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Shalom Eliahou

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Divsets, numerical semigroups and Wilf's conjecture

Shalom Eliahou

Abstract

Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity $m = \min(S \setminus \{0\})$ and conductor $c = \max(\mathbb{Z} \setminus S) + 1$. Let P be the set of primitive elements, i.e. minimal generators, of S , and let L be the set of elements of S which are smaller than c . Wilf's conjecture (1978) states that the inequality $|P||L| \geq c$ must hold. The conjecture has been shown to hold in case $|P| \geq m/2$ by Sammartano in 2012, and subsequently in case $|P| \geq m/3$ by the author in 2020. The main result in this paper is that Wilf's conjecture holds in case $|P| \geq m/4$ when m divides c . The proof uses *divsets* X , i.e. finite divisor-closed sets of monomials, as abstract models of numerical semigroups, and proceeds with estimates of the vertex-maximal matching number of the associated graph $G(X)$.

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1 Introduction

A *numerical semigroup* is a cofinite submonoid S of \mathbb{N} , i.e. a subset containing 0, stable under addition and with finite complement $\mathbb{N} \setminus S$. Equivalently, it is a set of the form $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$ where a_1, \dots, a_n are positive integers with $\gcd(a_1, \dots, a_n) = 1$, called *generators* of S . The least such n is usually denoted $e = e(S)$ and called the *embedding dimension* of S . The *multiplicity* of S is $m = m(S) = \min S^*$, where $S^* = S \setminus \{0\}$. The *Frobenius number* of S is $F = F(S) = \max(\mathbb{Z} \setminus S)$ and the *conductor* of S is $c = c(S) = F + 1$, satisfying $c + \mathbb{N} \subseteq S$ and minimal with respect to that property. The *genus* of S is $g = g(S) = |\mathbb{N} \setminus S|$, its number of gaps. We partition S as $S = L \sqcup R$, where $L = L(S) = \{a \in S \mid a < F(S)\}$ and $R = R(S) = \{a \in S \mid a > F(S)\}$, the *left part* and *right part* of S , respectively.

A *primitive element* of S is an element $a \in S^* \setminus (S^* + S^*)$, i.e. an element of S^* which is not the sum of two elements of S^* . We denote by $P = P(S)$ the set of primitive elements of S , and by $D = D(S) = S^* + S^*$ the set of *decomposable elements* of S . It is easy to see that P is contained in $[m, c - 1 + m] \cap \mathbb{N}$ and hence is finite, and is the unique minimal generating set of S . Thus $|P| = e(S)$.

One of the main open problems on numerical semigroups is the following conjecture, first raised as a question by Wilf [15].

Conjecture 1.1 (Wilf, 1978). *Let S be a numerical semigroup. Then $|P(S)||L(S)| \geq c(S)$.*

Denoting $W(S) = |P(S)||L(S)| - c(S)$, Wilf's conjecture amounts to

$$(1) \quad W(S) \geq 0.$$

See [2] for a survey on the conjecture up to 2018. Among many available partial results, we shall need here the following ones, grouped for convenience in a single statement.

Theorem 1.2. *Let S be a numerical semigroup. Then S satisfies Wilf's conjecture if either $|P| \leq 3$, or $|P| \geq m/3$, or $c \leq 3m$.*

The solution in case $|P| = 2$ is due to Sylvester [14]; its extension to $|P| \leq 3$ is due to Fröberg et al. [10]. The solution in case $|P| \geq m/2$ is due to Sammartano [13]; its extension to $|P| \geq m/3$ is achieved in [7]. Finally, the case $c \leq 3m$ is settled in [6].

Notation 1.3. *For $a, b \in \mathbb{Z}$, we denote by $\llbracket a, b \rrbracket = [a, b] \cap \mathbb{Z}$ the integer interval they span.*

The main result in this paper extends the case $|P| \geq m/3$ in Wilf's conjecture as follows.

Theorem 1.4. *Let S be a numerical semigroup such that $|P| \geq m/4$ and $c \in m\mathbb{N}$. Then S satisfies Wilf's conjecture.*

What motivates the added hypothesis $c \in m\mathbb{N}$? As it happens, the proofs of Wilf's conjecture in either case $c \leq 3m$ [6] or $|P| \geq m/3$ [7] can be significantly shortened when $c \in m\mathbb{N}$. Moreover, the first five instances of the very rare "near-misses in Wilf's conjecture" all belong to this case [8]. These facts lead us to consider the case $c \in m\mathbb{N}$ as a priority in research on Wilf's conjecture. Indeed, we believe that if the conjecture fails, then it will already fail in case $c \in m\mathbb{N}$. Whence the following terminology.

Definition 1.5. *A numerical semigroup S is special if its conductor c is a multiple of its multiplicity m , i.e. if $c \in m\mathbb{N}$.*

For instance, the *ordinary* (or *superficial*) numerical semigroup $O_m = \{0\} \cup (m + \mathbb{N})$ is special since it satisfies $c = m$.

See the reference books [11, 12] for more information on numerical semigroups.

1.1 Contents

In Section 2, we introduce divsets as abstract models of Apéry sets of numerical semigroups. In Section 3, we recall a few things about the depth and the functions $W(S), W_0(S)$. In Section 4, we introduce and study the graph of a divset. In Section 5, we start focusing on special numerical semigroups. In Section 6, we settle Wilf's conjecture for special numerical semigroups modeled by divsets of degree 2. In Section 7, we consider the relevant divsets of degree at least 3 and conclude the proof of the main theorem. A short Section 8 contains some concluding remarks.

2 Divsets

In this section, we introduce divsets, namely finite divisor-closed sets of monomials, and use them as abstract multiplicative models of numerical semigroups or, more to the point, of their Apéry sets.

2.1 Basic definitions

Given n commuting variables x_1, \dots, x_n , we denote by $\mathcal{M} = \mathcal{M}(x_1, \dots, x_n)$ the set

$$\mathcal{M} = \langle x_1, \dots, x_n \rangle = \{x_1^{a_1} \cdots x_n^{a_n} \mid (a_1, \dots, a_n) \in \mathbb{N}^n\}$$

of monomials in these variables.

Notation 2.1. The degree of $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}$ is $\deg(u) = \sum_i a_i$, the standard one. For $d \in \mathbb{N}$, we denote by $\mathcal{M}_d = \langle x_1, \dots, x_n \rangle_d$ the subset of monomials of degree d in \mathcal{M} .

Definition 2.2. A divset in \mathcal{M} is a finite subset $X \subset \mathcal{M}$ which is stable under taking divisors. That is, for all $u \in X$ and $v \in \mathcal{M}$, if $v|u$ then $v \in X$.

Said otherwise, a divset is a finite *downset* or *order ideal* in \mathcal{M} under divisibility.

Example 2.3. $X = \{x_1^3, x_1^2 x_2, x_1^2, x_1 x_2, x_1, x_2, 1\}$ is a divset in $\mathcal{M} = \langle x_1, x_2 \rangle$.

Definition 2.4. Let X be a divset. The degree of X is

$$\deg(X) = \max\{\deg(u) \mid u \in X\}.$$

Notation 2.5. Given a finite subset $U \subset \mathcal{M}$, we denote by $[U]$ the set of divisors of the elements of U , i.e. $[U] = \{v \in \mathcal{M} \mid \exists u \in U, v|u\}$. We call $[U]$ the divset spanned by U .

For instance, in Example 2.3, we have $X = \{x_1^3, x_1^2 x_2, x_1^2, x_1 x_2, x_1, x_2, 1\} = [x_1^3, x_1^2 x_2]$.

Notation 2.6. Let X be a divset. We denote by $\max(X)$ the maximal elements of X under divisibility. That is, $\max(X)$ is the set of those $u \in X$ which do not divide any $v \in X \setminus \{u\}$.

Clearly, a divset X is spanned by $\max(X)$, i.e. $X = [\max(X)]$.

Notation 2.7. Let X be a divset in \mathcal{M} . For $d \in \mathbb{N}$, we set $X_d = X \cap \mathcal{M}_d = \{u \in X \mid \deg(u) = d\}$.

For instance, we have $X_0 = \mathcal{M}_0 = \{1\}$ and $X_1 \subseteq \mathcal{M}_1 = \{x_1, \dots, x_n\}$.

Notation 2.8. Let X be a divset. We denote by $D(X) = \{u \in X \mid \deg(u) \geq 2\}$, i.e. the set of decomposable monomials in X . Thus $X = \{1\} \sqcup X_1 \sqcup D(X)$. We set $X^* = X \setminus \{1\}$.

Remark 2.9. Let $X \subset \mathcal{M}(x_1, \dots, x_{n-1})$ be a divset. Let x_n be a new variable. Then $X \cup \{x_n\}$ is a divset in $\mathcal{M}(x_1, \dots, x_n)$. Conversely, if $x_n \in \max(X)$, then $X \setminus \{x_n\}$ is a divset in $\mathcal{M}(x_1, \dots, x_{n-1})$.

2.2 The Apéry set

Let S be a numerical semigroup of multiplicity m and conductor c . Its Apéry set contains key information on its structure. We briefly recall its definition and basic properties.

Definition 2.10. *The Apéry set of S with respect to m is defined as $A = \text{Ap}(S, m) = S \setminus (m + S)$. We denote $A^* = A \setminus \{0\}$.*

It is well known that $\text{Ap}(S, m)$ has exactly m elements, one for each class mod m . More precisely, we have

$$\text{Ap}(S, m) = \{w_0, \dots, w_{m-1} \mid w_i = \min(S \cap (i + m\mathbb{N}))\}.$$

For instance, $\min(A) = w_0 = 0$ and $\max(A) = w_{c-1} = c - 1 + m$. Hence

$$\{0, c - 1 + m\} \subseteq A \subseteq \llbracket 0, c - 1 + m \rrbracket.$$

Moreover, denoting $P^* = P \setminus \{m\}$, it follows from the definition that

$$(2) \quad P^* \subseteq A^*.$$

The numerical semigroup S is completely determined by its Apéry set A via the formula

$$(3) \quad S = \bigsqcup_{a \in A} (a + m\mathbb{N}).$$

Thus, crucial data of a numerical semigroup S may be read on its Apéry set $A = \text{Ap}(S, m)$, e.g. its multiplicity m and conductor c , since $m = |A|$ and $c = \max(A) - m + 1$.

2.3 Divsets models of numerical semigroups

Here we use divsets X as abstract models of numerical semigroups S , indeed of Apéry sets $\text{Ap}(S, m)$, via specific maps $f : X \rightarrow S$.

Definition 2.11. *Let S be a numerical semigroup and let X be a divset. A map $f : X \rightarrow S$ is a morphism if $f(uv) = f(u) + f(v)$ for all $u, v \in X$.*

Of course $f(1) = 0$ for any such morphism, as $f(1) = f(1 \cdot 1) = f(1) + f(1)$.

Definition 2.12. *Let S be a numerical semigroup of multiplicity m . Let $A = \text{Ap}(S, m)$. A divset model of S is a divset X with an injective morphism $f : X \rightarrow S$ such that $f(X) \subseteq A$ and $f(D(X)) = A \cap D$.*

In particular, for all $a \in A \cap D$, i.e. such that $a = a_1 + a_2$ for some $a_1, a_2 \in A^*$, there are unique monomials $u, u_1, u_2 \in X^*$ such that

$$(4) \quad f(u) = a, f(u_1) = a_1, f(u_2) = a_2,$$

and then of course $u = u_1 u_2$.

Remark 2.13. *The purpose of a divset model of S is to capture the structure of $A \cap D$, the set of decomposable elements of the Apéry set of S . Indeed, understanding the structure of $A \cap D$ is one of the keys to advances on Wilf's conjecture $W(S) \geq 0$, as suggested by Proposition 3.9 below. Thus, despite the name, a divset model of S is really an abstract multiplicative model of its Apéry set A , and more specifically of its decomposable elements.*

Proposition 2.14. *Let S be a numerical semigroup of multiplicity m and $A = \text{Ap}(S, m)$. Let $f : X \rightarrow S$ be a divset model of S . Then $|X| \leq m$, $f(X_1) \subseteq P^*$ and $A \setminus f(X) \subseteq P^*$.*

Proof. Since f is injective and $f(X) \subseteq A$, we have $|X| = |f(X)| \leq |A| = m$. Moreover $f(X^*) \subseteq A^*$ since $f(1) = 0$. Let $u \in X_1$. If $f(u) \notin P^*$ then $f(u) \in A \cap D = A^* \setminus P^*$. Since $A \cap D = f(D(X))$ by definition, there exists $v \in D(X)$ such that $f(u) = f(v)$. Hence $u = v$ since f is injective, a contradiction since $\deg(u) = 1$ and $\deg(v) \geq 2$. Finally, since $0 \in f(X)$ and $A \cap D \subseteq f(X)$, we have $A \setminus f(X) \subseteq A^* \setminus (A \cap D) = P^*$. \square

Proposition 2.15. *Let S be a numerical semigroup. Then S admits a divset model.*

Proof. It is well known and easy to verify that, for any decomposable Apéry element $a \in A \cap D$, if $a = s_1 + s_2$ with $s_1, s_2 \in S^*$, then necessarily $s_1, s_2 \in A^*$. For any $a \in A \cap D$, consider the unique decomposition $a = p_{i_1} + \cdots + p_{i_d}$ into primitive elements $p_{i_j} \in P^*$ which is of minimal length d and lexicographically minimal in that length. Clearly, the same minimality properties also hold for any nonempty subsum of $p_{i_1} + \cdots + p_{i_d}$.

Let $\{p_1, \dots, p_n\} \subseteq P^*$ be the set of all primitive elements involved in the respective minimal decompositions of the elements in $A \cap D$. Let $X_1 = \{x_1, \dots, x_n\}$ be a set of n commuting variables. We set $u(p_i) = x_i$ for all $1 \leq i \leq n$. Then, for all $a \in A \cap D$, we associate to a the monomial $u(a)$ of degree d in $\{x_1, \dots, x_n\}$ mirroring, in multiplicative notation, the minimal decomposition $a = p_{i_1} + \cdots + p_{i_d}$. Let

$$X = \{1\} \sqcup X_1 \sqcup \{u(a) \mid a \in A \cap D\}.$$

By the remark on subsums above, if $a = a_1 + a_2$ with $a_1, a_2 \in A^*$, then $u(a) = u(a_1)u(a_2)$. Hence X is a divset. Let $f : X \rightarrow S$ be the unique morphism induced by $f(x_i) = p_i$ for all $i = 1, \dots, n$. Then f is an injective morphism, $f(X) \subseteq A$ and $f(D(X)) = A \cap D$ by construction. Hence $f : X \rightarrow S$ is a divset model of S , as desired. \square

Example 2.16. *Let $S = \langle 5, 6, 9 \rangle$. Then $m = 5$ and $A^* = \{w_1, w_2, w_3, w_4\} = \{6, 12, 18, 9\}$ with $w_i = \min(S \cap (i + 5\mathbb{N}))$ for $1 \leq i \leq 4$. Let*

$$\begin{aligned} X &= [x_1^2, x_2^2] = \{x_1^2, x_2^2, x_1, x_2\}, \\ X' &= [x_1^3, x_2] = \{x_1^3, x_1^2, x_1, x_2\}. \end{aligned}$$

The morphisms $f : X \rightarrow S$ and $f' : X' \rightarrow S$, both induced by $x_1 \mapsto 6, x_2 \mapsto 9$, yield two distinct divset models for S , since then $x_1^2 \mapsto 12$ and $x_1^3, x_2^2 \mapsto 18$.

Remark 2.17. One advantage of divsets X as abstract models of Apéry sets A is that for any $u \in X^*$, a decomposition $u = vw$ with $v, w \in X^*$ is unique up to order, whereas in A^* , decompositions $a = b + c$ are seldom unique in general as seen in the above example. Moreover, with divsets X one can use the terminology of monomials such as degree, divisibility and so on, notions which are less intuitive in additive notation.

Remark 2.18. In a divset model $f : X \rightarrow S$, by removing from X variables $x_i \in X_1$ not dividing any $u \in D(X)$, the resulting subset $X' \subseteq X$ is still a divset and the restriction $f' : X' \rightarrow S$ of f to X' is still a divset model since $D(X') = D(X)$.

Recall that for a divset model $f : X \rightarrow S$, we have $|X| \leq m$ where $m = |A|$ is the multiplicity of S . The case $|X| < m$ may occur if some primitive elements of S are not involved in the decompositions of the elements of $A \cap D$ as sums of primitive elements, and are thus harmlessly ignored by X . This may happen, for instance, for the right primitive elements, i.e. the elements of $P \setminus L$. Whereas the inclusion $f(X_1) \subseteq P^*$ might be strict, a key property of X is that it models *all* elements of $A \cap D$, as shown by the condition $f(D(X)) = A \cap D$. Note the equivalences

$$|X| = m \iff f(X_1) = P^* \iff f(X) = A.$$

3 Depth, $W(S)$, $W_0(S)$

We recall here some material for later use. Throughout this section, S is a numerical semi-group of multiplicity m and conductor c .

Definition 3.1 (See [9]). *The depth of S is the positive integer*

$$q = \lceil c/m \rceil.$$

Thus $c = qm - \rho$ where $\rho \in \llbracket 0, m-1 \rrbracket$.

In fact $\rho \in \llbracket 0, m-2 \rrbracket$. For otherwise, if $\rho = m-1$ then $c \equiv 1 \pmod{m}$, an absurdity since then its Frobenius number $F = c-1 = \max(\mathbb{Z} \setminus S)$ would be a multiple of m .

3.1 The depth function

We keep the same notation as above, namely $c = qm - \rho$ where $\rho \in \llbracket 0, m-2 \rrbracket$. As in [6], we partition S as $S = \sqcup_{i \in \mathbb{N}} S_i$, where for all $i \in \mathbb{N}$,

$$S_i = S \cap \llbracket im - \rho, (i+1)m - \rho - 1 \rrbracket.$$

In particular, we have

$$S_0 = \{0\}, S_1 = \llbracket m, 2m - \rho - 1 \rrbracket \cap S, \dots, S_{q-1} = \llbracket c - m, c - 1 \rrbracket \cap S, S_q = \llbracket c, c - 1 + m \rrbracket.$$

This gives rise to the following function.

Definition 3.2. The depth function $\delta : S \rightarrow \mathbb{Z}$ is defined for all $x \in S$ by

$$(5) \quad \delta(x) = i \iff x \in S_{q-i}.$$

Equivalently, $\delta(x)$ is the unique integer such that

$$(6) \quad x + \delta(x)m \in \llbracket c, c - 1 + m \rrbracket,$$

i.e. $\delta(x) = \lceil \frac{c-x}{m} \rceil$.

The function δ assumes the following values. Recall that $L = S \cap \llbracket 0, c - 1 \rrbracket$.

Lemma 3.3. For all $x \in S$, we have

$$\begin{aligned} \delta(x) = q &\iff x = 0, \\ \delta(x) \in \llbracket 1, q - 1 \rrbracket &\iff x \in L \setminus \{0\}, \\ \delta(x) = 0 &\iff x \in \llbracket c, c - 1 + m \rrbracket, \\ \delta(x) \leq -1 &\iff x \geq c + m. \end{aligned}$$

Proof. Straightforward from the definition. □

We shall need the following estimates from [7, Proposition 6].

Proposition 3.4. For all $x, y \in S$, we have

$$(7) \quad \delta(x+y) + q + 1 \geq \delta(x) + \delta(y) \geq \delta(x+y) + q - \min(\rho, 1).$$

Proof (Outline). First note that for all $k, l \geq 1$, we have

$$S_k + S_l \subset S_{k+l-\min(\rho, 1)} \cup S_{k+l} \cup S_{k+l+1}.$$

Set $\delta(x) = i, \delta(y) = j$. By (5), this means $x \in S_{q-i}, y \in S_{q-j}$. Hence

$$x + y \in S_{2q-i-j-\min(\rho, 1)} \cup S_{2q-i-j} \cup S_{2q-i-j+1}.$$

By (5) again, this means $-q + i + j - 1 \leq \delta(x+y) \leq -q + i + j + \min(\rho, 1)$. The claimed inequalities follow. □

3.2 Total depth

Definition 3.5. Given a finite subset $E \subset S$, the total depth of E is

$$\tau(E) = \sum_{x \in E} \delta(x).$$

Lemma 3.6. Let $A = \text{Ap}(S, m)$, $A^* = A \setminus \{0\}$ and $D = S^* + S^*$. Then

- (1) $\delta(A^*) \subseteq \llbracket 0, q-1 \rrbracket$,
- (2) $m = |P| + |A \cap D|$,
- (3) $|L| = \tau(A) = \tau(A^*) + q$.

Proof. Since $A^* \subseteq \llbracket m, c-1+m \rrbracket$, it follows from Lemma 3.3 that $\delta(A^*) \subseteq \llbracket 0, q-1 \rrbracket$. We have $|A^*| = m-1$ and $A^* = (A \cap P) \sqcup (A \cap D)$. Also $P = (A \cap P) \sqcup \{m\}$. Hence $|P| = |A \cap P| + 1$ and so

$$m = |A^*| + 1 = |A \cap D| + |A \cap P| + 1 = |A \cap D| + |P|.$$

We have $L = S \cap \llbracket 0, c-1 \rrbracket = \{0\} \sqcup (S \cap \llbracket m, c-1 \rrbracket)$. Moreover, for all $a \in A$, we have

$$L \cap (a + m\mathbb{N}) = a + \llbracket 0, \delta(a) - 1 \rrbracket m.$$

Hence $|L| = \sum_{a \in A} \delta(a) = \tau(A)$. Since $\delta(0) = q$, the formula $|L| = q + \tau(A^*)$ follows. \square

3.3 The numbers $W(S), W_0(S)$

The numbers $W(S), W_0(S)$ attached to the numerical semigroup S were introduced in [6]. The alternate notation $E(S) = W_0(S)$ was subsequently used in [1] and elsewhere. Recall the notation $D = S^* + S^*$ and $P = S^* \setminus D$.

Notation 3.7. We denote $D_q = D \cap S_q = D \cap \llbracket c, c-1+m \rrbracket$, and

$$\begin{aligned} W(S) &= |P||L| - c, \\ W_0(S) &= |P \cap L||L| - |A \cap D_q| + \rho. \end{aligned}$$

Thus, Wilf's conjecture amounts to the inequality

$$(8) \quad W(S) \geq 0$$

for all numerical semigroups S . The interest of $W_0(S)$ stems from the inequality

$$W(S) \geq W_0(S).$$

Therefore, if $W_0(S) \geq 0$ then S satisfies Wilf's conjecture in a somewhat stronger sense. For instance, the following result is proved in [6].

Theorem 3.8. *Let S be a numerical semigroup such that $c \leq 3m$. Then $W_0(S) \geq 0$.*

There are cases where $W_0(S) \leq -1$, but those are extremely rare. See [1, 8] for more details. Note finally the equivalence $W_0(S) = W(S) \iff P \subseteq L$.

3.4 New formulas

The following formulas exhibit a closer relationship between $W(S)$ and $W_0(S)$ than the original defining ones in Notation 3.7. In particular, they better show that $W_0(S)$ only takes $P \cap L$ into account and ignores $P \setminus L$. The symbols A^*, P, D keep the same meaning as above. We further denote $P_q = P \cap S_q = P \cap \llbracket c, c-1+m \rrbracket$, $p_q = |P_q|$ and $d_q = |D_q| = |D \cap \llbracket c, c-1+m \rrbracket|$.

Proposition 3.9. *Let S be a numerical semigroup. Then*

$$\begin{aligned} W(S) &= |P|\tau(A^*) - |A \cap D|q + \rho, \\ W_0(S) &= |P \cap L|\tau(A^*) - |A \cap D|q + \rho. \end{aligned}$$

Proof. We have

$$\begin{aligned} W(S) &= |P||L| - c \\ &= |P|(\tau(A^*) + q) - qm + \rho \\ &= |P|(\tau(A^*) + q) - q(|P| + |A \cap D|) + \rho \\ &= |P|\tau(A^*) - |A \cap D|q + \rho. \end{aligned}$$

Moreover,

$$\begin{aligned} W(S) &= |P||L| - c \\ &= (|P \cap L| + p_q)|L| - qm + \rho \\ &= (|P \cap L| + p_q)|L| - q(p_q + d_q) + \rho \\ &= |P \cap L||L| - qd_q + \rho + p_q(|L| - q) \\ &= W_0(S) + p_q(|L| - q) \\ &= W_0(S) + p_q\tau(A^*). \quad \square \end{aligned}$$

Corollary 3.10. *We have*

$$W(S) - W_0(S) = |P_q|\tau(A^*).$$

Proof. Since $|P| = |P \cap L| + |P_q|$, the proposition implies

$$\begin{aligned} W(S) - W_0(S) &= (|P| - |P \cap L|)\tau(A^*) \\ &= |P_q|\tau(A^*). \quad \square \end{aligned}$$

Corollary 3.11. *We have*

$$\begin{cases} W_0(S) \leq W(S), \\ W_0(S) = W(S) \iff P \subseteq L. \end{cases}$$

Proof. Indeed, since $P \subseteq S_1 \sqcup \dots \sqcup S_q$, we have $P_q = \emptyset \iff P \subseteq L$. □

Corollary 3.12. *If S is special of depth q , then $W(S) = |P|\tau(A^*) - |A \cap D|q$. □*

4 The graph of a divset

Let X be a divset in $\mathcal{M} = \mathcal{M}(x_1, \dots, x_n)$. Recall the notation $X^* = X \setminus \{1\}$. We canonically associate to X a graph $G = G(X)$ defined as follows. An *edge* in G is a pair $\{u_1, u_2\} \subseteq X^*$ such that $u_1 u_2 \in X^*$. This defines the edge set $E(G)$. The set $V(G)$ of vertices of G is defined as the set of the extremities of the edges. That is, $V(G) = \{u \in X^* \mid \exists v \in X^*, uv \in X^*\}$. We denote $V(X) = V(G)$ and $E(X) = E(G)$. For enhanced readability, we shall denote edges by (u_1, u_2) rather than $\{u_1, u_2\}$, even though we still consider them as undirected edges. The graph G has no multiple edges, but it may contain loops, namely all pairs (u, u) such that $u, u^2 \in X^*$.

An important measure of X in the sequel is the *vertex-maximal matching number* of the graph $G(X)$, defined below.

Notation 4.1. We set $\text{vm}(X) =$ the largest number of vertices in a matching in $G(X)$, i.e. of vertices touched by pairwise disjoint edges $(u_i, v_i) \in E(G)$.

We now consider a union of divsets.

Lemma 4.2. Let X, Y be divsets in \mathcal{M} . Then $X \cup Y$ is a divset. Moreover, if $X \cap Y = \{1\}$ then $\text{vm}(X \cup Y) = \text{vm}(X) + \text{vm}(Y)$.

Proof. Since both X, Y are divisor-closed, the same holds for $X \cup Y$. Moreover, if $X \cap Y = \{1\}$, then $X^* \cap Y^* = \emptyset$ and so $G(X \cup Y) = G(X) \sqcup G(Y)$. Hence the matchings in $G(X \cup Y)$ are exactly the disjoint unions of a matching in $G(X)$ and a matching in $G(Y)$. \square

4.1 The function $\mu(u_1, \dots, u_r)$

A divset X being spanned by its maximal monomials, it makes sense to introduce a useful auxiliary notation for $\text{vm}(X)$ in terms of spanning monomials.

Notation 4.3. Given monomials $u_1, \dots, u_r \in \mathcal{M}$, we set $\mu(u_1, \dots, u_r) = \text{vm}(X)$, where $X = [u_1, \dots, u_r]$ is the divset spanned by the u_i 's.

Determining $\mu(u_1, \dots, u_r)$ is difficult in general, due to its intimate relationship with matching numbers in graphs. Nevertheless, the case $r = 1$ is straightforward.

Proposition 4.4. Let $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M} \setminus \{1\}$. Then $\mu(u) = \prod_{i=1}^n (a_i + 1) - 2$.

Proof. Let $X = [u]$ be the divset of divisors of u . Then $\mu(u) = \text{vm}(X)$, i.e. the maximal number of vertices touched by a matching of the graph $G = G(X)$. Now $|X| = \prod_{i=1}^n (a_i + 1)$, i.e. the number of divisors of u . These divisors may be regrouped in independent edges of the form $(v, u/v)$ where v is a divisor of u such that $0 \leq \deg(v) \leq \deg(u/v)$, and v is lexicographically smaller than or equal to u/v if $\deg(v) = \deg(u/v)$. The pair $(1, u)$ must be

discounted since 1 is not a vertex of G by definition. The other pairs constitute a matching of G covering the whole of $X \setminus \{1, u\}$. Hence $\mu(u) = \text{vm}(X) = |X \setminus \{1, u\}| = \prod_{i=1}^n (a_i + 1) - 2$. \square

Notation 4.5. For $u \in \mathcal{M}(x_1, \dots, x_n)$, the support of u is $\text{supp}(u) = \{x_i, x_i | u\}$. More generally, for any subset $T \subseteq \mathcal{M}(x_1, \dots, x_n)$, we set $\text{supp}(T) = \cup_{u \in T} \text{supp}(u)$.

Corollary 4.6. Let $u \in \mathcal{M}(x_1, \dots, x_n) \setminus \{1\}$. If $\mu(u) \leq 5$ then $|\text{supp}(u)| \leq 2$. More precisely, if $\mu(u) \leq 5$ then $u \in \{1, x_1, \dots, x_1^6, x_1 x_2, x_1^2 x_2\}$ up to permutation of the variables.

Proof. Clearly, if v_1 divides v_2 in \mathcal{M} , then $\mu(v_1) \leq \mu(v_2)$. Assume $\mu(u) \leq 5$. Since $\mu(x_1 x_2 x_3) = 6$, it follows that $|\text{supp}(u)| \leq 2$. If $u = x_1^a$, then $0 \leq \mu(u) \leq 5 \iff 1 \leq a \leq 6$. And if $u = x_1^a x_2^b$ with $a \geq b \geq 1$, since $\mu(u) = (a+1)(b+1) - 2$, we have $\mu(u) \leq 5 \iff b = 1$ and $a \leq 2$. \square

Corollary 4.7. Let $X \subset \mathcal{M}(x_1, \dots, x_n)$ be a divset such that $\text{vm}(X) \leq 5$. Then $\deg(X) \leq 6$ and $X_k \subseteq \{x_1^k\}$ for $k = 4, 5, 6$ up to permutation. Moreover, $|\text{supp}(u)| \leq 2$ for all $u \in X$.

Proof. Let $u \in X$ be such that $\deg(u) \geq 4$. Since $\mu(u) \leq \text{vm}(X) \leq 5$, the preceding corollary implies $u \in \{x_1^4, x_1^5, x_1^6\}$. In particular, $\deg(X) \leq 6$. Moreover, since $\mu(x_1^4, x_2^4) \geq 6$ due to the matching $\{(x_1, x_1^3), (x_2, x_2^3), (x_1^2, x_1^2), (x_2^2, x_2^2)\}$ on 6 vertices in $G(X)$, it follows that $X_k \subseteq \{x_1^k\}$ for all $k = 4, 5, 6$ up to permutation of the variables, as stated. Finally, let $u \in X$. Since $\mu(u) \leq \text{vm}(X) \leq 5$, it follows from Corollary 4.6 that $|\text{supp}(u)| \leq 2$. \square

We shall need the following bounds in the sequel, mostly in case $|\text{supp}(u)| = 1$ or 2.

Proposition 4.8. Let $u \in \mathcal{M}(x_1, \dots, x_n) \setminus \{1\}$. Then $\mu(u) \geq |\text{supp}(u)|(\deg(u) - 1)$.

Proof. Let $u = \prod_{i=1}^n x_i^{a_i}$ with $a_i \in \mathbb{N}$ for all i . Then $\mu(u) = \prod_{i=1}^n (a_i + 1) - 2$. Let us define $f(u) = |\text{supp}(u)|(\deg(u) - 1)$. We shall prove $\mu(u) \geq f(u)$. Without loss of generality, we may and will assume $|\text{supp}(u)| = n$, i.e. $a_i \geq 1$ for all $1 \leq i \leq n$. In that case, the inequality $\mu(u) \geq f(u)$ translates to

$$(9) \quad \prod_{i=1}^n (a_i + 1) - 2 \geq n \left(\sum_{i=1}^n a_i - 1 \right).$$

We proceed by induction on n . If $n = 1$ the inequality is an equality. Assume now $n \geq 2$ and the inequality true for $n - 1$. As u is divisible by $x_1 \cdots x_n$ by hypothesis, we proceed by induction on the degree of $u / (x_1 \cdots x_n)$. If $u = x_1 \cdots x_n$ then (9) reduces to

$$2^n \geq n(n - 1) + 2.$$

This holds for all $n \geq 1$, whence (9) holds for $x_1 \cdots x_n$. Let now $u = \prod_{i=1}^n x_i^{a_i}$ with $a_i \geq 1$ for all i and assume that (9) holds for u . Up to symmetry, it suffices to prove that (9) holds for ux_n . Let $v = \prod_{i=1}^{n-1} x_i^{a_i}$, so that $u = vx_n^{a_n}$. As easily seen by direct computation, we have

$$\begin{aligned} \mu(ux_n) - \mu(u) &= \mu(v) + 2, \\ f(ux_n) - f(u) &= n. \end{aligned}$$

Thus, in order to prove that (9) holds for ux_n , it suffices to show that

$$(10) \quad \mu(v) + 2 \geq n.$$

By the induction hypothesis, and since $\deg(v) \geq n - 1$, we have

$$\begin{aligned} \mu(v) &\geq f(v) \\ &= (n - 1)(\deg(v) - 1) \\ &\geq (n - 1)(n - 2). \end{aligned}$$

Hence (10) holds since $(n - 1)(n - 2) \geq n - 2$ and we are done. \square

5 Focus on the special case

Throughout this section, S denotes a special numerical semigroup of multiplicity m , conductor c and depth $q = \lceil c/m \rceil = c/m$. As above, $A = \text{Ap}(S, m)$ and $A^* = A \setminus \{0\}$.

5.1 A lower bound on $W(S)$

With Wilf's conjecture $W(S) \geq 0$ in the background, the next result yields a lower bound on $W(S)$ via a divset model X of S . Recalling Notation 4.1, to X we associate $\text{vm}(X)$, i.e. the vertex-maximal matching number of the graph $G(X)$. Recall also the formula

$$W(S) = |P|\tau(A^*) - |A \cap D|q$$

of Proposition 3.9, since $\rho = qm - c = 0$ here.

Theorem 5.1. *Let S be a special numerical semigroup of depth q . Let X be a divset model of S and let $n = |X_1|$. Then*

- (1) $\tau(A^*) \geq \text{vm}(X)q/2$,
- (2) $W(S) \geq ((n + 1)\text{vm}(X) - 2|D(X)|)q/2$.

Proof. By hypothesis, there is an injective morphism $f : X \rightarrow S$ such that $f(X) \subseteq A$ and $f(D(X)) = A \cap D$. Let $G = G(X) = (V, E)$ be the graph of X . Set $k = \text{vm}(X)$. Hence, there is a subset $M \subseteq V \subseteq X^*$ of cardinality $|M| = k$ such that

$$M = \bigsqcup_{i=1}^{\ell} z_i,$$

where $z_1, \dots, z_\ell \in E$ are pairwise disjoint edges – here considered as subsets of V . Loops are allowed, so $|z_i| \in \{1, 2\}$ for all i . We have

$$(11) \quad |M| = k = |z_1| + \dots + |z_\ell|.$$

Let $z \in \{z_1, \dots, z_\ell\}$. Then $z = \{u_1, u_2\}$ with $u_1, u_2 \in X^*$ such that $u_1 u_2 \in X^*$. Let

$$a_1 = f(u_1), a_2 = f(u_2), a = f(u_1 u_2).$$

Then $a = a_1 + a_2$ since f is a morphism, and $a \in A^*$ since $f(X^*) \subseteq A^*$. We have $\delta(a) \geq 0$ since $\tau(A^*) \subseteq \llbracket 0, q-1 \rrbracket$ by Lemma 3.6. Hence

$$\delta(a_1) + \delta(a_2) \geq q + \delta(a_1 + a_2) = q + \delta(a) \geq q$$

by the right inequality in (7) and the value $\rho = 0$. We have $f(z) = \{a_1, a_2\}$ and so

$$(12) \quad \tau(f(z)) = \delta(a_1) + \delta(a_2) \geq q.$$

If $|z| = 1$, then $a_1 = a_2$ and $\delta(a_1) \geq q/2$. Therefore, whether $|z| = 1$ or $|z| = 2$, in either case we have

$$\tau(f(z)) \geq |z|q/2.$$

Since this holds for all $z \in \{z_1, \dots, z_\ell\}$, it follows that

$$\tau(f(\{z_1, \dots, z_\ell\})) = \sum_{i=1}^{\ell} \delta(f(z_i)) \geq \left(\sum_{i=1}^{\ell} |z_i| \right) q/2.$$

Since $\sum_{i=1}^{\ell} |z_i| = |M| = k$ by (11), it follows that $\tau(f(\{z_1, \dots, z_\ell\})) \geq kq/2$. And since

$$f(\{z_1, \dots, z_\ell\}) \subseteq f(X^*) \subseteq A^*,$$

we conclude that

$$\tau(A^*) \geq \tau(f(X^*)) \geq \tau(f(\{z_1, \dots, z_\ell\})) \geq kq/2 = \text{vm}(X)q/2,$$

as desired.

It remains to prove the formula $W(S) \geq ((n+1)\text{vm}(X) - 2|D(X)|)q/2$. By Proposition 3.9, we have $W(S) = |P|\tau(A^*) - |A \cap D|q + \rho$. Since $\tau(A^*) \geq \text{vm}(X)q/2$ and $\rho = 0$, this yields $W(S) \geq (|P|\text{vm}(X)/2 - |A \cap D|)q$. Finally, since $|P| \geq |X_1| + 1 = n + 1$ and $|A \cap D| = |f(D(X))| = |D(X)|$, the desired inequality follows. \square

Corollary 5.2. *Let S be a special numerical semigroup such that $|P| \geq m/4$. Let X be a divset model of S . If $\text{vm}(X) \geq 6$ then S satisfies Wilf's conjecture.*

Proof. Assume $\text{vm}(X) \geq 6$. Let q be the depth of S , so that $c = qm$. Theorem 5.1 implies $\tau(A^*) \geq \text{vm}(X)q/2 \geq 3q$, whence $|L| \geq 4q$ since $|L| = \tau(A^*) + q$ by Lemma 3.6 (3). Thus $|P||L| \geq (m/4)(4q) = qm = c$, i.e. $W(S) \geq 0$ as claimed. \square

5.2 Tame divsets

Definition 5.3. Let X be a divset and let $n = |X_1|$. We say that X is tame if

$$2|D(X)| \leq (n+1) \text{vm}(X),$$

and wild otherwise.

Proposition 5.4. Let S be a special numerical semigroup. Let $f: X \rightarrow S$ be a divset model of S . If X is tame, then $W_0(S) \geq 0$ and hence S satisfies Wilf's conjecture.

Proof. Set $X_1 = \{x_1, \dots, x_n\}$. Since X is tame, we have $(n+1) \text{vm}(X) \geq 2|D(X)|$.

Claim. We have $f(V_1) \subseteq P^* \cap L$. Indeed, $f(V_1) \subseteq f(X_1) \subseteq P^*$ by Proposition 2.14. Moreover, if $x_i \in V_1$, there exists $u \in X^*$ such that $x_i u \in X^*$. Hence

$$f(x_i) + f(u) = f(x_i u) \in f(X^*) \subseteq A^* \subseteq \llbracket m, c-1+m \rrbracket.$$

Since $f(u) \in A^*$ whence $f(u) \geq m$, it follows that $f(x_i) \leq c-1$, i.e. $f(x_i) \in L$. This settles the claim.

As $c = qm$ with $q \geq 2$, we have $P \cap L = (P^* \cap L) \sqcup \{m\}$. Since f is injective, the claim implies

$$|P \cap L| = |P^* \cap L| + 1 \geq |f(V_1)| + 1 \geq |X_1| + 1 = n + 1.$$

Now $|A \cap D| = |f(D(X))| = |D(X)|$, and $\tau(A^*) \geq \text{vm}(X)q/2$ by Theorem 5.1.

Summarizing, and using the formula for $W_0(S)$ in Proposition 3.9, we have

$$\begin{aligned} W_0(S) &= |P \cap L| \tau(A^*) - |A \cap D| q \\ &\geq (n+1) \text{vm}(X) q/2 - |A \cap D| q \\ &= ((n+1) \text{vm}(X) - 2|D(X)|) q/2. \end{aligned}$$

Now $(n+1) \text{vm}(X) - 2|D(X)| \geq 0$ since X is tame. Hence $W_0(S) \geq 0$. □

Almost all of the divsets involved in the proof of the main result in this paper turn out to be tame. But here is an example of a wild divset.

Example 5.5. Let $X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3]$, i.e. the set of all monomials of degree at most 3 in x_1, x_2 . Then $|X^*| = |X_1| + |D(X)| = 2 + (3+4) = 9$. Here $n = |V_1| = 2$ and $|D(X)| = 7$. Moreover, $\text{vm}(X) = 4$ as witnessed by the vertex-maximal matching $\{(x_1, x_1^2), (x_2, x_2^2)\}$ of the graph $G(X)$. Hence

$$(n+1) \text{vm}(X) - 2|D(X)| = 3 \cdot 4 - 2 \cdot 7 = -2,$$

so that X is wild. However, it can be shown that for any special numerical semigroup S modeled by X , one has $|P| \geq 7$, whence

$$W(S) = |P| \tau(A^*) - |A \cap D| q \geq 7 \cdot \text{vm}(X) q/2 - |D(X)| q = 14q - 7q = 7q.$$

Thus, all such numerical semigroups S satisfy Wilf's conjecture. See also the end of Section 7.6. Interestingly, those S include the five smallest numerical semigroups satisfying $W_0(S) \leq -1$ as described in [8].

Remark 5.6. It follows from Corollary 5.2 that, in order to settle Wilf's conjecture for special numerical semigroups S satisfying $|P| \geq m/4$, we need only consider divsets X satisfying $\text{vm}(X) \leq 5$. In the sequel, we will classify and analyze all divsets X satisfying this strong restriction on $\text{vm}(X)$. But before that, let us settle the independent case $\deg(X) = 2$.

6 The case $\deg(X) = 2$

We prove here that divsets of degree 2 are tame. By Proposition 5.4, this establishes Wilf's conjecture for all special numerical semigroups S admitting a divset model X of degree 2, i.e. such that $A \cap D = P^* + P^*$. For the proof, we need the following lemma about the classical matching number in graphs, i.e. the maximum number of independent edges.

Lemma 6.1. *Let $H = (H_1, H_2)$ be a simple bipartite graph without isolated vertices, with matching number k . Then*

$$|E(H)| \leq k \cdot \max(|H_1|, |H_2|).$$

Proof. We may assume $|H_1| \leq |H_2|$. Then $k \leq |H_1|$. Let M be a maximal matching with k edges, say

$$M = \{(x_i, y_i) \mid x_i \in H_1, y_i \in H_2, 1 \leq i \leq k\}.$$

Claim. $|H_1| = k$. Assume not. Let then $x_{k+1} \in H_1 \setminus \{x_1, \dots, x_k\}$. Since H does not have isolated vertices, it follows that x_{k+1} has a neighbour $z \in H_2$. Now necessarily $z \in \{y_1, \dots, y_k\}$, for otherwise there would be a new edge (x_{k+1}, z) independent of M , which would then yield a matching of cardinality $k+1$, a contradiction. Up to renumbering, we may assume $z = y_k$, i.e.

$$(x_{k+1}, y_k) \in E.$$

As $|H_2| \geq |H_1| \geq k+1$, there is a vertex $y_{k+1} \in H_2 \setminus \{y_1, \dots, y_k\}$. Since y_{k+1} is not isolated, it has a neighbour $z \in H_1$. But as in the preceding reasoning, we have $z \in \{x_1, \dots, x_k\}$, for otherwise the edge (z, y_{k+1}) would be independent of M , a contradiction. Up to renumbering, we may assume $z = x_k$. Thus

$$(x_k, y_{k+1}) \in E.$$

But then, by suppressing the edge (x_k, y_k) of M and replacing it by the two independent edges $(x_{k+1}, y_k), (x_k, y_{k+1})$, we obtain a matching

$$M' = M \sqcup \{(x_{k+1}, y_k), (x_k, y_{k+1})\} \setminus \{(x_k, y_k)\}$$

of cardinality $k+1$, a contradiction. This finishes the proof of the claim, whence

$$|H_1| = k.$$

Since $H = (H_1, H_2)$ is bipartite, we have $|E| \leq |H_1||H_2| = k|H_2|$, as claimed. \square

Theorem 6.2. *Let X be a divset of degree 2, i.e. such that $D(X) = X_2$. Then X is tame.*

Proof. Let V be the vertex set of the graph $G(X)$. Since the vertices of $G(X)$ are the strict factors of the elements of $D(X)$ by construction, and since $D(X) = X_2$ by hypothesis, it follows that $V \subseteq X_1$. Without loss of generality, we may assume $V = X_1$, since the elements of $X_1 \setminus V$ are ignored by $G(X)$. Set $V = X_1 = \{x_1, \dots, x_n\}$. Then $|D(X)| \leq n(n+1)/2$. Let $M \subseteq D(X)$ be a maximal set of pairwise coprime monomials. Denote by $V_2 \subseteq V_1$ the set of variables involved in M . Let $n_2 = |V_2|$. Since M corresponds to a matching in $G(X)$ with vertex set V_2 , we have

$$\text{vm}(X) \geq n_2.$$

Case 1. $V_2 = V_1$. In this case we are done. Indeed, we then have $n_2 = n$, so that $\text{vm}(X) \geq n$.

And since $|D(X)| \leq n(n+1)/2$, it follows that

$$(n+1)\text{vm}(X) - 2|D(X)| \geq (n+1)n - 2n(n+1)/2 = 0,$$

as desired.

Case 2. Assume $V_2 \neq V_1$. Let $V_3 = V_1 \setminus V_2$ and $n_3 = |V_3| = n - n_2$. By maximality of M , every monomial $u = x_i x_j \in D(X)$ with $i \leq j$ satisfies $x_i \in V_2$ or $x_j \in V_2$. Let $N \subseteq D(X)$ be a largest possible subset of pairwise coprime monomials $x_i x_j$ with $x_i \in V_2, x_j \in V_3$. Let $k = |N|$. Then k is the matching number of $G(X)$, i.e. the largest possible cardinality of a set of independent edges. We have

$$k \leq \min(n_2, n_3)$$

and $k \geq 1$ since $V_3 \neq \emptyset$. Each $x_i x_j \in N$ independently contributes a summand 2 to $\text{vm}(X)$. Thus

$$(13) \quad \text{vm}(X) \geq (n_2 - k) + 2k = n_2 + k.$$

Next, we claim that

$$(14) \quad |D(X)| \leq n_2(n_2 + 1)/2 + k \max(n_2, n_3).$$

Indeed, as V_3 induces the empty subgraph in $G(X)$, every monomial in $D(X)$ has its support in $[V_2, V_2]$ or $[V_2, V_3]$. Now the preceding lemma implies that the number of monomials with support in $[V_2, V_3]$ is less than or equal to $k \max(n_2, n_3)$. This proves (14). Combined with (13), this yields

$$(n+1)\text{vm}(X) - 2|D(X)| \geq (n+1)(n_2 + k) - n_2(n_2 + 1) - 2k \max(n_2, n_3).$$

It remains to show that the right-hand side is non-negative. Since $n = n_2 + n_3$, we have

$$\begin{aligned} (n+1)(n_2 + k) - n_2(n_2 + 1) - 2k \max(n_2, n_3) &= n_2(n+1) - n_2(n_2 + 1) + k(n+1 - 2 \max(n_2, n_3)) \\ &= n_2 n_3 + k(n_2 + n_3 + 1 - 2 \max(n_2, n_3)) \\ &= n_2 n_3 + k(\min(n_2, n_3) - \max(n_2, n_3) + 1) \end{aligned}$$

since $n_2 + n_3 = \min(n_2, n_3) + \max(n_2, n_3)$. But for any $k \leq a \leq b$, we have

$$\begin{aligned} ab + k(a - b + 1) &= k(a + 1) + b(a - k) \\ &\geq k(a + 1). \end{aligned}$$

Hence $(n + 1)(n_2 + k) - n_2(n_2 + 1) - 2k \max(n_2, n_3) \geq k(\min(n_2, n_3) + 1) \geq 0$, as desired. \square

Corollary 6.3. *Let S be a special numerical semigroup such that $A \cap D \subseteq P^* + P^*$. Then S satisfies Wilf's conjecture.*

Proof. Let $f : X \rightarrow S$ be a divset model. Let $G = (V, E)$ be the graph of X . By Remark 2.18, we may assume $X_1 \subseteq V$. Moreover, since $f(D(X)) = A \cap D \subseteq P^* + P^*$, we may assume that $D(X) \subseteq X_1 \cdot X_1$, i.e. that $\deg(X) = 2$. Whence X is tame by Theorem 6.2 and so $W_0(S) \geq 0$ by Proposition 5.4. \square

7 The case $\deg(X) \geq 3$

The case $\deg(X) = 2$ having been settled in Section 6, from here on we only consider divsets X satisfying $\deg(X) \geq 3$. Moreover, towards our main result, we only need consider divsets X such that $\text{vm}(X) \leq 5$ as observed in Remark 5.6. Hence we may assume $\deg(X) \leq 6$. Indeed, if $\deg(X) \geq 7$ then $\text{vm}(X) \geq 6$ by Proposition 4.4, whence Remark 5.6 applies. We shall also need Corollary 4.6, according to which

$$(15) \quad \mu(u) \leq 5 \implies u \in \{1, x_1, \dots, x_1^6, x_1 x_2, x_1^2 x_2\}$$

for all $u \in \mathcal{M}(x_1, \dots, x_n)$ up to permutation of the variables. Thus, a divset $X \subset \mathcal{M}(x_1, \dots, x_n)$ such that $\text{vm}(X) \leq 5$ only contains monomials of the above form.

In this section, all statements and proofs on divsets X in $\mathcal{M}(x_1, \dots, x_n)$ are understood *up to permutation of the variables x_i* , almost always tacitly so; occasionally, we may say “up to permutation”. Moreover, for bounds on $\text{vm}(X)$ and on $\mu(u)$ for $u \in X$, we shall constantly use Lemma 4.2, Proposition 4.4 and Proposition 4.8, here again almost always tacitly so. We shall also constantly use that $\text{vm}(X) \geq \text{vm}(Y)$ for any subdivset $Y \subseteq X$, including the particular case $\text{vm}(X) \geq \mu(u)$ for any $u \in X$.

7.1 On divsets X satisfying $\text{vm}(X) \leq 5$

As observed above, for our main result we only need to consider divsets X such that $\text{vm}(X) \leq 5$. Here is a first restriction used very often in the sequel.

Lemma 7.1. *Let $X \subset \mathcal{M}(x_1, \dots, x_n)$ be a divset such that $\text{vm}(X) \leq 5$. Then either $|\text{supp}(X_3)| \leq 2$ or $X_3 \subseteq x_1^2 X_1$ up to permutation. In particular, X_3 contains at most two cubes.*

Proof. Assume that $|\text{supp}(X_3)| \geq 3$ and that X_3 contains two monomials u_1, u_2 such that x_1^2 divides u_1 but not u_2 . Therefore, up to permutation of the variables, X_3 contains one of the following subsets:

$$\{x_1^3, x_2^3, x_3^3\}, \{x_1^3, x_2^2 x_3\}, \{x_1^2 x_2, x_1 x_3^2\}, \{x_1 x_2^2, x_1 x_3^2\}.$$

However, as $\text{vm}(X) \leq 5$ by hypothesis, none of these subsets is allowed in X . Indeed, in each case we have a matching covering 6 vertices:

- $\mu(x_1^3, x_2^3, x_3^3) = 6$ due to the matching $(x_1, x_1^2), (x_2, x_2^2), (x_3, x_3^2)$, or by Lemma 4.2.
- $\mu(x_1^3, x_2^2 x_3) = 6$ due to the matching $(x_1, x_1^2), (x_2, x_2 x_3), (x_3, x_3^2)$.
- $\mu(x_1^2 x_2, x_1 x_3^2) = 6$ due to the matching $(x_1, x_1 x_2), (x_2, x_1^2), (x_3, x_1 x_3)$.
- $\mu(x_1 x_2^2, x_1 x_3^2) = 6$ due to the matching $(x_1, x_1^2), (x_2, x_1 x_2), (x_3, x_1 x_3)$.

This concludes the proof. □

The relevant cases $\text{vm}(X) = 2, 3, 4, 5$ will be successively considered in the next sections.

7.2 The case $\text{vm}(X) = 2$

Proposition 7.2. *Let X be a divset in $\mathcal{M}(x_1, \dots, x_n)$ such that $\text{vm}(X) = 2$ and $\deg(X) \geq 3$. Then $|D(X)| \leq n + 1$ and X is tame.*

Proof. Let $u \in X$ be such that $\deg(u) \geq 3$. We have $\mu(u) \leq \text{vm}(X) = 2$. On the other hand, the inequality $\mu(u) \geq |\text{supp}(u)|(\deg(u) - 1)$ of Proposition 4.8 implies $\mu(u) \geq \deg(u) - 1 \geq 2$. Thus $\mu(u) = 2$, $\deg(u) = 3$ and $u = x_1^3$ up to permutation. Hence $D(X) = \{x_1^3\} \sqcup X_2$ and, since $\text{vm}(X) = 2$, all monomials in X_2 must be divisible by x_1 . That is, up to permutation we have

$$X = [x_1^3, x_1 x_2, \dots, x_1 x_k]$$

for some $k \leq n$. Hence $|D(X)| = |X_3| + |X_2| = 1 + k \leq n + 1$. Therefore X is tame since $2|D(X)| = 2(k + 1) \leq (n + 1)\text{vm}(X)$. □

7.3 The case $\text{vm}(X) = 3$

Proposition 7.3. *Let X be a divset in $\mathcal{M}(x_1, \dots, x_n)$ such that $\text{vm}(X) = 3$ and $\deg(X) \geq 3$. Then $|D(X)| \leq n + 2$ and X is tame.*

Proof. We have $\deg(X) \leq 4$. Indeed, monomials u of degree $\deg(u) \geq 5$ satisfy $\mu(u) \geq 4$ by Proposition 4.8, and hence are forbidden in X as $\text{vm}(X) = 3$.

Case 1: $\deg(X) = 3$. The set X_3 cannot contain $x_1^2 x_2$ since $\mu(x_1^2 x_2) = 4 > \text{vm}(X)$. Similarly, X_3 cannot contain $\{x_1^3, x_2^3\}$ since $\mu(x_1^3, x_2^3) = 4$. It follows that $X_3 = \{x_1^3\}$. We now consider

X_2 . It contains x_1^2 since X is divisor-closed. As $\mu(x_1^3, x_2x_3) = 4$ and $\mu(x_1^3, x_2^2, x_3^2) = 4$, we must have

$$X_2 \subseteq \{x_1^2, x_1x_2, \dots, x_1x_n, x_2^2\}.$$

Equality is possible. Indeed, let $X = [x_1^3, x_1x_2, \dots, x_1x_n, x_2^2]$. Then $\text{vm}(X) = 3$, since a matching of $G(X)$ on 3 vertices is given by the edges $\{(x_1, x_1^2), (x_2, x_2)\}$, and this matching is vertex-maximal since every edge of $G(X)$ besides (x_2, x_2) touches x_1 . Concluding the case $\text{vm}(X) = 3$ and $\deg(X) = 3$, we have

$$|D(X)| = |X_3| + |X_2| \leq 1 + (n + 1).$$

It follows that X is tame since $2|D(X)| \leq 2(n + 2) \leq 3(n + 1) = \text{vm}(X)(n + 1)$.

Case 2: $\deg(X) = 4$. Then $X_4 = \{x_1^4\}$ by Corollary 4.6 and (15). As above, we still have $X_3 = \{x_1^3\}$ and $X_2 \subseteq \{x_1^2, x_1x_2, \dots, x_1x_n, x_2^2\}$. But here, x_2^2 is excluded from X since $\mu(x_1^4, x_2^2) = 4$. It follows that $X \subseteq [x_1^4, x_1x_2, \dots, x_1x_n]$. Equality is possible, since then $\text{vm}(X) = 3$ as witnessed by the vertex-maximal matching $\{(x_1, x_2), (x_1^2, x_1^2)\}$. Here

$$|D(X)| = |X_4| + |X_3| + |X_2| \leq 1 + 1 + (n) = n + 2$$

again, whence X is tame. □

Corollary 7.4. *Let S be a special numerical semigroup such that $|P| \geq m/3$. Then $W(S) \geq 0$.*

This result is already known in the general case, without assuming that S is special [7]. But the proof below in the special case $c \in m\mathbb{N}$ is much shorter than in the general case as given in [7].

Proof. We proceed as in the proof of Corollary 5.2. Let X be a divset model of S . If $\text{vm}(X) \geq 4$, then Theorem 5.1 implies $\tau(A^*) \geq \text{vm}(X)q/2 \geq 2q$, whence $|L| \geq 3q$ by Lemma 3.6 (3), implying $|P||L| \geq (m/3)(3q) = qm = c$, i.e. $W(S) \geq 0$ as desired. Therefore, we only need consider the case $\text{vm}(X) = 3$, which is settled above. Since X is then tame, it follows from Proposition 5.4 that $W(S) \geq 0$. □

7.4 The case $\text{vm}(X) = 4$

Proposition 7.5. *Let X be a divset on n variables such that $\text{vm}(X) = 4$. Then $|D(X)| \leq 2n + 3$.*

Proof. Given $d \geq 3$, let X be a divset of maximal cardinality such that $X_1 = \{x_1, \dots, x_n\}$, $\deg(X) = d$ and $\text{vm}(X) = 4$. We have $d \leq 5$, for otherwise, if $u \in X$ satisfies $\deg(u) \geq 6$, then $\text{vm}(X) \geq \mu(u) \geq 5$.

Case 1: $d = 3$. We claim that $|D(X)| = 2n + 3$. We distinguish three subcases.

Case 1.1: $|\text{supp}(X_3)| = 1$. Then $X_3 = \{x_1^3\}$. We now consider X_2 . We have $X_2 \setminus x_1X_1 \neq \emptyset$, since $\text{vm}(\{\{x_1^3\} \cup x_1X_1\}) = 2$ only. Let $u \in X_2 \setminus x_1X_1$ with $|\text{supp}(u)|$ maximal, namely 1 or 2.

If $|\text{supp}(u)| = 1$ then $u = x_2^2$ up to permutation, and $X_2 \setminus x_1 X_1 = \{x_2^2, x_3^2\}$ in order to attain $\text{vm}(X) = 4$ precisely. In that case we have $X = [x_1^3, x_1 x_2, \dots, x_1 x_n, x_2^2, x_3^2]$. It satisfies all the constraints, including $\text{vm}(X) = 4$. Here $|X_3| = 1$ and $|X_2| = n + 2$, so that $|D(X)| = n + 3$.

If $|\text{supp}(u)| = 2$ then $u = x_2 x_3$ up to permutation. The equality $\mu(x_1^3, x_2 x_3) = 4 = \text{vm}(X)$ implies $X_2 \subseteq \{x_1, x_2, x_3\} X_1$, for the presence in X_2 of any monomial not divisible by x_1 , x_2 or x_3 would result in $\text{vm}(X) > 4$. If $n = 3$, the maximal cardinality under these conditions is achieved by

$$X = [x_1^3, x_1 x_2, x_1 x_3, x_2^2, x_2 x_3, x_3^2].$$

In this case $|D(X)| = 7 = 2n + 1$. If $n \geq 4$, the maximal cardinality is achieved by

$$X = [x_1^3, x_1 x_2, \dots, x_1 x_n, x_2^2, x_2 x_3, \dots, x_2 x_n].$$

In this case $|X_2| = 2n - 1$, whence $|D(X)| = 2n$.

Case 1.2: $|\text{supp}(X_3)| = 2$. Then $x_1^2 x_2 \in X_3$, and since $\mu(x_1^2 x_2) = 4 = \mu(x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3)$, we have $X_3 = \{x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3\}$ by maximality of $|X|$. Lemma 4.2 then implies that every other monomial $u \in D(X)$ is divisible by x_1 or x_2 . The maximal cardinality under these constraints is achieved by

$$X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1 x_3, \dots, x_1 x_n, x_2 x_3, \dots, x_2 x_n].$$

For instance $\text{vm}(X) = 4$ as desired, since every edge of $G(X)$ touches $\{x_1, x_2\}$. Here $|X_3| = 4$ and $|X_2| = 2n - 1$. It follows that $|D(X)| = |X_3| + |X_2| = 2n + 3$.

Case 1.3: $|\text{supp}(X_3)| \geq 3$. It then follows from Lemma 7.1 that $X_3 \subseteq x_1^2 X_1$ with $n \geq 3$. Hence we may assume

$$X_3 = \{x_1^3, x_1^2 x_2, \dots, x_1^2 x_n\}$$

with $n \geq 3$. We have $\text{vm}(X) = \mu(x_1^2 x_2) = 4$. Hence all $u \in D(X)$ are divisible by x_1 or x_2 , since otherwise $\text{vm}(X) > 4$ would result. Hence $X_3 \subseteq \{x_1^3, x_1^2 x_2, \dots, x_1^2 x_n\}$. The maximal cardinality under the present conditions is achieved by

$$X = [x_1^3, x_1^2 x_2, \dots, x_1^2 x_n].$$

For instance $\text{vm}(X) = 4$, since every edge of $G(X)$ touches $\{x_1, x_1^2\}$. Here $|X_3| = |X_2| = n$. It follows that $|D(X)| = 2n$.

Case 2: $d = 4$. Then $X_4 = \{x_1^4\}$ by Corollary 4.6 and (15). Since $\mu(x_1^4) = 3$ and $\text{vm}(X) = 4$, there is at most one monomial $u \in D(X)$ not divisible by x_1 and it must satisfy $\mu(u) = 1$. Thus $u = x_2^2$ up to permutation, and u is actually allowed in X . Indeed, the maximal cardinality under the present conditions is achieved by

$$X = [x_1^4, x_1^2 x_2, \dots, x_1^2 x_n, x_2^2].$$

We do have $\text{vm}(X) = 4$ since every edge of $G(X)$ except (x_2, x_2) touches $\{x_1, x_1^2\}$. Here $|X_4| = 1$, $|X_3| = n$ and $|X_2| = n + 1$. It follows that $|D(X)| = 2n + 2$.

Case 3: $d = 5$. Then $X_5 = \{x_1^5\}$ and $X_4 = \{x_1^4\}$ for the same reasons as above. Since $\mu(x_1^5) = 4 = \text{vm}(X)$, it follows that every $u \in D(X)$ must be divisible by x_1 . The monomial $u = x_1x_2^2$ is forbidden in X_3 since x_2^2 is forbidden in X . However, the $x_1^2x_i$ are allowed. Indeed, the maximal cardinality under the present conditions is achieved by

$$X = [x_1^5, x_1^2x_2, \dots, x_1^2x_n].$$

We do have $\text{vm}(X) = 4$ since every edge of $G(X)$ touches $\{x_1, x_1^2\}$. Here $|X_5| = |X_4| = 1$ and $|X_3| = |X_2| = n$. It follows that $|D(X)| = 2n + 2$. This concludes the proof of the proposition. \square

Corollary 7.6. (of the proof) *Let X be a divset on n variables such that $\text{vm}(X) = 4$. Then X is tame, except for*

$$(16) \quad X = [x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, \dots, x_1x_n, x_2x_3, \dots, x_2x_n].$$

Proof. Since $\text{vm}(X) = 4$, it follows from the definition that X is tame if and only if $|D(X)| \leq 2n + 2$. The proof of Proposition 7.5 shows that $|D(X)| \leq 2n + 2$, i.e. X is tame, except for the divset X given by (16), which satisfies $|D(X)| = 2n + 3$. \square

Remark 7.7. *For the wild X given by (16), we shall provide in Section 7.6 an ad-hoc proof that the numerical semigroups modeled by this X still satisfy Wilf's conjecture.*

7.5 The case $\text{vm}(X) = 5$

Proposition 7.8. *Let X be a divset on n variables such that $\text{vm}(X) = 5$. Then $|D(X)| \leq 2n + 4$ and X is tame. Up to permutation of the variables, the only case for which $|D(X)| = 2n + 4$ is $X = [x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, \dots, x_1x_n, x_2x_3, \dots, x_2x_n, x_3^2]$.*

Proof. Given $d \geq 3$, we shall consider divsets $T \subseteq \mathcal{M}(x_1, \dots, x_n)$ satisfying the conditions

$$(17) \quad \deg(T) = d, \text{vm}(T) = 5.$$

We have $d \leq 6$, for otherwise, if $u \in T$ satisfies $\deg(u) \geq 7$, then $\text{vm}(T) \geq \mu(u) \geq 6$.

Corollary 4.6 and (15) imply that T_3 only contains monomials of the form $x_i^2x_j$ with $1 \leq i, j \leq n$. We have $\mu(x_1^3, x_2^3, x_3^3) = 6$. Hence T_3 contains at most two cubes. Moreover, we have $T_4 \subseteq \{x_1^4\}$ and $T_5 \subseteq \{x_1^5\}$.

Let X be a divset of maximal cardinality such that $X_1 = \{x_1, \dots, x_n\}$ and satisfying (17). To determine $|X|$, we start by determining all minimal divsets Y in $\mathcal{M}(x_1, \dots, x_n)$ satisfying (17). Then, for each such Y , we determine all maximal extensions $Z \supseteq Y$ satisfying (17). The maximal cardinality attained by those Z will yield $|X|$. We shall find $|D(X)| = 2n + 4$, as desired.

Case 1: $d = 3$. The minimal divsets Y satisfying (17) are:

$$Y = [x_1^3, x_2^3, x_3^2], [x_1^3, x_2^2, x_3^2, x_4^2], [x_1^3, x_2^2, x_3 x_4], [x_1^2 x_2, x_3^2].$$

Indeed, if Y_3 contains two cubes, then since $\mu(x_1^3, x_2^3) = 4$, only a square x_i^2 with $i \geq 3$ can be minimally adjoined to get $\text{vm}(Y) = 5$ exactly. Similarly, if Y_3 contains a single cube, then since $\mu(x_1^3) = 2$, the only ways to minimally reach $\text{vm}(Y) = 5$ is to adjoin $\{x_2^2, x_3^2, x_4^2\}$ or $\{x_2^2, x_3 x_4\}$. Finally, if Y_3 contains no cube, it must contain $x_1^2 x_2$ and then, since $\mu(x_1^2 x_2) = 4$, the only way to minimally raise it to $\mu(Y) = 5$ exactly is to adjoin x_3^2 .

Let us now seek, for each such Y , its maximal extensions $Z \subset \mathcal{M}(x_1, \dots, x_n)$ subject to (17).

- For $Y = [x_1^3, x_2^3, x_3^2]$ and $n \geq 3$, the unique maximal extension Z subject to (17) is

$$(18) \quad Z = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1 x_3, \dots, x_1 x_n, x_2 x_3, \dots, x_2 x_n, x_3^2].$$

An instance of a vertex-maximal matching of $G(Z)$ is given by the three independent edges (x_1, x_1^2) , (x_2, x_2^2) , (x_3, x_3) . We do have $\text{vm}(Z) = 5$ since every edge of $G(Z)$ except (x_3, x_3) touches a vertex in $\{x_1, x_2\}$. Here $Z_3 = \{x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3\}$ and $X_2 = x_1 X_1 \cup x_2 X_1 \cup \{x_3^2\}$. Thus $|Z_3| = 4$ and $|Z_2| = 2n$, whence

$$|D(Z)| = |Z_2| + |Z_3| = 2n + 4.$$

- For $Y = [x_1^3, x_2^2, x_3^2, x_4^2]$ and $n \geq 4$, the unique maximal extension Z subject to (17) is

$$(19) \quad Z = [\{x_1^3\} \cup \langle x_1, x_2, x_3, x_4 \rangle_2 \cup \{x_1 x_5, \dots, x_1 x_n\}].$$

First note that adjoining to Y any monomial u of degree at least 2 with support in $\{x_5, \dots, x_n\}$ is not allowed, since it would yield $\text{vm}(Y) \geq 6$. Let $Y' = [\langle x_1, x_2, x_3, x_4 \rangle_2 \cup \{x_1 x_5, \dots, x_1 x_n\}]$. For this extension Y' of $Y \setminus \{x_1^3\}$, we have $\text{vm}(Y') = 4$ since every edge of $G(Y')$ touches a vertex in $\{x_1, x_2, x_3, x_4\}$. The adjunction of x_1^3 to Y' , giving Z , allows the new vertex x_1^2 in the resulting graph. For instance, a vertex-maximal matching of $G(Z)$ is given by the independent edges (x_1, x_1^2) , (x_2, x_3) , (x_4, x_4) . Here $|Z_3| = 1$ and $|Z_2| = \binom{5}{2} + (n-4) = n+6$, whence

$$|D(Z)| = |Z_2| + |Z_3| = n + 7 \leq 2n + 3$$

since $n \geq 4$.

- The case $Y = [x_1^3, x_2^2, x_3 x_4]$ is similar. The same analysis as above shows that, given $n \geq 4$, the unique maximal extension Z subject to (17) is again the one in (19).

- Finally, for $Y = [x_1^2 x_2, x_3^2]$ and $n \geq 3$, the unique maximal extension Z subject to (17) is

$$(20) \quad Z = \begin{cases} [x_1^3, x_1^2 x_2, x_1^2 x_3, \dots, x_1^2 x_n, x_3^2] & \text{if } n \geq 4, \\ [x_1^3, x_1^2 x_2, x_1^2 x_3, x_2^2, x_2 x_3, x_3^2] & \text{if } n = 3. \end{cases}$$

Indeed, first note that $\mu(x_1^2x_2, x_1x_3^2) = 6$, as witnessed by the three independent edges (x_1^2, x_2) , (x_1, x_1x_2) , (x_3, x_1x_3) . Similarly, $\mu(x_1^2x_2, x_3^2, x_1x_2^2) = 6$ and $\mu(x_1^2x_2, x_i^2x_j) \geq 6$ for $i, j \geq 2$. Thus, the only monomials allowed in Z_3 are the $x_i^2x_j$ for $i \geq 1$. Let $Y = [x_1^3, x_1^2x_2, x_1^2x_3, \dots, x_1^2x_n]$. Then $\text{vm}(Y) = 4$ since $\text{vm}(x_1^2x_2) = 4$ and every edge of $G(Y)$ touches a vertex in $\{x_1, x_1^2\}$. Since $Z = Y \cup \{x_3^2\}$, we conclude that $\text{vm}(Z) = 5$ thanks to the new independent loopy edge (x_3, x_3) .

– If $n \geq 4$, no new monomial of degree 2 may be added to Z_2 without augmenting $\text{vm}(Z)$, since $\mu(x_1^2x_4, x_2^2, x_3^2) = \mu(x_1^2x_4, x_2x_3) = 6$ for instance. Thus in this case, we have $|Z_3| = n$ and $Z_2 = x_1\{x_1, \dots, x_n\} \cup \{x_3^2\}$, so that $|Z_2| = n + 1$ and

$$|D(Z)| = |Z_2| + |Z_3| = 2n + 1.$$

– For $n = 3$, we may still adjoin $\{x_2^2, x_2x_3\}$ without augmenting $\text{vm}(Y)$, and the unique maximal extension of Y in this case is $Z = [x_1^3, x_1^2x_2, x_1^2x_3, x_2^2, x_2x_3, x_3^2]$. We then have

$$|D(Z)| = |Z_2| + |Z_3| = 6 + 3 = 9 = 2n + 3.$$

We have now described all maximal divsets $Z \subset \mathcal{M}(x_1, \dots, x_n)$ satisfying (17). The conclusion is that $|D(Z)| \leq 2n + 4$ in all cases, with equality attained exclusively by the case in (18). We conclude that the divset X of maximal cardinality satisfying (17) is

$$X = [x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, \dots, x_1x_n, x_2x_3, \dots, x_2x_n, x_3^2]$$

up to permutation. It satisfies $|D(X)| = 2n + 4$.

Case 2: $d = 4$. By Corollary 4.6 and (15), we have $X_4 = \{x_1^4\}$. Let $X' = X \setminus \{x_1^4\}$. Then $\text{vm}(X') \leq \text{vm}(X) = 5$ and $\text{deg}(X') = 3$. Moreover, $|D(X)| = |D(X')| + 1$. We have $|D(X')| \leq 2n + 4$ by the above results. If $|D(X')| < 2n + 4$ then $|D(X)| \leq 2n + 4$ and we are done. If $|D(X')| = 2n + 4$, then X' is as in (18), i.e.

$$X' = [x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, \dots, x_1x_n, x_2x_3, \dots, x_2x_n, x_3^2].$$

But the case $X' \cup \{x_1^4\}$ is not eligible since $\mu(x_1^4, x_2^3, x_3^2) = 3 + 2 + 1 = 6$. However, suppressing the last x_3^2 makes

$$X'' = [x_1^4, x_1^2x_2, x_1x_2^2, x_2^3, x_1x_3, \dots, x_1x_n, x_2x_3, \dots, x_2x_n]$$

eligible since every edge besides (x_1^2, x_1^2) touches $\{x_1, x_2\}$.

Summarizing, if $\text{vm}(X) = 5$ and $\text{deg}(X) = 4$ then $|D(X)| \leq 2n + 4$, as desired.

Case 3: $d = 5$. By the above, we have $X_5 = \{x_1^5\}$, $X_4 = \{x_1^4\}$ and $\{x_1^3\} \subseteq X_3$. We have $\mu(x_1^5, x_2^3) = \mu(x_1^5, x_1x_2^2) = 6$. But $\mu(x_1^5, x_1^2x_2, \dots, x_1^2x_n) = 4$ since every edge touches $\{x_1, x_1^2\}$. By adding x_2^2 , we obtain $X = [x_1^5, x_1^2x_2, \dots, x_1^2x_n, x_2^2]$, which satisfies $\mu(X) = 5$ and is maximal. It is unique up to permutation. We have $|D(X)| = |X_5| + |X_4| + |X_3| + |X_2| = 1 + 1 + n + n + 1 = 2n + 3$. Summarizing, if $\text{vm}(X) = 5$ and $\text{deg}(X) = 5$ then $|D(X)| \leq 2n + 3$.

Case 4: $d = 6$. By the above, we have $X_6 = \{x_1^6\}$, $X_5 = \{x_1^5\}$, $X_4 = \{x_1^4\}$ and $\{x_1^3\} \subseteq X_3$. We have $\mu(x_1^6, x_2^2) = 6$. Hence $X_2 \subseteq x_1 X_1$. We have $\mu(x_1^6, x_1 x_2, \dots, x_1 x_n) = 5$ since every edge besides (x_1^3, x_1^3) touches $\{x_1, x_1^2\}$. It is unique up to permutation. We have $|D(X)| = |X_6| + |X_5| + |X_4| + |X_3| + |X_2| = 1 + 1 + 1 + n + n = 2n + 3$. Summarizing, if $\text{vm}(X) = 5$ and $\text{deg}(X) = 6$ then $|D(X)| \leq 2n + 3$.

Having covered all relevant cases $3 \leq \text{deg}(X) \leq 6$, the proof that $\text{vm}(X) = 5$ implies $|D(X)| \leq 2n + 4$ is now complete. \square

7.6 A wild case

Let X be a divset in n variables. If $\text{vm}(X) = 4$ then X is tame if and only if $|D(X)| \leq 2n + 2$. But for the exceptional divset X given by (16), namely

$$(21) \quad X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1 x_3, \dots, x_1 x_n, x_2 x_3, \dots, x_2 x_n],$$

we have $\text{vm}(X) = 4$ and $|D(X)| = 2n + 3$ as seen in Corollary 7.6. Since X is wild, we need a separate proof that the numerical semigroups modeled by X satisfy Wilf's conjecture.

It would suffice to show that for $m = |X| = 3n + 4$, there is no morphism $f : X \rightarrow \mathbb{Z}/m\mathbb{Z}$. We can prove so much. But here we will present another proof, using the *level function* on a numerical semigroup S , which is closely linked to the depth function δ on S .

Definition 7.9. *Let S be a numerical semigroup of multiplicity m , conductor c and depth $q = \lceil c/m \rceil = \delta(0)$. The level function on S is the function*

$$\lambda : S \rightarrow \mathbb{N}$$

defined by $\lambda(s) = q - \delta(s) = \lceil s/m \rceil$ for all $s \in S$.

Thus $\lambda(0) = 0$ and, for all $i \in \mathbb{Z}$,

$$\lambda(s) = i \iff s \in \llbracket im - \rho, (i+1)m - \rho - 1 \rrbracket,$$

where $\rho = qm - c \in \llbracket 0, m - 2 \rrbracket$. In particular, for the Apéry set $A = \text{Ap}(S, m)$ and $A^* = A \setminus \{0\}$, since $A^* \subseteq \llbracket m + 1, c - 1 + m \rrbracket$, we have

$$(22) \quad \lambda(A^*) \subseteq \llbracket 1, q \rrbracket.$$

We shall need the following estimates from [7]. They are a straightforward consequence of the analogous estimates in Proposition 3.4 for the function δ .

Proposition 7.10. *Let S be as above. For all $s_1, s_2 \in S$, we have $\lambda(s_1 + s_2) \in \llbracket \lambda(s_1) + \lambda(s_2) - 1, \lambda(s_1) + \lambda(s_2) + 1 \rrbracket$. Moreover, if $c \in m\mathbb{N}$, then $\lambda(s_1 + s_2) \in \llbracket \lambda(s_1) + \lambda(s_2), \lambda(s_1) + \lambda(s_2) + 1 \rrbracket$.*

Proof. Straightforward from Proposition 3.4. \square

We now deal with the wild divset X given by (21). We start with the case $n \geq 3$. The case $n = 2$, namely for $X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3]$, will be dealt with separately afterwards.

Proposition 7.11. *Let $X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1 x_3, \dots, x_1 x_n, x_2 x_3, \dots, x_2 x_n]$ with $n \geq 3$. Let S be a numerical semigroup modeled by X such that $c = qm$. Then S satisfies Wilf's conjecture.*

Proof. We may and will assume $q \geq 4$, since by Theorem 3.8, Wilf's conjecture holds in case $c \leq 3m$, i.e. when $q \leq 3$. We will show that $W_0(S) \geq 3$. Let $A = \text{Ap}(S, m)$ and $A^* = A \setminus \{0\}$. Then A^* is essentially given by the additive version of X^* , namely

$$A^* = \{a, b, c_3, \dots, c_n; 2a, a+b, 2b, a+c_3, \dots, a+c_n, b+c_3, \dots, b+c_n; 3a, 2a+b, a+2b, 3b\},$$

where $a \leftrightarrow x_1$, $b \leftrightarrow x_2$ and $c_i \leftrightarrow x_i$ for all $3 \leq i \leq n$. Let us record some initial observations:

- $|A^*| = (n+3) + 2(n-2) + 4 = 3n+3$.
- $|A \cap D| = |A^*| - n = 2n+3$.
- $\{a, b, c_3, \dots, c_n\} \subseteq P \cap L$ by Proposition 2.14. Hence $|P \cap L| \geq n+1$ since $m \in P \cap L$.

Denote $\Lambda(A^*) = \sum_{x \in A^*} \lambda(x)$. Then

$$\Lambda(A^*) = |A^*|q - \tau(A^*)$$

since $\tau(A^*) = \sum_{x \in A^*} \delta(x)$ and $\lambda(x) = q - \delta(x)$ for all $x \in S$. We first seek a lower bound on $\tau(A^*)$ via an upper bound on $\Lambda(A^*)$. For that, we use the inequalities given by Proposition 7.10 in the special case $c = qm$, namely

- 1) $\lambda(x+y) \leq \lambda(x) + \lambda(y) + 1$,
- 2) $\lambda(x+y) \geq \lambda(x) + \lambda(y)$

for all $x, y \in S$. Without loss of generality, we may assume $\lambda(a) \leq \lambda(b)$. By 1), it follows that

$$\begin{cases} \lambda(2a) \leq 2\lambda(a) + 1 \leq 2\lambda(b) + 1, \\ \lambda(a+b) \leq \lambda(a) + \lambda(b) + 1 \leq 2\lambda(b) + 1, \\ \lambda(2b) \leq 2\lambda(b) + 1. \end{cases}$$

Since $b+c_i \in A^*$, we have $\lambda(b+c_i) \leq q$ by (22). It then follows from 2) that

$$q \geq \lambda(b+c_i) \geq \lambda(b) + \lambda(c_i),$$

whence $\lambda(c_i) \leq q - \lambda(b)$ for all $3 \leq i \leq n$. We now convert these upper bounds on λ into lower bounds on δ . Using $\delta(x) = q - \lambda(x)$ and so $\delta(a) \geq \delta(b)$, we get:

$$\begin{cases} \delta(a) + \delta(b) \geq 2\delta(b) = 2(q - \lambda(b)), \\ \delta(2a), \delta(a+b), \delta(2b) \geq q - 2\lambda(b) - 1, \\ \delta(c_i) \geq \lambda(b) \text{ for all } 3 \leq i \leq n. \end{cases}$$

Inserting these bounds in $\tau(A^*) \geq \delta(a) + \delta(b) + \delta(2a) + \delta(a+b) + \delta(2b) + \sum_{i=3}^n \delta(c_i)$ yields

$$\begin{aligned}\tau(A^*) &\geq 2(q - \lambda(b)) + 3(q - 2\lambda(b) - 1) + (n-2)\lambda(b) \\ &= 5q + (n-10)\lambda(b) - 3.\end{aligned}$$

Hence, using the formula for $W_0(S)$ in Proposition 3.9 for $\rho = qm - c = 0$, we have

$$\begin{aligned}W_0(S) &\geq (n+1)\tau(A^*) - |A \cap D|q \\ &\geq (n+1)(5q + (n-10)\lambda(b) - 3) - (2n+3)q \\ &= (3n+2)q + (n+1)(n-10)\lambda(b) - 3(n+1).\end{aligned}$$

We now use $q \geq \lambda(3b) \geq 3\lambda(b)$, the inequality $q \geq \lambda(3b)$ following from $3b \in A^*$ and (22).

Case 1. $\lambda(b) = 1$. Here we use our stronger hypothesis $q \geq 4$. This yields $W_0(S) \geq 4(3n+2) + (n+1)(n-10) - 3(n+1) = n^2 - 5$. Hence $W_0(S) \geq 4$ since $n \geq 3$ by assumption.

Case 2. $\lambda(b) \geq 2$. Here we only use $q \geq 3\lambda(b)$. This yields

$$\begin{aligned}W_0(S) &\geq (3(3n+2) + (n+1)(n-10))\lambda(b) - 3(n+1) \\ &= (n^2 - 4)\lambda(b) - 3(n+1) \\ &\geq 2n^2 - 3n - 11.\end{aligned}$$

Hence $W_0(S) \geq 9$ if $n \geq 4$.

Case $n = 3, \lambda(b) \geq 3$. The above estimate $W_0(S) \geq (n^2 - 4)\lambda(b) - 3(n+1)$ yields

$$W_0(S) \geq 15 - 12 = 3.$$

Case $n = 3, \lambda(b) = 2$. It remains to consider the case $n = 3$ and $\lambda(b) = 2$. The inequality $W_0(S) \geq (3n+2)q + (n+1)(n-10)\lambda(b) - 3(n+1)$ obtained earlier yields here

$$W_0(S) \geq 11q - 68.$$

Recall $q \geq 3\lambda(b)$, whence $q \geq 6$ here. If $q \geq 7$ we are done: $W_0(S) \geq 9$.

It remains to consider one last subcase, namely $n = 3, \lambda(b) = 2, q = 6$. Since $n = 3$, we have

$$A^* = \{a, b, c, 2a, a+b, 2b, a+c, b+c, 3a, 2a+b, a+2b, 3b\}.$$

We have $q = 3\lambda(b) \leq \lambda(3b) \leq q$. Hence $\lambda(3b) = q = 3\lambda(b)$. This implies $\lambda(2b) = 2\lambda(b)$. Indeed, we have $2\lambda(b) \leq \lambda(2b) \leq 2\lambda(b) + 1$, and if we had $\lambda(2b) = 2\lambda(b) + 1$, it would imply $\lambda(3b) \geq \lambda(b) + \lambda(2b) \geq 3\lambda(b) + 1$, a contradiction. Hence $\lambda(2a), \lambda(a+b), \lambda(2b) \leq 2\lambda(b)$. Thus

$$\begin{aligned}\tau(A^*) &\geq \delta(a) + \delta(b) + \delta(2a) + \delta(a+b) + \delta(2b) + \delta(c) \\ &\geq q - \lambda(a) + q - \lambda(b) + q - \lambda(2a) + q - \lambda(a+b) + q - \lambda(2b) + q - \lambda(c) \\ &\geq 5q - 8\lambda(b) + q - \lambda(c).\end{aligned}$$

Now $q \geq \lambda(b+c) \geq \lambda(b) + \lambda(c)$, so that $q - \lambda(c) \geq \lambda(b)$. Hence $\tau(A^*) \geq 5q - 7\lambda(b) = 8\lambda(b)$. Thus

$$\begin{aligned} W_0(S) &\geq (n+1)\tau(A^*) - (2n+3)q \\ &\geq 4\tau(A^*) - 9q \\ &\geq 32\lambda(b) - 27\lambda(b) \\ &= 5\lambda(b). \end{aligned}$$

Hence $W_0(S) \geq 5\lambda(b) = 10$. □

Finally, let us consider the case $n = 2$ for X given by (21) and already seen in Example 5.5.

Proposition 7.12. *Let $X = [x_1^3, x_1^2x_2, x_1x_2^2, x_2^3]$. Let S be a numerical semigroup modeled by X such that $c = qm$. Then S satisfies Wilf's conjecture.*

Proof. In the present case, for some specific S , the number $W_0(S)$ may actually take the value -1 . However, we can still show $W(S) \geq 0$. Indeed, up to possibly irrelevant primitive elements, we have

$$A = \{0, a, b, 2a, a+b, 2b, 3a, 2a+b, a+2b, 3b\}.$$

Since $\delta(3a), \delta(3b) \geq 0$, it follows that $\delta(a) + \delta(2a) \geq q$ and $\delta(b) + \delta(2b) \geq q$. Hence $\tau(A^*) \geq 2q$. Since $|A \cap D| = 7$, we have

$$\begin{aligned} W(S) &\geq |P|\tau(A^*) - |A \cap D|q \\ &\geq (2|P| - 7)q. \end{aligned}$$

Since we may assume $|P| \geq 4$ by Theorem 1.2, we conclude that $W(S) \geq q$ and we are done. □

7.7 Main theorem

We are now in a position to prove our main result.

Theorem 7.13. *Let S be a special numerical semigroup such that $|P| \geq m/4$. Then S satisfies Wilf's conjecture, i.e. $W(S) \geq 0$.*

Proof. Let X be a divset model of S . If $\deg(X) = 1$, then $D(X) = \emptyset$, in which case Theorem 5.1 (2) implies $W(S) \geq (n+1)\text{vm}(X)q/2 \geq 0$. If $\deg(X) = 2$, then X is tame as shown in Section 6, whence $W_0(S) \geq 0$ by Proposition 5.4.

Assuming now $\deg(X) \geq 3$, we switch to considerations on $\text{vm}(X)$. Since X contains a monomial u of degree 3, it follows from the inequality $\mu(u) \geq |\text{supp}(u)|(\deg(u) - 1)$ of Proposition 4.8 that $\mu(u) \geq 2$, whence $\text{vm}(X) \geq 2$. If $\text{vm}(X) \geq 6$ then S satisfies Wilf's conjecture by Corollary 5.2. The remaining cases $\text{vm}(X) = 2, 3, 4, 5$ are dealt with in Section 7.2, 7.3, 7.4 and 7.5, respectively. All divsets X in these cases, with the sole exception of (16), are shown to be tame in Propositions 7.2, 7.3, 7.8 and Corollary 7.6, implying

$W_0(S) \geq 0$ by Proposition 5.4. The wild case (16) is treated in Section 7.6. For it, the fact that S satisfies Wilf’s conjecture is proved in Proposition 7.11 for $n \geq 3$ and Proposition 7.12 for $n = 2$. \square

8 Concluding comments

(1) We have shown that Wilf’s conjecture holds for special numerical semigroups S satisfying $|P| \geq m/4$. Attempting to extend that result to the case $|P| \geq m/5$ with the same method would require two steps. The first one is to classify all divsets X such that $\text{vm}(X) \leq 7$, versus $\text{vm}(X) \leq 5$ in the present paper. This step should be doable. However, more wild divsets will emerge. The second step, probably the more demanding one, consists in finding ad-hoc proofs, as in Section 7.6, that the numerical semigroups modeled by these wild divsets do satisfy Wilf’s conjecture. Methods from additive combinatorics may be needed for this task, for instance to prove that a wild divset X of cardinality m cannot be embedded in the cyclic group $\mathbb{Z}/m\mathbb{Z}$.

(2) Proving Wilf’s conjecture in the general case $|P| \geq m/4$, i.e. without the hypothesis $c \in m\mathbb{N}$ as in this paper, is probably achievable with the same method, yet at the cost of more technical details to take care of.

(3) In [4], Wilf’s conjecture is shown by computer and new ideas to hold up to genus $g \leq 100$, a significant jump from the previously available verification up to $g \leq 66$. See [3] for an early source of these new ideas. Still in [4], the verification of Wilf’s conjecture in the special case $c \in m\mathbb{N}$ is pushed up to genus $g \leq 120$ by computer and using the main result of this paper.

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Author's address

Shalom Eliahou

Univ. Littoral Côte d'Opale, UR 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France and CNRS, FR 2037, France.

eliahou@univ-littoral.fr