Sur la gerbe de Kaletha, la courbe et l’ensemble de Kottwitz
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Let $E|E$ be the degree $n$ unramified extension and $X_n := X \otimes_E E_n$. Let $X'_n := X \otimes_E E(\pi^{1/n})$. There are two morphisms

$$X_n \to \mathfrak{X} \text{ and } X'_n \to \mathfrak{X}.$$ 

I think the following is true.

**Proposition 1.** The category of vector bundles on $\mathfrak{X}$ is equivalent to the category of vector bundles $\mathcal{E}$ on $X_n$ and $\mathcal{E}'$ on $X'_n$ together with an isomorphism of their pullback to $X \otimes_E E_n(\pi^{1/n})$.

The point would be the following. Consider

$$X_n \times_{\mathfrak{X}} X_n \to X_n \times_{\mathfrak{X}} X_n = \coprod_{\tau} X_n$$

where $\tau \in \text{Gal}(E_n|E)$. Now one has, with respect to this decomposition,

$$X_n \times_{\mathfrak{X}} X_n = \coprod_{\tau} T_\tau$$

where $T_\tau \to X_n$ is the $\mu_n$-torsor of isomorphisms between $\tau^* \mathcal{O}_{X_n}(1)$ and $\mathcal{O}_{X_n}(1)$ as $n$-roots of $\mathcal{O}_{X}(1)$. Now, the isomorphism class of this torsor lies in

$$H^1(X_n, \mu_n) = E^*_n/(E^*_n)^n$$

via the identification with $H^1(E_n, \mu_n)$ and Kummer theory.

**Lemma 1.** If $\tau = \sigma^i$, the preceding torsor is identified with the torsor of $n$-roots of $\pi^{-i}$.

**Proof.** The functor from $\mu_n$-torsors over Spec$(E_n)$ toward the one over $X_n$ is an equivalence. Via the functors $\varphi_{mod} \to \text{Bun}_X$ and $\varphi_n_{mod} \to \text{Bun}_X$, the base change functor $\text{Bun}_X \to \text{Bun}_{X_n}$ is identified with $(D, \varphi) \to (D, \varphi^n)$. Moreover the identification $\tau^* \mathcal{E}(D, \varphi^n) \sim \mathcal{E}(D, \varphi^n)$ is given by $\varphi^i \otimes 1 : (D, \varphi^n) \otimes_{E_n, \varphi^i} E_n \sim (D, \varphi^n)$.

Now, $\mathcal{O}_{X_n}(1)$ corresponds to the isocrystal $(L, \pi^{-1} \varphi^n)$ together with the identification $(L, \pi^{-1} \varphi^n)^{\otimes n} = (L, (\pi^{-1} \varphi)^n)$. Moreover $\tau^* \mathcal{O}_{X_n}(1)$ corresponds to the same but with the preceding identification multiplied by $\pi^{-i}$. Thus, our torsor sends the étale $E$-algebra $A$ to the set of automorphisms of $(L, \pi^{-1} \varphi^n) \otimes_E A$ that respect the preceding identifications. $\square$

Thus,

$$T_\tau = X \otimes_E E_n[T]/(T^n - \pi^{-i}),$$

and in particular, $T_\sigma = X \otimes_E E_n(\pi^{1/n})$.

Now, we have to look at

$$X_n \times_{\mathfrak{X}} X_n \times_{\mathfrak{X}} X_n \to \coprod_{\tau_1, \tau_2} X_n$$

which, I think, is equal to

$$\coprod_{\tau_1, \tau_2} T_{\tau_1} \times T_{\tau_2}$$

together with morphisms from $T_{\tau_1, \tau_2}$ toward $T_{\tau_1}$, $T_{\tau_2}$ and $T_{\tau_1 \tau_2}$. Try to conclude using this! Maybe I’m wrong, I don’t know, the cocyle condition has to be written carefully.