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Divsets, numerical semigroups and Wilf's conjecture

Shalom Eliahou

Abstract

Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity $m = \min(S \setminus \{0\})$ and conductor $c = \max(\mathbb{Z} \setminus S) + 1$. Let P be the set of primitive elements, i.e. minimal generators, of S , and let L be the set of elements of S which are smaller than c . Wilf's conjecture (1978) states that the inequality $|P||L| \geq c$ holds. The conjecture has been shown to hold in case $|P| \geq m/2$ by Sammartano in 2012, and subsequently in case $|P| \geq m/3$ by the author in 2020. The main result in this paper is that Wilf's conjecture holds in case $|P| \geq m/4$ if m divides c .

0 Caution

This as yet¹ incomplete document will be gradually completed in successive versions during Fall 2023.

1 Introduction

A *numerical semigroup* is a cofinite submonoid S of \mathbb{N} , i.e. a subset containing 0, stable under addition and with finite complement $\mathbb{N} \setminus S$. Equivalently, it is a set of the form $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$ where a_1, \dots, a_n are positive integers with $\gcd(a_1, \dots, a_n) = 1$, called *generators* of S . The least such n is usually denoted $e = e(S)$ and called the *embedding dimension* of S . The *multiplicity* of S is $m = m(S) = \min S^*$, where $S^* = S \setminus \{0\}$. The *Frobenius number* of S is $F = F(S) = \max(\mathbb{Z} \setminus S)$ and the *conductor* of S is $c = c(S) = F + 1$, satisfying $c + \mathbb{N} \subseteq S$ and minimal with respect to that property. The *genus* of S is $g = g(S) = |\mathbb{N} \setminus S|$, its number of gaps. We partition S as $S = L \sqcup R$, where $L = L(S) = \{a \in S \mid a < F(S)\}$ and $R = R(S) = \{a \in S \mid a > F(S)\}$, the *left part* and *right part* of S , respectively.

A *primitive element* of S is an element $a \in S^* \setminus (S^* + S^*)$, i.e. an element of S^* which is not the sum of two elements of S^* . We denote by $P = P(S)$ the set of primitive elements of S , and by $D = D(S) = S^* + S^*$ the set of *decomposable elements* of S . It is easy to see that P is contained in $[m, c + m - 1] \cap \mathbb{N}$ and hence is finite, and is the unique minimal generating set of S . Thus $|P| = e(S)$.

¹November 7th, 2023

One of the main open problems on numerical semigroups is the following conjecture, first raised as a question by Wilf [22].

Conjecture 1.1 (Wilf, 1978). *Let S be a numerical semigroup. Then $|P(S)||L(S)| \geq c(S)$.*

See [5] for a survey on the conjecture up to 2018. Among many available partial results, we shall need here the following ones, grouped for convenience in a single statement.

Theorem 1.2. *Let S be a numerical semigroup. Then S satisfies Wilf's conjecture if either $|P| \leq 3$, or $|P| \geq m/3$, or $c \leq 3m$.*

The solution in case $|P| = 2$ is due to Sylvester [21]; its extension to $|P| \leq 3$ is due to Fröberg et al. [13]. The solution in case $|P| \geq m/2$ is due to Sammartano [20]; its extension to $|P| \geq m/3$ is achieved in [10]. Finally, the case $c \leq 3m$ is settled in [9].

Notation 1.3. *For $a, b \in \mathbb{Z}$, we denote by $\llbracket a, b \rrbracket = [a, b] \cap \mathbb{Z}$ the integer interval they span.*

1.1 Special numerical semigroups

The main result in this paper extends the above case $|P| \geq m/3$ in Wilf's conjecture as follows.

Theorem 1.4. *Let S be a numerical semigroup such that $|P| \geq m/4$ and $c \in m\mathbb{N}$. Then S satisfies Wilf's conjecture.*

What motivates the added hypothesis $c \in m\mathbb{N}$? As it happens, the proofs of Wilf's conjecture in either case $c \leq 3m$ [9] or $|P| \geq m/3$ [10] can be significantly shortened when $c \in m\mathbb{N}$. Moreover, the first five instances of the very rare “near-misses in Wilf's conjecture” all belong to this case [11]. These facts lead us to consider the case $c \in m\mathbb{N}$ as a priority in research on Wilf's conjecture. Indeed, we believe that if the conjecture fails, then it will already fail in case $c \in m\mathbb{N}$. Whence the following terminology.

Definition 1.5. *A numerical semigroup S is special if its conductor c is a multiple of its multiplicity m .*

For instance, the *ordinary* (or *superficial*) numerical semigroup $O_m = \{0\} \cup (m + \mathbb{N})$ is special since it satisfies $c = m$.

1.2 Contents

In Section 2, we introduce divsets as abstract models of Apéry sets. In Section 3, we recall some needed material about the depth and the functions $W(S), W_0(S)$. In Section 4, we start focusing on special numerical semigroups. In Section 5, we settle Wilf's conjecture for special numerical semigroups modeled by a divset of degree 2. In Section 6 – to be completed – we consider the case of divsets of degree at least 3 and conclude the proof of the main theorem.

2 Divsets

In this section, we recall what is the Apéry set of a numerical semigroup and introduce divsets as abstract models thereof.

2.1 The Apéry set

Let S be a numerical semigroup of multiplicity m and conductor c . Its Apéry set contains key information on its structure.

Definition 2.1. *The Apéry set of S with respect to m is defined as $A = \text{Ap}(S, m) = S \setminus (m + S)$. We denote $A^* = A \setminus \{0\}$.*

It is well known that $\text{Ap}(S, m)$ has exactly m elements, one for each class mod m . More precisely, we have

$$\text{Ap}(S, m) = \{w_0, \dots, w_{m-1} \mid w_i = \min(S \cap (i + m\mathbb{N}))\}.$$

For instance, $\min(A) = w_0 = 0$ and $\max(A) = w_{c-1} = c + m - 1$. Hence $A \subseteq \llbracket 0, c + m - 1 \rrbracket$. Moreover, denoting $P^* = P \setminus \{m\}$, it follows from the definition that

$$(1) \quad P^* \subseteq A^*.$$

The set S is completely determined by its Apéry set A via the formula

$$(2) \quad S = \bigsqcup_{a \in A} (a + m\mathbb{N}).$$

2.2 Basic definitions

Throughout this section, for given $n \geq 1$, we denote by

$$\mathcal{M} = \mathcal{M}(x_1, \dots, x_n) = \{x_1^{a_1} \cdots x_n^{a_n} \mid (a_1, \dots, a_n) \in \mathbb{N}^n\}$$

the set of monomials in n commuting variables x_1, \dots, x_n .

Notation 2.2. *The degree of $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}$ is the standard one, namely $\deg(u) = \sum_i a_i$. For $d \in \mathbb{N}$, we denote by $\mathcal{M}_d \subset \mathcal{M}$ the subset of monomials of degree d .*

Definition 2.3. *A divset in \mathcal{M} is a finite subset $X \subset \mathcal{M}$ which is stable under taking divisors. That is, for all $u \in X$ and $v \in \mathcal{M}$, if $v \mid u$ then $v \in X$.*

Said otherwise, a divset is a finite *downset* or *order ideal* in \mathcal{M} under divisibility.

Example 2.4. $X = \{x_1^3, x_1^2 x_2, x_1^2, x_1 x_2, x_1, x_2, 1\}$ is a divset in $\mathcal{M}(x_1, x_2)$.

Definition 2.5. Let X be a divset. We define the degree of X as

$$\deg(X) = \max\{\deg(u) \mid u \in X\}.$$

Notation 2.6. Given a subset of $U \subseteq \mathcal{M}$ of monomials, we denote by $[U]$ the set of divisors of the elements of U . That is, $[U] = \{v \in \mathcal{M} \mid \exists u \in U, v|u\}$.

In the above example $X = \{x_1^3, x_1^2x_2, x_1^2, x_1x_2, x_1, x_2, 1\}$, we have $X = [x_1^3, x_1^2x_2]$.

Notation 2.7. Let X be a divset. We denote by $\max(X)$ the maximal elements of X under divisibility. That is, $\max(X)$ is the set of those $u \in X$ which do not divide any $v \in X \setminus \{u\}$.

Clearly, a divset X is completely determined by $\max(X)$, namely as $X = [\max(X)]$.

Notation 2.8. Let X be a divset in \mathcal{M} . For $d \in \mathbb{N}$, we set $X_d = X \cap \mathcal{M}_d = \{u \in X \mid \deg(u) = d\}$.

For instance, we have $X_0 = \mathcal{M}_0 = \{1\}$ and $X_1 \subseteq \mathcal{M}_1 = \{x_1, \dots, x_n\}$.

Notation 2.9. Let X be a divset. We denote by $D(X) = \{u \in X \mid \deg(u) \geq 2\}$ the set of decomposable monomials in X . Thus $X = \{1\} \sqcup X_1 \sqcup D(X)$. We set $X^* = X \setminus \{1\}$.

2.3 The graph of a divset

Let X be a divset. We canonically associate to X a graph $G = G(X)$ defined as follows. An edge in G is a pair $\{u_1, u_2\}$ with $u_1, u_2 \in X^*$ such that $u_1u_2 \in X^*$. This defines the edge set $E(G)$. The set $V(G)$ of vertices of G is defined as the set of the extremities of the edges. That is, $V(G) = \{u \in X^* \mid \exists v \in X^*, uv \in X^*\}$. We denote $V(X) = V(G)$ and $E(X) = E(G)$. The graph G has no multiple edges, but it may contain loops, namely all pairs $\{u, u\}$ such that $u, u^2 \in X^*$.

An important measure of X in the sequel is the *vertex-maximal matching number* of the graph $G(X)$, defined below.

Notation 2.10. We set $\text{vm}(X) =$ the largest cardinality of a subset $Y \subseteq X^*$ being the vertex set of a matching in $G(X)$, i.e. where Y is the union of pairwise disjoint pairs $\{u, v\} \subseteq X^*$ such that $uv \in X^*$.

2.4 Modeling Apéry sets

Here we use divsets as abstract models of Apéry sets via specific maps $f : X \rightarrow S$ from a divset X to a numerical semigroup S .

Definition 2.11. Let S be a numerical semigroup and X a divset. A map $f : X \rightarrow S$ is a morphism if $f(uv) = f(u) + f(v)$ for all $u, v \in X$.

Of course $f(1) = 0$ for any such morphism, as $f(1) = f(1 \cdot 1) = f(1) + f(1)$.

Definition 2.12. *Let S be a numerical semigroup of multiplicity m . Let $A = \text{Ap}(S, m)$. A divset model of S is a divset X with an injective morphism $f : X \rightarrow S$ such that $f(X) \subseteq A$ and $f(D(X)) = A \cap D$.*

In particular, for all $a \in A \cap D$, i.e. such that $a = a_1 + a_2$ with $a, a_1, a_2 \in A^*$, there are unique monomials $u, u_1, u_2 \in X^*$ such that

$$(3) \quad f(u) = a, f(u_1) = a_1, f(u_2) = a_2.$$

Despite the name, a divset model of S is really an abstract multiplicative model of its Apéry set A , up to primitive elements not involved as summands of elements in $A \cap D$.

Proposition 2.13. *Let S be a numerical semigroup of multiplicity m and $A = \text{Ap}(S, m)$. Let $f : X \rightarrow S$ be a divset model of S . Then $|X| \leq m$, $f(X_1) \subseteq P^*$ and $A \setminus f(X) \subseteq P^*$.*

Proof. Since f is injective and $f(X) \subseteq A$, we have $|X| = |f(X)| \leq |A| = m$. Moreover $f(X^*) \subseteq A^*$ since $f(1) = 0$. Let $u \in X_1$. If $f(u) \notin P^*$ then $f(u) \in A \cap D = A^* \setminus P^*$. Since $A \cap D = f(D(X))$ by definition, there exists $v \in D(X)$ such that $f(u) = f(v)$. Hence $u = v$ since f is injective, a contradiction since $\deg(u) = 1$ and $\deg(v) \geq 2$. Finally, since $0 \in f(X)$ and $A \cap D \subseteq f(X)$, we have $A \setminus f(X) \subseteq A^* \setminus (A \cap D) = P^*$. \square

Proposition 2.14. *Let S be a numerical semigroup. Then S admits a divset model.*

Proof. It is well known and easy to verify that, for any decomposable Apéry element $a \in A \cap D$, if $a = s_1 + s_2$ with $s_1, s_2 \in S^*$, then necessarily $s_1, s_2 \in A^*$. For any $a \in A \cap D$, consider the unique decomposition $a = p_{i_1} + \cdots + p_{i_d}$ into primitive elements $p_{i_j} \in P^*$ which is lexicographically minimal of minimal length d . Clearly, for any nonempty subsum of $p_{i_1} + \cdots + p_{i_d}$, the same minimality properties hold.

Let $\{p_1, \dots, p_n\} \subseteq P^*$ be the set of all primitive elements involved in the respective minimal decompositions of the elements in $A \cap D$. Let $X_1 = \{x_1, \dots, x_n\}$ be a set of n commuting variables. We set $u(p_i) = x_i$ for all $1 \leq i \leq n$. Then, for all $a \in A \cap D$, we associate to a the monomial $u(a)$ of degree d in $\{x_1, \dots, x_n\}$ mirroring, in multiplicative notation, the minimal decomposition $a = p_{i_1} + \cdots + p_{i_d}$. Let

$$X = \{1\} \sqcup X_1 \sqcup \{u(a) \mid a \in A \cap D\}.$$

By the remark on subsums above, if $a = a_1 + a_2$ with $a_1, a_2 \in A^*$, then $u(a) = u(a_1)u(a_2)$. Hence X is a divset. Let $f : X \rightarrow S$ be the unique morphism induced by $f(x_i) = p_i$ for all $i = 1, \dots, n$. Then f is an injective morphism, $f(X) \subseteq A$ and $f(D(X)) = A \cap D$ by construction. Hence $f : X \rightarrow S$ is a divset model of S , as desired. \square

Example 2.15. Let $S = \langle 5, 6, 9 \rangle$. Then $m = 5$ and $A^* = \{w_1, w_2, w_3, w_4\} = \{6, 12, 18, 9\}$ with $w_i = \min(S \cap (i + 5\mathbb{N}))$ for $1 \leq i \leq 4$. Let

$$\begin{aligned} X &= [x_1^2, x_2^2] = \{x_1^2, x_2^2, x_1, x_2\}, \\ X' &= [x_1^3, x_2] = \{x_1^3, x_1^2, x_1, x_2\}. \end{aligned}$$

The morphisms $f : X \rightarrow S$ and $f' : X' \rightarrow S$ both induced by $x_1 \mapsto 6, x_2 \mapsto 9$ yield two distinct divset models for S , as easily verified using $18 = 2 \cdot 9 = 3 \cdot 6$.

Remark 2.16. One advantage of divsets X as abstract models of Apéry sets A is that for any $u \in X^*$, a decomposition $u = vw$ with $v, w \in X^*$ is unique up to order, whereas in A^* , decompositions $a = b + c$ are seldom unique in general as seen in the above example. Moreover, with divsets X one can use the terminology of monomials such as degree, divisibility and so on, notions which are less intuitive in additive notation.

Remark 2.17. In a divset model $f : X \rightarrow S$, by removing from X variables $x_i \in X_1$ not dividing any $u \in D(X)$, the resulting subset $X' \subseteq X$ is still a divset and the restriction $f' : X' \rightarrow S$ of f to X' is still a divset model since $D(X') = D(X)$. Thus, we may and will assume without loss of generality that $X_1 \subseteq V$, where $V = V(X)$ is the vertex set of the associated graph $G(X)$.

Recall that for a divset model $f : X \rightarrow S$ where S is of multiplicity m , we have $|X| \leq m$. The case of equality deserves a name.

Definition 2.18. Let S be a numerical semigroup of multiplicity m . A full divset model of S is a divset model $f : X \rightarrow S$ such that $|X| = m$.

Note that, for a full divset model $f : X \rightarrow S$, we have $f(X_1) = P^*$ and so $f(X) = A$.

3 Depth, $W(S)$, $W_0(S)$

We recall here some needed material for later use. Throughout this section, S is a numerical semigroup of multiplicity m and conductor c .

Definition 3.1 (See [12]). The depth of S is the positive integer

$$q = \lceil c/m \rceil.$$

Thus $c = qm - \rho$ where $\rho \in \llbracket 0, m-1 \rrbracket$.

In fact $\rho \in \llbracket 0, m-2 \rrbracket$. For otherwise, if $\rho = m-1$ then $c \equiv 1 \pmod{m}$, an absurdity since then its Frobenius number $F = c-1 = \max(\mathbb{Z} \setminus S)$ would be a multiple of m .

3.1 The depth function

We keep the same notation as above, namely $c = qm - \rho$ where $\rho \in \llbracket 0, m-2 \rrbracket$. As in [9], we partition S as $S = \sqcup_{i \in \mathbb{N}} S_i$, where for all $i \in \mathbb{N}$,

$$S_i = S \cap \llbracket im - \rho, (i+1)m - \rho - 1 \rrbracket.$$

In particular, we have

$$\begin{aligned} S_0 &= \{0\}, \\ S_1 &= \llbracket m, 2m - \rho - 1 \rrbracket \cap S, \\ &\dots \\ S_{q-1} &= \llbracket c - m, c - 1 \rrbracket \cap S, \\ S_q &= \llbracket c, c + m - 1 \rrbracket. \end{aligned}$$

This gives rise to the following function.

Definition 3.2. The depth function $\delta : S \rightarrow \mathbb{Z}$ is defined for all $x \in S$ by

$$(4) \quad \delta(x) = i \iff x \in S_{q-i}.$$

Equivalently, $\delta(x)$ is the unique integer such that

$$(5) \quad x + \delta(x)m \in \llbracket c, c + m - 1 \rrbracket,$$

$$\text{i.e. } \delta(x) = \lceil \frac{c-x}{m} \rceil.$$

The function δ assumes the following values. Recall that $L = S \cap \llbracket 0, c - 1 \rrbracket$.

Lemma 3.3. For all $x \in S$, we have

$$\begin{aligned} \delta(x) = q &\iff x = 0, \\ \delta(x) \in \llbracket 1, q-1 \rrbracket &\iff x \in L \setminus \{0\}, \\ \delta(x) = 0 &\iff x \in \llbracket c, c + m - 1 \rrbracket, \\ \delta(x) \leq -1 &\iff x \geq c + m. \end{aligned}$$

Proof. Straightforward from the definition. □

We shall need the following estimates from [10, Proposition 6].

Proposition 3.4. For all $x, y \in S$, we have

$$(6) \quad \delta(x+y) + q + 1 \geq \delta(x) + \delta(y) \geq \delta(x+y) + q - \min(\rho, 1).$$

Proof (Outline). First note that for all $k, l \geq 1$, we have

$$S_k + S_l \subset S_{k+l-\min(\rho, 1)} \cup S_{k+l} \cup S_{k+l+1}.$$

Set $\delta(x) = i, \delta(y) = j$. By (4), this means $x \in S_{q-i}, y \in S_{q-j}$. Hence

$$x + y \in S_{2q-i-j-\min(\rho, 1)} \cup S_{2q-i-j} \cup S_{2q-i-j+1}.$$

By (4) again, this means $-q + i + j - 1 \leq \delta(x+y) \leq -q + i + j + \min(\rho, 1)$. The claimed inequalities follow. □

3.2 Total depth

Definition 3.5. Given a finite subset $E \subset S$, the total depth of E is

$$\delta(E) = \sum_{x \in E} \delta(x).$$

Lemma 3.6. Let $A = \text{Ap}(S, m)$, $A^* = A \setminus \{0\}$ and $D = S^* + S^*$. Then

$$\begin{aligned} \delta(A^*) &\subseteq \llbracket 0, q-1 \rrbracket, \\ m &= |P| + |A \cap D|, \\ |L| &= \delta(A) = \delta(A^*) + q. \end{aligned}$$

Proof. Since $A^* \subseteq \llbracket m, c+m-1 \rrbracket$, it follows from Lemma 3.3 that $\delta(A^*) \subseteq \llbracket 0, q-1 \rrbracket$. We have $|A^*| = m-1$ and $A^* = (A \cap P) \sqcup (A \cap D)$. Also $P = (A \cap P) \sqcup \{m\}$. Hence $|P| = |A \cap P| + 1$ and so

$$m = |A^*| + 1 = |A \cap D| + |A \cap P| + 1 = |A \cap D| + |P|.$$

We have $L = S \cap \llbracket 0, c-1 \rrbracket = \{0\} \sqcup (S \cap \llbracket m, c-1 \rrbracket)$. Moreover, for all $a \in A$, we have

$$L \cap (a + m\mathbb{N}) = a + \llbracket 0, \delta(a) - 1 \rrbracket m.$$

Hence $|L| = \sum_{a \in A} \delta(a) = \delta(A)$. Since $\delta(0) = q$, the formula $|L| = q + \delta(A^*)$ follows. \square

3.3 On the numbers $W(S), W_0(S)$

The numbers $W(S), W_0(S)$ attached to the numerical semigroup S were introduced in [9]. The alternate notation $E(S) = W_0(S)$ was subsequently proposed in [4] and elsewhere. Recall the notation $D = S^* + S^*$ and $P = S^* \setminus D$.

Notation 3.7. We denote $D_q = D \cap S_q = D \cap \llbracket c, c+m-1 \rrbracket$, and

$$\begin{aligned} W(S) &= |P||L| - c, \\ W_0(S) &= |P \cap L||L| - |A \cap D_q| + \rho. \end{aligned}$$

Thus, Wilf's conjecture amounts to the inequality

$$(7) \quad W(S) \geq 0$$

for all numerical semigroups S . The interest of $W_0(S)$ stems from the inequality

$$W(S) \geq W_0(S).$$

Therefore, if $W_0(S) \geq 0$ then S satisfies Wilf's conjecture in a somewhat stronger sense. For instance, the following result is proved in [9].

Theorem 3.8. Let S be a numerical semigroup such that $c \leq 3m$. Then $W_0(S) \geq 0$.

There are cases where $W_0(S) \leq -1$, but those are extremely rare. See [4, 11] for more details. Note finally that if $P \subseteq L$ then $W_0(S) = W(S)$.

3.4 New formulas

The following formulas exhibit a closer relationship between $W(S)$ and $W_0(S)$ than from the original ones. The symbols A^*, P, D keep the same meaning as above. We further denote $P_q = P \cap S_q$, $p_q = |P_q|$ and $d_q = |D_q|$.

Proposition 3.9. *Let S be a numerical semigroup. Then*

$$\begin{aligned} W(S) &= |P|\delta(A^*) - |A \cap D|q + \rho, \\ W_0(S) &= |P \cap L|\delta(A^*) - |A \cap D|q + \rho. \end{aligned}$$

Proof. We have

$$\begin{aligned} W(S) &= |P||L| - c \\ &= |P|(\delta(A^*) + q) - qm + \rho \\ &= |P|(\delta(A^*) + q) - q(|P| + |A \cap D|) + \rho \\ &= |P|\delta(A^*) - |A \cap D|q + \rho. \end{aligned}$$

Moreover,

$$\begin{aligned} W(S) &= |P||L| - c \\ &= (|P \cap L| + p_q)|L| - qm + \rho \\ &= (|P \cap L| + p_q)|L| - q(p_q + d_q) + \rho \\ &= |P \cap L||L| - qd_q + \rho + p_q(|L| - q) \\ &= W_0(S) + p_q(|L| - q) \\ &= W_0(S) + p_q\delta(A^*). \quad \square \end{aligned}$$

Corollary 3.10. *We have*

$$W(S) - W_0(S) = |P_q|\delta(A^*).$$

Proof. Since $|P| = |P \cap L| + |P_q|$, the proposition implies

$$\begin{aligned} W(S) - W_0(S) &= (|P| - |P \cap L|)\delta(A^*) \\ &= |P_q|\delta(A^*). \quad \square \end{aligned}$$

Corollary 3.11. *We have*

$$\begin{cases} W_0(S) \leq W(S), \\ W_0(S) = W(S) \iff P \subseteq L. \end{cases}$$

Proof. Indeed, since $P \subseteq S_1 \sqcup \cdots \sqcup S_q$, we have $P_q = \emptyset \iff P \subseteq L$. □

Corollary 3.12. *If S is special of depth q , then $W(S) = |P|\delta(A^*) - |A \cap D|q$. □*

4 Focus on the special case

Throughout this section, S denotes a special numerical semigroup of multiplicity m , conductor c and depth $q = \lceil c/m \rceil = c/m$, i.e. with $\rho = 0$. As above, $A = \text{Ap}(S, m)$ and $A^* = A \setminus \{0\}$.

Proposition 4.1. *Let S be a special numerical semigroup of depth q . Let X be a divset model of S and let $n = |X_1|$. Then*

$$\begin{aligned}\delta(A^*) &\geq \text{vm}(X)q/2, \\ W(S) &\geq ((n+1)\text{vm}(X) - 2|D(X)|)q/2.\end{aligned}$$

Proof. By hypothesis, there is an injective morphism $f : X \rightarrow S$ such that $f(X) \subseteq A$ and $f(D(X)) = A \cap D$. Let $G = G(X) = (V, E)$ be the graph of X . Denote $k = \text{vm}(X)$. Hence, there is a subset $M \subseteq V \subseteq X^*$ of cardinality $|M| = k$ such that

$$M = \bigsqcup_{i=1}^{\ell} z_i,$$

where $z_1, \dots, z_\ell \in E$ are pairwise disjoint edges. Loops are allowed, so $|z_i| \in \{1, 2\}$ for all i . We have

$$(8) \quad |M| = k = |z_1| + \dots + |z_\ell|.$$

Let $z \in \{z_1, \dots, z_\ell\}$. Then $z = \{u_1, u_2\}$ with $u_1, u_2 \in X^*$ such that $u_1 u_2 \in X^*$. Let

$$a_1 = f(u_1), a_2 = f(u_2), a = f(u_1 u_2).$$

Then $a = a_1 + a_2$ since f is a morphism, and $a \in A^*$ since $f(X^*) \subseteq A^*$. We have $\delta(a) \geq 0$ since $\delta(A^*) \subseteq \llbracket 0, q-1 \rrbracket$ by Lemma 3.6. Hence

$$\delta(a_1) + \delta(a_2) \geq q + \delta(a_1 + a_2) = q + \delta(a) \geq q$$

by the right inequality in (6) and the value $\rho = 0$. We have $f(z) = \{a_1, a_2\}$ and so

$$\delta(f(z)) = \delta(a_1) + \delta(a_2) \geq q.$$

If $|z| = 1$ then $a_1 = a_2$ and $\delta(a_1) \geq q/2$, whereas if $|z| = 2$ then $\delta(f(z)) = \delta(a_1) + \delta(a_2) \geq q$. Summarizing, we have

$$\delta(f(z)) \geq |z|q/2$$

for all $z \in \{z_1, \dots, z_\ell\}$. Hence

$$\delta(f(\{z_1, \dots, z_\ell\})) = \sum_{i=1}^{\ell} \delta(f(z_i)) \geq \left(\sum_{i=1}^{\ell} |z_i| \right) q/2.$$

Since $\sum_{i=1}^{\ell} |z_i| = |M| = k$ by (8), it follows that $\delta(f(\{z_1, \dots, z_\ell\})) \geq kq/2$. And since

$$f(\{z_1, \dots, z_\ell\}) \subseteq f(X^*) \subseteq A^*,$$

we conclude that

$$\delta(A^*) \geq \delta(f(X^*)) \geq \delta(f(\{z_1, \dots, z_\ell\})) \geq kq/2,$$

as desired.

It remains to prove the formula $W(S) \geq ((n+1)\text{vm}(X) - 2|D(X)|)q/2$. By Proposition 3.9, we have $W(S) = |P|\delta(A^*) - |A \cap D|q + \rho$. Since $\delta(A^*) \geq \text{vm}(X)q/2$ and $\rho = 0$, this yields $W(S) \geq (|P|\text{vm}(X)/2 - |A \cap D|)q$. Finally, since $|P| \geq |X_1| + 1 = n + 1$ and $|A \cap D| = |f(D(X))| = |D(X)|$, the desired inequality follows. \square

Corollary 4.2. *Let S be a special numerical semigroup of depth q . Let X be a divset model of S . If $\text{vm}(X) \geq 2k$ then $\delta(A^*) \geq kq$.*

Proof. By Proposition 4.1 and the hypothesis, we have $\delta(A^*) \geq \text{vm}(X)q/2 \geq kq$. \square

Corollary 4.3. *Let S be a special numerical semigroup of depth q such that $|P| \geq m/4$. Let X be a divset model of S . If $\text{vm}(X) \geq 6$ then S satisfies Wilf's conjecture.*

Proof. It follows from the previous corollary that $\delta(A^*) \geq 3q$. By Lemma 3.6, it follows that $|L| \geq 4q$. Hence $|P||L| \geq (m/4)(4q) = qm = c$, i.e. $W(S) \geq 0$ as claimed. \square

4.1 Tame divsets

Definition 4.4. *Let X be a divset. Let $G(X) = (V, E)$ be the graph of X . Set $V_1 = V \cap X_1$ and $n = |V_1|$. We say that X is tame if*

$$(n+1)\text{vm}(X) \geq 2|D(X)|.$$

We say that X is wild if it is not tame, i.e. if $(n+1)\text{vm}(X) < 2|D(X)|$.

Proposition 4.5. *Let S be a special numerical semigroup of depth $q \geq 2$. If S admits a tame divset model X , then $W_0(S) \geq 0$ and hence S satisfies Wilf's conjecture.*

Proof. Let $f : X \rightarrow S$ be a divset model with X tame. Let $G(X) = (V, E)$ be the graph of X and $n = |V_1| = |V \cap X_1|$. Since S is tame, we have $(n+1)\text{vm}(X) \geq 2|D(X)|$.

Claim. We have $f(V_1) \subseteq P^* \cap L$. Indeed, we have $f(V_1) \subseteq f(X_1) \subseteq P^*$ by Proposition 2.13. Moreover, if $x_i \in V_1$, there exists $u \in X^*$ such that $x_i u \in X^*$. Hence

$$f(x_i) + f(u) = f(x_i u) \in f(X^*) \subseteq A^* \subseteq \llbracket m, c + m - 1 \rrbracket.$$

Since $f(u) \in A^*$ hence $f(u) \geq m$, it follows that $f(x_i) \leq c - 1$, i.e. $f(x_i) \in L$. This settles the claim.

As $c = qm$ with $q \geq 2$, we have $P \cap L = (P^* \cap L) \sqcup \{m\}$. Since f is injective, the claim implies

$$|P \cap L| = |P^* \cap L| + 1 \geq |f(V_1)| + 1 \geq |X_1| + 1 \geq n + 1.$$

Now $|A \cap D| = |f(D(X))| = |D(X)|$, and $\delta(A^*) \geq \text{vm}(X)q/2$ by Proposition 4.1.

Summarizing, and using the formula for $W_0(S)$ in Proposition 3.9, we have

$$\begin{aligned} W_0(S) &= |P \cap L| \delta(A^*) - |A \cap D| q \\ &\geq (n + 1) \text{vm}(X) q / 2 - |A \cap D| q \\ &= ((n + 1) \text{vm}(X) - 2|D(X)|) q / 2. \end{aligned}$$

Now $(n + 1) \text{vm}(X) - 2|D(X)| \geq 0$ since X is tame. Hence $W_0(S) \geq 0$. \square

Almost all of the divsets involved in the next sections turn out to be tame. But here is an example of a wild divset.

Example 4.6. Let $X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3]$, i.e. the set of all monomials of degree at most 3 in x_1, x_2 . Then $|X^*| = |X_1| + |D(X)| = 2 + (3 + 4) = 9$. Here $n = |V_1| = 2$ and $|D(X)| = 7$. Moreover, $\text{vm}(X) = 4$ as witnessed by the vertex-maximal matching $\{\{x_1, x_1^2\}, \{x_2, x_2^2\}\}$ of the graph $G(X)$. Hence

$$(n + 1) \text{vm}(X) - 2|D(X)| = 3 \cdot 4 - 2 \cdot 7 = -2,$$

so that X is wild. However, it can be shown that for any special numerical semigroup S modeled by X , one has $|P| \geq 7$, whence

$$W(S) = |P| \delta(A^*) - |A \cap D| q \geq 7 \cdot \text{vm}(X) q / 2 - |D(X)| q = 14q - 7q = 7q.$$

Thus, all such numerical semigroups S satisfy Wilf's conjecture. Interestingly, those S include the five smallest numerical semigroups satisfying $W_0(S) \leq -1$ as described in [11].

4.2 Proof strategy

It follows from Corollary 4.3 that, in order to settle Wilf's conjecture for special numerical semigroups S satisfying $|P| \geq m/4$, it suffices to consider divsets X such that $\text{vm}(X) \leq 5$. Thus, we need to classify all such divsets.

To start with, since a divset is determined by its maximal elements, we introduce the following auxiliary notation.

Notation 4.7. Let $u_1, \dots, u_r \in \mathcal{M}$ and $X = [u_1, \dots, u_r]$. We set $\mu(u_1, \dots, u_r) = \text{vm}(X)$.

Determining $\mu(u_1, \dots, u_r)$ is difficult in general, due to its intimate relationship with matching numbers in graphs. Nevertheless, the case $r = 1$ is straightforward.

Lemma 4.8. *Let $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}$. Then $\mu(u) = \prod_{i=1}^n (a_i + 1) - 2$.*

Proof. Let $X = [u]$, the divset of divisors of u . Then $\mu(u) = \text{vm}(X)$, i.e. the maximal number of vertices touched by a matching of the graph $G(X)$. Now $|X| = \prod_{i=1}^n (a_i + 1)$, the number of divisors of u . These divisors may be regrouped in independent edges of the form $\{v, u/v\}$ where v is a divisor of u such that $0 \leq \deg(v) \leq \deg(u/v)$, and v is lexicographically smaller than or equal to u/v if $\deg(v) = \deg(u/v)$. The pair $\{1, u\}$ must be discounted since 1 is not a vertex of G by definition. The other pairs constitute a matching of G covering the whole of $X \setminus \{0, u\}$. Hence $\mu(u) = \text{vm}(X) = |X \setminus \{0, u\}| = \prod_{i=1}^n (a_i + 1) - 2$. \square

Coupled with more combinatorial arguments, this basic result will allow us to classify all divsets X such that $\text{vm}(X) \leq 5$, as needed. Then in each case, we shall show that all special numerical semigroups S modeled by those divsets and such that $|P| \geq m/4$ satisfy Wilf's conjecture, thereby proving the main result in this paper.

5 The case $\deg(X) = 2$

We prove here that divsets of degree 2 are tame. This establishes Wilf's conjecture for all special numerical semigroups S such that $A \cap D \subseteq P^* + P^*$. In order to do so, we need the following lemma about the classical *matching number* in graphs, namely the maximum number of independent edges.

Lemma 5.1. *Let $H = (H_1, H_2)$ be a simple bipartite graph without isolated vertices, with matching number k . Then*

$$|E(H)| \leq k \cdot \max(|H_1|, |H_2|).$$

Proof. We may assume $|H_1| \leq |H_2|$. Then $k \leq |H_1|$. Let M be a maximal matching with k edges, say

$$M = \{(x_i, y_i) \mid x_i \in H_1, y_i \in H_2, 1 \leq i \leq k\}.$$

Claim. $|H_1| = k$. Assume not. Let then $x_{k+1} \in H_1 \setminus \{x_1, \dots, x_k\}$. Since H does not have isolated vertices, it follows that x_{k+1} has a neighbour $z \in H_2$. Now necessarily $z \in \{y_1, \dots, y_k\}$, for otherwise there would be a new edge (x_{k+1}, z) independent of M , which would then yield a matching of cardinality $k + 1$, a contradiction. Up to renumbering, we may assume $z = y_k$, i.e.

$$(x_{k+1}, y_k) \in E.$$

As $|H_2| \geq |H_1| \geq k + 1$, there is a vertex $y_{k+1} \in H_2 \setminus \{y_1, \dots, y_k\}$. Since y_{k+1} is not isolated, it has a neighbour $z \in H_1$. But as in the preceding reasoning, we have $z \in \{x_1, \dots, x_k\}$, for otherwise the edge (z, y_{k+1}) would be independent of M , a contradiction. Up to renumbering, we may assume $z = x_k$. Thus

$$(x_k, y_{k+1}) \in E.$$

But then, by suppressing the edge (x_k, y_k) of M and replacing it by the two independent edges $(x_{k+1}, y_k), (x_k, y_{k+1})$, we obtain a matching

$$M' = M \sqcup \{(x_{k+1}, y_k), (x_k, y_{k+1})\} \setminus \{(x_k, y_k)\}$$

of cardinality $k + 1$, a contradiction. This finishes the proof of the claim, whence

$$|H_1| = k.$$

Since $H = (H_1, H_2)$ is bipartite, we have $|E| \leq |H_1||H_2| = k|H_2|$, as claimed. \square

Proposition 5.2. *Let X be a divset of degree 2, i.e. such that $D(X) = X_2$. Then X is tame, i.e. $(n + 1) \text{vm}(X) \geq 2|D(X)|$, where $n = |V \cap X_1|$ and V is the vertex set of the graph $G(X)$.*

Proof. Let $V_1 = V \cap X_1 = \{x_1, \dots, x_n\}$, the set of variables dividing the monomials in $D(X) = X_2$. We have $V = V_1$ since if $u \in V$, there exists $v \in V$ such that $uv \in D(X) = X_2$, implying $\deg(u) = \deg(v) = 1$. Clearly $|D(X)| \leq n(n + 1)/2$. Let $M \subseteq D(X)$ be a maximal set of pairwise coprime monomials. Denote by $V_2 \subseteq V_1$ the set of variables involved in M . Let $n_2 = |V_2|$. Since M corresponds to a matching in $G(X)$ with vertex set V_2 , we have

$$\text{vm}(X) \geq n_2.$$

Case 1. $V_2 = V_1$. In this case we are done. Indeed, we then have $n_2 = n$, so that $\text{vm}(X) \geq n$.

And since $|D(X)| \leq n(n + 1)/2$, it follows that

$$(n + 1) \text{vm}(X) - 2|D(X)| \geq (n + 1)n - 2n(n + 1)/2 = 0,$$

as desired.

Case 2. Assume $V_2 \neq V_1$. Let $V_3 = V_1 \setminus V_2$ and $n_3 = |V_3| = n - n_2$. By maximality of M , every monomial $u = x_i x_j \in D(X)$ with $i \leq j$ satisfies $x_i \in V_2$ or $x_j \in V_2$. Let $N \subseteq D(X)$ be a largest possible subset of pairwise coprime monomials $x_i x_j$ with $x_i \in V_2, x_j \in V_3$. Let $k = |N|$. We have $k \leq \min(n_2, n_3)$ and $k \geq 1$ since $V_3 \neq \emptyset$. Each $x_i x_j \in N$ independently contributes a summand 2 to $\text{vm}(X)$. Thus

$$(9) \quad \text{vm}(X) \geq (n_2 - k) + 2k = n_2 + k.$$

Next, we claim that

$$(10) \quad |D(X)| \leq n_2(n_2 + 1)/2 + k \max(n_2, n_3).$$

Indeed, as V_2 induces the empty subgraph in $G(X)$, every monomial in $D(X)$ has its support in $[V_1, V_1]$ or $[V_1, V_2]$. Now the preceding lemma implies that the number of monomials with support in $[V_1, V_2]$ is less than or equal to $k \max(n_2, n_3)$. This proves (10). Combined with (9), this yields

$$(n+1)\text{vm}(X) - 2|D(X)| \geq (n+1)(n_2+k) - n_2(n_2+1) - 2k\max(n_2, n_3).$$

It remains to show that the right-hand side is non-negative. Since $n = n_2 + n_3$, we have

$$\begin{aligned} (n+1)(n_2+k) - n_2(n_2+1) - 2k\max(n_2, n_3) &= n_2(n+1) - n_2(n_2+1) + k(n+1 - 2\max(n_2, n_3)) \\ &= n_2n_3 + k(n_2 + n_3 + 1 - 2\max(n_2, n_3)) \\ &= n_2n_3 + k(\min(n_2, n_3) - \max(n_2, n_3) + 1) \end{aligned}$$

since $n_2 + n_3 = \min(n_2, n_3) + \max(n_2, n_3)$. But for any $k \leq a \leq b$, we have

$$\begin{aligned} ab + k(a-b+1) &= k(a+1) + b(a-k) \\ &\geq k(a+1). \end{aligned}$$

Hence $(n+1)(n_2+k) - n_2(n_2+1) - 2k\max(n_2, n_3) \geq k(\min(n_2, n_3) + 1) \geq 0$, as desired. \square

Corollary 5.3. *Let S be a special numerical semigroup such that $A \cap D \subseteq P^* + P^*$. Then S satisfies Wilf's conjecture.*

Proof. Let $f : X \rightarrow S$ be a divset model. Let $G = (V, E)$ be the graph of X . By Remark 2.17, we may assume $X_1 \subseteq V$. Moreover, since $f(D(X)) = A \cap D \subseteq P^* + P^*$, we may assume that $D(X) \subseteq X_1 \cdot X_1$, i.e. that $\deg(X) = 2$. We conclude with Propositions 4.5 and 5.2. \square

6 The case $\deg(X) \geq 3$

The case $\deg(X) = 2$ being settled, from here on we only consider divsets X satisfying $\deg(X) \geq 3$. [...]

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