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# Divsets, numerical semigroups and Wilf's conjecture

Shalom Eliahou

## Abstract

Let  $S \subseteq \mathbb{N}$  be a numerical semigroup with multiplicity  $m = \min(S \setminus \{0\})$  and conductor  $c = \max(\mathbb{Z} \setminus S) + 1$ . Let  $P$  be the set of primitive elements, i.e. minimal generators, of  $S$ , and let  $L$  be the set of elements of  $S$  which are smaller than  $c$ . Wilf's conjecture (1978) states that the inequality  $|P||L| \geq c$  holds. The conjecture has been shown to hold in case  $|P| \geq m/2$  by Sammartano in 2012, and subsequently in case  $|P| \geq m/3$  by the author in 2020. The main result in this paper is that Wilf's conjecture holds in case  $|P| \geq m/4$  if  $m$  divides  $c$ .

## 0 Caution

This as yet<sup>1</sup> incomplete document will be gradually completed in successive versions during Fall 2023.

## 1 Introduction

A *numerical semigroup* is a cofinite submonoid  $S$  of  $\mathbb{N}$ , i.e. a subset containing 0, stable under addition and with finite complement  $\mathbb{N} \setminus S$ . Equivalently, it is a set of the form  $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$  where  $a_1, \dots, a_n$  are positive integers with  $\gcd(a_1, \dots, a_n) = 1$ , called *generators* of  $S$ . The least such  $n$  is usually denoted  $e = e(S)$  and called the *embedding dimension* of  $S$ . The *multiplicity* of  $S$  is  $m = m(S) = \min S^*$ , where  $S^* = S \setminus \{0\}$ . The *Frobenius number* of  $S$  is  $F = F(S) = \max(\mathbb{Z} \setminus S)$  and the *conductor* of  $S$  is  $c = c(S) = F + 1$ , satisfying  $c + \mathbb{N} \subseteq S$  and minimal with respect to that property. The *genus* of  $S$  is  $g = g(S) = |\mathbb{N} \setminus S|$ , its number of gaps. We partition  $S$  as  $S = L \sqcup R$ , where  $L = L(S) = \{a \in S \mid a < F(S)\}$  and  $R = R(S) = \{a \in S \mid a > F(S)\}$ , the *left part* and *right part* of  $S$ , respectively.

A *primitive element* of  $S$  is an element  $a \in S^* \setminus (S^* + S^*)$ , i.e. an element of  $S^*$  which is not the sum of two elements of  $S^*$ . We denote by  $P = P(S)$  the set of primitive elements of  $S$ , and by  $D = D(S) = S^* + S^*$  the set of *decomposable elements* of  $S$ . It is easy to see that  $P$  is contained in  $[m, c + m - 1] \cap \mathbb{N}$  and hence is finite, and is the unique minimal generating set of  $S$ . Thus  $|P| = e(S)$ .

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<sup>1</sup>November 7th, 2023

One of the main open problems on numerical semigroups is the following conjecture, first raised as a question by Wilf [22].

**Conjecture 1.1** (Wilf, 1978). *Let  $S$  be a numerical semigroup. Then  $|P(S)||L(S)| \geq c(S)$ .*

See [5] for a survey on the conjecture up to 2018. Among many available partial results, we shall need here the following ones, grouped for convenience in a single statement.

**Theorem 1.2.** *Let  $S$  be a numerical semigroup. Then  $S$  satisfies Wilf's conjecture if either  $|P| \leq 3$ , or  $|P| \geq m/3$ , or  $c \leq 3m$ .*

The solution in case  $|P| = 2$  is due to Sylvester [21]; its extension to  $|P| \leq 3$  is due to Fröberg et al. [13]. The solution in case  $|P| \geq m/2$  is due to Sammartano [20]; its extension to  $|P| \geq m/3$  is achieved in [10]. Finally, the case  $c \leq 3m$  is settled in [9].

**Notation 1.3.** *For  $a, b \in \mathbb{Z}$ , we denote by  $\llbracket a, b \rrbracket = [a, b] \cap \mathbb{Z}$  the integer interval they span.*

## 1.1 Special numerical semigroups

The main result in this paper extends the above case  $|P| \geq m/3$  in Wilf's conjecture as follows.

**Theorem 1.4.** *Let  $S$  be a numerical semigroup such that  $|P| \geq m/4$  and  $c \in m\mathbb{N}$ . Then  $S$  satisfies Wilf's conjecture.*

What motivates the added hypothesis  $c \in m\mathbb{N}$ ? As it happens, the proofs of Wilf's conjecture in either case  $c \leq 3m$  [9] or  $|P| \geq m/3$  [10] can be significantly shortened when  $c \in m\mathbb{N}$ . Moreover, the first five instances of the very rare “near-misses in Wilf's conjecture” all belong to this case [11]. These facts lead us to consider the case  $c \in m\mathbb{N}$  as a priority in research on Wilf's conjecture. Indeed, we believe that if the conjecture fails, then it will already fail in case  $c \in m\mathbb{N}$ . Whence the following terminology.

**Definition 1.5.** *A numerical semigroup  $S$  is special if its conductor  $c$  is a multiple of its multiplicity  $m$ .*

For instance, the *ordinary* (or *superficial*) numerical semigroup  $O_m = \{0\} \cup (m + \mathbb{N})$  is special since it satisfies  $c = m$ .

## 1.2 Contents

In Section 2, we introduce divsets as abstract models of Apéry sets. In Section 3, we recall some needed material about the depth and the functions  $W(S), W_0(S)$ . In Section 4, we start focusing on special numerical semigroups. In Section 5, we settle Wilf's conjecture for special numerical semigroups modeled by a divset of degree 2. In Section 6 – to be completed – we consider the case of divsets of degree at least 3 and conclude the proof of the main theorem.

## 2 Divsets

In this section, we recall what is the Apéry set of a numerical semigroup and introduce divsets as abstract models thereof.

### 2.1 The Apéry set

Let  $S$  be a numerical semigroup of multiplicity  $m$  and conductor  $c$ . Its Apéry set contains key information on its structure.

**Definition 2.1.** *The Apéry set of  $S$  with respect to  $m$  is defined as  $A = \text{Ap}(S, m) = S \setminus (m + S)$ . We denote  $A^* = A \setminus \{0\}$ .*

It is well known that  $\text{Ap}(S, m)$  has exactly  $m$  elements, one for each class mod  $m$ . More precisely, we have

$$\text{Ap}(S, m) = \{w_0, \dots, w_{m-1} \mid w_i = \min(S \cap (i + m\mathbb{N}))\}.$$

For instance,  $\min(A) = w_0 = 0$  and  $\max(A) = w_{c-1} = c + m - 1$ . Hence  $A \subseteq \llbracket 0, c + m - 1 \rrbracket$ . Moreover, denoting  $P^* = P \setminus \{m\}$ , it follows from the definition that

$$(1) \quad P^* \subseteq A^*.$$

The set  $S$  is completely determined by its Apéry set  $A$  via the formula

$$(2) \quad S = \bigsqcup_{a \in A} (a + m\mathbb{N}).$$

### 2.2 Basic definitions

Throughout this section, for given  $n \geq 1$ , we denote by

$$\mathcal{M} = \mathcal{M}(x_1, \dots, x_n) = \{x_1^{a_1} \cdots x_n^{a_n} \mid (a_1, \dots, a_n) \in \mathbb{N}^n\}$$

the set of monomials in  $n$  commuting variables  $x_1, \dots, x_n$ .

**Notation 2.2.** *The degree of  $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}$  is the standard one, namely  $\deg(u) = \sum_i a_i$ . For  $d \in \mathbb{N}$ , we denote by  $\mathcal{M}_d \subset \mathcal{M}$  the subset of monomials of degree  $d$ .*

**Definition 2.3.** *A divset in  $\mathcal{M}$  is a finite subset  $X \subset \mathcal{M}$  which is stable under taking divisors. That is, for all  $u \in X$  and  $v \in \mathcal{M}$ , if  $v \mid u$  then  $v \in X$ .*

Said otherwise, a divset is a finite *downset* or *order ideal* in  $\mathcal{M}$  under divisibility.

**Example 2.4.**  $X = \{x_1^3, x_1^2 x_2, x_1^2, x_1 x_2, x_1, x_2, 1\}$  is a divset in  $\mathcal{M}(x_1, x_2)$ .

**Definition 2.5.** Let  $X$  be a divset. We define the degree of  $X$  as

$$\deg(X) = \max\{\deg(u) \mid u \in X\}.$$

**Notation 2.6.** Given a subset of  $U \subseteq \mathcal{M}$  of monomials, we denote by  $[U]$  the set of divisors of the elements of  $U$ . That is,  $[U] = \{v \in \mathcal{M} \mid \exists u \in U, v|u\}$ .

In the above example  $X = \{x_1^3, x_1^2 x_2, x_1^2, x_1 x_2, x_1, x_2, 1\}$ , we have  $X = [x_1^3, x_1^2 x_2]$ .

**Notation 2.7.** Let  $X$  be a divset. We denote by  $\max(X)$  the maximal elements of  $X$  under divisibility. That is,  $\max(X)$  is the set of those  $u \in X$  which do not divide any  $v \in X \setminus \{u\}$ .

Clearly, a divset  $X$  is completely determined by  $\max(X)$ , namely as  $X = [\max(X)]$ .

**Notation 2.8.** Let  $X$  be a divset in  $\mathcal{M}$ . For  $d \in \mathbb{N}$ , we set  $X_d = X \cap \mathcal{M}_d = \{u \in X \mid \deg(u) = d\}$ .

For instance, we have  $X_0 = \mathcal{M}_0 = \{1\}$  and  $X_1 \subseteq \mathcal{M}_1 = \{x_1, \dots, x_n\}$ .

**Notation 2.9.** Let  $X$  be a divset. We denote by  $D(X) = \{u \in X \mid \deg(u) \geq 2\}$  the set of decomposable monomials in  $X$ . Thus  $X = \{1\} \sqcup X_1 \sqcup D(X)$ . We set  $X^* = X \setminus \{1\}$ .

## 2.3 The graph of a divset

Let  $X$  be a divset. We canonically associate to  $X$  a graph  $G = G(X)$  defined as follows. An edge in  $G$  is a pair  $\{u_1, u_2\}$  with  $u_1, u_2 \in X^*$  such that  $u_1 u_2 \in X^*$ . This defines the edge set  $E(G)$ . The set  $V(G)$  of vertices of  $G$  is defined as the set of the extremities of the edges. That is,  $V(G) = \{u \in X^* \mid \exists v \in X^*, uv \in X^*\}$ . We denote  $V(X) = V(G)$  and  $E(X) = E(G)$ . The graph  $G$  has no multiple edges, but it may contain loops, namely all pairs  $\{u, u\}$  such that  $u, u^2 \in X^*$ .

An important measure of  $X$  in the sequel is the *vertex-maximal matching number* of the graph  $G(X)$ , defined below.

**Notation 2.10.** We set  $\text{vm}(X) =$  the largest cardinality of a subset  $Y \subseteq X^*$  being the vertex set of a matching in  $G(X)$ , i.e. where  $Y$  is the union of pairwise disjoint pairs  $\{u, v\} \subseteq X^*$  such that  $uv \in X^*$ .

## 2.4 Modeling Apéry sets

Here we use divsets as abstract models of Apéry sets via specific maps  $f : X \rightarrow S$  from a divset  $X$  to a numerical semigroup  $S$ .

**Definition 2.11.** Let  $S$  be a numerical semigroup and  $X$  a divset. A map  $f : X \rightarrow S$  is a morphism if  $f(uv) = f(u) + f(v)$  for all  $u, v \in X$ .

Of course  $f(1) = 0$  for any such morphism, as  $f(1) = f(1 \cdot 1) = f(1) + f(1)$ .

**Definition 2.12.** *Let  $S$  be a numerical semigroup of multiplicity  $m$ . Let  $A = \text{Ap}(S, m)$ . A divset model of  $S$  is a divset  $X$  with an injective morphism  $f : X \rightarrow S$  such that  $f(X) \subseteq A$  and  $f(D(X)) = A \cap D$ .*

In particular, for all  $a \in A \cap D$ , i.e. such that  $a = a_1 + a_2$  with  $a, a_1, a_2 \in A^*$ , there are unique monomials  $u, u_1, u_2 \in X^*$  such that

$$(3) \quad f(u) = a, f(u_1) = a_1, f(u_2) = a_2.$$

Despite the name, a divset model of  $S$  is really an abstract multiplicative model of its Apéry set  $A$ , up to primitive elements not involved as summands of elements in  $A \cap D$ .

**Proposition 2.13.** *Let  $S$  be a numerical semigroup of multiplicity  $m$  and  $A = \text{Ap}(S, m)$ . Let  $f : X \rightarrow S$  be a divset model of  $S$ . Then  $|X| \leq m$ ,  $f(X_1) \subseteq P^*$  and  $A \setminus f(X) \subseteq P^*$ .*

*Proof.* Since  $f$  is injective and  $f(X) \subseteq A$ , we have  $|X| = |f(X)| \leq |A| = m$ . Moreover  $f(X^*) \subseteq A^*$  since  $f(1) = 0$ . Let  $u \in X_1$ . If  $f(u) \notin P^*$  then  $f(u) \in A \cap D = A^* \setminus P^*$ . Since  $A \cap D = f(D(X))$  by definition, there exists  $v \in D(X)$  such that  $f(u) = f(v)$ . Hence  $u = v$  since  $f$  is injective, a contradiction since  $\deg(u) = 1$  and  $\deg(v) \geq 2$ . Finally, since  $0 \in f(X)$  and  $A \cap D \subseteq f(X)$ , we have  $A \setminus f(X) \subseteq A^* \setminus (A \cap D) = P^*$ .  $\square$

**Proposition 2.14.** *Let  $S$  be a numerical semigroup. Then  $S$  admits a divset model.*

*Proof.* It is well known and easy to verify that, for any decomposable Apéry element  $a \in A \cap D$ , if  $a = s_1 + s_2$  with  $s_1, s_2 \in S^*$ , then necessarily  $s_1, s_2 \in A^*$ . For any  $a \in A \cap D$ , consider the unique decomposition  $a = p_{i_1} + \dots + p_{i_d}$  into primitive elements  $p_{i_j} \in P^*$  which is lexicographically minimal of minimal length  $d$ . Clearly, for any nonempty subsum of  $p_{i_1} + \dots + p_{i_d}$ , the same minimality properties hold.

Let  $\{p_1, \dots, p_n\} \subseteq P^*$  be the set of all primitive elements involved in the respective minimal decompositions of the elements in  $A \cap D$ . Let  $X_1 = \{x_1, \dots, x_n\}$  be a set of  $n$  commuting variables. We set  $u(p_i) = x_i$  for all  $1 \leq i \leq n$ . Then, for all  $a \in A \cap D$ , we associate to  $a$  the monomial  $u(a)$  of degree  $d$  in  $\{x_1, \dots, x_n\}$  mirroring, in multiplicative notation, the minimal decomposition  $a = p_{i_1} + \dots + p_{i_d}$ . Let

$$X = \{1\} \sqcup X_1 \sqcup \{u(a) \mid a \in A \cap D\}.$$

By the remark on subsums above, if  $a = a_1 + a_2$  with  $a_1, a_2 \in A^*$ , then  $u(a) = u(a_1)u(a_2)$ . Hence  $X$  is a divset. Let  $f : X \rightarrow S$  be the unique morphism induced by  $f(x_i) = p_i$  for all  $i = 1, \dots, n$ . Then  $f$  is an injective morphism,  $f(X) \subseteq A$  and  $f(D(X)) = A \cap D$  by construction. Hence  $f : X \rightarrow S$  is a divset model of  $S$ , as desired.  $\square$

**Example 2.15.** Let  $S = \langle 5, 6, 9 \rangle$ . Then  $m = 5$  and  $A^* = \{w_1, w_2, w_3, w_4\} = \{6, 12, 18, 9\}$  with  $w_i = \min(S \cap (i + 5\mathbb{N}))$  for  $1 \leq i \leq 4$ . Let

$$\begin{aligned} X &= [x_1^2, x_2^2] = \{x_1^2, x_2^2, x_1, x_2\}, \\ X' &= [x_1^3, x_2] = \{x_1^3, x_1^2, x_1, x_2\}. \end{aligned}$$

The morphisms  $f : X \rightarrow S$  and  $f' : X' \rightarrow S$  both induced by  $x_1 \mapsto 6, x_2 \mapsto 9$  yield two distinct divset models for  $S$ , as easily verified using  $18 = 2 \cdot 9 = 3 \cdot 6$ .

**Remark 2.16.** One advantage of divsets  $X$  as abstract models of Apéry sets  $A$  is that for any  $u \in X^*$ , a decomposition  $u = vw$  with  $v, w \in X^*$  is unique up to order, whereas in  $A^*$ , decompositions  $a = b + c$  are seldom unique in general as seen in the above example. Moreover, with divsets  $X$  one can use the terminology of monomials such as degree, divisibility and so on, notions which are less intuitive in additive notation.

**Remark 2.17.** In a divset model  $f : X \rightarrow S$ , by removing from  $X$  variables  $x_i \in X_1$  not dividing any  $u \in D(X)$ , the resulting subset  $X' \subseteq X$  is still a divset and the restriction  $f' : X' \rightarrow S$  of  $f$  to  $X'$  is still a divset model since  $D(X') = D(X)$ . Thus, we may and will assume without loss of generality that  $X_1 \subseteq V$ , where  $V = V(X)$  is the vertex set of the associated graph  $G(X)$ .

Recall that for a divset model  $f : X \rightarrow S$  where  $S$  is of multiplicity  $m$ , we have  $|X| \leq m$ . The case of equality deserves a name.

**Definition 2.18.** Let  $S$  be a numerical semigroup of multiplicity  $m$ . A full divset model of  $S$  is a divset model  $f : X \rightarrow S$  such that  $|X| = m$ .

Note that, for a full divset model  $f : X \rightarrow S$ , we have  $f(X_1) = P^*$  and so  $f(X) = A$ .

### 3 Depth, $W(S)$ , $W_0(S)$

We recall here some needed material for later use. Throughout this section,  $S$  is a numerical semigroup of multiplicity  $m$  and conductor  $c$ .

**Definition 3.1** (See [12]). The depth of  $S$  is the positive integer

$$q = \lceil c/m \rceil.$$

Thus  $c = qm - \rho$  where  $\rho \in \llbracket 0, m-1 \rrbracket$ .

In fact  $\rho \in \llbracket 0, m-2 \rrbracket$ . For otherwise, if  $\rho = m-1$  then  $c \equiv 1 \pmod{m}$ , an absurdity since then its Frobenius number  $F = c-1 = \max(\mathbb{Z} \setminus S)$  would be a multiple of  $m$ .

### 3.1 The depth function

We keep the same notation as above, namely  $c = qm - \rho$  where  $\rho \in \llbracket 0, m-2 \rrbracket$ . As in [9], we partition  $S$  as  $S = \sqcup_{i \in \mathbb{N}} S_i$ , where for all  $i \in \mathbb{N}$ ,

$$S_i = S \cap \llbracket im - \rho, (i+1)m - \rho - 1 \rrbracket.$$

In particular, we have

$$\begin{aligned} S_0 &= \{0\}, \\ S_1 &= \llbracket m, 2m - \rho - 1 \rrbracket \cap S, \\ &\dots \\ S_{q-1} &= \llbracket c - m, c - 1 \rrbracket \cap S, \\ S_q &= \llbracket c, c + m - 1 \rrbracket. \end{aligned}$$

This gives rise to the following function.

**Definition 3.2.** The depth function  $\delta : S \rightarrow \mathbb{Z}$  is defined for all  $x \in S$  by

$$(4) \quad \delta(x) = i \iff x \in S_{q-i}.$$

Equivalently,  $\delta(x)$  is the unique integer such that

$$(5) \quad x + \delta(x)m \in \llbracket c, c + m - 1 \rrbracket,$$

i.e.  $\delta(x) = \lceil \frac{c-x}{m} \rceil$ .

The function  $\delta$  assumes the following values. Recall that  $L = S \cap \llbracket 0, c - 1 \rrbracket$ .

**Lemma 3.3.** For all  $x \in S$ , we have

$$\begin{aligned} \delta(x) = q &\iff x = 0, \\ \delta(x) \in \llbracket 1, q-1 \rrbracket &\iff x \in L \setminus \{0\}, \\ \delta(x) = 0 &\iff x \in \llbracket c, c + m - 1 \rrbracket, \\ \delta(x) \leq -1 &\iff x \geq c + m. \end{aligned}$$

*Proof.* Straightforward from the definition. □

We shall need the following estimates from [10, Proposition 6].

**Proposition 3.4.** For all  $x, y \in S$ , we have

$$(6) \quad \delta(x+y) + q + 1 \geq \delta(x) + \delta(y) \geq \delta(x+y) + q - \min(\rho, 1).$$

*Proof (Outline).* First note that for all  $k, l \geq 1$ , we have

$$S_k + S_l \subset S_{k+l-\min(\rho, 1)} \cup S_{k+l} \cup S_{k+l+1}.$$

Set  $\delta(x) = i, \delta(y) = j$ . By (4), this means  $x \in S_{q-i}, y \in S_{q-j}$ . Hence

$$x+y \in S_{2q-i-j-\min(\rho, 1)} \cup S_{2q-i-j} \cup S_{2q-i-j+1}.$$

By (4) again, this means  $-q + i + j - 1 \leq \delta(x+y) \leq -q + i + j + \min(\rho, 1)$ . The claimed inequalities follow. □



## 3.2 Total depth

**Definition 3.5.** Given a finite subset  $E \subset S$ , the total depth of  $E$  is

$$\delta(E) = \sum_{x \in E} \delta(x).$$

**Lemma 3.6.** Let  $A = \text{Ap}(S, m)$ ,  $A^* = A \setminus \{0\}$  and  $D = S^* + S^*$ . Then

$$\begin{aligned} \delta(A^*) &\subseteq \llbracket 0, q-1 \rrbracket, \\ m &= |P| + |A \cap D|, \\ |L| &= \delta(A) = \delta(A^*) + q. \end{aligned}$$

*Proof.* Since  $A^* \subseteq \llbracket m, c+m-1 \rrbracket$ , it follows from Lemma 3.3 that  $\delta(A^*) \subseteq \llbracket 0, q-1 \rrbracket$ . We have  $|A^*| = m-1$  and  $A^* = (A \cap P) \sqcup (A \cap D)$ . Also  $P = (A \cap P) \sqcup \{m\}$ . Hence  $|P| = |A \cap P| + 1$  and so

$$m = |A^*| + 1 = |A \cap D| + |A \cap P| + 1 = |A \cap D| + |P|.$$

We have  $L = S \cap \llbracket 0, c-1 \rrbracket = \{0\} \sqcup (S \cap \llbracket m, c-1 \rrbracket)$ . Moreover, for all  $a \in A$ , we have

$$L \cap (a + m\mathbb{N}) = a + \llbracket 0, \delta(a) - 1 \rrbracket m.$$

Hence  $|L| = \sum_{a \in A} \delta(a) = \delta(A)$ . Since  $\delta(0) = q$ , the formula  $|L| = q + \delta(A^*)$  follows.  $\square$

## 3.3 On the numbers $W(S), W_0(S)$

The numbers  $W(S), W_0(S)$  attached to the numerical semigroup  $S$  were introduced in [9]. The alternate notation  $E(S) = W_0(S)$  was subsequently proposed in [4] and elsewhere. Recall the notation  $D = S^* + S^*$  and  $P = S^* \setminus D$ .

**Notation 3.7.** We denote  $D_q = D \cap S_q = D \cap \llbracket c, c+m-1 \rrbracket$ , and

$$\begin{aligned} W(S) &= |P||L| - c, \\ W_0(S) &= |P \cap L||L| - |A \cap D_q| + \rho. \end{aligned}$$

Thus, Wilf's conjecture amounts to the inequality

$$(7) \quad W(S) \geq 0$$

for all numerical semigroups  $S$ . The interest of  $W_0(S)$  stems from the inequality

$$W(S) \geq W_0(S).$$

Therefore, if  $W_0(S) \geq 0$  then  $S$  satisfies Wilf's conjecture in a somewhat stronger sense. For instance, the following result is proved in [9].

**Theorem 3.8.** Let  $S$  be a numerical semigroup such that  $c \leq 3m$ . Then  $W_0(S) \geq 0$ .

There are cases where  $W_0(S) \leq -1$ , but those are extremely rare. See [4, 11] for more details. Note finally that if  $P \subseteq L$  then  $W_0(S) = W(S)$ .

### 3.4 New formulas

The following formulas exhibit a closer relationship between  $W(S)$  and  $W_0(S)$  than from the original ones. The symbols  $A^*, P, D$  keep the same meaning as above. We further denote  $P_q = P \cap S_q$ ,  $p_q = |P_q|$  and  $d_q = |D_q|$ .

**Proposition 3.9.** *Let  $S$  be a numerical semigroup. Then*

$$\begin{aligned} W(S) &= |P|\delta(A^*) - |A \cap D|q + \rho, \\ W_0(S) &= |P \cap L|\delta(A^*) - |A \cap D|q + \rho. \end{aligned}$$

*Proof.* We have

$$\begin{aligned} W(S) &= |P||L| - c \\ &= |P|(\delta(A^*) + q) - qm + \rho \\ &= |P|(\delta(A^*) + q) - q(|P| + |A \cap D|) + \rho \\ &= |P|\delta(A^*) - |A \cap D|q + \rho. \end{aligned}$$

Moreover,

$$\begin{aligned} W(S) &= |P||L| - c \\ &= (|P \cap L| + p_q)|L| - qm + \rho \\ &= (|P \cap L| + p_q)|L| - q(p_q + d_q) + \rho \\ &= |P \cap L||L| - qd_q + \rho + p_q(|L| - q) \\ &= W_0(S) + p_q(|L| - q) \\ &= W_0(S) + p_q\delta(A^*). \quad \square \end{aligned}$$

**Corollary 3.10.** *We have*

$$W(S) - W_0(S) = |P_q|\delta(A^*).$$

*Proof.* Since  $|P| = |P \cap L| + |P_q|$ , the proposition implies

$$\begin{aligned} W(S) - W_0(S) &= (|P| - |P \cap L|)\delta(A^*) \\ &= |P_q|\delta(A^*). \quad \square \end{aligned}$$

**Corollary 3.11.** *We have*

$$\begin{cases} W_0(S) \leq W(S), \\ W_0(S) = W(S) \iff P \subseteq L. \end{cases}$$

*Proof.* Indeed, since  $P \subseteq S_1 \sqcup \cdots \sqcup S_q$ , we have  $P_q = \emptyset \iff P \subseteq L$ . □

**Corollary 3.12.** *If  $S$  is special of depth  $q$ , then  $W(S) = |P|\delta(A^*) - |A \cap D|q$ . □*

## 4 Focus on the special case

Throughout this section,  $S$  denotes a special numerical semigroup of multiplicity  $m$ , conductor  $c$  and depth  $q = \lceil c/m \rceil = c/m$ , i.e. with  $\rho = 0$ . As above,  $A = \text{Ap}(S, m)$  and  $A^* = A \setminus \{0\}$ .

**Proposition 4.1.** *Let  $S$  be a special numerical semigroup of depth  $q$ . Let  $X$  be a divset model of  $S$  and let  $n = |X_1|$ . Then*

$$\begin{aligned}\delta(A^*) &\geq \text{vm}(X)q/2, \\ W(S) &\geq ((n+1)\text{vm}(X) - 2|D(X)|)q/2.\end{aligned}$$

*Proof.* By hypothesis, there is a injective morphism  $f : X \rightarrow S$  such that  $f(X) \subseteq A$  and  $f(D(X)) = A \cap D$ . Let  $G = G(X) = (V, E)$  be the graph of  $X$ . Denote  $k = \text{vm}(X)$ . Hence, there is a subset  $M \subseteq V \subseteq X^*$  of cardinality  $|M| = k$  such that

$$M = \bigsqcup_{i=1}^{\ell} z_i,$$

where  $z_1, \dots, z_\ell \in E$  are pairwise disjoint edges. Loops are allowed, so  $|z_i| \in \{1, 2\}$  for all  $i$ . We have

$$(8) \quad |M| = k = |z_1| + \dots + |z_\ell|.$$

Let  $z \in \{z_1, \dots, z_\ell\}$ . Then  $z = \{u_1, u_2\}$  with  $u_1, u_2 \in X^*$  such that  $u_1 u_2 \in X^*$ . Let

$$a_1 = f(u_1), a_2 = f(u_2), a = f(u_1 u_2).$$

Then  $a = a_1 + a_2$  since  $f$  is a morphism, and  $a \in A^*$  since  $f(X^*) \subseteq A^*$ . We have  $\delta(a) \geq 0$  since  $\delta(A^*) \subseteq \llbracket 0, q-1 \rrbracket$  by Lemma 3.6. Hence

$$\delta(a_1) + \delta(a_2) \geq q + \delta(a_1 + a_2) = q + \delta(a) \geq q$$

by the right inequality in (6) and the value  $\rho = 0$ . We have  $f(z) = \{a_1, a_2\}$  and so

$$\delta(f(z)) = \delta(a_1) + \delta(a_2) \geq q.$$

If  $|z| = 1$  then  $a_1 = a_2$  and  $\delta(a_1) \geq q/2$ , whereas if  $|z| = 2$  then  $\delta(f(z)) = \delta(a_1) + \delta(a_2) \geq q$ . Summarizing, we have

$$\delta(f(z)) \geq |z|q/2$$

for all  $z \in \{z_1, \dots, z_\ell\}$ . Hence

$$\delta(f(\{z_1, \dots, z_\ell\})) = \sum_{i=1}^{\ell} \delta(f(z_i)) \geq \left( \sum_{i=1}^{\ell} |z_i| \right) q/2.$$

Since  $\sum_{i=1}^{\ell} |z_i| = |M| = k$  by (8), it follows that  $\delta(f(\{z_1, \dots, z_\ell\})) \geq kq/2$ . And since

$$f(\{z_1, \dots, z_\ell\}) \subseteq f(X^*) \subseteq A^*,$$

we conclude that

$$\delta(A^*) \geq \delta(f(X^*)) \geq \delta(f(\{z_1, \dots, z_\ell\})) \geq kq/2,$$

as desired.

It remains to prove the formula  $W(S) \geq ((n+1)\text{vm}(X) - 2|D(X)|)q/2$ . By Proposition 3.9, we have  $W(S) = |P|\delta(A^*) - |A \cap D|q + \rho$ . Since  $\delta(A^*) \geq \text{vm}(X)q/2$  and  $\rho = 0$ , this yields  $W(S) \geq (|P|\text{vm}(X)/2 - |A \cap D|)q$ . Finally, since  $|P| \geq |X_1| + 1 = n + 1$  and  $|A \cap D| = |f(D(X))| = |D(X)|$ , the desired inequality follows.  $\square$

**Corollary 4.2.** *Let  $S$  be a special numerical semigroup of depth  $q$ . Let  $X$  be a divset model of  $S$ . If  $\text{vm}(X) \geq 2k$  then  $\delta(A^*) \geq kq$ .*

*Proof.* By Proposition 4.1 and the hypothesis, we have  $\delta(A^*) \geq \text{vm}(X)q/2 \geq kq$ .  $\square$

**Corollary 4.3.** *Let  $S$  be a special numerical semigroup of depth  $q$  such that  $|P| \geq m/4$ . Let  $X$  be a divset model of  $S$ . If  $\text{vm}(X) \geq 6$  then  $S$  satisfies Wilf's conjecture.*

*Proof.* It follows from the previous corollary that  $\delta(A^*) \geq 3q$ . By Lemma 3.6, it follows that  $|L| \geq 4q$ . Hence  $|P||L| \geq (m/4)(4q) = qm = c$ , i.e.  $W(S) \geq 0$  as claimed.  $\square$

## 4.1 Tame divsets

**Definition 4.4.** *Let  $X$  be a divset. Let  $G(X) = (V, E)$  be the graph of  $X$ . Set  $V_1 = V \cap X_1$  and  $n = |V_1|$ . We say that  $X$  is tame if*

$$(n+1)\text{vm}(X) \geq 2|D(X)|.$$

*We say that  $X$  is wild if it is not tame, i.e. if  $(n+1)\text{vm}(X) < 2|D(X)|$ .*

**Proposition 4.5.** *Let  $S$  be a special numerical semigroup of depth  $q \geq 2$ . If  $S$  admits a tame divset model  $X$ , then  $W_0(S) \geq 0$  and hence  $S$  satisfies Wilf's conjecture.*

*Proof.* Let  $f : X \rightarrow S$  be a divset model with  $X$  tame. Let  $G(X) = (V, E)$  be the graph of  $X$  and  $n = |V_1| = |V \cap X_1|$ . Since  $S$  is tame, we have  $(n+1)\text{vm}(X) \geq 2|D(X)|$ .

**Claim.** We have  $f(V_1) \subseteq P^* \cap L$ . Indeed, we have  $f(V_1) \subseteq f(X_1) \subseteq P^*$  by Proposition 2.13. Moreover, if  $x_i \in V_1$ , there exists  $u \in X^*$  such that  $x_i u \in X^*$ . Hence

$$f(x_i) + f(u) = f(x_i u) \in f(X^*) \subseteq A^* \subseteq \llbracket m, c + m - 1 \rrbracket.$$

Since  $f(u) \in A^*$  hence  $f(u) \geq m$ , it follows that  $f(x_i) \leq c - 1$ , i.e.  $f(x_i) \in L$ . This settles the claim.

As  $c = qm$  with  $q \geq 2$ , we have  $P \cap L = (P^* \cap L) \sqcup \{m\}$ . Since  $f$  is injective, the claim implies

$$|P \cap L| = |P^* \cap L| + 1 \geq |f(V_1)| + 1 \geq |X_1| + 1 \geq n + 1.$$

Now  $|A \cap D| = |f(D(X))| = |D(X)|$ , and  $\delta(A^*) \geq \text{vm}(X)q/2$  by Proposition 4.1.

Summarizing, and using the formula for  $W_0(S)$  in Proposition 3.9, we have

$$\begin{aligned} W_0(S) &= |P \cap L| \delta(A^*) - |A \cap D| q \\ &\geq (n + 1) \text{vm}(X) q / 2 - |A \cap D| q \\ &= ((n + 1) \text{vm}(X) - 2|D(X)|) q / 2. \end{aligned}$$

Now  $(n + 1) \text{vm}(X) - 2|D(X)| \geq 0$  since  $X$  is tame. Hence  $W_0(S) \geq 0$ .  $\square$

Almost all of the divsets involved in the next sections turn out to be tame. But here is an example of a wild divset.

**Example 4.6.** Let  $X = [x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3]$ , i.e. the set of all monomials of degree at most 3 in  $x_1, x_2$ . Then  $|X^*| = |X_1| + |D(X)| = 2 + (3 + 4) = 9$ . Here  $n = |V_1| = 2$  and  $|D(X)| = 7$ . Moreover,  $\text{vm}(X) = 4$  as witnessed by the vertex-maximal matching  $\{\{x_1, x_1^2\}, \{x_2, x_2^2\}\}$  of the graph  $G(X)$ . Hence

$$(n + 1) \text{vm}(X) - 2|D(X)| = 3 \cdot 4 - 2 \cdot 7 = -2,$$

so that  $X$  is wild. However, it can be shown that for any special numerical semigroup  $S$  modeled by  $X$ , one has  $|P| \geq 7$ , whence

$$W(S) = |P| \delta(A^*) - |A \cap D| q \geq 7 \cdot \text{vm}(X) q / 2 - |D(X)| q = 14q - 7q = 7q.$$

Thus, all such numerical semigroups  $S$  satisfy Wilf's conjecture. Interestingly, those  $S$  include the five smallest numerical semigroups satisfying  $W_0(S) \leq -1$  as described in [11].

## 4.2 Proof strategy

It follows from Corollary 4.3 that, in order to settle Wilf's conjecture for special numerical semigroups  $S$  satisfying  $|P| \geq m/4$ , it suffices to consider divsets  $X$  such that  $\text{vm}(X) \leq 5$ . Thus, we need to classify all such divsets.

To start with, since a divset is determined by its maximal elements, we introduce the following auxiliary notation.

**Notation 4.7.** Let  $u_1, \dots, u_r \in \mathcal{M}$  and  $X = [u_1, \dots, u_r]$ . We set  $\mu(u_1, \dots, u_r) = \text{vm}(X)$ .

Determining  $\mu(u_1, \dots, u_r)$  is difficult in general, due to its intimate relationship with matching numbers in graphs. Nevertheless, the case  $r = 1$  is straightforward.

**Lemma 4.8.** *Let  $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathcal{M}$ . Then  $\mu(u) = \prod_{i=1}^n (a_i + 1) - 2$ .*

*Proof.* Let  $X = [u]$ , the divset of divisors of  $u$ . Then  $\mu(u) = \text{vm}(X)$ , i.e. the maximal number of vertices touched by a matching of the graph  $G(X)$ . Now  $|X| = \prod_{i=1}^n (a_i + 1)$ , the number of divisors of  $u$ . These divisors may be regrouped in independent edges of the form  $\{v, u/v\}$  where  $v$  is a divisor of  $u$  such that  $0 \leq \deg(v) \leq \deg(u/v)$ , and  $v$  is lexicographically smaller than or equal to  $u/v$  if  $\deg(v) = \deg(u/v)$ . The pair  $\{1, u\}$  must be discounted since 1 is not a vertex of  $G$  by definition. The other pairs constitute a matching of  $G$  covering the whole of  $X \setminus \{0, u\}$ . Hence  $\mu(u) = \text{vm}(X) = |X \setminus \{0, u\}| = \prod_{i=1}^n (a_i + 1) - 2$ .  $\square$

Coupled with more combinatorial arguments, this basic result will allow us to classify all divsets  $X$  such that  $\text{vm}(X) \leq 5$ , as needed. Then in each case, we shall show that all special numerical semigroups  $S$  modeled by those divsets and such that  $|P| \geq m/4$  satisfy Wilf's conjecture, thereby proving the main result in this paper.

## 5 The case $\deg(X) = 2$

We prove here that divsets of degree 2 are tame. This establishes Wilf's conjecture for all special numerical semigroups  $S$  such that  $A \cap D \subseteq P^* + P^*$ . In order to do so, we need the following lemma about the classical *matching number* in graphs, namely the maximum number of independent edges.

**Lemma 5.1.** *Let  $H = (H_1, H_2)$  be a simple bipartite graph without isolated vertices, with matching number  $k$ . Then*

$$|E(H)| \leq k \cdot \max(|H_1|, |H_2|).$$

*Proof.* We may assume  $|H_1| \leq |H_2|$ . Then  $k \leq |H_1|$ . Let  $M$  be a maximal matching with  $k$  edges, say

$$M = \{(x_i, y_i) \mid x_i \in H_1, y_i \in H_2, 1 \leq i \leq k\}.$$

**Claim.**  $|H_1| = k$ . Assume not. Let then  $x_{k+1} \in H_1 \setminus \{x_1, \dots, x_k\}$ . Since  $H$  does not have isolated vertices, it follows that  $x_{k+1}$  has a neighbour  $z \in H_2$ . Now necessarily  $z \in \{y_1, \dots, y_k\}$ , for otherwise there would be a new edge  $(x_{k+1}, z)$  independent of  $M$ , which would then yield a matching of cardinality  $k + 1$ , a contradiction. Up to renumbering, we may assume  $z = y_k$ , i.e.

$$(x_{k+1}, y_k) \in E.$$

As  $|H_2| \geq |H_1| \geq k + 1$ , there is a vertex  $y_{k+1} \in H_2 \setminus \{y_1, \dots, y_k\}$ . Since  $y_{k+1}$  is not isolated, it has a neighbour  $z \in H_1$ . But as in the preceding reasoning, we have  $z \in \{x_1, \dots, x_k\}$ , for otherwise the edge  $(z, y_{k+1})$  would be independent of  $M$ , a contradiction. Up to renumbering, we may assume  $z = x_k$ . Thus

$$(x_k, y_{k+1}) \in E.$$

But then, by suppressing the edge  $(x_k, y_k)$  of  $M$  and replacing it by the two independent edges  $(x_{k+1}, y_k), (x_k, y_{k+1})$ , we obtain a matching

$$M' = M \sqcup \{(x_{k+1}, y_k), (x_k, y_{k+1})\} \setminus \{(x_k, y_k)\}$$

of cardinality  $k + 1$ , a contradiction. This finishes the proof of the claim, whence

$$|H_1| = k.$$

Since  $H = (H_1, H_2)$  is bipartite, we have  $|E| \leq |H_1||H_2| = k|H_2|$ , as claimed.  $\square$

**Proposition 5.2.** *Let  $X$  be a divset of degree 2, i.e. such that  $D(X) = X_2$ . Then  $X$  is tame, i.e.  $(n + 1) \text{vm}(X) \geq 2|D(X)|$ , where  $n = |V \cap X_1|$  and  $V$  is the vertex set of the graph  $G(X)$ .*

*Proof.* Let  $V_1 = V \cap X_1 = \{x_1, \dots, x_n\}$ , the set of variables dividing the monomials in  $D(X) = X_2$ . We have  $V = V_1$  since if  $u \in V$ , there exists  $v \in V$  such that  $uv \in D(X) = X_2$ , implying  $\deg(u) = \deg(v) = 1$ . Clearly  $|D(X)| \leq n(n + 1)/2$ . Let  $M \subseteq D(X)$  be a maximal set of pairwise coprime monomials. Denote by  $V_2 \subseteq V_1$  the set of variables involved in  $M$ . Let  $n_2 = |V_2|$ . Since  $M$  corresponds to a matching in  $G(X)$  with vertex set  $V_2$ , we have

$$\text{vm}(X) \geq n_2.$$

**Case 1.**  $V_2 = V_1$ . In this case we are done. Indeed, we then have  $n_2 = n$ , so that  $\text{vm}(X) \geq n$ .

And since  $|D(X)| \leq n(n + 1)/2$ , it follows that

$$(n + 1) \text{vm}(X) - 2|D(X)| \geq (n + 1)n - 2n(n + 1)/2 = 0,$$

as desired.

**Case 2.** Assume  $V_2 \neq V_1$ . Let  $V_3 = V_1 \setminus V_2$  and  $n_3 = |V_3| = n - n_2$ . By maximality of  $M$ , every monomial  $u = x_i x_j \in D(X)$  with  $i \leq j$  satisfies  $x_i \in V_2$  or  $x_j \in V_2$ . Let  $N \subseteq D(X)$  be a largest possible subset of pairwise coprime monomials  $x_i x_j$  with  $x_i \in V_2, x_j \in V_3$ . Let  $k = |N|$ . We have  $k \leq \min(n_2, n_3)$  and  $k \geq 1$  since  $V_3 \neq \emptyset$ . Each  $x_i x_j \in N$  independently contributes a summand 2 to  $\text{vm}(X)$ . Thus

$$(9) \quad \text{vm}(X) \geq (n_2 - k) + 2k = n_2 + k.$$

Next, we claim that

$$(10) \quad |D(X)| \leq n_2(n_2 + 1)/2 + k \max(n_2, n_3).$$

Indeed, as  $V_2$  induces the empty subgraph in  $G(X)$ , every monomial in  $D(X)$  has its support in  $[V_1, V_1]$  or  $[V_1, V_2]$ . Now the preceding lemma implies that the number of monomials with support in  $[V_1, V_2]$  is less than or equal to  $k \max(n_2, n_3)$ . This proves (10). Combined with (9), this yields

$$(n+1)\text{vm}(X) - 2|D(X)| \geq (n+1)(n_2+k) - n_2(n_2+1) - 2k\max(n_2, n_3).$$

It remains to show that the right-hand side is non-negative. Since  $n = n_2 + n_3$ , we have

$$\begin{aligned} (n+1)(n_2+k) - n_2(n_2+1) - 2k\max(n_2, n_3) &= n_2(n+1) - n_2(n_2+1) + k(n+1 - 2\max(n_2, n_3)) \\ &= n_2n_3 + k(n_2 + n_3 + 1 - 2\max(n_2, n_3)) \\ &= n_2n_3 + k(\min(n_2, n_3) - \max(n_2, n_3) + 1) \end{aligned}$$

since  $n_2 + n_3 = \min(n_2, n_3) + \max(n_2, n_3)$ . But for any  $k \leq a \leq b$ , we have

$$\begin{aligned} ab + k(a - b + 1) &= k(a + 1) + b(a - k) \\ &\geq k(a + 1). \end{aligned}$$

Hence  $(n+1)(n_2+k) - n_2(n_2+1) - 2k\max(n_2, n_3) \geq k(\min(n_2, n_3) + 1) \geq 0$ , as desired.  $\square$

**Corollary 5.3.** *Let  $S$  be a special numerical semigroup such that  $A \cap D \subseteq P^* + P^*$ . Then  $S$  satisfies Wilf's conjecture.*

*Proof.* Let  $f : X \rightarrow S$  be a divset model. Let  $G = (V, E)$  be the graph of  $X$ . By Remark 2.17, we may assume  $X_1 \subseteq V$ . Moreover, since  $f(D(X)) = A \cap D \subseteq P^* + P^*$ , we may assume that  $D(X) \subseteq X_1 \cdot X_1$ , i.e. that  $\deg(X) = 2$ . We conclude with Propositions 4.5 and 5.2.  $\square$

## 6 The case $\deg(X) \geq 3$

The case  $\deg(X) = 2$  being settled, from here on we only consider divsets  $X$  satisfying  $\deg(X) \geq 3$ . [...]

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