



HAL
open science

Divsets, numerical semigroups and Wilf's conjecture

Shalom Eliahou

► **To cite this version:**

| Shalom Eliahou. Divsets, numerical semigroups and Wilf's conjecture. 2023. hal-04234167v2

HAL Id: hal-04234167

<https://hal.science/hal-04234167v2>

Preprint submitted on 28 Nov 2023 (v2), last revised 24 Apr 2024 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Divsets, numerical semigroups and Wilf's conjecture

Shalom Eliahou

Abstract

Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity $m = \min(S \setminus \{0\})$ and conductor $c = \max(\mathbb{Z} \setminus S) + 1$. Let P be the set of primitive elements, i.e. minimal generators, of S , and let L be the set of elements of S which are smaller than c . Wilf's conjecture (1978) states that the inequality $|P||L| \geq c$ holds. The conjecture has been shown to hold in case $|P| \geq m/2$ by Sammartano in 2012, and subsequently in case $|P| \geq m/3$ by the author in 2020. The main result in this paper is that Wilf's conjecture holds in case $|P| \geq m/4$ if m divides c .

0 Caution

This as yet¹ incomplete document will be gradually completed in successive versions during Fall 2023.

1 Introduction

A *numerical semigroup* is a cofinite submonoid S of \mathbb{N} , i.e. a subset containing 0, stable under addition and with finite complement $\mathbb{N} \setminus S$. Equivalently, it is a set of the form $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$ where a_1, \dots, a_n are positive integers with $\gcd(a_1, \dots, a_n) = 1$, called *generators* of S . The least such n is usually denoted $e = e(S)$ and called the *embedding dimension* of S . The *multiplicity* of S is $m = m(S) = \min S^*$, where $S^* = S \setminus \{0\}$. The *Frobenius number* of S is $F = F(S) = \max(\mathbb{Z} \setminus S)$ and the *conductor* of S is $c = c(S) = F + 1$, satisfying $c + \mathbb{N} \subseteq S$ and minimal with respect to that property. The *genus* of S is $g = g(S) = |\mathbb{N} \setminus S|$, its number of gaps. We partition S as $S = L \sqcup R$, where $L = L(S) = \{a \in S \mid a < F(S)\}$ and $R = R(S) = \{a \in S \mid a > F(S)\}$, the *left part* and *right part* of S , respectively.

A *primitive element* of S is an element $a \in S^* \setminus (S^* + S^*)$, i.e. an element of S^* which is not the sum of two elements of S^* . We denote by $P = P(S)$ the set of primitive elements of S , and by $D = D(S) = S^* + S^*$ the set of *decomposable elements* of S . It is easy to see that P is contained in $[m, c + m - 1] \cap \mathbb{N}$ and hence is finite, and is the unique minimal generating set of S . Thus $|P| = e(S)$.

¹November 7th, 2023

One of the main open problems on numerical semigroups is the following conjecture, first raised as a question by Wilf [22].

Conjecture 1.1 (Wilf, 1978). *Let S be a numerical semigroup. Then $|P(S)||L(S)| \geq c(S)$.*

See [5] for a survey on the conjecture up to 2018. Among many available partial results, we shall need here the following ones, grouped for convenience in a single statement.

Theorem 1.2. *Let S be a numerical semigroup. Then S satisfies Wilf's conjecture if either $|P| \leq 3$, or $|P| \geq m/3$, or $c \leq 3m$.*

The solution in case $|P| = 2$ is due to Sylvester [21]; its extension to $|P| \leq 3$ is due to Fröberg et al. [13]. The solution in case $|P| \geq m/2$ is due to Sammartano [20]; its extension to $|P| \geq m/3$ is achieved in [10]. Finally, the case $c \leq 3m$ is settled in [9].

Notation 1.3. *For $a, b \in \mathbb{Z}$, we denote by $\llbracket a, b \rrbracket = [a, b] \cap \mathbb{Z}$ the integer interval they span.*

1.1 Special numerical semigroups

The main result in this paper extends the above case $|P| \geq m/3$ in Wilf's conjecture as follows.

Theorem 1.4. *Let S be a numerical semigroup such that $|P| \geq m/4$ and $c \in m\mathbb{N}$. Then S satisfies Wilf's conjecture.*

What motivates the added hypothesis $c \in m\mathbb{N}$? As it happens, the proofs of Wilf's conjecture in either case $c \leq 3m$ [9] or $|P| \geq m/3$ [10] can be significantly shortened when $c \in m\mathbb{N}$. Moreover, the first five instances of the very rare “near-misses in Wilf's conjecture” all belong to this case [11]. These facts lead us to consider the case $c \in m\mathbb{N}$ as a priority in research on Wilf's conjecture. Indeed, we believe that if the conjecture fails, then it will already fail in case $c \in m\mathbb{N}$. Whence the following terminology.

Definition 1.5. *A numerical semigroup S is special if its conductor c is a multiple of its multiplicity m .*

For instance, the *ordinary* (or *superficial*) numerical semigroup $O_m = \{0\} \cup (m + \mathbb{N})$ is special since it satisfies $c = m$.

1.2 Contents

In Section 2, we introduce divsets as abstract models of Apéry sets. In Section 3, we recall some needed material about the depth and the functions $W(S), W_0(S)$. In Section 4, we start focusing on special numerical semigroups. In Section 5, we settle Wilf's conjecture for special numerical semigroups modeled by a divset of degree 2. In Section 6 – to be completed – we consider the case of divsets of degree at least 3 and conclude the proof of the main theorem.

2 Divsets

In this section, we recall what is the Apéry set of a numerical semigroup and introduce divsets as abstract models thereof.

2.1 The Apéry set

Let S be a numerical semigroup of multiplicity m and conductor c . Its Apéry set contains key information on its structure.

Definition 2.1. *The Apéry set of S with respect to m is defined as $A = \text{Ap}(S, m) = S \setminus (m + S)$. We denote $A^* = A \setminus \{0\}$.*

It is well known that $\text{Ap}(S, m)$ has exactly m elements, one for each class mod m . More precisely, we have

$$\text{Ap}(S, m) = \{w_0, \dots, w_{m-1} \mid w_i = \min(S \cap (i + m\mathbb{N}))\}.$$

For instance, $\min(A) = w_0 = 0$ and $\max(A) = w_{c-1} = c + m - 1$. Hence $A \subseteq \llbracket 0, c + m - 1 \rrbracket$. Moreover, denoting $P^* = P \setminus \{m\}$, it follows from the definition that

$$(1) \quad P^* \subseteq A^*.$$

The set S is completely determined by its Apéry set A via the formula

$$(2) \quad S = \bigsqcup_{a \in A} (a + m\mathbb{N}).$$

2.2 Basic definitions

Throughout this section, for given $n \geq 1$, we denote by

$$\mathcal{M} = \mathcal{M}(x_1, \dots, x_n) = \{x_1^{a_1} \cdots x_n^{a_n} \mid (a_1, \dots, a_n) \in \mathbb{N}^n\}$$

the set of monomials in n commuting variables x_1, \dots, x_n .

Notation 2.2. *The degree of $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}$ is the standard one, namely $\deg(u) = \sum_i a_i$. For $d \in \mathbb{N}$, we denote by $\mathcal{M}_d \subset \mathcal{M}$ the subset of monomials of degree d .*

Definition 2.3. *A divset in \mathcal{M} is a finite subset $X \subset \mathcal{M}$ which is stable under taking divisors. That is, for all $u \in X$ and $v \in \mathcal{M}$, if $v \mid u$ then $v \in X$.*

Said otherwise, a divset is a finite *downset* or *order ideal* in \mathcal{M} under divisibility.

Example 2.4. $X = \{x_1^3, x_1^2 x_2, x_1^2, x_1 x_2, x_1, x_2, 1\}$ is a divset in $\mathcal{M}(x_1, x_2)$.

Definition 2.5. Let X be a divset. We define the degree of X as

$$\deg(X) = \max\{\deg(u) \mid u \in X\}.$$

Notation 2.6. Given a subset of $U \subseteq \mathcal{M}$ of monomials, we denote by $[U]$ the set of divisors of the elements of U . That is, $[U] = \{v \in \mathcal{M} \mid \exists u \in U, v|u\}$.

In the above example $X = \{x_1^3, x_1^2x_2, x_1^2, x_1x_2, x_1, x_2, 1\}$, we have $X = [x_1^3, x_1^2x_2]$.

Notation 2.7. Let X be a divset. We denote by $\max(X)$ the maximal elements of X under divisibility. That is, $\max(X)$ is the set of those $u \in X$ which do not divide any $v \in X \setminus \{u\}$.

Clearly, a divset X is completely determined by $\max(X)$, namely as $X = [\max(X)]$.

Notation 2.8. Let X be a divset in \mathcal{M} . For $d \in \mathbb{N}$, we set $X_d = X \cap \mathcal{M}_d = \{u \in X \mid \deg(u) = d\}$.

For instance, we have $X_0 = \mathcal{M}_0 = \{1\}$ and $X_1 \subseteq \mathcal{M}_1 = \{x_1, \dots, x_n\}$.

Notation 2.9. Let X be a divset. We denote by $D(X) = \{u \in X \mid \deg(u) \geq 2\}$ the set of decomposable monomials in X . Thus $X = \{1\} \sqcup X_1 \sqcup D(X)$. We set $X^* = X \setminus \{1\}$.

2.3 The graph of a divset

Let X be a divset. We canonically associate to X a graph $G = G(X)$ defined as follows. An edge in G is a pair $\{u_1, u_2\}$ with $u_1, u_2 \in X^*$ such that $u_1u_2 \in X^*$. This defines the edge set $E(G)$. The set $V(G)$ of vertices of G is defined as the set of the extremities of the edges. That is, $V(G) = \{u \in X^* \mid \exists v \in X^*, uv \in X^*\}$. We denote $V(X) = V(G)$ and $E(X) = E(G)$. The graph G has no multiple edges, but it may contain loops, namely all pairs $\{u, u\}$ such that $u, u^2 \in X^*$.

An important measure of X in the sequel is the *vertex-maximal matching number* of the graph $G(X)$, defined below.

Notation 2.10. We set $\text{vm}(X) =$ the largest cardinality of a subset $Y \subseteq X^*$ being the vertex set of a matching in $G(X)$, i.e. where Y is the union of pairwise disjoint pairs $\{u, v\} \subseteq X^*$ such that $uv \in X^*$.

2.4 Modeling Apéry sets

Here we use divsets as abstract models of Apéry sets via specific maps $f : X \rightarrow S$ from a divset X to a numerical semigroup S .

Definition 2.11. Let S be a numerical semigroup and X a divset. A map $f : X \rightarrow S$ is a morphism if $f(uv) = f(u) + f(v)$ for all $u, v \in X$.

Of course $f(1) = 0$ for any such morphism, as $f(1) = f(1 \cdot 1) = f(1) + f(1)$.

Definition 2.12. *Let S be a numerical semigroup of multiplicity m . Let $A = \text{Ap}(S, m)$. A divset model of S is a divset X with an injective morphism $f : X \rightarrow S$ such that $f(X) \subseteq A$ and $f(D(X)) = A \cap D$.*

In particular, for all $a \in A \cap D$, i.e. such that $a = a_1 + a_2$ with $a, a_1, a_2 \in A^*$, there are unique monomials $u, u_1, u_2 \in X^*$ such that

$$(3) \quad f(u) = a, f(u_1) = a_1, f(u_2) = a_2.$$

Despite the name, a divset model of S is really an abstract multiplicative model of its Apéry set A , up to primitive elements not involved as summands of elements in $A \cap D$.

Proposition 2.13. *Let S be a numerical semigroup of multiplicity m and $A = \text{Ap}(S, m)$. Let $f : X \rightarrow S$ be a divset model of S . Then $|X| \leq m$, $f(X_1) \subseteq P^*$ and $A \setminus f(X) \subseteq P^*$.*

Proof. Since f is injective and $f(X) \subseteq A$, we have $|X| = |f(X)| \leq |A| = m$. Moreover $f(X^*) \subseteq A^*$ since $f(1) = 0$. Let $u \in X_1$. If $f(u) \notin P^*$ then $f(u) \in A \cap D = A^* \setminus P^*$. Since $A \cap D = f(D(X))$ by definition, there exists $v \in D(X)$ such that $f(u) = f(v)$. Hence $u = v$ since f is injective, a contradiction since $\deg(u) = 1$ and $\deg(v) \geq 2$. Finally, since $0 \in f(X)$ and $A \cap D \subseteq f(X)$, we have $A \setminus f(X) \subseteq A^* \setminus (A \cap D) = P^*$. \square

Proposition 2.14. *Let S be a numerical semigroup. Then S admits a divset model.*

Proof. It is well known and easy to verify that, for any decomposable Apéry element $a \in A \cap D$, if $a = s_1 + s_2$ with $s_1, s_2 \in S^*$, then necessarily $s_1, s_2 \in A^*$. For any $a \in A \cap D$, consider the unique decomposition $a = p_{i_1} + \dots + p_{i_d}$ into primitive elements $p_{i_j} \in P^*$ which is lexicographically minimal of minimal length d . Clearly, for any nonempty subsum of $p_{i_1} + \dots + p_{i_d}$, the same minimality properties hold.

Let $\{p_1, \dots, p_n\} \subseteq P^*$ be the set of all primitive elements involved in the respective minimal decompositions of the elements in $A \cap D$. Let $X_1 = \{x_1, \dots, x_n\}$ be a set of n commuting variables. We set $u(p_i) = x_i$ for all $1 \leq i \leq n$. Then, for all $a \in A \cap D$, we associate to a the monomial $u(a)$ of degree d in $\{x_1, \dots, x_n\}$ mirroring, in multiplicative notation, the minimal decomposition $a = p_{i_1} + \dots + p_{i_d}$. Let

$$X = \{1\} \sqcup X_1 \sqcup \{u(a) \mid a \in A \cap D\}.$$

By the remark on subsums above, if $a = a_1 + a_2$ with $a_1, a_2 \in A^*$, then $u(a) = u(a_1)u(a_2)$. Hence X is a divset. Let $f : X \rightarrow S$ be the unique morphism induced by $f(x_i) = p_i$ for all $i = 1, \dots, n$. Then f is an injective morphism, $f(X) \subseteq A$ and $f(D(X)) = A \cap D$ by construction. Hence $f : X \rightarrow S$ is a divset model of S , as desired. \square

Example 2.15. Let $S = \langle 5, 6, 9 \rangle$. Then $m = 5$ and $A^* = \{w_1, w_2, w_3, w_4\} = \{6, 12, 18, 9\}$ with $w_i = \min(S \cap (i + 5\mathbb{N}))$ for $1 \leq i \leq 4$. Let

$$\begin{aligned} X &= [x_1^2, x_2^2] = \{x_1^2, x_2^2, x_1, x_2\}, \\ X' &= [x_1^3, x_2] = \{x_1^3, x_1^2, x_1, x_2\}. \end{aligned}$$

The morphisms $f : X \rightarrow S$ and $f' : X' \rightarrow S$ both induced by $x_1 \mapsto 6, x_2 \mapsto 9$ yield two distinct divset models for S , as easily verified using $18 = 2 \cdot 9 = 3 \cdot 6$.

Remark 2.16. One advantage of divsets X as abstract models of Apéry sets A is that for any $u \in X^*$, a decomposition $u = vw$ with $v, w \in X^*$ is unique up to order, whereas in A^* , decompositions $a = b + c$ are seldom unique in general as seen in the above example. Moreover, with divsets X one can use the terminology of monomials such as degree, divisibility and so on, notions which are less intuitive in additive notation.

Remark 2.17. In a divset model $f : X \rightarrow S$, by removing from X variables $x_i \in X_1$ not dividing any $u \in D(X)$, the resulting subset $X' \subseteq X$ is still a divset and the restriction $f' : X' \rightarrow S$ of f to X' is still a divset model since $D(X') = D(X)$. Thus, we may and will assume without loss of generality that $X_1 \subseteq V$, where $V = V(X)$ is the vertex set of the associated graph $G(X)$.

Recall that for a divset model $f : X \rightarrow S$ where S is of multiplicity m , we have $|X| \leq m$. The case of equality deserves a name.

Definition 2.18. Let S be a numerical semigroup of multiplicity m . A full divset model of S is a divset model $f : X \rightarrow S$ such that $|X| = m$.

Note that, for a full divset model $f : X \rightarrow S$, we have $f(X_1) = P^*$ and so $f(X) = A$.

3 Depth, $W(S)$, $W_0(S)$

We recall here some needed material for later use. Throughout this section, S is a numerical semigroup of multiplicity m and conductor c .

Definition 3.1 (See [12]). The depth of S is the positive integer

$$q = \lceil c/m \rceil.$$

Thus $c = qm - \rho$ where $\rho \in \llbracket 0, m-1 \rrbracket$.

In fact $\rho \in \llbracket 0, m-2 \rrbracket$. For otherwise, if $\rho = m-1$ then $c \equiv 1 \pmod{m}$, an absurdity since then its Frobenius number $F = c-1 = \max(\mathbb{Z} \setminus S)$ would be a multiple of m .

3.1 The depth function

We keep the same notation as above, namely $c = qm - \rho$ where $\rho \in \llbracket 0, m-2 \rrbracket$. As in [9], we partition S as $S = \sqcup_{i \in \mathbb{N}} S_i$, where for all $i \in \mathbb{N}$,

$$S_i = S \cap \llbracket im - \rho, (i+1)m - \rho - 1 \rrbracket.$$

In particular, we have

$$\begin{aligned} S_0 &= \{0\}, \\ S_1 &= \llbracket m, 2m - \rho - 1 \rrbracket \cap S, \\ &\dots \\ S_{q-1} &= \llbracket c - m, c - 1 \rrbracket \cap S, \\ S_q &= \llbracket c, c + m - 1 \rrbracket. \end{aligned}$$

This gives rise to the following function.

Definition 3.2. The depth function $\delta : S \rightarrow \mathbb{Z}$ is defined for all $x \in S$ by

$$(4) \quad \delta(x) = i \iff x \in S_{q-i}.$$

Equivalently, $\delta(x)$ is the unique integer such that

$$(5) \quad x + \delta(x)m \in \llbracket c, c + m - 1 \rrbracket,$$

$$\text{i.e. } \delta(x) = \lceil \frac{c-x}{m} \rceil.$$

The function δ assumes the following values. Recall that $L = S \cap \llbracket 0, c - 1 \rrbracket$.

Lemma 3.3. For all $x \in S$, we have

$$\begin{aligned} \delta(x) = q &\iff x = 0, \\ \delta(x) \in \llbracket 1, q-1 \rrbracket &\iff x \in L \setminus \{0\}, \\ \delta(x) = 0 &\iff x \in \llbracket c, c + m - 1 \rrbracket, \\ \delta(x) \leq -1 &\iff x \geq c + m. \end{aligned}$$

Proof. Straightforward from the definition. □

We shall need the following estimates from [10, Proposition 6].

Proposition 3.4. For all $x, y \in S$, we have

$$(6) \quad \delta(x+y) + q + 1 \geq \delta(x) + \delta(y) \geq \delta(x+y) + q - \min(\rho, 1).$$

Proof (Outline). First note that for all $k, l \geq 1$, we have

$$S_k + S_l \subset S_{k+l-\min(\rho, 1)} \cup S_{k+l} \cup S_{k+l+1}.$$

Set $\delta(x) = i, \delta(y) = j$. By (4), this means $x \in S_{q-i}, y \in S_{q-j}$. Hence

$$x+y \in S_{2q-i-j-\min(\rho, 1)} \cup S_{2q-i-j} \cup S_{2q-i-j+1}.$$

By (4) again, this means $-q + i + j - 1 \leq \delta(x+y) \leq -q + i + j + \min(\rho, 1)$. The claimed inequalities follow. □

3.2 Total depth

Definition 3.5. Given a finite subset $E \subset S$, the total depth of E is

$$\delta(E) = \sum_{x \in E} \delta(x).$$

Lemma 3.6. Let $A = \text{Ap}(S, m)$, $A^* = A \setminus \{0\}$ and $D = S^* + S^*$. Then

$$\begin{aligned} \delta(A^*) &\subseteq \llbracket 0, q-1 \rrbracket, \\ m &= |P| + |A \cap D|, \\ |L| &= \delta(A) = \delta(A^*) + q. \end{aligned}$$

Proof. Since $A^* \subseteq \llbracket m, c+m-1 \rrbracket$, it follows from Lemma 3.3 that $\delta(A^*) \subseteq \llbracket 0, q-1 \rrbracket$. We have $|A^*| = m-1$ and $A^* = (A \cap P) \sqcup (A \cap D)$. Also $P = (A \cap P) \sqcup \{m\}$. Hence $|P| = |A \cap P| + 1$ and so

$$m = |A^*| + 1 = |A \cap D| + |A \cap P| + 1 = |A \cap D| + |P|.$$

We have $L = S \cap \llbracket 0, c-1 \rrbracket = \{0\} \sqcup (S \cap \llbracket m, c-1 \rrbracket)$. Moreover, for all $a \in A$, we have

$$L \cap (a + m\mathbb{N}) = a + \llbracket 0, \delta(a) - 1 \rrbracket m.$$

Hence $|L| = \sum_{a \in A} \delta(a) = \delta(A)$. Since $\delta(0) = q$, the formula $|L| = q + \delta(A^*)$ follows. \square

3.3 On the numbers $W(S), W_0(S)$

The numbers $W(S), W_0(S)$ attached to the numerical semigroup S were introduced in [9]. The alternate notation $E(S) = W_0(S)$ was subsequently proposed in [4] and elsewhere. Recall the notation $D = S^* + S^*$ and $P = S^* \setminus D$.

Notation 3.7. We denote $D_q = D \cap S_q = D \cap \llbracket c, c+m-1 \rrbracket$, and

$$\begin{aligned} W(S) &= |P||L| - c, \\ W_0(S) &= |P \cap L||L| - |A \cap D_q| + \rho. \end{aligned}$$

Thus, Wilf's conjecture amounts to the inequality

$$(7) \quad W(S) \geq 0$$

for all numerical semigroups S . The interest of $W_0(S)$ stems from the inequality

$$W(S) \geq W_0(S).$$

Therefore, if $W_0(S) \geq 0$ then S satisfies Wilf's conjecture in a somewhat stronger sense. For instance, the following result is proved in [9].

Theorem 3.8. Let S be a numerical semigroup such that $c \leq 3m$. Then $W_0(S) \geq 0$.

There are cases where $W_0(S) \leq -1$, but those are extremely rare. See [4, 11] for more details. Note finally that if $P \subseteq L$ then $W_0(S) = W(S)$.

3.4 New formulas

The following formulas exhibit a closer relationship between $W(S)$ and $W_0(S)$ than from the original ones. The symbols A^*, P, D keep the same meaning as above. We further denote $P_q = P \cap S_q$, $p_q = |P_q|$ and $d_q = |D_q|$.

Proposition 3.9. *Let S be a numerical semigroup. Then*

$$\begin{aligned} W(S) &= |P|\delta(A^*) - |A \cap D|q + \rho, \\ W_0(S) &= |P \cap L|\delta(A^*) - |A \cap D|q + \rho. \end{aligned}$$

Proof. We have

$$\begin{aligned} W(S) &= |P||L| - c \\ &= |P|(\delta(A^*) + q) - qm + \rho \\ &= |P|(\delta(A^*) + q) - q(|P| + |A \cap D|) + \rho \\ &= |P|\delta(A^*) - |A \cap D|q + \rho. \end{aligned}$$

Moreover,

$$\begin{aligned} W(S) &= |P||L| - c \\ &= (|P \cap L| + p_q)|L| - qm + \rho \\ &= (|P \cap L| + p_q)|L| - q(p_q + d_q) + \rho \\ &= |P \cap L||L| - qd_q + \rho + p_q(|L| - q) \\ &= W_0(S) + p_q(|L| - q) \\ &= W_0(S) + p_q\delta(A^*). \quad \square \end{aligned}$$

Corollary 3.10. *We have*

$$W(S) - W_0(S) = |P_q|\delta(A^*).$$

Proof. Since $|P| = |P \cap L| + |P_q|$, the proposition implies

$$\begin{aligned} W(S) - W_0(S) &= (|P| - |P \cap L|)\delta(A^*) \\ &= |P_q|\delta(A^*). \quad \square \end{aligned}$$

Corollary 3.11. *We have*

$$\begin{cases} W_0(S) \leq W(S), \\ W_0(S) = W(S) \iff P \subseteq L. \end{cases}$$

Proof. Indeed, since $P \subseteq S_1 \sqcup \cdots \sqcup S_q$, we have $P_q = \emptyset \iff P \subseteq L$. □

Corollary 3.12. *If S is special of depth q , then $W(S) = |P|\delta(A^*) - |A \cap D|q$. □*

4 Focus on the special case

Throughout this section, S denotes a special numerical semigroup of multiplicity m , conductor c and depth $q = \lceil c/m \rceil = c/m$, i.e. with $\rho = 0$. As above, $A = \text{Ap}(S, m)$ and $A^* = A \setminus \{0\}$.

Proposition 4.1. *Let S be a special numerical semigroup of depth q . Let X be a divset model of S and let $n = |X_1|$. Then*

$$\begin{aligned}\delta(A^*) &\geq \text{vm}(X)q/2, \\ W(S) &\geq ((n+1)\text{vm}(X) - 2|D(X)|)q/2.\end{aligned}$$

Proof. By hypothesis, there is an injective morphism $f : X \rightarrow S$ such that $f(X) \subseteq A$ and $f(D(X)) = A \cap D$. Let $G = G(X) = (V, E)$ be the graph of X . Denote $k = \text{vm}(X)$. Hence, there is a subset $M \subseteq V \subseteq X^*$ of cardinality $|M| = k$ such that

$$M = \bigsqcup_{i=1}^{\ell} z_i,$$

where $z_1, \dots, z_\ell \in E$ are pairwise disjoint edges. Loops are allowed, so $|z_i| \in \{1, 2\}$ for all i . We have

$$(8) \quad |M| = k = |z_1| + \dots + |z_\ell|.$$

Let $z \in \{z_1, \dots, z_\ell\}$. Then $z = \{u_1, u_2\}$ with $u_1, u_2 \in X^*$ such that $u_1 u_2 \in X^*$. Let

$$a_1 = f(u_1), a_2 = f(u_2), a = f(u_1 u_2).$$

Then $a = a_1 + a_2$ since f is a morphism, and $a \in A^*$ since $f(X^*) \subseteq A^*$. We have $\delta(a) \geq 0$ since $\delta(A^*) \subseteq \llbracket 0, q-1 \rrbracket$ by Lemma 3.6. Hence

$$\delta(a_1) + \delta(a_2) \geq q + \delta(a_1 + a_2) = q + \delta(a) \geq q$$

by the right inequality in (6) and the value $\rho = 0$. We have $f(z) = \{a_1, a_2\}$ and so

$$\delta(f(z)) = \delta(a_1) + \delta(a_2) \geq q.$$

If $|z| = 1$ then $a_1 = a_2$ and $\delta(a_1) \geq q/2$, whereas if $|z| = 2$ then $\delta(f(z)) = \delta(a_1) + \delta(a_2) \geq q$. Summarizing, we have

$$\delta(f(z)) \geq |z|q/2$$

for all $z \in \{z_1, \dots, z_\ell\}$. Hence

$$\delta(f(\{z_1, \dots, z_\ell\})) = \sum_{i=1}^{\ell} \delta(f(z_i)) \geq \left(\sum_{i=1}^{\ell} |z_i| \right) q/2.$$

Since $\sum_{i=1}^{\ell} |z_i| = |M| = k$ by (8), it follows that $\delta(f(\{z_1, \dots, z_\ell\})) \geq kq/2$. And since

$$f(\{z_1, \dots, z_\ell\}) \subseteq f(X^*) \subseteq A^*,$$

we conclude that

$$\delta(A^*) \geq \delta(f(X^*)) \geq \delta(f(\{z_1, \dots, z_\ell\})) \geq kq/2,$$

as desired.

It remains to prove the formula $W(S) \geq ((n+1)\text{vm}(X) - 2|D(X)|)q/2$. By Proposition 3.9, we have $W(S) = |P|\delta(A^*) - |A \cap D|q + \rho$. Since $\delta(A^*) \geq \text{vm}(X)q/2$ and $\rho = 0$, this yields $W(S) \geq (|P|\text{vm}(X)/2 - |A \cap D|)q$. Finally, since $|P| \geq |X_1| + 1 = n + 1$ and $|A \cap D| = |f(D(X))| = |D(X)|$, the desired inequality follows. \square

Corollary 4.2. *Let S be a special numerical semigroup of depth q . Let X be a divset model of S . If $\text{vm}(X) \geq 2k$ then $\delta(A^*) \geq kq$.*

Proof. By Proposition 4.1 and the hypothesis, we have $\delta(A^*) \geq \text{vm}(X)q/2 \geq kq$. \square

Corollary 4.3. *Let S be a special numerical semigroup of depth q such that $|P| \geq m/4$. Let X be a divset model of S . If $\text{vm}(X) \geq 6$ then S satisfies Wilf's conjecture.*

Proof. It follows from the previous corollary that $\delta(A^*) \geq 3q$. By Lemma 3.6, it follows that $|L| \geq 4q$. Hence $|P||L| \geq (m/4)(4q) = qm = c$, i.e. $W(S) \geq 0$ as claimed. \square

4.1 Tame divsets

Definition 4.4. *Let X be a divset. Let $G(X) = (V, E)$ be the graph of X . Set $V_1 = V \cap X_1$ and $n = |V_1|$. We say that X is tame if*

$$(n+1)\text{vm}(X) \geq 2|D(X)|.$$

We say that X is wild if it is not tame, i.e. if $(n+1)\text{vm}(X) < 2|D(X)|$.

Proposition 4.5. *Let S be a special numerical semigroup of depth $q \geq 2$. If S admits a tame divset model X , then $W_0(S) \geq 0$ and hence S satisfies Wilf's conjecture.*

Proof. Let $f : X \rightarrow S$ be a divset model with X tame. Let $G(X) = (V, E)$ be the graph of X and $n = |V_1| = |V \cap X_1|$. Since S is tame, we have $(n+1)\text{vm}(X) \geq 2|D(X)|$.

Claim. We have $f(V_1) \subseteq P^* \cap L$. Indeed, we have $f(V_1) \subseteq f(X_1) \subseteq P^*$ by Proposition 2.13. Moreover, if $x_i \in V_1$, there exists $u \in X^*$ such that $x_i u \in X^*$. Hence

$$f(x_i) + f(u) = f(x_i u) \in f(X^*) \subseteq A^* \subseteq \llbracket m, c + m - 1 \rrbracket.$$

Since $f(u) \in A^*$ hence $f(u) \geq m$, it follows that $f(x_i) \leq c - 1$, i.e. $f(x_i) \in L$. This settles the claim.

As $c = qm$ with $q \geq 2$, we have $P \cap L = (P^* \cap L) \sqcup \{m\}$. Since f is injective, the claim implies

$$|P \cap L| = |P^* \cap L| + 1 \geq |f(V_1)| + 1 \geq |X_1| + 1 \geq n + 1.$$

Now $|A \cap D| = |f(D(X))| = |D(X)|$, and $\delta(A^*) \geq \text{vm}(X)q/2$ by Proposition 4.1.

Summarizing, and using the formula for $W_0(S)$ in Proposition 3.9, we have

$$\begin{aligned} W_0(S) &= |P \cap L| \delta(A^*) - |A \cap D| q \\ &\geq (n + 1) \text{vm}(X) q / 2 - |A \cap D| q \\ &= ((n + 1) \text{vm}(X) - 2|D(X)|) q / 2. \end{aligned}$$

Now $(n + 1) \text{vm}(X) - 2|D(X)| \geq 0$ since X is tame. Hence $W_0(S) \geq 0$. \square

Almost all of the divsets involved in the next sections turn out to be tame. But here is an example of a wild divset.

Example 4.6. Let $X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3]$, i.e. the set of all monomials of degree at most 3 in x_1, x_2 . Then $|X^*| = |X_1| + |D(X)| = 2 + (3 + 4) = 9$. Here $n = |V_1| = 2$ and $|D(X)| = 7$. Moreover, $\text{vm}(X) = 4$ as witnessed by the vertex-maximal matching $\{\{x_1, x_1^2\}, \{x_2, x_2^2\}\}$ of the graph $G(X)$. Hence

$$(n + 1) \text{vm}(X) - 2|D(X)| = 3 \cdot 4 - 2 \cdot 7 = -2,$$

so that X is wild. However, it can be shown that for any special numerical semigroup S modeled by X , one has $|P| \geq 7$, whence

$$W(S) = |P| \delta(A^*) - |A \cap D| q \geq 7 \cdot \text{vm}(X) q / 2 - |D(X)| q = 14q - 7q = 7q.$$

Thus, all such numerical semigroups S satisfy Wilf's conjecture. Interestingly, those S include the five smallest numerical semigroups satisfying $W_0(S) \leq -1$ as described in [11].

4.2 Proof strategy

It follows from Corollary 4.3 that, in order to settle Wilf's conjecture for special numerical semigroups S satisfying $|P| \geq m/4$, it suffices to consider divsets X such that $\text{vm}(X) \leq 5$. Thus, we need to classify all such divsets.

To start with, since a divset is determined by its maximal elements, we introduce the following auxiliary notation.

Notation 4.7. Let $u_1, \dots, u_r \in \mathcal{M}$ and $X = [u_1, \dots, u_r]$. We set $\mu(u_1, \dots, u_r) = \text{vm}(X)$.

Determining $\mu(u_1, \dots, u_r)$ is difficult in general, due to its intimate relationship with matching numbers in graphs. Nevertheless, the case $r = 1$ is straightforward.

Lemma 4.8. *Let $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}$. Then $\mu(u) = \prod_{i=1}^n (a_i + 1) - 2$.*

Proof. Let $X = [u]$, the divset of divisors of u . Then $\mu(u) = \text{vm}(X)$, i.e. the maximal number of vertices touched by a matching of the graph $G(X)$. Now $|X| = \prod_{i=1}^n (a_i + 1)$, the number of divisors of u . These divisors may be regrouped in independent edges of the form $\{v, u/v\}$ where v is a divisor of u such that $0 \leq \deg(v) \leq \deg(u/v)$, and v is lexicographically smaller than or equal to u/v if $\deg(v) = \deg(u/v)$. The pair $\{1, u\}$ must be discounted since 1 is not a vertex of G by definition. The other pairs constitute a matching of G covering the whole of $X \setminus \{0, u\}$. Hence $\mu(u) = \text{vm}(X) = |X \setminus \{0, u\}| = \prod_{i=1}^n (a_i + 1) - 2$. \square

Coupled with more combinatorial arguments, this basic result will allow us to classify all divsets X such that $\text{vm}(X) \leq 5$, as needed. Then in each case, we shall show that all special numerical semigroups S modeled by those divsets and such that $|P| \geq m/4$ satisfy Wilf's conjecture, thereby proving the main result in this paper.

5 The case $\deg(X) = 2$

We prove here that divsets of degree 2 are tame. This establishes Wilf's conjecture for all special numerical semigroups S such that $A \cap D \subseteq P^* + P^*$. In order to do so, we need the following lemma about the classical *matching number* in graphs, namely the maximum number of independent edges.

Lemma 5.1. *Let $H = (H_1, H_2)$ be a simple bipartite graph without isolated vertices, with matching number k . Then*

$$|E(H)| \leq k \cdot \max(|H_1|, |H_2|).$$

Proof. We may assume $|H_1| \leq |H_2|$. Then $k \leq |H_1|$. Let M be a maximal matching with k edges, say

$$M = \{(x_i, y_i) \mid x_i \in H_1, y_i \in H_2, 1 \leq i \leq k\}.$$

Claim. $|H_1| = k$. Assume not. Let then $x_{k+1} \in H_1 \setminus \{x_1, \dots, x_k\}$. Since H does not have isolated vertices, it follows that x_{k+1} has a neighbour $z \in H_2$. Now necessarily $z \in \{y_1, \dots, y_k\}$, for otherwise there would be a new edge (x_{k+1}, z) independent of M , which would then yield a matching of cardinality $k + 1$, a contradiction. Up to renumbering, we may assume $z = y_k$, i.e.

$$(x_{k+1}, y_k) \in E.$$

As $|H_2| \geq |H_1| \geq k + 1$, there is a vertex $y_{k+1} \in H_2 \setminus \{y_1, \dots, y_k\}$. Since y_{k+1} is not isolated, it has a neighbour $z \in H_1$. But as in the preceding reasoning, we have $z \in \{x_1, \dots, x_k\}$, for otherwise the edge (z, y_{k+1}) would be independent of M , a contradiction. Up to renumbering, we may assume $z = x_k$. Thus

$$(x_k, y_{k+1}) \in E.$$

But then, by suppressing the edge (x_k, y_k) of M and replacing it by the two independent edges $(x_{k+1}, y_k), (x_k, y_{k+1})$, we obtain a matching

$$M' = M \sqcup \{(x_{k+1}, y_k), (x_k, y_{k+1})\} \setminus \{(x_k, y_k)\}$$

of cardinality $k + 1$, a contradiction. This finishes the proof of the claim, whence

$$|H_1| = k.$$

Since $H = (H_1, H_2)$ is bipartite, we have $|E| \leq |H_1||H_2| = k|H_2|$, as claimed. \square

Proposition 5.2. *Let X be a divset of degree 2, i.e. such that $D(X) = X_2$. Then X is tame, i.e. $(n + 1) \text{vm}(X) \geq 2|D(X)|$, where $n = |V \cap X_1|$ and V is the vertex set of the graph $G(X)$.*

Proof. Let $V_1 = V \cap X_1 = \{x_1, \dots, x_n\}$, the set of variables dividing the monomials in $D(X) = X_2$. We have $V = V_1$ since if $u \in V$, there exists $v \in V$ such that $uv \in D(X) = X_2$, implying $\deg(u) = \deg(v) = 1$. Clearly $|D(X)| \leq n(n + 1)/2$. Let $M \subseteq D(X)$ be a maximal set of pairwise coprime monomials. Denote by $V_2 \subseteq V_1$ the set of variables involved in M . Let $n_2 = |V_2|$. Since M corresponds to a matching in $G(X)$ with vertex set V_2 , we have

$$\text{vm}(X) \geq n_2.$$

Case 1. $V_2 = V_1$. In this case we are done. Indeed, we then have $n_2 = n$, so that $\text{vm}(X) \geq n$.

And since $|D(X)| \leq n(n + 1)/2$, it follows that

$$(n + 1) \text{vm}(X) - 2|D(X)| \geq (n + 1)n - 2n(n + 1)/2 = 0,$$

as desired.

Case 2. Assume $V_2 \neq V_1$. Let $V_3 = V_1 \setminus V_2$ and $n_3 = |V_3| = n - n_2$. By maximality of M , every monomial $u = x_i x_j \in D(X)$ with $i \leq j$ satisfies $x_i \in V_2$ or $x_j \in V_2$. Let $N \subseteq D(X)$ be a largest possible subset of pairwise coprime monomials $x_i x_j$ with $x_i \in V_2, x_j \in V_3$. Let $k = |N|$. We have $k \leq \min(n_2, n_3)$ and $k \geq 1$ since $V_3 \neq \emptyset$. Each $x_i x_j \in N$ independently contributes a summand 2 to $\text{vm}(X)$. Thus

$$(9) \quad \text{vm}(X) \geq (n_2 - k) + 2k = n_2 + k.$$

Next, we claim that

$$(10) \quad |D(X)| \leq n_2(n_2 + 1)/2 + k \max(n_2, n_3).$$

Indeed, as V_2 induces the empty subgraph in $G(X)$, every monomial in $D(X)$ has its support in $[V_1, V_1]$ or $[V_1, V_2]$. Now the preceding lemma implies that the number of monomials with support in $[V_1, V_2]$ is less than or equal to $k \max(n_2, n_3)$. This proves (10). Combined with (9), this yields

$$(n+1)\text{vm}(X) - 2|D(X)| \geq (n+1)(n_2+k) - n_2(n_2+1) - 2k \max(n_2, n_3).$$

It remains to show that the right-hand side is non-negative. Since $n = n_2 + n_3$, we have

$$\begin{aligned} (n+1)(n_2+k) - n_2(n_2+1) - 2k \max(n_2, n_3) &= n_2(n+1) - n_2(n_2+1) + k(n+1 - 2 \max(n_2, n_3)) \\ &= n_2n_3 + k(n_2 + n_3 + 1 - 2 \max(n_2, n_3)) \\ &= n_2n_3 + k(\min(n_2, n_3) - \max(n_2, n_3) + 1) \end{aligned}$$

since $n_2 + n_3 = \min(n_2, n_3) + \max(n_2, n_3)$. But for any $k \leq a \leq b$, we have

$$\begin{aligned} ab + k(a-b+1) &= k(a+1) + b(a-k) \\ &\geq k(a+1). \end{aligned}$$

Hence $(n+1)(n_2+k) - n_2(n_2+1) - 2k \max(n_2, n_3) \geq k(\min(n_2, n_3) + 1) \geq 0$, as desired. \square

Corollary 5.3. *Let S be a special numerical semigroup such that $A \cap D \subseteq P^* + P^*$. Then S satisfies Wilf's conjecture.*

Proof. Let $f : X \rightarrow S$ be a divset model. Let $G = (V, E)$ be the graph of X . By Remark 2.17, we may assume $X_1 \subseteq V$. Moreover, since $f(D(X)) = A \cap D \subseteq P^* + P^*$, we may assume that $D(X) \subseteq X_1 \cdot X_1$, i.e. that $\deg(X) = 2$. We conclude with Propositions 4.5 and 5.2. \square

6 The case $\deg(X) \geq 3$

The case $\deg(X) = 2$ being settled, from here on we only consider divsets X satisfying $\deg(X) \geq 3$. [...]

References

- [1] M. BRAS-AMORÓS, Fibonacci-like behavior of the number of numerical semigroups of a given genus, *Semigroup Forum* 76 (2008) 379–384.
- [2] W. BRUNS, P. GARCIA-SANCHEZ, C. O'NEILL, D. WILBURNE, Wilf's conjecture in fixed multiplicity. arXiv 1903.04342 [math.CO], 2019.
- [3] G. CHARTRAND, L. LESNIAK, P. ZHANG, *Graphs & digraphs*. Sixth edition. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2016. xii+628 pp. ISBN: 978-1-4987-3576-6.
- [4] M. DELGADO, On a question of Eliahou and a conjecture of Wilf, *Math. Z.* 288 (2018) 595–627.

- [5] M. DELGADO, Conjecture of Wilf: a survey. To appear in Numerical Semigroups - IMNS 2018, Springer INdAM Series 40, ISBN : 978-3-030-40821-3. arXiv 1902.03461 [math.CO].
- [6] M. DELGADO, Trimming the numerical semigroups tree to probe Wilf's conjecture to higher genus. arXiv 1910.12377 [math.CO], 2019.
- [7] M. DELGADO, S. ELIAHOU AND J. FROMENTIN, A verification of Wilf's conjecture up to genus 100. 2023. <https://hal.science/hal-04236367>
- [8] M. DELGADO, P.A. GARCÍA-SÁNCHEZ AND J. MORAIS, "Numericalsgps": a GAP package on numerical semigroups. <http://www.gap-system.org/Packages/numericalsgps.html>
- [9] S. ELIAHOU, Wilf's conjecture and Macaulay's theorem, J. Eur. Math. Soc. 20 (2018) 2105–2129. DOI 10.4171/JEMS/807.
- [10] S. ELIAHOU, A graph-theoretic approach to Wilf's conjecture. Elec. J. Combin. 27(2) (2020), #P2.15, 31 pp. <https://doi.org/10.37236/9106>.
- [11] S. ELIAHOU AND J. FROMENTIN, Near-misses in Wilf's conjecture, Semigroup Forum 98 (2019) 285-298. DOI 10.1007/s00233-018-9926-5.
- [12] S. ELIAHOU AND J. FROMENTIN, Gapsets and numerical semigroups, J. Combin. Theory Ser. A 169 (2020), 105129, 19 pp. DOI 10.1016/j.jcta.2019.105129.
- [13] R. FRÖBERG, C. GOTTLIEB AND R. HÄGGKVIST, On numerical semigroups, Semigroup Forum 35 (1987) 63–83.
- [14] J. FROMENTIN AND F. HIVERT, Exploring the tree of numerical semigroups, Math. Comp. 85 (2016) 2553–2568.
- [15] N. KAPLAN, Counting numerical semigroups by genus and some cases of a question of Wilf, J. Pure Appl. Algebra 216 (2012) 1016–1032.
- [16] N. KAPLAN AND L. YE, The proportion of Weierstrass semigroups, J. Algebra 373 (2013) 377–391.
- [17] A. MOSCARIELLO AND A. SAMMARTANO, On a conjecture by Wilf about the Frobenius number, Math. Z. 280 (2015) 47–53.
- [18] J.L. RAMÍREZ ALFONSÍN, The Diophantine Frobenius problem. Oxford Lecture Series in Mathematics and its Applications 30, Oxford University Press, Oxford, 2005.

- [19] J.C. ROSALES AND P.A. GARCÍA-SÁNCHEZ, Numerical semigroups. *Developments in Mathematics*, 20. Springer, New York, 2009.
- [20] A. SAMMARTANO, Numerical semigroups with large embedding dimension satisfy Wilf's conjecture, *Semigroup Forum* 85 (2012) 439–447.
- [21] J.J. SYLVESTER, Mathematical questions with their solutions, *Educational Times* 41 (1884) 21.
- [22] H. WILF, A circle-of-lights algorithm for the money-changing problem, *Amer. Math. Monthly* 85 (1978) 562–565.
- [23] A. ZHAI, Fibonacci-like growth of numerical semigroups of a given genus, *Semigroup Forum* 86 (2013) 634–662.

Author's address

Shalom Eliahou

Univ. Littoral Côte d'Opale, UR 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62100 Calais, France and CNRS, FR 2037, France.

eliahou@univ-littoral.fr