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DYNAMICS OF AUTOMORPHISM GROUPS OF PROJECTIVE SURFACES: CLASSIFICATION, EXAMPLES AND OUTLOOK

SERGE CANTAT AND ROMAIN DUJARDIN

ABSTRACT. We first present an overview of our previous work on the dynamics of subgroups of automorphism groups of compact complex surfaces, together with a selection of open problems and new classification results. Then, we study two families of examples in depth: the first one comes from folding plane pentagons, and the second one is a family of groups introduced by Jérémy Blanc, which exhibits interesting new dynamical features.

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INTRODUCTION

This article is the sixth of a series dedicated to the dynamics of groups of automorphisms of compact complex surfaces [18, 26, 24, 25, 23]. Our purpose is to review our previous work and to enrich it with new examples, applications, and open problems. Let us briefly summarize its contents and new features.

In the first two sections, we offer a detailed presentation of our results, illustrated with a large number of open questions.

A standing assumption in our previous papers is that the surfaces into consideration are projective. It is natural to question this assumption and to relate the classification of surfaces to the dynamics of their groups of automorphisms. This is dealt with in the first part of the paper: in § 3.3, we show that only projective surfaces can carry “non-elementary” automorphism groups and we describe a few examples on Hopf surfaces in § 3.4.

Then, we explain how our theory applies to the following geometric examples:

(1) The first one comes from classical Euclidean geometry, and is given by folding plane pentagons of given side length along their diagonals. Surprisingly enough, this gives rise to a group action on a K3 surface, which is reminiscent of the Wehler family of examples, which has been a thread in our work. These examples are studied in Part 2. Section 5 describes the underlying algebraic geometry of the problem, which goes back to the work of Darboux [35] on quadrilaterals and elliptic functions. The ergodic theory of random pentagon foldings is analyzed in Section 6, in the spirit of the work of Benoist and Hulin [7, 8].

(2) In Part 3 (Sections 7 to 10), we focus on groups of automorphisms introduced by Blanc in [12]. Each of these examples is determined by the choice of a plane cubic curve $C \subset \mathbb{P}_{\mathbb{C}}^2$, an integer $m \geq 3$, and m points q_1, \dots, q_m on C . To each q_j , one associates a Jonquières involution, that fixes C pointwise and preserves the pencil of lines through q_j . The group generated by these m involutions lifts to a group of automorphisms on a rational surface X which is obtained by blowing-up $5m$ points of C . A key property of these automorphism groups is that they preserve a singular volume form with poles along the strict transform C_X of C and whose total mass is infinite. Using our previous results, we prove that, *for an appropriate choice of the points q_i , the only ergodic stationary measures on $X(\mathbf{R})$ are fixed points and are contained in $C_X(\mathbf{R})$* . In particular, random orbits almost surely converge, on average, to the curve $C_X(\mathbf{R})$. Thus, as we shall explain, this behavior differs strongly from that of automorphisms of non-rational surfaces.

1. OVERVIEW AND OPEN PROBLEMS

In this section, we provide a detailed overview of our former results, with short introductions to their proofs, as well as a description of our main new results and examples.

1.1. From orbit closures to stationary measures. Let X be a compact space and Γ be an infinite group of homeomorphisms of X . We make the standing assumption that Γ is countable; this will actually not be a restriction in the cases of interest to this paper. Our general aim is to study the dynamics of such an action. In particular we wish to address the following usual problems:

- (Pb1) describe the orbit closures $\overline{\Gamma \cdot x}$, for $x \in X$;
(Pb2) study the finite orbits of Γ

$$(1.1) \quad \text{Per}(\Gamma) = \{x; \Gamma \cdot x \text{ is finite}\};$$

by definition, points of $\text{Per}(\Gamma)$ will be called **Γ -periodic points**.

- (Pb3) classify Γ -invariant measures, that is, probability measures μ on X such that $f_*\mu = \mu$ for all $f \in \Gamma$;
(Pb4) describe the asymptotic distribution of Γ -orbits.

Observe that the last question is not properly formulated until a specific way of going to infinity in Γ (in other words, a notion of “time”) has been described. A common choice for this is to fix a probability measure ν on Γ such that $\langle \text{Supp}(\nu) \rangle = \Gamma$ and to explore Γ by walking at random according to ν . Then the notion of asymptotic distribution of the orbit of $x \in X$ may either refer to the asymptotics of the orbital averages

$$(1.2) \quad \int \delta_{f_{n-1} \cdots f_0(x)} d\nu(f_0) \cdots d\nu(f_{n-1}),$$

or to the time averages along random trajectories

$$(1.3) \quad \frac{1}{n} \sum_{k=1}^n \delta_{f_{n-1} \cdots f_0(x)}$$

for $\nu^{\mathbb{N}}$ -almost every $(f_n)_{n \geq 0}$. In both cases, understanding the limit points essentially boils down to the following problem:

- (Pb5) classify ν -stationary probability measures.

Of course, (Pb5) subsumes (Pb3). To understand the meaning and relevance of (Pb5), let us first recall that a probability measure μ on X is ν -**stationary** if

$$(1.4) \quad \mu = \nu * \mu := \int f_*\mu d\nu(f).$$

Since the measure in Equation (1.2) is the n -th convolution $\nu^{*n} * \delta_x$, any limit point of this sequence of measures is ν -stationary. Breiman’s ergodic theorem shows that the same is true for the random empirical measures in Equation (1.3) (see [11, §3.2]). In particular, if K is a compact Γ -invariant subset of X , then starting from $x \in K$ one constructs ν -stationary measures with support in K ; these may also be obtained by applying a fixed point theorem to the operator $\mu \mapsto \nu * \mu$ acting on probability measures on K . As a consequence, taking $K = \overline{\Gamma \cdot x}$, we see that a solution to (Pb5) is also useful for (Pb1). This approach, using ergodic theoretic methods to study orbit closures (that is, use (Pb5) and (Pb3) to study (Pb1)), is now commonplace in this area of research.

So, the set of stationary measures on X is a non-empty, compact and convex subset of the set of probability measures on X . It contains Γ -invariant probability measures, but in many situations invariant measures fail to exist. Thus, stationary measures can be viewed as the correct analogues of invariant measures when studying (large) groups of transformations instead of cyclic groups (i.e. the iterations of a single homeomorphism). In this respect, it may seem hopeless at first sight to classify all stationary measures, but, in fact, they often satisfy rigidity

properties which make such a classification feasible. Results in homogeneous dynamics, in particular the work of Ratner [78, 77, 58] and Benoist and Quint [9], illustrate perfectly this line of thought. Similar phenomenon also appear in non-homogeneous dynamics, notably in the work of Eskin and Mirzakhani [51] and Brown and Rodriguez Hertz [16]. Our work is strongly influenced by these former results and the methods developed to reach them.

Remark 1.1. For ν -stationary (resp. Γ -invariant) measures, there is a notion of ergodicity: such a measure is ergodic if it is an extremal point of the compact convex set of ν -stationary (resp. Γ -invariant) measures. It is a non-trivial result that these two notions coincide, that is a ν -stationary measure μ is ergodic if and only if every almost Γ -invariant subset A (i.e. $\mu(f(A)\Delta A) = 0$ for every $f \in \Gamma$) has measure 0 or 1. Hence, in the classification problems (Pb5) and (Pb3), it is enough to restrict to ergodic measures.

1.2. Non-elementary groups of automorphisms and the classification of surfaces. In this paper, X will be a compact complex surface. We denote by $\text{Aut}(X)$ its group of holomorphic diffeomorphisms, which we call **automorphisms**, even when X is not algebraic. The group Γ will be contained in $\text{Aut}(X)$.

It would also be natural to consider actions by automorphisms on quasi-projective or affine surfaces, for instance by polynomial automorphisms of \mathbf{C}^2 , or even by birational transformations. Extending our results to this more general setting or to higher dimensions is an important challenge, which would undoubtedly lead to serious difficulties. References include [19, 79, 57] (see also Example 1.6, Remark 1.17, Question 1.14, and Theorem 2.2 below).

Two related constraints will be imposed on Γ . Firstly, Γ must be sufficiently large to expect some measure rigidity property. For instance, if Γ is abelian, its stationary measures are automatically invariant; and if Γ is generated by an automorphism with positive entropy, there are uncountably many invariant ergodic measures, without any hope of classifying them. Secondly, the action of Γ on the cohomology of X must also be “sufficiently large”. To explain this, recall that $\text{Aut}(X)$ is a complex Lie group and, assuming X to be Kähler, a theorem of Fujiki and Lieberman asserts that the connected component $\text{Aut}(X)^0$ of $\text{Aut}(X)$ is a subgroup of finite index in the kernel of the homomorphism

$$(1.5) \quad \text{Aut}(X) \ni f \mapsto f^* \in \text{GL}(H^2(X; \mathbf{Z}))$$

that describes the action on the cohomology. Then, using the invariance of the intersection form on $H^2(X; \mathbf{Z})$, the Hodge index theorem, and the Tits alternative, one obtains easily (see [21, 26]) the equivalence of the following properties for any subgroup $\Gamma \subset \text{Aut}(X)$

- (a) the image Γ^* of Γ in $\text{GL}(H^2(X; \mathbf{Z}))$ is not virtually Abelian;
- (b) the image Γ^* of Γ in $\text{GL}(H^2(X; \mathbf{Z}))$ contains a pair of linear maps (f^*, g^*) generating a non-Abelian free group;
- (c) Γ contains a pair of automorphisms (f, g) with positive topological entropy generating a non-Abelian free group.

Moreover, the classification of compact Kähler surfaces implies that if these properties are satisfied for some $\Gamma \subset \text{Aut}(X)$, then either X is a torus or $\text{Aut}(X)^0$ is trivial. From this, one deduces that, up to finite index, only two distinct regimes need to be studied:

- (1) the case where Γ is contained in $\text{Aut}(X)^0$,
- (2) the case where Γ satisfies the three equivalent properties (a), (b), (c).

The first case boils down to homogeneous dynamics, the main example being given by subgroups of $\text{PGL}_3(\mathbf{C})$ acting by linear projective transformations on $\mathbb{P}^2(\mathbf{C})$. Thus, we focus on the second case that is, we shall impose that Γ satisfies (a), (b), (c).

Even if our understanding of non-Kähler compact complex surfaces is less satisfactory, we shall see in Section 3 that these properties are also relevant in this more general context. So, we introduce the following

Definition 1.2. Let X be a compact complex surface. A subgroup Γ of $\text{Aut}(X)$ is **non-elementary** if Γ^* contains a non-Abelian free group, otherwise, it is **elementary**.

A first important result, proven below in Section 3, says that the existence of a non-elementary subgroup $\Gamma \subset \text{Aut}(X)$ forces X to be projective, hence, by the GAGA principle, it also forces Γ to act by regular algebraic transformations. Consequently, the study of non-elementary actions belongs to algebraic dynamics.

Theorem 1.3. *Let X be a compact complex surface such that there exists a non-elementary subgroup $\Gamma \subset \text{Aut}(X)$. Then X is projective and it must be a blow-up of \mathbb{P}^2 , a K3 surface, an Enriques surface, or an Abelian surface*

A consequence is that, unless X is a torus, $\text{Aut}(X)^0$ is trivial, hence $\text{Aut}(X)$ is discrete, and the countability assumption on Γ is automatically satisfied. As suggested by this theorem, examples of non-elementary group actions are scarce and rely on algebro-geometric constructions. Besides the following two classes of examples, several new ones will be discussed in this paper.

Example 1.4. The first examples are found on (certain) compact tori $X = \mathbf{C}^2/\Lambda$. Recall that every automorphism of such a surface is induced by an affine transformation $x \in X \mapsto Ax + B$, where $A \in \text{GL}_2(\mathbf{C})$ preserves Λ and $B \in \mathbf{C}^2$. By Theorem 1.3, if $\text{Aut}(X)$ is non-elementary, then X must be projective, that is, be an Abelian surface.

More generally, a **Kummer group** (X, Γ) is, by definition, a compact complex surface X endowed with a subgroup Γ of $\text{Aut}(X)$, such that there exists a torus $X' = \mathbf{C}^2/\Lambda$, a subgroup $\Gamma' \subset \text{Aut}(X')$, and a generically finite, dominant rational map $\varphi: X' \rightarrow X$ that semi-conjugates Γ' to Γ (i.e. $\varphi \circ \Gamma' = \Gamma \circ \varphi$). We refer to [24, §4] for details and a classification of such examples. Kummer examples can be analyzed with tools from homogeneous dynamics, and as we will see, they often stand out for their exceptional, distinctive properties, somewhat similar to those of monomial, Tchebychev, and Lattès mappings in one-dimensional dynamics (see [20]).

Example 1.5. Consider the family of all K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (equivalently, smooth surfaces of tri-degree $(2, 2, 2)$); they depend on 26 parameters. If X is such a surface, the three natural projections onto $\mathbb{P}^1 \times \mathbb{P}^1$ are dominant morphisms of degree 2, and each of them determines a regular involution of X (which exchanges the two points in the fibers). Thus, X comes with a group $\Gamma \subset \text{Aut}(X)$, generated by this involutions. For a general choice of X , Γ is non-elementary. We shall refer to these examples (X, Γ) as **Wehler examples**. These were studied thoroughly in our work (see [26, 24, 23]). The family of pentagon folding groups introduced in Part 2 is in many ways reminiscent from the Wehler family.

Example 1.6. Let us mention an example which is unfortunately excluded by our assumptions, because the surface is not compact (when compactifying the surface, the automorphisms become birational transformations). Consider the affine surface $M_0 \subset \mathbf{C}^3$ defined by

$$(1.6) \quad x^2 + y^2 + z^2 = 3xyz.$$

It is a cubic surface, endowed with three natural 2-to-1 ramified covers onto \mathbf{C}^2 , obtained by forgetting one of the variables, each of which defining a regular involution on M_0 . The dynamics of the group generated by these involutions was studied by Markov in his 1879 thesis, in relation to the study of binary quadratic forms and Diophantine approximation, and M_0 is now called *the Markov surface*. It can be considered as a degenerate Wehler example. We refer to [19, 40, 79, 57] and [80] in this volume for a dynamical viewpoint on these surfaces.

1.3. Further remarks: parabolic elements, fields of definition, irrational surfaces. From now on we will assume that $\Gamma \subset \text{Aut}(X)$ is non-elementary and therefore that X is projective. Our strongest results feature an additional hypothesis on Γ , namely the existence of parabolic elements. The vocabulary is as follows. An automorphism $f \in \text{Aut}(X)$ is **elliptic** if f^* is a finite order element of $\text{GL}(H^2(X; \mathbf{Z}))$. It is **loxodromic** if the spectral radius λ_f of f^* is > 1 ; in that case, λ_f is the only eigenvalue of f^* with modulus > 1 , it is simple, and $\log(\lambda_f)$ is equal to the topological entropy of $f: X \rightarrow X$. Otherwise, f is **parabolic**; this means that some iterate $(f^*)^k$, for some $k \geq 1$, is unipotent and distinct from the identity. In this case, there is a unique genus 1 fibration $\pi: X \rightarrow B$ which is f -invariant; this means that there is an automorphism f_B of the Riemann surface B such that $\pi \circ f = f_B \circ \pi$; moreover, f_B has finite order, except if X is a torus \mathbf{C}^2/Λ . Hence, assuming X is not a torus and changing f into a positive iterate, f preserves every fiber, each smooth fiber is a genus 1 curve, and f acts upon it as a translation; such maps are referred to as **Halphen twists**. The analysis of these parabolic automorphisms leads to rich structures and is connected to the field of discrete integrable systems (see [25, §3] for an account, and [47] for a thorough treatment).

With this vocabulary, Γ is non-elementary if and only if it contains a pair of loxodromic elements (f, g) that generates a non-Abelian free group. The hypothesis that Γ contains a parabolic element is of a different nature –it is analogous, in Ratner’s theory, to the existence of unipotent elements. If Γ contains two parabolic elements preserving distinct fibrations, then Γ is non-elementary; conversely, if Γ contains a parabolic element and is non-elementary, then it contains two parabolic elements with distinct invariant fibrations. We refer to [21, 26] for these results.

Our results may also depend on the field of definition. When X is projective, X and Γ are defined by polynomial equations and formulas with coefficients in a subfield of \mathbf{C} . Some of our results require that X and Γ be defined over \mathbf{R} , in which case one might restrict the dynamics to the real part $X(\mathbf{R}) \subset X$. Some require that X and Γ are defined over $\overline{\mathbf{Q}}$.

Another thing that the reader should keep in mind comes from the classification of surfaces. By Theorem 1.3, surfaces for which $\text{Aut}(X)$ is non-elementary fall into two types.

– If X is rational, then X is a blow-up of the plane \mathbb{P}^2 at at least 10 points; as we shall see, a non-elementary subgroup $\Gamma \subset \text{Aut}(X)$ may or may not preserve a continuous volume form.

– If X is not rational, then X is a blow-up of a K3, Enriques, or Abelian surface X_0 ; moreover, by the uniqueness of the minimal model, the group $\text{Aut}(X)$ is obtained by pulling back of a subgroup of $\text{Aut}(X_0)$. On X_0 , there is a natural $\text{Aut}(X_0)$ -invariant volume form, induced by the triviality of the canonical bundle K_{X_0} or its square $K_{X_0}^{\otimes 2}$. This volume form lifts to X as an $\text{Aut}(X)$ -invariant form which vanishes only along the exceptional divisor of the birational morphism $X \rightarrow X_0$.

Some of our dynamical results do require the existence of such an invariant volume form, a hypothesis that might fail only for rational surfaces. Note that if X is irrational and defined over \mathbf{R} , there is an $\text{Aut}(X_{\mathbf{R}})$ -invariant volume form on $X(\mathbf{R})$ as well.

1.4. Stiffness. Let us now come back to our five initial problems. As explained in the previous sections, we only deal with non-elementary group actions on projective surfaces. We refer to [26, §10.3] for elementary actions on compact Kähler surfaces and to § 3.4 below for some remarks on the non-Kähler case.

We say that a probability measure μ on X is **Zariski diffuse** if it gives no mass to proper Zariski closed subsets of X . Ergodic stationary or invariant measures which are not Zariski diffuse are easily analyzed (see [26, Prop. 10.6]), so we focus on Zariski diffuse measures.

In certain situations, (Pb5) is actually equivalent to (Pb3); this happens whenever all ν -stationary measures are invariant: this is the **stiffness** (or ν -stiffness) property of Furstenberg [56]. An obvious obstruction to stiffness is when there is no Γ -invariant measure at all. The overall philosophy of [26] is the converse: *non-elementary group actions on compact complex surfaces admitting a natural Zariski diffuse invariant measure tend to be stiff* (in fact, this “principle” also applies to elementary groups). In this respect, recall from § 1.3, that if X is not rational, then, working on the minimal model X_0 of X , there is a natural invariant volume form.

Let us be more specific. Let $\Gamma \subset \text{Aut}(X)$ be a non-elementary group of automorphisms of a complex projective surface. Fix a probability measure ν on Γ satisfying the *generating condition*

$$(S) \quad \text{Supp}(\nu) \text{ generates } \Gamma$$

and the *moment condition*

$$(M) \quad \int \left(\log \|f\|_{C^1(X)} + \log \|f^{-1}\|_{C^1(X)} \right) d\nu(f) < +\infty.$$

Condition (S) is natural if we want the random dynamics to faithfully describe the group action and necessary. Condition (M) is necessary to apply the tools of smooth ergodic theory; note that by the Cauchy estimates, it automatically implies the same finiteness for the C^k norm. Pick an ergodic ν -stationary measure μ . Then μ admits two Lyapunov exponents $\lambda^- \leq \lambda^+$. These exponents are defined by applying the Oseledets theorem fiberwise to the (non-invertible) dynamical system associated to the random dynamics:

$$(1.7) \quad F_+ : \begin{array}{ccc} \Omega \times X & \longrightarrow & \Omega \times X \\ (\omega, x) & \longmapsto & (\sigma\omega, f_\omega^1(x)) \end{array}$$

where $\Omega = \text{Aut}(X)^{\mathbf{N}}$, $\omega = (f_n)_{n \geq 0}$, σ is the shift and $f_\omega^n = f_{n-1} \circ \cdots \circ f_0$ (so that $f_\omega^1 = f_0$).

If $\lambda^- < 0 < \lambda^+$, one says that μ is **hyperbolic**. Then, Oseledets theorem provides a measurable line field of stable directions $E_\omega^s(x)$ defined for $\nu^{\mathbf{N}} \times \mu$ -almost every (ω, x) . We say that

E^s is **non-random** if it does not depend on ω ; more precisely, if for μ -almost every x , there is a line $E^s(x) \subset T_x X$ such that $E_\omega^s(x) = E^s(x)$ for $\nu^{\mathbf{N}}$ -almost every trajectory ω . Our first key result is the following (see [26, Thms C and 9.1]).

Theorem 1.7. *Let $\Gamma \subset \text{Aut}(X)$ be a non-elementary group of automorphisms of a (necessarily projective) compact complex surface. Let ν be a probability measure on Γ satisfying Conditions (S) and (M), and let μ be a Zariski diffuse, ergodic, and hyperbolic ν -stationary measure. If the field of Oseledets stable directions is non-random, then μ is Γ -invariant.*

When $\lambda^+ \geq \lambda^- \geq 0$, the *invariance principle* of Crauel [33] (this terminology is due to Avila-Viana [3]) directly implies that μ is invariant. Thus, if Γ preserves a volume form, we have $\lambda^+ + \lambda^- = 0$, and for the stiffness problem we just have to consider hyperbolic measures.

Now, assume in addition that X and Γ are defined over \mathbf{R} , that $X(\mathbf{R})$ is non-empty, and that Γ preserves an area form on $X(\mathbf{R})$; more generally, we can consider a Γ -invariant, smooth, totally real surface $\Sigma \subset X$ on which Γ preserves an area form. Then, we can apply the results of Brown and Rodriguez Hertz [16], which assert that for a volume preserving action on a 2-dimensional real surface, the randomness of the stable line field $E_\omega^s(x)$ implies the invariance of μ (see [16, Thm 3.4]). From this and the previous considerations, we get:

Theorem 1.8. *Let $\Gamma \subset \text{Aut}(X)$ be a non-elementary group of automorphisms of a compact complex surface. Let ν be a probability measure on Γ satisfying Conditions (S) and (M). Let $\Sigma \subset X$ be a Γ -invariant, smooth, totally real surface that supports a Γ -invariant area form. Then, any Zariski diffuse ν -stationary measure on Σ is Γ -invariant.*

We expect that similar techniques will provide the same result on the complex surface X , without restricting to a real submanifold. Indeed, the following result ought to be true: *under the assumptions of Theorem 1.7, if Γ preserves a smooth volume form on X , then every Zariski diffuse ν -stationary measure is Γ -invariant.* In particular, this automatic invariance of Zariski diffuse stationary measures should hold for K3 surfaces. At the time of writing these lines, new results have been announced by Brown, Eskin, Filip and Rodriguez-Hertz [15], which should complete this story.

Notes on the proof.– The proof of Theorem 1.7 occupies most of [26]. It relies on the following chain of ideas: if E^s is non-random then either (1) μ satisfies a strong form of zero fiber entropy (see Section 2 for this notion) which makes it invariant or (2) the (global) stable manifolds $W_\omega^s(x)$, obtained from Pesin theory, are non-random as well. Now, almost every stable manifold $W_\omega^s(x)$ is biholomorphic to \mathbf{C} and, to any non-constant entire curve $\mathbf{C} \rightarrow X$, one can associate its so-called Ahlfors-Nevannlina currents (obtained as limits of integration on large disks in \mathbf{C}). Thus, to almost every point (ω, x) is associated a set of closed positive currents. In this dynamical situation, we show that there is in fact a unique current T_ω^s associated to $W_\omega^s(x)$ and that T_ω^s depends only on ω . Thus, if the stable manifolds are non-random, the T_ω^s are non-random as well, hence so are their cohomology classes $[T_\omega^s] \in H^2(X, \mathbf{R})$. But these cohomology classes can be analyzed by looking at the action of Γ on $H^2(X; \mathbf{R})$. Using Furstenberg's theory of random products of matrices (applied to the action of (Γ^*, ν) on $H^2(X, \mathbf{R})$) and the non-elementary assumption, we prove that these classes $[T_\omega^s]$ should in fact depend non-trivially on ω . This contradiction rules out case (2), and the theorem follows.

1.5. A new instance of stiffness. Theorem 1.8 requires the existence of an invariant area form. In Part 3, we study a family of examples on rational surfaces, first defined by Blanc in [12], that do not preserve any smooth volume form. The following theorem summarizes the main properties of our examples:

Theorem 1.9. *Let C be a smooth, connected, real cubic curve in \mathbb{P}^2 . Let $k \geq 4$ be an integer. One can find a set of $5k$ points of $C(\mathbf{C})$, consisting of $3k$ real points and k pairs of complex conjugate points, such that, after blowing up the plane at these $5k$ points, one gets the following objects:*

- (1) *a rational surface X , in which the strict transform C_X of C is a curve of genus 1;*
- (2) *a non-elementary group $\Gamma \subset \text{Aut}(X)$ isomorphic to the free product $\ast_1^k \mathbf{Z}/2\mathbf{Z}$;*
- (3) *a rational 2-form Ω_X that does not vanish, has poles of order 1 along C_X , and is Γ -invariant;*
- (4) *X , Γ , C_X , and Ω_X are defined over \mathbf{R} ; the total volume of $\Omega_X \wedge \overline{\Omega_X}$ is infinite and, after restriction to $X(\mathbf{R})$, the total area of Ω_X is also infinite.*

The dynamics on $X(\mathbf{C})$ satisfies:

- (5) *every orbit $\Gamma \cdot x \subset X(\mathbf{C})$ is infinite, except for $x \in C_X(\mathbf{C})$, in which case $\Gamma \cdot x = \{x\}$;*
- (6) *$(\Omega_X \wedge \overline{\Omega_X})$ -almost every Γ -orbit is dense in $X(\mathbf{C})$.*

And for the real dynamics on $X(\mathbf{R})$ we get:

- (7) *every infinite orbit in $X(\mathbf{R})$ is dense in $X(\mathbf{R})$;*
- (8) *if ν is a probability measure on Γ satisfying Conditions (S) and (M), then every ergodic ν -stationary measure on $X(\mathbf{R})$ is a Dirac mass δ_x at some point $x \in C_X(\mathbf{R})$; such stationary measures have zero Lyapunov exponents;*
- (9) *the action of Γ on $H^1(X(\mathbf{R}); \mathbf{R})$ has a positive Lyapunov exponent (in the sense of Furstenberg's theory).*

The last property contains in particular the existence of automorphisms $f \in \Gamma$ with an eigenvalue > 1 on $H^1(X(\mathbf{R}); \mathbf{R})$: such automorphisms have positive entropy in the real locus $X(\mathbf{R})$. The proof occupies all of Part 3 and uses most of our previous results. In particular, the data (C and the points that we blow up) must be defined over $\overline{\mathbf{Q}}$.

It is interesting to compare this family of examples with the following ones coming from homogeneous dynamics.

Example 1.10. Consider the group $\text{SL}_2(\mathbf{R})$, acting by linear projective transformations on the real projective line $\mathbb{P}^1(\mathbf{R})$. The diagonal action on $\mathbb{P}^1(\mathbf{R}) \times \mathbb{P}^1(\mathbf{R})$ preserves the form

$$(1.8) \quad \frac{dx \wedge dy}{(x - y)^2}.$$

This meromorphic section of the canonical bundle $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ does not vanish and has a double pole along the diagonal; its total area is infinite. Now, consider $\Gamma = \text{SL}_2(\mathbf{Z}) \subset \text{SL}_2(\mathbf{R})$ acting diagonally on $\mathbb{P}^1(\mathbf{R}) \times \mathbb{P}^1(\mathbf{R})$. The closure of every Γ -orbit is equal to $\mathbb{P}^1(\mathbf{R}) \times \mathbb{P}^1(\mathbf{R})$ or to Δ . If ν is a probability measure on Γ satisfying Condition (S), then there is a unique ν -stationary measure μ_ν on $\mathbb{P}^1(\mathbf{R}) \times \mathbb{P}^1(\mathbf{R})$, this measure is supported on Δ , and it is not invariant. Moreover,

the action of Γ on $\pi_1(\mathbb{P}^1(\mathbf{R}) \times \mathbb{P}^1(\mathbf{R})) \simeq \mathbf{Z}^2$ is trivial, and the entropy of every $f \in \Gamma$ vanishes. The standard action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{R}^2 \subset \mathbb{P}^2(\mathbf{R})$ provides a similar example, for which the origin is fixed and rational points have discrete orbits in \mathbf{R}^2 .

Notes on the proof.– The mechanism for the stiffness property in Theorem 1.9 is quite different from that of Theorem 1.8. This time, we use Theorem 1.7 and [16], together with the existence of a singular invariant volume to prove by contradiction that all ergodic stationary probability measure μ have vanishing Lyapunov exponents. So by the invariance principle all stationary measures are invariant, and the existence of parabolic elements in the group allows for a complete classification (see Theorem 1.13 below).

All the examples of non-elementary groups encountered so far admit an invariant probability measure.

Question 1.11. *Does there exist a non-elementary group action on a rational projective surface without any invariant probability measure?*

Such an example could be found only on a rational surface. A natural candidate would be Lesieutre’s tri-Coble examples, briefly described in § 4.4 below.

Another question, which is somehow dual to the previous one, is the following:

Question 1.12. *If $\Gamma \subset \mathrm{Aut}(X)$ is a non-elementary subgroup preserving a smooth volume form, does Γ admit a hyperbolic stationary measure on X (for some probability measure on Γ satisfying (S) and (M))?*

If Γ possesses a parabolic element, then by [23, Thm 1.3], the invariant volume itself is hyperbolic. In the general case, if there is no hyperbolic measure, then by the invariance principle, every stationary measure is invariant, so this is really a question about the classification of invariant measures (more precisely, about the second line of the table in Section 2). Note that without the volume preserving assumption, Theorem 1.9 shows that the answer to this question (on $X(\mathbf{R})$) is negative. Note also the connection with the following fundamental open problem from conservative dynamics: *if f is a volume-preserving loxodromic automorphism on a compact complex surface X , does the invariant volume have positive measure theoretic entropy?*

These questions, as well as the proof of Theorem 1.9 are a source of motivation for the classification of invariant measures, which we discuss now.

1.6. Classification of invariant measures. Probability measures invariant under a non-elementary group are subject to so many constraints that it is reasonable to hope for a complete classification. Still, in its full generality, this problem remains out of reach for the moment¹.

Theorem 1.13. *Let $\Gamma \subset \mathrm{Aut}(X)$ be a non-elementary group of automorphisms of a projective surface containing a parabolic element. Let μ be an ergodic, Zariski diffuse, Γ -invariant measure. There exists a proper Γ -invariant subvariety $Z \subset X$ such that:*

¹To illustrate the difficulty of this kind of question, it may be interesting to recall that Furstenberg’s famous problem asking for a classification of measures on the circle invariant under $\times 2$ and $\times 3$ is still open to date (this is however quite different from our setting since it concerns an Abelian semigroup).

- either μ is a measure with a smooth positive density in $X \setminus Z$;
- or there exists a Γ -invariant totally real surface embedded in $X \setminus Z$ such that μ is a measure with a smooth positive density on Σ .

This result, obtained in [18, 25], shows that *if Γ is non-elementary and contains a parabolic element, the classification problem for invariant measures is completely solved* (see [25, Thm A] for a precise statement). This classification comes with an almost complete description of orbit closures (which is most satisfactory under additional assumptions, see § 1.9). A consequence of this classification and of the results described in Section 1.7 is that *if Γ does not preserve any curve, there are only finitely many ergodic Γ -invariant measures, unless X is a torus*.

Notes on the proof.– The first input to prove Theorem 1.13 is that if $h \in \Gamma$ is a parabolic automorphism preserving the fibers of a genus 1 fibration $\pi: X \rightarrow B$, then h acts as a uniquely ergodic translation on most fibers of π (it also acts periodically on a dense, countable set of fibers). Thus, any h -invariant measure can be described in terms of its disintegration along the fibers of π . From this it follows rather easily that any ergodic Zariski diffuse Γ -invariant measure satisfies a local homogeneity property (i.e. some local invariance under connected groups of translations), which implies that it is either absolutely continuous with respect to the Lebesgue measure on X , or to the Lebesgue measure on an immersed totally real surface Σ . The delicate issue is to show, in the latter situation, that Σ globalizes to an embedded submanifold outside some algebraic subset Z .

When Γ does not contain parabolic elements, the classification of invariant measures is still at an early stage, and is arguably the main open problem in our program. It can be subdivided in several sub-problems of independent interest; since the discussion at this point becomes rather technical, we defer it to Section 2. Let us just mention one open question, which we formulate in the easiest context of automorphisms of the affine plane.

Question 1.14. *Can one construct two polynomial automorphisms f and g of \mathbf{C}^2 such that*

- (i) *f and g fix the origin o ;*
- (ii) *there is a germ of holomorphic diffeomorphism $\varphi: (\mathbf{C}^2, o) \rightarrow (\mathbf{C}^2, o)$ that conjugates f and g simultaneously to two elements of $\mathrm{SU}_2(\mathbf{C})$;*
- (iii) *the group generated by f and g is a non-Abelian free group.*

Our expectation is that such an example does not exist. If such an example were to be found, it would have an invariant probability measure for each small sphere centered at the origin.

1.7. Finite orbits and arithmetic dynamics. For a non-elementary group Γ , the existence of a finite orbit is an overdetermined property. Indeed, if $f \in \mathrm{Aut}(X)$ is a loxodromic element, then its periodic points of sufficiently large period form a countable subset of X , and any Γ -periodic point must be f -periodic for every $f \in \Gamma$. Since Γ is non-elementary, it contains two loxodromic automorphisms f and g satisfying no algebraic relation (see § 1.2). In this case it is expected that $\mathrm{Per}(f) \cap \mathrm{Per}(g)$ is finite, or contained in a curve of $\langle f, g \rangle$ -periodic points

Example 1.15. To implement these ideas, a first approach is to work in families and prove generic results. In the family of Wehler examples (see Example 1.5), we proved in [24, Thm

A] that a very general member has no periodic orbit. Such a result is still open for other natural families (see Questions 4.2 and 6.6 below). In this paper, we develop new methods to get similar results for pentagon foldings (see Part 2) and Blanc's examples (see Part 3).

A deeper problem is to deal with specific, individual examples. Then the existence of infinitely many finite orbits for a non-elementary group falls into the setting of *unlikely intersection problems* (see [87]), and it will not be a surprise to the experts that methods from arithmetic geometry arise here. For instance, the following result was established in [24, Thm B].

Theorem 1.16. *Let X be a projective surface defined over $\overline{\mathbf{Q}}$. Let $\Gamma \subset \text{Aut}(X)$ be a non-elementary group, defined over $\overline{\mathbf{Q}}$ and containing a parabolic element. If $\text{Per}(\Gamma)$ is Zariski dense, then (X, Γ) is a Kummer example.*

Thus, when X is not a Kummer surface, $\text{Per}(\Gamma)$ is the union of a Γ -invariant curve, together with finitely many isolated points².

Notes on the proof.— Fix a probability measure ν on Γ with finite support generating Γ . First, we construct a height function h_ν on $X(\overline{\mathbf{Q}})$ such that $\sum_f \nu(f)h \circ f = \lambda_\nu h$ for some $\lambda > 1$. It follows that Γ -periodic points have zero height. Assuming the existence of a Zariski dense set of periodic orbits, and using Yuan's equidistribution theorem for small points, we construct a Γ -invariant measure with special properties. To conclude, we use the classification of invariant measures described in Theorem 1.13, and a rigidity result proved in [27]. The existence of parabolic elements is used at several places in the proof, notably to apply Theorem 1.13 (see [24, §6.3] for a discussion). Corvaja, Tsimerman and Zannier [31] recently proved a closely related theorem which also requires parabolic elements. Their proof is based on different ideas (o-minimal geometry, variation of the canonical height for families of elliptic curves, etc).

Remark 1.17. In [54], Wehler examples with a finite but large number of periodic points are constructed. In the closely related case of Markov type surfaces

$$(1.9) \quad x^2 + y^2 + z^2 = xyz + ax + by + cz + d,$$

where $(a, b, c, d) \in \mathbf{C}^4$ is a parameter, all possible finite orbits have been classified (see [70]). The only parameter with infinitely many periodic orbits is $(0, 0, 0, 4)$; the corresponding surface is the quotient of the multiplicative group $\mathbf{C}^\times \times \mathbf{C}^\times$ by the involution $(u, v) \mapsto (1/u, 1/v)$, and the group action on $M_{(0,0,0,4)}$ lifts to the monomial action of $\text{GL}_2(\mathbf{Z})$; finite orbits correspond to torsion points. Thus, the situation is perfectly similar to that of Kummer examples, but for a multiplicative torus.

Surprisingly enough, Theorem 1.16 can be used to prove the same result when X and Γ are defined over \mathbf{C} ; in that case we must suppose (for technical reasons) that Γ does not preserve any curve, and the existence of infinitely many periodic orbits then implies that X is a torus (see [24, Thm C]). Related, stronger, results were obtained for polynomial automorphisms of \mathbf{C}^2 , and more generally affine surfaces [48, 1]. This motivates the following question:

²Here, we use the following fact: if Γ acts by automorphisms on a curve C with a Zariski dense set of periodic points, then the image of Γ in $\text{Aut}(C)$ is finite. Indeed, C has only finitely many irreducible components, and if a subgroup Γ_0 acts on some irreducible component C_i by fixing a subset F_i such that the Euler characteristic of $C_i \setminus F_i$ is negative, then the restriction of Γ_0 to C_i is finite.

Question 1.18. *Can we classify pairs of automorphisms (f, g) of positive entropy on a compact projective surface having a Zariski dense set of common periodic points? Or having the same maximal entropy measure?*

Note that in the second question the common maximal entropy measure would be a $\langle f, g \rangle$ -invariant measure, so we are back to the theme of § 1.6. If one can show that this measure is smooth, then the main result of [27] implies that $(X, \langle f, g \rangle)$ is a Kummer example; the proof of Theorem 1.16 relies on this fact.

In one-dimensional algebraic dynamics, unlikely intersection problems are better understood. For instance, one can produce *uniform bounds* on the number of common periodic points (instead of orbits) for pairs of distinct quadratic maps $z \mapsto z^2 + c$, $z \mapsto z^2 + c'$: we refer to [38, 39] for precise results. Here is a version of this problem for non-elementary groups of automorphisms:

Question 1.19. *Is there a uniform bound for $\# \text{Per}(\Gamma)$ in the family of (smooth) Wehler examples?*

Note that for a general parameter in the Wehler family there is no invariant curve, so $\text{Per}(\Gamma)$ is indeed finite (see [26, Prop. 3.6]). Examples of Wehler groups with $\# \text{Per}(\Gamma) \geq 288$ were constructed in [54] (see Remark 1.17). On the other hand, some singular Wehler examples are Kummer hence give rise to a dense set of periodic orbits (see [20, Ex. 8.3]).

Let us conclude this section with a last question which may be considered as a variation on a prediction of Kawaguchi and Silverman [63]. Assume that X and Γ are defined over $\overline{\mathbf{Q}}$ and that Γ is non-elementary. Fix a polarization H of X , denote by $h: X(\overline{\mathbf{Q}}) \rightarrow \mathbf{R}_+$ a Weil height associated to H . For every $f \in \text{Aut}(X)$, let

$$(1.10) \quad \deg_H(f) = H \cdot (f^*H).$$

Fix a point $x \in X(\overline{\mathbf{Q}})$ with a Zariski dense orbit, and set

$$(1.11) \quad N(x; R) := \# \{y \in \Gamma \cdot x ; h(y) \leq R\}.$$

Kawaguchi and Silverman suggested that the Zariski density of $\Gamma \cdot x$ should imply that $h(f(x))$ is of the same order of magnitude as $\deg(f)$. More precisely, introduce the counting function

$$(1.12) \quad N_{\deg}(R) := \# \{f \in \Gamma ; \deg(f) \leq R\},$$

and ask:

Question 1.20. *If $\Gamma \subset \text{Aut}(X)$ is a non-elementary automorphism group defined over $\overline{\mathbf{Q}}$, and $x \in X$ has a Zariski dense orbit, does $N(x; R)$ grow like $N_{\deg}(R)$?*

The group Γ acts on a hyperbolic space $\mathbb{H}_X \subset H^{1,1}(X; \mathbf{R})$ (see Section 3), and $\log(\deg_H(f))$ is comparable to the distance in \mathbb{H}_X between the cohomology class $(H \cdot H)^{-1/2}[H]$ and its image by f^* . From this, we see that $N_{\deg}(R)$ is a classical counting function for orbits of discrete isometry groups in hyperbolic geometry. As a consequence, $N_{\deg}(R)$ grows like R^α , where α is the Hausdorff dimension of the limit set of the isometry group $\Gamma^* \subset \text{Isom}(\mathbb{H}_X)$ (see [5, 41]). Question 1.20 was answered positively by Baragar in [5] for some K3 examples; this problem is also well understood for Markov surfaces (see [57] for an account). The interested reader should also consult [52].

1.8. Equidistribution and uniform expansion. Up to now, we addressed Problems (Pb2), (Pb3), (Pb5), i.e. the study of stationary and invariant measures. We asserted in § 1.1 that classifying stationary measures is the key step to understand the asymptotic, stochastic distribution of orbits, or more precisely the limiting behavior of the averages (1.2) and (1.3). This assertion, while certainly correct, hides a final difficulty. Imagine an ideal situation where the set of ν -stationary measures is completely described, and its extremal points are given by some smooth (or Zariski diffuse) measure μ_0 and a finite or countable set of finite orbits. If $x \in X$ is a general point, the limit points of the averages (1.2) and (1.3) are convex combinations of these ergodic measures. The problem is to decide which combinations do arise; this is where uniform expansion comes into play. Let ν be a probability measure on $\text{Aut}(X)$, and set $\Gamma = \langle \text{Supp}(\nu) \rangle$. We say that ν is **uniformly expanding** if there exists $c > 0$ and an integer n_0 such that for every $x \in X$ and every $v \in T_x X \setminus \{0\}$,

$$(1.13) \quad \int \log \frac{\|D_x f(v)\|}{\|v\|} d\nu^{(n_0)}(f) \geq c,$$

where $\nu^{(n)}$ denotes the n^{th} convolution power of ν , and the norm is relative to some given Riemannian metric on X . This notion was introduced in non-linear random dynamics in [73, 30], notably to get equidistribution results analogous to those obtained in homogeneous dynamics in [50, 10]. Now, suppose that ν is uniformly expanding and satisfies

$$(M_+) \quad \text{there exists } p > 1 \text{ such that } \int \left(\log \|f\|_{C^1(X)} + \log \|f^{-1}\|_{C^1(X)} \right)^p d\nu(f) < +\infty.$$

Let F be a finite Γ -orbit. Then, if $x \in X$ has an infinite orbit, any cluster value of the sequence of measures in (1.2) or (1.3) gives zero mass to F (see [23, Thm 4.3]). Indeed, the uniform expansion condition makes F repelling on average, an idea which is formalized by the notion of *Margulis function*. So in the ideal situation described above, every infinite orbit equidistributes towards the smooth stationary measure μ_0 . Moreover, in the real volume-preserving setting, where two natural Zariski diffuse ergodic measures exist, given respectively by an invariant volume form on X and an invariant area form on $X(\mathbf{R})$, we can also decide which orbits are equidistributed with respect to the first or the second measure (see [23, §4.3]).

In homogeneous dynamics, establishing uniform expansion boils down to an expansion property for a random product of matrices. In a non-linear context, the situation is more delicate; fortunately, an abstract ergodic criterion for expansion was devised in [73, 30], which fits well with our holomorphic setting. This leads to a neat necessary and sufficient condition for uniform expansion on non-rational surfaces (see [23, Thm 1.5]):

Theorem 1.21. *Let X be a compact complex surface which is not rational. Let $\Gamma \subset \text{Aut}(X)$ be a non-elementary group containing parabolic elements. Let ν be a probability measure on $\text{Aut}(X)$ satisfying Conditions (S) and (M). Then, ν is uniformly expanding if and only if the following two conditions hold:*

- (1) every finite Γ -orbit is uniformly expanding;
- (2) there is no Γ -invariant algebraic curve.

Again, we assume that Γ contains a parabolic element because the classification of invariant measures is used in the proof.

In many interesting situations, we have a family $(X_\lambda; \Gamma_\lambda)_{\lambda \in \Lambda}$ of non-elementary automorphism groups, parameterized by some quasi-projective manifold. Then, as in Example 1.15, we developed tools to exclude the existence of proper, Zariski closed invariant subsets *for very general parameters* $\lambda \in \Lambda$. Hence, for these parameters, uniform expansion holds. In addition, Condition 1.13 is robust under C^1 perturbations, so this dense subset of uniformly expanding parameters is automatically promoted to an open and dense subset. In Part 2, we apply these ideas to the dynamics of random pentagon foldings.

Another interesting consequence of uniform expansion is that a volume preserving uniformly expanding action on a (connected, compact, real or complex) surface is automatically ergodic with respect to the invariant volume. This follows from a Hopf-type argument (see [46, 30]). An interesting question (which goes beyond complex dynamics) is whether such an ergodicity can be made quantitative (for uniformly expanding actions). To formalize this question, denote by $d\text{vol}$ the invariant volume form. Then, consider the Markov operator

$$(1.14) \quad P_\nu : \varphi \mapsto \int \varphi \circ f \, d\nu(f)$$

acting on some Banach space of continuous functions. Then, the question is to determine the speed of convergence of P_ν^n towards $\int \varphi \, d\text{vol}$. Is it exponential? A positive answer seems beyond reach for the moment, since it is open even in the homogeneous case, except for linear maps on tori, for which the Fourier techniques of [14] provide effective estimates.

1.9. Back to orbit closures. Putting together the results of §§ 1.4, 1.6, 1.7 and 1.8, we arrive at a complete understanding of orbit closures, under rather strong assumptions (see [23, Thm. 10.1]).

Theorem 1.22. *Let X be a compact complex surface which is not rational. Let $\Gamma \subset \text{Aut}(X)$ be a non-elementary group containing parabolic elements, and satisfying conditions (1) and (2) of Theorem 1.21. Then there exists a finite set F and a real-analytic (possibly singular) totally real surface Σ , both Γ -invariant, such that*

- if $x \in F$ then $\Gamma \cdot x$ is finite;
- if $x \in \Sigma \setminus F$, then $\overline{\Gamma \cdot x}$ is a union of components of Σ ;
- otherwise $\overline{\Gamma \cdot x} = X$.

Under less stringent hypotheses, but still assuming that Γ contains parabolic elements, weaker results on orbit closures are obtained in [25, §8], which deserve further study.

Problem 1.23. *Complete the classification of orbit closures when Γ is non-elementary and contains a parabolic element.*

Comments.– In [25, §8], we classified closed, Γ -invariant subsets F whose accumulation locus $\text{Acc}(F)$ is not contained in some explicit invariant algebraic subset STang_Γ . Thus, when there is no proper invariant algebraic subset, or more generally when uniform expansion holds, this leads to a complete classification. In the general case we must understand the situation where $\text{Acc}(\Gamma \cdot x) \subset \text{STang}_\Gamma$. It is easy to see that in this situation x must be g -periodic for each parabolic $g \in \Gamma$. In this case one expects that $x \in \text{Per}(\Gamma)$, but this result is not yet available (see [31] for partial results). See Theorem 10.5 below for a worked out example.

2. APPENDIX: CLASSIFICATION OF INVARIANT MEASURES

This section is a complement to § 1.6 and may be skipped on a first reading. We discuss what needs to be done to describe invariant probability measures for non-elementary groups that, a priori, do not contain parabolic elements.

2.1. Reducing the number of cases. We fix a complex projective surface X and a non-elementary subgroup Γ of $\text{Aut}(X)$. Let μ be a Zariski diffuse, ergodic, Γ -invariant probability measure. It is useful to introduce a probability measure ν on Γ satisfying Conditions (S) and (M) in order to speak about the Lyapunov exponents of μ ; since they depend on ν , we denote them by $\lambda^-(\nu) \leq \lambda^+(\nu)$.

A preliminary observation is that the exponents cannot be simultaneously positive (resp. negative), because in such a case a classical Pesin-theoretic argument implies that μ would be atomic. So $\lambda^-(\nu) \leq 0 \leq \lambda^+(\nu)$ and if $\lambda^-(\nu) = \lambda^+(\nu)$, then both exponents vanish. This gives three distinct possibilities: either $\lambda^-(\nu) < 0 < \lambda^+(\nu)$, or exactly one of the exponents vanish, or both of them vanish. We shall also distinguish three possibilities, depending on the number of invariant line fields: (1) no invariant line field, or pair of invariant line fields; (2) there exists a measurable Γ -invariant line field $E \subset TX$; (3) there exists an invariant measurable pair of line fields $E_1, E_2 \subset TX$. This leads to the following table of 12 possibilities:

	no invariant line field or pair of line fields	$\exists E \leq TX$ invariant line field	$\exists E_1, E_2 \leq TX$ pair of invariant line fields
$\exists \nu, \lambda^+(\nu) = \lambda^-(\nu) = 0$	A1	A2	A3
$\forall \nu, \lambda^+(\nu) = \lambda^-(\nu) = 0$	B1	B2	B3
$\exists \nu, \lambda^- < \lambda^+ \ \& \ \lambda^- \lambda^+ = 0$	C1	C2	C3
$\exists \nu, \lambda^+(\nu) < 0 < \lambda^-(\nu)$	D1	D2	D3

Let us discuss this table and the relationship between its entries. Note that the third row of this table does not appear in the volume preserving case.

2.1.1. The first column. The case when both exponents vanish (the first line) is analyzed in Section 7 of [23]. In case A1, Theorems 7.2 and 7.3 of [23] imply that the cocycle given by the action of Γ on the projectivization of the tangent space is measurably reducible to a compact group. In particular A1 implies B1.

The proof of Theorem 5.1 in [16] (see §13.2.4) shows that case C1 reduces to case D1.

2.1.2. The second column. When there is a measurable invariant line field $x \mapsto E(x)$, the Lyapunov exponent in the direction of E , which is one of $\lambda^-(\nu)$ or $\lambda^+(\nu)$, is given by the explicit formula

$$(2.1) \quad \lambda_E(\nu) = \int \log \|Df|_{E(x)}\| d\mu(x) d\nu(f) = \int \lambda_E(f, \mu) d\nu(f),$$

where $\lambda_E(f, \mu)$ is the integrated Lyapunov exponent of μ as an f -invariant measure (note that μ is not necessarily f -ergodic). The second exponent admits also a simple expression, because

it is equal to $\lambda^-(\nu) + \lambda^+(\nu) - \lambda_E(\nu)$, and

$$(2.2) \quad \lambda^-(\nu) + \lambda^+(\nu) = \int \log |\text{Jac} f(x)| d\mu(x) d\nu(f),$$

where Jac is the Jacobian determinant with respect to any given smooth volume form on X .

Let us show that A2 reduces to B2, C2 or D2. There are two possibilities: either $\lambda_E(f, \mu) = 0$ for every f , or not. In the first case we are in situation B2. In the second, we can change ν to put more weight on an element with $\lambda_E(f, \mu) \neq 0$ to impose $\lambda_E(\nu) \neq 0$; doing so, we fall in C2 or D2 (moreover, in C2 the non-vanishing exponent is along the invariant line field).

2.1.3. *The third column.* There are two possibilities: either every $f \in \Gamma$ preserves (resp. exchanges) the directions E_1 and E_2 almost everywhere, or not. In the first case, there is an index 2 subgroup preserving both directions, and we are in a special case of column 2. Otherwise, the directions are intertwined by the dynamics, and we claim that we are in case B3 (this argument is a variation on [16, Lem. 5.2]). Indeed, note that by definition there is a set of positive, hence full, μ -measure where $E_1(x) \neq E_2(x)$, and let

$$(2.3) \quad \Lambda(\nu) = \frac{1}{2} \int (\log \|Df|_{E_1(x)}\| + \log \|Df|_{E_2(x)}\|) d\mu(x) d\nu(f).$$

By ergodicity, for $(\nu^{\mathbb{N}} \times \mu)$ -almost every (ω, x) and every $v \in E_1(x) \cup E_2(x) \setminus \{0\}$,

$$(2.4) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \log \|Df_\omega^n(x)\| = \Lambda(\nu).$$

If $\lambda^+(\nu) = \lambda^-(\nu)$, then, by the preliminary remark, both exponents vanish. If $\lambda^-(\nu) < \lambda^+(\nu)$, then by Oseledets' theorem, for μ -a.e. x there is a line $E^s(x)$ such that if $v \notin E^s(x)$, then $\frac{1}{n} \log \|Df_\omega^n(x)\| \rightarrow \lambda^+$. Thus $E^s(x) \notin \{E_1(x), E_2(x)\}$ and $\Lambda = \lambda^+$. But now there is a set of 3 invariant directions, so the projective tangent action recurs to a compact set (see the proof of Theorem 7.3 in [23]) and we conclude that $\lambda^+(\nu) = \lambda^-(\nu) = 0$, a contradiction.

2.1.4. *Summary.* After these reductions, the situation is summarized in the following table. Colored cells do not need to be treated because they reduce to other ones.

	no invariant line field or pair of line fields	$\exists E \leq TX$ invariant line field	$\exists E_1, E_2 \leq TX$ pair of invariant line fields
$\exists \nu, \lambda^+(\nu) = \lambda^-(\nu) = 0$			
$\forall \nu, \lambda^+(\nu) = \lambda^-(\nu) = 0$	B1: Fatou behavior?	B2	B3
$\exists \nu, \lambda^+ < \lambda^- \ \& \ \lambda^- \lambda^+ = 0$		C2	
$\exists \nu, \lambda^+(\nu) < 0 < \lambda^-(\nu)$	D1: μ homogeneous?	D2: μ not Z. diffuse?	

2.2. **Comments.** Let us now analyze some of the remaining cases. We first note that we do not know any example, nor even a possible model for the dynamics, for the cases B2, B3, and C2 (in C2 it would make sense to further distinguish the cases whether the Lyapunov exponent in the invariant direction vanishes or not).

- B1: in that case, [23, Thm. 7.3] shows that the cocycle describing the action of Γ on the projectivized tangent bundle of X is measurably reducible to a compact subgroup of $\text{GL}_2(\mathbb{C})$. This suggests a Fatou-type behavior for the dynamics. *Must an invariant measure of type B1 be*

contained in the Fatou set of the Γ -action? As far as we know, there is no known example of a Fatou domain for a non-elementary group action by automorphisms on a projective surface. See Question 1.14 for a related problem.

- D1: μ is hyperbolic and there is no invariant line field. The techniques of [16] show that in this case μ has some homogeneity properties along stable and unstable manifolds. This implies in particular that μ has positive fiber entropy (as a ν -stationary measure). In a nutshell, recall that the fiber entropy can be defined by

$$(2.5) \quad h_\mu(X, \nu) = h_{\nu^{\mathbb{N}} \times \mu}(F_+) - h_{\nu^{\mathbb{N}}}(\sigma)$$

and it quantifies the relative entropy of F_+ in the fibers of the projection $\Omega \times X \rightarrow X$ (of course this definition needs to be adapted when $h_{\nu^{\mathbb{N}}}(\sigma) = \infty$, see [26, §7.6] for a brief presentation). *In the volume preserving (rational) case, it should be expected that, under assumption D1, μ is either absolutely continuous or absolutely continuous along some real analytic surface Σ . In the general case, this has to be replaced by a SRB property.* Partial results in this spirit include [16, Thm 5.1] and [26, Thm 11.1] for the real case. In the general complex case, such a result seems to be out of reach for the moment, even using [15]. The most delicate point would be to construct Σ from an information on the stable and unstable conditionals of μ only.

- D2: μ is hyperbolic and there is an invariant line field. Since μ is Γ -invariant, the natural extension of $(F_+, \nu^{\mathbb{N}} \times \mu)$ is $(F, \nu^{\mathbb{Z}} \times \mu)$, where F is the skew product over the 2-sided shift associated to the random dynamical system (X, ν, μ) (see [26, §7.1]). For this invertible dynamical system, the invariant line field corresponds to either E^s or E^u . Note that by the invariance of μ , F^{-1} is the skew product map associated to the random dynamical system induced by the reversed measure $\check{\nu}$, defined by $\check{\nu}(f) = \nu(f^{-1})$. Therefore, if $E = E^u$, E is the stable Oseledets bundle associated to the random dynamical system $(X, \check{\nu}, \mu)$. From this discussion, we conclude that in either case we may assume that $E = E^s$, i.e. the field of Oseledets stable directions is non-random. Theorem 9.1 in [26] then asserts that $h_\mu(X, \nu) = 0$, and furthermore by [23, Thm. B.1], $h_\mu(f) = 0$ for every $f \in \Gamma$. In other words, μ is a common zero entropy measure for all group elements. We expect that such a situation does not happen, so the conclusion should be that a Zariski diffuse invariant measure cannot satisfy D2.

2.3. A dual point of view and an example. Let us remark that another way of formulating this problem is to start with a loxodromic automorphism f of X and some f -invariant measure μ and to ask for a description of the stabilizer of μ in $\text{Aut}(X)$ (cf. [16, Thm 5.1] and [26, Thm 11.1]). Here is one instance of this problem:

Question 2.1. *Let f be a loxodromic automorphism of a complex projective surface X , and let μ_f be its unique invariant measure of maximal entropy. Is the stabilizer of μ in $\text{Aut}(X)$ virtually Abelian?*

To conclude, we answer a similar question in the specific case of polynomial automorphisms of the plane. One goal here is to illustrate some dynamical similarities between affine and projective surfaces. Before stating our result, recall that the Jacobian determinant $\text{Jac}(f)$ of such an automorphism f is constant; when $\text{Jac}(f) = \pm 1$, the Lebesgue measure $\text{Leb}_{\mathbb{R}^2}$ is invariant and f is said to be conservative.

Theorem 2.2. *Let f be a polynomial automorphism of \mathbf{R}^2 . Let μ be an ergodic f -invariant measure with positive entropy supported on \mathbf{R}^2 . If $g \in \text{Aut}(\mathbf{R}^2)$ satisfies $g_*\mu = \mu$, then:*

- (a) *either f and g are conservative and μ is the restriction of $\text{Leb}_{\mathbf{R}^2}$ to a Borel set of positive measure invariant under f and g ;*
- (b) *or the group generated by f and g is solvable and virtually cyclic; in particular, there exists $(n, m) \in \mathbf{Z}^2 \setminus \{(0, 0)\}$ such that $f^n = g^m$.*

Remark 2.3. With the techniques developed in [19], the same result should apply to the dynamics of $\text{Out}(\mathbb{F}_2)$ acting on the real part of the character surfaces of the once punctured torus.

Sketch of Proof. Since the proof is a direct adaptation of [26, Thm 11.1], we briefly explain the argument and leave the details to the reader.

Set $\Gamma = \langle f, g \rangle$. Since its entropy is positive, f is of Hénon type in the sense of [67]: it is conjugate to a composition of generalized Hénon maps, as in [53, Thm. 2.6]. Thus, the support of μ is a compact subset of \mathbf{R}^2 , because the basins of attraction of the line at infinity for f and f^{-1} , respectively, cover the complement of a compact subset of \mathbf{C}^2 . In addition μ must be Zariski diffuse, because its entropy is positive.

Let w be an element of Γ and set $h = wfw^{-1}$; then h is also of Hénon type. We follow the proof of [26, Thm 11.1], which leads us to three cases.

Case 3 is treated exactly in the same way and implies that μ is absolutely continuous. This implies that f is conservative and, μ being ergodic for f , it must be the restriction of $\text{Leb}_{\mathbf{R}^2}$ to some Γ -invariant subset.

In Cases 1 and 2, arguing as in [26, Thm 11.1] and keeping the same notation, we arrive at $W^s(h, x) = W^s(f, x)$ or $W^u(f, x)$ on a set of positive measure. For a Hénon type automorphism of \mathbf{C}^2 , the closure of any stable manifold is equal to the forward Julia set J^+ , and J^+ carries a unique positive closed current T^+ of mass 1 relative to the Fubini Study form in $\mathbb{P}^2(\mathbf{C})$ (see [85]). So we infer that $T_h^+ = T_f^+$ or $T_h^+ = T_f^-$; as a consequence, the Green functions of f and h satisfy $G_h^+ = G_f^+$ or $G_h^+ = G_f^-$, respectively.

The group $\text{Aut}(\mathbf{C}^2)$ is the amalgamated product the affine and the elementary subgroups along their intersection. Let T be the associated Bass-Serre tree. Each $u \in \text{Aut}(\mathbf{C}^2)$ gives rise to an isometry u_* of T and, u is of Hénon type if and only if u_* is loxodromic (its axis $\text{Geo}(u_*)$ is the unique u_* -invariant geodesic, and u_* acts as a translation along it). From [67, Thm. 5.4], $G_h^+ = G_f^+$ implies $\text{Geo}(h_*) = \text{Geo}(f_*)$; changing f into f^{-1} , $G_h^+ = G_f^-$ gives $\text{Geo}(h_*) = \text{Geo}(f_*^{-1}) = \text{Geo}(f_*)$ because $\text{Geo}(f_*^{-1}) = \text{Geo}(f_*)$. Since $w_* \text{Geo}(f_*) = \text{Geo}(h_*)$, we see that Γ preserves $\text{Geo}(f_*)$; so, all $u \in \Gamma$ of Hénon type satisfy $\text{Geo}(u_*) = \text{Geo}(f_*)$. From [67, Prop. 4.10], we conclude that Γ is solvable and virtually cyclic. \square

Part 1. Classification and first examples

3. SURFACES ADMITTING NON-ELEMENTARY GROUPS OF AUTOMORPHISMS

3.1. More Kähler vocabulary. Let X be a compact Kähler surface. Recall from § 1.1 that a subgroup $\Gamma \subset \text{Aut}(X)$ is non-elementary when its image $\Gamma^* \subset \text{GL}(H^*(X, \mathbf{Z}))$ contains a non-Abelian free subgroup. By Hodge theory, Γ acts on $H^{1,1}(X, \mathbf{R})$ by preserving the intersection form, which is of Minkowski type; thus Γ acts by isometries on the associated hyperbolic space \mathbb{H}_X , which is the component of the hyperboloid

$$(3.1) \quad \{u \in H^{1,1}(X, \mathbf{R}); \langle u|u \rangle = 1\}$$

containing the class of a Kähler form. Then Γ is non-elementary in the sense of § 1.2 if, and only if the induced subgroup of $\text{Isom}(\mathbb{H}_X)$ is non-elementary in the sense of hyperbolic geometry (this is the original definition from [26, §2]). The classification of automorphisms in the elliptic, parabolic, and loxodromic types, as described in Section 1.3, corresponds exactly to the three possible types of isometries of hyperbolic spaces. By theorems of Gromov and Yomdin, $f \in \text{Aut}(X)$ has positive entropy if and only if f is loxodromic: indeed, the translation length of f^* on \mathbb{H}_X is equal to the topological entropy of $f: X \rightarrow X$ and to the logarithm of the spectral radius $\lambda(f)$ of f^* on $H^{1,1}(X; \mathbf{R})$ (see [21]). By a ping pong argument (see Lemma 3.1 below), we can add the following characterization to the equivalent conditions (a)-(c) of § 1.2.

(d) Γ contains a non-Abelian free group all of whose elements (distinct from id_X) have positive entropy (i.e. are loxodromic).

3.2. Non-elementary groups of automorphisms on general surfaces. Let M be a compact manifold. Exactly as in the complex case, let us say that a group Γ of homeomorphisms of M is non-elementary if its image Γ^* in $\text{GL}(H^*(M; \mathbf{Z}))$ contains a non-Abelian free subgroup.

Lemma 3.1. *Let M be a compact manifold, and Γ be a non-elementary subgroup of $\text{Diff}^\infty(M)$. Then Γ contains a non-Abelian free group Γ_0 such that the topological entropy of every $f \in \Gamma_0 \setminus \{\text{id}\}$ is positive.*

Proof. We split the proof in two steps. The first one concerns groups of matrices, the second one is where topological entropy enters into place.

Step 1.- *The image Γ^* of Γ in $\text{GL}(H^*(M; \mathbf{Z}))$ contains a free subgroup Γ_1^* , such that every element of $\Gamma_1^* \setminus \{\text{id}\}$ has spectral radius larger than 1.*

The proof uses basic ideas involved in Tits's alternative, here in the simple case of subgroups of $\text{GL}_n(\mathbf{Z})$. Let N be the rank of $H_{t.f.}^*(M; \mathbf{Z})$, where *t.f.* stands for "torsion free". Fix a basis of this free \mathbf{Z} -module. Then Γ^* determines a subgroup of $\text{GL}_N(\mathbf{Z})$. Our assumption implies that the derived subgroup of Γ^* contains a non-Abelian free group Γ_0^* of rank 2.

If all (complex) eigenvalues of all elements of Γ_0^* have modulus ≤ 1 , then by Kronecker's lemma all of them are roots of unity. This implies that Γ_0^* contains a finite index nilpotent subgroup (see Proposition 2.2 and Corollary 2.4 of [6]), contradicting the existence of a non-Abelian free subgroup. Thus, there is an element f^* in Γ_0^* with a complex eigenvalue of modulus $\alpha > 1$. Let m be the number of eigenvalues of f^* of modulus α , counted with multiplicities.

Consider the linear representation of Γ_0^* on $\bigwedge^m H^*(M; \mathbf{C})$; the action of f^* on this space has a unique dominant eigenvalue, of modulus α^m ; the corresponding eigenline determines an attracting fixed point for f^* in the projective space $\mathbb{P}(\bigwedge^m H^*(M; \mathbf{C}))$; the action of f^* on the topological space $\mathbb{P}(\bigwedge^m H^*(M; \mathbf{C}))$ is proximal. Let

$$(3.2) \quad \{0\} = W_0 \subset W_1 \subset \cdots \subset W_k \subset W_{k+1} = \bigwedge^m H^*(M; \mathbf{C})$$

be a Jordan-Hölder sequence for the representation of Γ^* : the subspaces W_i are invariant and the induced representation of Γ^* on W_{i+1}/W_i is irreducible for all $0 \leq i \leq k$. Let V be the quotient space W_{i+1}/W_i in which the eigenvalue of f^* of modulus α^m appears. Since Γ_0^* is contained in the derived subgroup of Γ , the linear transformation of V induced by f^* has determinant 1; thus, $\dim(V) \geq 2$. Now, we can apply Lemma 3.9 of [6] to (a finite index, Zariski connected subgroup of) $\Gamma_0^*|_V$: changing f if necessary, both $f^*|_V$ and $(f^{-1})^*|_V$ are proximal, and there is an element g^* in Γ^* that maps the attracting fixed points a_f^+ and $a_f^- \in \mathbb{P}(V)$ of $f^*|_V$ and $(f^*|_V)^{-1}$ to two distinct points (i.e. $\{a_f^+, a_f^-\} \cap \{g^*(a_f^+), g^*(a_f^-)\} = \emptyset$). Then, by the ping-pong lemma, large powers of f^* and $g^* \circ f^* \circ (g^*)^{-1}$ generate a non-Abelian free group $\Gamma_1^* \subset \Gamma^*$ such that each element $h^* \in \Gamma_1^* \setminus \{\text{id}\}$ has an attracting fixed point in $\mathbb{P}(V)$. This implies that every element of $\Gamma_1^* \setminus \{\text{id}\}$ has an eigenvalue of modulus > 1 in $H^*(M; \mathbf{C})$.

Step 2.- Since Γ_1^* is free, there is a free subgroup $\Gamma_1 \subset \Gamma$ such that the homomorphism $\Gamma_1 \rightarrow \Gamma_1^*$ is an isomorphism. By Yomdin's theorem [86], all elements of $\Gamma_1 \setminus \{\text{id}\}$ have positive entropy, and we are done. \square

Theorem 3.2. *Let M be a compact complex surface such that there exists a non-elementary subgroup $\Gamma \subset \text{Aut}(M)$. Then M is Kähler.*

Proof. By Lemma 3.1, M admits an automorphism of positive topological entropy. It was shown in [17] that this property implies that M is Kähler. \square

3.3. Projectivity.

Theorem 3.3. *Let X be a compact complex surface and Γ be a non-elementary subgroup of $\text{Aut}(X)$. Then X is projective, and is birationally equivalent to a rational surface, an Abelian surface, a K3 surface, or an Enriques surface.*

The next sections will provide examples of non-elementary groups of automorphisms for each of these four classes of surfaces.

It follows from this classification that when X is not rational, there is a canonical volume form preserved by Γ ; moreover, such a form induces an invariant volume form on $X(\mathbf{R})$ when the action is by real automorphisms (see [25, Rmk 2.3]). This constraint has deep consequences on the dynamics of Γ .

Let us prove Theorem 3.3. If Γ is a non-elementary group of automorphisms on X , Theorem 3.2 asserts that X must be Kähler. Then the last assertion of the theorem readily follows from the classification given in [21, Thm 10.1]. What remains to show is that a compact Kähler surface admitting a non-elementary automorphism group is projective. The proof follows closely the arguments given in [21] and [81].

Lemma 3.4 (see also [81], Thm. 2.2). *Let f be a loxodromic automorphism of a compact Kähler surface X . The following properties are equivalent:*

- (1) *on $H^{2,0}(X; \mathbf{C})$, f^* acts by multiplication by a root of unity;*
- (2) *X is projective.*

Remark 3.5. If X supports a loxodromic automorphism, then $\dim H^{2,0}(X, \mathbf{C}) \leq 1$. When this dimension equals 1, that is when X is a torus or K3 surface, $H^{2,0}(X; \mathbf{C})$ is generated by a holomorphic 2-form Ω_X that does not vanish and satisfies $\int_X \Omega_X \wedge \overline{\Omega_X} = 1$. It is unique up to multiplication by a complex number of modulus 1. So for every f , we can write $f^*\Omega_X = J(f)\Omega_X$, and the Jacobian determinant

$$(3.3) \quad f \in \text{Aut}(X) \mapsto J(f) \in \mathbb{U}_1$$

defines a unitary character on the group $\text{Aut}(X)$. It follows that the first condition of Lemma 3.4 can be reformulated as:

- (1') *either $H^{2,0}(X; \mathbf{C}) = 0$ or $J(f)$ is a root of unity.*

Moreover, by Kodaira's embedding theorem, X is projective when $H^{2,0}(X; \mathbf{C}) = 0$.

Proof of Lemma 3.4. The characteristic polynomial χ_f of $f^*: H_{t,f}^2(X; \mathbf{Z}) \rightarrow H_{t,f}^2(X; \mathbf{Z})$ is a monic polynomial with integer coefficients. Since f is loxodromic, f^* has a real eigenvalue $\lambda(f) > 1$. Besides $\lambda(f)$ and $\lambda(f)^{-1}$, all other roots of χ_f have modulus 1, so $\lambda(f)$ is a reciprocal quadratic integer or a Salem number (see § 2.4.3 of [21] for more details). Thus, the decomposition of χ_f into irreducible factors can be written as

$$(3.4) \quad \chi_f(t) = S_f(t) \times R_f(t) = S_f(t) \times \prod_{i=1}^m C_{f,i}(t)$$

where S_f is a Salem polynomial or a reciprocal quadratic polynomial, and the $C_{f,i}$ are cyclotomic polynomials. In particular if ξ is an eigenvalue of f^* and a root of unity, we see that ξ is a root of $R_f(t)$ but not of $S_f(t)$.

The subspace $H^{2,0}(\mathbf{C}) \subset H^2(X; \mathbf{C})$ is $\text{Aut}(X)$ -invariant. By Hodge index theorem, the hermitian form $\Omega \in H^{2,0}(\mathbf{C}) \mapsto \int_X \Omega \wedge \overline{\Omega}$ is positive definite and $\text{Aut}(X)$ -invariant. Thus, all eigenvalues of f^* on $H^{2,0}(\mathbf{C})$ have modulus 1. Note, furthermore, that if an eigenvalue of $f^*|_{H^{2,0}(X; \mathbf{C})}$ is not a root of unity, then it is a root of S_f .

Assume that all eigenvalues of f^* on $H^{2,0}(X; \mathbf{C})$ are roots of unity. Then $\text{Ker}(S_f(f^*)) \subset H^2(X; \mathbf{R})$ is a f^* -invariant subspace of $H^{1,1}(X; \mathbf{R})$. This subspace is defined over \mathbf{Q} and is of Minkowski type; in particular, it contains integral classes of positive self-intersection. Thus, by the Kodaira embedding theorem, X is projective.

Conversely, assume that X is projective. The Néron-Severi group $\text{NS}(X; \mathbf{Q}) \subset H^{1,1}(X; \mathbf{R})$ is f^* -invariant and contains vectors of positive self-intersection; so, by the description of the linear action of Γ given in [26, Prop. 2.8], $\text{NS}(X; \mathbf{R})$ contains all isotropic lines associated to loxodromic automorphisms. Now, any f^* invariant subspace defined over \mathbf{Q} and containing the eigenspace associated to $\lambda(f)$ contains $\text{Ker}(S_f(f^*))$, hence $\text{Ker}(S_f(f^*)) \subset \text{NS}(X; \mathbf{Q})$. In particular, $\text{Ker}(S_f(f^*))$ does not intersect $H^{2,0}(X; \mathbf{C})$, which is invariant, and we conclude that all eigenvalues of f^* on $H^{2,0}(X; \mathbf{C})$ are roots of unity. \square

Lemma 3.6. *Let X be a compact Kähler surface. If X is not projective, then $\text{Aut}(X)^*$ is virtually Abelian and if it contains a loxodromic element it is virtually cyclic.*

Proof. Let $\Gamma \subset \text{Aut}(X)$ be a free subgroup (possibly reduced to a cyclic group or to $\{\text{id}\}$) such that every $f \in \Gamma \setminus \{\text{id}\}$ is loxodromic. Since X is not projective, Lemma 3.4 and Remark 3.5 show that $h^{2,0}(X) = 1$ and that Γ acts faithfully by scalar multiplication on $H^{2,0}(X; \mathbf{C})$. Indeed otherwise the kernel of $\Gamma \rightarrow \text{GL}(H^{2,0}(X; \mathbf{C}))$ would contain loxodromic elements and X would be projective. From this, we deduce that Γ has rank at most 1, hence $\text{Aut}(X)$ is elementary. To conclude, we refer to Theorem 3.2 of [21], which says that there are two possibilities: either $\text{Aut}(X)^*$ contains a finite index, cyclic subgroup generated by a loxodromic automorphism; or $\text{Aut}(X)^*$ contains a finite index, Abelian subgroup, all of whose non-trivial elements are parabolic (permuting the fibers of a genus 1 fibration on X). \square

We can now conclude the proof of Theorem 3.3: indeed we already know that X is Kähler by Theorem 3.2, and then it is projective by Lemma 3.6.

3.4. Remarks on the non-Kähler case. Consider the Hopf surface X_α obtained by taking the quotient of $\mathbf{C}^2 \setminus \{(0, 0)\}$ by the group of homotheties $(x, y) \mapsto (\alpha^m x, \alpha^m y)$, for some $\alpha \in \mathbf{C}^\times$ with $|\alpha| < 1$. Taking the quotient of $\mathbf{C}^2 \setminus \{(0, 0)\}$ by all homotheties, we get the projective line $\mathbb{P}^1(\mathbf{C})$: this yields a fibration $\eta: X_\alpha \rightarrow \mathbb{P}^1(\mathbf{C})$ with fibers isomorphic to the genus 1 curve $E_\alpha = \mathbf{C}^\times / \langle \alpha^{\mathbf{Z}} \rangle$.

Recall that a subgroup $\Gamma \subset \text{SL}_2(\mathbf{C})$ is **non-elementary** if it contains a non-Abelian free group and is not relatively compact; equivalently, Γ induces a non-elementary group of isometries of the hyperbolic space \mathbb{H}^3 , whose boundary is $\mathbb{S}^2 \simeq \mathbb{P}^1(\mathbf{C})$. By definition the **limit set** $\text{Lim}(\Gamma)$ is the closure of the set of fixed points of loxodromic elements of Γ .

Let us fix such a non-elementary group Γ . Let ν be a probability measure on $\text{SL}_2(\mathbf{C})$, whose support generates Γ as a closed semigroup. It follows from Furstenberg's theory of random products of matrices that $\text{Lim}(\Gamma)$ is the unique closed, minimal Γ -invariant subset of $\mathbb{P}^1(\mathbf{C})$, and there is a unique ν -stationary measure $\mu_{\mathbb{P}^1(\mathbf{C})}$ on $\mathbb{P}^1(\mathbf{C})$; moreover, the support of $\mu_{\mathbb{P}^1(\mathbf{C})}$ coincides with $\text{Lim}(\Gamma)$.

Now, consider the action of Γ on X_α induced by the natural action of Γ on \mathbf{C}^2 . The fibration $\eta: X_\alpha \rightarrow \mathbb{P}^1(\mathbf{C})$ is Γ -equivariant. Any $f \in \Gamma$ acts by scalar multiplication along the fibers of $\mathbf{C}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{P}^1(\mathbf{C})$. Since the multiplication $z \mapsto \beta z$ induces a translation on the elliptic curve E_α , the action of Γ on X_α is an isometric extension of the action on $\mathbb{P}^1(\mathbf{C})$, so we are in the setting of [59]. From this we obtain: *if Γ is Zariski dense in $\text{SL}_2(\mathbf{C})$, viewed as a real Lie group, then X_α supports a unique minimal invariant subset Λ_X and a unique ν -stationary measure μ_X ; this measure is not Γ -invariant; both Λ_X and μ_X are invariant under the action of \mathbf{C}^\times on X_α by homotheties; and the marginal $\eta_* \mu_X$ of μ_X is equal to $\mu_{\mathbb{P}^1(\mathbf{C})}$ ⁽³⁾.*

³With the notation of [59], the KAN^+ decomposition of $\text{SL}_2(\mathbf{C})$ can be chosen in such a way that $A = \mathbf{R}_+^\times$ is the group of diagonal matrices with real positive coefficients, and its centralizer M in the maximal compact subgroup K is \mathbb{S}^1 , the group of diagonal matrices with eigenvalues of modulus 1. Then, the product MA is just \mathbf{C}^\times , the group of complex diagonal matrices in $\text{SL}_2(\mathbf{C})$. In particular, this group is connected (it corresponds to the group C in [59]). Thus, our assertions follows from the main theorems stated on the second page of [59].

Now, take $\alpha \in \mathbf{R}_+^\times$ and assume that Γ_ν is a non-elementary subgroup of $\mathrm{SL}_2(\mathbf{R})$ (in particular, Γ_ν is not Zariski dense in the real Lie group $\mathrm{SL}_2(\mathbf{C})$). The limit set $\mathrm{Lim}(\Gamma)$ is contained in $\mathbb{P}^1(\mathbf{R})$; thus, the ν -stationary measures on X_α are supported in the preimage of $\mathbb{P}^1(\mathbf{R})$ by the fibration η . There is, in that case, a one parameter family of such measures, parametrized by an angle $\theta \in \mathbf{R}/\mathbf{Z}$. Indeed, the plane $\mathbf{R}^2 \subset \mathbf{C}^2$ is Γ -invariant, and the projection of $\mathbf{R}^2 \setminus \{(0, 0)\}$ in X_α supports a unique ν -stationary measure $\mu_{X(\mathbf{R})}$. Then, all the stationary measures are obtained by “rotating” $\mu_{X(\mathbf{R})}$ by $(x, y) \mapsto \exp(2i\pi\theta) \cdot (x, y)$.

Remark 3.7. Let X be a Hopf or a Inoue surface. According to the description of $\mathrm{Aut}(X)$ by Namba, Matumoto and Nakagawa (see [28, 75]), either $\mathrm{Aut}(X)$ is virtually solvable, or X is a Hopf surface obtained as a quotient of its universal cover $\mathbf{C}^2 \setminus \{(0, 0)\}$ by a group of homotheties

$$(3.5) \quad \Phi_{m,n}(x, y) = (\alpha^m \xi^n x, \alpha^m \xi^n y)$$

where $\alpha \in \mathbf{C}^\times$ has modulus < 1 , ξ is a root of unity (of order q for some $q \geq 1$), and $(m, n) \in \mathbf{Z} \times \mathbf{Z}/q\mathbf{Z}$. Taking a finite cover brings us back to the case $\xi = 1$, which we just described.

Remark 3.8. If X is a Kodaira fibration, then X comes with an $\mathrm{Aut}(X)$ -invariant fibration on an elliptic curve with elliptic fibers. In this case every stationary measure is invariant. This follows for instance from the fact that such actions are distal (see [56, Thm. 3.5]).

These remarks do not exhaust all possible non-Kähler compact complex surfaces. Indeed, there are other examples of surfaces in class VII. Moreover, the classification of VII_0 surfaces is not complete yet (even, as far as we know, if we assume that $\mathrm{Aut}(X)$ is not virtually nilpotent).

Question 3.9. *Let X be a non-Kähler compact complex surface. Suppose there is a probability measure ν on $\mathrm{Aut}(X)$ such that X supports a ν -stationary measure which is not Γ_ν -invariant and has a Zariski dense support. Then must X be a Hopf surface?*

4. FROM ENRIQUES TO RATIONAL SURFACES

In this Section we start with by describing two families of surfaces with non-elementary automorphism groups, obtained by taking quotients of K3 surfaces by an involution. For Enriques surfaces, the involution is fixed point free; this is not the case for Coble surfaces. We briefly mention the examples of Blanc, whose detailed study is the purpose of Part 3, and conclude with an example of Lesieutre of a non-elementary group on a rational surface without invariant curve. Coble, Blanc, and Lesieutre surfaces are all rational, but their dynamical features happen to be quite different: this stems from the existence or non-existence of a global invariant measure.

4.1. Enriques (see [32, 45]). Recall that Enriques surfaces are quotients of K3 surfaces by fixed point free involutions. According to Horikawa and Kondō ([60, 61, 66]), the moduli space \mathcal{M}_E of complex Enriques surfaces is a rational quasi-projective variety of dimension 10. An Enriques surface X is nodal if it contains a smooth rational curve; such rational curves have self-intersection -2 , and are called nodal-curves or (-2) -curves. Nodal Enriques surfaces form a set of codimension 1 in \mathcal{M}_E .

For any Enriques surface X , the lattice $(\mathrm{NS}(X; \mathbf{Z}), q_X)$ is isomorphic to the orthogonal direct sum $E_{10} = U \oplus E_8(-1)$ ⁽⁴⁾. Let $W_X \subset \mathrm{O}(\mathrm{NS}(X; \mathbf{Z}))$ be the subgroup generated by reflexions about classes u such that $u^2 = -2$, and $W_X(2)$ be the subgroup of W_X acting trivially on $\mathrm{NS}(X; \mathbf{Z})$ modulo 2. Both W_X and $W_X(2)$ have finite index in $\mathrm{O}(\mathrm{NS}(X; \mathbf{Z}))$. The following result is due independently to Nikulin and Barth and Peters (see [45] for details and references).

Theorem 4.1. *If X is an Enriques surface which is not nodal,*

- (1) *the homomorphism $\mathrm{Aut}(X) \ni f \mapsto f^* \in \mathrm{GL}(H^2(X, \mathbf{Z}))$ is injective,*
- (2) *its image satisfies $W_X(2) \subset \mathrm{Aut}(X)^* \subset W_X$.*

In particular, $\mathrm{Aut}(X)^$ is a finite index subgroup in $\mathrm{O}(\mathrm{NS}(X; \mathbf{Z}))$, thus $\mathrm{Aut}(X)^*$ is a lattice in the rank 1 Lie group $\mathrm{O}(\mathrm{NS}(X; \mathbf{R})) \simeq \mathrm{O}_{1,9}(\mathbf{R})$ and it acts irreducibly on $\mathrm{NS}(X; \mathbf{R})$.*

This implies that $\mathrm{Aut}(X)$ is non-elementary, contains parabolic automorphisms, and does not preserve any curve (because $\mathrm{Aut}(X)$ does not have a fixed point in $H^2(X; \mathbf{R})$). From these properties, and from [25], we obtain a classification of $\mathrm{Aut}(X)$ -invariant probability measures on unnodal complex Enriques surfaces; from [24], we know that $\mathrm{Aut}(X)$ has only finitely many finite orbits. On the other hand, we expect that generically all orbits are infinite:

Question 4.2. *Is it true that if $X \in \mathcal{M}_E$ is a general (resp. very general) Enriques surface, then $\mathrm{Aut}(X)$ does not have any finite orbit?*

If the answer is positive, then one could also apply the main results of [23] to describe the distribution of random orbits on (real, unnodal) Enriques surfaces.

4.2. Coble (see [22]). In this article, a **Coble surface** is, by definition, obtained by blowing up the ten nodes of a general rational sextic curve $C_0 \subset \mathbb{P}^2$. The result is a rational surface X with a large group of automorphisms. To be precise, consider the canonical class $k_X \subset \mathrm{NS}(X; \mathbf{Z})$; since $K_{\mathbb{P}^2}$ is $\mathcal{O}_{\mathbb{P}^2}(-3)$ and since we blow up the nodes of C_0 , one gets

$$(4.1) \quad -2k_X = [C] \quad \text{and} \quad k_X = -3\mathbf{e}_0 + \sum_{i=1}^{10} \mathbf{e}(p_i)$$

where C is the strict transform of C_0 in X , and $(\mathbf{e}_0, \mathbf{e}(p_1), \dots, \mathbf{e}(p_{10}))$ is the basis of $\mathrm{NS}(X)$ given by the classes of the total transform of a line and the exceptional divisors E_i obtained by blowing up the ten double points p_i of C_0 . The orthogonal complement k_X^\perp is a lattice of dimension 10, isomorphic to E_{10} , and we define $W_X(2)$ exactly in the same way as for Enriques surfaces. Then, $\mathrm{Aut}(X)^*$ preserves the decomposition $k_X \oplus k_X^\perp$. As before we say that X is unnodal when it does not contain any smooth rational curve of self-intersection -2 . If X is unnodal, the $\mathrm{Aut}(X)^*$ contains $W_X(2)$, in particular it is non-elementary (see [22], Theorem 3.5).

Let us explain this result in a more explicit way (see [22]). Let C_0 be a general rational sextic as before. Choose one of the double points of C_0 , say p_i , and consider the cubic curve

⁴Here, U is the standard 2-dimensional Minkowski lattice, (\mathbf{Z}^2, x_1x_2) , and E_8 is the root lattice given by the corresponding Dynkin diagram; so $E_8(-1)$ is negative definite, and E_{10} has signature $(1, 9)$ (see [32, Chap. II]). Also, in this paper $\mathrm{NS}(X; \mathbf{Z})$ denotes the torsion free part of the Néron-Severi group, which is sometimes denoted by $\mathrm{Num}(X; \mathbf{Z})$ in the literature on Enriques surfaces.

C_i containing the remaining 9 points. Then, C_0 and $2C_i$ generate a pencil of sextic curves, with base points at the p_j , $j \neq i$. Blowing up these 9 points, we obtain a surface X_i with a genus 1 fibration $\pi_i: X_i \rightarrow \mathbb{P}^1_{\mathbb{C}}$; the sextic C_0 lifts to a nodal fiber of π_i and $2C_i$ gives a smooth multiple fiber. The 9 exceptional divisors E_j , $j \neq i$, determine 9 multisections of π_i , each of degree 2. Each pair (j, k) , $j, k \neq i$, defines an automorphism g_{jk} of X_i , acting by translation along the fibers of π_i : the translation on the fiber $X_{i,b} := \pi_i^{-1}(b)$ is by the divisor of degree 0 defined by

$$(4.2) \quad \tau_{jk} := (E_k - E_j)_{X_{i,b}} \in \text{Pic}^0(X_{i,b}).$$

For a general choice of C_0 , the g_{jk} generate a free Abelian group A_i of rank 8; its elements, except the identity, are parabolic automorphisms of X_i preserving the fibration π_i (hence also the singular point p_i of the strict transform of C_0 in X_i). Thus, blowing-up p_i , the group A_i lifts to a subgroup of $\text{Aut}(X)$. In this way we obtain 10 copies A_i of \mathbf{Z}^8 in $\text{Aut}(X)$, acting as parabolic groups with respect to distinct genus 1 fibrations. In particular, $\text{Aut}(X)$ is non-elementary. Note that the strict transform of C_0 is $\text{Aut}(X)$ -invariant and is contained in a fiber of π_i for each index i .

Also, Coble surfaces are degenerations of Enriques surfaces; thus, Coble surfaces share many features of Enriques surfaces, but there are also interesting differences. For instance, $\text{Aut}(X)$ preserves the class k_X , and this class is non-trivial. Moreover, there is a holomorphic section of $-2K_X$ vanishing exactly along the strict transform $C \subset X$ of the rational sextic curve C_0 ; this means that there is a meromorphic section $\Omega_X = \xi(x, y)(dx \wedge dy)^2$ of $2K_X$ that does not vanish and has a simple pole along C . Thus, the formula

$$(4.3) \quad \text{vol}_X(U) = \int_U |\xi(x, y)| dx \wedge dy \wedge d\bar{x} \wedge d\bar{y} = \int_U |\xi(x, y)| (idx \wedge d\bar{x}) \wedge (idy \wedge d\bar{y})$$

determines a measure $\text{vol}_X = \Omega_X^{1/2} \wedge \overline{\Omega_X^{1/2}}$. The total mass of this measure is finite. Indeed, if locally $C = \{x = 0\}$ then $\xi(x, y) = \eta(x, y)/x$ where η is regular; thus, $|\xi| = |\eta| |x|^{-1}$ is locally integrable because $\frac{1}{r^\alpha}$ is integrable with respect to $rdrd\theta$ when $\alpha < 2$. We may assume that it is a probability after multiplying Ω_X by some adequate constant, and this measure is $\text{Aut}(X)$ -invariant, because vol_X is uniquely determined by the complex structure (see also Remark 4.3 below). In particular the ergodic theory of Coble examples can be studied with the techniques of [26].

4.3. Blanc. Another family of examples was introduced by Blanc in [12]. Here we describe them informally, a more detailed presentation will be given in Part 3.

Start with a smooth cubic curve $C \subset \mathbb{P}^2$. If q is a point of C , the Jonquières involution associated to (C, q) is the birational involution σ_q of \mathbb{P}^2 characterized by the following properties: it fixes C pointwise and it preserves the pencil of lines through q . The indeterminacy points of σ_q are q and the four tangency points of C with this pencil, which may be “infinitely near” q . Thus, the indeterminacies of σ_q are resolved by blowing-up points of C , possibly several times. After such a sequence of blow-ups σ_q becomes an automorphism of a rational surface that fixes pointwise the strict transform of C . In particular, if we blow-up further points of this curve, σ_q lifts to an automorphism of the new surface.

Pick a finite number of points $q_i \in C_0$, $i = 1, \dots, k$, and resolve simultaneously the indeterminacies of the Jonquières involutions $\sigma_i := \sigma_{q_i}$ determined by the q_i . The result is a rational surface X , together with a subgroup $\Gamma := \langle s_1, \dots, s_k \rangle$ of $\text{Aut}(X)$. Blanc proves in [12, 13] that:

(1) *there are no relations between these involutions, that is, Γ is a free product*

$$\langle s_1, \dots, s_k \rangle \simeq \bigstar_{i=1}^k \mathbf{Z}/2\mathbf{Z};$$

(2) *the composition of two distinct involutions $s_i \circ s_j$ is parabolic;*

(3) *the composition of three distinct involutions is loxodromic.*

In addition, there is a meromorphic section Ω_X of K_X with a simple pole along the strict transform of C_0 , however, contrary to Coble surfaces, the form $\text{vol}_X := \Omega_X \wedge \overline{\Omega_X}$ is not integrable (its total mass is infinite). This observation will play a crucial role in Part 3.

Remark 4.3. If $\Gamma \subset \text{Aut}(X)$ is generated by involutions and there is a meromorphic form Ω such that $f^*\Omega = \xi(f)\Omega$ for every $f \in \Gamma$, then $\xi(f) = \pm 1$: this is the case for Blanc's examples or general Coble surfaces, since $W_X(2)$ is also generated by involutions (see [45]).

4.4. Lesieutre. In [69], Lesieutre constructs rational surfaces X with the following properties: $\text{Aut}(X)$ contains three involutions τ_i , $i = 1, 2, 3$, such that the group $\Gamma := \langle \tau_1, \tau_2, \tau_3 \rangle \subset \text{Aut}(X)$ and $f := \tau_1 \circ \tau_2 \circ \tau_3$ satisfy

(1) *X and Γ are defined over \mathbf{Q} ;*

(2) *Γ is non-elementary, and isomorphic to $(\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z}) \star (\mathbf{Z}/2\mathbf{Z})$;*

(3) *Γ does not contain any parabolic element;*

(4) *f is loxodromic and does not preserve any curve.*

In particular, $\text{Aut}(X)$ does not preserve any rational section of K_X (moreover, $-K_X$ is not pseudo-effective).

Thus, the situation is quite different from Enriques, Coble and Blanc surfaces, since there is neither parabolic automorphism nor invariant algebraic volume form. Our results are so far not powerful enough to describe the finite orbits or stationary measures for such an example.

Part 2. Pentagon folding and dynamics on K3 surfaces

5. THE SPACE OF PENTAGONS AND THE FOLDING GROUPS

The automorphism groups of Wehler surfaces were discussed at length in our previous papers. Here we describe another family of K3 surfaces with a non-elementary group action, coming from the geometry of pentagons in the euclidean plane.

Remark 5.1. The “folding” terminology is borrowed from [7, 8], which was a source of motivation for these results. There is a vast amount of literature on length-preserving transformations in spaces of polygons, notably motivated by algorithmic questions: see for instance Section 5.3.2 in the monograph [37] where our folding transformations are referred to as “flips”.

5.1. Spaces of plane pentagons. Let $\ell = (\ell_0, \dots, \ell_4) \in \mathbf{R}_{>0}^5$ be a 5-tuple of positive real numbers such that there exists a pentagon with side lengths ℓ_i ; this imposes a condition on the ℓ_i , defined by explicit inequalities, and we say that ℓ is **admissible** if this condition is satisfied. Here a pentagon is just an ordered set of points $(a_i)_{i=0, \dots, 4}$ in the Euclidean plane \mathbf{R}^2 , such that $\text{dist}(a_i, a_{i+1}) = \ell_i$ for $i = 0, \dots, 4$ (with $a_5 = a_0$ by definition, in other words we consider indices modulo 5); pentagons are not assumed to be convex, and two distinct sides $[a_i, a_{i+1}]$ and $[a_j, a_{j+1}]$ may intersect at a point which is not one of the a_i 's.

Let $\text{Pent}(\ell)$ be the set of pentagons with side lengths $(\ell_i)_{i=0}^4$. Note that $\text{Pent}(\ell)$ is naturally a real algebraic variety, defined by polynomial equations of the form $\text{dist}(a_i, a_{i+1})^2 = \ell_i^2$.

For every i , a_i is one of the two intersection points $\{a_i, a'_i\}$ of the circles of respective centers a_{i-1} and a_{i+1} and radii ℓ_{i-1} and ℓ_i . The transformation exchanging these two points a_i and a'_i , while keeping the other vertices fixed, defines an involution of $\text{Pent}(\ell)$, that we denote by s_{i-1} (this choice for the index will be convenient later). Geometrically, it corresponds to folding (or reflecting) the pentagon along the diagonal $(a_{i-1}a_{i+1})$. It commutes with the action of the group $\text{SO}_2(\mathbf{R}) \times \mathbf{R}^2$ of positive isometries of the plane, hence, it induces an involution σ_{i-1} on the quotient space

$$(5.1) \quad \text{Pent}^0(\ell) = \text{Pent}(\ell) / (\text{SO}_2(\mathbf{R}) \times \mathbf{R}^2).$$

Each element of $\text{Pent}^0(\ell)$ admits a unique representative with $a_0 = (0, 0)$ and $a_1 = (\ell_0, 0)$, so as before $\text{Pent}^0(\ell)$ is a real algebraic variety, which is easily seen to be of dimension 2 (see [34, 84]). We will see below that, when it is smooth, it is a real K3 surface. The five involutions σ_i act by algebraic diffeomorphisms on this surface, and for a general choice of lengths, the group generated by these involutions is non-elementary.

Remark 5.2. If we consider quadrilateral instead of pentagons, the corresponding space

$$(5.2) \quad \text{Quad}^0(\ell_0, \ell_1, \ell_2, \ell_3)$$

is a curve of genus 1 and the involutions typically generate an infinite dihedral group. The corresponding dynamical system, both on the space of quadrilaterals and the space of quadrilaterals

modulo isometries, was studied in depth in [49, 7, 8]. With n -gons, $n \geq 4$, one would get Calabi-Yau manifolds of dimension $n - 3$ ⁵.

5.2. Algebraic geometry of $\text{Pent}^0(\ell)$. To analyze the algebraic structure and geometry of $\text{Pent}^0(\ell)$, we view a plane pentagon with side lengths ℓ_0, \dots, ℓ_4 modulo translations as the data of a 5-tuple of vectors $(v_i)_{i=0, \dots, 4}$ in \mathbf{R}^2 (identified with \mathbf{C}) of respective length ℓ_i such that $\sum_i v_i = 0$. Write $v_i = \ell_i t_i$ with $|t_i| = 1$. Then the action of $\text{SO}_2(\mathbf{R})$ can be identified to the diagonal multiplicative action of $\mathbb{U} = \{\alpha \in \mathbf{C} ; |\alpha| = 1\}$ on the t_i :

$$(5.3) \quad \alpha \cdot (t_0, \dots, t_4) = (\alpha t_0, \dots, \alpha t_4).$$

Now, following Darboux [35], we consider the surface X in $\mathbb{P}_{\mathbf{C}}^4$ defined by the equations

$$(5.4) \quad \begin{cases} \ell_0 z_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3 + \ell_4 z_4 = 0 \\ \ell_0/z_0 + \ell_1/z_1 + \ell_2/z_2 + \ell_3/z_3 + \ell_4/z_4 = 0 \end{cases}$$

where $[z_0 : \dots : z_4]$ is some fixed choice of homogeneous coordinates, and the second equation must be multiplied by $z_0 z_1 z_2 z_3 z_4$ to obtain a homogeneous equation of degree 4.

Remark 5.3. This surface is isomorphic to the Hessian of a cubic surface (see [44, §9]). More precisely, consider a cubic surface $S \subset \mathbb{P}_{\mathbf{C}}^3$ whose equation F can be written in Sylvester's pentahedral form, that is, as a sum $F = \sum_{i=0}^4 \lambda_i F_i^3$ for some complex numbers λ_i and linear forms F_i with $\sum_{i=0}^4 F_i = 0$. By definition, its Hessian surface H_F is defined by $\det(\partial_i \partial_j F) = 0$. Then, using the linear forms F_i to embed H_F in $\mathbb{P}_{\mathbf{C}}^4$, we obtain the surface defined by the pair of equations $\sum_{i=0}^4 z_i = 0$ and $\sum_{i=0}^4 \frac{1}{\lambda_i z_i} = 0$. Thus, H_F is our surface X , for $\ell_i^2 = \lambda_i$. We refer to [43, 36, 42, 82] for an introduction to these surfaces and their birational transformations.

For completeness, let us directly prove some of its basic properties.

Lemma 5.4. *Let $\ell = (\ell_0, \dots, \ell_4)$ be an element of $(\mathbf{C}^*)^5$. The surface $X_\ell \subset \mathbb{P}_{\mathbf{C}}^4$ defined by the system (5.4) has 10 singularities at the points q_{ij} determined by the system of equations $\ell_i z_i + \ell_j z_j = 0$, $z_k = z_l = z_m = 0$ with $i < j$ and $\{i, j, k, l, m\} = \{0, 1, 2, 3, 4\}$. In the complement of these ten isolated singularities, X_ℓ is smooth if and only if*

$$(5.5) \quad \sum_{i=0}^4 \varepsilon_i \ell_i \neq 0 \quad \forall \varepsilon_i \in \{\pm 1\}.$$

Note that for positive ℓ_i 's, violation of condition (5.5) means that there exist a degenerate pentagon with lengths ℓ_i .

Notation.— We shall use the notation X instead of X_ℓ when the dependence on ℓ is not crucial.

Proof. We first look for singularities in the complement of the hyperplanes $z_i = 0$, and work in the chart $z_0 = 1$. Then, if we substitute $z_4 = -(\ell_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3)/\ell_4$ in the second equation of (5.4), we obtain an affine equation of X in the chart $z_0 = 1$, namely:

$$(5.6) \quad \frac{\ell_1}{z_1} + \frac{\ell_2}{z_2} + \frac{\ell_3}{z_3} - \frac{\ell_4^2}{\ell_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3} + \ell_0 = 0.$$

⁵This follows from computations which are similar to the ones used to prove Lemma 5.4 and Lemma 5.5 below, the difference being that the singularities of X are not isolated when $n \geq 6$.

The singularities are determined by the system of equations $z_1^2 = z_2^2 = z_3^2 = \ell_4^{-2}(\ell_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3)^2$. So, by symmetry, at a singularity where none of the coordinates vanishes we must have $z_i = \varepsilon_i z$ for some $\varepsilon_i = \pm 1$ and a common factor $z \neq 0$; this is precisely Condition (5.5).

Looking for singularities with one coordinate equal to 0, say $z_1 = 0$ in the chart $z_0 = 1$, we obtain the system of equations

$$(5.7) \quad \begin{cases} 0 = (\ell_0 z_2 z_3 + \ell_3 z_2 + \ell_2 z_3)(\ell_0 + \ell_2 z_2 + \ell_3 z_3) + (\ell_1^2 - \ell_4^2) z_2 z_3 \\ 0 = \ell_1 z_3 (\ell_0 + 2\ell_2 z_2 + \ell_3 z_3) \\ 0 = \ell_1 z_2 (\ell_0 + \ell_2 z_2 + 2\ell_3 z_3) \end{cases}$$

together with $\ell_0 + \ell_2 z_2 + \ell_3 z_3 + \ell_4 z_4 = 0$ and $\ell_1 z_2 z_3 z_4 = 0$ (in particular, z_2, z_3 or z_4 must vanish). The solutions of this system are given by $z_1 = z_2 = z_3 = 0$, which gives the point $q_{04} = [\ell_4 : 0 : 0 : 0 : -\ell_0]$, or $z_1 = z_2 = 0$ and $\ell_0 + \ell_3 z_3 = 0$, which corresponds to $q_{03} = [\ell_3 : 0 : 0 : -\ell_0 : 0]$, or $z_1 = z_3 = 0$ which gives q_{02} , or $z_1 = z_4 = 0$ but then either $z_2 = 0$ or $z_3 = 0$ and we end up again with q_{02} and q_{03} . The result follows by symmetry. \square

Lemma 5.5. *If $\ell \in (\mathbf{C}^*)^5$ satisfies Condition (5.5), then the ten singularities are simple nodes (Morse singularities) and the surface X is a (singular) K3 surface: a minimal resolution \hat{X} of X is a K3 surface, which is obtained by blowing-up its ten nodes, thereby creating ten rational (-2) -curves.*

Proof. Working in the chart $z_0 = 1$ and replacing z_4 by $-(\ell_0 + \ell_1 z_1 + \ell_2 z_2 + \ell_3 z_3)/\ell_4$, the quadratic term of the equation of X at the singularity $(z_1, z_2, z_3) = (0, 0, 0)$ is $(-\ell_0/\ell_4)Q$, where

$$(5.8) \quad Q(z_1, z_2, z_3) = \ell_1 z_2 z_3 + \ell_2 z_1 z_3 + \ell_3 z_1 z_2$$

is a non-degenerate quadratic form (its determinant is $2\ell_1 \ell_2 \ell_3 \neq 0$). So locally X is holomorphically equivalent to the quadratic cone $\{Q = 0\}$, hence to a quotient singularity $(\mathbf{C}^2, 0)/\eta$ with $\eta(x, y) = (-x, -y)$. The minimal resolution of such a singularity is obtained by a simple blow-up of the ambient space, the exceptional divisor being a (-2) -curve in the smooth surface \hat{X} . The adjunction formula shows that there is a holomorphic 2-form Ω_X on the regular part of X ; locally, Ω_X lifts to an η -invariant form Ω'_X on $\mathbf{C}^2 \setminus \{0\}$, which by the Hartogs theorem extends across the origin to a non-vanishing 2-form. To recover \hat{X} , one can first blow-up \mathbf{C}^2 at the origin and then take the quotient by (the lift of) η : a simple calculation shows that Ω'_X determines a non-vanishing 2-form on \hat{X} . After such a surgery is done at the ten nodes, \hat{X} is a smooth surface with a non-vanishing section of $K_{\hat{X}}$; since it contains at least ten rational curves, it cannot be an Abelian surface, so it must be a K3 surface. \square

Remark 5.6. Let L_{ij} be the line defined by the equations $z_i = 0, z_j = 0, \ell_0 z_0 + \dots + \ell_4 z_4 = 0$; each of these ten lines is contained in X , each of them contains 3 singularities of X (namely q_{kl}, q_{lm}, q_{km} with obvious notations), and each singularity is contained in three of these lines. If one projects them on a plane, the ten lines L_{ij} form a Desargues configuration (see [42, 43]).

5.3. The real part. All this works for any choice of complex numbers $\ell_i \neq 0$. When the ℓ_i are real, X is endowed with two real structures. First, one can consider the complex conjugation

$$(5.9) \quad c: [z_i] \mapsto [\bar{z}_i]$$

on $\mathbb{P}^4(\mathbb{C})$ and restrict it to X : this gives a first antiholomorphic involution c_X . Another one is

$$(5.10) \quad s_X: [z_i] \mapsto [1/\bar{z}_i].$$

To be more precise, consider first, the quartic birational involution $J \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^4)$ defined by $J([z_i]) = [1/z_i]$; J preserves X , it determines a birational transformation $J_X \in \text{Bir}(X)$, and on \hat{X} it becomes an automorphism because every birational transformation of a K3 surface is regular. Moreover, J commutes to c . Thus, $s_X = J_X \circ c_X$ determines a second antiholomorphic involution $s_{\hat{X}}$ of \hat{X} . In what follows, we denote by (X, s_X) this real structure (even if it would be better to study it on \hat{X}); its real part is the fixed point set of s_X , i.e. the set of points in $X(\mathbb{C})$ with coordinates of modulus 1: the real part does not contain any of the singularities of X , this is why we prefer to stay in X rather than lift everything to \hat{X} . In conclusion, if $(\ell_i) \in (\mathbf{R}_+^*)^5$, with the real structure defined by s_X , the real part of X coincides with $\text{Pent}^0(\ell_0, \dots, \ell_4)$.

Remark 5.7. When $\ell_i > 0$ for every i , a complete description of the possible homeomorphism types for the real locus (in the smooth and singular cases) is given in [34]: *in the smooth case, it is an orientable surface of genus $g = 0, \dots, 4$ or the disjoint union of two tori; if one includes singular surfaces, one gets a total of 19 topological types*. In particular if $\text{Pent}^0(\ell)$ is disconnected, it is the disjoint union of two tori. The space of possible side lengths can be tessellated in cells corresponding to smooth surfaces $\text{Pent}^0(\ell)$, with walls corresponding to singular surfaces. Cells are encoded by a 2×2 ‘‘code-matrix’’ in [34, Table 4]. With this viewpoint, the disconnected surfaces correspond to exactly three cells (see Figure 1 below).

Remark 5.8. The involution J preserves X and the two real structures (X, c_X) and (X, s_X) . It lifts to a fixed point free involution \hat{J}_X on \hat{X} , and \hat{X}/\hat{J}_X is an Enriques surface. On pentagons, J corresponds to the symmetry $(x, y) \in \mathbf{R}^2 \mapsto (x, -y)$ that reverses orientation. Thus we see that the space of pentagons modulo affine isometries is an Enriques surface. When X acquires an eleventh singularity which is fixed by J_X , then \hat{X}/\hat{J}_X becomes a Coble surface: see [42, §5] for nice explicit examples. This happens for instance when all lengths are 1, except one which is equal to 2 (this corresponds to $t = 1/4$ in [42, §5.2]).

5.4. Involution and the folding groups. Let us express the folding transformations in coordinates. Given $i \neq j$ in $\{0, \dots, 4\}$ (consecutive or not) we define an involution $(t_i, t_j) \mapsto (t'_i, t'_j)$ preserving the vector $\ell_i t_i + \ell_j t_j$ by taking the symmetric of t_i and t_j with respect to the line directed by $\ell_i t_i + \ell_j t_j$. In coordinates, $t'_k = u/t_k$ for some u of modulus 1, and equating $\ell_i t_i + \ell_j t_j = \ell_i t'_i + \ell_j t'_j$ one obtains

$$(5.11) \quad (t'_i, t'_j) = \left(\frac{u}{t_i}, \frac{u}{t_j} \right), \text{ with } u = \frac{\ell_i t_i + \ell_j t_j}{\ell_i t_i^{-1} + \ell_j t_j^{-1}}.$$

Observe that these computations also make sense when the ℓ_i are complex numbers, or when we replace the t_i by the complex numbers z_i . This defines a birational involution $\sigma_{ij}: X \dashrightarrow X$,

$$(5.12) \quad \sigma_{ij}[z_0 : \dots : z_4] = [z'_0 : \dots : z'_4]$$

with $z'_k = z_k$ if $k \neq i, j$, $z'_i = v z_j$, and $z'_j = v z_i$ with $v = (\ell_i z_i + \ell_j z_j)/(\ell_i z_j + \ell_j z_i)$. Again, since every birational self-map of a K3 surface is an automorphism, these involutions σ_{ij} are elements of $\text{Aut}(\hat{X})$ that commute with the antiholomorphic involution $s_{\hat{X}}$; hence, they

generate a subgroup of $\text{Aut}(\hat{X}; s_{\hat{X}})$. Thus we have constructed a family of projective surfaces \hat{X} , depending on a parameter $\ell \in \mathbb{P}^4(\mathbf{C})$, endowed with a group of automorphisms generated by involutions.

For consecutive sides, i.e when $j = i + 1$ modulo 5 (in the next few lines, all indices are considered modulo 5) $\sigma_{i,i+1}$ corresponds to the folding transformation described in § 5.1 and denoted by σ_i there. We define the **folding group** Γ (resp. the **extended folding group** Γ^{ext}) to be the group generated by the 5 folding involutions $\sigma_{i,i+1}$ (resp. the group generated by all 10 involutions σ_{ij}). Likewise, for given $m \in \{0, \dots, 4\}$ we introduce the subgroup Γ_m (resp. Γ_m^{ext}) stabilizing the side m , that is the group generated by the 3 folding involutions $\sigma_{i,i+1}$ such that $m \notin \{i, i+1\}$ (resp. by the 6 involutions σ_{ij} with $m \notin \{i, j\}$).

Remark 5.9. Pick a singular point q_{ij} , and project X from that point onto a plane, say the plane $\{z_i = 0\}$ in the hyperplane $P = \{\ell_0 z_0 + \dots + \ell_4 z_4 = 0\}$. One gets a 2-to-1 cover $X \rightarrow \mathbb{P}_{\mathbf{C}}^2$, ramified along a sextic curve (this curve is the union of two cubics, see [82]). The involution σ_{ij} permutes the points in the fibers of this 2 to 1 cover: if x is a point of X , the line joining q_{ij} and x intersects X in the third point $\sigma_{ij}(x)$. The singularity q_{ij} is an indeterminacy point, mapped by σ_{ij} to the opposite line L_{ij} .

Proposition 5.10. *For a general parameter $\ell \in (\mathbf{C}^*)^5$:*

- (1) X is a K3 surface with ten nodes, which admits two real structures c_X and s_X when $\ell \in \mathbb{P}^4(\mathbf{R})$;
- (2) if i, j, k are three distinct indices (modulo 5), then $\sigma_{ij} \circ \sigma_{jk}$ is a parabolic transformation on \hat{X} ; its invariant fibration is induced by $\pi_{lm}: [z_0 : \dots : z_4] \mapsto [z_l : z_m]$ where l and m are the complementary indices (i.e. $\{i, j, k, l, m\} = \{0, 1, 2, 3, 4\}$);
- (3) if i, j, k , and l are four distinct indices (modulo 5), then σ_{ij} commutes to σ_{kl} .
- (4) the folding group Γ (resp. the extended folding group Γ^{ext}) is a non-elementary subgroup of $\text{Aut}(\hat{X}; s_{\hat{X}})$ that does not preserve any algebraic curve;
- (5) likewise, the subgroup Γ_m stabilizing the side m is non-elementary, and its invariant curves in \hat{X} are contained in the total transform of the lines L_{ml} for $l \neq m$ (see Remark 5.6).

Remark 5.11. In [42], Dolgachev computes the action of σ_{ij} on $\text{NS}(\hat{X})$. This contains a proof of this proposition. He also describes, up to finite index, the Coxeter group generated by the σ_{ij} . The automorphism groups of \hat{X} and of the Enriques surface \hat{X}/\hat{J}_X are described in [43] and [83].

The next example shows that the folding groups can be elementary for certain parameters.

Example 5.12. Say that a pentagon is *equilateral* if $\ell_0 = \ell_1 = \ell_2 = \ell_3 = \ell_4$. Let $X_{\text{eq}}(\mathbf{R})$ be the surface of all equilateral pentagons, modulo rotations, translations, and dilatations. It is connected and of genus 4. On X_{eq} , the group generated by the involutions is finite and isomorphic to \mathfrak{S}_5 , because $\sigma_{i,j}(t_i, t_j) = (t_j, t_i)$ (see Equation 5.11). So, this highly symmetric case is also highly degenerate.

Proof of Proposition 5.10. We already established Assertion (1) in the previous lemmas. For Assertion (2), denote by l, m the indices for which $\{i, j, k, l, m\} = \{0, \dots, 4\}$, and consider the

linear projection $\pi_{lm}: \mathbb{P}^4(\mathbf{C}) \dashrightarrow \mathbb{P}^1(\mathbf{C})$ defined by $[z_0 : \dots : z_4] \mapsto [z_l : z_m]$. The fibers of π_{lm} are the hyperplanes containing the plane $\{z_l = z_m = 0\}$, which intersects X on the line L_{lm} . This line is a common component of the pencil of curves cut out by the fibers of π_{lm} on X , and the mobile part of this pencil determines a fibration $\pi_{lm}|_X: X \rightarrow \mathbb{P}^1$ whose fibers are the plane cubics

$$(5.13) \quad (\ell_l z_l + \ell_m z_m)(\ell_m z_l + \ell_l z_m) z_i z_j z_k = z_l z_m (\ell_i z_j z_k + \ell_j z_i z_k + \ell_k z_i z_j)(\ell_i z_i + \ell_j z_j + \ell_k z_k),$$

with $[z_l : z_m]$ fixed. The general member of this fibration is a smooth cubic, hence a curve of genus 1.

Then σ_{ij} and σ_{jk} preserve $\pi_{lm}|_X$, and along the general fiber of $\pi_{lm}|_X$ each of them is described by Remark 5.9; for instance, $\sigma_{ij}(x)$ is the third point of intersection of the cubic with the line (q_{ij}, x) . Thus, writing such a cubic as $\mathbf{C}/\Lambda_{[z_l:z_m]}$, σ_{ij} acts as $z \mapsto -z + b_{ij}$, for some $b_{ij} \in \mathbf{C}/\Lambda_{[z_l:z_m]}$ that depends on $[z_l : z_m]$ and the parameter ℓ ; it has four fixed points on the cubic curve, which are the points of intersection of the cubic (5.13) with the hyperplanes $z_i = z_j$ and $z_i = -z_j$; equivalently, the line (q_{ij}, x) is tangent to the cubic at these four points.

By Lemma 5.14 below, either $\sigma_{ij} \circ \sigma_{jk}$ is of order ≤ 66 (in fact of order ≤ 12 because it preserves $\pi_{lm}|_X$ fiber-wise), or it is parabolic. In other words, the locus in the parameter space where $\sigma_{ij} \circ \sigma_{jk}$ is not parabolic is defined by the equation $(\sigma_{ij} \circ \sigma_{jk})^{12} = \text{id}$. Since there do exist pentagons for which $\sigma_{ij} \circ \sigma_{jk}$ is of infinite order (indeed, this reduces to the corresponding fact for quadrilaterals, see Example 5.13 below), we conclude that $\sigma_{ij} \circ \sigma_{jk}$ is parabolic for general ℓ .

Assertion (3) follows directly from the fact that σ_{ij} changes the coordinates z_i and z_j but keeps the other three fixed.

To prove Assertion (4), we see that for a general parameter ℓ , Γ contains two such parabolic automorphisms associated to distinct fibrations π_{lm} and $\pi_{l'm'}$ so it is non-elementary (this follows from Theorem 3.2 in [21]). To show that Γ does not preserve any curve in \hat{X} , assume by way of contradiction that $E \subset \hat{X}$ be a Γ -periodic irreducible curve, and denote by F its image in $\mathbb{P}_{\mathbf{C}}^4$ under the projection $\hat{X} \rightarrow X$. If F is a point, it is one of the singularities q_{ij} . Note that Γ acts transitively on the singularities of X : given any pair of singularities (q, q') , there is an element of Γ which is well defined at q and q' and maps q to q' . Thus, we can assume that $j = i + 1$, and changing E into its image under (the lift of) $\sigma_{ij} \in \Gamma$ the curve F becomes the line L_{ij} . So, we may assume that F is an irreducible curve. Now, the orbit of F is periodic under the action of the parabolic automorphisms $g_i = \sigma_{ij} \circ \sigma_{jk}$, with $j = i + 1$ and $k = i + 2$ modulo 5. Since the invariant curves of a parabolic automorphism are contained in the fibers of its invariant fibration, we deduce that F is contained in the fibers of each of the projections π_{lm} with $m = l + 1$, which is impossible. So there is no invariant curve.

The corresponding statement for Γ^{ext} follows immediately.

The reasoning for (5) is similar. Without loss of generality, assume $m = 0$. Then again Γ_m is non-elementary since it contains the parabolic elements $\sigma_{12} \circ \sigma_{23}$ and $\sigma_{23} \circ \sigma_{34}$ (with distinct associated fibrations π_{04} and π_{01}). Reasoning as above shows that if $E \subset \hat{X}$ is any Γ -periodic irreducible curve projecting to a curve in X , then its image F in X is contained in a fiber of each of the projections $\pi_{01}, \pi_{02}, \pi_{03}$ and π_{04} . So we conclude that $F \subset \{z_0 = 0\}$, but then the equation of X forces another coordinate to vanish, and we conclude that F is one of the L_{0l} . \square

Example 5.13. Let us give some geometric explanations for Assertion (2) of Proposition 5.10. Choose $(l, m) = (1, 2)$, and normalize the pentagons so that $a_0 = (0, 0)$ and $t_0 = 1$, which means that $a_1 = (\ell_0, 0)$. In homogeneous coordinates, this corresponds to the normalization $[1 : z_1 : z_2 : z_3 : z_4]$ with $z_i = t_i$. The pentagons contained in a fiber of $\pi_{12}|_X$ have three fixed vertices, namely a_0 , a_1 and a_2 . The remaining free vertices a_3 and a_4 move along the circles centered at a_2 and a_0 , of respective radii ℓ_2 and ℓ_4 , with the constraint $a_3a_4 = \ell_3$. These circles are two conics, the fiber is an elliptic curve which is a 2-to-1 cover of each of these two conics, the involutions σ_{23} and σ_{34} preserve these fibers, and $\sigma_{23} \circ \sigma_{34}$ is a translation on the elliptic curve. Forgetting the vertex a_1 , we obtain a quadrilateral (a_0, a_2, a_3, a_4) , and one recovers the transformations described in [7]. The side lengths of this quadrilateral are ℓ_2 , ℓ_3 , ℓ_4 and $\|\overrightarrow{a_0a_2}\|$, hence the translation vector (which only depend on these lengths) varies non-trivially when deforming the pentagon, which corresponds to the twisting property of parabolic transformations.

Lemma 5.14. *Let X be a K3 or Enriques surface, and $\pi: X \rightarrow B$ be a genus 1 fibration. If $g \in \text{Aut}(X)$ maps some fiber F of π to a fiber of π , then g preserves the fibration and either g is parabolic or it is periodic of order ≤ 66 .*

Proof. Since g maps F to some fiber F' , it maps the complete linear system $|F|$ to $|F'|$, but both linear systems are made of the fibers of π . So g preserves the fibration and is not loxodromic. If g is not parabolic it is elliptic, and its action on cohomology has finite order since it preserves $H^2(X, \mathbf{Z})$. On a K3 or Enriques surface every holomorphic vector field vanishes identically, so $\text{Aut}(X)^0$ is trivial and the kernel of the homomorphism $\text{Aut}(X) \ni f \mapsto f^*$ is finite (see [21, Theorem 2.6]); as a consequence, any elliptic automorphism has finite order. The upper bound on the order of g was obtained in [64]. \square

6. RANDOM FOLDINGS AND ERGODIC THEORY

We have now gathered enough geometric information to draw some dynamical consequences on the dynamics of pentagon folding.

6.1. Dynamics on $\text{Pent}^0(\ell)$. Recall that the folding groups Γ , Γ^{ext} and Γ_m were defined in § 5.4. Recall also that a parameter $\ell \in \mathbf{R}_{>0}^5$ is admissible when it corresponds to at least one pentagon.

Recall from Remark 5.7 that $X(\mathbf{R}) \simeq \text{Pent}^0(\ell)$ can be disconnected, in which case it is the disjoint union of two tori. This happens in 3 of the 19 possible configurations listed in [34], the shapes of which are sketched in Figure 1. Such a pentagon cannot be deformed continuously to its reflection along the horizontal side, hence the configuration space is disconnected. On the other hand,

- (1) folding it along its longest diagonal maps it into the other component, so Γ and Γ^{ext} act transitively on the set of components of $\text{Pent}^0(\ell)$;
- (2) the involutions preserving the horizontal side preserve each component of $\text{Pent}^0(\ell)$, so Γ_m preserves each component of $\text{Pent}^0(\ell)$ for $m = 0$.

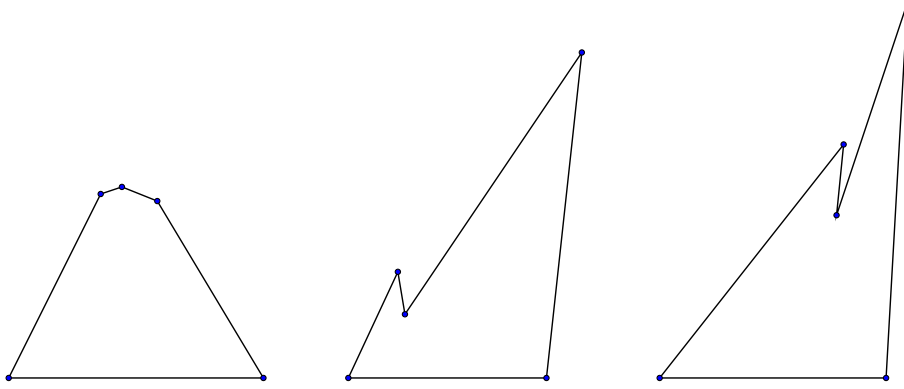


FIGURE 1. Pentagons with disconnected configuration spaces (respectively corresponding to the codes $(\begin{smallmatrix} \circ & \circ \\ \circ & \circ \end{smallmatrix})$, $(\begin{smallmatrix} \subseteq & \emptyset \\ \subseteq & \emptyset \end{smallmatrix})$ and $(\begin{smallmatrix} \Xi & \Xi \\ \Xi & \Xi \end{smallmatrix})$ in [34, Table 4])

In the next two statements, “general” means that the conclusions of Proposition 5.10 are satisfied.

Proposition 6.1. *For a general admissible parameter $\ell \in \mathbf{R}_{>0}^5$, the action of Γ (resp. of Γ^{ext}) on $\text{Pent}^0(\ell)$ is ergodic with respect to its natural volume form. Likewise, for $m \in \{0, \dots, 4\}$ the action of Γ_m is ergodic on each Γ_m -orbit of connected components of $\text{Pent}^0(\ell)$.*

Proof. Since Γ has no invariant curve and acts transitively on the set of components of $\text{Pent}^0(\ell)$, the result follows from case (c) of [25, Thm A]. For Γ_m , the same argument applies, upon adding the observation that the Γ_m -invariant curves do not intersect $\text{Pent}^0(\ell)$. \square

Theorem 6.2. *Let $\ell \in \mathbf{R}_{>0}^5$ be a general admissible parameter. Fix a probability measure ν on Γ (resp. Γ^{ext}) satisfying the moment condition (M), and whose support generates Γ (resp. Γ^{ext}). Then all ν -stationary measures on $\text{Pent}^0(\ell)$ are invariant, and the ergodic invariant measures are given by:*

- finitely many periodic orbits;
- $\text{vol}_{\text{Pent}^0(\ell)}$.

In particular, the set of ν -stationary measures is a finite dimensional simplex.

Proof. This follows from Proposition 6.1, the stiffness theorem of [26], and the finiteness of the set of finite orbits established in [24, Thm C]. \square

Remark 6.3. The classification of stationary measures also holds for Γ_m , except that in this case we cannot apply [24, Thm C] to infer the finiteness of the set of periodic orbits, because of the existence of invariant curves.

The random ergodic theorem then implies that for $\text{vol}_{\text{Pent}^0(\ell)}$ -almost every pentagon $x \in \text{Pent}^0(\ell)$, the sequence of empirical measures $\frac{1}{n} \sum_{k=1}^n \delta_{f_\omega^k(x)}$ is almost surely equidistributed on $\text{Pent}^0(\ell)$. As explained in § 1.8, to deduce the more precise result that this random equidistribution holds for every pentagon with infinite orbit we need some information about periodic orbits. This is where it is useful to work with the extended folding group.

Theorem 6.4. *For a very general admissible parameter $\ell \in \mathbf{R}_{>0}^5$, every orbit of Γ^{ext} in $\text{Pent}^0(\ell)$ is infinite. Hence, for such a parameter, the only ergodic Γ^{ext} -invariant probability measure on $\text{Pent}^0(\ell)$ is the natural volume.*

We thus obtain the following equidistribution result (the moment condition (M_+) was defined in § 1.8):

Corollary 6.5. *Fix a probability measure ν on Γ^{ext} satisfying the moment condition (M_+) and generating Γ^{ext} . There is an open and dense subset of full measure in the set of admissible parameters ℓ such that for any $x \in \text{Pent}^0(\ell)$: either $\Gamma^{\text{ext}} \cdot x$ is finite or for $\nu^{\mathbf{N}}$ -almost every ω ,*

$$\frac{1}{n} \sum_{k=1}^n \delta_{f_{\omega}^k(x)} \rightarrow \text{vol}_{\text{Pent}^0(\ell)}$$

as $n \rightarrow +\infty$.

Proof. By [23, Thm 1.5], at a very general parameter the action of Γ^{ext} is uniformly expanding. Since uniform expansion is an open property, it holds on a dense open set. Then the equidistribution result follows from [23, Thm 10.4] \square

Proof of Theorem 6.4. By Theorem 6.2, it suffices to prove the first assertion. If this assertion were not correct, then, arguing exactly as in [24, Thm A], we would find a finite index subgroup $\Gamma' \subset \Gamma^{\text{ext}}$ such that the algebraic set

$$(6.1) \quad \mathcal{Z} = \{(\ell, x) \in \mathbf{C}^5 \times X_{\ell}(\mathbf{C}), x \in X_{\ell}(\mathbf{C}), \forall f \in \Gamma', f(x) = x\}$$

has a Zariski dense projection to \mathbf{C}^5 . Since Γ^{ext} does not preserve any curve in X_{ℓ} (for a general ℓ), then so does Γ' , and as a consequence, the natural projection $\pi: \mathcal{Z} \rightarrow \mathbf{C}^5$ is generically finite. Then, there exists a Zariski dense open subset $W \subset \mathbf{C}^5$ such that above W , π is a finite unramified cover and the surfaces \hat{X}_{ℓ} are all smooth.

Since $\mathbf{R}_{>0}^5$ is Zariski dense in \mathbf{C}^5 , the intersection $W_{>0}(\mathbf{R}) := W \cap \mathbf{R}_{>0}^5$ is the complement of a proper Zariski closed subset of $\mathbf{R}_{>0}^5$. Reducing this open set slightly (by cutting out additional hypersurfaces if necessary), we may assume that every connected component of $W_{>0}(\mathbf{R})$ is simply connected. If W_i is such a connected component (for the euclidean topology), and if (ℓ, x) is a point of \mathcal{Z} with projection $\ell \in W_i$, then there is a unique continuous (algebraic) section

$$(6.2) \quad \ell' \in W_i \mapsto (\ell', x') \in \mathcal{Z}$$

of π defined on W_i that maps ℓ to (ℓ, x) . This will be referred to as the continuous continuation of the Γ' -fixed point x . Moreover, under our contradiction hypothesis, we may choose W_i and (ℓ, x) such that the section defined in (6.2) takes its values x' in $\text{Pent}^0(\ell')$ (i.e. in the real part of the complex surface $X_{\ell'}$ for the real structure defined in (5.10)). We fix such a pair $(W_i, (\ell, x))$, where x corresponds to a normalized pentagon (a_i) (i.e. $a_0 = (0, 0)$, $a_1 = (\ell_0, 0)$).

Recall that a planar polygon is said to be degenerate, or flat, if it is contained in a line (ie. its vertices are collinear). Since we can choose ℓ as we wish in the open set W_i , we may assume that

$$(6.3) \quad \text{no line contains three of the vertices } a_i.$$

In particular, x itself is not flat.

The triangle $(a_2a_3a_4)$ and the quadrilaterals $a_0a_2a_3a_4$ (with side lengths $\ell_5 := \|\overrightarrow{a_0a_2}\|$, ℓ_2 , ℓ_3 , ℓ_4) and $a_1a_2a_3a_4$ (with side lengths ℓ_1 , ℓ_2 , ℓ_3 , and $\ell_6 := \|\overrightarrow{a_1a_4}\|$) are non-degenerate. These quadrilaterals are periodic for the respective transformations $\sigma_{23} \circ \sigma_{34}$ and $\sigma_{12} \circ \sigma_{23}$. Therefore, by the Darboux alternative for quadrilaterals (see [35, 7, 62] and also Example 5.13), their side lengths satisfy a non-trivial relation. More precisely, given the lengths ℓ_1, \dots, ℓ_4 , there exists a countable set $D_5 = D_5(\ell_2, \ell_3, \ell_4)$ and $D_6 = D_6(\ell_1, \ell_2, \ell_3)$ such that

$$(6.4) \quad \ell_5 \in D_5(\ell_2, \ell_3, \ell_4) \quad \text{and} \quad \ell_6 \in D_6(\ell_1, \ell_2, \ell_3).$$

We now deform the pentagon by varying ℓ_0 while keeping the other side lengths fixed; this gives a variation $\ell' = (\ell'_0, \ell_1, \dots, \ell_4)$ of ℓ in W_i , and a continuation $x' = (a'_i) \in \mathcal{Z}$ of x (with the normalization $a'_0 = (0, 0)$ and $a'_1 = (\ell_0, 0)$). Let us show that for ℓ' near ℓ , the continuation x' of x is geometrically determined, in a unique way, by the length conditions (6.4). Since $\|\overrightarrow{a'_0a'_2}\|$ varies continuously and must stay in D_5 , it is constant, hence the triangle $a'_0a'_1a'_2$ has side lengths ℓ'_0 , ℓ_1 and ℓ_5 which defines it uniquely up to isometry. Since the initial triangle $a_0a_1a_2$ is not flat, then the point a'_2 is uniquely determined as a continuation of a_2 . Similarly, the triangle $a'_0a'_1a'_4$ determines a'_4 . Finally, since the lengths ℓ_2 and ℓ_3 are fixed and the triangle $a_2a_3a_4$ is not flat, the continuation a'_3 is uniquely determined. In conclusion, the periodicity of x' under the action of $\sigma_{23} \circ \sigma_{34}$ and of $\sigma_{12} \circ \sigma_{23}$ determines x' as ℓ_0 varies; thus, in what follows we can forget about \mathcal{Z} : the section (ℓ', x') is, in fact, given by the geometric construction we have just described.

To reach the desired contradiction, we now argue that the periodicity of x' under the parabolic automorphism $\sigma_{13} \circ \sigma_{34}$ creates an additional rigidity that cannot be satisfied (we might use $\sigma_{12} \circ \sigma_{24}$ instead). Indeed, $\sigma_{13} \circ \sigma_{34}$ acts on the vectors $v_1 = \overrightarrow{a_1a_2}$, $v_3 = \overrightarrow{a_3a_4}$ and $v_4 = \overrightarrow{a_4a_0}$, so it can be seen as a transformation of the “virtual” quadrilateral $a_0b_1a_3a_4$, where b_1 is such that $\overrightarrow{a_3b_1} = \overrightarrow{a_2a_1}$. Since x' is a Γ^{ext} -periodic point of $\text{Pent}^0(\ell')$ for all ℓ' , the quantity $\|v_1 + v_3 + v_4\| = \|\overrightarrow{a_0b_1}\|$ must be constant; more precisely, it does not vary with ℓ'_0 on a neighborhood of ℓ . But this function depends algebraically on the parameters, and any neighborhood of ℓ in W_i is Zariski dense in \mathbf{C}^5 , so we conclude that this function actually does not depend on ℓ'_0 . Thus, to reach the desired contradiction, one just needs to contemplate Figure 2. \square

Since we use transformations like $\sigma_{13} \circ \sigma_{34}$ in the proof, it is necessary to work with Γ^{ext} . The analogous result for Γ remains elusive ⁽⁶⁾.

Question 6.6.

- (1) *Is it true that for a very general set of lengths $\ell \in \mathbb{P}^4(\mathbf{C})$, every orbit of Γ (resp. Γ_m) in $\text{Pent}^0(\ell)$ (resp. in X_ℓ) is infinite?*
- (2) *Is there a dense open set of parameters $\ell \in \mathbb{P}^4(\mathbf{R})$ for which every finite orbit is uniformly expanding?*
- (3) *Can we replace “very general” by “general” in Theorem 6.4 and in Question (1)?*

⁶We actually do not know any example of a “general” periodic pentagon, that is a pentagon $x \in \text{Pent}^0(\ell)$ with a finite Γ -orbit, for which ℓ satisfies the conditions of Proposition 5.10.

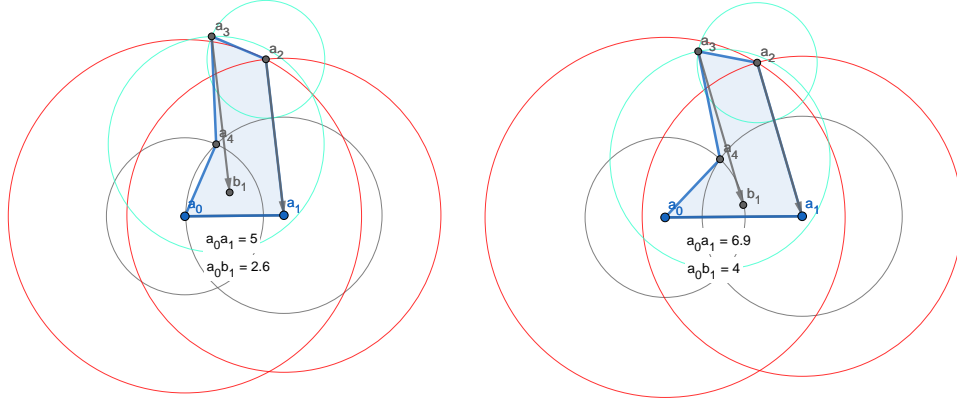


FIGURE 2. Proof of Theorem 6.4. The corresponding circles have the same radius on the two figures, and serve for the construction of a_2 , a_3 and a_4 (for instance on both pictures the red circle on the left is centered at a_0 and has radius ℓ_5 and the red circle on the right is centered at a_1 with radius ℓ_2). When ℓ_0 varies the distance a_0b_1 varies as well. (Realized with GeoGebra and available at <https://www.geogebra.org/m/edydwafb>).

Let us expand a little bit on this. As seen above, a positive answer to the first question implies a positive answer to the second one. On the other hand, even without excluding the existence of a persistent periodic orbit for Γ , it might still be possible to show that the action of Γ on $\text{Pent}^0(\ell)$ is uniformly expanding generically (this point of view is developed in [23, §9]). Indeed, by [23], it is enough to show that at such a persistent periodic orbit, for a general parameter ℓ , we can find parabolic elements in Γ that exhibit a non-trivial twisting in different directions. Geometrically, this would require to understand how the rotation numbers associated to sub-quadrilaterals vary when deforming the pentagon (an explicit formula for this rotation number is given in [7, 62]).

6.2. Dynamics on $\text{Pent}^1(\ell)$. Recall that \mathbb{U} denotes the unit circle in $\mathbf{R}^2 \simeq \mathbf{C}$. To an element $u \in \mathbb{U}$ corresponds a unique rotation R_u (centered at the origin); in complex coordinates, it just corresponds to multiplication by u , and its inverse is $R_u^{-1} = R_{\bar{u}}$.

For $\ell = (\ell_0, \dots, \ell_4) \in \mathbf{R}_{>0}^5$, let us introduce the space of pentagons modulo translations

$$(6.5) \quad \text{Pent}^1(\ell) = \text{Pent}(\ell)/\mathbf{R}^2.$$

With notation as in § 5.1, every element \tilde{x} of $\text{Pent}^1(\ell)$ admits a unique representative with $a_0 = (0, 0)$. Then, $\overline{a_0 a_1} = \ell_0 t_0$, for some unit vector $t_0 \in \mathbb{U}$ and if we apply the rotation $R_{\bar{t}_0}$, we get a normalized pentagon x (with $a_0 = (0, 0)$ and $a_1 = (\ell_0, 0)$). This shows that $\text{Pent}^1(\ell)$ is a trivial circle bundle over $\text{Pent}^0(\ell)$:

$$(6.6) \quad \text{Pent}^1(\ell) \simeq \text{Pent}^0(\ell) \times \mathbb{U}.$$

The reciprocal diffeomorphism is obtained as follows. Let (x, u) be an element of $\text{Pent}^0(\ell) \times \mathbb{U}$.

- To x , one associates its normalized pentagon, that is, the unique pentagon (a_i) in its $\text{SO}(2) \times \mathbf{R}^2$ orbit corresponding to x such that $a_0 = (0, 0)$ and $a_1 = (\ell_0, 0)$.

– then one rotates it by R_u to get an element of $\text{Pent}^1(\ell)$.

As a real algebraic variety, $\text{Pent}^1(\ell)$ is the real locus, with respect to the involution

$$(6.7) \quad s: (z_i) \mapsto (1/\bar{z}_i),$$

of the variety defined in \mathbf{C}^5 by the system of Equations (5.4).

The involutions σ_{ij} and the groups Γ and Γ_m are defined exactly as before, with the same formulas; to avoid confusion with the action on Pent^0 , we may denote the involutions by $\tilde{\sigma}_{ij}$ and the corresponding pentagons by \tilde{x} . In the coordinates $(x, u) \in \text{Pent}^0(\ell) \times \mathbb{U}$, $\tilde{\sigma}_{ij}$ is given by

$$(6.8) \quad \begin{cases} \tilde{\sigma}_{ij}(x, u) = (\sigma_{ij}(x), u), & \text{if } 0 \notin \{i, j\} \\ \tilde{\sigma}_{0j}(x, u) = (\sigma_{0j}(x), h_{0j}(x)u) \end{cases}$$

where $x \mapsto h_{0j}(x)$ takes values in \mathbb{U} . To justify the second line, simply observe that σ_{ij} commutes with the rotation R_v , i.e. with $(x, u) \mapsto (x, uv)$; thus, $\tilde{\sigma}_{0j}(x, u) = R_u \tilde{\sigma}_{0j}(x, 1)$.

Instead of choosing a_0 as a base point (which is translated to the origin), we could choose any of the five vertices a_m , $m = 0, \dots, 4$. This provides five different identifications $\text{Pent}^1(\ell) \simeq \text{Pent}^0(\ell) \times \mathbb{U}$; for each of them, we denote by $\vartheta_m: \text{Pent}^1(\ell) \rightarrow \mathbb{U}$ the projection onto the second factor (in other words, for a pentagon \tilde{x} , $\vartheta_m(\tilde{x})$ is the angle $\angle(\overrightarrow{a_m a_{m+1}}, (1, 0))$). Changing from the basepoint a_i to the basepoint a_j yields a change of coordinates of the form $(x, u) \mapsto (x, R_{\alpha_{i,j}(x)} u)$, where $\alpha_{i,j}(x)$ is the unit vector with angle $\angle(\overrightarrow{a_i a_{i+1}}, \overrightarrow{a_j a_{j+1}})$. So,

$$(6.9) \quad \text{vol}_{\text{Pent}^1(\ell)} := \text{vol}_{\text{Pent}^0(\ell)} \times \text{Leb}_{\mathbb{U}}$$

defines a Γ -invariant volume.

Proposition 6.7. *The action of Γ is ergodic on $\text{Pent}^1(\ell)$ with respect to the natural volume $\text{vol}_{\text{Pent}^1(\ell)}$.*

Proof. The argument is borrowed from Chivet's master's thesis [29]. Let B be a Borel set of positive volume such that for every $\gamma \in \Gamma$,

$$(6.10) \quad \text{vol}_{\text{Pent}^1(\ell)}(B \Delta \gamma^{-1}(B)) = 0.$$

Pick an index $m \in \{0, \dots, 4\}$, and note that Γ_m preserves the fibers of ϑ_m . Since B is almost Γ_m -invariant and the Γ_m -action on $\text{Pent}^0(\ell)$ is ergodic, we get that B is ϑ_m -saturated, which means that every fiber of ϑ_m intersects B on a set of zero or full measure for $\text{vol}_{\text{Pent}^0(\ell)}$.

Let us fix a value of m , say $m = 0$ and work in the system of coordinates (x, u) associated to this choice; in these coordinates, R_v is of the form $(x, u) \mapsto (x, uv)$. Set

$$(6.11) \quad B_0 = \left\{ u \in \mathbb{U}; \text{vol}_{\text{Pent}^0(\ell)}(B \cap \vartheta_0^{-1}(u)) = 1 \right\}.$$

If we push $\text{vol}_{\text{Pent}^1(\ell)}$ onto \mathbb{U} by ϑ_0 , one gets the Lebesgue measure, and B_0 has a positive Lebesgue measure; below, we shall consider Lebesgue density points u_0 of B_0 .

Now fix another value of m , say $m = 1$. Let F be any fiber of ϑ_1 . Since $\vartheta_0|_F: F \rightarrow \mathbb{U}$ admits a regular value, there exists $\delta > 0$ and an open set $W \subset F$ such that $\vartheta_0|_W$ is a submersion onto an interval of length 2δ . In addition, since any other fiber of ϑ_1 is of the form $R_v(F)$, we deduce

that there exists an open set $V \in \text{Pent}^0(\ell)$ such that, for any $u_0 \in \mathbb{U}$, there exists a neighborhood $N(u_0) \subset \mathbb{U}$ such that, for every $u \in N(u_0)$ and every $x \in V$, there exists $W_x \in \vartheta_1^{-1}(\vartheta_1(x, u))$ such that $\vartheta_0|_{W_x}$ realizes a submersion $W_x \rightarrow]u - \delta, u + \delta[$.

Fix any density point u_0 of B_0 , so that there exists $A \subset B_0 \cap N(u_0)$ of positive Lebesgue measure. For every $u \in A$, $B \cap (V \times \{u\})$ is of full $\text{vol}_{\text{Pent}^0(\ell)}$ mass in $V \times \{u\}$. By Fubini's theorem, and the fact that the B is ϑ_1 -saturated, there is a set of positive measure $B' \subset A \times V$ such that for any $(x, u) \in B'$, the ϑ_1 -fiber of (x, u) is contained in B (modulo a set of measure zero). Projecting back by ϑ_0 and using the fact that B is ϑ_0 -saturated, we conclude that $]u_0 - \delta, u_0 + \delta[\subset B_0$ (modulo a set of measure zero). From this we easily conclude that $B_0 = \mathbb{U}$ (modulo a set of measure zero) and, applying the Fubini theorem again completes the proof. \square

Theorem 6.8. *Let $\ell \in \mathbf{R}_{>0}^5$ be a general parameter. Fix a probability measure ν on Γ satisfying the moment condition (M) and whose support generates Γ . Then, the ergodic, ν -stationary measures are*

- atomic measures on Γ -periodic orbits⁽⁷⁾;
- measures of the form

$$\left(\sum_{x \in F} \delta_x \right) \times \text{Leb}_{\mathbb{U}}$$

where F is a finite orbit for the action of Γ on $\text{Pent}^0(\ell)$;

- the measure $\text{vol}_{\text{Pent}^1(\ell)}$.

In particular, every ν -stationary measure on $\text{Pent}^1(\ell)$ is invariant.

If ν is a probability measure on Γ^{ext} satisfying condition (M) and generating Γ^{ext} , and if ℓ is very general, then the only ergodic ν -stationary measure on $\text{Pent}^1(\ell)$ is $\text{vol}_{\text{Pent}^1(\ell)}$.

In the latter case, we could also state an equidistribution result in the spirit of Corollary 6.5. The core of the proof is the following stationary version of a celebrated argument due to Furstenberg [55], which we state here in a general form.

Lemma 6.9. *Consider a random dynamical system $(X, (f_\omega), \nu, \mu)$, where X is a compact metric space, ν is a Borel probability measure on $\text{Homeo}(X)$, and μ is an ergodic ν -stationary measure on X . Let G be a compact group, with Haar measure λ . Consider a G -extension of this random dynamical system, by transformations of $X \times G$ of the form*

$$F_\omega : (x, g) \mapsto (f_\omega(x), h_\omega(x) \cdot g)$$

Then for this extension, $\mu \times \lambda$ is ν -stationary, and if it is ergodic, it is the unique stationary measure projecting down to μ .

Proof. The stationarity of $\mu \times \lambda$ is an easy exercise. Let $\tilde{\mu}$ be an ergodic, ν -stationary probability measure on $X \times G$ with marginal $(\pi_X)_* \tilde{\mu} = \mu$. By the random ergodic theorem, $\tilde{\mu}$ -almost every

⁷Such periodic orbits are even rarer than periodic orbits on $\text{Pent}^0(\ell)$ (we do not know any example). So we strongly believe that there exists a dense, Zariski open subset $W \subset \mathbf{R}_{>0}^5$ such that for $\ell \in W$, Γ does not have any finite orbit in $\text{Pent}^1(\ell)$ (see Question 6.6).

(x, g) is $\tilde{\mu}$ -generic: this means that for $\nu^{\mathbf{N}}$ -almost every ω ,

$$(6.12) \quad \frac{1}{n} \sum_{k=0}^{n-1} \delta_{F_{\omega}^k(x,g)} \xrightarrow{n \rightarrow \infty} \tilde{\mu}.$$

Let $R_h : (x, g) \mapsto (x, g \cdot h)$ denote the right translation induced by $h \in G$. Note that F_{ω} commutes with R_h and $\mu \times \lambda$ is R_h -invariant. From this we infer that if (x, g) is $\mu \times \lambda$ -generic, then all points in the fiber $\{x\} \times G$ are $\mu \times \lambda$ -generic. By Fubini's theorem, it follows that $\tilde{\mu}$ -almost every point is in fact $\mu \times \lambda$ -generic, hence $\tilde{\mu} = \mu \times \lambda$, as asserted. \square

We shall also need the following well known lemma.

Lemma 6.10. *Let Γ be a group of rotations of the unit circle and ν be a measure such that $\langle \text{Supp}(\nu) \rangle = \Gamma$.*

- *If Γ is infinite, the Lebesgue measure is the only ν -stationary measure.*
- *If Γ is finite, the ergodic stationary measures are supported by its finite orbits.*

Proof. Let us prove the first assertion (the second one is obvious). Since Γ is infinite, it is a dense subgroup of $\text{SO}(2)$. Since $\text{SO}(2)$ is Abelian, by the Choquet-Deny theorem, every stationary measure is invariant. Now, if μ is Γ -invariant, it is $\text{SO}(2)$ -invariant too, by density and dominated convergence. Thus, μ is the Lebesgue measure. \square

Proof of Theorem 6.8. By Equation (6.8), the dynamics on $\text{Pent}^1(\ell)$ is a \mathbb{U} -extension of the one on $\text{Pent}^0(\ell)$. Theorem 6.2 implies that if ν generates Γ , any ergodic ν -stationary measure $\tilde{\mu}$ on $\text{Pent}^1(\ell)$ either projects to a finite orbit or to $\text{vol}_{\text{Pent}^0(\ell)}$. In the latter case, Proposition 6.7 and Lemma 6.9 imply that $\tilde{\mu} = \text{vol}_{\text{Pent}^1(\ell)}$. Assume now that

$$(6.13) \quad (\pi_{\text{Pent}^0(\ell)})_* \tilde{\mu} = \frac{1}{|F|} \sum_{x \in F} \delta_x$$

for some finite orbit F . In this situation we can also use Lemma 6.9 to classify stationary measures. Restricting to a finite index subgroup, endowed with the induced measure (see [11, Chap. 5]) reduces the problem to the case where $F = \{x_0\}$. Then the classification follows from Lemma 6.10.

The second statement of the theorem follows similarly, using Theorem 6.4 instead of 6.2. \square

6.3. Dynamics on $\text{Pent}(\ell)$. We can finally derive some information about the dynamics of random foldings on $\text{Pent}(\ell)$, which is a \mathbf{R}^2 -extension of the dynamics on $\text{Pent}^1(\ell)$. Indeed $\text{Pent}(\ell)$ can be identified to $\text{Pent}^1(\ell) \times \mathbf{R}^2$ by choosing a preferred vertex, say a_0 , and translating it to the origin. Doing so, we obtain a diffeomorphism

$$(6.14) \quad \text{Pent}(\ell) \ni (a_i) \longmapsto (\tilde{x}, a_0) = ((v_i), a_0) \in \text{Pent}^1(\ell) \times \mathbf{R}^2, \text{ where } v_i = \overline{a_i a_{i+1}}.$$

We already introduced the involution s_i in § 5.1, which descends to $\tilde{\sigma}_{i,i+1}$ (indices modulo 5) on $\text{Pent}^1(\ell)$ and to $\sigma_{i,i+1}$ on $\text{Pent}^0(\ell)$. With the identification given by Equation (6.14), we

obtain:

$$(6.15) \quad \begin{cases} s_i((a_i)) = (\tilde{\sigma}_{i,i+1}(\tilde{x}), a_0) \text{ for } i \neq 4 \\ s_4((a_i)) = (\tilde{\sigma}_{40}(\tilde{x}), a_0 + \tilde{\sigma}_{40}(v_4) - v_4) \end{cases}$$

By definition the group Γ is the group generated by the involutions s_i (we are slightly abusing notation here).

To find an involution descending to $\tilde{\sigma}_{ij}$ when $j \neq i + 1$, different options can be chosen, depending on the vertices that remain fixed under the involution. First, observe that if $j \neq i + 1$ we can always choose i, j so that $j = i + 2$ modulo 5 (so the vectors v_i and v_{i+2} will change while the others remain fixed). In analogy with the case of s_i , we define r_i such that $r_i(a_i) = a_i$ and $[r_i(P)] = \sigma_{i,i+2}([P])$, so that r_i fixes a_i, a_{i+3} and a_{i+4} and moves a_{i+1} and a_{i+2} (another option would have been to leave a_{i+2} fixed). It can be expressed in coordinates in $\text{Pent}(\ell)$ as in (6.15). We define Γ^{ext} acting on $\text{Pent}(\ell)$ by $\Gamma^{\text{ext}} := \langle s_i, r_i, i = 0 \dots 5 \rangle$.

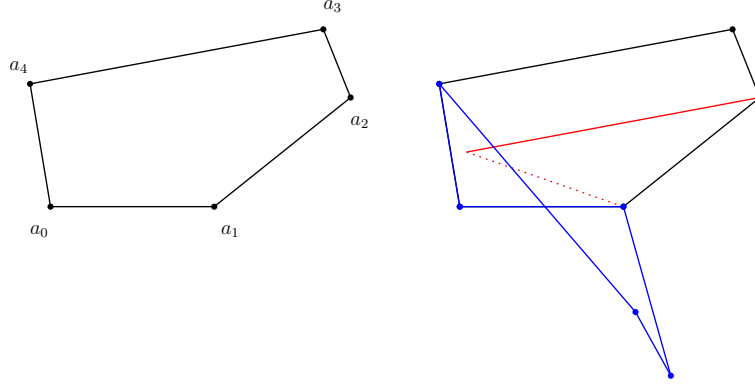


FIGURE 3. The black pentagon on the left is folded to the blue pentagon on the right by the lift r_1 of $\sigma_{1,3}$. The red segment is parallel to $[a_3, a_4]$ and the red dotted segment gives the direction along which the vectors v_1 and v_3 are reflected.

In [7, 8], the authors study in detail how a quadrilateral drifts in the plane under successive foldings. Fix a probability measure ν on Γ^{ext} . From a pentagon $P_0 = (a_i)$ with $a_0 = (0, 0)$ and a sequence $\omega = (f_n) \in (\Gamma^{\text{ext}})^{\mathbf{N}}$ of folding instructions, we obtain the following random sequence of pentagons $P_n \in \text{Pent}(\ell)$:

$$(6.16) \quad P_n(\omega) = f_{n-1} \cdots f_0(P_0).$$

Note that the parameter space for the starting point is $\text{Pent}^1(\ell)$.

Proposition 6.11.

- (1) Let $\ell \in \mathbf{R}^5$ be a general parameter. Assume that ν satisfies the moment condition (M) and generates Γ . Then, for $\text{vol}_{\text{Pent}^1(\ell)}$ -almost every P_0 , the linear drift of the sequence (P_n) vanishes, that is

$$\lim_{n \rightarrow \infty} \frac{1}{n} \text{dist}(P_n(\omega), (0, 0)) = 0$$

for $\nu^{\mathbf{N}}$ -almost every ω .

(2) Assume now that ℓ is very general, ν satisfies the moment condition (M_+) , and ν generates Γ^{ext} . Then the same conclusion holds for every $P_0 \in \text{Pent}^1(\ell)$.

Proof. To study the drift it is enough to study the location of the point a_0 . For this we use the coordinates given by the identification $\text{Pent}(\ell) \simeq \text{Pent}^1(\ell) \times \mathbf{R}^2$ and write $P_n(\omega) = ([P_n(\omega)], a_0(P_n(\omega)))$. From Equation (6.15) and its analogue for the r_i , we see that there exists a function $w : \Gamma^{\text{ext}} \times \text{Pent}^1(\ell) \rightarrow \mathbf{R}^2$ such that $a_0(f(P)) = a_0(P) + w(f, [P])$. This function is continuous and is a cocycle: $w(fg, [P]) = w(f, g[P]) + w(g, [P])$. We then obtain

$$(6.17) \quad a_0(P_n(\omega)) = \sum_{k=0}^{n-1} w(f_k, [P_k(\omega)]).$$

For Assertion (1), we apply the Birkhoff ergodic theorem to the skew product

$$(6.18) \quad F : (\omega, x) \mapsto (\sigma(\omega), f_\omega(x))$$

(where σ is the shift on $(\Gamma^{\text{ext}})^{\mathbf{N}}$) and to the ergodic measure $\nu^{\mathbf{N}} \times \text{vol}_{\text{Pent}^1(\ell)}$. This shows that for $\text{vol}_{\text{Pent}^1(\ell)}$ -almost every P_0 and $\nu^{\mathbf{N}}$ -almost every ω ,

$$(6.19) \quad \frac{1}{n} \sum_{k=0}^{n-1} w(f_k, [P_k(\omega)]) \xrightarrow{n \rightarrow \infty} \int w(\gamma, P) d\nu(\gamma) d\text{vol}_{\text{Pent}^1(\ell)}(P).$$

Now for every $\gamma \in \Gamma^{\text{ext}}$, the invariance of $\text{vol}_{\text{Pent}^1(\ell)}$ under rotations implies that

$$(6.20) \quad \int w(\gamma, P) d\text{vol}_{\text{Pent}^1(\ell)}(P) = 0,$$

and the result follows.

For Assertion (2), ℓ being very general, we can assume that $\text{vol}_{\text{Pent}^1(\ell)}$ is the unique stationary measure on $\text{Pent}^1(\ell)$ (see Theorem 6.8). The argument will be the same as for Assertion (1), except that we shall need a more sophisticated limit theorem that makes use of this unique ergodicity. As already observed, w is a cocycle and, by unique ergodicity, it has a unique average in the sense of [11, §3.3.2]. The analogue of formula (6.15) for extended foldings implies that there exists $C(\ell) \leq 2 \sum \ell_i$ such that that for any $\gamma \in \Gamma^{\text{ext}}$

$$(6.21) \quad w_{\text{sup}}(\gamma) := \sup_{[P] \in \text{Pent}^1(\ell)} \|w(\gamma, [P])\| \leq C(\ell) \text{length}(\gamma),$$

where the length $\text{length}(\gamma)$ is relative to the given generators of Γ^{ext} . The moment condition implies that

$$(6.22) \quad \int_{\Gamma^{\text{ext}}} w_{\text{sup}}(\gamma) d\nu(\gamma) < \infty,$$

thus we can apply the Law of Large Cocycles [11, Thm 3.9], which completes the proof. \square

It is natural to ask for a better understanding of the asymptotic behavior of $\text{dist}(P_n(\omega), (0, 0))$. Intuitively, by Equation (6.17), the first vertex $a_0(P_n(\omega))$ should behave like a random walk in the plane, the steps of which are random vectors distributed according to some explicit measure supported by a bounded disk. Under appropriate non-degeneracy properties of this measure,

such a random walk escapes with speed $O(\sqrt{n})$ and, after rescaling, converges to a Brownian motion (for some positive definite covariance form). Numerical experiments indicate that $\text{dist}(P_n(\omega), (0, 0))$ indeed behaves like \sqrt{n} .

Question 6.12. *Does $a_0(P_n(\omega))$ satisfy a central limit theorem? Does $\frac{1}{\sqrt{n}}a_0(P_n(\omega))$ converge to a Brownian motion?*

As far as we know, this question is already open for random foldings of quadrilaterals. An even simpler variant is to take a triangle and reflect it randomly along one of its sides. This last problem falls into the setting of random iteration of Euclidean isometries in which case the result is known (see e.g. [2] and references therein). To establish such a result, it is likely that some estimates would be needed for the speed of the ergodicity of the base dynamics in Pent^0 , as discussed in § 1.8 (cf. [11, Chap. 11 and 12]).

Part 3. Ergodic theory of Blanc's examples

7. ACTION ON COHOMOLOGY IN THE COMPLEX SURFACE

7.1. Setting and volume form. Consider a smooth cubic curve $C \subset \mathbb{P}_{\mathbb{C}}^2$ and a point $q \in C$. Denote by $\sigma_q: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$ the birational involution defined by the following properties: it fixes C pointwise and it preserves the pencil of lines through q . If L is a general line through q , the restriction of σ_q to L is defined as follows: beside q , L intersects C in two other points p and p' ; identifying $L \setminus \{p, p'\}$ to $\mathbb{P}^1 \setminus \{0, \infty\}$, with affine coordinate z , $\sigma_q|_L$ corresponds to $z \mapsto -z$. By definition, σ_q is the **Jonquières involution** associated to the pencil of lines through q and fixing C pointwise.

Now, fix a positive integer k , and k distinct points q_i on C . Each q_i gives rise to a Jonquières involution $\sigma_i := \sigma_{q_i}$. It has five base points, namely q_i and the four points $p_{i,j} \in C$ such that the line $(q_i p_{i,j})$ is tangent to C at $p_{i,j}$. One of the $p_{i,j}$ may be infinitely near (above q_i) if q_i is an inflexion point, but for simplicity we shall assume that:

(Hyp1) none of the q_i is an inflexion point of C ;

(Hyp2) the points q_i and $p_{i,j}$ for $1 \leq i \leq k$ and $1 \leq j \leq 4$ are pairwise distinct.

In fact, (Hyp2) implies (Hyp1) since otherwise one of the $p_{i,j}$ would coincide with q_i (as a point of the plane, i.e. $p_{i,j}$ would be infinitely near q_i). We shall denote by X the rational surface obtained by blowing up the $5k$ points q_i and $p_{i,j}$ and by $\pi: X \rightarrow \mathbb{P}_{\mathbb{C}}^2$ the natural projection.

We also consider the surface X_i obtained by fixing i and blowing up the base points q_i and $p_{i,j}$ of σ_i , creating five exceptional divisors $E(q_i)$, and $E(p_{i,j})$ for $1 \leq j \leq 4$. Doing so, we get a birational morphism $\pi_i: X_i \rightarrow \mathbb{P}_{\mathbb{C}}^2$, and a distinguished basis for its Néron-Severi group:

- e_0 will denote the class of the total transform of a line $L \subset \mathbb{P}_{\mathbb{C}}^2$;
- e_{q_i} the class of $E(q_i)$;
- and $e_{p_{i,j}}$ the class of $E(p_{i,j})$.

This basis is orthogonal for the intersection form, $e_0^2 = 1$, and $e_{q_i}^2 = e_{p_{i,j}}^2 = -1$. Abusing slightly, the same notation will be used for the classes of (the total transform of) these curves in the surface X .

The involution σ_i lifts to an involutive automorphism $\tilde{\sigma}_i$ of X_i , and then all the $\tilde{\sigma}_i$ lift to automorphisms of X , which we still denote by $\tilde{\sigma}_i$. Indeed, since $\tilde{\sigma}_i$ fixes the strict transform of C in X_i pointwise, it preserves the exceptional divisors obtained by blowing up the q_ℓ and $p_{\ell,j}$ for $\ell \neq i$. According to [12], the subgroup $\Gamma \subset \text{Aut}(X)$ generated by the $\tilde{\sigma}_i$ is a free product of k copies of $\mathbf{Z}/2\mathbf{Z}$, i.e.

$$(7.1) \quad \Gamma \simeq \mathbf{Z}/2\mathbf{Z} * \cdots * \mathbf{Z}/2\mathbf{Z},$$

acting faithfully on the Néron-Severi group $\text{NS}(X)$. Since the canonical bundle of $\mathbb{P}^2(\mathbb{C})$ is $\mathcal{O}(-3)$ and we only blow-up points of C , we obtain the following properties.

(1) The strict transform C_X of C in X is fixed pointwise by the group Γ . Its self-intersection is equal to

$$(7.2) \quad C_X^2 = 9 - 5k,$$

so it is negative when $k \geq 2$. The strict transform C_{X_i} of C in X_i has self-intersection $C_{X_i}^2 = 4$.

(2) Assume $k \geq 2$. The curve C_X is the unique member of the linear system $|C_X|$, and its cohomology class is equal to the anticanonical class $-k_X$. In particular, C_X is invariant under the action of $\text{Aut}(X)$. More precisely, C being a cubic curve, there is a meromorphic 2-form Ω_0 on $\mathbb{P}_{\mathbb{C}}^2$ that does not vanish and has a simple pole along C . Now, since we only blow up smooth points of C , an easy local computation shows that the pull back

$$(7.3) \quad \Omega_X := \pi^* \Omega_0$$

is a meromorphic 2-form that does not vanish and has a simple pole along C_X . We shall fix such a form Ω_X ; it is almost unique: on X , any non-vanishing meromorphic 2-form is proportional to Ω_X .

(3) The total volume of the singular volume form $\Omega_X \wedge \overline{\Omega_X}$ is infinite:

$$(7.4) \quad \int_X \Omega_X \wedge \overline{\Omega_X} = +\infty.$$

We set

$$(7.5) \quad \text{vol}_X^\infty = \Omega_X \wedge \overline{\Omega_X},$$

which we identify to a positive measure on X .

(4) Let $\text{Jac}_\Omega: \text{Aut}(X) \rightarrow \mathbf{C}^*$ denote the homomorphism such that

$$(7.6) \quad \forall f \in \Gamma, f^* \Omega = \text{Jac}_\Omega(f) \Omega$$

(cf. Remark 4.3). We have

$$(7.7) \quad \text{Jac}_\Omega(\Gamma) = \{1\}.$$

To see this, consider the involution σ_1 and the pencil of lines through q_1 . Let r be a point of C such that $(q_1 r)$ intersects C transversely; then, σ_1 determines a germ of diffeomorphism fixing r . Since this germ has order 2 and fixes C pointwise, we can linearize it⁽⁸⁾. Since C is pointwise fixed, there is a small euclidean neighborhood of r and holomorphic coordinates (x, y) on it, in which $r = (0, 0)$, $\sigma_1(x, y) = (x, -y)$ and $C = \{y = 0\}$. In these coordinates, the form Ω_X is equal to

$$\varphi(x, y) \frac{dx \wedge dy}{y}$$

for some non-vanishing holomorphic function φ . Then, $\text{Jac}_\Omega(\sigma_1) = \varphi(x, -y)/\varphi(x, y)$. Evaluating this ratio at r , we get $\text{Jac}_\Omega(\sigma_1) = 1$. Thus, $\text{Jac}_\Omega(\sigma_i) = 1$ for $i = 1, \dots, k$, and $\text{Jac}_\Omega(\Gamma) = \{1\}$.

(5) If C and the q_i are defined over $\overline{\mathbf{Q}}$, then X and Γ are also defined over $\overline{\mathbf{Q}}$. Indeed, if K is the number field over which C and the q_i are defined then, for each i , the $p_{i,j}$ are defined over some quartic extension of K ; thus, X and Γ are defined over some extension L of K of degree $[L : K] \leq 4k$.

7.2. Finer description of the involution σ_i .

⁸Indeed, its differential L at the origin in the system of coordinates mentioned just below is $L(x, y) = (x, -y)$, and $\psi \circ \sigma_1 = L \circ \psi$, where $\psi = \frac{1}{2}(\text{id} + L \circ \sigma_1)$

7.2.1. *Invariant fibration.* For each $i = 1, \dots, k$, the pencil of lines through q_i gives rise to a rational fibration

$$(7.8) \quad \eta_{q_i} : X_i \rightarrow \mathbb{P}_{\mathbb{C}}^1.$$

The singular fibers of η_{q_i} correspond to the total transform of the lines $(q_i p_{i,j})$ (each of them is made of two rational curves). In X , we obtain k fibrations $\eta_{q_i} : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$, each of them with $5k - 1 = 4 + 5(k - 1)$ singular fibers.

Let $L_{i,j} \subset X_i$ be the strict transform of the line $(q_i p_{i,j})$. Since $(q_i p_{i,j})$ is tangent to C at $p_{i,j}$, the three curves C_{X_i} , $E(p_{i,j})$, and $L_{i,j}$ have a unique common point

$$(7.9) \quad \tilde{p}_{i,j} := C_{X_i} \cap E(p_{i,j}) = L_{i,j} \cap E(p_{i,j})$$

in X_i . The involution $\tilde{\sigma}_i$ permutes $E(p_{i,j})$ and $L_{i,j}$ and fixes this intersection point $\tilde{p}_{i,j}$. This can be seen as follows: if L is the strict transform of a general line through q , then $\tilde{\sigma}_i^*([L]) = [L]$. It follows that $\tilde{\sigma}_i^*([L_{i,j}] + [E(p_{i,j})]) = [L_{i,j}] + [E(p_{i,j})]$. Now, σ_i contracts $\pi_i(L_{i,j})$ onto $p_{i,j}$, so $\tilde{\sigma}_i(L_{i,j}) = cE(p_{i,j})$ for some integer $c \geq 1$; and we infer that $c = 1$ because $\tilde{\sigma}_i^*$ is an involution. Thus, $\tilde{\sigma}_i(L_{i,j}) = E(p_{i,j})$ and $\tilde{\sigma}_i(E(p_{i,j})) = L_{i,j}$.

7.2.2. *Action on $E_{p_{i,j}}$.* Let x be a point of $E(p_{i,j})$. A first possibility is that $\tilde{\sigma}_i(x)$ belongs to $\text{Exc}(\pi_i)$ (more precisely $\tilde{\sigma}_i(x) \in E(q_i) \cup E(p_{i,j})$). This happens precisely when

- $\tilde{\sigma}_i(x) = x$, and in that case $x = \tilde{p}_{i,j} = L_{i,j} \cap E(p_{i,j})$;
- or $\tilde{\sigma}_i(x)$ is the intersection point of $L_{i,j}$ and $E(q_i)$, and in that case x is the intersection of $E(p_{i,j})$ with the conic D_i (see below § 7.2.4).

Otherwise, $\tilde{\sigma}_i(x)$ is a point of $L_{i,j}$ that does not belong to the exceptional set $\text{Exc}(\pi_i)$.

7.2.3. *Blow-ups of points of C .* Now, let $r \in C$ be distinct from the base points of σ_i . The line (qr) is σ_i -invariant, and σ_i acts as $z \mapsto -z$ on it, by fixing $0 \simeq r$ and $\infty \simeq r'$, where r' is the third point of intersection of (qr) with C . Thus, when r is blown up, the action of $\tilde{\sigma}_i$ on the exceptional divisor $E(r)$ is an involution with exactly two fixed points: a fixed point corresponding to the tangent line $T_r C$, and a fixed point corresponding to the line (qr) (or more precisely to $T_r(qr)$).

Remark 7.1. By (Hyp2), q_i does not coincide with q_l or a $p_{l,j}$ when $l \neq i$. Thus, the intersection point between $E(q_l)$ and the strict transform of $(q_l p_{l,j})$ is never fixed by $\tilde{\sigma}_i$: it is mapped to another point of $E(q_l)$.

7.2.4. *Action on Néron-Severi, and the conic D_i .* According to [12], the action of $\tilde{\sigma}_i$ on the Néron-Severi group of X_i is given, in the basis $(\mathbf{e}_0, \mathbf{e}_q, \mathbf{e}_{p_1}, \dots, \mathbf{e}_{p_4})$, by the matrix

$$(7.10) \quad \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ -2 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

For instance, the second column means that the exceptional divisor $E(q_i)$ is mapped to the strict transform \tilde{D}_i of a plane conic D_i which goes through the points q_i and $p_{i,j}$ with multiplicity 1; its class in $\text{NS}(X_i; \mathbf{Z})$ is $2\mathbf{e}_0 - \mathbf{e}_q - \mathbf{e}_{p_1} - \dots - \mathbf{e}_{p_4}$. This conic intersects C in 6 points, counted with multiplicity, and \tilde{D}_i intersects C_{X_i} in exactly one point, which must be fixed under the action of $\tilde{\sigma}_i$. On the other hand, since $\tilde{\sigma}_i$ fixes C_{X_i} pointwise, the point

$$(7.11) \quad \tilde{q}_i := E(q_i) \cap C_{X_i}$$

must be fixed under the action of $\tilde{\sigma}_i$ ⁽⁹⁾. Since $\tilde{\sigma}_i(E(q_i)) = \tilde{D}_i$, we conclude that $\tilde{D}_i \cap C_{X_i} = \tilde{q}_i$, thus D_i is tangent to C at q_i . This argument also shows that the only points of $E(q_i)$ which are mapped into the exceptional set of π_i (resp. π) are \tilde{q}_i and the intersection points of $E(q_i)$ with the $L_{i,j}$: the point $E(q_i) \cap L_{i,j}$ is mapped to $L_{i,j} \cap E(p_{i,j})$, a point which corresponds to the tangent direction of D_i at $p_{i,j}$ (cf. § 7.2.2).

We already stated without proof the fact that Γ is isomorphic to the free product of the $\tilde{\sigma}_i$. Blanc obtains this result by using (7.10) and proving that the action of Γ on $\text{NS}(X; \mathbf{Z})$ is faithful and its image is such a free product $\mathbf{Z}/2\mathbf{Z} * \dots * \mathbf{Z}/2\mathbf{Z}$. Thus we get:

Lemma 7.2. *When $k \geq 3$, the group $\Gamma \subset \text{Aut}(X)$ is non-elementary.*

7.3. Invariant curves ($k \geq 3$). With the matrix from Equation (7.10) at hand, we deduce that all fixed points of $\tilde{\sigma}_i^*$ acting on $\text{NS}(X_i)$ (resp. $\text{NS}(X_i; \mathbf{R})$) are of the form

$$(7.12) \quad u = d\mathbf{e}_0 - (d - 2m)\mathbf{e}_{q_i} - m \sum_{j=1}^4 \mathbf{e}_{p_{i,j}}$$

for some pair of integers (d, m) (resp. pair of real numbers (d, m)). From now on we set

$$(7.13) \quad \Sigma_i := \sum_{j=1}^4 \mathbf{e}_{p_{i,j}} \in \text{NS}(X_i; \mathbf{Z}) \quad (\text{resp. } \in \text{NS}(X; \mathbf{Z})).$$

Lemma 7.3. *If $k \geq 3$ and (Hyp1) and (Hyp2) are satisfied, the only (reduced) effective curve $U \subset X$ which is invariant under the action of Γ is the curve C_X .*

Proof. Let $u \in \text{NS}(X)$ be the class of an invariant curve U . Then by Equation (7.12) its expression in the basis $(\mathbf{e}_0, \mathbf{e}_{q_1}, \dots, \mathbf{e}_{q_k}, \mathbf{e}_{p_{1,1}}, \dots, \mathbf{e}_{p_{1,4}}, \mathbf{e}_{p_{2,1}}, \dots, \mathbf{e}_{p_{k,4}})$ is of the form

$$(7.14) \quad u = d\mathbf{e}_0 - (d - 2m_1)\mathbf{e}_{q_1} - \dots - (d - 2m_k)\mathbf{e}_{q_k} - m_1\Sigma_1 - \dots - m_k\Sigma_k$$

for some integers d and m_i . The strict transform of a general line through q_i , whose class is $\mathbf{e}_0 - \mathbf{e}_{q_i}$, must intersect U non-negatively. This implies that $m_i \geq 0$. Similarly, $d \geq 0$ because a general line intersects U non-negatively.

The equality $[C_X] = 3\mathbf{e}_0 - \mathbf{e}_{q_1} - \Sigma_1 - \dots - \mathbf{e}_{q_k} - \Sigma_k$ implies

$$(7.15) \quad U \cdot C_X = -(k - 3)d - 2(m_1 + \dots + m_k).$$

Now, assuming that C_X is not an irreducible component of U , we obtain

$$(7.16) \quad 0 \leq -(k - 3)d - 2(m_1 + \dots + m_k).$$

⁹We abuse notation for convenience and use similar notations (like $L_{i,j}$, $\tilde{p}_{i,j}$, \tilde{q}_i) for objects defined in X and X_i . For instance, \tilde{q}_i also stands for $\tilde{q}_i := E(q_i) \cap C_X$.

Since $k \geq 3$, we infer that $k = 3$ and $m_i = 0$ for all i , therefore $u = d\mathbf{e}_0 - d(\mathbf{e}_{q_1} + \cdots + \mathbf{e}_{q_k})$. From this it follows that the intersection of U with the strict transform of a general line through q_i is 0 for all i . This means that U is the strict transform of a plane curve of degree d and is mapped to a point by each of the fibrations η_{q_i} (see §7.2). This is a contradiction, so C_X must be a component of U ; in particular, $\overline{U \setminus C_X}$ is also invariant. Repeating this argument with $U \setminus C_X$ finishes the proof. \square

7.4. Parabolic automorphisms. From now on, let us add the following hypothesis

(Hyp3) For any $i \neq j$, the line $(q_i q_j)$ does not contain any of the points $p_{l,m}$ for $1 \leq l \leq k$, $1 \leq m \leq 4$, nor q_n for $1 \leq n \leq k$ and $n \neq i, j$.

The strict transform $M_{i,j}$ of the line $(q_i q_j)$ is invariant under $\Gamma_{i,j} := \langle \sigma_i, \sigma_j \rangle$. Furthermore this line $(q_i q_j)$ intersects C in a third point and (Hyp3) assures that this point is none of the points that we blow-up.

Assume for a few lines that $k = 2$, so that X is obtained by blowing-up the points q_r and $p_{r,j}$ for $r = 1, 2$. Then $[M_{1,2}] = \mathbf{e}_0 - \mathbf{e}_{q_1} - \mathbf{e}_{q_2}$ and $M_{1,2} \cdot C_X = 1$. The class $u_{1,2}$ of $C_X + M_{1,2}$ satisfies

$$(7.17) \quad u_{1,2} = 4\mathbf{e}_0 - 2\mathbf{e}_{q_1} - 2\mathbf{e}_{q_2} - \Sigma_1 - \Sigma_2;$$

it is invariant under $\Gamma_{1,2}$, i.e. under $\tilde{\sigma}_1^*$ and $\tilde{\sigma}_2^*$, and it is isotropic.

Lemma 7.4. *Assume $k = 2$ and the three hypotheses (Hyp1-Hyp3) are satisfied.*

- (1) *If v is the class of an \mathbf{R} -divisor with non-negative self intersection and v is invariant under $\tilde{\sigma}_1^*$ and $\tilde{\sigma}_2^*$ then v is a multiple of $u_{1,2}$.*
- (2) *The composition $g_{1,2} = \sigma_1 \circ \sigma_2$ is a parabolic automorphism of X that preserves the isotropic class $u_{1,2}$.*
- (3) *The invariant genus 1 pencil of $g_{1,2}$ is given by the pencil of plane quartic curves going through the $p_{i,j}$ with multiplicity 1 and through q_i with multiplicity 2, for $i = 1, 2$.*

Before proving this lemma, let us state the following immediate corollary.

Corollary 7.5. *If $k \geq 2$ and the hypotheses (Hyp1-Hyp3) are satisfied, then*

- (1) *each of the automorphisms $g_{i,j} = \tilde{\sigma}_i \circ \tilde{\sigma}_j$, $i \neq j$, is parabolic;*
- (2) *the invariant genus 1 fibration of $g_{i,j}$ corresponds to the linear system of plane quartics going through q_i and q_j with multiplicity 2 and through the $p_{i,l}$ and $p_{j,l}$ with multiplicity 1 (for $l = 1, \dots, 4$).*

Proof of Lemma 7.4. Every invariant class can be written as $v = d\mathbf{e}_0 - (d - 2m_1)\mathbf{e}_{q_1} - m_1\Sigma_1 - (d - 2m_2)\mathbf{e}_{q_2} - m_2\Sigma_2$. These classes form a 3-dimensional subspace of $\text{NS}(X; \mathbf{R})$ on which the intersection form is non-positive and degenerate: its kernel is generated by $u_{1,2}$. So, if $v^2 \geq 0$, v is proportional to $u_{1,2}$; more precisely, $m_1 = m_2$ and $v = m_1 u_{1,2}$.

In the group $\langle \tilde{\sigma}_1, \tilde{\sigma}_2 \rangle$, $g_{1,2}$ generates a cyclic, normal subgroup of index 2. Thus, the fixed point set of $g_{1,2}^*$ in $\text{NS}(X; \mathbf{R})$ is invariant under $\langle \tilde{\sigma}_1, \tilde{\sigma}_2 \rangle$. If $g_{1,2}$ were elliptic, this set of fixed points would intersect the set of classes $v \in \mathbb{H}_X \subset \text{NS}(X; \mathbf{R})$ on a non-empty convex subset F of the hyperbolic space \mathbb{H}_X . This convex set F would be invariant under the action of $\tilde{\sigma}_1^*$, and

the involution $\tilde{\sigma}_1^*$ would have a fixed point in F ; such a fixed point would also be fixed by $\tilde{\sigma}_2^*$ because $\tilde{\sigma}_2 = \tilde{\sigma}_1 \circ g_{1,2}$; thus, it is fixed by $\langle \tilde{\sigma}_1^*, \tilde{\sigma}_2^* \rangle$, in contradiction with the first assertion. We conclude that $g_{1,2}$ is not elliptic. Since it preserves the isotropic class $u_{1,2}$, it is parabolic¹⁰.

Let us now prove the third property. Set $\mathcal{M}_{1,2} = \mathcal{O}(C_X + M_{1,2})$ and $\mathcal{L}_{1,2} = \mathcal{O}(M_{1,2})$ (viewed as line bundles or as sheaves on X). Since X is rational we have $\chi(\mathcal{O}_X) = 1$. Since $M_{1,2}$ is a smooth rational curve with $M_{1,2}^2 = -1$, the Riemann-Roch theorem gives

$$(7.18) \quad h^0(X, \mathcal{L}_{1,2}) - h^1(X, \mathcal{L}_{1,2}) + h^2(X, \mathcal{L}_{1,2}) = 1.$$

We have $h^0(X, \mathcal{L}_{1,2}) = 1$ because $M_{1,2}$ is irreducible and has negative self-intersection, and $h^2(X, \mathcal{L}_{1,2}) = 0$ by Serre duality; thus, $h^1(X, \mathcal{L}_{1,2}) = 0$. Now, looking at the restriction of $\mathcal{M}_{1,2}$ to C_X , we get the long exact sequence

$$(7.19) \quad H^0(X, \mathcal{L}_{1,2}) \rightarrow H^0(X, \mathcal{M}_{1,2}) \rightarrow H^0(C_X, \mathcal{M}_{1,2}|_{C_X}) \rightarrow H^1(X, \mathcal{L}_{1,2}) \rightarrow \dots$$

By what we know of the $H^i(X, \mathcal{L}_{1,2})$ this gives

$$(7.20) \quad \mathbf{C} \rightarrow H^0(X, \mathcal{M}_{1,2}) \rightarrow H^0(C_X, \mathcal{M}_{1,2}|_{C_X}) \rightarrow 0.$$

Let us identify C_X to C via the projection $X \rightarrow \mathbb{P}_{\mathbf{C}}^2$ and fix an inflexion point o of C ; the divisor obtained by intersecting a line with C is equivalent to $3o$. In restriction to $C_X \simeq C$, $\mathcal{M}_{1,2}$ is given by the divisor $12o - 2q_1 - 2q_2 - p_{1,1} - \dots - p_{2,4}$, of degree 0. According to § 7.2.4, there is a conic D_1 tangent to C at q_1 and passing through the $p_{1,j}$; thus, $2q_1 + p_{1,1} + \dots + p_{1,4} = 6o$ on C ; similarly, $2q_2 + p_{2,1} + \dots + p_{2,4} = 6o$. Hence $12o - 2q_1 - 2q_2 - p_{1,1} - \dots - p_{2,4} = 0$, which means that $\mathcal{M}_{1,2}|_{C_X}$ is the trivial line bundle $\mathcal{O}(C_X)$. Thus, $h^0(C_X, \mathcal{M}_{1,2}|_{C_X}) = 1$ and Equation (7.20) gives $h^0(X, \mathcal{M}_{1,2}) = 2$.

In other words, the linear system $|C_X + M_{1,2}|$ is a pencil of curves. The general member D of this linear system is irreducible, because otherwise C_X or $M_{1,2}$ would be a fixed component, then we could write $D = D_0 + C_X$ (resp. $D_0 + M_{1,2}$) for some movable curve D_0 , but since $C_X^2 = M_{1,2}^2 = -1$, this curve D_0 would simultaneously satisfy $D_0^2 = -1$, a contradiction. Now, since the self-intersection of $C_X + M_{1,2}$ is 0, the elements of $|C_X + M_{1,2}|$ are disjoint and form a fibration. Finally, since they intersect trivially C_X and $C_X = -k_X$, the genus formula shows that they have genus 1. \square

8. FINITE ORBITS AND INVARIANT MEASURES

We keep notation as in the previous sections. In § 8.2, we prove that when $k \geq 4$, under general assumptions on the q_i , every orbit of Γ outside C_X is infinite. Then we discard the possibility of Γ -invariant measures in § 8.3. This relies heavily on the results of [24, 25].

8.1. Finite orbits outside C_X : a finiteness result. By a **generalized Kummer surface**, we mean a desingularization of a quotient A/G where A is an Abelian surface, G is a subgroup of $\text{Aut}(A)$, and the set of points $x \in A$ with a non-trivial stabilizer in G is a finite subset F_G of A (see [25, 27]).

¹⁰Alternatively, one can compute the product of the matrices for $\tilde{\sigma}_1^*$ and $\tilde{\sigma}_2^*$, and check that some power of it is unipotent, but not the identity.

Proposition 8.1. *The surface X is not a generalized Kummer surface.*

Proof. Assume, by contradiction, that X is a generalized Kummer surface. Then, there exists a birational morphism $\varepsilon: X \rightarrow X_0$ onto a singular surface X_0 , an Abelian surface A , and a finite subgroup G of $\text{Aut}(A)$ as above, such that X_0 is isomorphic to A/G ; the singularities of X_0 correspond to the finite set $F_G \subset A$ (see [24, §4]). In particular, the singularities of X_0 are of quotient type, and the genus 1 curve C_X cannot be contracted by ε ; set $C_{X_0} = \varepsilon(C_X)$ and let C_A be the preimage of C_{X_0} in A under the quotient map $\eta: A \rightarrow A/G = X_0$. As a consequence, the meromorphic 2-form Ω_X induces a meromorphic 2-form Ω_{X_0} on the regular part of X_0 , hence a meromorphic 2-form $\eta^*\Omega_{X_0}$ on $A \setminus F_G$; this 2-form has poles along C_A because the quotient map η is a local isomorphism on $C_A \setminus F_G$. Let Ω_A be a holomorphic 2-form on A with $\int_A \Omega_A \wedge \overline{\Omega_A} = 1$. Then, there is a holomorphic function $\varphi: A \setminus F_G \rightarrow \mathbf{C}$ such that $\Omega_A = \varphi \eta^* \Omega_{X_0}$. By the Hartogs theorem, this function φ extends holomorphically to A ; and because it vanishes along C_A , it is identically zero. This contradiction shows that X is not a Kummer surface. \square

Proposition 8.2. *Suppose $k \geq 3$. If C and the q_i are defined over $\overline{\mathbf{Q}}$, then Γ has at most finitely many finite orbits in $X(\mathbf{C}) \setminus C_X$.*

Proof. By Lemma 7.4, the group Γ contains parabolic elements. If the curve C and the points q_i are defined over $\overline{\mathbf{Q}}$, then X and the $\tilde{\sigma}_i$ are all defined over some number field K . Since X is not a Kummer surface, (X, Γ) is not a Kummer group and Theorem B of [24] implies that the union $\text{Per}(\Gamma)$ of all finite orbits of Γ is not Zariski dense. Let $\overline{\text{Per}(\Gamma)}$ be its Zariski closure: it is made of a one-dimensional part $\text{Per}^1(\Gamma)$ that contains C_X , together with a sporadic finite set $\text{Per}^0(\Gamma)$. Since $\text{Per}^1(\Gamma)$ is Γ -invariant, Lemma 7.3 shows that $\text{Per}^1(\Gamma) = C_X$. This concludes the proof. \square

8.2. Finite orbits outside C_X : non-existence. By (Hyp2), q_i does not coincide with q_l or a $p_{l,j}$ with $l \neq i$. Thus, the points of intersection between $E(q_l)$ and the strict transform $L_{l,j}$ of $(q_l p_{l,j})$ is never fixed by $\tilde{\sigma}_i$: it is mapped to another point of $E(q_l)$ (see § 7.2.3). To get more rigidity, let us add a stronger hypothesis

(Hyp4) For $i \neq l$, the involution $\tilde{\sigma}_i$ acts on $E(q_l)$, by mapping the four points corresponding to the directions $(q_l p_{l,j})$ to four other points (i.e. $(\tilde{\sigma}_i)_* T_{q_l}(q_l p_{l,j}) \neq T_{q_l}(q_l p_{l,j'})$ for any i , $l \neq i$, and $1 \leq j, j' \leq 4$).

To check that this condition is satisfied for a general choice of points, note that for fixed q_l and varying q_i , $\tilde{\sigma}_i$ acts upon $E(q_l)$ as an involution fixing a fixed point (corresponding to the line $T_{q_l}C$) and a mobile one (corresponding to the line $(q_l q_i)$). Fix a coordinate on $E(q_l)$ in which $T_{q_l}C = \infty$. Then the induced involution is of the form $z \mapsto -z + 2c(q_i)$, where $c(q_i)$ is the coordinate of the mobile point. For fixed a and b in $E(q_l)$, taken among the points $L_{l,j} \cap E(q_l)$, the relation $\tilde{\sigma}_i(a) = b$ reads $a + b = 2c(q_i)$; such a relation is not satisfied for a general choice of q_i .

Proposition 8.3. *Let $C \subset \mathbb{P}_{\overline{\mathbf{Q}}}^2$ be a smooth cubic curve, and let $k \geq 4$ be an integer. Consider the set \mathcal{H}_k of k -tuples $(q_1, q_2, \dots, q_k) \in C(\overline{\mathbf{Q}})$ satisfying hypotheses (Hyp1) to (Hyp4). Fix such*

a k -tuple (q_1, q_2, \dots, q_k) ; then, for $q'_k \in C(\overline{\mathbf{Q}})$ outside a finite set, the pair (X, Γ) determined by $(q_1, q_2, \dots, q_{k-1}, q'_k)$ does not have any finite orbit, except for the points of C_X , which are fixed.

Lemma 8.4. *Assume $k \geq 2$. If x does not belong to C_X , there exists an element f of Γ such that $\pi(f(x)) \notin C$.*

Proof. Observe that if $\pi(\Gamma(x)) \subset C$, then $\Gamma(x) \subset \text{Exc}(\pi) \setminus C_X$. If $x \notin \text{Exc}(\pi)$, we are done. Otherwise x belongs to some $E(p_{i,j})$, or to some $E(q_i)$.

In the first case, assume for concreteness that $x \in E(p_{1,1})$ and set $y = \tilde{\sigma}_1(x)$. By § 7.2.2, if $y \in \text{Exc}(\pi)$, then y is the intersection point of the strict transform $L_{1,1}$ of the line $(p_{1,1}q_1)$ either with $E(q_1)$ or with $E(p_{1,1})$. The first possibility does not happen because $L_{1,1} \cap E(q_1)$ is in C_X and is fixed by $\tilde{\sigma}_1$ (see § 7.2.1). In the second possibility, by (Hyp4), $\tilde{\sigma}_2(y)$ is not on one of the strict transforms of the lines $(q_1p_{1,l})$, thus by § 7.2.2 again, $\tilde{\sigma}_1(\tilde{\sigma}_2(y)) \notin \text{Exc}(\pi)$ and we are done.

In the second case, assume for concreteness that $x \in E(q_1)$ and, again, set $y = \tilde{\sigma}_1(x)$. By § 7.2.4, y belongs to \tilde{D}_1 . If $y \in \text{Exc}(\pi)$, then either $y \in \tilde{D}_1 \cap E(p_{1,j})$ for some j , and we are back to the previous case, or $y \in \tilde{D}_1 \cap E(q_1)$ and then $x = \tilde{q}_1 \in C_X$ is fixed, contradicting our assumptions. \square

Proof of Proposition 8.3. Consider the subgroup Γ_3 of Γ generated by $\tilde{\sigma}_1, \tilde{\sigma}_2$, and $\tilde{\sigma}_3$. By Proposition 8.2, Γ_3 has only finitely many finite orbits outside C_X . By the previous lemma, each of these is the orbit $\Gamma(x_j)$ of some $x_j \notin C_X \cup \text{Exc}(\pi)$. Let F be the union of these finite orbits $\Gamma(x_j)$, and set $F' = F \setminus (C_X \cup \text{Exc}(\pi))$. For a general choice of $q_k \in C$, the lines $(q_k x)$ for $x \in \pi(F')$ are pairwise distinct, and x does not belong to the lines $(q_k p_{k,j})$, for $1 \leq j \leq 4$, nor to the conic D_{q_k} from § 7.2.4 (see Lemma 8.5 below). Thus, $\sigma_k(\pi(F')) \cap \pi(F') = \emptyset$, and the orbit of $\tilde{\sigma}_k(z)$ under Γ_3 is infinite for any z in F' . This shows that Γ does not have any finite orbit, except for its fixed point set, which coincides with C_X . \square

Lemma 8.5. *Fix $x \in \mathbb{P}^2 \setminus C$. Then for general $q \in C$, x does not belong to the conic D_q .*

Proof. Indeed, suppose that x is in D_q for every q in some Zariski dense open subset U of C . By continuity, the same holds for $U = C$. Choose $q \in C$ such that (xq) is tangent to C at q and q is not an inflexion point of C ¹¹. By § 7.2.4, D_q is tangent to C , hence to (xq) , at q , so $D_q \cap (xq) = \{q\}$. This is a contradiction. \square

¹¹Such a point always exists. Otherwise, looking at the linear projection from C to $\mathbb{P}^1 \simeq \mathbb{P}(T_x \mathbb{P}^2)$, which is a ramified cover of degree 3, one sees that there would be three inflexion lines meeting in x . So, assume p, q, r are inflexion points with $T_p C \cap T_q C \cap T_r C = \{x\}$. Put C in Weierstrass form $x_2^2 x_3 = x_1^3 + b x_1 x_3^2 + c x_3^3$, with $r = [0 : 1 : 0]$, $T_r C = \{x_3 = 0\}$, and $x = [1 : d : 0]$ for some $d \in \mathbf{C}$. The lines passing through x have equations $x_2 = d x_1 + e x_3$. Such a line intersects C when

$$x_1^3 - d^2 x_1^2 x_3 + (b - 2de)x_1 x_3^2 + (c - e^2)x_3^3 = 0,$$

and it is an inflexion line when this cubic equation coincides with $(x_1 - \tau x_3)^3$ for some triple root τ . This implies that $3\tau = d^2$, $3\tau^2 = b - 2de$, and $e = (d^4 - 3b)/6d$, which means that e is determined by d ; in other words, there is at most one inflexion line of slope d (plus the line at infinity), and it is impossible, for a smooth cubic curve C , to have three inflexion lines with a common point.

8.3. Invariant measures.

Theorem 8.6. *Fix a smooth cubic curve $C \subset \mathbb{P}^2$ defined over $\overline{\mathbf{Q}}$. Assume that $k \geq 4$, and consider a k -tuple of points $(q_1, q_2, \dots, q_k) \in C(\overline{\mathbf{Q}})$ satisfying hypotheses (Hyp1) to (Hyp4). Then, for $q'_k \in C(\overline{\mathbf{Q}})$ outside a proper real analytic curve, the pair (X, Γ) determined by $(q_1, q_2, \dots, q_{k-1}, q'_k)$ does not have any invariant probability measure except for the probability measures supported on the fixed point set C_X .*

Proof. We have to show that every ergodic, invariant probability measure is supported on C_X . According to Lemma 7.3, there is no invariant curve except C_X . Thus, from Proposition 8.3, it suffices to exclude the existence of invariant measures giving no mass to proper Zariski closed subsets. Let μ be such a measure. By Theorem A of [25], there are two possibilities:

- (1) either μ is supported on a totally real, real analytic subset $\Sigma \subset X$, and μ is absolutely continuous with respect to the 2-dimensional Lebesgue measure on Σ , with a real analytic density along the smooth locus of Σ ;
- (2) or the support of μ is the whole surface X , and μ is absolutely continuous with respect to any smooth volume form, with a positive and real analytic density outside some invariant algebraic subset Z .

Let us exclude the second case. We already know that C_X is the only invariant algebraic subset, hence $\mu = \varphi \text{vol}_X^\infty$ for some real analytic function $\varphi: X \setminus C_X \rightarrow \mathbf{R}_+$. The ergodicity of μ and the invariance of vol_X^∞ imply that φ is constant. But then, by Property (3) of § 7.1, $\mu(X) = \infty$, which is a contradiction.

To rule out the first case, we argue as follows. The real (singular) surface Σ is invariant under the action of $\Gamma_{k-1} = \langle \tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_{k-1} \rangle$, and supports an invariant probability measure with a smooth density. According to Theorem C of [25], there are only finitely many surfaces of this type. We denote by $\Sigma_{k-1} \subset X$ the union of these real analytic subsets: it is the maximal, 2-dimensional, real analytic subset of X that supports a Γ_{k-1} -invariant probability measure with full support. To conclude, it suffices to show that, after perturbation of q_k , the surface Σ_{k-1} is not $\tilde{\sigma}_k$ -invariant.

For this, denote by z a smooth point of Σ_{k-1} with $\pi(z) \notin C$. As r varies along C , the point $\sigma_r(\pi(z))$ describes a complex algebraic curve. This curve cannot be contained in $\pi(\Sigma_{k-1})$, because Σ_{k-1} is totally real. Thus, the set $B_{k-1} = \{r \in C; \sigma_r(z) \in \Sigma_{k-1}\}$ is a proper real analytic subset of C . Then, we choose a point $q'_k \in C \setminus B_{k-1}$ such that (q_1, \dots, q'_k) satisfies (Hyp1) to (Hyp4) and the conclusion of Proposition 8.3, and we are done. \square

Remark 8.7. Since the proof of the theorem goes by breaking down all possible invariant totally real surfaces, the argument does not apply to the real case. Another argument will be given for the real setting in Theorem 10.3 below.

We believe that under the assumptions of Theorem 8.6, every stationary measure is invariant. More precisely, every ergodic stationary measure should have both Lyapunov exponents zero, therefore be invariant. In the next sections, we establish this result for some real examples.

9. REAL CONSTRUCTION

In this section, we construct examples for which X and the σ_i are defined over the real numbers, $X(\mathbf{R})$ is obtained by blowing-up 12 points of $\mathbb{P}^2(\mathbf{R})$ (while $X(\mathbf{C})$ is obtained by blowing up 20 points), and the action of Γ on the 1-dimensional homology of $X(\mathbf{R})$ admits a positive Lyapunov exponent.

9.1. Topology of real rational surfaces. Let $X_{\mathbf{R}}$ be a real projective surface obtained by blowing up a points r_i of $\mathbb{P}^2(\mathbf{R})$, as well as b pairs of complex conjugate points $\{s_j, \bar{s}_j\} \subset \mathbb{P}^2(\mathbf{C}) \setminus \mathbb{P}^2(\mathbf{R})$. We shall look at the first homology group of $X(\mathbf{R})$ with integral or rational coefficients. For simplicity, consider some homogeneous coordinate system $[x : y : z]$ on $\mathbb{P}^2_{\mathbf{R}}$, and suppose that the points r_i are not contained in the line at infinity $L_{\infty} = \{z = 0\}$: they are contained in the affine plane $\mathbb{A}^2(\mathbf{R})$ of points $[x : y : 1]$ with $(x, y) \in \mathbf{R}^2$. We endow this plane \mathbf{R}^2 with the usual, counterclockwise, orientation.

For each index i , we denote by U_i a small disk centered at r_i , and we orient its boundary $C_i := \partial U_i$ in the clockwise direction. We suppose that the U_i are pairwise disjoint. The exceptional divisor E_i obtained by blowing up r_i is a projective line. Then $E_i(\mathbf{R})$ is a circle, that we orient in such a way that $C_i = 2E_i$ modulo homotopy (see Figure 5). Let V be the complement of the U_i in \mathbf{R}^2 , with the orientation induced by the orientation of \mathbf{R}^2 . We orient L_{∞} in such a way that $\partial V = 2L_{\infty} + \sum_i C_i$. Then in $H^1(X(\mathbf{R}), \mathbf{Z})$,

$$2\ell_{\infty} + 2 \sum_i \mathbf{e}_i = 0$$

where $\mathbf{e}_i = [E_i(\mathbf{R})]$ is the homology class of $E_i(\mathbf{R})$ and $\ell_{\infty} = [L_{\infty}(\mathbf{R})]$ is the homology class of $L_{\infty}(\mathbf{R})$. Then, $H^1(X(\mathbf{R}), \mathbf{Z})$ is isomorphic to $\mathbf{Z}^a \oplus \mathbf{Z}/2\mathbf{Z}$, and $H^1(X(\mathbf{R}), \mathbf{Q})$ is isomorphic to \mathbf{Q}^a . More precisely, a basis of $H^1(X(\mathbf{R}), \mathbf{Q})$ is provided by the classes $(\mathbf{e}_1, \dots, \mathbf{e}_a)$.

If L is any line in $\mathbb{P}^2(\mathbf{R})$ which is not vertical, we orient L from left to right; in other words, one can parametrize L by $x \in \mathbf{R} \mapsto [x : \alpha x + \beta : 1]$ for some α, β in \mathbf{R} , and this parametrization is compatible with orientations. Letting U_L^+ be the open half-plane above L , its boundary in $\mathbb{P}^2(\mathbf{R})$ is made of L and the line at infinity. In $X(\mathbf{R})$, this gives

$$(9.1) \quad \ell_{\infty} + \ell + 2 \sum_{r_i \in U_L^+} \mathbf{e}_i + \sum_{r_j \in L} \mathbf{e}_j = 0,$$

(see Figure 6 for a local picture of a blow-up at a boundary of a domain) hence

$$(9.2) \quad \ell = \sum_{r_i \text{ strictly below } L} \mathbf{e}_i - \sum_{r_i \text{ strictly above } L} \mathbf{e}_i$$

in $H^1(X(\mathbf{R}); \mathbf{Q})$. This formula works even if some of the r_i are contained in L .

9.2. Action of one Jonquières involution. Consider a smooth real cubic curve $C \subset \mathbb{P}^2_{\mathbf{R}}$ such that $C(\mathbf{R})$ is connected. We assume that C is in Weierstrass form $y^2 = x^3 + ux^2 + vx + w$ and we orient $C(\mathbf{R})$ from bottom to top, i.e. from negative values of y to positive ones.

Let q be a point of $C(\mathbf{R})$ which is not an inflexion point. Let σ be the Jonquières involution associated to (C, q) , as in Sections 7.1 and 7.2 above. Besides q , the four remaining base points

p_j of σ are made of two real points p_1, p_2 and two complex conjugate points $\{p_3, p_4 = \overline{p_3}\}$; here, we shall assume that the position of q, p_1 , and p_2 are as on Figure 4 below.

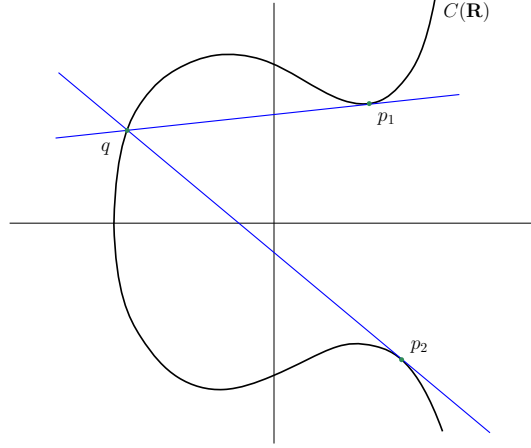


FIGURE 4.

Denote by X the surface obtained by blowing up the base points of σ , by $\pi: X \rightarrow \mathbb{P}^2$ the natural morphism, and by $\tilde{\sigma}$ the automorphism $\pi^{-1} \circ \sigma \circ \pi$ of X . To describe the action of $\tilde{\sigma}$ on $H^1(X(\mathbf{R}); \mathbf{Q})$, we use the following facts:

— the involution $\tilde{\sigma}$ permutes E_{p_1} and the strict transform $L_{qp_1} \subset X$ of the line $(qp_1) \subset \mathbb{P}^2(\mathbf{R})$. More precisely, L_{qp_1} , E_{p_1} , and C_X have a unique common point \tilde{p}_1 . This point is fixed by $\tilde{\sigma}$ and the differential $D\tilde{\sigma}_{\tilde{p}_1}$ can be described with the help of Figure 5. There is a basis of $T_{\tilde{p}_1}X$ given by vectors u and v which are respectively tangent to E_{p_1} and L_{qp_1} and are compatible with their orientations; moreover, after scaling v by some positive factor, we may assume that $-u + v$ is tangent to $C_X(\mathbf{R})$ (and is compatible with its orientation). Then, the matrix of $D\tilde{\sigma}_{\tilde{p}_1}$ in this basis is

$$(9.3) \quad D\tilde{\sigma}_{\tilde{p}_1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Thus, $\tilde{\sigma}(E_1(\mathbf{R})) = -L_{qp_1}(\mathbf{R})$, where the minus sign means that the orientation is reversed. In homology, this gives

$$(9.4) \quad \tilde{\sigma}^* \mathbf{e}_{p_1} = -\ell_{qp_1} = -\mathbf{e}_{p_2}.$$

Here, for the second equality we used formula (9.2) and the fact that the unique point outside (qp_1) is the point p_2 which is below it. When we shall blow up more points, extra terms will be added.

— the picture is different at p_2 (because the concavity of C is reversed), and we obtain

$$(9.5) \quad \tilde{\sigma}^* \mathbf{e}_{p_2} = \ell_{qp_2} = -\mathbf{e}_{p_1}.$$

— the “image” of q by σ is the conic D_q that goes through q, p_1, p_2 , and the points $\{p_3, p_4 = \overline{p_3}\}$. It is tangent to C at q ; its real part is an ellipse, which we orient in the clockwise direction. Now,

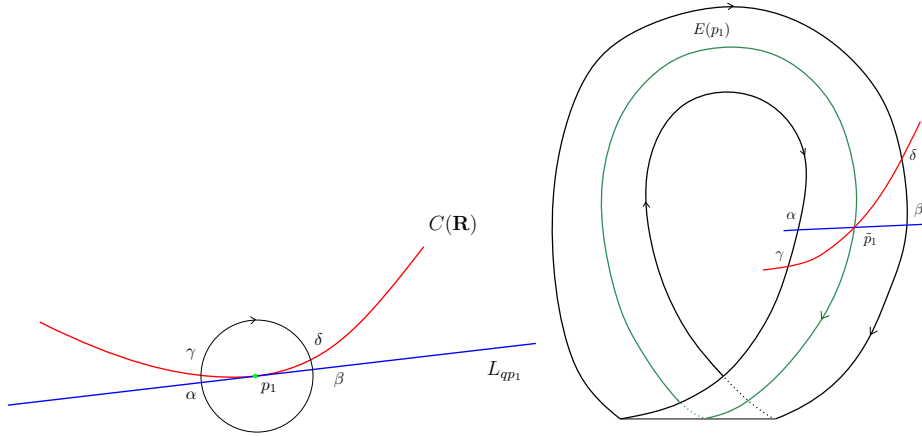


FIGURE 5. This Möbius band is obtained from a blow-up of a small disk around p_1 ; its boundary is the preimage of the circle bounding this disk. On the right, the green curve is the exceptional divisor $E(p_1)$; the blue line is the strict transform of (qp_1) ; the red curve is the strict transform of C . The involution $\tilde{\sigma}$ fixes the point of intersection of these three curves, permuting the green and blue curves. The orientations are the ones defined previously.

$D_q(\mathbf{R})$ bounds a disk Ω , more precisely, the boundary of Ω is $-D_q(\mathbf{R})$ (i.e. $D_q(\mathbf{R})$ but with the anti-clockwise direction). Taking the preimage of Ω in $X(\mathbf{R})$, this gives

$$(9.6) \quad \partial\Omega = -D_q(\mathbf{R}) + E_{p_1}(\mathbf{R}) + E_{p_2}(\mathbf{R}) + E_q(\mathbf{R})$$

$$(9.7) \quad [D_q(\mathbf{R})] = \mathbf{e}_q + \mathbf{e}_{p_1} + \mathbf{e}_{p_2}$$

and then we obtain

$$(9.8) \quad \tilde{\sigma}^* \mathbf{e}_q = \mathbf{e}_q + \mathbf{e}_{p_1} + \mathbf{e}_{p_2}.$$

Altogether, in the basis $(\mathbf{e}_q, \mathbf{e}_{p_1}, \mathbf{e}_{p_2})$ of $H^1(X(\mathbf{R}); \mathbf{Q})$, the matrix for $\tilde{\sigma}^*$ is

$$(9.9) \quad \tilde{\sigma}^* = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

9.3. Action of three Jonquières involutions. We now move on to the case when three involutions σ_i are considered, each of them attached to a point q_i of $C(\mathbf{R})$. We suppose that the relative position of the points q_i and $p_{i,j}$ are in the following order along C (from bottom to top):

$$(9.10) \quad p_{1,2}, p_{2,2}, p_{3,2}, q_3, q_2, q_1, p_{1,1}, p_{2,1}, p_{3,1};$$

to obtain such a configuration, start with q_1 as in Figure 4, then choose $q_2 \in C$ slightly below q_1 and q_3 slightly below q_2 . Now, X is the blow up of the plane at the fifteen points $q_i, p_{i,j}$, $1 \leq i \leq 3, 1 \leq j \leq 4$, and the lifts of the σ_i to X are denoted $\tilde{\sigma}_i$.

To compute the action of $\tilde{\sigma}_1$ on $H^1(X(\mathbf{R}); \mathbf{Q})$, we remark that the relative positions of the q_i and $p_{i,j}$, impose the following properties:

– the points $p_{2,1}$ and $p_{3,1}$ are above the line $(q_1 p_{1,1})$, and the points $p_{1,2}$, $p_{2,2}$, $p_{3,2}$, q_3 , and q_2 are below it. Thus,

$$(9.11) \quad \tilde{\sigma}_1^* \mathbf{e}_{p_{1,1}} = -\mathbf{e}_{p_{1,2}} - \mathbf{e}_{q_2} + \mathbf{e}_{p_{2,1}} - \mathbf{e}_{p_{2,2}} - \mathbf{e}_{q_3} + \mathbf{e}_{p_{3,1}} - \mathbf{e}_{p_{3,2}}.$$

– similarly,

$$(9.12) \quad \tilde{\sigma}_1^* \mathbf{e}_{p_{1,2}} = -\mathbf{e}_{p_{1,1}} + \mathbf{e}_{q_2} - \mathbf{e}_{p_{2,1}} + \mathbf{e}_{p_{2,2}} + \mathbf{e}_{q_3} - \mathbf{e}_{p_{3,1}} + \mathbf{e}_{p_{3,2}}.$$

– the ellipse $D_{q_1}(\mathbf{R})$ bounds an open set V_1 that contains q_2 , q_3 , $p_{2,2}$, and $p_{3,2}$ in its interior. Thus, if we cut out small disks centered at those four points from V_1 , and take its boundary in $X(\mathbf{R})$, we obtain the equality

$$(9.13) \quad [D_{q_1}(\mathbf{R})] = \mathbf{e}_{q_1} + \mathbf{e}_{p_{1,1}} + \mathbf{e}_{p_{1,2}} + 2\mathbf{e}_{q_2} + 2\mathbf{e}_{p_{2,2}} + 2\mathbf{e}_{q_3} + 2\mathbf{e}_{p_{3,2}}.$$

– if r is a point from $C(\mathbf{R})$ that is not one of q_1 , $p_{1,1}$, or $p_{1,2}$, and if one blows up that point, the curve E_r is fixed by the lift $\tilde{\sigma}_1$. The line $(q_1 r)$ and its strict transform $L_{q_1, r}$ are also invariant. Since r is not one of the $p_{1,j}$, $(q_1 r)$ and C intersect transversely at r : along C , σ_1 is the identity, and along $(q_1 r)$, σ_1 is conjugate to $z \mapsto -z$, fixing r and another point r' of C . Thus, on the blow up $E_r(\mathbf{R})$, we see that $\tilde{\sigma}_1$ reverses the orientation (see Figure 6). This gives

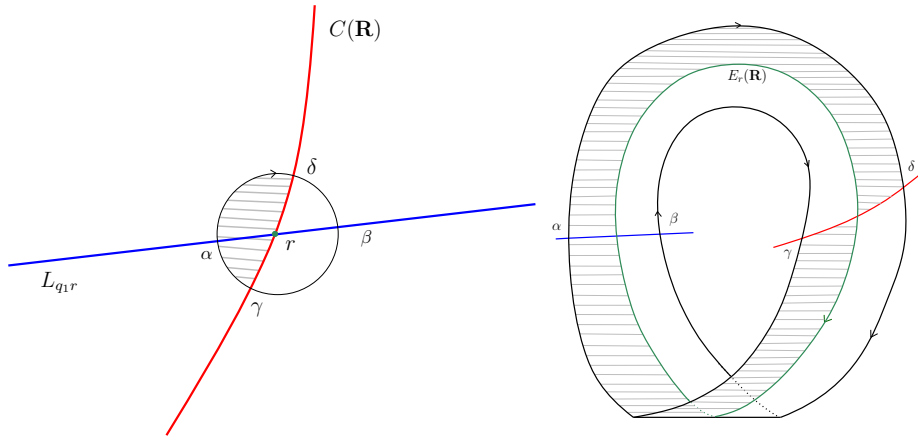


FIGURE 6. The involution exchanges the hatched and the plain sides of C , so in the blow-up it reverses the orientation of $E_r(\mathbf{R})$

$$(9.14) \quad \tilde{\sigma}_1^*(\mathbf{e}_r) = -\mathbf{e}_r$$

for $r \in \{q_2, p_{2,1}, p_{2,2}, q_3, p_{3,1}, p_{3,2}\}$.

Altogether, in the basis $(\mathbf{e}_{q_1}, \mathbf{e}_{p_{1,1}}, \mathbf{e}_{p_{1,2}}, \mathbf{e}_{q_2}, \mathbf{e}_{p_{2,1}}, \mathbf{e}_{p_{2,2}}, \mathbf{e}_{q_3}, \mathbf{e}_{p_{3,1}}, \mathbf{e}_{p_{3,2}})$ of $H^1(X(\mathbf{R}); \mathbf{Q})$, we obtain the following matrices for $\tilde{\sigma}_1^*$

$$(9.15) \quad \tilde{\sigma}_1^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Similarly, we obtain

$$(9.16) \quad \tilde{\sigma}_2^* = \begin{pmatrix} -1 & 0 & 0 & 2 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 1 & 0 & 0 & -1 \end{pmatrix},$$

$$(9.17) \quad \tilde{\sigma}_3^* = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

9.4. Positive Lyapunov exponent. Let Γ^* be the image of $\Gamma = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ in $\mathrm{GL}(H^1(X; \mathbf{Z}))$. This group preserves a three dimensional subspace, on which it acts by multiplication by ± 1 . The quotient is given by the linear map

$$(9.18) \quad (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \mapsto (x_1, x_2 - x_3, x_4, x_5 - x_6, x_7, x_8 - x_9).$$

On the quotient space, which we denote by V , the involutions act by the following matrices.

$$(9.19) \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & -1 & 0 \\ -2 & 2 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$(9.20) \quad A_2 = \begin{pmatrix} -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 \\ 0 & 0 & -2 & 2 & 0 & -1 \end{pmatrix}.$$

$$(9.21) \quad A_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We shall denote by $\bar{\Gamma}^*$ the subgroup of $\mathrm{SL}_6(\mathbf{Z})$ generated by these involutions, and by G the Zariski closure of $\bar{\Gamma}^*$ in SL_6 .

Lemma 9.1. *The following properties are satisfied:*

- (1) *The group $\bar{\Gamma}^*$ contains a non-Abelian free group.*
- (2) *The group G is semi-simple.*
- (3) *The action of $\bar{\Gamma}^*$ on $V \otimes \mathbf{C}$ is strongly irreducible.*
- (4) *If ν is a probability measure on Γ that satisfies Conditions (S) and (M), then ν has a positive Lyapunov exponent on $H^1(X(\mathbf{R}); \mathbf{R})$.*

From this lemma and the calculation of the spectral radius

Proof. We shall need the following facts (computations of characteristic polynomials and Galois groups were done with sagemath):

(a) The element $f = A_1 A_2 A_3$ has characteristic polynomial

$$(9.22) \quad P_f(t) = t^6 - 4t^5 - 3t^4 - 2t^3 + 5t^2 + 2t + 1$$

whose factorization in $\mathbf{Q}[t]$ is $P(t) = (t - 1) \times (t^5 - 3t^4 - 6t^3 - 8t^2 - 3t - 1)$. In particular, f has six distinct eigenvalues, and only one of them is rational, namely 1. The only other real eigenvalue is $\lambda_f \simeq 4.679$, then there are two complex conjugate eigenvalues of modulus strictly between 1 and λ_f , and two of modulus < 1 . Moreover, for every $k \geq 1$, f^k has also 6 distinct eigenvalues (see below an argument for the similar case of g).

(b) The element $g = A_1A_2A_3A_2A_1A_3A_1A_3A_1A_2A_3A_2A_1A_2A_3$ has characteristic polynomial

$$(9.23) \quad P_g(t) = t^6 - 24t^5 - 83t^4 - 122t^3 - 35t^2 + 22t + 1$$

which is irreducible in $\mathbf{Q}[t]$. In particular, g has six distinct eigenvalues. Four of them are real, with a single one of maximal absolute value $\lambda_g > 1$, and there are two complex conjugate eigenvalues of modulus > 1 . Moreover, for every $k \geq 1$, g^k has also 6 distinct eigenvalues.

To prove this last point, it suffices to show that λ_g^k is an algebraic number of degree 6 for every $k \geq 1$. For this, one computes the Galois group of the splitting field F of P_g : it is isomorphic to the symmetric group \mathfrak{S}_6 , so $[F : \mathbf{Q}] = 6!$. The degree 6 extension $\mathbf{Q}(\lambda_g)$ of \mathbf{Q} is the subfield of F fixed by a subgroup $H \subset \mathfrak{S}_6$ of index 6. Note that λ_g^k can not be rational for any $k \geq 1$, because since λ_g is a unit, λ_g^k would be equal to 1, in contradiction with $\lambda_g > 1$. Thus, if λ_g^k had degree < 6 , there would be an intermediate extension $K \subset \mathbf{Q}(\lambda_g)$ of degree $d = 2$ or 3 . This extension would be the fixed field of a group $G \subset \mathfrak{S}_6$ of index d . For $d = 3$ such a group does not exist. For $d = 2$ we get $G = \mathfrak{A}_6$ and \mathfrak{A}_6 would contain H as a subgroup of index 3; this is a contradiction, since the largest maximal subgroup of \mathfrak{A}_6 has index 6.

(c) The eigenvector of f corresponding to the eigenvalue 1 is defined over \mathbf{Q} . Thus, it is in general position with respect to the eigenvectors of g (i.e. it is not contained in any proper g -invariant subspace of V , because such a subspace would be rational, thereby producing a rational factor of P_g)

To show that $\bar{\Gamma}^*$ contains a non-Abelian free group, note that the eigenvector of f for the leading eigenvalue $\lambda^+(f) \simeq 4.679$ is not mapped to another eigenvector of f by A_1 . Thus by the ping-pong lemma, if we set $h = A_1 \circ f \circ A_1^{-1} = A_2A_3A_1$, then the group generated by f and h contains a non-Abelian free subgroup of $\text{GL}(V)$ all of whose elements $\neq \text{Id}$ have an eigenvalue > 1 (see Lemma 3.1). So, at this stage we know that Γ is non-elementary.

Consider the connected component of the identity $G^o \subset G$. The intersection $\bar{\Gamma}_0^* := \bar{\Gamma}^* \cap G^o(\mathbf{C})$ is a finite index subgroup of $\bar{\Gamma}^*$ and is a Zariski dense subgroup of G^o . Since $\bar{\Gamma}^*$ is contained in $\text{SL}_m(\mathbf{Z})$, the linear algebraic groups G and G^o are both defined over \mathbf{Q} . Let R be the solvable radical of G^o and let U be its unipotent radical (see [74], Chap. 6.h, page 135); they are defined over \mathbf{Q} and are characteristic subgroups of G^o ; in particular, they are normal in G . Let $F \subset V$ be the fixed point set of U . This vector subspace is defined over \mathbf{Q} , its dimension is positive, and it is g -invariant. Since the characteristic polynomial of g is irreducible over \mathbf{Q} , we infer that $F = V$ and $U = \{\text{id}_V\}$. This implies that R is a torus; over \mathbf{C} , R is diagonalizable and $R(\mathbf{C})$ isomorphic to $(\mathbf{C}^\times)^r$ for some $r \leq 5$. The group G acts by conjugacy on R ; since the automorphism group of R is discrete (isomorphic to $\text{GL}_r(\mathbf{Z})$) and G^o is connected, we deduce that R is central in G^o . In particular, R commutes with f^k and with g^k if k is chosen to insure that f^k and g^k are in G^o . Thus, R being connected, each of the sixth complex eigenlines of f^k is R -invariant, and the same holds for the eigenlines of g^k . By Property (c) above, this implies that R is made of homotheties, and since $R \subset \text{SL}_6$ we deduce that R is trivial. Thus G^o is semi-simple, and so is G .

By the first property, $G^o(\mathbf{R})$ is not bounded, so at least one of its semisimple factors is a (non-Abelian) non-compact almost simple real Lie group.

Now, $V \otimes \mathbf{C}$ is a direct sum of irreducible representations V_i of G^o . Choose $k \geq 1$ such that $f^k \in G^o$. One of the V_i , say V_1 , must contain the eigenline of f^k corresponding to the eigenvalue 1; thus, V_1 contains an element of $V(\mathbf{Q})$, and since $\bar{\Gamma}_0^*$ is Zariski dense in G^o and is defined over \mathbf{Z} , we deduce that, in fact, V_1 is defined over \mathbf{Q} . By Property (b), $V_1 = V$, and G^o acts irreducibly on V . Thus, $\bar{\Gamma}_0^*$ acts strongly irreducibly on V , and so does $\bar{\Gamma}^*$.

Finally, since the action of $\bar{\Gamma}^*$ on V is strongly irreducible and unbounded, Theorem 3.31 of [11] shows the positivity of the first Lyapunov exponent in V , hence in $H^1(X(\mathbf{R}); \mathbf{R})$, and the proof is complete. \square

Putting together all the results in this section, we have established the following result.

Theorem 9.2. *Let C be a smooth, real, plane cubic whose real part $C(\mathbf{R})$ is connected. There exists a non-empty open subset $U \subset C(\mathbf{R})^3$ such that if (q_1, q_2, q_3) belongs to U , and ν is a probability measure on the corresponding group $\Gamma = \langle \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3 \rangle$, generating Γ and with a finite first moment, then the top Lyapunov exponent of the action of Γ on $H^1(X(\mathbf{R}), \mathbf{R})$ is positive.*

We now extend this theorem to the case of 4 points instead of 3. So, we blow-up one more point $q_4 \in C(\mathbf{R})$, with q_4 between q_1 and q_3 , and we still denote by X the surface and by Γ the group generated by the 4 involutions. The cohomology group $H^1(X(\mathbf{R}); \mathbf{Q})$ of the new surface $X(\mathbf{R})$ has dimension 12, with coordinates (x_1, \dots, x_{12}) ; the subspace defined by

$$(9.24) \quad x_2 = x_3, x_5 = x_6, x_8 = x_9, x_{11} = x_{12}, x_1 = x_4 = x_7 = x_{10} = 0$$

is invariant, and the restriction of Γ to this subspace factors through a finite group. Let W be the quotient space, which is of dimension 8.

Choose an index $j \in \{1, 2, 3, 4\}$ and consider the subgroup Γ_j of Γ generated by the $\tilde{\sigma}_i$ with $i \neq j$. Then, there is a finite index subgroup Γ_j^o of Γ_j whose action on $W \otimes \mathbf{C}$ is reducible: there is an invariant subspace T_j of dimension 2 (on which Γ_j acts diagonally with eigenvalues equal to ± 1), and the quotient space W/T_j is a strongly irreducible representation V_j , defined over \mathbf{Q} , and of dimension 6 (it is isomorphic to the representation V studied in Lemma 9.1). Pick g_j in Γ_j^o satisfying Property (b) of the proof of Lemma 9.1. Then g_j preserves a unique subspace W_j of W of dimension 6, defined over \mathbf{Q} , which projects surjectively onto V_j .

Now, take a finite index subgroup Γ' of Γ and let $K \subset W$ be a Γ' -invariant subspace. Changing the Γ_j^o into finite index subgroups, we may assume that they preserve K . Projecting K to V_j , Lemma 9.1 implies that either K is contained in T_j (and then $\dim(K) \leq 2$), or that K is mapped onto V_j (and then $\dim(K) \geq 6$). If K is contained in one of the T_j , one checks that its projection onto some other V_i is non-trivial, which gives a contradiction. It follows that the projection of K onto each V_j is surjective. Since K is g_j -invariant, it contains W_j , and finally we obtain that $K = W$. Thus, the action of Γ on W is strongly irreducible, and we get:

Theorem 9.3. *Let C be a smooth, real, plane cubic whose real part $C(\mathbf{R})$ is connected. There exists a non-empty open subset $U \subset C(\mathbf{R})^4$ such that if (q_1, q_2, q_3, q_4) belongs to U , and ν is a probability measure on the corresponding group $\Gamma = \langle \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\sigma}_4 \rangle$, generating Γ and with a finite first moment, then the top Lyapunov exponent of the action of Γ on $H^1(X(\mathbf{R}), \mathbf{R})$ is positive.*

10. DYNAMICS ON THE REAL SURFACE

In this section we complete the proof of Theorem 1.9.

10.1. Preliminaries from ergodic theory.

Proposition 10.1. *Let X be a complex projective surface (resp. a real projective surface). Let Γ be a non-elementary subgroup of $\text{Aut}(X)$ (resp. of $\text{Aut}(X_{\mathbf{R}})$) containing a parabolic element. Then the action of Γ is ergodic with respect to the Lebesgue measure on X (resp. on $X(\mathbf{R})$).*

Proof. This result is contained in [18, 25] but it was not explicitly stated there, so we provide details in the case of complex surfaces. We only deal with the case where X is not Abelian, in which case Γ is automatically countable. We closely follow [25, §4] and freely use the vocabulary from that paper. Let $A \subset X$ be a measurable subset such that $\text{vol}_X(A\Delta\gamma^{-1}A) = 0$, where vol_X is the probability measure associated to some Kähler form on X . Replacing A by $\bigcap_{\gamma \in \Gamma} \gamma^{-1}A$, we may assume that A is Γ -invariant. Assuming that $\text{vol}_X(A) > 0$, we have to show that $\text{vol}_X(A) = 1$. As in [25, §4.3.1], Γ contains a “special” subgroup $\langle g, h \rangle$, freely generated by two independent Halphen twists. Denote by $\pi_g, \pi_h : X \rightarrow \mathbb{P}^1$ the associated invariant fibrations. Recall that for almost every $w \in \mathbb{P}^1$, the action of g (resp. h) on $\pi_g^{-1}(w)$ (resp. $\pi_h^{-1}(w)$) is uniquely ergodic, and the unique invariant measure is the Haar measure. Since A is g -invariant, there is a subset $B_g \in \mathbb{P}^1$ of positive measure such that $\text{vol}_X(A\Delta\pi_g^{-1}(B_g)) = 0$. Consequently A intersects any π_h -fiber on a set of positive Haar measure, and then using the h -invariance, $A = X$ up to a set of volume 0. \square

Proposition 10.2. *Let X be a compact Kähler surface. Let ν be a probability measure on $\text{Aut}(X)$ that satisfies the moment condition (M) and let μ be an ergodic ν -stationary measure. Let η be a non-trivial meromorphic 2-form on X such that*

- (i) $\int \log^+ |\text{Jac}_\eta(f)(x)| d\mu(x) d\nu(f) < +\infty$;
- (ii) μ gives zero mass to the set of zeroes and poles of η .

Then the Lyapunov exponents of μ satisfy:

$$\lambda^- + \lambda^+ = \int \log(|\text{Jac}_\eta f(x)|^2) d\mu(x) d\nu(f).$$

In particular, if η is invariant, $\lambda^- + \lambda^+ = 0$.

Before starting the proof, let us recall some notation from § 1.4: we denote $\Omega = \text{Aut}(X)^{\mathbf{N}}$ and F_+ the skew-product transformation of $\Omega \times X$ associated to the random dynamics, acting as the one-sided shift on Ω and by automorphisms on X . The measure $\nu^{\mathbf{N}} \times \mu$ is F_+ invariant. One may also consider the 2-sided shift $\vartheta : \text{Aut}(X)^{\mathbf{Z}} \rightarrow \text{Aut}(X)^{\mathbf{Z}}$ and the corresponding skew-product F on $\text{Aut}(X)^{\mathbf{Z}} \times X$, defined by $F(\xi, x) = (\vartheta\xi, f_0(x))$, where $\xi = (f_i)_{i \in \mathbf{Z}}$. The natural extension of $\nu^{\mathbf{N}} \times \mu$ will be denoted by m : it is invariant and ergodic and its projection on $\text{Aut}(X)^{\mathbf{Z}}$ is $\nu^{\mathbf{Z}}$. Beware that m differs from $\nu^{\mathbf{Z}} \times \nu$ unless μ is invariant.

Proof. Fix a Kähler metric κ_0 on X . Fix a trivialization of the tangent bundle TX , given by a measurable family of linear isomorphisms $L(x) : T_x X \rightarrow \mathbf{C}^2$ such that (a) $\det(L(x)) = 1$ and

(b) $1/c \leq \|L(x)\| + \|L(x)^{-1}\| \leq c$, for some constant $c > 1$; here, the determinant is relative to a volume form vol_X on X and the standard volume form on \mathbf{C}^2 , and the norm is with respect to $(\kappa_0)_x$ on $T_x X$ and the standard euclidean metric on \mathbf{C}^2 .

For $(\xi, x) \in \text{Aut}(X)^{\mathbf{Z}} \times X$ and $n \geq 0$, the differential $D_x f_\xi^n$ is expressed in this trivialization as a matrix

$$(10.1) \quad A^{(n)}(\xi, x) = L(f_\xi^n(x)) \circ D_x f_\xi^n \circ L(x)^{-1}.$$

Denote by $\chi_n^-(\xi, x) \leq \chi_n^+(\xi, x)$ the singular values of $A^{(n)}(\xi, x)$. Then m -almost surely, $\frac{1}{n} \log \chi_n^\pm(\xi, x)$ converges towards λ^\pm as n goes to $+\infty$.

The form $\eta \wedge \bar{\eta}$ can be written $\eta \wedge \bar{\eta} = \varphi(x) \text{vol}_X$ for some function $\varphi: X \rightarrow [0, +\infty]$. Locally, one can write $\eta = h(x) dx_1 \wedge dx_2$ where (x_1, x_2) are local holomorphic coordinates and h is a meromorphic function; then $\varphi(x) \text{vol}_X = |h(x)|^2 dx_1 \wedge dx_2 \wedge d\bar{x}_1 \wedge d\bar{x}_2$. The jacobian Jac_η satisfies

$$(10.2) \quad |\text{Jac}_\eta(f)(x)|^2 = \frac{\varphi(f(x))}{\varphi(x)} \text{Jac}_{\text{vol}}(f)(x)$$

for every $f \in \text{Aut}(X)$ and $x \in X$. Using $\det(L(x)) = 1$, we get

$$(10.3) \quad \det(A^{(n)}(\xi, x)) = \text{Jac}_{\text{vol}}(f_\xi^n)(x),$$

and then

$$(10.4) \quad \frac{1}{n} \log \chi_n^-(\xi, x) + \frac{1}{n} \log \chi_n^+(\xi, x) = \frac{2}{n} \log |\text{Jac}_\eta f_\xi^n(x)| - \frac{1}{n} \log(\varphi(f_\xi^n(x))/\varphi(x)).$$

By the Oseledets theorem, the left hand side of (10.4) converges almost surely to $\lambda^- + \lambda^+$. Since the Jacobian Jac_η is multiplicative along orbits, i.e. $\text{Jac}_\eta f_\xi^n(x) = \prod_{k=0}^{n-1} \text{Jac}_\eta f_{\vartheta^k \xi}(f_\xi^k x)$, the integrability condition and the ergodic theorem imply that, almost surely,

$$(10.5) \quad \begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log |\text{Jac}_\eta f_\xi^n(x)| &= \int \log |\text{Jac}_\eta f_\xi^1(x)| dm(\xi, x) \\ &= \int \log |\text{Jac}_\eta f_\omega^1(x)| dm_+(\omega, x) \\ &= \int \log |\text{Jac}_\eta f(x)| d\mu(x) d\nu(f). \end{aligned}$$

Let $\text{div}(\eta)$ be the set of zeroes and poles of η . Since μ is ergodic and does not charge $\text{div}(\eta)$, we deduce that for m -almost every (ξ, x) , there is a sequence (n_j) such that $f_\xi^{n_j}(x)$ stays at positive distance from $\text{div}(\eta)$; along such a sequence, $\log |\varphi(f_\xi^{n_j}(x))/\varphi(x)|$ stays bounded, and the right hand side of (10.4) tends to $2 \int \log |\text{Jac}_\eta f(x)| d\mu(x) d\nu(f)$. This concludes the proof. \square

10.2. Stiffness.

Theorem 10.3. *Let C be a smooth, real, plane cubic which is defined over $\overline{\mathbf{Q}}$ and whose real part $C(\mathbf{R})$ is connected. One can find four points q_1, q_2, q_3, q_4 in $C(\mathbf{R})$ such that the following properties hold. Let X be the Blanc surface constructed in § 7 and let $\Gamma = \langle \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\sigma}_4 \rangle \subset \text{Aut}(X_{\mathbf{R}})$ be the group generated by the four Jonquières involutions.*

Let ν be any probability measure on Γ which satisfies Conditions (S) and (M). Then, any ergodic ν -stationary measure on $X(\mathbf{R})$ is the Dirac mass δ_x at some point $x \in C_X(\mathbf{R})$. Such measures have vanishing Lyapunov exponents.

In fact, the q_i can be chosen in some explicit open subset of $C(\mathbf{R} \cap \overline{\mathbf{Q}})^4$.

Proof. We first choose the points (q_1, \dots, q_4) in $C(\mathbf{R} \cap \overline{\mathbf{Q}})$ such that the properties (Hyp1-4) hold and such that (q_1, q_2, q_3) satisfies the assumption of Theorem 9.2. Then we move q_4 to q'_4 as in Proposition 8.3 to make sure that the conclusions of this proposition are satisfied.

Step 1.— Here we show that every invariant ergodic probability measure is a Dirac mass δ_x for some $x \in C_X(\mathbf{R})$. The argument is similar to that of Theorem 8.6. First, the invariant meromorphic 2-form is real and induces a volume form $\text{vol}_{X(\mathbf{R})}^\infty$, which is not locally integrable along $C_X(\mathbf{R})$. By Lemma 7.3 and Proposition 8.3 every proper Zariski closed invariant subset is contained in C_X (hence fixed pointwise). Thus, by [25], if μ is any ergodic invariant measure on $X(\mathbf{R})$ then either μ is a Dirac mass on $C_X(\mathbf{R})$ or $\mu = \varphi \text{vol}_{X(\mathbf{R})}^\infty$, for some real analytic function φ on $X(\mathbf{R}) \setminus C_X(\mathbf{R})$. To exclude this second possibility, note that $C(\mathbf{R})$ is connected and has a unique point at infinity, so $\mathbb{P}^2(\mathbf{R}) \setminus C(\mathbf{R})$ is connected, and so is $X(\mathbf{R}) \setminus C_X(\mathbf{R})$. Then, the ergodicity of μ and the invariance of $\text{vol}_{X(\mathbf{R})}^\infty$ imply that φ is constant μ -almost everywhere; since φ is analytic, it must be locally constant, and since $X(\mathbf{R}) \setminus C_X(\mathbf{R})$ is connected, φ is constant; but then $\mu(X(\mathbf{R})) = \infty$, which is absurd.

Step 2.— Now, pick a probability measure ν on Γ satisfying (S) and (M). Let μ be an ergodic, ν -stationary measure on $X(\mathbf{R})$.

Lemma 10.4. *The measure μ cannot be absolutely continuous with respect to the Lebesgue measure on $X(\mathbf{R})$.*

Proof. Equivalently, let us show that μ is not absolutely continuous with respect to the invariant infinite volume $\text{vol}_{X(\mathbf{R})}^\infty$. For notational ease, in this proof we write X for $X(\mathbf{R})$ (actually the result holds for both the complex and the real variety). Reasoning by contradiction, we assume that there is a function $\xi: X \rightarrow \mathbf{R}_+$ such that $\xi \in L^1(X, \text{vol}_X^\infty)$ and $\mu = \xi \text{vol}_X^\infty$. The stationarity of μ and the invariance of vol_X^∞ under the action of Γ give

$$(10.6) \quad \xi = \int_{\Gamma} \xi \circ f^{-1} d\nu(f)$$

vol_X^∞ -almost everywhere. For $M \geq 0$, we set $\xi_M = \min(\xi, M)$ and obtain

$$(10.7) \quad \xi_M \geq \int_{\Gamma} \xi_M \circ f^{-1} d\nu(f)$$

because the minimum of two harmonic functions is subharmonic. Then

$$(10.8) \quad \int_X \xi_M d\text{vol}_X^\infty \geq \int_{\Gamma} \int_X \xi_M \circ f^{-1} d\text{vol}_X^\infty d\nu(f)$$

$$(10.9) \quad = \int_X \xi_M d\text{vol}_X^\infty$$

because vol_X^∞ is invariant and ξ_M is integrable. This shows that the Inequality (10.7) is in fact an equality vol_X^∞ -almost everywhere. Thus, ξ and ξ_M are both ν -harmonic.

By Markov's inequality, there exists $M > 0$ such that

$$(10.10) \quad 0 < \text{vol}_X^\infty(\{\xi > M\}) < +\infty.$$

Given such an M , the real number

$$(10.11) \quad \alpha := \int_{\{\xi > M\}} (\xi - M) d\text{vol}_X^\infty = \int_{\{\xi > M\}} (\xi - \xi_M) d\text{vol}_X^\infty$$

satisfies $0 < \alpha \leq 1$ because μ is a probability measure. If $\alpha < 1$, μ could be decomposed as a convex combination $\mu = \alpha\mu_M^+ + (1 - \alpha)\mu_M^-$, with

$$(10.12) \quad \mu_M^+ = \alpha^{-1}(\xi - \xi_M)\text{vol}_X^\infty \quad \text{and} \quad \mu_M^- = (1 - \alpha)^{-1}\xi_M\text{vol}_X^\infty.$$

Such a decomposition must be trivial, because the harmonicity of ξ_M and of ξ imply that μ_M^- and μ_M^+ are stationary measures, while μ is assumed to be ergodic. Thus, $\alpha = 1$, for any such M . This shows that there is measurable subset $A \subset X$ such that

- (a) $\text{vol}_X^\infty(A) \in]0, \infty[$;
- (b) $\xi = \text{vol}_X^\infty(A)^{-1}1_A$, where 1_A is the characteristic function of A .

The stationarity relation for ξ implies that A is $\text{Supp}(\nu)$ -almost invariant, hence Γ -invariant, up to a subset of zero volume. Proposition 10.1 then implies that $A = X(\mathbf{R})$ up to a subset of zero volume. This is a contradiction because μ is a probability measure and $\text{vol}_X^\infty(X) = +\infty$. \square

Step 3.— Let μ be an ergodic ν -stationary measure as in Step 2. Assume by way of contradiction that μ is not invariant.

A first observation is that $\mu(C_X(\mathbf{R})) = 0$, so μ is Zariski diffuse. By Proposition 10.2, $\lambda^+ + \lambda^- = 0$. Therefore, Crauel's invariance principle implies that μ must be hyperbolic. Thus, by Theorem 1.7 and [16, Thm. 3.4], μ is a fiberwise SRB measure. Specifically, this means that working in $\text{Aut}(X(\mathbf{R}))^{\mathbf{Z}} \times X(\mathbf{R})$, and considering a measurable partition \mathcal{P}^u subordinate to the family of local Pesin unstable manifolds, then for m -almost every (ξ, x) , the conditional $m_{(\xi, x)}^u := m(\cdot | \mathcal{P}^u(\xi, x))$ is absolutely continuous with respect to the Lebesgue measure on $W^u(\xi, x)$ (see [16] and [26, §7] for details on these concepts, and the paragraph following Proposition 10.2 for the notation).

The same construction can be done with a local stable partition \mathcal{P}^s to get stable conditional measures $m_{(\xi, x)}^s$. These conditional measures admit a pointwise Hausdorff dimension at (ξ, x) which is defined almost everywhere and is constant by ergodicity. We denote these dimensions by $\dim(\mu^{u/s})$. In this context, the analogue of the Ledrappier-Young formula holds¹² and asserts that for m -almost every (ξ, x) ,

$$(10.13) \quad h_\mu(X, \nu) = \lambda^+ \dim^u(\mu) = |\lambda^-| \dim^s(\mu),$$

where $h_\mu(X, \nu)$ is the fiber entropy (see Section 2 for a brief account on this notion). Since $\dim^u(\mu) = 1$ and $\lambda^+ = |\lambda^-|$, we conclude that $\dim^s(\mu) = 1$ as well. As in deterministic

¹²The first equality in formula (10.13) is proven in [76]. For random dynamical systems there is a dissymmetry between the future and the past, because F^{-1} is not the skew product map associated to a independent, identically distributed, random dynamical system. So we can not just consider F^{-1} to get the second inequality. Fortunately, only minor adaptations are required for this: they are described in the paragraphs following Theorem 2.1 in [72] (see also [65] for a unified discussion, with additional pointers to the literature

dynamics, this implies that μ^s is absolutely continuous with respect to the Lebesgue measure along stable manifolds¹³. Thus, m has absolutely continuous conditionals along both stable and unstable manifolds.

The next important property is the absolute continuity of the local stable and unstable laminations, which follows the lines of the classical deterministic case. A detailed treatment for the stable lamination is given in [71, Chap. III], and a unified treatment for the stable and unstable laminations (with less details) is in [65, Thm 2.2.12].

At this stage we can directly adapt Theorems 5.1 and 5.5 of [68], which implies that the conditionals of m along the fibers $\{\xi\} \times X(\mathbf{R})$ are almost surely absolutely continuous with respect to the Lebesgue measure, and we conclude that μ itself is absolutely continuous. But Lemma 10.4 asserts that this is impossible. This contradiction shows that μ is invariant. Thus, applying Step 2, we conclude that every ergodic stationary measure is a Dirac mass δ_x at some point $x \in C_X(\mathbf{R})$.

Step 4.– It remains to show that the point masses on C_X , viewed as stationary measures, have zero Lyapunov exponents. This is elementary: let $x \in X(\mathbf{R})$ (possibly among the \tilde{q}_i) and let (v_1, v_2) be a basis of $T_x X$ such that v_1 is tangent to C_X . Then $(\tilde{\sigma}_i)_*(v_1) = v_1$ so the matrix of $(\tilde{\sigma}_i)_*$ in the basis (v_1, v_2) is upper triangular, and the other eigenvalue of this upper-triangular matrix is -1 (cf. the proof of Formula (7.7)). So any product of such matrices is of the form $\begin{pmatrix} 1 & * \\ 0 & \pm 1 \end{pmatrix}$ and the result follows. \square

10.3. Orbit closures.

Theorem 10.5. *Let C, q_1, \dots, q_4, X and Γ be as in Theorem 10.3. Then every $x \in X(\mathbf{R}) \setminus C_X(\mathbf{R})$ has a dense orbit in $X(\mathbf{R})$.*

Proof. In [25], we defined an invariant algebraic subset STang_Γ , which is the union of the maximal invariant curve and a finite set, and we proved that for every $x \in X(\mathbf{C})$, either $\Gamma \cdot x$ is dense in $X(\mathbf{C})$, or $\text{Acc}(\Gamma \cdot x) \setminus \text{STang}_\Gamma$ is locally equal to some Γ -invariant real surface. In our situation, $\text{STang}_\Gamma = C_X$, because C_X is the maximal invariant curve and every orbit outside C_X is infinite. For $x \in X(\mathbf{R}) \setminus C_X(\mathbf{R})$, we deduce that if $\text{Acc}(\Gamma \cdot x) \setminus C_X(\mathbf{R})$ is non-empty, then it is open (and closed) in $X(\mathbf{R}) \setminus C_X(\mathbf{R})$, so by connectedness $\Gamma \cdot x$ is dense in $X(\mathbf{R})$. To sum up, all we need to show is that $\text{Acc}(\Gamma \cdot x) \setminus C_X(\mathbf{R}) \neq \emptyset$ for every $x \in X(\mathbf{R}) \setminus C_X(\mathbf{R})$. For this, we use the structure of the invariant fibrations of parabolic elements in Γ .

Step 1.– Geometry of the invariant fibration of $g_{ii'}$.

Fix two distinct indices, say $i = 1$ and $i' = 2$. In this step we study the geometry of the invariant fibration π_{12} of $g_{12} = \tilde{\sigma}_1 \circ \tilde{\sigma}_2$ near C_X .

Let us first work over \mathbf{C} . Recall from § 7.4 that the fibration comes from the pencil of quadrics going through q_1 and q_2 with multiplicity 2 and through the $p_{i,j}$ with multiplicity 1 for $i = 1, 2$. In the surface X_{12} obtained by blowing up these 10 points (as in Lemma 7.4), it corresponds to the linear system $|C_{X_{12}} + M_{12}|$, where M_{12} is the strict transform of the line $(q_1 q_2)$. We denote by $\pi_{12}: X_{12} \rightarrow \mathbb{P}^1$ this fibration, and fix an affine coordinate z on \mathbb{P}^1 such that $\pi_{12}(C_{X_{12}} \cup M_{12}) = 0$ and $\pi_{12}(X_{12}(\mathbf{R})) \subset \mathbb{P}^1(\mathbf{R})$.

¹³This is proven for the unstable direction in [71], and the adaptation to the stable direction is explained in [4].

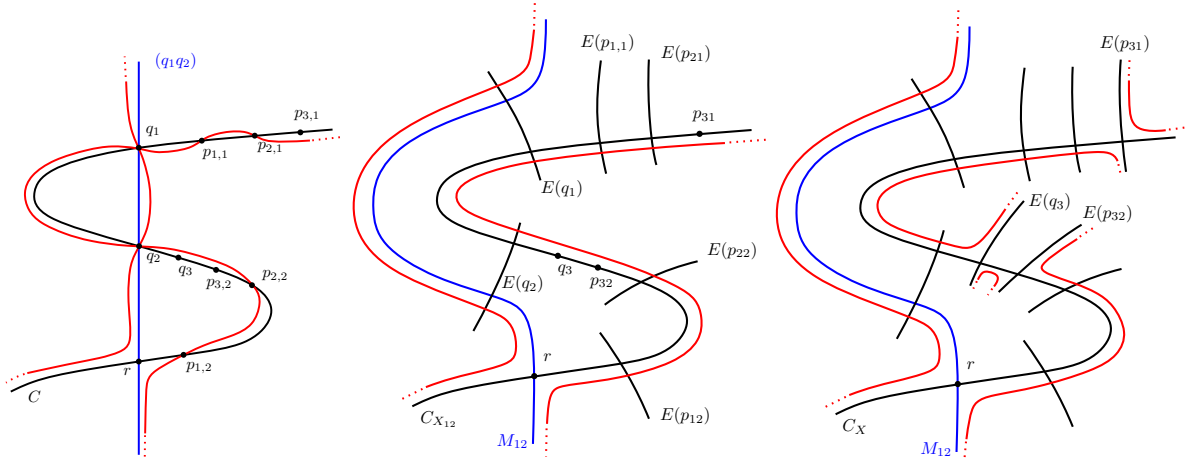


FIGURE 7. Schematic view of a real fiber of π_{12} (in red) close to the singular fiber on \mathbb{P}^2 (left), on X_{12} (middle) and on X (right). All components (including the red fiber, except on the left) are topological circles.

Let r be the third intersection point of C and $(q_1 q_2)$; we also denote by r its incarnation in X_{12} or X (because r is not blown up). It is the only intersection point of $C_{X_{12}}$ and M_{12} .

Now, M_{12} is a smooth rational curve with self-intersection -1 , so it can be blown down to get a new, smooth projective surface Y . The fibration π_{12} gives a genus 1 fibration $\pi_Y : Y \rightarrow \mathbb{P}^1$ and $C_{X_{12}}$ is a smooth fiber C_Y of π_Y . In a small tubular neighborhood of C_Y , the fibration π_Y is a submersion (otherwise, C_Y would be a multiple fiber, and C_X would have multiplicity > 1 in $C_{X_{12}} + M_{12}$). The curve M_{12} is contracted to a point $r_Y \in Y$, and M_{12} is the exceptional divisor of the blow-up of Y at r_Y . Thus, the geometry of $\pi_{12} : X_{12} \rightarrow \mathbb{P}^1$ near M_{12} is the geometry of a smooth foliation after a blow-up, and:

- (1) π_{12} has a Morse singularity at the point $r \in X_{12}$; there are local coordinates such that $\pi_{12}(x, y) = xy$, with the two coordinate axes corresponding to C_X and M_{12} , respectively;
- (2) if b is close to 0, the fiber $\pi_{12}^{-1}(b)$ is close to $C_{X_{12}} \cup M_{12}$; it is the pull-back of a smooth fiber of π_Y close to C_Y ; as b approaches 0, $\pi_{12}^{-1}(b)$ converges towards $C_{X_{12}} \cup M_{12}$ in the Hausdorff topology, and in the C^1 topology in the complement of r ;
- (3) in the real surface, $C_{X_{12}}(\mathbf{R})$ is a topological circle with a Möbius band as tubular neighborhood, and so is $M_{12}(\mathbf{R})$; the smooth fibers of π_{12} in $X(\mathbf{R})$ near $C_{X_{12}}(\mathbf{R}) \cup M_{12}(\mathbf{R})$ are topological circles, turning once around $C_{X_{12}}(\mathbf{R})$ and around $M_{12}(\mathbf{R})$.

If we blow-up the remaining points to construct $X(\mathbf{R})$, $C_{X_{12}}(\mathbf{R}) \cup M_{12}(\mathbf{R})$ is replaced by its total transform: we add 6 topological circles corresponding to the exceptional divisors obtained by blowing up $q_3, q_4, p_{3,1}, \dots, p_{4,2}$, and the picture near each of these circles is similar to the one near M_{12} (see Figure 7 for a visual illustration).

Step 2.– Dynamics of g_{12} on smooth fibers.

If we fix an affine coordinate z on (q_1q_2) such that $r = \infty$, $q_2 = 0$ and $q_1 = 1$, then $\sigma_2(z) = -z$ and $\sigma_1(z) = 2 - z$. So $\sigma_1 \circ \sigma_2(z) = z + 2$, that is, on M_{12} g_{12} acts as a parabolic transformation with fixed point at r .

For $b \in \mathbb{P}^1(\mathbf{R})$ near 0, denote by $Y_b(\mathbf{R})$ the real part of $\pi_Y^{-1}(b)$; thus, $Y_0(\mathbf{R})$ is the smooth fiber $C_Y(\mathbf{R})$. The complex curve $Y_b(\mathbf{C})$ is smooth of genus 1; it is a quotient \mathbf{C}/L_b , where $L_b = \mathbf{Z} + \mathbf{Z}\tau(b)$ is a lattice in \mathbf{C} and $Y_b(\mathbf{R})$ corresponds to \mathbf{R}/\mathbf{Z} (because $C_Y(\mathbf{R})$ and the fibers near it are connected). Thus, in a neighborhood N of 0, we get a real analytic map τ with values in the upper half plane, and, a map $\varphi_b: t \in \mathbf{R}/\mathbf{Z} \rightarrow \varphi_b(t) \in Y(\mathbf{R})$ parametrizing $Y_b(\mathbf{R})$, depending analytically on $(b, t) \in N \times \mathbf{R}/\mathbf{Z}$. Pulling back φ_b by the natural birational map $\varpi_{12}: X_{12} \rightarrow Y$ (resp. $X \rightarrow Y$), we get a family of parametrizations $\tilde{\varphi}_b$ of the fibers $X_{12,b}$ (resp. X_b) of π_{12} , for $b \in N \setminus \{0\}$. Since g_{12} acts by translation along the curves Y_b (resp. X_b , $b \neq 0$), and g_{12} is the identity on C_X , we see that φ_b conjugates $g_{12}|_{Y_b}$ to a rotation of the circle with angle $\alpha(b)$ converging towards 0 as b goes to 0.

Step 3.– Conclusion.

Pick $x \in X(\mathbf{R}) \setminus C_X(\mathbf{R})$ such that $\Gamma \cdot x$ accumulates $C_X(\mathbf{R})$; fix a sequence (x_j) of distinct points of $\Gamma \cdot x$ converging to $C_X(\mathbf{R})$, and set $b_j = \pi_{12}(x_j)$. Step 1 shows that $X_{b_j}(\mathbf{R})$ converges in the Hausdorff topology to $C_X \cup M_{12} \cup E_{q_3} \cup \dots \cup E_{p_{4,2}}$; this convergence holds in the C^1 topology away from the singular points. Take an arbitrary point $y_0 \in M_{12}(\mathbf{R}) \setminus \{r\}$ and consider the segment $[y_0, g_{12}(y_0)]$ in $M_{12}(\mathbf{R})$. By the description of g_{12} in the first paragraph of Step 2, $[y_0, g_{12}(y_0)]$ is disjoint from r . If I is a small interval containing y_0 and transverse to the fibration π_{12} , then the intervals I and $J := g_{12}(I)$ intersect each fiber X_b near M_{12} transversely into two points y_b and $g_{12}(y_b)$; as b goes to 0, the segment $[y_b, g_{12}(y_b)] \subset X_b(\mathbf{R})$ converges towards the segment $[y_0, g_{12}(y_0)] \subset M_{12}$.

The following lemma is elementary and left to the reader.

Lemma 10.6. *Let $R: \mathbf{R}/\mathbf{Z} \rightarrow \mathbf{R}/\mathbf{Z}$ be a rotation of angle $-1/2 < \alpha < 1/2$. Fix $t \in \mathbf{R}/\mathbf{Z}$ and let K be the shortest closed segment joining t to $R(t)$. Then for any $s \in \mathbf{R}/\mathbf{Z}$, the R -orbit of s intersects K .*

This lemma shows that the orbit of x_j under g_{12} must intersect $[y_{b_j}, g_{12}(y_{b_j})]$, hence so does $\Gamma \cdot x$. Taking $j \rightarrow \infty$, this implies that $\Gamma \cdot x$ accumulates $[y_0, g_{12}(y_0)]$, which is contained in $M_{12}(\mathbf{R}) \setminus C_X(\mathbf{R})$. As explained before Step 1, this completes the proof of the theorem. \square

10.4. Conclusion of the proof of Theorem 1.9. We pick C and the q_i as in Theorem 10.3. Assertion (1) follows from the smoothness of C and the genus formula, and Assertion (2) is a theorem of Blanc. Assertions (3) and (4) on Ω_X are described in Section 3. Property (5) follows from Proposition 8.3. Assertion (6) on the generic density of orbits is an elementary consequence of Proposition 10.1: indeed for any non-empty open set U , the set of $x \in X$ such that $(\Gamma \cdot x) \cap U = \emptyset$ has zero Lebesgue measure. The statement (7) on orbit closures is Theorem 10.5. The stiffness property (8) is established in Theorem 10.3, and finally, Assertion (9) is Theorem 9.3. \square

We conclude the paper with an open question. By Breiman's ergodic theorem, for every $x \in X(\mathbf{R}) \setminus C(\mathbf{R})$ and ν^N -almost every ω , any cluster value of the sequence of empirical measures $\frac{1}{n} \sum_{k=1}^n \delta_{f_\omega^k(x)}$ is a probability measure on C , a priori depending on x , ω and a choice of subsequence. The question is about the complexity of the set of limiting measures:

Question 10.7. *Which probability measures do arise in this way? Do the sequences of empirical measures typically converge or, on the contrary, does "historic behavior" occur?*

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