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Sur la gerbe de Kaletha, la courbe et l'ensemble de Kottwitz

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Let $E_n|E$ be the degree n unramified extension and $X_n := X \otimes_E E_n$. Let $X'_n := X \otimes_E E(\pi^{1/n})$. There are two morphisms

$$X_n \to \mathfrak{X} \text{ and } X'_n \to \mathfrak{X}.$$

I think the following is true.

Proposition 1. *The category of vector bundles on $\mathfrak{X}$ is equivalent to the category of vector bundles $\mathscr{E}$ on X_n and $\mathscr{E}'$ on X'_n together with an isomorphism of their pullback to $X \otimes_E E_n(\pi^{1/n})$.*

The point would be the following. Consider

$$X_n \times_{\mathfrak{X}} X_n \longrightarrow X_n \times_X X_n = \coprod_\tau X_n$$

where $\tau \in \mathrm{Gal}(E_n|E)$. Now one has, with respect to this decomposition,

$$X_n \times_{\mathfrak{X}} X_n = \coprod_\tau T_\tau$$

where $T_\tau \to X_n$ is the μ_n-torsor of isomorphisms between $\tau^*\mathcal{O}_{X_n}(1)$ and $\mathcal{O}_{X_n}(1)$ as n-roots of $\mathcal{O}_X(1)$. Now, the isomorphism class of this torsor lies in

$$H^1(X_n, \mu_n) = E_n^*/(E_n^*)^n$$

via the identification with $H^1(E_n, \mu_n)$ and Kummer theory.

Lemma 1. *If $\tau = \sigma^i$, the preceding torsor is identified with the torsor of n-roots of π^{-i}.*

Proof. The functor from μ_n-torsors over $\mathrm{Spec}(E_n)$ toward the one over X_n is an equivalence. Via the functors $\varphi\text{-}\mathrm{mod}_L \to \mathrm{Bun}_X$ and $\varphi^n\text{-}\mathrm{mod}_L \to \mathrm{Bun}_{X_n}$, the base change functor $\mathrm{Bun}_X \to \mathrm{Bun}_{X_n}$ is identified with $(D, \varphi) \mapsto (D, \varphi^n)$. Moreover the identification $\tau^*\mathscr{E}(D, \varphi^n) \xrightarrow{\sim} \mathscr{E}(D, \varphi^n)$ is given by $\varphi^i \otimes 1 : (D, \varphi^n) \otimes_{E_n, \sigma^i} E_n \xrightarrow{\sim} (D, \varphi^n)$.

Now, $\mathcal{O}_{X_n}(1)$ corresponds to the isocrystal $(L, \pi^{-1}\varphi^n)$ together with the identification $(L, \pi^{-1}\varphi^n)^{\otimes n} = (L, (\pi^{-1}\varphi)^n)$. Moreover $\tau^*\mathcal{O}_{X_n}(1)$ corresponds to the same but with the preceding identification multiplied by π^{-i}. Thus, our torsor sends the étale E-algebra A to the set of automorphisms of $(L, \pi^{-1}\varphi^n) \otimes_E A$ that respect the preceding identifications. $\square$

Thus,

$$T_\tau = X \otimes_E E_n[T]/(T^n - \pi^{-i}),$$

and in particular, $T_\sigma = X \otimes_E E_n(\pi^{1/n})$.

Now, we have to look at

$$X_n \times_{\mathfrak{X}} X_n \times_{\mathfrak{X}} X_n \longrightarrow \coprod_{\tau_1, \tau_2} X_n$$

which, I think, is equal to

$$\coprod_{\tau_1, \tau_2} T_{\tau_1} \times T_{\tau_2}$$

together with morphisms from T_{τ_1, τ_2} toward T_{τ_1}, T_{τ_2} and $T_{\tau_1\tau_2}$. Try to conclude using this ! Maybe I'm wrong, I don't know, the cocyle condition has to be written carefully.