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# Divsets, numerical semigroups and Wilf's conjecture

October 9, 2023

## Abstract

Let  $S \subseteq \mathbb{N}$  be a numerical semigroup with multiplicity  $m = \min(S \setminus \{0\})$  and conductor  $c = \max(\mathbb{N} \setminus S) + 1$ . Let  $P$  be the set of primitive elements of  $S$ , and let  $L$  be the set of elements of  $S$  which are smaller than  $c$ . Wilf's conjecture (1978) states that the inequality  $|P||L| \geq c$  always hold. The conjecture has been shown to hold in case  $|P| \geq m/2$  by Sammartano in 2012, and subsequently in case  $|P| \geq m/3$  by the author in 2020. The main result in this paper is that Wilf's conjecture holds in case  $|P| \geq m/4$  with  $c \in m\mathbb{N}$ .

## 0 Caution

This document is the starting point of a full paper to be gradually completed in successive versions within the next few weeks of Fall 2023.

## 1 Introduction

A *numerical semigroup* is a cofinite submonoid  $S$  of  $\mathbb{N}$ , i.e. a subset containing 0, stable under addition and with finite complement  $\mathbb{N} \setminus S$ . Equivalently, it is a set of the form  $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$  where  $a_1, \dots, a_n$  are positive integers with  $\gcd(a_1, \dots, a_n) = 1$ , called *generators* of  $S$ . The least such  $n$  is usually denoted  $e$  and called the *embedding dimension* of  $S$ , see below.

Let  $S$  be a numerical semigroup and  $S^* = S \setminus \{0\}^*$ . A *primitive element* of  $S$  is an element  $a \in S^* \setminus (S^* + S^*)$ , i.e. an element of  $S^*$  which is not the sum of two elements of  $S^*$ . We denote by  $P = P(S)$  the set of primitive elements of  $S$ , and by  $D(S) = S^* \setminus P(S) = S^* + S^*$  the set of *decomposable elements* of  $S$ . It is easy to see that  $P$  is finite and is the unique minimal generating set of  $S$ . The *embedding dimension* of  $S$  is  $e = e(S) = |P|$ , the *multiplicity* of  $S$  is  $m = m(S) = \min S^*$ , the *Frobenius number* of  $S$  is  $F = F(S) = \max(\mathbb{Z} \setminus S)$  and the *conductor* of  $S$  is  $c = c(S) = F + 1$ , satisfying  $c + \mathbb{N} \subseteq S$  and minimal with respect to that property. The *genus* of  $S$  is  $g = g(S) = |\mathbb{N} \setminus S|$ , its number of gaps. We partition  $S$  as  $S = L \sqcup R$ , where  $L = L(S) = \{a \in S \mid a < F(S)\}$  and  $R = R(S) = \{a \in S \mid a > F(S)\}$ , the *left part* and *right part* of  $S$ , respectively.

Currently, one of the main open problems on numerical semigroups is the following conjecture, first stated as a question by Wilf [21].

**Conjecture 1.1** (Wilf, 1978). *Let  $S$  be a numerical semigroup. Then  $|P||L| \geq c$ .*

See [5] for a survey on the conjecture up to 2018. Among many partial results, the conjecture has been shown to hold in case  $|P| \geq m/2$  by Sammartano in 2012 [19], and subsequently in case  $|P| \geq m/3$  by the author in 2020 [9]. The main result in this paper is as follows.

**Theorem 1.2.** *Let  $S$  be a numerical semigroup such that  $c \in m\mathbb{N}$  and  $|P| \geq m/4$ . Then  $S$  satisfies Wilf's conjecture.*

The motivation to focus on the case  $c \in m\mathbb{N}$  is justified in Section 2.

## 2 Special numerical semigroups

Given a numerical semigroup  $S$ , consider the Euclidean division of its conductor  $c$  by its multiplicity  $m$  with nonpositive remainder:

$$(1) \quad c = qm - \rho, \quad 0 \leq \rho \leq m - 1.$$

As in [11], we call the integer  $q = \lceil c/m \rceil$  the *depth* of  $S$ . The particular case  $c = qm$ , i.e. with  $\rho = 0$ , might well be the heart of Wilf's conjecture. Indeed, the proofs of Wilf's conjecture in either case  $c \leq 3m$  [8] or  $|P| \geq m/3$  [9] can be significantly shortened when  $\rho = 0$ . Moreover, the first five instances of the very rare "near-misses in Wilf's conjecture" all belong to this case [10].

These facts lead us to believe that if Wilf's conjecture fails for some exotic numerical semigroups, then it already fails in cases where  $\rho = 0$ . This justifies introducing the following terminology.

**Definition 2.1.** *A numerical semigroup  $S$  is special if  $c \in m\mathbb{N}$ , where  $m$  is its multiplicity and  $c$  is its conductor.*

For instance, the ordinary numerical semigroup  $O_m = \{0\} \cup (m + \mathbb{N})$  is special since it satisfies  $c = m$ . The above discussion leads us to consider the special case as a priority in research on Wilf's conjecture.

In this context, we extend here the validity of Wilf's conjecture to the case  $|P| \geq m/4$  for special numerical semigroups. More generally, using the notation in (1), our main result in this paper reads as follows.

**Theorem 2.2.** *Let  $S$  be a numerical semigroup satisfying  $|P| \geq m/4$ . Then  $|P||L| \geq c - m + \rho$ . Moreover, if  $S$  is special, then  $|P||L| \geq c$ , i.e.  $S$  satisfies Wilf's conjecture.*

### 3 The Apéry set

**Notation 3.1.** Let  $S$  be a numerical semigroup of multiplicity  $m$ . We denote by  $A = AP(S, m)$  the Apéry set of  $S$  with respect to  $m$ , defined as

$$A = S \setminus (m + S).$$

We set  $A^* = A \setminus \{0\}$ .

**Definition 3.2.** Let  $S$  be a numerical semigroup of multiplicity  $m \geq 2$  and conductor  $c$ . The level function of  $S$  is  $\lambda : S \rightarrow \mathbb{N}$  defined by  $\lambda(s) = \lceil s/m \rceil$  for all  $s \in S$ . Thus  $\lambda(s) = l$  if and only if  $s \in S \cap [(l-1)m + 1, lm]$ .

**Proposition 3.3.** Let  $S$  be a numerical semigroup with level function  $\lambda : S \rightarrow \mathbb{N}$ . Then  $\lambda(s_1 + s_2) \in \{\lambda(s_1) + \lambda(s_2) - 1, \lambda(s_1) + \lambda(s_2), \lambda(s_1) + \lambda(s_2) + 1\}$  for all  $s_1, s_2 \in S$ .

*Proof.* Let  $l_i = \lambda(s_i)$  for  $i = 1, 2$ . Then  $s_i \in S \cap [(l_i - 1)m + 1, l_i m]$  for  $i = 1, 2$ . Hence  $s_1 + s_2 \in S \cap [(l_1 + l_2 - 2)m + 2, (l_1 + l_2 + 2)m]$ . Etc.  $\square$

**Definition 3.4.** Let  $S$  be a numerical semigroup of depth  $q \geq 1$ . The depth function of  $S$  is the function  $\delta : S \rightarrow \mathbb{Z}$  defined by  $\delta(s) = q - \lambda(s)$  for all  $s \in S$ .

**Notation 3.5.** Given a finite subset  $X \subset S$ , we denote by  $\tau(X)$  its total depth, i.e.  $\tau(X) = \sum_{x \in X} \delta(x)$ .

**Proposition 3.6.** Assume  $S$  is special of depth  $q$ . Then  $W(S) = |P| \tau(A^*) - |A \cap D| q$ .

### 4 Divsets

**Notation 4.1.** Let  $x_1, \dots, x_n$  be commuting variables. We denote by  $M = \{x_1^{a_1} \cdots x_n^{a_n} \mid (a_1, \dots, a_n) \in \mathbb{N}^n\}$  the set of all monomials in  $x_1, \dots, x_n$ . For  $d \in \mathbb{N}$ , we denote by  $M_d \subset M$  the subset of monomials  $u \in M$  of degree  $d$ .

**Definition 4.2.** A divset is a finite set  $X$  of monomials in  $n$  variables  $x_1, \dots, x_n$  which is stable under taking divisors. That is, for all  $u \in X$ , if  $v \in M$  and  $v|u$  then  $v \in X$ .

**Example 4.3.**  $X = \{x_1^3, x_1^2 x_2, x_1^2, x_1 x_2, x_1, x_2, 1\}$  is a divset in the variables  $x_1, x_2$ .

**Notation 4.4.** Given a set of  $U$  of monomials in  $\{x_1, \dots, x_n\}$ , we denote by  $[U]$  the set of divisors of the elements of  $U$ . That is,  $[U] = \{v \in M \mid \exists u \in U, v|u\}$ .

In the above example  $X = \{x_1^3, x_1^2 x_2, x_1^2, x_1 x_2, x_1, x_2, 1\}$ , we have  $X = [x_1^3, x_1^2 x_2]$ .

**Definition 4.5.** Let  $X$  be a divset. We define the degree of  $X$  as

$$\deg(X) = \max\{\deg(u) \mid u \in X\}.$$

**Notation 4.6.** Let  $X$  be a divset. We denote by  $\max(X)$  the maximal elements of  $X$  under divisibility, i.e. the set of those  $u \in X$  which do not divide any  $v \in X \setminus \{u\}$ .

Clearly, a divset  $X$  is completely determined by  $\max(X)$ , as  $X = [\max(X)]$ .

**Notation 4.7.** Let  $X$  be a divset on the variables  $\{x_1, \dots, x_n\}$ . For all  $d \geq 0$ , we denote  $X_d = X \cap M_d = \{u \in X \mid \deg(u) = d\}$ .

For instance, we have  $X_0 = \{1\}$  and  $X_1 = \{x_1, \dots, x_n\}$ .

**Notation 4.8.** Let  $X$  be a divset. We denote by  $D(X) = \{u \in X \mid \deg(u) \geq 2\}$ , i.e. the set of decomposable monomials in  $X$ . Thus  $X = \{1\} \sqcup X_1 \sqcup D(X)$ . We denote  $X^* = X \setminus \{1\}$ .

## 4.1 The graph of a divset

Let  $X$  be a divset. We canonically associate to  $X$  a graph  $G = G(X)$  defined as follows. An edge in  $G$  is a pair  $\{u_1, u_2\}$  with  $u_1, u_2 \in X^*$  such that  $u_1 u_2 \in X$ . This defines the edge set  $E(G)$ . The set  $V(G)$  of vertices of  $G$  is defined as the set of the extremities of the edges. Said otherwise, we have  $V(G) = \{u \in X^* \mid \exists v \in X^*, uv \in X\}$ .

An important measure of  $X$  in the sequel is the *vertex-maximal matching number* of the graph  $G(X)$ , defined below.

**Notation 4.9.** We set  $\text{vm}(X) =$  the largest cardinality of a subset  $Y \subseteq X^*$  obtained as the set of vertices of a matching in  $G(X)$ , i.e. a union of pairwise disjoint pairs  $\{u, v\} \subseteq Y$  such that  $uv \in X$ .

**Definition 4.10.** Let  $X$  be a divset on  $n$  variables. We say that  $X$  is tame if

$$(n + 1) \text{vm}(X) \geq 2|D(X)|.$$

We say that  $X$  wild if it is not tame.

The link with Wilf's conjecture will become clear in the next section.

## 4.2 Divsets and numerical semigroups

In a numerical semigroup  $S$  of multiplicity  $m$ , its Apéry set with respect to  $m$  carries key information on the structure of  $S$ .

**Notation 4.11.** Let  $S$  be a numerical semigroup of multiplicity  $m$ . We denote by  $A = AP(S, m)$  the Apéry set of  $S$  with respect to  $m$ , defined as

$$A = S \setminus (m + S).$$

We set  $A^* = A \setminus \{0\}$ .

It is well known that  $AP(S, m)$  has exactly  $m$  elements, one for each class mod  $m$ . More precisely, we have

$$AP(S, m) = \{w_0, \dots, w_{m-1} \mid w_i = \min(S \cap (i + m\mathbb{N}))\}.$$

It turns out that divsets can be considered as a sort of abstract model for Apéry sets. This will be made more precise below.

**Definition 4.12.** *Let  $S$  be a numerical semigroup and  $X$  a divset. A morphic map from  $X$  to  $S$  is a map  $f : X \rightarrow S$  such that  $f(1) = 0$  and  $f(uv) = f(u) + f(v)$  for all  $u, v \in X$ .*

**Definition 4.13.** *Let  $S$  be a numerical semigroup of multiplicity  $m$ . Let  $A = AP(S, m)$ . A divset representation of  $S$  is a injective morphic map  $f : X \rightarrow S$ , where  $X$  is a divset of cardinality  $|X| = m$  and such that  $f(X) = A$ .*

**Proposition 4.14.** *Let  $S$  be a numerical semigroup of multiplicity  $m$ . Then  $S$  admits at least one divset representation.*

*Proof.* The key point is the well known fact that, for any Apéry element  $a \in A^*$ , if there is a decomposition  $a = s_1 + s_2$  with  $s_1, s_2 \in S^*$ , then necessarily  $s_1, s_2 \in A^*$ .  $\square$

**Proposition 4.15.** *Let  $S$  be a numerical semigroup such that  $c \in m\mathbb{N}$  and which is represented by a tame divset  $X$ . Then  $S$  satisfies Wilf's conjecture.*

*Proof.* We have  $\tau(X^*) \geq vm(X)q/2$ . Hence

$$\begin{aligned} W_0(S) &= |P \cap L| \tau(X^*) - |X \cap D| q \\ &\geq (n+1) vm(X) q/2 - |X \cap D| q \\ &= ((n+1) vm(X) - 2|X \cap D|) q/2. \end{aligned}$$

Since  $X$  is tame by hypothesis, we have  $(n+1) vm(X) - 2|X \cap D| \geq 0$ . Hence  $W_0(S) \geq 0$ .  $\square$

## 5 The case $\deg(X) = 2$

We prove here that divsets of degree 2 are tame. This establishes Wilf's conjecture for all special numerical semigroups  $S$  such that  $D(S) \subseteq P + P$ . In order to do so, we need the following lemma about the classical *matching number* in graphs, namely the maximum number of independent edges.

**Lemma 5.1.** *Let  $H = (H_1, H_2)$  be a simple bipartite graph without isolated vertices, with matching number  $k$ . Then*

$$|E(H)| \leq k \cdot \max(|H_1|, |H_2|).$$

*Proof.* We may assume  $|H_1| \leq |H_2|$ . Then  $k \leq |H_1|$ . Let  $M$  be a maximal matching with  $k$  edges, say

$$M = \{(x_i, y_i) \mid x_i \in H_1, y_i \in H_2, 1 \leq i \leq k\}.$$

**Claim.**  $|H_1| = k$ . Assume not. Let then  $x_{k+1} \in H_1 \setminus \{x_1, \dots, x_k\}$ . Since  $H$  does not have isolated vertices, it follows that  $x_{k+1}$  has a neighbour  $z \in H_2$ . Now necessarily  $z \in \{y_1, \dots, y_k\}$ , for otherwise there would be a new edge  $(x_{k+1}, z)$  independent of  $M$ , which would then yield a matching of cardinality  $k + 1$ , a contradiction. Up to renumbering, we may assume  $z = y_k$ , i.e.

$$(x_{k+1}, y_k) \in E.$$

As  $|H_2| \geq |H_1| \geq k + 1$ , there is a vertex  $y_{k+1} \in H_2 \setminus \{y_1, \dots, y_k\}$ . Since  $y_{k+1}$  is not isolated, it has a neighbour  $z \in H_1$ . But as in the preceding reasoning, we have  $z \in \{x_1, \dots, x_k\}$ , for otherwise the edge  $(z, y_{k+1})$  would be independent of  $M$ , a contradiction. Up to renumbering, we may assume  $z = x_k$ . Thus

$$(x_k, y_{k+1}) \in E.$$

But then, by suppressing the edge  $(x_k, y_k)$  of  $M$  and replacing it by the two independent edges  $(x_{k+1}, y_k), (x_k, y_{k+1})$ , we obtain a matching

$$M' = M \sqcup \{(x_{k+1}, y_k), (x_k, y_{k+1})\} \setminus \{(x_k, y_k)\}$$

of cardinality  $k + 1$ , a contradiction. This finishes the proof of the claim, whence

$$|H_1| = k.$$

Since  $H = (H_1, H_2)$  is bipartite, we have  $|E| \leq |H_1||H_2| = k|H_2|$ , as claimed.  $\square$

**Proposition 5.2.** *Let  $X$  be a divset of degree 2, i.e. such that  $D(X) = X_2$ . Then  $X$  is tame, i.e.  $(n + 1) \text{vm}(X) \geq 2|D(X)|$ , where  $n = |X_1|$ .*

*Proof.* Let  $V = \{x_1, \dots, x_n\}$ , the set of variables dividing the monomials of  $X \cap D$ . Then  $|X \cap D| \leq n(n + 1)/2$ . Let  $M \subseteq X \cap D$  be a maximal set of monomials of degré 2 and pairwise coprime. Denote  $V_1 \subseteq V$  the set of variables involved in  $M$ . Let  $V_2 = V \setminus V_1$ ,  $n_1 = |V_1|$  and  $n_2 = |V_2| = n - n_1$ . By construction, we have

$$\tau(V_1) \geq n_1 q / 2.$$

**Case 1.**  $V_1 = V$ . In this case we are done. Indeed, we then have  $\tau(V) = \tau(V_1) \geq nq/2$ , and since  $|X \cap D| \leq n(n + 1)/2$ , it follows that

$$\begin{aligned} W(S) &= |P|\tau(V) - |X \cap D|q \\ &\geq (n + 1)nq/2 - n(n + 1)/2 \cdot q \\ &\geq 0. \end{aligned}$$

**Case 2.** Assume  $V_1 \neq V$ , i.e.  $V_2 \neq \emptyset$ . By maximality of  $M$ , it follows that every monomial  $u = x_i x_j \in X \cap D$  with  $i \leq j$  satisfies  $x_i \in V_1$ . Let  $k$  be the maximum number of monomials  $u = x_i x_j$  with  $x_i \in V_1$ ,  $x_j \in V_2$  and pairwise coprime. We have  $k \leq \min(n_1, n_2)$ , but also  $k \geq 1$  since  $V_2 \neq \emptyset$ . For each such pair  $(v_1, v_2) \in V_1 \times V_2$ , i.e. with  $v_1 v_2 \in X \cap D$ , we have  $\delta(v_1) + \delta(v_2) \geq q$ . This already contributes  $kq$  to  $\tau(X)$ . Thus

$$(2) \quad \tau(X) \geq (n_1 - k)q/2 + kq.$$

Next, we claim that

$$(3) \quad |X \cap D| \leq n_1(n_1 + 1)/2 + k \max(n_1, n_2).$$

Indeed, as  $V_2$  is the empty graph, every monomial in  $X \cap D$  has its support in  $[V_1, V_1]$  or  $[V_1, V_2]$ . But the preceding lemma implies that the number of monomials with support in  $[V_1, V_2]$  is less than or equal to  $k \max(n_1, n_2)$ . This proves (3).

Now,  $W(S) \geq (n + 1)\tau(V) - |X \cap D|q$ . Hence, by (2) and (3), we have

$$\begin{aligned} W(S) &\geq (n + 1)((n_1 - k)/2 + k)q - n_1(n_1 + 1)/2 - k \max(n_1, n_2)q \\ &= ((n + 1)(n_1 + k) - n_1(n_1 + 1) - 2k \max(n_1, n_2))q/2. \end{aligned}$$

It remains to show that  $(n + 1)(n_1 + k) - n_1(n_1 + 1) - 2k \max(n_1, n_2) \geq 0$ . We have

$$\begin{aligned} (n + 1)(n_1 + k) - n_1(n_1 + 1) - 2k \max(n_1, n_2) &= n_1(n + 1) - n_1(n_1 + 1) + k(n + 1 - 2 \max(n_1, n_2)) \\ &= n_1 n_2 + k(n_1 + n_2 + 1 - 2 \max(n_1, n_2)) \\ &= n_1 n_2 + k(\min(n_1, n_2) - \max(n_1, n_2) + 1) \end{aligned}$$

since  $n_1 + n_2 = \min(n_1, n_2) + \max(n_1, n_2)$ . But if  $k \leq a \leq b$ , we have

$$\begin{aligned} ab + k(a - b + 1) &= k(a + 1) + b(a - k) \\ &\geq k(a + 1). \end{aligned}$$

It follows that  $(n + 1)(n_1 + k) - n_1(n_1 + 1) - 2k \max(n_1, n_2) \geq k(\min(n_1, n_2) + 1)$ .  $\square$

## References

- [1] M. BRAS-AMORÓS, Fibonacci-like behavior of the number of numerical semigroups of a given genus, *Semigroup Forum* 76 (2008) 379–384.
- [2] W. BRUNS, P. GARCIA-SANCHEZ, C. O’NEILL, D. WILBURNE, Wilf’s conjecture in fixed multiplicity. arXiv 1903.04342 [math.CO], 2019.
- [3] G. CHARTRAND, L. LESNIAK, P. ZHANG, *Graphs & digraphs*. Sixth edition. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2016. xii+628 pp. ISBN: 978-1-4987-3576-6.



- [4] M. DELGADO, On a question of Eliahou and a conjecture of Wilf, *Math. Z.* 288 (2018) 595–627.
- [5] M. DELGADO, Conjecture of Wilf: a survey. To appear in *Numerical Semigroups - IMNS 2018*, Springer INdAM Series 40, ISBN : 978-3-030-40821-3. arXiv 1902.03461 [math.CO].
- [6] M. DELGADO, Trimming the numerical semigroups tree to probe Wilf’s conjecture to higher genus. arXiv 1910.12377 [math.CO], 2019.
- [7] M. DELGADO, P.A. GARCÍA-SÁNCHEZ AND J. MORAIS, “Numericalsgps”: a GAP package on numerical semigroups. <http://www.gap-system.org/Packages/numericalsgps.html>
- [8] S. ELIAHOU, Wilf’s conjecture and Macaulay’s theorem, *J. Eur. Math. Soc.* 20 (2018) 2105–2129. DOI 10.4171/JEMS/807.
- [9] S. ELIAHOU, A graph-theoretic approach to Wilf’s conjecture. *Elec. J. Combin.* 27(2) (2020), #P2.15, 31 pp. <https://doi.org/10.37236/9106>.
- [10] S. ELIAHOU AND J. FROMENTIN, Near-misses in Wilf’s conjecture, *Semigroup Forum* 98 (2019) 285-298. DOI 10.1007/s00233-018-9926-5.
- [11] S. ELIAHOU AND J. FROMENTIN, Gapsets and numerical semigroups, *J. Combin. Theory Ser. A* 169 (2020), 105129, 19 pp. DOI 10.1016/j.jcta.2019.105129.
- [12] R. FRÖBERG, C. GOTTLIEB AND R. HÄGGKVIST, On numerical semigroups, *Semigroup Forum* 35 (1987) 63–83.
- [13] J. FROMENTIN AND F. HIVERT, Exploring the tree of numerical semigroups, *Math. Comp.* 85 (2016) 2553–2568.
- [14] N. KAPLAN, Counting numerical semigroups by genus and some cases of a question of Wilf, *J. Pure Appl. Algebra* 216 (2012) 1016–1032.
- [15] N. KAPLAN AND L. YE, The proportion of Weierstrass semigroups, *J. Algebra* 373 (2013) 377–391.
- [16] A. MOSCARIELLO AND A. SAMMARTANO, On a conjecture by Wilf about the Frobenius number, *Math. Z.* 280 (2015) 47–53.
- [17] J.L. RAMÍREZ ALFONSÍN, *The Diophantine Frobenius problem*. Oxford Lecture Series in Mathematics and its Applications 30, Oxford University Press, Oxford, 2005.

- [18] J.C. ROSALES AND P.A. GARCÍA-SÁNCHEZ, Numerical semigroups. *Developments in Mathematics*, 20. Springer, New York, 2009.
- [19] A. SAMMARTANO, Numerical semigroups with large embedding dimension satisfy Wilf's conjecture, *Semigroup Forum* 85 (2012) 439–447.
- [20] J.J. SYLVESTER, Mathematical questions with their solutions, *Educational Times* 41 (1884) 21.
- [21] H. WILF, A circle-of-lights algorithm for the money-changing problem, *Amer. Math. Monthly* 85 (1978) 562–565.
- [22] A. ZHAI, Fibonacci-like growth of numerical semigroups of a given genus, *Semigroup Forum* 86 (2013) 634–662.

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