPOSITION 5.1. *There exists a v-cover $(T_i \to \mathcal{M}_Z)_i$ where for all i,*

(1) T_i is affinoid perfectoid and $T_i \to S$ is Zariski closed in an affinoid perfectoid space that is étale over a perfectoid ball $\mathbb{B}^{d_i,1/p^\infty}_S$,
(2) $T_i \to \mathcal{M}_Z$ is ℓ-coho. sm. and f.s..

→ using the "quasi-projectivity assumption" for $Z \to X_S$ this is reduced to proving the result for $Z = \mathbb{P}^n_{X_S}$ in which case this is an exercise about BC spaces using the explicit formula

for $\mathcal{M}_{\mathbb{P}^n_{X_S}}$.

We can now prove our Theorem.

THEOREM 5.2 (First step in the proof of the Jacobian criterion). *The étale sheaf $\mathbb{F}_\ell$ on $\mathcal{M}_Z^{sm}$ is ULA relatively to the morphism $\mathcal{M}_Z^{sm} \to S$.*

$\to$ Apply Propositions 3.1 and 5.1 together with, of course, Theorem 4.1.

CHAPTER 11

11th Lecture: August 4 2023

We now switch to the deformation to the normal cone argument to finish the proof of the Jacobian criterion.

1. Background on the deformation to the normal cone (see Fulton's book)

1.1. Normal cones. Let

$$i : Y \hookrightarrow X$$

be a closed immersion of schemes defined by the ideal $\mathfrak{I}$. Recall that the normal cone to i is

$$C_Y X = \mathrm{Spec}\Big(\bigoplus_{k\geq 0} \mathfrak{I}^k/\mathfrak{I}^{k+1}\Big) \longrightarrow Y.$$

When i is a regular immersion the associated normal bundle is associated to the vector bundle $(\mathfrak{I}/\mathfrak{I}^2)^\vee$ on Y and one has

$$C_Y X = \mathbb{V}\Big((\mathfrak{I}/\mathfrak{I}^2)^\vee\Big)$$

the geometric realization of this vector bundle.

$\to$ the notion of normal cone generalized thus the notion of the normal vector bundle associated to a regular embedding.

$\to$ it is called a cone because it is the spectrum of a graded quasi-coherent algebra and is thus equipped with a $\mathbb{G}_m$-action

$$\begin{tikzcd} C_Y X \arrow[loop right, "\mathbb{G}_m"] \arrow[d, shift left] \\ Y \arrow[u, bend left, "0"] \end{tikzcd}$$

where 0 is the zero section of this cone given by $(C_Y X)^{\mathbb{G}_m} \xrightarrow{\sim} Y$.

$\to$ thus although in general, when i is not a regular immersion, $C_Y X \to Y$ is not (the geometric realization of) a vector bundle, it is still a cone like any vector bundle.

We will use later the following construction. Let S be a scheme and

$$C = \mathrm{Spec}(\mathcal{A}) \longrightarrow S$$

be a cone, that is to say $\mathcal{A}$ is a graded quasi-coherent $\mathcal{O}_S$-algebra, $\mathcal{A} = \bigoplus_{k\geq 0} \mathcal{A}_k$ with $\mathcal{A}_0 = \mathcal{O}_S$. We define its projective completion as

$$\begin{aligned}\overline{C} &= \operatorname{Proj}\Big(\mathcal{A}_\bullet \underbrace{\otimes_{\mathcal{O}_S}}_{\substack{\text{graded}\\ \text{tensor product}}} \mathcal{O}_S[T]\Big) \longrightarrow S \\ &= (C \times \mathbb{A}^1) \smallsetminus \{0\}/\mathbb{G}_m\end{aligned}$$

where the action of $\mathbb{G}_m$ is the diagonal one and $\{0\} = (C \times \mathbb{A}^1)^{\mathbb{G}_m}$ is the origin of the cone $C \times \mathbb{A}^1$.

There is then a diagram

$$\begin{tikzcd} C \arrow[r, hook, "\text{open}"] \arrow[dr] & \overline{C} \arrow[d] & \overline{C} \smallsetminus C = (C \smallsetminus \{0\})/\mathbb{G}_m \arrow[l, hook'] \arrow[dl] \\ & S & \end{tikzcd}$$

that is to say $\overline{C}$ is $\mathbb{G}_m$-equivariant a compactification of C obtained by adding

$$(C \smallsetminus \{0\})/\mathbb{G}_m = \operatorname{Proj}(\mathcal{A}_\bullet)$$

at ∞ where $\{0\} \hookrightarrow C$ is the origin of the cone, $\operatorname{Spec}(\mathcal{A}_0) \hookrightarrow \operatorname{Spec}(\mathcal{A})$.

If our cone is the geometric realization of a vector bundle $\mathscr{E}$,

$$C = \mathbb{V}(\mathscr{E}),$$

then

$$\overline{C} = \mathbb{P}(\mathscr{E} \oplus \mathcal{O}_S) \supset \overline{C} \smallsetminus C = \mathbb{P}(\mathscr{E}).$$

Example 1.1. *When we have a closed immersion of smooth S-schemes*

$$\begin{tikzcd} Y \arrow[rr, hook, "i"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$

there is an exact sequence of vector bundles on Y

$$0 \longrightarrow T_{Y/S} \longrightarrow i^*T_{X/S} \longrightarrow (\mathfrak{I}/\mathfrak{I}^2)^\vee \longrightarrow 0.$$

Remark 1.2. *The terminology normal bundle comes from the fact that if (M, g) is a Riemannian manifold and $i : N \hookrightarrow M$ a closed submanifold then the metric g allows us to identify the normal bundle with the orthogonal of T_N inside i^*T_M with respect to the metric g, giving a splitting of the sequence of $\mathcal{C}^\infty$-vector bundles $0 \to T_N \to i^*T_N \to \mathcal{N}_{N/M} \to 0$. At the end, for $n \in N$, the normal bundle at n is the set of tangents vectors in $(TM)_n$ that are orthogonal to $(TN)_n$.*

Remark 1.3. *In the context of the preceding remark, let*

$$\exp : TM \longrightarrow M$$

(that is well defined only in a nbd. of the zero section in general if (M, g) is not complete) be the map that sends the tangent vector $X \in (TM)_m$ to $\gamma(1)$ where γ is the unique geodesic satisfying $\gamma(0) = m$ and $\gamma'(0) = X$.

One can find a neighborhood U of the zero section of $\mathcal{N}_{N/M} = (TN)^{\perp} \subset i^*TM$ such that the map

$$\exp_{|U} : U \longrightarrow M$$

is an isomorphism onto an open neighborhood of N inside M. This is what's called a tubular neighborhood of N inside M. The deformation to the normal cone is an algebraic analog of this construction.

1.2. The deformation to the normal cone.

1.2.1. *What we want.* The deformation to the normal cone is a $\mathbb{G}_m$-equivariant family of closed immersions parametrized by $\mathbb{A}^1$

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r, bend left] & Y \times \mathbb{A}^1 \arrow[r, hook] & W \arrow[d] \arrow[loop right, "\mathbb{G}_m"] \\
& & X \times \mathbb{A}^1 \arrow[d] \arrow[loop right, "\mathbb{G}_m"] \\
& & \mathbb{A}^1 \arrow[loop right, "\mathbb{G}_m"]
\end{tikzcd}$$

satisfying:

(1) The following diagram

$$\begin{tikzcd}
Y \times \mathbb{A}^1 \arrow[r, hook] \arrow[dr, hook, "i \times \mathrm{Id}_{\mathbb{A}^1}"'] & W \arrow[d] \\
& X \times \mathbb{A}^1
\end{tikzcd}$$

commutes.

(2) The morphism

$$\begin{tikzcd}
W \arrow[d] \\
\mathbb{A}^1
\end{tikzcd}$$

is flat i.e. we have a flat family parametrized by $\mathbb{A}^1$.

(3) Its restriction to $\mathbb{G}_m = \mathbb{A}^1 \smallsetminus \{0\}$ is identified with

$$\begin{tikzcd}
Y \times \mathbb{G}_m \arrow[r, hook] & W_{|\mathbb{G}_m} \arrow[d, "\simeq"] \\
& X \times \mathbb{G}_m
\end{tikzcd}$$

and thus the family of immersions $Y \hookrightarrow W_t$ for $t \in \mathbb{A}^1 \smallsetminus \{0\}$ is the trivial family associated to $i : Y \hookrightarrow X$.

(4) Its fiber at $0 \in \mathbb{A}^1$ is identified with

$$\begin{tikzcd} Y \arrow[r, hook, "\substack{\text{zero section}\\ \text{of the normal cone}}"] & W_0 = C_Y X \arrow[d, "\substack{\text{projection to } Y\\ \text{composed wt. } i}"] \\ & X \end{tikzcd}$$

and thus the immersion $Y \hookrightarrow W_0$ is identified with the inclusion of the zero section of the normal cone.

1.2.2. *The construction.* This is defined in the following way. Let Z be the blow-up of $Y \times \{0\}$ inside $X \times \mathbb{A}^1$,

$$\begin{tikzcd} & Z \arrow[d, "\text{blow-up}"] \\ Y \times \{0\} \arrow[r, hook] & X \times \mathbb{A}^1. \end{tikzcd}$$

We have

$$Z = \operatorname{Proj}\Big(\bigoplus_{k \geq 0} \sum_{i=0}^{k} \mathfrak{I}^{k-i} T^i \mathcal{O}_X[T]\Big).$$

The closed immersion $Y \times \mathbb{A}^1 \hookrightarrow Z$ is defined by the universal property of the blow-up: via the morphism $Y \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ the pull-back of $Y \times \{0\}$ inside $X \times \mathbb{A}^1$ is a Cartier divisor.

It is evident that over $\mathbb{G}_m$, the immersion $Y \times \mathbb{A}^1 \hookrightarrow Z$ is identified with $Y \times \mathbb{G}_m \hookrightarrow X \times \mathbb{G}_m$.

The fiber over $\{0\}$ is

$$\operatorname{Proj}\Big(\mathcal{O}_X \bigoplus (\mathfrak{I} \oplus \mathcal{O}_X/\mathfrak{I}.t) \bigoplus \cdots \bigoplus (\mathfrak{I}^k \oplus \mathfrak{I}^{k-1}/\mathfrak{I}^k.t \oplus \cdots \oplus \mathfrak{I}/\mathfrak{I}^2.t^{k-1} \oplus \mathcal{O}_X/\mathfrak{I}.t^k) \bigoplus \cdots\Big).$$

Here we use the bigraded ring

$$\mathcal{A} = \bigoplus_{k,l \in \mathbb{N}} \mathcal{A}_{k,l}$$

where

- $\mathcal{A}_{0,l} = \mathfrak{I}^l$,
- $\mathcal{A}_{k,l} = \mathfrak{I}^l/\mathfrak{I}^{l+1}$ if $k > 0$.

Then, if $\operatorname{Tot}\mathcal{A}$ is the graded ring such that

$$(\operatorname{Tot}\mathcal{A})_d = \bigoplus_{k+l=d} \mathcal{A}_{k,l},$$

the fiber over $\{0\}$ is

$$\operatorname{Proj}\big(\operatorname{Tot}\mathcal{A}\big).$$

There are two graded ideals in $\operatorname{Tot}\mathcal{A}$,

$$\mathfrak{a} = \bigoplus_{k>0, l\geq 0} \mathcal{A}_{k,l}$$

and

$$\mathfrak{b} = \bigoplus_{l\geq 0} \mathfrak{I}\mathcal{A}_{0,0}$$

contained in the augmentation ideal $(\operatorname{Tot}\mathcal{A})^+$. One has

$$\mathfrak{a}\mathfrak{b} = (0)$$

and thus

$$\operatorname{Proj}\big(\operatorname{Tot}\mathcal{A}\big) = V^+(\mathfrak{a}) \cup V^+(\mathfrak{b}).$$

From this formula we see that the fiber of Z over $\{0\}$ is a union of $B_Y X$ the blow-up of Y inside X and $\overline{C_Y X}$ the projective completion of the normal cone $C_Y X$. There is an identification

$$\overline{C_Y X} \smallsetminus C_Y X = (C_Y X \smallsetminus \{0\})/\mathbb{G}_m.$$

This is identified with the exceptional Cartier divisor $E_Y X \subset B_Y X$ of the blow-up of Y inside X and our fiber at $\{0\}$ is then

$$\boxed{Z_0 = B_Y X \coprod_{E_Y X = \overline{C_Y X} \smallsetminus C_Y X} \overline{C_Y X}.}$$

One then sets

$$\boxed{W = Z \smallsetminus \underbrace{B_Y X}_{\substack{\text{closed subset}\\ \text{of the fiber at } \{0\}}}.}$$

as an open subscheme of Z.

1.2.3. *An explicit formula.* One can show the following.

Proposition 1.4. *Let*

$$\mathcal{A} = \bigoplus_{n<0} \mathfrak{I}^{-n} t^n \bigoplus \mathcal{O}_X[t] = \mathcal{O}_X\left[\tfrac{\mathfrak{I}}{t}, t\right] \subset \mathcal{O}_X[t, t^{-1}]$$

as a quasi-coherent $\mathcal{O}_X$-algebra. The morphism $\mathcal{O}_X[t] \to \mathcal{A}$ identifies

$$\mathit{Spec}(\mathcal{A}) \longrightarrow X \times \mathbb{A}^1$$

with

$$W \longrightarrow X \times \mathbb{A}^1.$$

In particular, the morphism $W \to X \times \mathbb{A}^1$ is affine.

1.3. Some classical applications.

1.3.1. *Verdier's specialization to the normal cone.* Let k be an algebraically closed field and

$$i : Y \hookrightarrow X$$

be a closed immersion of finite type k-schemes. Let Λ be a finite ring killed by a power of ℓ invertible in k. Recall that an étale constructible sheaf $\mathscr{F}$ of Λ-modules on the cone $C_Y X$ is said to be *monodromic* if for all $\lambda \in \mathbb{G}_m(k)$, if $m_\lambda : C_Y X \to X_Y X$ is the action of λ, then

$$m_\lambda^* \mathscr{F} \simeq \mathscr{F}.$$

This is equivalent to say that for all $x \in C_Y X$, if $w_x : \mathbb{G}_m \to C_Y X$ is the morphism $\lambda \mapsto \lambda.x$ then $w_x^* \mathscr{F}$ is a moderate étale local system of Λ-modules.

Let

$$D_c^b(C_Y X, \Lambda)^{mon}$$

be the category of bounded étale complexes of Λ-modules on $C_Y X$ with constructible monodromic cohomology.

Verdier defines a factorization

$$\begin{tikzcd} D_c^b(X,\Lambda) \arrow[rr, "i^*"] \arrow[dr, "\mathrm{Sp}_Y X"'] & & D_c^b(Y,\Lambda) \\ & D_c^b(C_Y X,\Lambda)^{mon} \arrow[ur, "0^*"'] & \end{tikzcd}$$

where $0 : Y \hookrightarrow C_Y X$ is the zero section of the cone. This is called the specialization to the normal cone.

This is defined in the following way. If $\mathrm{pr} : X \times \mathbb{G}_m \longrightarrow X$ is the projection we use the diagram

$$\begin{tikzcd} C_Y X \arrow[r, hook] \arrow[d] & W \arrow[d] & X \times \mathbb{G}_m \arrow[l, hook'] \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{A}_k^1 & \mathbb{G}_{m,k} \arrow[l, hook'] \end{tikzcd}$$

Then, for $A \in D_c^b(X, \Lambda)$,

$$\mathrm{Sp}_Y X(A) := R\Psi_{\bar{\eta}}\big(\mathrm{pr}^* A\big)$$

where $\bar{\eta}$ is any geometric point over the generic point of $\mathbb{A}_k^1$.

The fact that $0^* \mathrm{Sp}_Y X(A) = i^* A$ and that $\mathrm{Sp}_Y X(A)$ is monodromic is a theorem of Verdier.

1.3.2. *Microlocalization.* When $i : Y \hookrightarrow X$ is a regular immersion, the cone $C_Y X$ is a vector bundle

$$N_Y X = \mathbb{V}\big((\mathfrak{I}/\mathfrak{I}^2)^\vee\big) \longrightarrow Y,$$

the normal bundle. We can look at the dual vector bundle

$$N_Y^* X = \mathbb{V}\big(\mathfrak{I}/\mathfrak{I}^2\big) \longrightarrow Y.$$

There is then a microlocalization functor

$$\mu = \underbrace{\mathscr{F}}_{\substack{\text{Fourier}\\ \text{transform}}} \circ \mathrm{Sp}_Y X : D^b_c(X, \Lambda) \longrightarrow D^b_c(N^*_Y X, \Lambda).$$

For example, if X is smooth over $\mathrm{Spec}(k)$, applying this to Y the diagonal of $X \times X$, we can define

$$\mu\mathscr{H}om(A, B) = \mu(R\mathscr{H}om(p_1^* A, p_2^! B)) \in D^b_c(T^* X, \Lambda).$$

1.3.3. *Gysin maps.* See Fulton's book.

2. Proof of the Jacobian criterion of smoothness

Recall: we have proven that if

$$\begin{array}{c} \mathcal{M}_Z^{sm} \\ \big\downarrow f \\ S \end{array}$$

then $\mathbb{F}_\ell$ is f-ULA. It remains to prove that

$$Rf^! \mathbb{F}_\ell$$

is invertible.

2.1. First reduction. It suffices to prove that for any morphism

$$g : S' \longrightarrow \mathcal{M}_Z^{sm}$$

with S' a strictly totally disconnected perfectoid space,

$$g^* Rf^! \mathbb{F}_\ell$$

is invertible. For this, recall that $\mathbb{F}_\ell$ f-ULA implies that the formation of $Rf^! \mathbb{F}_\ell$ is compatible with base change/S. In particular, in the diagram

$$\begin{tikzcd} \mathcal{M}^{sm}_{Z \times_{X_S} X_{S'}} \arrow[r, equal] & \mathcal{M}_Z^{sm} \times_S S' \arrow[r] \arrow[d, "f'"] & \mathcal{M}_Z^{sm} \arrow[d, "f"] \\ & S' \arrow[r, "f \circ g"] \arrow[u, bend left, "s"] & S, \end{tikzcd}$$

where the section s is $(g, \mathrm{Id}_{S'})$, we have

$$g^* Rf^! \mathbb{F}_\ell = s^* Rf'^! \mathbb{F}_\ell.$$

We thus have to prove that

$$s^* Rf'^! \mathbb{F}_\ell$$

is invertible. Up to replacing S by S' we are thus reduced to proving that for any section s of $f : \mathcal{M}_Z^{sm} \to S$,

$$s^* Rf^! \mathbb{F}_\ell$$

is invertible when S is a strictly totally disconnected perfectoid space.

2.2. Construction of the deformation: 1st step. The section s corresponds to a section

$$\begin{tikzcd} Z \arrow[d] \\ X_S \arrow[u, bend right] \end{tikzcd}$$

Let us suppose, to simplify (and this case is sufficient for the application in the article where $Z = (G/P)^{ad} \times X_S$), that, if $S = \mathrm{Spa}(R, R^+)$, via the GAGA morphism of ringed spaces

$$X_S \longrightarrow \mathfrak{X}_{R,R^+}$$

one has

$$Z = \mathfrak{Z}^{ad}$$

where $\mathfrak{Z} \to \mathfrak{X}_{R,R^+}$ is quasi-projective and smooth as a morphism of schemes. The section $X_S \hookrightarrow Z$ then corresponds to a section of $\mathfrak{Z} \to \mathfrak{X}_{R,R^+}$ and we can perform a deformation to the normal cone for this schematical section and then adify it.

At the end we obtain a $\mathbb{G}_m$-equivariant embedding

$$\begin{tikzcd} X_S \times \mathbb{A}^1 \arrow[r, hook] & W \arrow[d] \\ & Z \times \mathbb{A}^1 \end{tikzcd}$$

where

(1) when restricted to $Z \times \mathbb{G}_m$ this is

$$\begin{tikzcd} X_S \times \mathbb{G}_m \arrow[r, hook] & Z \times \mathbb{G}_m \arrow[d, "\mathrm{Id}"] \\ & Z \times \mathbb{G}_m \end{tikzcd}$$

(2) The fiber at $\{0\}$ is

$$\begin{tikzcd} X_S \arrow[r, hook] & C_{X_S} Z \arrow[d] \\ & Z \end{tikzcd}$$

Recall that

$$\mathcal{M}_{\mathbb{A}^1 \times X_S} = \underline{E}_S$$

and we have the formula $M_{Z_1 \times Z_2} = \mathcal{M}_{Z_1} \times_S \mathcal{M}_{Z_2}$.

We can now consider the associated diagram

$$\begin{tikzcd} \mathcal{M}_W \arrow[d, shift right, "g"'] \\ S \times \underline{E} \arrow[u, shift right, bend right, "\sigma"'] \end{tikzcd}$$

This is equivariant with respect to the action of $\underline{E}^\times$.

At the end one has an $\underline{E}^\times$-equivariant diagram with cartesian squares

$$\begin{tikzcd}
BC(s^*T_{Z/X_S}) \arrow[r, hook] \arrow[d, shift left] & \mathcal{M}_W^{sm} \arrow[d, shift right, "g"'] & \mathcal{M}_Z^{sm} \times \underline{E} \smallsetminus \{0\} \arrow[l, hook'] \arrow[d, shift right, "f"'] \\
S \arrow[u, bend left, "\text{zero section}"] \arrow[r, hook, "\mathrm{Id}_S \times \{0\}"] & S \times \underline{E} \arrow[u, shift right, "\sigma"'] & S \times \underline{E} \smallsetminus \{0\} \arrow[l, hook'] \arrow[u, shift right, "s \times \mathrm{Id}_{\underline{E} \smallsetminus \{0\}}"']
\end{tikzcd}$$

2.3. Construction of the deformation: 2nd step. Up to replacing W by a quasi-compact open nbd. of the section $X_S \times \mathbb{B}^1 \hookrightarrow W$ we can suppose we have an $\underline{\mathcal{O}_E} \smallsetminus \{0\}$-equivariant diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, shift left] & \mathcal{M}_W \arrow[d, shift right, "g"'] & \Delta \arrow[l, hook'] \arrow[d, shift right, "f"'] \\
S \arrow[u, bend left, "\text{zero section}"] \arrow[r, hook, "\mathrm{Id}_S \times \{0\}"] & S \times \underline{\mathcal{O}_E} \arrow[u, shift right, "\sigma"'] & S \times \underline{\mathcal{O}_E} \smallsetminus \{0\} \arrow[l, hook'] \arrow[u, shift right, "s \times \mathrm{Id}_{\underline{\mathcal{O}_E} \smallsetminus \{0\}}"']
\end{tikzcd}$$

with $\mathcal{M}_W = \mathcal{M}_W^{sm}$ that is quasi-compact and U an open nbd. of the zero section of $BC(s^*T_{Z/X_S})$. Pulling-back the situation to $\pi^{\mathbb{N} \cup \{+\infty\}} \subset \mathcal{O}_E$ we obtain a $\pi^{\mathbb{N}}$-equivariant diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, shift left] & \mathcal{M}_W \arrow[d, shift right, "g"'] & \Delta \arrow[l, hook'] \arrow[d, shift right, "f"'] \\
S \arrow[u, bend left, "\text{zero section}"] \arrow[r, hook, "\mathrm{Id}_S \times \{0\}"] & S \times \underline{\pi^{\mathbb{N} \cup \{+\infty\}}} \arrow[u, shift right, "\sigma"'] & S \times \underline{\pi^{\mathbb{N}}} \arrow[l, hook'] \arrow[u, shift right, "s \times \mathrm{Id}_{\underline{\pi^{\mathbb{N}}}}"']
\end{tikzcd}$$

2.4. Proof of the Jacobian criterion. Let

$$A = Rg^! \mathbb{F}_\ell \in D_{\text{ét}}(\mathcal{M}_W, \mathbb{F}_\ell)$$

that is $\pi^{\mathbb{N}}$-equivariant. If $i : U \hookrightarrow \mathcal{M}_W^{sm}$, since $\mathbb{F}_\ell$ is g-ULA,

$$i^* A \in D_{\text{ét}}(U, \mathbb{F}_\ell)$$

is étale locally on S isomorphic to $\mathbb{F}_\ell[2d]$ with the trivial $\pi^{\mathbb{N}}$-equivariant structure, where $d = \deg(s^*T_{Z/X_S})$. Since S is strictly totally disconnected we can fix an $E^\times$-equivariant isomorphism

$$\mathbb{F}_\ell[2d] \xrightarrow{\sim} i^* A.$$

Now, we have

$$\varinjlim_{N \geq 0} H^{-2d}(\underbrace{g^{-1}(S \times \underline{\pi^{\mathbb{N}_{\geq N} \cup \{+\infty\}}})}_{\mathcal{M}_W^{\geq N}}, A) \xrightarrow{\sim} H^{-2d}(U, i^* A).$$

We can thus find an integer $N \geq 0$ and a $\pi^{\mathbb{N}}$-equivariant morphism

$$\mathbb{F}_\ell[2d] \longrightarrow A_{|\mathcal{M}_{\overline{W}}^{\geq N}}$$

inducing the given equivariant isomorphism $\mathbb{F}_\ell[2d] \xrightarrow{\sim} i^*A$. Let

$$B \in D_{\text{ét}}(\mathcal{M}_{\overline{W}}^{\geq N}, \mathbb{F}_\ell)$$

be a cone of this morphism ans an $\pi^{\mathbb{N}}$-equivariant object

$$\mathbb{F}_\ell[2d] \longrightarrow A_{|\mathcal{M}_{\overline{W}}^{\geq N}} \longrightarrow B \xrightarrow{+1} .$$

Since $\mathbb{F}_\ell[2d]$ and A (as the Verdier dual of $\mathbb{F}_\ell$, see the stability property of the ULA property under Verdier duality) are g-ULA, B is g-ULA. In particular its Verdier dual is g-ULA and

$$Rg_!\mathbb{D}(B)$$

is constructible, where $\mathbb{D} = \mathbb{D}_{\mathcal{M}_{\overline{W}}^{\geq N}/S\times\underline{\pi^{\mathbb{N}_{\geq N}\cup\{+\infty\}}}}$. Since its fiber at $S = S \times \pi^{+\infty}$ is zero we deduce that, up to replacing N by a bigger integer, we can suppose that

$$Rg_!\mathbb{D}(B) = 0.$$

Applying Verdier duality and using the biduality of B we obtain

$$Rg_*B = 0.$$

We not note $\Delta^{(n)}$ for $n \in \mathbb{N}_{\geq N}$ the fiber of g over $S \times \pi^n$. The $\pi^{\mathbb{N}}$-equivariance property implies that we have

$$\Delta^{(n+1)} \subset \Delta^{(n)}$$

with

$$\bigcap_{n\geq N} \Delta^{(n)} = s(S).$$

We thus have if $f^{(n)} : \Delta^{(n)} \to S$, and C is the fiber of B at $S \times \pi^{\mathbb{N}}$: for all $n \geq N$

$$Rf_*^{(n)}C_{|\mathcal{M}_Z^{(n)}} = 0.$$

We deduce that

$$s^*C = \varinjlim_{n\geq N} Rf_*^{(n)}C = 0$$

and thus

$$\mathbb{F}_\ell[2d] \xrightarrow{\sim} s^*Rf^!\mathbb{F}_\ell.$$

This finishes the proof.

CHAPTER 12

12th Lecture: August 15 2023

1. Local Shimura varieties

1.1. Definition.

1.1.1. *The tower over the reflex field.* Let G over E be as before. Let $[b] \in B(G)$ and $\{\mu\}$ be a conjugacy class cocharacters of $G_{\overline{E}}$ where we have fixed $\overline{E}$ an algebraic closure of E. Let $E_\mu|E$ be the field of definition of $\{\mu\}$, $E_\mu \subset \overline{E}$, and $\breve{E}_\mu$ the completion of the maximal unramified extension of E_μ. For $K \subset G(\mathbb{Q}_p)$ compact open let

$$\mathcal{M}_K(G, b, \mu) \longrightarrow \mathrm{Spa}(\breve{E}_\mu)^\diamond$$

be the associated local Shimura variety. If

$$\mathcal{M}(G, b, \mu) = \varprojlim_K \mathcal{M}_K(G, b, \mu),$$

The v-sheaf $\mathcal{M}(G, b, \mu)$ sends S an $\overline{\mathbb{F}}_q$-perf. space to an untilt $S^\sharp$ over $\breve{E}_\mu$ and an isomorphism

$$\mathscr{E}_{1|X_S \smallsetminus S^\sharp} \xrightarrow{\sim} \mathscr{E}_{b|X_S \smallsetminus S^\sharp}$$

that is meromorphic along the Cartier divisor $S^\sharp \hookrightarrow X_S$ and is of type $\leq \mu$ geometrically fiberwise over S.

This is representable by a locally spatial diamond compactifiable of finite dim. trg. over $\mathrm{Spa}(\breve{E}_\mu)$. There are two morphisms

$$\begin{tikzcd}
 & \mathcal{M}(G, b, \mu) \arrow[loop above, "\underline{G(E)} \times \widetilde{G}_b"] \arrow[dl, "\pi_{dR}"'] \arrow[dr, "\pi_{HT}"] & \\
\mathrm{Gr}_G^{\leq \mu, a} \arrow[ur, bend left, "\underline{G(E)}"] & & \mathrm{Gr}_G^{\leq \mu^{-1}, a} \arrow[ul, bend right, "\widetilde{G}_b"']
\end{tikzcd}$$

where

(1) $\mathcal{M}_K(G, b, \mu) = \underline{K} \backslash \mathcal{M}(G, b, \mu)$,
(2) Gr_G is the B_{dR}-affine Grassmanian and $\mathrm{Gr}^{\leq \mu}$ the closed Schubert cell defined by μ, a spatial diamond proper over $\mathrm{Spa}(\breve{E}_\mu)^\diamond$,

(3) $\widetilde{G}_b \to *$ is the group of automorphisms of $\mathscr{E}_b$ that sends $S \to *$ to $\mathrm{Aut}(\mathscr{E}_{b|X_S})$, this is a semi-direct product
$$\widetilde{G}_b = \underline{G_b(E)} \ltimes \widetilde{G}_b^\circ,$$
(4) π_{dR} is $\widetilde{G}_b$-equivariant and a $\underline{G(E)}$-torsor onto its image, an open subset, the so-called admissible open subset
(5) π_{HT} is $\underline{G(E)}$-equivariant and a $\widetilde{G}_b$-torsor over its image, the sub-locally spatial diamond of $\mathrm{Gr}_G^{\leq \mu^{-1}}$ defined by a locally closed generalizing subset of $|\,\mathrm{Gr}_G^{\leq \mu^{-1}}\,|$. We still call it the admissible subset.

1.1.2. *The Frobenius action.* The preceding picture descends from $\mathrm{Spa}(\breve{E}_\mu)$ to
$$\mathrm{Div}^1_{E_\mu} = \mathrm{Spa}(\breve{E}_\mu)^\diamond / \varphi_{E_\mu}^{\mathbb{Z}}.$$
In fact, given any degree 1 effective divisor D on X_{S,E_μ} its norm $D' = N_{E_\mu/E}$ is a degree 1 Cartier divisor on $X_S := X_{S,E}$ and we can speak about modifications
$$\mathscr{E}_{1|X_S \smallsetminus D'} \xrightarrow{\sim} \mathscr{E}_{b|X_S \smallsetminus D'}.$$
The moduli space $\mathcal{M}_K(G,b,\mu)$ thus descends via $\mathrm{Spa}(\breve{E}_\mu)^\diamond \to \mathrm{Div}^1_{E_\mu}$ to a moduli space

$$\mathrm{Sht}_K(G,b,\mu) \longrightarrow \mathrm{Div}^1_{E_\mu}$$
and more generally
$$\mathrm{Sht}(G,b,\mu) \longrightarrow \mathrm{Div}^1_{E_\mu}.$$

1.1.3. *Coefficients.* The composite
$$\mathcal{M}(G,b,\mu) \xrightarrow{\pi_{dR}} \mathrm{Gr}_G^{\leq \mu} \longrightarrow [L^+G \backslash \mathrm{Gr}_G^{\leq \mu}] \hookrightarrow [L^+G \backslash \mathrm{Gr}_G^{\leq \mu}] = [L^+G \backslash LG / L^+G]$$
and
$$\begin{tikzcd}
\mathcal{M}(G,b,\mu) \arrow[r, "\pi_{HT}"] & \mathrm{Gr}_G^{\leq \mu^{-1}} \arrow[r] & {[L^+G \backslash \mathrm{Gr}_G^{\leq \mu^{-1}}]} \arrow[r, hook] & {[L^+G \backslash \mathrm{Gr}_G]} \arrow[d, equal] \\
& & & {[L^+G \backslash LG / L^+G]} \arrow[d, "\simeq", "g \mapsto g^{-1}"'] \\
& & & {[L^+G \backslash LG / L^+G]}
\end{tikzcd}$$
coincide. This descends to a morphism over $\mathrm{Div}^1_{E_\mu}$

$$\mathrm{Sht}(G,b,\mu) \longrightarrow Hck_G^{\leq \mu}$$

where $Hck_G^{\leq \mu} \to \mathrm{Div}^1_{E_\mu}$ is the closed Schubert strata in the local Hecke stack. This descends to a morphism
$$\big[\underline{G(E)} \times \widetilde{G}_b \backslash \mathrm{Sht}(G,b,\mu)\big] \longrightarrow Hck_G^{\leq \mu}.$$

The geometric Satake correspondence allows us to define a perverse ULA (relative to the morphism toward $\mathrm{Div}^1_{E_\mu}$) sheaf

$$j_{!*}\Lambda \in D_{\text{ét}}(Hck^{\leq\mu}, \Lambda)$$

where j is the inclusion of the open Schubert cell. This corresponds to the representation of $\widehat{G} \rtimes W_{E_\mu}$ with weight μ.

By pull-back we obtain a on object

$$\boxed{S_\mu \in D_{\text{ét}}\Big(\big[\underline{G(E)} \times \widetilde{G}_b \backslash \mathrm{Sht}(G, b, \mu)\big], \Lambda\Big)}$$

1.2. Cohomology.

1.2.1. *The equivariant cohomology complex.* One has

$$\mathrm{Div}^1_{E_\mu} = \mathrm{Spa}(\breve{E}_\mu)^\diamond / \varphi^{\mathbb{Z}}_{E_\mu} = [\mathrm{Spa}(\mathbb{C}_p^\flat)/\underline{W_{E_\mu}}].$$

There is thus a morphism

$$\mathrm{Div}^1_{E_\mu} \longrightarrow [*/\underline{W_{E_\mu}}].$$

Lemma 1.1 (Drinfeld lemma; particular case). *Pull-back induces an equivalence*

$$D(G_b(E) \times W_{E_\mu}, \Lambda) \xrightarrow{\sim} D_{\text{ét}}([*/\widetilde{G}_b] \times [*/W_{E_\mu}], \Lambda) \xrightarrow{\sim} D_{\text{ét}}([*/\widetilde{G}_b] \times \mathrm{Div}^1_{E_\mu}], \Lambda).$$

1.2.2. *The theorem.* Here is the theorem we want to prove. Let

$$f_K : \big[\widetilde{G}_b \backslash \mathrm{Sht}_K(G, b, \mu)\big] \longrightarrow [*/\widetilde{G}_b] \times \mathrm{Div}^1_{E_\mu}$$

Theorem 1.2. *For all compact open subset K of $G(E)$*

$$Rf_{K!}S_\mu \in D(G_b(E), \Lambda)^{BW_{E_\mu}}$$

is a compact object in $D\big(G_b(E), \Lambda\big)$ i.e. an object in the thick triangulated sub-category of $D\big(G_b(E), \Lambda\big)$ generated by the c-$\mathrm{Ind}_{K'}^{G_b(E)}\Lambda$ when K' goes through the set of compact open pro-p subgroups of $G_b(E)$.

2. Local charts using the Jacobian criterion

2.1. Construction of the local charts. Let $[b] \in B(G)$. Suppose, to simplify that G is quasi-split (if not everytuing works since $G \times X_S$ is a quasi-split group scheme over X_S). Let $[\nu_b] \in X_*(A)^+_{\mathbb{Q}}$, M_b its centralizer, a standard Levi, and P_b the standard parabolic subgroup associated. We note b_{M_b} for the canonical reduction of b to M_b.

Definition 2.1. *We note $\mathcal{M}_b$ the small v-stack associating to S a P_b-bundle $\mathscr{E}$ on X_S such that geometrically fiberwise on S, $\mathscr{E} \overset{P_b}{\times} M_b$ is isomorphic to $\mathscr{E}_{b_{M_b}}$.*

There is a cartesian square

$$\begin{tikzcd} \mathcal{M}_b \arrow[r, hook] \arrow[d] & \mathrm{Bun}_{P_b} \arrow[d] \\ [\,*\,/\underline{G_b(E)}] \arrow[r, hook] & \mathrm{Bun}_{M_b} \end{tikzcd}$$

where the right down map sends a P_b-torsor $\mathscr{E}$ on X_S to $\mathscr{E} \overset{P_b}{\times} M_b$. Let

$$\mathrm{Bun}^{\circ}_{P_b} \subset \mathrm{Bun}_{P_b}$$

be the open sub-stack such that $\mathrm{Bun}^{\circ}_{P_b}(S)$ is the groupoid of P_b-bundles $\mathscr{E}$ on X_S such that the vector bundle $\mathscr{E} \overset{P_b,\mathrm{Ad}}{\times} \mathrm{Lie}\, G/\mathrm{Lie}\, P_b$ has geometrically fiberwise on $S > 0$ HN slopes. The weights of $[\nu_b] \in X_* a(A)^+_{\mathbb{Q}}$ on $\mathrm{Lie}\, gG/\mathrm{Lie}\, P_b$ are < 0. FRom this we deduce that

$$\mathcal{M}_b \subset \mathrm{Bun}^{\circ}_{P_b}.$$

Now, the Jacobian criterion of smoothness implies that the morphism

$$\mathrm{Bun}^{\circ}_{P_b} \longrightarrow \mathrm{Bun}_G$$

is cohomologically smooth. We deduce a diagram

$$\begin{tikzcd} \widetilde{\mathcal{M}}_b \arrow[r] \arrow[d, shift left] & \mathcal{M}_b \arrow[r, "\ell\text{-coho. smooth}"] \arrow[d, shift left] \arrow[l, bend right, "\underline{G_b(E)}"'] & \mathrm{Bun}_G \\ * \arrow[r] \arrow[u, shift left] & [\,*\,/\underline{G_b(E)}] \arrow[u, shift left] & \end{tikzcd}$$

where the square is cartesian and defines $\widetilde{\mathcal{M}}_b$. The left vertical section if given by the inclusion $M_p \subset P_b$. Let $K \subset G_b(E)$ be compact open pro-p. We obtain an ℓ-coho. sm. morphism

$$f^b_K : \ \underline{K}\backslash\widetilde{\mathcal{M}}_b \xrightarrow[\text{étale}]{\text{separated}} \mathcal{M}_b \xrightarrow[\text{smooth}]{\ell\text{-coho.}} \mathrm{Bun}_G$$

2.2. Properties of $\widetilde{M}_b$. The following are two key points of the local charts constructed.

THEOREM 2.2. *The v-sheaf $\widetilde{\mathcal{M}}_b$ satisfies*

(1) $\widetilde{\mathcal{M}}_b \smallsetminus \{*\}$ is a spatial diamond

(2) If $i : \{*\} \hookrightarrow \widetilde{\mathcal{M}}_b$, for any $A \in D_{\text{ét}}(\widetilde{\mathcal{M}}_b, \Lambda)$ one has

$$R\Gamma(\widetilde{\mathcal{M}}_b, A) \xrightarrow{\sim} i^* A.$$

3. Some compact generators of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$

DEFINITION 3.1. *For $[b] \in B(G)$ and K a compact open pro-p subgroup of $G_b(E)$ define*

$$A_K^b = Rf_{K!}^b \, Rf_K^{b!} \, \Lambda \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda).$$

EXAMPLE 3.2. *If b is basic then*

$$A_K^b = i_!^b \Big(c - \mathrm{Ind}_K^{G_b(E)} \Lambda \Big)$$

and thus A_K^b corresponds in this case to the standard generator $c - \mathrm{Ind}_K^{G_b(E)} \Lambda$ of $D(G_b(E), \Lambda)$.

PROPOSITION 3.3. *The following is satisfied*

(1) For any $B \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ one has

$$Hom(A_K^b, B) = (i^b)^* B.$$

(2) The collection $\big(A_K^b\big)_{[b],K}$ is a set of compact generators of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$.

$\to$ In particular,

$$D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^\omega = \text{thick triangulated sub-cat. generated by } \big(A_K^b\big)_{[b],K}.$$

4. The compactness criterion

Here is the main theorem.

THEOREM 4.1 (Compacity criterion). *An object $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ is compact iff $\{|b| \mid (i^b)^* A \neq 0\}$ is finite (i.e. A is supported on a qc. open subset of Bun_G) and for all $[b] \in B(G)$,*

$$(i^b)^* A \in D(G_b(E), \Lambda)$$

is compact (i.e. lies in the thick triangulated sub-category generated by the collection $\big(c - \mathrm{Ind}_K^{G_b(E)} \Lambda\big)_{\substack{K \subset G_b(E) \\ \text{open pro-}p}}$).

Proof. Its is evident that A compact implies it is supported on a qc open subset. Let $U \subset \mathrm{Bun}_G$ be such a qc open subset and $A \in D_{\text{ét}}(U, \Lambda)$. Choose $[b] \in |U|$ a closed point. Let $j : U \smallsetminus \{[b]\} \hookrightarrow U$. There is an exact triangle

$$i^b_* i^{b*} A \longrightarrow A \longrightarrow j_! j^* A \xrightarrow{+1} .$$

The functors i^b_* and $j_!$ have right adjoints that commute with arbitrary direct sums (in fact $R(i^b)^!$ is isomorphic to a shift of $(i^b)^*$ since U and $[*/\widetilde{G}_b]$ are ℓ-cohomologically smooth). Thus, at the end we just need to prove, by induction on the cardinality of $|U|$, that if A is compact then j^*A is compact in $D_{\text{ét}}(U \smallsetminus \{[b]\}, \Lambda)$. Since $D_{\text{ét}}(U, \Lambda)^\omega$ is the thick triangulated sub-category generated by the $A^{b'}_{K'}$ with $[b'] \in |U|$ is suffices to prove that for $[b'] \in |U|$ and $K' \subset G_{b'}(E)$ open pro-p,

$$j^* A^{b'}_{K'}$$

is compact. If $[b'] \neq [b]$ one has $j^* A^{[b']}_{K'} = A^{[b']}_{K'}$ and the result is evident. If $[b'] = [b]$ one has

$$j^* A^{[b]}_K = R f^{b,\circ}_{K!} \, R(f^{b,\circ}_K)^! \Lambda$$

that is compact since $\widetilde{\mathcal{M}}^\circ_b := \widetilde{\mathcal{M}}_b \smallsetminus \{*\}$ is spatial and thus quasi-compact. □

→ the key point of this proof is to prove that j^* sends compact objects to compact objects. This is absolutely not evident since Rj_* does not commute with arbitrary direct sums in general since, as said before, Bun_G being not quasi-separated, j is not quasi-compact in general.

5. Stability of compact objects under the action of Hecke correspondences

5.1. Hecke correspondences. Let us consider the 2-category with one object whose 1-morphisms are $\mathrm{Rep}_\Lambda(\widehat{G})$ with composition given by $\otimes_\Lambda$ and whose 2-morphisms are the usual morphisms in $\mathrm{Rep}_\Lambda(\widehat{G})$. There is a morphism of 2-categories from this 2-category to $\mathcal{C}_*$ the category of correspondences over $*$. This is given by the geometric Satake correspondence.

5.2. Stability of compact objects. The following result is formal.

Proposition 5.1. *For any $W \in \mathrm{Rep}(\widehat{G})$, $T_W \in End(D_{\text{ét}}(\mathrm{Bun}_G, \Lambda))$ sends compact objects to compact objects;*

$$T_W : D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^\omega \longrightarrow D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)^\omega.$$

Proof. This is a consequence of the fact that

$$T : \mathrm{Rep}(\widehat{G}) \longrightarrow \mathrm{End}(D_{\text{ét}}(\mathrm{Bun}_G, \Lambda))$$

is a monoidal functor, $T_{W_1 \otimes W_2} = T_{W_1} \circ T_{W_2}$. Thus, by application of T to $\mathbf{1} \to \check{W} \otimes W$ and $\check{W} \otimes W \to \mathbf{1}$ on obtains that $T_{\check{W}}$ is a right adjoint to T_W. Since $T_{\check{W}}$ commutes with arbitrary direct sums we deduce the result. □

6. Proof of the finitess result

It suffices to verify that

$$Rf_{K!}S_\mu \in D(G_b(E), \Lambda)$$

is identified with

$$(i^b)^* T_\mu \Big(i^1_! c\text{-}\mathrm{Ind}_K^{G(E)} \Lambda \Big).$$

Bibliography