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Heteroclinic for a 6-dimensional reversible system occurring in orthogonal domain walls in convection

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Abstract

A six-dimensional reversible normal form system occurs in B nard-Rayleigh convection between parallel planes, when we look for domain walls intersecting orthogonally (see Buffoni et al [1]). We prove analytically the existence, local uniqueness, and analyticity in parameters, of a heteroclinic connection between two equilibria, each corresponding to a system of convective rolls. We prove that the 3-dimensional unstable manifold of one equilibrium, intersects transversally the 3-dimensional stable manifold of the other equilibrium, both manifolds lying on a 5-dimensional invariant manifold. We also study the linearized operator along the heteroclinic, allowing to prove (in another paper) the persistence under perturbation, of the heteroclinic obtained in [1].

Key words: Reversible dynamical systems, Invariant manifolds, Bifurcations, Heteroclinic connection, Domain walls in convection

1 Introduction and Results

Let us study the following reversible system in \mathbb{R}^6

$$\begin{aligned} A^{(4)} &= A(1 - A^2 - gB^2) \\ B'' &= \varepsilon^2 B(-1 + gA^2 + B^2), \end{aligned} \tag{1}$$

where the coordinates in \mathbb{R}^6 are $Z = (A_0, A_1, A_2, A_3, B_0, B_1) = (A, A', A'', A''', B, B')$. This system occurs in the search for domain walls intersecting orthogonally, in a fluid dynamic problem such as the B nard-Rayleigh convection between parallel horizontal plates (see subsection 1.1 and all details in [1]). The heteroclinic we are looking for, corresponds to the connection between rolls on one side and rolls oriented orthogonally on the other side.

We would like to find analytically a heteroclinic connection ($g > 1$, ε small) such that

$$\begin{aligned} A_*(x), B_*(x) &> 0, \\ (A_*(x), B_*(x)) &\rightarrow \begin{cases} (1, 0) \text{ as } x \rightarrow -\infty \\ (0, 1) \text{ as } x \rightarrow +\infty \end{cases}. \end{aligned}$$

By a variational argument Boris Buffoni et al [1] prove the existence of such a heteroclinic orbit, for any $g > 1$, and ε small enough. This type of elegant proof does not unfortunately allow to prove the persistence of such heteroclinic curve under reversible perturbations of the vector field. This is our motivation for producing analytic arguments, proving such an existence, uniqueness and smoothness in parameters (ε, g) of this orbit, however for limited values of g , fortunately including physical interesting ones. Then we study the linearized operator along the heteroclinic curve, allowing to attack the problem of existence of orthogonal domain walls in convection (forthcoming paper).

After some basic consideration on the system, a first part of the paper (sections 4, 5, 6) is devoted to the proof of Theorem 1. Then section 7, is devoted to the study of the linearized operator along the heteroclinic, some properties of which are necessary for the forthcoming proof of existence of orthogonal domain walls in convection.

We set $\delta = (g-1)^{1/2}$. The idea here might be to use the arc of equilibria $A^2 + B^2 = 1$, which exists for $\delta = 0$, connecting end points $M_- = (1, 0)$ and $M_+ = (0, 1)$, and to prove that for suitable values of δ , the 3-dimensional unstable manifold of M_- intersects transversally the 3-dimensional stable manifold of M_+ , both staying on a 5 dimensional invariant manifold \mathcal{W}_δ . However, for $\delta = 0$ the situation in M_+ is very degenerated, with a quadruple 0 eigenvalue for the linearized operator, while it is only a double eigenvalue in M_- . Then we are not able to prove, for δ close to 0, that the 3-dimensional unstable manifold of M_- exists from $B = 0$ until B reaches a value close enough to 1.

The strategy here consists to keep in mind that, after changing the coordinate x in $\bar{x} = \varepsilon x$, the limit $\varepsilon \rightarrow 0$ of the system (1) gives a non C^1 heteroclinic solution such that (i) for x running from $-\infty$ to 0, then (A_0, B_0) varies from $(1, 0)$ to $(0, \frac{1}{\sqrt{g}})$ on the ellipse $A_0^2 + gB_0^2 = 1$, while (ii) for x running from 0 to $+\infty$, then (A_0, B_0) varies from $(0, \frac{1}{\sqrt{g}})$ to $(0, 1)$ in satisfying the differential equation (see the first integral (3)).

$$B'_0 = \frac{\varepsilon}{\sqrt{2}}(1 - B_0^2).$$

The major difficulty in the proof of Theorem 1 is to prove the existence of the 3-dim unstable manifold of M_- until A_0 reaches a neighborhood of 0, and to prove the existence of the 3-dim stable manifold of M_+ until B_0 reaches a neighborhood of $1/\sqrt{g} = 1/\sqrt{1 + \delta^2}$. The usual proofs of existence of such invariant manifolds give only local results, so we need to use here a first integral of the system, expressing that both manifolds lie on a 5-dimensional invariant

manifold, and then we are able to extend sufficiently the domain of existence of these manifolds. Indeed we prove the following

Theorem 1 *Let us choose $0 < \delta_0 < 1/3$, then for $\delta_0 \leq \delta \leq 0.825$, η_0 such that for ε small enough with $\alpha = \varepsilon^{2/7}$, where $0 < \alpha = [(1 + \delta^2)\eta_0^2 - 1]^{1/2}$, the 3-dim unstable manifold of M_- intersects transversally the 3-dim stable manifold of M_+ . The connecting curve which is obtained is locally unique (it is the only curve for this intersection). Moreover its dependency in parameters (ε, δ) is analytic. In addition we have $B(x)$ and $B'(x) > 0$ on $(-\infty, +\infty)$, the principal part of $B(x)$ being given*

i) for $x \in (-\infty, 0]$, by

$$\begin{aligned} B_0(x) &= \frac{1}{(1 + \frac{\delta^2}{2})^{1/2} \cosh(x_0 - \varepsilon\delta x)}, \\ \cosh x_0 &= \frac{1}{B_{00}(1 + \frac{\delta^2}{2})^{1/2}}, \\ B_{00} &= B_0(0) = (1 - \eta_0^2\delta^2)^{1/2}, \end{aligned}$$

ii) for $x \in [0, +\infty)$, by

$$B_0(x) = \frac{\tanh(\varepsilon x/\sqrt{2}) + B_{00}}{1 + B_{00} \tanh(\varepsilon x/\sqrt{2})}.$$

For $x \rightarrow -\infty$ we have $(A_0 - 1, A_1, A_2, A_3, B_0, B_1) \rightarrow 0$ at least as $e^{\varepsilon\delta x}$, while for $x \rightarrow +\infty$, $(A_0, A_1, A_2, A_3) \rightarrow 0$ at least as $-\sqrt{\frac{\varepsilon}{2}}x$, and $(B_0 - 1, B_1) \rightarrow 0$ at least as $e^{-\sqrt{2}\varepsilon x}$.

Remark 2 *From the form of system (1) and Lemma 10 it should be noticed that*

$$A_0 - 1 \rightarrow 0 \text{ as } e^{2\varepsilon\delta x} \text{ while } B_0 \rightarrow 0 \text{ as } e^{\varepsilon\delta x} \text{ for } x \rightarrow -\infty.$$

In section 4 we prove at Lemma 10 the existence of the unstable manifold of $M_- = (1, 0)$ until a neighborhood of $(A_0, B_0) = (0, 1/\sqrt{1 + \delta^2})$. Here there is no restriction on the choice of δ , except $\delta \geq \delta_0 > 0$.

In section 5 we prove at Lemma 17 the existence of the stable manifold of $M_+ = (0, 1)$ until (backward direction) a neighborhood of $(0, 1/\sqrt{1 + \delta^2})$. Here there is a restriction $\delta \leq 0.95$, for being able to reach the end point.

In section 6 we need the restriction $\delta \leq 0.825$ for saving a transverse intersection of the two manifolds. This ends the proof of Theorem 1.

In section 7 we give in Lemma 23 the properties of the linearized operator along the heteroclinic, which are necessary to prove a persistence result under a reversible perturbation for the heteroclinic in the 8-dimension space (with $B \in \mathbb{C}$).

Remark 3 *It should be noticed that we show that, in the middle of the heteroclinic, $A_0(0) \sim \alpha\delta$ which is very close to 0. The choice of $\alpha = \varepsilon^{2/7}$ results from*

the proof of Theorem 1. If, one day, we are able to choose a much smaller α , for example $\alpha = \sqrt{\varepsilon}$ this would imply important improvements for the proof of orthogonal domain walls (however see Remark 16).

Remark 4 *Using symmetries of the system: $A \mapsto \pm A$, $B \mapsto \pm B$ and reversibility symmetry: $(A(x), B(x)) \mapsto (A(-x), B(-x))$, we find 8 heteroclinics. Two are connecting M_- to M_+ with opposite dynamics, two others connect $-M_-$ to M_+ , two connect M_- to $-M_+$, and two connect $-M_-$ to $-M_+$. The one which interests us is the only one connecting M_- to M_+ with the dynamics running from M_- to M_+ .*

1.1 Origin of system (1)

The Bénard-Rayleigh convection problem is a classical problem in fluid mechanics. It concerns the flow of a three-dimensional viscous fluid layer situated between two horizontal parallel plates and heated from below. Upon increasing the difference of temperature between the two plates, the simple conduction state loses stability at a critical value of the temperature difference corresponding to a critical value \mathcal{R}_c of the Rayleigh number. Beyond the instability threshold, a convective regime develops in which patterns are formed, such as convective rolls, hexagons, or squares. Observed patterns are often accompanied by defects.

Mathematically, the governing equations are the Navier-Stokes equations coupled with an equation for the temperature, and completed by boundary conditions at the two plates. Observed patterns are then found as particular steady solutions of these equations. Very recently, the existence of orthogonal domain walls has been studied by [1], where the authors handle the full governing equations, showing that the study leads to a small perturbation of the reduced system of amplitude equations (1).

Starting from a formulation of the steady governing equations as an infinite-dimensional dynamical system in which the horizontal coordinate x plays the role of evolutionary variable (spatial dynamics), a center manifold reduction is performed, which leads to a 12-dimensional reduced reversible dynamical system (reducing to 8-dimensional after restricting to solutions with reflection symmetry $y \rightarrow -y$). A normal form for this reduced system is obtained, for which, after an appropriate rescaling of the normal form, the principal part is the system (1), with $B \in \mathbb{C}$, and B^2 replaced by $|B|^2$. The truncation leading to (1) allows to take B real, since its phase does not play any role at this level. Solutions of the system (1) provide leading order approximations of solutions of the full governing equations. In particular, the equilibrium $(A_0, B_0) = (0, 1)$ of the system (1) gives an approximation of convection rolls (in the x direction) bifurcating for Rayleigh numbers $\mathcal{R} > \mathcal{R}_c$ close to \mathcal{R}_c , whereas the equilibrium $(A_0, B_0) = (1, 0)$ of the system (1) gives the same convection rolls (in the y direction) rotated by an angle $\pi/2$ with the phase fixed by the imposed reflection symmetry. A heteroclinic orbit connecting these two equilibria provides then an approximation of orthogonal domain walls (see Figure 1). The parameter ε in

(1) is such that ε^4 is proportional to $\mathcal{R}^{1/2} - \mathcal{R}_c^{1/2}$. The parameter $g > 1$ in (1) is function of the Prandtl number, while other parameters, which only appear in higher orders, are the wave numbers of the rolls, close to the critical value.

Remark 5 Values of δ such that $0.476 \leq \delta$ include values obtained for δ in the Bénard-Rayleigh convection problem where g is function of the Prandtl number \mathcal{P} (see [3]). With rigid-rigid, rigid-free, or free-free boundaries the minimum values of g are respectively ($g_{\min} = 1.227, 1.332, 1.423$) corresponding to $\delta_{\min} = 0.476, 0.576, 0.650$. The restriction in Theorem 1 corresponds to $1 < g \leq 1.680$. The eligible values for the Prandtl number are respectively $\mathcal{P} > 0.7404, > 0.9125, > 1.332$.

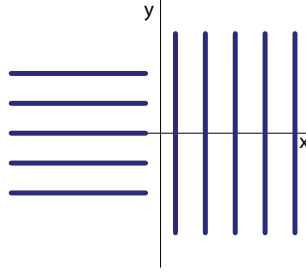


Figure 1: Orthogonal domain wall

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2 Global invariant manifold \mathcal{W}_δ

The first observation is that we have the first integral

$$\varepsilon^2(A'^2)'' - 3\varepsilon^2 A''^2 - B'^2 + \frac{\varepsilon^2}{2}(A^2 + B^2 - 1)^2 + \varepsilon^2 \delta^2 A^2 B^2 = 0, \quad (2)$$

i.e.

$$2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2 - B_1^2 + \frac{\varepsilon^2}{2}(A_0^2 + B_0^2 - 1)^2 + \varepsilon^2 \delta^2 A_0^2 B_0^2 = 0 \quad (3)$$

This defines a 5-dimensional invariant manifold \mathcal{W}_δ valid for any $\delta > 0$, which contains the heteroclinic curve that we are looking for. The singular points of this manifold are given by

$$\begin{aligned} A_1 &= A_2 = A_3 = B_1 = 0, \\ 0 &= A_0(A_0^2 + (1 + \delta^2)B_0^2 - 1), \\ 0 &= B_0((1 + \delta^2)A_0^2 + B_0^2 - 1). \end{aligned}$$

For $\delta > 0$, and since $(A_0, B_0) = (0, 0)$ or $(\pm(\delta^2 + 2)^{-1/2}, \pm(\delta^2 + 2)^{-1/2})$ do not belong to \mathcal{W}_δ , we only find the singular points

$$\begin{aligned} (A_0, B_0) &= (\pm 1, 0), \\ (A_0, B_0) &= (0, \pm 1). \end{aligned} \tag{4}$$

For $\delta = 0$, all singular points belong to a circle of singular points:

$$A_0^2 + B_0^2 = 1. \tag{5}$$

3 Linear study of the dynamics

3.1 Case $\delta > 0$ ($g > 1$)

3.1.1 Neighborhood of $M_- = (1, 0)$

The eigenvalues of the linearized operator at M_- are such that $\lambda^4 = -2$ or $\lambda^2 = \varepsilon^2 \delta^2$, hence

$$\begin{aligned} \pm 2^{-1/4}(1 \pm i), \\ \pm \varepsilon \delta. \end{aligned}$$

This gives a 3-dimensional unstable manifold, and a 3-dimensional stable manifold.

3.1.2 Neighborhood of $M_+ = (0, 1)$

The eigenvalues of the linearized operator at M_+ are such that $\lambda^4 = -\delta^2$ or $\lambda^2 = 2\varepsilon^2$, hence

$$\begin{aligned} \pm 2^{-1/2}(1 \pm i)\delta', \\ \pm \varepsilon \sqrt{2}, \quad \delta' = \sqrt{\delta}. \end{aligned}$$

This gives again a 3-dimensional unstable manifold and a 3-dimensional stable manifold.

All this implies that the 3-dimensional unstable manifold starting at M_- is included into the 5-dimensional manifold \mathcal{W}_δ , as well as the 3-dimensional stable manifold starting at M_+ is included into the 5-dimensional manifold \mathcal{W}_δ . This gives a good hope that these two manifolds intersect along a heteroclinic curve...provided that they still exist "far" from the end points M_+ and M_- . The idea is to show that this occurs when δ is not too small.

3.2 Linear dynamics near the arc of equilibria for $\delta = 0$

Let us denote the family of fixed points as

$$A_*^2 + B_*^2 = 1, \quad B_* \geq 0,$$

which is an arc of circle of equilibria in the (A_0, B_0) plane. The linearized operator along this family is

$$\mathbf{L}_* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2A_*^2 & 0 & 0 & 0 & -2A_*B_* & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2\varepsilon^2 A_* B_* & 0 & 0 & 0 & 2\varepsilon^2 B_*^2 & 0 \end{pmatrix},$$

and its eigenvalues are

$$0 \text{ double,}$$

and λ such that

$$\lambda^4 - 2\varepsilon^2 B_*^2 \lambda^2 + 2A_*^2 = 0.$$

The discriminant is

$$\Delta = \varepsilon^4 B_*^4 - 2A_*^2 = \varepsilon^4 B_*^4 + 2B_*^2 - 2,$$

it results that

i) for $0 < B_*^2 < B_c^2 = \frac{1}{\varepsilon^4}(-1 + \sqrt{1 + 2\varepsilon^4}) \simeq 1 - \varepsilon^4/2$, i.e. $A_* > A_c \simeq \frac{\varepsilon^2}{\sqrt{2}}$, we have $\Delta < 0$ and 4 complex eigenvalues

$$\lambda_0, \overline{\lambda_0}, -\lambda_0, -\overline{\lambda_0},$$

while

ii) for $B_c < B_* < 1$, i.e. $A_* < A_c$, we have $\Delta > 0$ and 4 real eigenvalues

$$\begin{aligned} \lambda_{\pm}^2 &= \varepsilon^2 B_*^2 \pm \sqrt{\varepsilon^4 B_*^4 - 2A_*^2} > 0, \\ \text{two} &> 0, \text{ two} < 0, \end{aligned}$$

and

iii) for $B_* = B_c$, i.e. $A_* = A_c$, $\Delta = 0$ and two double eigenvalues

$$\lambda = \pm \varepsilon B_*.$$

In all cases there are two stable directions and two well determined unstable directions, depending smoothly on (A_*, B_*) . The eigenvector belonging to the eigenvalue 0 is *tangent to the family of equilibria*.

Let us sum up the situation for the system (1) with $\delta = 0$, linearized along the arc of equilibria:

for $0 < B_* < B_c$ we have a two-dimensional invariant subspace, associated with two eigenvalues $\lambda_0, \overline{\lambda_0}$, and a two-dimensional invariant subspace associated with $-\lambda_0$ and $-\overline{\lambda_0}$.

When B_* reaches B_c the eigenvalues λ_0 and $\overline{\lambda_0}$ merge in a double eigenvalue εB_* which splits in two real positive eigenvalues λ_+, λ_- when $B_c < B_* < 1$. The same phenomenon occurs for eigenvalues $-\lambda_0$ and $-\overline{\lambda_0}$ which merge in $-\varepsilon B_*$ and then split in $-\lambda_+, -\lambda_-$. The smooth dependence in B_* of the two-dimensional invariant spaces (general result of perturbation theory of linear operators [6]) shows that these two-dimensional subspaces stay transverse to each other all along the curve $A_*^2 + B_*^2 = 1$. We sum up these properties in the following

Lemma 6 *The system (1) admits, for $\delta = 0$ an arc of equilibria $(A_0, B_0) = (A_*, B_*)$, $A_*^2 + B_*^2 = 1$. Along this arc of equilibria, the family of 2-dimensional stable manifolds and the family of 2-dimensional unstable manifolds belong to the 5-dimensional manifold \mathcal{W}_0 (singular along the arc of equilibria). These families constitute two 3-dimensional manifolds intersecting each other transversally along the open arc (without end point M_+).*

Remark 7 *Notice that the limit points $M_- = (1, 0)$ and $M_+ = (0, 1)$ have a degenerate situation, because of the multiple 0 eigenvalue. The above transversality is weaker and weaker as $(A_*, B_*) \rightarrow (0, 1)$. A "serious" study is needed since it is not obvious, for example, that the 3-dim stable manifold of M_+ is obtained for $\delta \neq 0$ by a simple perturbation of the family of 2-dimensional stable manifolds along the arc $A_*^2 + B_*^2 = 1$, for $A_* > 0$.*

4 Unstable manifold of M_-

4.1 Change of coordinates

Let us fix $0 < \delta_0 \leq 1/3$, and $\delta_1 > 1$, we assume, from now on

$$0 \leq B_0 \leq \sqrt{1 - \eta_0^2 \delta^2}, \quad \eta_0 > \frac{1}{\sqrt{1 + \delta^2}} = \frac{1}{\sqrt{g}},$$

$$\alpha \stackrel{def}{=} (\eta_0^2(1 + \delta^2) - 1)^{1/2}, \quad \frac{\varepsilon^2}{\alpha^2} \leq \delta_0 \leq \delta \leq \delta_1, \quad (6)$$

and let us define new coordinates

$$Z = (\widetilde{A}_* + \widetilde{A}_0, A_1, A_2, A_3, B_0, B_1)^t \quad (7)$$

where $A_0 = \widetilde{A}_*$ cancels A'_3 with

$$\widetilde{A}_*^2 \stackrel{def}{=} 1 - (1 + \delta^2)B_0^2, \quad \widetilde{A}_* \geq \delta\alpha.$$

In the following α is a "small parameter", the relative size of which, with respect to ε is precized later.

Remark 8 *The occurrence of \widetilde{A}_* is linked with a formal computation of an expansion of the heteroclinic in powers of ε , which gives \widetilde{A}_* as the principal part of A_0 , valid for $B_0 < (1 + \delta^2)^{-1/2} = 1/\sqrt{g}$. The hope is to build the unstable manifold until this limit value.*

Remark 9 *We choose the conditions on δ , $\delta_0 \leq \delta \leq \delta_1$ in the purpose to include known computed values of the coefficient $g = 1 + \delta^2$, in the convection problems, with different boundary conditions (see [3]).*

We prove below the main result of this section:

Lemma 10 For ε small enough, α small enough and for $0 < \delta_0 < 1/3$, and

$$\begin{aligned}\delta &\in [\delta_0, \delta_1], \quad \alpha^2 = \eta_0^2(1 + \delta^2) - 1, \\ \varepsilon^2 &\leq \delta_0 \alpha^2, \quad \varepsilon = \alpha^{7/2},\end{aligned}$$

the 3-dimensional unstable manifold of M_- exists for

$$0 \leq B_0(x) \leq (1 - \eta_0^2 \delta^2)^{1/2}, \quad x \in (-\infty, 0].$$

It sits in \mathcal{W}_g , is analytic in (ε, δ) , and

$$\begin{aligned}A_0 &= \widetilde{A}_* + B_0 \mathcal{O}\left(\frac{\varepsilon \delta}{\widetilde{A}_*^{1/2}} e^{\varepsilon \delta x}\right) \\ A_1 &= B_0 \mathcal{O}\left(\frac{\varepsilon \delta}{\widetilde{A}_*} e^{\varepsilon \delta x}\right) \\ A_2 &= B_0 \mathcal{O}(\varepsilon \delta e^{\varepsilon \delta x}) \\ A_3 &= B_0 \mathcal{O}(\varepsilon \delta e^{\varepsilon \delta x}),\end{aligned}$$

where

$$0 \leq 1 - \widetilde{A}_* \leq c B_0^2, \quad \widetilde{A}_*(0) = 1, \quad \widetilde{A}_* \geq \delta \alpha.$$

Moreover, as $x \rightarrow -\infty$, $A_0 - \widetilde{A}_*$, A_1, A_2, A_3 are bounded by $c \varepsilon \delta e^{2\varepsilon \delta x}$, and B_0 , resp. B_1 by $c e^{\varepsilon \delta x}$, resp. $c \varepsilon e^{\varepsilon \delta x}$, where c is a constant independent of ε, δ .

Remark 11 We observe that A_0 reaches a value close to 0 since \widetilde{A}_* reaches $\delta \alpha$ which is close to 0, while B_0 reaches $(1 - \eta_0^2 \delta^2)^{1/2}$ which is close to $1/(1 + \delta^2)^{1/2} = 1/\sqrt{g}$, not close to 1.

The system (1) becomes

$$\begin{aligned}\widetilde{A}_0' &= A_1 + \frac{(1 + \delta^2) B_0}{\widetilde{A}_*} B_1 \\ A_1' &= A_2 \\ A_2' &= A_3 \\ A_3' &= -2 \widetilde{A}_*^2 \widetilde{A}_0 - 3 \widetilde{A}_* \widetilde{A}_0^2 - \widetilde{A}_0^3 \\ B_0' &= B_1 \\ B_1' &= \varepsilon^2 \delta^2 B_* (\widetilde{A}_*^2 - B_0^2) + 2 \varepsilon^2 (1 + \delta^2) \widetilde{A}_* B_0 \widetilde{A}_0 + \varepsilon^2 (1 + \delta^2) B_0 \widetilde{A}_0^2,\end{aligned}\tag{8}$$

Now, we define the linear operator

$$\mathbf{L}_\delta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \frac{(1 + \delta^2) B_0}{\widetilde{A}_*} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2 \widetilde{A}_*^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 \varepsilon^2 (1 + \delta^2) \widetilde{A}_* B_0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\tag{9}$$

for which 0 is a double eigenvalue, and such that the non zero eigenvalues satisfy

$$\lambda^4 - 2\varepsilon^2 B_0^2 (1 + \delta^2)^2 \lambda^2 + 2\widetilde{A}_*^2 = 0. \quad (10)$$

The discriminant is

$$\Delta' = \varepsilon^4 B_0^4 (1 + \delta^2)^4 - 2\widetilde{A}_*^2.$$

Our assumption $B_0 \leq \sqrt{1 - \eta_0^2 \delta^2}$ and $\frac{\varepsilon^2}{\alpha^2} \leq \delta \leq \delta_1$, in addition with the constraint

$$\frac{1}{\alpha} \geq (1 + \delta^2)^2. \quad (11)$$

implies

$$-\Delta' \geq \widetilde{A}_*^2.$$

Then we have two pairs of complex eigenvalues

$$\lambda_{\pm}^2 = \varepsilon^2 B_0^2 (1 + \delta^2)^2 \pm i\sqrt{-\Delta'}.$$

The idea is to find new coordinates able to manage a new linear operator in the form of two independent blocs

$$\begin{pmatrix} \pm\lambda_r & \lambda_i \\ -\lambda_i & \pm\lambda_r \end{pmatrix} \quad (12)$$

for which the eigenvalues are

$$\pm\lambda_r \pm i\lambda_i,$$

where

$$\begin{aligned} 2\lambda_r^2 &= \sqrt{2\widetilde{A}_*} + \varepsilon^2 B_0^2 (1 + \delta^2)^2 \\ 2\lambda_i^2 &= \sqrt{2\widetilde{A}_*} - \varepsilon^2 B_0^2 (1 + \delta^2)^2 \\ \lambda_r^2 - \lambda_i^2 &= \varepsilon^2 B_0^2 (1 + \delta^2)^2 \\ \lambda_r^2 + \lambda_i^2 &= \sqrt{2\widetilde{A}_*} \\ 4\lambda_r^2 \lambda_i^2 &= -\Delta'. \end{aligned} \quad (13)$$

We choose a form of the linear operator as (12) for being able to have good estimates for the monodromy operator associated with the linear operator, the coefficients of which are functions of $B_0 \in [0, \sqrt{1 - \eta_0^2 \delta^2}]$ (see Appendix A.1).

4.2 Estimates for the eigenvalues

First, notice that (13) and

$$\alpha \leq (1 + \delta^2)^{-2}$$

imply

$$\lambda_r \lambda_i \geq \frac{\widetilde{A}_*}{2},$$

$$2^{1/4}\widetilde{A}_*^{1/2} \geq \lambda_r \geq \frac{\widetilde{A}_*^{1/2}}{2^{1/4}} \geq \frac{\alpha^{1/2}}{2^{1/4}}\sqrt{\delta}, \quad (14)$$

$$\frac{1}{2^{3/4}}\widetilde{A}_*^{1/2} \leq \lambda_i \leq \frac{\widetilde{A}_*^{1/2}}{2^{1/4}}. \quad (15)$$

4.3 New coordinates

The eigenvector and generalized eigenvector for the eigenvalue 0 are :

$$Z_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{A}_* \\ 0 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 0 \\ -(1+\delta^2)B_0 \\ 0 \\ 0 \\ 0 \\ \widetilde{A}_* \end{pmatrix}.$$

Now we denote by

$$V_r^+ \pm i\lambda_i V_i^+, \quad V_r^- \pm i\lambda_i V_i^-$$

the eigenvectors belonging respectively to the eigenvalues

$$\lambda_r \pm i\lambda_i, \quad -\lambda_r \pm i\lambda_i$$

then we define

$$V_r^+ = \begin{pmatrix} -\frac{\lambda_r(\lambda_r^2 - 3\lambda_i^2)}{2\widetilde{A}_*^2} \\ 1 \\ \lambda_r \\ \lambda_r^2 - \lambda_i^2 \\ -\frac{\lambda_r(\lambda_r^2 - \lambda_i^2)}{(1+\delta^2)B_0 A_*} \\ -\frac{(\lambda_r^2 - \lambda_i^2)^2}{(1+\delta^2)B_0 A_*} \end{pmatrix}, \quad V_i^+ = \begin{pmatrix} -\frac{3\lambda_r^2 - \lambda_i^2}{2\widetilde{A}_*^2} \\ 0 \\ 1 \\ 2\lambda_r \\ -\frac{(\lambda_r^2 - \lambda_i^2)}{(1+\delta^2)B_0 A_*} \\ -\frac{2\lambda_r(\lambda_r^2 - \lambda_i^2)}{(1+\delta^2)B_0 A_*} \end{pmatrix},$$

and we define new coordinates as

$$\begin{pmatrix} \widetilde{A}_0 \\ A_1 \\ A_2 \\ A_3 \\ 0 \\ B_1 \end{pmatrix} = B_0(x_1 V_r^+ + x_2 \lambda_i V_i^+ + y_1 V_r^- + y_2 \lambda_i V_i^- + z_0 Z_0 + z_1 Z_1).$$

We observe that after eliminating z_0 , we still have 6 coordinates, including B_0 as one of the new coordinates.

Remark 12 We notice that we put B_0 in front of the new coordinates, as this results from the analysis, and shorten the computations.

We have now

$$\begin{aligned}
\widetilde{A}_0 &= -B_0 \frac{\lambda_r(\lambda_r^2 - 3\lambda_i^2)}{2\widetilde{A}_*^2} (x_1 - y_1) - B_0 \frac{\lambda_i(3\lambda_r^2 - \lambda_i^2)}{2\widetilde{A}_*^2} (x_2 + y_2) \\
A_1 &= B_0(x_1 + y_1) - (1 + \delta^2)B_0^2 z_1 \\
A_2 &= \lambda_r B_0(x_1 - y_1) + \lambda_i B_0(x_2 + y_2) \\
A_3 &= (\lambda_r^2 - \lambda_i^2)B_0(x_1 + y_1) + 2\lambda_r \lambda_i B_0(x_2 - y_2) \\
0 &= -\frac{(\lambda_r^2 - \lambda_i^2)}{(1 + \delta^2)B_0 \widetilde{A}_*} A_2 + \widetilde{A}_* B_0 z_0 \\
B_1 &= -\frac{(\lambda_r^2 - \lambda_i^2)}{(1 + \delta^2)B_0 \widetilde{A}_*} A_3 + \widetilde{A}_* B_0 z_1,
\end{aligned} \tag{16}$$

which needs to be inverted. We obtain

$$\begin{aligned}
B_0 x_1 &= \frac{(\lambda_r^2 + \lambda_i^2)}{4\lambda_r} \widetilde{A}_0 + \frac{3\lambda_r^2 - \lambda_i^2}{4\lambda_r(\lambda_r^2 + \lambda_i^2)} A_2 \\
&\quad + \frac{A_1}{2} + \frac{(1 + \delta^2)B_0}{2\widetilde{A}_*} B_1 + \frac{(\lambda_r^2 - \lambda_i^2)}{2\widetilde{A}_*^2} A_3,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\lambda_i B_0 x_2 &= -\frac{(\lambda_r^2 + \lambda_i^2)}{4} \widetilde{A}_0 - \frac{\lambda_r^2 - 3\lambda_i^2}{4(\lambda_r^2 + \lambda_i^2)} A_2 \\
&\quad - \frac{(\lambda_r^2 - \lambda_i^2)}{4\lambda_r} \left(A_1 + \frac{(1 + \delta^2)B_0}{\widetilde{A}_*} B_1 \right) + \frac{1}{4\lambda_r} \left(1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A}_*^2} \right) A_3,
\end{aligned} \tag{18}$$

$$\begin{aligned}
B_0 y_1 &= -\frac{(\lambda_r^2 + \lambda_i^2)}{4\lambda_r} \widetilde{A}_0 - \frac{3\lambda_r^2 - \lambda_i^2}{4\lambda_r(\lambda_r^2 + \lambda_i^2)} A_2 \\
&\quad + \frac{A_1}{2} + \frac{(1 + \delta^2)B_0}{2\widetilde{A}_*} B_1 + \frac{(\lambda_r^2 - \lambda_i^2)}{2\widetilde{A}_*^2} A_3,
\end{aligned} \tag{19}$$

$$\begin{aligned}
\lambda_i B_0 y_2 &= -\frac{(\lambda_r^2 + \lambda_i^2)}{4} \widetilde{A}_0 - \frac{\lambda_r^2 - 3\lambda_i^2}{4(\lambda_r^2 + \lambda_i^2)} A_2 \\
&\quad + \frac{(\lambda_r^2 - \lambda_i^2)}{4\lambda_r} \left(A_1 + \frac{(1 + \delta^2)B_0}{\widetilde{A}_*} B_1 \right) - \frac{1}{4\lambda_r} \left(1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A}_*^2} \right) A_3,
\end{aligned} \tag{20}$$

$$B_0 z_1 = \frac{(\lambda_r^2 - \lambda_i^2)}{(1 + \delta^2)B_0 \widetilde{A}_*^2} A_3 + \frac{1}{\widetilde{A}_*} B_1.$$

Let us now define

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, Y = X = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

then, for ε small enough, we obtain the following useful estimates

$$\begin{aligned} \frac{\widetilde{A}_*^{-1/2}}{2^{3/4}} &\leq \lambda_r, \lambda_i < 2^{1/4} \widetilde{A}_*^{-1/2}, \quad \widetilde{A}_* \geq \delta\alpha \geq \frac{\varepsilon^2}{\alpha}, \\ |\widetilde{A}_0| &\leq 3 \frac{B_0}{\widetilde{A}_*^{-1/2}} (|X| + |Y|), \\ |A_1| &\leq B_0 (|X| + |Y|) + 2B_0^2 |z_1|, \\ |A_2| &\leq 2B_0 \widetilde{A}_*^{-1/2} (|X| + |Y|), \\ |A_3| &\leq 2B_0 \widetilde{A}_* (|X| + |Y|), \\ |B_1| &\leq 3\varepsilon^2 B_0^2 (|X| + |Y|) + \widetilde{A}_* B_0 |z_1|. \end{aligned} \tag{21}$$

4.4 System with new coordinates

The system (8) written in the new coordinates is computed in Appendix A.2. It takes the following form

$$\begin{aligned} x'_1 &= f_1 + \lambda_r x_1 + \lambda_i x_2 \\ &+ B_1 \left[a_1 \widetilde{A}_0 + c_1 A_2 + d_1 A_3 + e_1 \frac{B_1}{B_0} - \frac{1}{B_0} x_1 \right] \\ &- \varepsilon^2 \frac{(1 + \delta^2)(2 - \delta^2) B_0}{2 \widetilde{A}_*} \widetilde{A}_0^{-2} - \varepsilon^2 \frac{(1 + \delta^2) B_0}{2 \widetilde{A}_*} \widetilde{A}_0^{-3}, \end{aligned} \tag{22}$$

$$\begin{aligned} x'_2 &= f_2 - \lambda_i x_1 + \lambda_r x_2 + B_1 \left[-a_2 \widetilde{A}_0 + b_2 A_1 + c_2 A_2 + d_2 A_3 + e_2 B_1 - \frac{1}{B_0} x_2 \right] \\ &- \frac{1}{4\lambda_r \lambda_i \widetilde{A}_* B_0} \left(3\widetilde{A}_*^{-2} - 2\varepsilon^4 B_0^4 (1 + \delta^2)^4 \right) \widetilde{A}_0^{-2} - \frac{1}{4\lambda_r \lambda_i B_0} \left(1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A}_*^2} \right) \widetilde{A}_0^{-3}. \end{aligned} \tag{23}$$

$$\begin{aligned} y'_1 &= f_1 - \lambda_r y_1 + \lambda_i y_2 + \\ &+ B_1 \left[-a_1 \widetilde{A}_0 - c_1 A_2 + d_1 A_3 + e_1 \frac{B_1}{B_0} - \frac{1}{B_0} y_1 \right] \\ &- \varepsilon^2 \frac{(1 + \delta^2)(2 - \delta^2) B_0}{2 \widetilde{A}_*} \widetilde{A}_0^{-2} - \varepsilon^2 \frac{(1 + \delta^2) B_0}{2 \widetilde{A}_*} \widetilde{A}_0^{-3}, \end{aligned} \tag{24}$$

$$\begin{aligned} y'_2 &= -f_2 - \lambda_i y_1 - \lambda_r y_2 + B_1 \left[-a_2 \widetilde{A}_0 - b_2 A_1 + c_2 A_2 - d_2 A_3 + e_2 B_1 - \frac{1}{B_0} y_2 \right] \\ &+ \frac{1}{4\lambda_r \lambda_i \widetilde{A}_* B_0} \left(3\widetilde{A}_*^{-2} - 2\varepsilon^4 B_0^4 (1 + \delta^2)^4 \right) \widetilde{A}_0^{-2} + \frac{1}{4\lambda_r \lambda_i B_0} \left(1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A}_*^2} \right) \widetilde{A}_0^{-3}, \end{aligned} \tag{25}$$

$$B'_0 = -\varepsilon^2(1 + \delta^2)B_0 \frac{A_3}{\widetilde{A}_*} + \widetilde{A}_* B_0 z_1,$$

with

$$f_1 = \frac{\varepsilon^2 \delta^2 B_0 (1 + \delta^2) (\widetilde{A}_*^2 - B_0^2)}{2\widetilde{A}_*},$$

$$f_2 = -\frac{\varepsilon^2 \delta^2 B_0 (1 + \delta^2) (\lambda_r^2 - \lambda_t^2) (\widetilde{A}_*^2 - B_0^2)}{4\lambda_r \lambda_t \widetilde{A}_*},$$

coefficients a_j, b_j, c_j, d_j, e_j are defined and estimated in Appendix A.2 in (78,79), (80,81,82), (83,84), (85,86). Here $\widetilde{A}_0, A_1, A_2, A_3, B_1$ should be replaced by their (linear) expressions (16) in coordinates $(x_1, x_2, y_1, y_2, z_1)$ with coefficients functions of B_0 . The system above should be completed by the differential equation for z_1 . In fact we replace the equation for z'_1 by the direct resolution of the first integral (3) with respect to z_1 (see below).

4.5 Resolution of (3) with respect of $z_1(X, Y, B_0)$

For extending the validity for the existence of the unstable manifold of M_- we need to replace the differential equation for z_1 in using instead the first integral (3). This leads to

$$B_1^2 = \left\{ \widetilde{A}_* B_0 z_1 - \varepsilon^2 \frac{B_0(1 + \delta^2)}{\widetilde{A}_*} A_3 \right\}^2 = 2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2 +$$

$$\frac{\varepsilon^2}{2} (-\delta^2 B_0^2 + 2\widetilde{A}_* \widetilde{A}_0 + \widetilde{A}_0^2)^2 + \varepsilon^2 \delta^2 (\widetilde{A}_* + \widetilde{A}_0)^2 B_0^2,$$

hence

$$\widetilde{A}_*^2 z_1^2 = \varepsilon^2 \delta^2 \widetilde{A}_*^2 \left(1 + \frac{\delta^2 B_0^2}{2\widetilde{A}_*^2}\right) + \frac{2\varepsilon^2}{B_0} A_3 (x_1 + y_1) - \frac{\varepsilon^4 (1 + \delta^2)^2}{\widetilde{A}_*^2} A_3^2 - \frac{\varepsilon^2}{B_0^2} A_2^2 +$$

$$+ \frac{2\varepsilon^2 \widetilde{A}_*^2}{B_0^2} \widetilde{A}_0^2 + \frac{2\varepsilon^2 \widetilde{A}_* \widetilde{A}_0^3}{B_0^2} + \frac{\varepsilon^2}{2B_0^2} \widetilde{A}_0^4, \quad (26)$$

where we may observe on the r.h.s., that

$$\frac{\delta^2}{2\widetilde{A}_*^2} < \frac{1}{2\alpha^2},$$

hence

$$\varepsilon^2 \delta^2 \leq \varepsilon^2 \delta^2 \left(1 + \frac{\delta^2 B_0^2}{2\widetilde{A}_*^2}\right) \leq \varepsilon^2 \delta^2 \left(1 + \frac{1}{2\alpha^2}\right),$$

which is independent of (X, Y) . Moreover there is no linear part in (X, Y) . For further estimates, we make a new scaling

$$(X, Y, z_1) = \varepsilon \delta (\overline{X}, \overline{Y}, \overline{z_1}). \quad (27)$$

We notice that (21) implies

$$\begin{aligned}
\left| \frac{2\varepsilon^2}{B_0} A_3(x_1 + y_1) \right| &\leq c\varepsilon^4 \delta^2 \widetilde{A}_* (|\overline{X}| + |\overline{Y}|)^2 \\
\left| \frac{\varepsilon^4(1 + \delta^2)^2}{\widetilde{A}_*^2} A_3^2 \right| &\leq c\varepsilon^6 \delta^2 (|\overline{X}| + |\overline{Y}|)^2 \\
\frac{\varepsilon^2}{B_0^2} A_2^2 &\leq c\varepsilon^4 \delta^2 \widetilde{A}_* (|\overline{X}| + |\overline{Y}|)^2 \\
\frac{2\varepsilon^2 \widetilde{A}_*^2}{B_0^2} A_0^2 &\leq c\varepsilon^4 \delta^2 \widetilde{A}_* (|\overline{X}| + |\overline{Y}|)^2 \\
\left| \frac{2\varepsilon^2 \widetilde{A}_*}{B_0^2} A_0^3 \right| &\leq \frac{c\varepsilon^5 \delta^3}{\widetilde{A}_*^{1/2}} (|\overline{X}| + |\overline{Y}|)^3 \\
\frac{\varepsilon^2}{2B_0^2} A_0^4 &\leq \frac{c\varepsilon^6 \delta^4}{\widetilde{A}_*^2} (|\overline{X}| + |\overline{Y}|)^4,
\end{aligned}$$

so that the factors in the estimates are such that

$$\frac{c\varepsilon^4 \delta^2 \widetilde{A}_*}{\varepsilon^2 \delta^2 \widetilde{A}_*^2} \leq \frac{c\varepsilon^2}{\widetilde{A}_*}, \quad \frac{c\varepsilon^6 \delta^4}{\varepsilon^2 \delta^2 \widetilde{A}_*^4} \leq c \frac{\varepsilon^2 \delta_0}{\widetilde{A}_*^2}, \quad \frac{c\varepsilon^5 \delta^3}{\varepsilon^2 \delta^2 \widetilde{A}_*^{5/2}} \leq c \frac{\varepsilon^{5/2} \delta^{3/4}}{\widetilde{A}_*^2},$$

c being independent of ε and δ . Now defining \overline{z}_{10} such that

$$1 \leq \overline{z}_{10}(B_0) \stackrel{def}{=} \left(1 + \frac{\delta^2 B_0^2}{2\widetilde{A}_*^2}\right)^{1/2} \leq \frac{1}{\alpha}, \quad \text{for } \alpha \leq 1/\sqrt{2}, \quad (28)$$

It results that

$$\overline{z}_1^2 = \overline{z}_{10}^2 + \mathcal{O}\left(\frac{\varepsilon^2}{\widetilde{A}_*} (|\overline{X}| + |\overline{Y}|)^2 + \frac{\varepsilon^{5/2} \delta^{3/4}}{\widetilde{A}_*^2} (|\overline{X}| + |\overline{Y}|)^3 + \frac{\varepsilon^2}{\widetilde{A}_*^2} (|\overline{X}| + |\overline{Y}|)^4\right)$$

and using

$$B_0^2 = \frac{1 - \widetilde{A}_*^2}{1 + \delta^2}$$

we also have

$$\frac{1}{\overline{z}_{10}^2} = \frac{2\widetilde{A}_*^2}{2\widetilde{A}_*^2 + \delta^2 B_0^2} \leq \frac{2(1 + \delta^2)\widetilde{A}_*^2}{\delta^2} \leq \frac{c\widetilde{A}_*^2}{\delta^2},$$

so that

$$\begin{aligned}
\overline{z}_1 &= \overline{z}_{10}(B_0) \left\{ 1 + \frac{\widetilde{A}_*^2}{\delta^2} \mathcal{O}\left(\frac{\varepsilon^2}{\widetilde{A}_*} (|\overline{X}| + |\overline{Y}|)^2 + \frac{\varepsilon^{5/2} \delta^{3/4}}{\widetilde{A}_*^2} (|\overline{X}| + |\overline{Y}|)^3 + \frac{\varepsilon^2}{\widetilde{A}_*^2} (|\overline{X}| + |\overline{Y}|)^4\right) \right\}^{1/2} \\
&= \overline{z}_{10}(B_0) \{1 + \mathcal{O}[\varepsilon^2 (|\overline{X}| + |\overline{Y}|)^2]\}^{1/2}, \quad \text{for } |\overline{X}| + |\overline{Y}| \leq \rho, \rho \text{ fixed,}
\end{aligned}$$

and taking the square root, we obtain (using $\frac{\varepsilon^2}{\alpha} \leq \varepsilon\sqrt{\delta_0}$)

$$\bar{z}_1 = \bar{z}_{10}(B_0) + \mathcal{Z}(\bar{X}, \bar{Y}, B_*) \quad (29)$$

with

$$\mathcal{Z}(\bar{X}, \bar{Y}, B_0) = \mathcal{O}(\varepsilon(|\bar{X}| + |\bar{Y}|)^2),$$

$\mathcal{Z}(\bar{X}, \bar{Y}, B_0)$ being defined in the ball

$$|\bar{X}| + |\bar{Y}| \leq \rho,$$

provided that ε is small enough and where ρ is of order 1, not necessarily small with respect to ε and α . Moreover \mathcal{Z} is analytic in its arguments and is at least quadratic in (\bar{X}, \bar{Y}) .

Since z_1 contains \bar{z}_{10} which is independent of (\bar{X}, \bar{Y}) , the new system has new "constant terms" and "linear terms", appearing as perturbations of the former ones.

4.6 System where z_1 is eliminated

The new system is computed in Appendix A.3. We obtain (notice that B_0 is in factor of the "constant" terms)

$$\begin{aligned} \bar{X}' &= \mathbf{L}_0 \bar{X} + B_0 \mathcal{F}_0 + \mathcal{L}_{01}(\bar{X}, \bar{Y}) + \mathcal{B}_{01}(\bar{X}, \bar{Y}), \\ \bar{Y}' &= \mathbf{L}_1 \bar{Y} + B_0 \mathcal{F}_1 + \mathcal{L}_{11}(\bar{X}, \bar{Y}) + \mathcal{B}_{11}(\bar{X}, \bar{Y}), \end{aligned} \quad (30)$$

where

$$\mathbf{L}_0 = \begin{pmatrix} \lambda_r & \lambda_i \\ -\lambda_i & \lambda_r \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} -\lambda_r & \lambda_i \\ -\lambda_i & -\lambda_r \end{pmatrix},$$

and with the following estimates, for terms independent of (\bar{X}, \bar{Y})

$$|\mathcal{F}_0| + |\mathcal{F}_1| \leq \frac{c\varepsilon}{\alpha^3}, \quad (31)$$

for terms which are linear in (\bar{X}, \bar{Y})

$$|\mathcal{L}_{01}(\bar{X}, \bar{Y})| + |\mathcal{L}_{11}(\bar{X}, \bar{Y})| \leq c \frac{\varepsilon}{\alpha^2} (|\bar{X}| + |\bar{Y}|), \quad (32)$$

and for terms at least quadratic in (\bar{X}, \bar{Y}) , choosing α small enough and for

$$|\bar{X}| + |\bar{Y}| \leq \rho,$$

we obtain

$$|\mathcal{B}_{01}(\bar{X}, \bar{Y})| + |\mathcal{B}_{11}(\bar{X}, \bar{Y})| \leq \frac{c\varepsilon}{\alpha^{1/2}} (|\bar{X}| + |\bar{Y}|)^2. \quad (33)$$

4.7 Integral formulation for solutions bounded as $x \rightarrow -\infty$

Let us introduce the monodromy operators associated with the linear operators $\mathbf{L}_0, \mathbf{L}_1$ which have non constant coefficients (functions of B_0 (see [2]):

$$\begin{aligned}\frac{\partial}{\partial x} S_0(x, s) &= \mathbf{L}_0 S_0(x, s), \quad S_0(x, s_1) S_0(s_1, s_2) = S_0(x, s_2), \quad S_0(x, x) = \mathbb{I}, \\ \frac{\partial}{\partial x} S_1(x, s) &= \mathbf{L}_1 S_1(x, s), \quad S_1(x, s_1) S_1(s_1, s_2) = S_1(x, s_2), \quad S_1(x, x) = \mathbb{I}.\end{aligned}$$

The coefficients of operators $\mathbf{L}_0, \mathbf{L}_1$ are functions of B_0 , so we need the Lemma 28 in Appendix A.1, with the following estimates, valid for $0 \leq B_0 \leq \sqrt{1 - \eta_0^2 \delta^2}$, $\alpha \leq (1 + \delta^2)^{-2}$:

$$\|\mathbf{S}_0(x, s)\| \leq e^{\sigma(x-s)}, \quad -\infty < x < s \leq 0, \quad (34)$$

$$\|\mathbf{S}_1(x, s)\| \leq e^{-\sigma(x-s)}, \quad -\infty < s < x \leq 0, \quad (35)$$

with

$$\sigma = \frac{\alpha^{1/2} \delta^{1/2}}{2^{1/4}}.$$

We are looking for solutions of (30) which stay bounded for $x \rightarrow -\infty$. Then, thanks to estimates (34) (35), the system (30) may be formulated as

$$\begin{aligned}\bar{X}(x) &= \mathbf{S}_0(x, 0) \bar{X}_0 + \int_0^x \mathbf{S}_0(x, s) G_0(s) ds \\ \bar{Y}(x) &= \int_{-\infty}^x \mathbf{S}_1(x, s) G_1(s) ds\end{aligned} \quad (36)$$

$$\begin{aligned}G_0(s) &\stackrel{def}{=} B_0 \mathcal{F}_0 + \mathcal{L}_{01}(\bar{X}, \bar{Y}) + \mathcal{B}_{01}(\bar{X}, \bar{Y}), \\ G_1(s) &\stackrel{def}{=} B_0 \mathcal{F}_1 + \mathcal{L}_{11}(\bar{X}, \bar{Y}) + \mathcal{B}_{11}(\bar{X}, \bar{Y})\end{aligned}$$

where \bar{X}, \bar{Y} and B_0 are bounded and continuous functions of s , B_0 tending towards 0 as $s \rightarrow -\infty$.

4.8 Strategy

The idea is

- i) solve (36) with respect to (\bar{X}, \bar{Y}) in function of (X_0, B_0) ;
- ii) solve the integro-differential equation for B_0 , with $B_0|_{x=0} = B_0(0)$.

Then the unstable manifold of M_- is given (see [2]) by $Y|_{x=0}, z_1|_{x=0}$ in terms of $X_0, B_0(0)$. The result will be valid for an interval $[0, \sqrt{1 - \eta_0^2 \delta^2}]$ for B_0 and it appears that A_0 is then very close to 0 at this end point. The hope is that this should allow to compute its intersection with the 3-dim stable manifold of M_+ which computation should be valid for B_0 in the interval $[\sqrt{1 - \eta_0^2 \delta^2}, 1]$.

4.9 Resolution for (\bar{X}, \bar{Y})

Let us define, for $\kappa > 0$

$$C_\kappa^0 = \{\bar{X} \in C^0(-\infty, 0]; \bar{X}(x)e^{-\kappa x} \text{ is bounded}\}$$

equiped with the norm

$$\|\bar{X}\|_\kappa = \sup_{(-\infty, 0)} |\bar{X}(x)e^{-\kappa x}|.$$

We observe that, provided that $\kappa < \sigma$

$$\begin{aligned} \left| \int_{-\infty}^x \mathbf{S}_1(x, s)e^{\kappa s} ds \right| &\leq \frac{e^{\kappa x}}{\kappa + \sigma} \\ |\mathbf{S}_0(x, 0)e^{-\kappa x}| &\leq e^{(\sigma - \kappa)x}, \quad x \leq 0, \\ \left| \int_0^x \mathbf{S}_0(x, s)e^{\kappa s} ds \right| &\leq \frac{e^{\kappa x}}{\sigma - \kappa}, \quad x \leq 0. \end{aligned}$$

Let us choose

$$\kappa \leq \frac{\sigma}{2},$$

then

$$\begin{aligned} \left| \int_{-\infty}^x \mathbf{S}_1(x, s)e^{\kappa s} ds \right| &\leq \frac{e^{\kappa x}}{\sigma} = 2^{1/4} \frac{e^{\kappa x}}{\alpha^{1/2} \delta^{1/2}}, \\ \left| \int_0^x \mathbf{S}_0(x, s)e^{\kappa s} ds \right| &\leq 2^{5/4} \frac{e^{\kappa x}}{\alpha^{1/2} \delta^{1/2}}, \quad x \leq 0. \end{aligned}$$

Let us assume that

$$\|B_0\|_\kappa \leq m$$

holds with m independent of ε , which needs to be proved at next subsection. Hence, the implicit function theorem applies for (\bar{X}, \bar{Y}) in the function space C_κ^0 , provided that we can choose $\kappa \leq \frac{\sigma}{2}$ and $\|\bar{X}\|_\kappa + \|\bar{Y}\|_\kappa \leq \rho$. Using the above estimates for coefficients, we obtain

$$|\bar{X}(x)e^{-\kappa x}| \leq |\bar{X}_0| + \frac{2^{5/4}}{\alpha^{1/2} \delta^{1/2}} \|B_0 \mathcal{F}_0 + \mathcal{L}_{01}(\bar{X}, \bar{Y}) + \mathcal{B}_{01}(\bar{X}, \bar{Y})\|_\kappa,$$

hence

$$\|\bar{X}\|_\kappa \leq |\bar{X}_0| + \frac{2^{5/4}}{\alpha^{1/2} \delta^{1/2}} \|B_0 \mathcal{F}_0 + \mathcal{L}_{01}(\bar{X}, \bar{Y}) + \mathcal{B}_{01}(\bar{X}, \bar{Y})\|_\kappa, \quad (37)$$

and in the same way

$$\|\bar{Y}\|_\kappa \leq \frac{2^{1/4}}{\alpha^{1/2} \delta^{1/2}} \|B_0 \mathcal{F}_1 + \mathcal{L}_{11}(\bar{X}, \bar{Y}) + \mathcal{B}_{11}(\bar{X}, \bar{Y})\|_\kappa. \quad (38)$$

Remark 13 *The choice of κ is governed by the behavior of $B_0(x)$ as $x \rightarrow -\infty$, which is studied at next subsection.*

For ε small enough, estimates on \mathcal{F}_1 , \mathcal{B}_{11} , (38) and $\|\overline{X}\|_\kappa + \|\overline{Y}\|_\kappa \leq \rho$, we obtain, with $S \stackrel{def}{=} \|\overline{X}\|_\kappa + \|\overline{Y}\|_\kappa$

$$S \leq |\overline{X}_0| + c \left[\frac{\varepsilon m}{\alpha^{7/2} \delta^{1/2}} + \frac{S\varepsilon}{\alpha^{5/2} \delta^{1/2}} + \frac{\varepsilon S \rho}{\alpha \delta^{1/2}} \right]$$

so that (using $\delta \geq \delta_0 > 0$) for

$$\varepsilon = \alpha^{7/2}, \quad (39)$$

$$\frac{\varepsilon}{\alpha^{5/2} \delta^{1/2}} + \frac{\varepsilon \rho}{\alpha \delta^{1/2}} \leq c\varepsilon^{2/7},$$

$$S(1 - c\varepsilon^{2/7}) \leq |\overline{X}_0| + cm,$$

which implies

$$S \leq (1 + c'\varepsilon^{2/7})|\overline{X}_0| + c'm,$$

which leads finally to

$$\|\overline{Y}\|_\kappa \leq c(m + \varepsilon^{2/7}|\overline{X}_0|), \quad (40)$$

$$\|\overline{X}\|_\kappa \leq (1 + c\varepsilon^{2/7})|\overline{X}_0| + cm, \quad (41)$$

where c is a number independent of ε , and $|\overline{X}_0| \leq \rho/2$, $\varepsilon = \alpha^{7/2}$ small enough, and we assume m bounded by a certain M of order 1, $S \leq \rho$, where ρ is fixed arbitrarily, of order 1.

4.10 Resolution for B_0

We intend to solve the part of our system for B_0 with $B_0(0) = B_0|_{x=0}$.

We notice from (16) and (21) that

$$\begin{aligned} B_1 &= \varepsilon \delta \widetilde{A}_* B_0 \{ \overline{z}_{10}(B_0) + \mathcal{Z}(\overline{X}, \overline{Y}, B_0) \} - \varepsilon^3 \delta (1 + \delta^2) \frac{B_0}{A_*} \overline{A}_3 \\ \overline{A}_3 &= B_0 [\varepsilon^2 B_0^2 (1 + \delta^2)^2 (\overline{x}_1 + \overline{y}_1) + 2\lambda_r \lambda_i (\overline{x}_2 - \overline{y}_2)], \\ \varepsilon^2 (1 + \delta^2) \frac{\overline{A}_3}{A_*} &\leq \frac{4\varepsilon^2}{A_*} (|\overline{X}| + |\overline{Y}|) \leq 4\varepsilon \sqrt{\delta_0} (|\overline{X}| + |\overline{Y}|), \end{aligned}$$

so that it is clear that (see above estimates for \mathcal{Z})

$$\overline{B}_1 > 0 \text{ for } B_0 \in (0, \sqrt{1 - \eta_0^2 \delta^2}), |\overline{X}| + |\overline{Y}| \leq \rho. \quad (42)$$

This is coherent with the study of the linearized system near M_- : Indeed the principal part of the differential equation for B_0 is

$$B_0' = \varepsilon \delta B_0 \widetilde{A}_* \overline{z}_{10}(B_0)$$

which may be integrated as

$$\begin{aligned} B_0^2 &= \frac{1}{\left(1 + \frac{\delta^2}{2}\right) \cosh^2(x_0 - \varepsilon \delta x)}, \\ \cosh x_0 &= \frac{1}{B_0(0) \left(1 + \frac{\delta^2}{2}\right)^{1/2}}, \end{aligned} \quad (43)$$

which satisfies $B_0 = 0$ for $x = -\infty$, and $B_0 = B_0(0)$ for $x = 0$. More precisely the differential equation for B_0 is now (after replacing $(\overline{X}, \overline{Y})$ by its expression found at previous subsection)

$$B_0' = \varepsilon \delta \widetilde{A}_* B_0 \overline{z_{10}}(B_0) [1 + f(B_0)] \quad (44)$$

where $f(B_0)$ is a non local analytic function of B_0 in C_κ^0 , such that

$$\|f(B_0)\|_\kappa \leq c\varepsilon\rho.$$

Remark 14 We may notice that we might replace $c\varepsilon\rho$ in the estimate above, by

$$c\varepsilon\rho e^{\kappa x} \rightarrow 0 \text{ as } x \rightarrow -\infty,$$

since X and $Y \in C_\kappa^0$.

We are looking for the solution such that $B_0 = 0$ for $x = -\infty$, and $B_0(0) \leq \sqrt{1 - \eta_0^2 \delta^2}$ for $x = 0$. We can rewrite (44) as

$$\frac{2B_0 B_0'}{\widetilde{A}_* \overline{z_{10}}(B_0)} = 2\varepsilon\delta[1 + f(B_0)]. \quad (45)$$

We now introduce the variable v :

$$v = \frac{1 - \sqrt{1 - \left(1 + \frac{\delta^2}{2}\right) B_0^2}}{1 + \sqrt{1 - \left(1 + \frac{\delta^2}{2}\right) B_0^2}}, \quad B_0^2 = \frac{1}{1 + \frac{\delta^2}{2}} \frac{4v}{(1+v)^2},$$

so that

$$(\ln v)' = 2\varepsilon\delta[1 + f(B_0)].$$

We observe that for x running from $-\infty$ to 0,

$$w = \ln v \text{ is increasing from } -\infty \text{ to } w_0 = \ln v_0 < 0.$$

Now let us define h continuous in its argument and such that

$$\begin{aligned} h(w) &= f(B_0), \\ B_0 &= \frac{1}{\left(1 + \frac{\delta^2}{2}\right)^{1/2}} \frac{2e^{w/2}}{(1 + e^w)}, \end{aligned}$$

and let us find an a priori estimate for the solution $B_0(x)$, for $x \in (-\infty, 0]$. We obtain by simple integration

$$\int_0^x \frac{w'(s)}{1+h(w)(s)} ds = 2\varepsilon\delta x.$$

For α small enough we have

$$1 - c\varepsilon\rho \leq \frac{1}{1+h(w)} \leq 1 + c\varepsilon\rho,$$

hence (since $w < w_0$, and $x < 0$)

$$(w_0 - w)(1 - c\varepsilon\rho) \leq -2\varepsilon\delta x \leq (w_0 - w)(1 + c\varepsilon\rho)$$

so that

$$\exp\left(\frac{-2\varepsilon\delta x}{1 + c\varepsilon\rho}\right) \leq e^{w_0 - w} \leq \exp\left(\frac{-2\varepsilon\delta x}{1 - c\varepsilon\rho}\right)$$

and

$$v_0 \exp\left(\frac{2\varepsilon\delta}{1 - c\varepsilon\rho}x\right) \leq v(x) \leq v_0 \exp\left(\frac{2\varepsilon\delta}{1 + c\varepsilon\rho}x\right).$$

It finally results that we obtain an a priori estimate for

$$B_0(x) = \mathcal{B}_0(\overline{X}_0, B_0(0))(x) \in C_\kappa^0, \quad (46)$$

$$\begin{aligned} \mathcal{B}_0(\overline{X}_0, B_0(0))(x) &= \frac{1}{\left(1 + \frac{\delta^2}{2}\right)^{1/2}} \frac{2\sqrt{v(x)}}{(1 + v(x))}, \quad x \in (-\infty, 0), \\ \frac{2\sqrt{v_0} \exp\left(\frac{\varepsilon\delta}{1 - c\varepsilon\rho}x\right)}{1 + v_0 \exp\left(\frac{2\varepsilon\delta}{1 - c\varepsilon\rho}x\right)} &\leq \left(1 + \frac{\delta^2}{2}\right)^{1/2} \mathcal{B}_0 \leq \frac{2\sqrt{v_0} \exp\left(\frac{\varepsilon\delta}{1 + c\varepsilon\rho}x\right)}{1 + v_0 \exp\left(\frac{2\varepsilon\delta}{1 + c\varepsilon\rho}x\right)}, \quad (47) \\ v_0 &= \frac{1 - \sqrt{1 - (1 + \frac{\delta^2}{2})B_0^2(0)}}{1 + \sqrt{1 - (1 + \frac{\delta^2}{2})B_0^2(0)}} < 1. \end{aligned}$$

It remains to notice that we can choose

$$\kappa = \frac{\varepsilon\delta}{1 + c\varepsilon\rho}$$

in the proof for $(\overline{X}, \overline{Y})$, which needs to satisfy

$$\kappa \leq \frac{\sigma}{2} = \frac{\alpha^{1/2}\sqrt{\delta}}{2^{5/4}}. \quad (48)$$

We have already chosen $\varepsilon = \alpha^{7/2}$ hence

$$\kappa \leq \varepsilon\delta = \delta\alpha^{7/2} \leq \frac{\alpha^{1/2}\sqrt{\delta}}{2^{5/4}}$$

for α small enough, and (48) is satisfied. The a priori estimate for B_0 allows to prove that there is a unique solution of the integro-differential equation (45) which satisfies the estimate (47) (see [2]). Since B_0 is in factor in $\widetilde{A}_0, A_1, A_2, A_3, B_1$ the behavior for $x \rightarrow -\infty$ of the coordinates of the unstable manifold, is governed by the behavior of B_0 . The estimates indicated in Lemma 10 results from (21), (27) and (28). This ends the proof of Lemma 10. In addition we have the following result (useful for the forthcoming persistence proof):

Corollary 15 *On the unstable manifold described in Lemma 10 we have the estimates*

$$\begin{aligned} |A_0 A_1| &\leq c\varepsilon e^{\varepsilon\delta x}, \quad x \in (-\infty, 0], \\ B_0^2(x) &\leq (B_{00}^2 + c\varepsilon)e^{2\varepsilon\delta x}. \end{aligned}$$

The first estimate is better than the one directly obtained with Lemma 10. For proving it, we notice that

$$A_0 A_1 = \widetilde{A}_* \widetilde{A}_*' + \widetilde{A}_0 \widetilde{A}_*' + \widetilde{A}_* A_1 + \widetilde{A}_0 A_1,$$

and from (21) we have

$$\begin{aligned} |\widetilde{A}_* A_1| &\leq B_0(|X| + |Y|) + 2B_0^2 \widetilde{A}_* |z_1| \\ &\leq B_0 \rho \varepsilon \delta + 2B_0 \varepsilon \delta \left[\left(\widetilde{A}_*^2 + \frac{\delta^2 B_0^2}{2} \right)^{1/2} + \mathcal{O}(\varepsilon) \right] \\ &\leq c\varepsilon e^{\varepsilon\delta x}. \end{aligned}$$

We also have directly

$$\begin{aligned} |\widetilde{A}_* \widetilde{A}_*' | &= 2(1 + \delta^2) B_0 B_0' \\ &\leq c\varepsilon e^{2\varepsilon\delta x}, \end{aligned}$$

and from Lemma 10

$$|\widetilde{A}_0 \widetilde{A}_*' | + |\widetilde{A}_0 A_1| \leq c e^{2\varepsilon\delta x} \frac{\varepsilon}{\sqrt{\alpha}} \frac{\varepsilon}{\alpha} \leq c\varepsilon e^{2\varepsilon\delta x}.$$

The second estimate results from (45), which implies for a certain $c > 0$

$$|(B_0^2(x)e^{-2\varepsilon\delta x})'| \leq c\varepsilon^2 \delta e^{\kappa x}, \quad \text{with } \kappa \sim \varepsilon\delta.$$

This leads to ($x < 0$)

$$\begin{aligned} B_0^2(x)e^{-2\varepsilon\delta x} - B_{00}^2 &\leq c\varepsilon^2 \delta \int_x^0 e^{\kappa\tau} d\tau \\ &\leq \frac{c\varepsilon^2 \delta}{\kappa}. \end{aligned}$$

Corollary 15 is proved.

Remark 16 *The above corollary allows to improve the estimates for the monodromy operators $S_0(x, s)$ and $S_1(x, s)$ as shown in Appendix A.1. This allows to avoid the division by $\sqrt{\alpha}$ in the estimates in subsection 4.9, and finally allows a better choice $\alpha = \varepsilon^{1/3}$ instead of $\varepsilon^{2/7}$. Since this does not lead to major changes in the sequel, we keep $\alpha = \varepsilon^{2/7}$ for simplicity.*

Let us define the hyperplane H_0

$$B_0 = (1 - \eta_0^2 \delta^2)^{1/2}.$$

4.11 Intersection of the stable manifold with H_0

We need to give precisely the intersection of the unstable manifold with the hyperplane $B_0 = \sqrt{1 - \eta_0^2 \delta^2}$. This gives a two-dimensional manifold lying in the 4-dimensional manifold $\mathcal{W}_g \cap H_0$. Taking into account of

$$\begin{aligned} \widetilde{A}_* &= \delta\alpha \\ \lambda_r, \lambda_i &\sim \frac{\delta^{1/2} \alpha^{1/2}}{2^{1/4}}, \quad \varepsilon = \alpha^{7/2}, \\ \widetilde{A}_* \overline{x}_{10} &\sim \frac{\delta}{\sqrt{2}} B_{00}, \\ |\overline{Y}(0)| &= \mathcal{O}(\varepsilon^{2/7} |\overline{X}_0| + B_{00}), \end{aligned}$$

we obtain a two-dimensional intersection which is tangent to a plane (parameters $\overline{x}_1, \overline{x}_2$) with principal part given by

$$\begin{aligned} A_0 &= \delta\alpha + \frac{\varepsilon \delta^{1/2}}{2^{3/4} \alpha^{1/2}} B_{00} (\overline{x}_1 - \overline{x}_2) \\ A_1 &= \varepsilon \delta B_{00} \overline{x}_1 - \frac{(1 + \delta^2)}{\sqrt{2}} B_{00}^3 \frac{\varepsilon \delta}{\alpha} \\ A_2 &= \frac{\delta^{3/2}}{2^{1/4}} B_{00} \alpha^{1/2} \varepsilon (\overline{x}_1 + \overline{x}_2) + \mathcal{O}(\alpha^{1/2} \varepsilon) \\ A_3 &= \sqrt{2} \delta^2 B_{00} \varepsilon \alpha \overline{x}_2 + \mathcal{O}(\alpha \varepsilon) \\ B_{00} &= \sqrt{1 - \eta_0^2 \delta^2} \sim (1 + \delta^2)^{-1/2}, \end{aligned} \tag{49}$$

with

$$|\overline{x}_1| + |\overline{x}_2| \leq \rho, \quad \delta_0 \leq \delta \leq \delta_1, \quad \varepsilon = \alpha^{7/2}, \quad \alpha^2 = \eta_0^2 (1 + \delta^2) - 1 > 0,$$

and where we do not write B_1 since we know that this manifold lies in the 5 dimensional manifold \mathcal{W}_g .

5 Stable manifold of M_+

We show the following

Lemma 17 For ε small enough, $\alpha = [\eta_0^2(1+\delta^2)-1]^{1/2}$ small enough, $\delta_0 \leq \delta \leq 0.95$, the 3-dimensional stable manifold of M_+ is included in the 5-dimensional manifold \mathcal{W}_g , it exists for A_0, A_1, A_2, A_3 in a ball of small radius η (independent of ε, α), is analytic in parameters (ε, δ) , and reaches $B_0(0) = B_{00} \stackrel{\text{def}}{=} \sqrt{1 - \eta_0^2 \delta^2}$. Moreover as $x \rightarrow +\infty$, $(A_0, A_1, A_2, A_3, B_0 - 1, B_1) \rightarrow 0$ as $\exp(-\sqrt{2}\varepsilon x)$,

$$z_0 + z_1 \stackrel{\text{def}}{=} \frac{B_0 - 1}{\delta^{1/2}} \simeq -\frac{(1 - B_{00})(1 - \tanh(\varepsilon x/\sqrt{2}))}{1 + B_{00} \tanh(\varepsilon x/\sqrt{2})}, \quad (50)$$

$$|z_0 + z_1| \leq 0.282.$$

Let us define

$$\delta' = \delta^{1/2},$$

and choose a new basis

$$V_r^- = \begin{pmatrix} 1 \\ -\frac{\delta'}{\sqrt{2}} \\ 0 \\ \frac{\delta'^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \quad V_i^- = \begin{pmatrix} 0 \\ -\frac{\delta'}{\sqrt{2}} \\ \delta'^2 \\ -\frac{\delta'^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix},$$

$$V_r^+ = \begin{pmatrix} 1 \\ \frac{\delta'}{\sqrt{2}} \\ 0 \\ -\frac{\delta'^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \quad V_i^+ = \begin{pmatrix} 0 \\ \frac{\delta'}{\sqrt{2}} \\ \delta'^2 \\ \frac{\delta'^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix},$$

$$W_1^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -\varepsilon\sqrt{2} \end{pmatrix}, \quad W_1^+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \varepsilon\sqrt{2} \end{pmatrix},$$

for defining new coordinates $(x_1, x_2, y_1, y_2, z_0, z_1)$ such that

$$Z = (0, 0, 0, 0, 1, 0)^t + \delta' x_1 V_r^- + \delta' x_2 V_i^- + \delta' y_1 V_r^+ + \delta' y_2 V_i^+ + \delta' z_0 W_1^- + \delta' z_1 W_1^+$$

$$\begin{aligned}
A_0 &= \delta'(x_1 + y_1) \\
A_1 &= -\frac{\delta'^2}{\sqrt{2}}(x_1 - y_1 + x_2 - y_2) \\
A_2 &= \delta'^3(x_2 + y_2) \\
A_3 &= \frac{\delta'^4}{\sqrt{2}}(x_1 - y_1 - x_2 + y_2) \\
B_0 &= 1 + \delta'(z_0 + z_1) \\
B_1 &= -\varepsilon\sqrt{2}\delta'(z_0 - z_1).
\end{aligned} \tag{51}$$

A simple resolution leads to

$$\begin{aligned}
x_1 &= \frac{A_0}{2\delta'} - \frac{A_1}{2\sqrt{2}\delta'^2} + \frac{A_3}{2\sqrt{2}\delta'^4} \\
x_2 &= -\frac{A_1}{2\sqrt{2}\delta'^2} + \frac{A_2}{2\delta'^3} - \frac{A_3}{2\sqrt{2}\delta'^4} \\
y_1 &= \frac{A_0}{2\delta'} + \frac{A_1}{2\sqrt{2}\delta'^2} - \frac{A_3}{2\sqrt{2}\delta'^4} \\
y_2 &= \frac{A_1}{2\sqrt{2}\delta'^2} + \frac{A_2}{2\delta'^3} + \frac{A_3}{2\sqrt{2}\delta'^4} \\
z_0 &= \frac{B_0 - 1}{2\delta'} - \frac{B_1}{2\varepsilon\delta'\sqrt{2}} \\
z_1 &= \frac{B_0 - 1}{2\delta'} + \frac{B_1}{2\varepsilon\delta'\sqrt{2}}.
\end{aligned}$$

Let us define

$$\begin{aligned}
A_0 &= \delta'(x_1 + y_1) = \delta'u \\
B_0 &= 1 + \delta'v,
\end{aligned} \tag{52}$$

then system (1) reads as

$$\begin{aligned}
A'_0 &= A_1, \\
A'_1 &= A_2, \\
A'_2 &= A_3, \\
A'_3 &= -A_0(\delta^2 + 2\delta^2v + \delta u^2 + (1 + \delta^2)v^2), \\
v' &= \frac{1}{\delta'}B_1, \\
B'_1 &= \varepsilon^2(1 + \delta'v)(2\delta'v + \delta v^2 + (1 + \delta^2)\delta u^2).
\end{aligned}$$

With variables (51) this gives

$$\begin{aligned}
x'_1 &= -\frac{\delta'}{\sqrt{2}}(x_1 + x_2) - \frac{\delta'ug(u, v)}{2\sqrt{2}}, \\
x'_2 &= \frac{\delta'}{\sqrt{2}}(x_1 - x_2) + \frac{\delta'ug(u, v)}{2\sqrt{2}},
\end{aligned}$$

$$\begin{aligned} y'_1 &= \frac{\delta'}{\sqrt{2}}(y_1 + y_2) + \frac{\delta'ug(u, v)}{2\sqrt{2}}, \\ y'_2 &= -\frac{\delta'}{\sqrt{2}}(y_1 - y_2) - \frac{\delta'ug(u, v)}{2\sqrt{2}}, \end{aligned}$$

$$z'_0 = -\varepsilon\sqrt{2}z_0 - \frac{\varepsilon\delta'}{2\sqrt{2}}f(u, v),$$

$$z'_1 = \varepsilon\sqrt{2}z_1 + \frac{\varepsilon\delta'}{2\sqrt{2}}f(u, v),$$

$$\begin{aligned} g(u, v) &= u^2 + 2\delta v + (1 + \delta^2)v^2 \\ f(u, v) &= 3v^2 + \delta'v^3 + (1 + \delta^2)(1 + \delta'v)u^2, \end{aligned}$$

where the linear part is as expected.

For finding the stable manifold of M_+ we put the system in an integral form, looking for solutions tending to 0 as $x \rightarrow +\infty$:

$$\begin{aligned} X(x) &= e^{-\mathbf{L}x}X_0 - \frac{\delta'}{2\sqrt{2}} \int_0^x e^{-\mathbf{L}(x-s)}u(s)G(u, v)(s)ds, \\ Y(x) &= -\frac{\delta'}{2\sqrt{2}} \int_x^{+\infty} e^{\mathbf{L}(x-s)}u(s)G(u, v)(s)ds, \end{aligned} \quad (53)$$

$$\begin{aligned} z_0(x) &= e^{-\varepsilon\sqrt{2}x}z_{00} - \frac{\varepsilon\delta'}{2\sqrt{2}} \int_0^x e^{-\varepsilon\sqrt{2}(x-s)}f(u, v)(s)ds, \\ z_1(x) &= -\frac{\varepsilon\delta'}{2\sqrt{2}} \int_x^{+\infty} e^{\varepsilon\sqrt{2}(x-s)}f(u, v)(s)ds, \end{aligned} \quad (54)$$

where

$$G = \begin{pmatrix} g \\ -g \end{pmatrix}, \quad \mathbf{L} = \frac{\delta'}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

We notice that

$$e^{\mathbf{L}x} = e^{\frac{\delta'x}{\sqrt{2}}} \begin{pmatrix} \cos \frac{\delta'x}{\sqrt{2}} & \sin \frac{\delta'x}{\sqrt{2}} \\ -\sin \frac{\delta'x}{\sqrt{2}} & \cos \frac{\delta'x}{\sqrt{2}} \end{pmatrix}, \quad (55)$$

$$\|e^{-\mathbf{L}x}\| \leq e^{-\frac{\delta'x}{\sqrt{2}}}, \quad x > 0,$$

hence, we have the estimates (for $x \geq 0$)

$$\begin{aligned} e^{\varepsilon x\sqrt{2}} \left\| \int_0^x e^{-\mathbf{L}(x-s)} e^{-\varepsilon s\sqrt{2}} ds \right\| &\leq \int_0^x e^{-\frac{(\delta'-2\varepsilon)(x-s)}{\sqrt{2}}} ds \leq \frac{\sqrt{2}}{\delta'-2\varepsilon}, \\ e^{\varepsilon x\sqrt{2}} \left\| \int_x^\infty e^{\mathbf{L}(x-s)} e^{-\varepsilon s\sqrt{2}} ds \right\| &\leq \int_x^\infty e^{\frac{(\delta'+2\varepsilon)(x-s)}{\sqrt{2}}} ds = \frac{\sqrt{2}}{\delta'+2\varepsilon}. \end{aligned}$$

We need a precise estimate on $|z_{00}|$, for being able to obtain an intersection of the stable manifold of M_+ with the unstable manifold of M_- computed previously at Lemma 10. We need to reach values of (X_0, z_{00}) such that

$$z_{00} + z_1(0) = -h(\delta) + \frac{\alpha^2 \delta^2}{2\delta'(1+\delta^2)} + \mathcal{O}(\alpha^4) < 0$$

$|X_0|$ close to 0,

with

$$h(\delta) = \frac{1}{\delta^{1/2}} \left(1 - \sqrt{1 - \frac{\delta^2}{1 + \delta^2}}\right),$$

and where

$$\eta_0 - \frac{1}{\sqrt{1 + \delta^2}} \sim \frac{\alpha^2}{2\sqrt{1 + \delta^2}}.$$

We notice that $|z_{00} + z_1(0)|$ is very closely below $h(\delta)$. Moreover, we notice that the maximum of $h(\delta)$ obtained for $\delta = \frac{1+\sqrt{5}}{2} \simeq 1.618$, is such that

$$h_{\max} = \left(\frac{1 + \sqrt{5}}{2}\right)^{-2} \simeq 0.382.$$

Remark 18 *The strategy is to first solve (53) with respect to (X, Y) in function of $(X_0, v) \in \mathbb{R}^2 \times C_{\varepsilon\sqrt{2}}^0$ (defined below) and then to solve the first integral (3) with respect to $(z_0 - z_1)$ in function of (X_0, v) , which gives an integro differential equation for v . It is then possible to solve this integro-differential equation for $v(x)$ in $C_{\varepsilon\sqrt{2}}^0$ in function of (X_0, z_{00}) . The stable manifold of M_+ is then obtained with (X, Y, z_0, z_1) as a function of (X_0, z_{00}) .*

Let us define for this section

$$C_{\kappa}^0 = \{\bar{X} \in C^0[0, +\infty); \bar{X}(x)e^{\kappa x} \text{ is bounded}\}$$

equipped with the norm

$$\|\bar{X}\|_{\kappa} = \sup_{(0, +\infty)} |\bar{X}(x)e^{\kappa x}|.$$

Using (55), the system (53,54) gives two scalar equations with unknown functions (u, v) . We obtain for $u(x)$:

$$u(x) = e^{-\frac{\delta'x}{\sqrt{2}}} u_0(x) - \frac{\delta'}{2} \int_0^{\infty} e^{-\frac{\delta'|x-s|}{\sqrt{2}}} \cos\left[\frac{\delta'|x-s|}{\sqrt{2}} - \frac{\pi}{4}\right] u(s) g(u, v)(s) ds, \quad (56)$$

with

$$u_0(x) = x_{10} \cos \frac{\delta'x}{\sqrt{2}} - x_{20} \sin \frac{\delta'x}{\sqrt{2}}.$$

Hence, we obtain the following estimate in $C^0_{\varepsilon\sqrt{2}}$,

$$\|u\|_{\varepsilon\sqrt{2}} \leq \|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}} + \frac{\sqrt{2}\delta}{(\delta - 4\varepsilon^2)} \|u\|_{\varepsilon\sqrt{2}} \|g\|_0.$$

Moreover we have

$$\begin{aligned} \|g\|_0 &\leq \|u\|_{\varepsilon\sqrt{2}}^2 + 2\delta \|v\|_{\varepsilon\sqrt{2}} + (1 + \delta^2) \|v\|_{\varepsilon\sqrt{2}}^2, \\ &\leq \|u\|_{\varepsilon\sqrt{2}}^2 + 3\|v\|_{\varepsilon\sqrt{2}} \end{aligned} \quad (57)$$

which, for

$$\begin{aligned} \delta &\leq 0.95 \\ \|v\|_{\varepsilon\sqrt{2}} &\leq h(\delta) = 0.282, \end{aligned}$$

gives

$$\sqrt{2}[2\delta\|v\|_{\varepsilon\sqrt{2}} + (1 + \delta^2)\|v\|_{\varepsilon\sqrt{2}}^2] < 0.972 < 1.$$

It results that for $\|v\|_{\varepsilon\sqrt{2}} \leq 0.282$ we can bound $\|u\|_{\varepsilon\sqrt{2}}$ as

$$0.028\|u\|_{\varepsilon\sqrt{2}} \leq \|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}} + \frac{\sqrt{2}\delta}{(\delta - 4\varepsilon^2)} \|u\|_{\varepsilon\sqrt{2}}^3$$

which, for

$$|X_0| \leq 0.0015$$

and ε small enough, leads to

$$\begin{aligned} \|u\|_{\varepsilon\sqrt{2}} &\leq 50\|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}} \leq 50|X_0|, \\ \|g\|_0 &\leq \frac{0.98}{\sqrt{2}}. \end{aligned} \quad (58)$$

Remark 19 *The fact that $(x_1 + y_1)$ is in factor of $g(x_1 + y_1, z_0 + z_1)$ for the estimates of $\|X\|_{\varepsilon\sqrt{2}}$ and $\|Y\|_{\varepsilon\sqrt{2}}$ is essential here, since this leaves a freedom on $\|z_0 + z_1\|_{\varepsilon\sqrt{2}}$. This is the main difference with section 4, and this allows to use the direct method for the stable manifold of M_+ , contrary to what we did for the computation of the unstable manifold of M_- . In fact, using the direct method for the unstable manifold would give a too small bound for B_0 , with no possibility of connection with the stable manifold of M_+ .*

Then we obtain

$$\begin{aligned} \|x_j\|_{\varepsilon\sqrt{2}} &\leq \|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}} + \frac{\delta'}{2(\delta' - 2\varepsilon)} \|u\|_{\varepsilon\sqrt{2}} \|g\| \\ &\leq \|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}} + \frac{1}{2\sqrt{2}} \|u\|_{\varepsilon\sqrt{2}} \\ &\leq \left(1 + \frac{25}{\sqrt{2}}\right) \|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}} = 18.7 \|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}} \end{aligned} \quad (59)$$

$$\begin{aligned} \|y_j\|_{\varepsilon\sqrt{2}} &\leq \frac{1}{2} \|u\|_{\varepsilon\sqrt{2}} (\|u\|_{\varepsilon\sqrt{2}}^2 + 3\|v\|_{\varepsilon\sqrt{2}}) \\ &\leq \frac{25}{\sqrt{2}} \|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}} = 17.7 \|X_0 e^{-\frac{\delta'x}{\sqrt{2}}}\|_{\varepsilon\sqrt{2}}, \end{aligned} \quad (60)$$

which is valid, as soon as ε is small enough and

$$|X_0| \leq 0.0015, \quad \delta_0 \leq \delta \leq 0.95, \quad \|z_0 + z_1\|_{\varepsilon\sqrt{2}} \leq 0.282$$

holds.

5.1 Using the first integral (3)

For extending the domain of validity for the stable manifold of M_+ , instead of using the differential equations for z_0 and z_1 we use the first integral (3):

$$B_1^2 = \frac{\varepsilon^2}{2}(B_0^2 - 1 + A_0^2)^2 + \varepsilon^2\delta^2 A_0^2 B_0^2 + 2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2,$$

hence

$$\begin{aligned} B_1^2 &= \frac{\varepsilon^2}{2}[(B_0^2 - 1)^2 + 2\delta(1 + \delta^2)u^2(2\delta'v + \delta v^2) \\ &\quad + \delta^2 u^4 + 8\delta^3(x_1 y_1 - x_2 y_2)]. \end{aligned}$$

Taking the square root gives the traces of the stable and of the unstable manifolds on \mathcal{W}_g . The stable manifold needs satisfy $B_1 = B'_0 > 0$, since $B_0 < 1$ for $x = 0$, $B_0 = 1$ for $x = \infty$, and the sign of B_1 does not change in the interval. Hence

$$B_1 = \frac{\varepsilon}{\sqrt{2}}(1 - B_0^2) \left(1 + \frac{2\delta'(1 + \delta^2)u^2}{(2v + \delta'v^2)} + \frac{\delta u^4 + 8\delta^3(x_1 y_1 - x_2 y_2)}{(2v + \delta'v^2)^2} \right)^{1/2} \quad (61)$$

and the estimates we found for $\|X\|_{\varepsilon\sqrt{2}}$ and $\|Y\|_{\varepsilon\sqrt{2}}$ may be used.

Remark 20 *We notice that this implies that $v < 0$, $v' > 0$, and $|v(x)|_{\max} = |z_{00} + z_1(0)|$ is then $\mathcal{O}(\alpha^2)$ close to $h(\delta)$.*

We observe that

$$B_1 = B'_0 = \frac{\varepsilon}{\sqrt{2}}[1 - B_0^2]$$

may be easily integrated ($B_0(0) = \sqrt{1 - \eta_0^2 \delta^2}$, $B_0(\infty) = 1$), and moreover leads to

$$z_0 - z_1 = \frac{1}{2}v(2 + \delta'v)$$

i.e.

$$z_1 = -\frac{\delta'}{4}(z_0 + z_1)^2 < 0 \quad (62)$$

which is the solution of (54) for $u = 0$. Let us show that the expression above for B_1 , is valid. Using $\|v\|_{\varepsilon\sqrt{2}} \leq 0.282$, and $\delta \leq 0.95$, we obtain

$$\begin{aligned} 1 - B_0^2 &= |\delta'v|(2 - |\delta'v|) \\ &\geq 1.725\delta'|v| \end{aligned}$$

so that

$$\begin{aligned} & \left| \frac{2\delta'(1+\delta^2)u^2}{(2v+\delta'v^2)} + \frac{\delta u^4 + 8\delta^2(x_1y_1 - x_2y_2)}{(2v+\delta'v^2)^2} \right| \\ & \leq \frac{4u^2 + 0.24|u|}{1.725|v|} + \frac{u^4 + 0.02u^2}{(1.725)^2v^2}. \end{aligned}$$

In using (59,60)

$$\begin{aligned} |8\delta^2(x_1y_1 - x_2y_2)| & \leq 160|X_0||u| (|u|^2 + 3|v|) \\ & \leq 480|X_0||u| \left(\frac{1}{40}|u| + |v| \right) \\ & \leq 0.72|u| \left(\frac{1}{40}|u| + |v| \right), \end{aligned}$$

and we also have

$$|8\delta^2(x_1y_1 - x_2y_2)| \leq 8|X||u|(0.075|u| + 3|v|).$$

We now assume that for $x \in (0, \infty)$

$$|u(x)| \leq 0.9|v(x)|, \quad (63)$$

which has to be checked at the end. Then

$$\left| \frac{2\delta'(1+\delta^2)u^2}{(2v+\delta'v^2)} + \frac{\delta u^4 + 8\delta^2(x_1y_1 - x_2y_2)}{(2v+\delta'v^2)^2} \right| \leq 0.68,$$

and also

$$\left| \frac{2\delta'(1+\delta^2)u^2}{(2v+\delta'v^2)} + \frac{\delta u^4 + 8\delta^2(x_1y_1 - x_2y_2)}{(2v+\delta'v^2)^2} \right| \leq 9.383(|X| + |Y|)$$

implying a smooth function for the square root, provided that we check a posteriori (63) and that $|X| + |Y|$ is small enough. It follows from (61) that $z_0 - z_1$ is a smooth function of (v, X, Y) for ε small enough,

$$\begin{aligned} \delta & < 0.95, \quad |v| \leq 0.282, \\ |X| + |Y| & \leq 0.9|v|, \quad |X| + |Y| \text{ small enough,} \end{aligned}$$

so that

$$\begin{aligned} z_0 - z_1 & = \frac{1}{2}v(2 + \delta'v)[1 + \mathcal{Z}(X, Y)], \\ |\mathcal{Z}(X, Y)| & \leq 10(|X| + |Y|). \end{aligned}$$

It results that

$$-\varepsilon\sqrt{2}(z_0 - z_1) = v' = -\varepsilon\sqrt{2}v\left(1 + \frac{\delta'}{2}v\right)[1 + \mathcal{Z}(X, Y)], \quad (64)$$

where X, Y are expressed in function of (X_0, v) (non local expression in v). Then we can integrate the integro-differential equation, as in section 4.10. We introduce the new variable w as

$$\begin{aligned} w' &= \frac{v'}{v(1 + (\delta'/2)v)}, \\ w &= \ln\left(\frac{-v}{1 + (\delta'/2)v}\right), \\ v &= -\frac{e^w}{1 + \frac{\delta'}{2}e^w}; \end{aligned}$$

w decreases from w_0 to $-\infty$ for $x \in (0, \infty)$, while v grows from $v_0 < 0$ to 0. We observe now that we have

$$\begin{aligned} |\mathcal{Z}(X(X_0, v), Y(X_0, v))| &\leq 10(|X| + |Y|) \leq 20(18.7 + 17.7)|X_0| \\ &\leq 740|X_0|, \end{aligned}$$

so that, expressed in term of w , we have

$$\begin{aligned} \mathcal{Z}(X(X_0, v(w)), Y(X_0, v(w))) &= h(w), \\ |h(w)| &\leq c|X_0| < 1/2 \end{aligned}$$

for $|X_0|$ small enough. We then obtain, by simple integration

$$\varepsilon\sqrt{2}x(1 - c|X_0|) \leq w_0 - w(x) \leq \varepsilon\sqrt{2}x(1 + c|X_0|).$$

Remark 21 *The constant c above may be replaced by*

$$ce^{-\varepsilon\sqrt{2}x}$$

since $|X|$ and $|Y|$ lie in $C_{\varepsilon\sqrt{2}}^0$.

We deduce the estimate

$$v_0 \frac{1 - \tanh\left(\frac{\varepsilon x(1-c|X_0|)}{\sqrt{2}}\right)}{1 + B_{00} \tanh\left(\frac{\varepsilon x(1-c|X_0|)}{\sqrt{2}}\right)} \leq v(x) \leq v_0 \frac{1 - \tanh\left(\frac{\varepsilon x(1+c|X_0|)}{\sqrt{2}}\right)}{1 + B_{00} \tanh\left(\frac{\varepsilon x(1+c|X_0|)}{\sqrt{2}}\right)} \quad (65)$$

where

$$v_0 = \frac{B_{00} - 1}{\delta'}, \quad B_{00} = B_0(0).$$

The a priori estimate for v obtained in (65) allows to prove (see [2]) the existence and uniqueness of a solution for (64), provided that (63) is satisfied on the whole interval $x \in [0, \infty)$. For checking this, we notice from (65) that

$$|v(x)| \geq |v_0|e^{-\varepsilon\sqrt{2}x}, \quad |u(x)| \leq 50|X_0|e^{-\frac{\delta'x}{\sqrt{2}}},$$

where the last estimate results from the bound for $\|g\|_0$ and an estimate for $u(x)e^{\frac{\delta'x}{\sqrt{2}}}$ from (56). Hence (63) is satisfied as soon as

$$50|X_0| \leq 0.9|v_0|,$$

which is OK for $|X_0|$ small enough. Lemma 17 is proved.

5.2 Intersection of the stable manifold with H_0

We need to compute the intersection of the 3-dimensional stable manifold of M_+ with the hyperplane H_0 defined by

$$B_0 = \sqrt{1 - \eta_0^2 \delta^2}. \quad (66)$$

We then obtain a 2-dimensional sub-manifold living in the 4-dimensional manifold $\mathcal{W}_g \cap H_0$. We have by construction

$$\begin{aligned} A_0 &= \delta^{1/2}(x_{10} + y_{10}), \\ A_1 &= -\frac{\delta}{\sqrt{2}}(x_{10} + x_{20} - y_{10} - y_{20}) \\ A_2 &= \delta^{3/2}(x_{20} + y_{20}) \\ A_3 &= \frac{\delta^2}{\sqrt{2}}(x_{10} - x_{20} - y_{10} + y_{20}), \end{aligned} \quad (67)$$

where y_{10} and y_{20} are expressed in function of $X_0 = (x_{10}, x_{20})$, with the restriction

$$|x_{10}| + |x_{20}| \leq \eta.$$

6 Intersection of the two manifolds

We need to see the intersection of the plane (49) tangent to the unstable manifold of M_- , with the plane tangent to the stable manifold of M_+ given by (67).

We then find a linear system with 4 unknowns $(\bar{x}_1^{(u)}, \bar{x}_2^{(u)}, x_{10}^{(s)}, x_{20}^{(s)})$, with the restrictions

$$|x_{10}^{(s)}| + |x_{20}^{(s)}| \leq \eta, \quad |\bar{x}_1^{(u)}| + |\bar{x}_2^{(u)}| \leq \rho.$$

We then have

$$\begin{aligned} (x_{10}^{(s)} + y_{10}^{(s)}) &= \delta' \alpha + \frac{\varepsilon}{2^{3/4} \alpha^{1/2}} B_{00} (\bar{x}_1^{(u)} - \bar{x}_2^{(u)}) \\ -(x_{10}^{(s)} + x_{20}^{(s)} - y_{10}^{(s)} - y_{20}^{(s)}) &= \sqrt{2} \varepsilon B_{00} \bar{x}_1^{(u)} - (1 + \delta^2) B_{00} \frac{\varepsilon}{\alpha} \\ (x_{20}^{(s)} + y_{20}^{(s)}) &= \frac{\alpha^{1/2} \varepsilon}{2^{1/4}} B_0 (\bar{x}_1^{(u)} + \bar{x}_2^{(u)}) \\ (x_{10}^{(s)} - x_{20}^{(s)} - y_{10}^{(s)} + y_{20}^{(s)}) &= 2\alpha \varepsilon B_0 \bar{x}_2^{(u)}, \end{aligned} \quad (68)$$

where we need to express $(y_{10}^{(s)}, y_{20}^{(s)})$ as a linear function of $(x_{10}^{(s)}, x_{20}^{(s)})$. Let us define

$$X_0^{(s)} = \begin{pmatrix} x_{10}^{(s)} \\ x_{20}^{(s)} \end{pmatrix}, \quad Y_0^{(s)} = \begin{pmatrix} y_{10}^{(s)} \\ y_{20}^{(s)} \end{pmatrix}, \quad \bar{X}^{(u)} = \begin{pmatrix} \bar{x}_1^{(u)} \\ \bar{x}_2^{(u)} \end{pmatrix},$$

then we have (with $c = \frac{1+\delta^2}{4} B_{00}$)

$$X_0^{(s)} = \begin{pmatrix} \frac{\delta'}{2} \alpha + c \frac{\varepsilon}{\alpha} \\ c \frac{\varepsilon}{\alpha} \end{pmatrix} + M_1 \bar{X}^{(u)},$$

$$Y_0^{(s)} = \begin{pmatrix} \frac{\delta'}{2}\alpha - c\frac{\varepsilon}{\alpha} \\ 0 \end{pmatrix} + \mathbf{M}_2 \bar{X}^{(u)},$$

with

$$\begin{aligned} \mathbf{M}_1 &= \frac{\varepsilon B_0 2^{1/4}}{4\alpha^{1/2}} \begin{pmatrix} 1 - 2^{1/4}\alpha^{1/2} & -1 + 2^{3/4}\alpha^{3/2} \\ -2^{1/4}\alpha^{1/2} + \sqrt{2}\alpha & \sqrt{2}\alpha - 2^{3/4}\alpha^{3/2} \end{pmatrix}, \\ \mathbf{M}_2 &= \frac{\varepsilon B_0 2^{1/4}}{4\alpha^{1/2}} \begin{pmatrix} 1 + 2^{1/4}\alpha^{1/2} & -1 - 2^{3/4}\alpha^{3/2} \\ 2^{1/4}\alpha^{1/2} + \sqrt{2}\alpha & \sqrt{2}\alpha + 2^{3/4}\alpha^{3/2} \end{pmatrix}. \end{aligned}$$

The matrix \mathbf{M}_2 is invertible with

$$\mathbf{M}_2^{-1} = \frac{4\alpha^{1/2}}{\varepsilon B_0 2^{1/4} \det(\mathbf{M}'_2)} \begin{pmatrix} \sqrt{2}\alpha + 2^{3/4}\alpha^{3/2} & 1 + 2^{3/4}\alpha^{3/2} \\ -2^{1/4}\alpha^{1/2} - \sqrt{2}\alpha & 1 + 2^{1/4}\alpha^{1/2} \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{M}'_2) &= [\sqrt{2}\alpha(1 + 2^{1/4}\alpha^{1/2})^2 + (1 - 2^{3/4}\alpha^{3/2})(2^{1/4}\alpha^{1/2} + \sqrt{2}\alpha)] \\ &= 2^{1/4}\alpha^{1/2} + 2\sqrt{2}\alpha + \mathcal{O}(\alpha^{3/2}). \end{aligned}$$

It results that

$$\begin{aligned} \mathbf{M}_1 \mathbf{M}_2^{-1} &\sim \begin{pmatrix} 1 & 2^{3/2}\alpha \\ -2^{3/2}\alpha & -1 \end{pmatrix} \\ X_0^{(s)} &\sim \begin{pmatrix} 1 & 2^{3/2}\alpha \\ -2^{3/2}\alpha & -1 \end{pmatrix} Y_0^{(s)} + \begin{pmatrix} \frac{\delta'}{2}\alpha \\ \frac{\delta'}{2}\alpha \end{pmatrix}. \end{aligned} \quad (69)$$

Equation (69) represents a 2-dim affine plane resulting from the 4-dim linear system expressing the intersection of the two manifolds. This gives a condition on coordinates of the stable manifold, and shows that this affine plane needs to intersect the tangent plane to the stable manifold given by (67) with $Y_0^{(s)}$ expressed as a linear function of $X_0^{(s)}$.

Let us show below (subsection 6.0.1) that in restricting a little δ , and for α small enough, then, for the tangent plane to the stable manifold at the intersection with H_0 , we have

$$|Y_0^{(s)}| \leq k |X_0^{(s)}|, \quad k < 1. \quad (70)$$

This gives bounds for the slope of the tangent plane to this intersection which passes through the origin in $(X^{(s)}, Y^{(s)})$: using (69), we see that

$$|X_0^{(s)}| \leq k |X_0^{(s)}| + \mathcal{O}(\alpha),$$

hence this tangent plane intersects transversally the affine plane (69), and defines a unique point $(X_0^{(s)}, Y_0^{(s)}) = \mathcal{O}(\alpha)$. This satisfies the constraint of order η on $(X_0^{(s)}, Y_0^{(s)})$. Then $\bar{X}^{(u)}$ is uniquely determined in using (68). Finally, from (49)

we obtain

$$\begin{aligned}
A_0(0) &\sim \alpha\delta \\
A_1(0) &\sim -\frac{(1+\delta^2)\delta}{\sqrt{2}}B_{00}^3\frac{\varepsilon}{\alpha} \\
A_2(0) &= \mathcal{O}(B_{00}\varepsilon\alpha^{1/2}) \\
A_3(0) &= \mathcal{O}(B_{00}\varepsilon\alpha^{1/2}).
\end{aligned}$$

This proves that the intersection between the unstable manifold of M_- and the stable manifold of M_+ is transverse while they both sit on \mathcal{W}_g and cross the hyperplane (66). Since it is the transverse intersection of two manifolds, depending analytically on parameters (ε, δ) , the resulting curve depends analytically on these parameters.

We observe that, along this intersection, and by construction, $B_1(x) = B'_0(x) > 0$. Its principal part on $(-\infty, 0]$ is given by (43) with $B_0(0) = B_{00} = \sqrt{1 - \eta_0^2\delta^2}$, and on $[0, +\infty)$ by (50).

The Theorem 1 is then proved. Moreover, since (51), (59), (60) hold we also have the following

Corollary 22 *For $x \in [0, +\infty)$ there exists $c > 0$ independent of ε, δ such that for the heteroclinic curve*

$$|A_0^{(m)}(x)| \leq c\alpha\delta e^{-\frac{\delta'x}{\sqrt{2}}}, \quad m = 0, 1, 2, 3.$$

6.0.1 Proof of (70)

The tangent plane to the stable manifold is given by Y_0 expressed with (53) for $x = 0$, $u(x)$ (function of X_0) given by (56), where u is replaced by 0 in $g(u, v)$ and $G(u, v)$. Then we obtain the estimates

$$\begin{aligned}
|Y_0| &\leq \frac{1}{2}\|u\|_0\|g\|_0, \\
\|u\|_0 &\leq |X_0| + \sqrt{2}\|u\|_0\|g\|_0,
\end{aligned}$$

hence

$$|Y_0| \leq \frac{1}{2} \frac{\|g\|_0}{1 - \sqrt{2}\|g\|_0} |X_0|,$$

where

$$\|g\|_0 \leq 2\delta\|v\|_0 + (1 + \delta^2)\|v\|_0^2.$$

Observing that $\|v\|_0$ is bounded by $h(\delta)$, it is easy to check that for

$$\delta \leq 0.825,$$

we obtain $h(\delta) = 0.2779$, so that

$$\|g\|_0 \leq 0.5218,$$

and

$$\frac{1}{2} \frac{\|g\|_0}{1 - \sqrt{2}\|g\|_0} \leq k < 1,$$

with $k = 0.9956$.

7 Study of the linearized operator

Let us redefine the heteroclinic connection we found at Theorem 1 as

$$(A_*(x), B_*(x)) \subset \mathbb{R}^2$$

with

$$1 < 1 + \delta_0^2 \leq g = 1 + \delta^2 \leq 1 + (0.825)^2,$$

and where we know that, for ε small enough

$$\begin{aligned} B_*(x) &> 0, \quad B'_*(x) > 0 \\ (A_*(x), B_*(x)) &\rightarrow \begin{cases} (1, 0) \text{ as } x \rightarrow -\infty \\ (0, 1) \text{ as } x \rightarrow +\infty \end{cases}, \end{aligned}$$

at least as $e^{\varepsilon\delta x}$ for $x \rightarrow -\infty$, and at least as $e^{-\sqrt{2}\varepsilon x}$ for $x \rightarrow +\infty$.

The system (1) is now considered with B_0 complex valued, so in (1) B^2 is replaced by $|B|^2$.

For being able to prove any persistence result under reversible perturbations of system (1) in $\mathbb{R}^4 \times \mathbb{C}^2$ we need to study the linearized operator at the above heteroclinic solution. We follow the lines of [3].

The linearized operator is given by

$$\begin{aligned} A^{(4)} &= (1 - 3A_*^2 - gB_*^2)A - gA_*B_*(B + \overline{B}), \\ B'' &= \varepsilon^2(-1 + gA_*^2 + 2B_*^2)B + 2\varepsilon^2gA_*B_*A + \varepsilon^2B_*^2\overline{B}. \end{aligned}$$

Taking real and imaginary parts for B :

$$B = C + iD,$$

we then obtain the linearized system

$$\begin{aligned} -A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C &= 0, \\ \frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A &= 0, \\ \frac{1}{\varepsilon^2}D'' + (1 - gA_*^2 - B_*^2)D &= 0. \end{aligned}$$

Notice that the equation for D decouples, so that we can split the linear operator in an operator \mathcal{M}_g acting on (A, C) and an operator \mathcal{L}_g acting on D :

$$\mathcal{M}_g \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C \\ \frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A \end{pmatrix},$$

$$\mathcal{L}_g D = \frac{1}{\varepsilon^2} D'' + (1 - gA_*^2 - B_*^2)D.$$

Let us define the Hilbert spaces

$$L_\eta^2 = \{u; u(x)e^{\eta|x|} \in L^2(\mathbb{R})\},$$

$$\mathcal{D}_0 = \{(A, C) \in H_\eta^4 \times H_\eta^2; A \in H_\eta^4, C \in \mathcal{D}_1\}$$

$$\mathcal{D}_1 = \{C \in H_\eta^2; \varepsilon^{-2} \|C''\|_{L_\eta^2} + \varepsilon^{-1} \|C'\|_{L_\eta^2} + \|C\|_{L_\eta^2} \stackrel{def}{=} \|C\|_{\mathcal{D}_1} < \infty\}$$

equipped with natural scalar products. Below, we prove the following

Lemma 23 *Except maybe for a set of isolated values of g , the kernel of \mathcal{M}_g in L_η^2 is one dimensional, span by (A'_*, B'_*) , and its range has codimension 1, L^2 -orthogonal to (A'_*, B'_*) . \mathcal{M}_g has a pseudo-inverse acting from L_η^2 to \mathcal{D}_0 for any $\eta > 0$ small enough, with bound independent of ε .*

The operator \mathcal{L}_g has a trivial kernel, and its range which has codimension 1, is L^2 -orthogonal to B_ ($B_* \notin L^2$). \mathcal{L}_g has a pseudo-inverse acting respectively from L_η^2 to \mathcal{D}_1 for $\eta > 0$ small enough, with bound independent of ε .*

Remark 24 *The above Lemma is useful for proving the persistence under reversible perturbations of our heteroclinic. This is done in a forthcoming paper and appears to be more difficult than the symmetric case solved in [3].*

7.1 Asymptotic operators

Let us define the operators obtained when $x = \pm\infty$:

$$\mathcal{M}_\infty^- \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} - 2A \\ \varepsilon^{-2} C'' - (g-1)C \end{pmatrix},$$

$$\mathcal{M}_\infty^+ \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} - (g-1)A \\ \varepsilon^{-2} C'' - 2C \end{pmatrix},$$

$$\mathcal{L}_\infty^- D = \varepsilon^{-2} D'' - (g-1)D,$$

$$\mathcal{L}_\infty^+ D = \varepsilon^{-2} D''.$$

Notice that all these operators are negative. Furthermore, their spectra in $L^2(\mathbb{R})$ are such that

$$\sigma(\mathcal{M}_\infty^-) = (-\infty, -c_-], \quad c_- = \max\{2, (g-1)\} > 0,$$

$$\sigma(\mathcal{M}_\infty^+) = (-\infty, -c_+], \quad c_+ = c_-,$$

$$\sigma(\mathcal{L}_\infty^-) = (-\infty, -(g-1)],$$

$$\sigma(\mathcal{L}_\infty^+) = (-\infty, 0].$$

Operators \mathcal{M}_g and \mathcal{L}_g are respectively relatively compact perturbations of the corresponding asymptotic operators \mathcal{M}_∞ and \mathcal{L}_∞ defined as

$$\mathcal{M}_\infty = \begin{cases} \mathcal{M}_\infty^-, & x < 0 \\ \mathcal{M}_\infty^+, & x > 0 \end{cases}, \quad \mathcal{L}_\infty = \begin{cases} \mathcal{L}_\infty^-, & x < 0 \\ \mathcal{L}_\infty^+, & x > 0 \end{cases},$$

Their essential spectrum, i.e. the set of $\lambda \in \mathbb{C}$ for which $\lambda - \mathcal{M}_g$ (resp. $\lambda - \mathcal{L}_g$) is not Fredholm with index 0, is equal to the essential spectrum of \mathcal{M}_∞ (resp. \mathcal{L}_∞) (see [6]). The latter spectra are found from the spectra of \mathcal{M}_∞^\pm and \mathcal{L}_∞^\pm :

$$\begin{aligned} \sigma_{ess}(\mathcal{M}_\infty) &= (-\infty, -c_+], \\ \sigma_{ess}(\mathcal{L}_\infty) &= (-\infty, 0]. \end{aligned}$$

In particular, this implies that 0 does not belong to the essential spectrum of \mathcal{M}_g , so that the operator \mathcal{M}_g is Fredholm with index 0. Moreover operators \mathcal{M}_∞ and \mathcal{L}_∞ are self adjoint negative operators in L^2 , and \mathcal{M}_∞ has a bounded inverse [6].

$$\|\mathcal{M}_\infty^{-1}\|_{L^2} \leq \frac{1}{c_+}.$$

This last property remains valid in exponentially weighted spaces, with weights $e^{\eta|x|}$, and η sufficiently small, since this acts as a small perturbation of the differential operator (see [5] section 3.1).

We show at section 7.3.1 that the kernel of \mathcal{M}_g is one-dimensional (except for a finite set of values of g), spanned by $(A'_*, B'_*) \stackrel{def}{=} U_*$ with a range orthogonal to U_* in L^2 . Let us define the projections Q_0 on U_*^\perp and P_0 on U_* , which are orthogonal projections in L^2 , then we need to solve in L_η^2

$$\mathcal{M}_g u = f$$

in decomposing

$$\begin{aligned} u &= zU_* + v, \quad v = Q_0 u, \\ (\mathcal{M}_\infty + \mathcal{A}_g)v &= Q_0 f \end{aligned}$$

and we need to satisfy the compatibility condition

$$\langle f, U_* \rangle = 0,$$

while z is arbitrary and we obtain for v :

$$(\mathbb{I} + \mathcal{M}_\infty^{-1} \mathcal{A}_g)v = \mathcal{M}_\infty^{-1} Q_0 f,$$

where the operator $\mathcal{M}_\infty^{-1} \mathcal{A}_g$ is now a compact operator for which -1 is not an eigenvalue, since $v \in U_*^\perp$. It results that there is a number c independent of ε such that

$$\|v\|_{L_\eta^2} \leq c \|f\|_{L_\eta^2}.$$

From the form of operator \mathcal{M}_g and using interpolation properties, we obtain for $v = (A, C)$

$$\|(A, C)\|_{\mathcal{D}_0} \leq c \|f\|_{L_\eta^2}$$

with a certain c independent of ε .

7.2 Properties of \mathcal{L}_g

Notice that \mathcal{L}_g is self adjoint in $L^2(\mathbb{R})$ and that

$$\mathcal{L}_g B_* = 0, \quad \text{but } B_* \notin L^2(\mathbb{R}).$$

This property allows to solve explicitly the equation $\mathcal{L}_g u = f \in L^2_\eta$ with respect to $u \in L^2_\eta$ (using variation of constants method), and shows that it has a unique solution, provided that

$$\int_{\mathbb{R}} f B_* dx = 0.$$

We obtain

$$\begin{aligned} u(x) &= \int_x^\infty \frac{\varepsilon^2 B_*(x)}{B_*^2(s)} F(s) ds \\ \text{with } F(s) &= \int_s^\infty f(\tau) B_*(\tau) d\tau \text{ for } s \geq 0 \\ &= - \int_{-\infty}^s f(\tau) B_*(\tau) d\tau \text{ for } s \leq 0. \end{aligned}$$

By Fubini's theorem we can write for $x \geq 0$

$$u(x) = \varepsilon^2 B_*(x) \int_x^\infty f(\tau) B_*(\tau) \left(\int_x^\tau \frac{ds}{B_*^2(s)} \right) d\tau$$

and, for $x \leq 0$

$$\begin{aligned} u(x) &= -\varepsilon^2 B_*(x) \int_{-\infty}^x f(\tau) B_*(\tau) \left(\int_x^0 \frac{ds}{B_*^2(s)} \right) d\tau \\ &\quad - \varepsilon^2 B_*(x) \int_x^0 f(\tau) B_*(\tau) \left(\int_\tau^0 \frac{ds}{B_*^2(s)} \right) d\tau. \end{aligned}$$

The asymptotic properties of $B_*(x)$ at $\pm\infty$ imply, for $x \geq 0$

$$|u(x)| e^{\eta x} \leq C \varepsilon^2 \int_x^\infty |f(\tau) e^{\eta \tau}| (\tau - x) e^{-\eta(\tau-x)} d\tau,$$

and for $x \leq 0$

$$\begin{aligned} |u(x)| e^{-\eta x} &\leq \frac{C \varepsilon^2}{2\varepsilon \delta} \int_{-\infty}^x |f(\tau) e^{-\eta \tau}| e^{-(\eta+\varepsilon\delta)(x-\tau)} d\tau \\ &\quad + \frac{C \varepsilon^2}{2\varepsilon \delta} \int_x^0 |f(\tau) e^{-\eta \tau}| e^{(\eta-\varepsilon\delta)(\tau-x)} d\tau. \end{aligned}$$

The bound

$$\|u\|_{L^2_\eta} \leq c_2 \|f\|_{L^2_\eta}$$

follows from classical convolution results between functions in L^2 and functions in L^1 , since

$$\begin{aligned}\int_{-\infty}^0 e^{(\eta-\varepsilon\delta)\tau} d\tau &= \frac{1}{\eta-\varepsilon\delta}, \\ \int_0^{\infty} \tau e^{-\eta\tau} d\tau &= \frac{1}{\eta^2}.\end{aligned}$$

Then, we choose $\eta = \frac{1}{2}\varepsilon\delta$, so that the pseudo-inverse of \mathcal{L}_g has a bounded inverse in L^2_η :

$$\|\widetilde{\mathcal{L}}_g^{-1}\| \leq c_2,$$

where c_2 is independent of ε . Using the form of \mathcal{L}_g we obtain easily

$$\|u\|_{\mathcal{D}_1} \leq c_3 \|f\|_{L^2_\eta}$$

with c_3 independent of ε .

Remark 25 *The choice made for η is such that*

$$\eta < \varepsilon\delta, \quad \eta < \varepsilon\sqrt{2},$$

for values of δ for which Theorem 1 is valid. This means that as $x \rightarrow -\infty$ ($A_ - 1, B_*$), and, as $x \rightarrow +\infty$ ($A_*, B_* - 1$) tend exponentially to 0 faster than $e^{-\eta|x|}$.*

In fact, \mathcal{L}_g has the same properties as the operator \mathcal{M}_i in the proof of Lemma 7.3 in [3], see also [4]: \mathcal{L}_g is Fredholm with index -1, when acting in L^2_η , for η small enough. \mathcal{L}_g has a trivial kernel, and its range is orthogonal to B_* , with the scalar product of $L^2(\mathbb{R})$.

7.3 Properties of \mathcal{M}_g

We saw that \mathcal{M}_g is Fredholm with index 0. Furthermore the derivative of the heteroclinic solution belongs to its kernel:

$$\begin{aligned}\mathcal{M}_g \begin{pmatrix} A'_* \\ B'_* \end{pmatrix} &= \begin{pmatrix} -A_*^{(5)} + A'_* - (A_*^3)' - gB_*^2 A'_* - gA_*(B_*^2)' \\ \varepsilon^{-2} B_*''' + [B'_* - gA_*^2 B'_* - (B_*^3)' - gB_*(A_*^2)'] \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\end{aligned}\tag{71}$$

We show below (see section 7.3.1) that the kernel of \mathcal{M}_g , is one dimensional, then this implies that the range of \mathcal{M}_g needs satisfy the orthogonality with only one element. In fact, because of selfadjointness in L^2 , the range of \mathcal{M}_g is orthogonal in $L^2(\mathbb{R})$ to

$$(A'_*, B'_*) \in L^2_\eta.$$

7.3.1 Dimension of $\ker \mathcal{M}_g$

Any element $\zeta(x)$ in the kernel lies, by definition, in L^2_η , hence $\zeta(x)$ tends towards 0 exponentially at $\pm\infty$. Near $x = \pm\infty$ the vector $\zeta(x) \sim \zeta_\pm(x)$ should verify

$$\mathcal{M}_\infty^\pm \zeta_\pm(x) = 0$$

where there are only 2 possible good dimensions (on each side). This gives a bound = 2 to the dimension of the kernel of \mathcal{M}_g . Let us show that *dimension 2 of $\ker \mathcal{M}_g$ implies non uniqueness of the heteroclinic*, which contradicts Theorem 1, hence the only possibility is that the dimension is one.

Let us choose arbitrarily g_0 and assume that the kernel of \mathcal{M}_{g_0} consists in

$$\zeta_0(x), \zeta_1(x)$$

where $\zeta_0 = (A'_*, B'_*)|_{g_0}$ and let us decompose a solution of (1) in the neighborhood of g_0 as

$$U = \mathbf{T}_a(U_*^{(g_0)} + a_1\zeta_1 + Y), \quad (72)$$

where \mathbf{T}_a represents the shift $x \mapsto x + a$, where $a, a_1 \in \mathbb{R}$, and Y belongs to a subspace transverse to $\ker \mathcal{M}_{g_0}$. Let us denote by \mathbf{Q}_0 and $\mathbf{P}_0 = \mathbb{I} - \mathbf{Q}_0$, projections, respectively on the range of \mathcal{M}_{g_0} , and on a complementary subspace (\mathbf{Q}_0 may be built in using the eigenvectors ζ_0^*, ζ_1^* of the adjoint operator $\mathcal{M}_{g_0}^*$). Let us denote by

$$\mathcal{F}(U, g) = 0$$

the system (1) where we look for an heteroclinic U for $g \neq g_0$. Then, we have

$$\begin{aligned} \mathcal{F}(U_*^{(g_0)}, g_0) &= 0, \\ D_U \mathcal{F}(U_*^{(g_0)}, g_0) &= \mathcal{M}_{g_0}, \end{aligned}$$

and since

$$\mathcal{M}_{g_0} \zeta_j = 0, \quad j = 0, 1,$$

using the equivariance under operator \mathbf{T}_a , we obtain (denoting $\mathcal{F}_0 = \mathcal{F}(U_*^{(g_0)}, g_0)$ and $[\cdot]^{(2)}$ the argument of a quadratic operator)

$$\begin{aligned} 0 &= \mathcal{M}_{g_0} Y + (g - g_0) \partial_g \mathcal{F}_0 + \frac{1}{2} D_{UU}^2 \mathcal{F}_0 [a_1 \zeta_1 + Y]^{(2)} + \\ &\quad + \mathcal{O}(|g - g_0|(|g - g_0| + |a_1| + \|Y\|) + \|Y\|^3). \end{aligned}$$

The projection \mathbf{Q}_0 of this equation allows to use the implicit function theorem to solve with respect to Y and then obtain a unique solution

$$Y = \mathcal{Y}(a_1, g),$$

with

$$\begin{aligned} \mathcal{Y} &= -(g - g_0) \widetilde{\mathcal{M}}_{g_0}^{-1} \mathbf{Q}_0 \partial_g \mathcal{F}_0 - \frac{1}{2} \widetilde{\mathcal{M}}_{g_0}^{-1} \mathbf{Q}_0 D_{UU}^2 \mathcal{F}_0 [a_1 \zeta_1]^{(2)} + \\ &\quad + \mathcal{O}(|g - g_0|(|g - g_0| + |a_1|) + |a_1|^3). \end{aligned}$$

Then projecting on the complementary space, (only one equation since we work in the subspace orthogonal to ζ_0^*), we may observe (see the proof below) that $\mathbf{P}_0 \partial_{g_0} \mathcal{F}_0 = 0$ and then obtain the "bifurcation" equation as

$$q(a_1, g - g_0) = \mathcal{O}(|g - g_0| + |a_1|)^3,$$

where the function q is quadratic in its arguments and

$$q|_{g=g_0} \zeta_1 = \frac{1}{2} \mathbf{P}_0 D_{UU}^2 \mathcal{F}_0 [a_1 \zeta_1]^{(2)}.$$

This equation is just at main order a second degree equation in a_1 depending on $g - g_0$. Provided that the discriminant is not 0, the generic number of solutions is 2 or 0. If the discriminant is 0 for $g = g_0$, we just go a little farther in g , and obtain a non zero discriminant, since the discriminant cannot stay = 0, because of the analyticity in g of the heteroclinic. This is true except for a set of isolated values of g . We can then use the implicit function theorem for finding corresponding solutions for the system with higher order terms. In fact we already know a solution, corresponding to $U_*^{(g)} = U_*^{(g_0)} + (g - g_0) \partial_g U_*^{(g_0)} + h.o.t.$ which corresponds to specific values for a_1 and Y , of order $\mathcal{O}(g - g_0)$. It then results that there is at least another solution of order $\mathcal{O}(g - g_0)$, so that there exists another heteroclinic, in the neighborhood of the known one (then in contradiction with Theorem 1).

Remark 26 *The above proof with only 1 dimension in the Kernel, provides $Y = -(g - g_0) \widetilde{\mathcal{M}}_{g_0}^{-1} \partial_g \mathcal{F}_0 + \mathcal{O}((g - g_0)^2)$, which gives a unique heteroclinic. Since we found only one heteroclinic, this shows that the kernel is of dimension 1.*

7.3.2 Proof of $\mathbf{P}_0 \partial_g \mathcal{F}_0 = 0$

Lemma 27 *Any (u, v) in the kernel of \mathcal{M}_g satisfies*

$$\int_{\mathbb{R}} A_* B_* (B_* u + A_* v) dx = 0,$$

and $\partial_g \mathcal{F}_0(U_*, g) = (A_* B_*^2, A_*^2 B_*)$ belongs to the range of \mathcal{M}_g , hence $\mathbf{P}_0 \partial_g \mathcal{F}_0 = 0$.

Proof.

Differentiating with respect to g the system (1) verified by the heteroclinic, we obtain

$$\mathcal{M}_g \begin{pmatrix} \partial_g A_* \\ \partial_g B_* \end{pmatrix} = \begin{pmatrix} A_* B_*^2 \\ A_*^2 B_* \end{pmatrix} = \partial_g \mathcal{F}_0(U_*, g),$$

hence $(A_* B_*^2, A_*^2 B_*)$ belongs to the range of \mathcal{M}_g . When $(u, v) \in \ker \mathcal{M}_g$, then $(u, v) \in \ker \mathcal{M}_g^*$ where $\mathcal{M}_g = \mathcal{M}_g^*$, when the adjoint is computed with the scalar product of L^2 , hence

$$\int_{\mathbb{R}} A_* B_* (B_* u + A_* v) dx = 0. \quad (73)$$

Hence, the eigenvectors ζ_0^*, ζ_1^* of the adjoint \mathcal{M}_g^* (the orthogonal of this 2-dimensional eigenspace is the range of \mathcal{M}_g), are orthogonal to $\partial_g \mathcal{F}_0 = (A_* B_*^2, A_*^2 B_*)|_{g_0}$ in L^2 .

A Appendix

A.1 Monodromy operator

Let us prove the estimate for the monodromy operators. We prove the following

Lemma 28 *For $\eta_0 \delta \leq A_* \leq 1$, and $\alpha^{-1} \geq (1 + \delta^2)^2$ and the following estimates hold*

$$\begin{aligned} \|\mathbf{S}_0(x, s)\| &\leq e^{\sigma(x-s)}, \quad -\infty < x < s \\ \|\mathbf{S}_1(x, s)\| &\leq e^{-\sigma(x-s)}, \quad -\infty < s < x \end{aligned}$$

with

$$\sigma = \frac{\alpha^{1/2} \delta^{1/2}}{2^{1/4}}.$$

We start with the system

$$\begin{aligned} x_1' &= \lambda_r x_1 + \lambda_i x_2 \\ x_2' &= -\lambda_i x_1 + \lambda_r x_2 \end{aligned}$$

where λ_r and λ_i are functions of x . When $\eta_0 \delta \leq A_* \leq 1$, $\alpha^{-1} \geq (1 + \delta^2)^2$, we have, for ε small enough (see (14))

$$\lambda_r \geq \frac{\alpha^{1/2} \delta^{1/2}}{2^{1/4}} = \sigma.$$

Now we have

$$(x_1^2 + x_2^2)' = 2\lambda_r(x_1^2 + x_2^2)$$

hence

$$(x_1^2 + x_2^2)(x) = e^{\int_s^x 2\lambda_r(\tau) d\tau} (x_1^2 + x_2^2)(s),$$

which, for $x < s$, leads to

$$\sqrt{(x_1^2 + x_2^2)(x)} \leq e^{\sigma(x-s)} \sqrt{(x_1^2 + x_2^2)(s)}.$$

The proof is then done for the operator \mathbf{S}_0 . The estimate for \mathbf{S}_1 is obtained in the same way.

Remark 29 *We have*

$$\mathbf{S}_0(x, s) = e^{\int_s^x \lambda_r(\tau) d\tau} \begin{pmatrix} \cos(\int_s^x \lambda_i(\tau) d\tau) & \sin(\int_s^x \lambda_i(\tau) d\tau) \\ -\sin(\int_s^x \lambda_i(\tau) d\tau) & \cos(\int_s^x \lambda_i(\tau) d\tau) \end{pmatrix}.$$

Remark 30 In fact, once we have proved Lemma 10, we are able to improve the estimate of the integral $\int_s^x \lambda_r(\tau) d\tau$ in using Corollary 15. Indeed this corollary leads to

$$\begin{aligned}\widetilde{A}_*^2 &\geq 1 - (1 + \delta^2)(B_{00}^2 + c\varepsilon)e^{2\varepsilon\delta x} \\ &\geq 1 - [1 - \alpha^2\delta^2 + c\varepsilon(1 + \delta^2)]e^{2\varepsilon\delta x} \\ &\geq 1 - (1 - c'\alpha^2)e^{2\varepsilon\delta x}\end{aligned}$$

provided that

$$\varepsilon < c'\alpha^2, \quad c' \leq \frac{\delta^2}{1 + c(1 + \delta^2)}.$$

Now, in using

$$(1 - a^4)^{1/4} \leq 3^{-1/4}(1 - a) \text{ for } a < 1.$$

this leads to ($x < 0$)

$$\begin{aligned}\int_x^0 \lambda_r(\tau) d\tau &\geq \frac{1}{2^{1/4}} \int_x^0 [1 - (1 - c'\alpha^2)e^{2\varepsilon\delta\tau}]^{1/4} d\tau \\ &\geq \frac{1}{6^{1/4}} \int_x^0 [1 - (1 - c'\alpha^2)^{1/4} e^{\frac{\varepsilon\delta\tau}{2}}] d\tau \\ &\geq \frac{1}{6^{1/4}} [|x| - (1 - c'\alpha^2)^{1/4} \frac{2}{\varepsilon\delta} (1 - e^{\frac{\varepsilon\delta x}{2}})] \\ &\geq \frac{1}{6^{1/4}} |x|,\end{aligned}$$

so that

$$e^{\int_s^x \lambda_r(\tau) d\tau} \leq e^{\frac{x-s}{6^{1/4}}}, \quad x < s < 0, \quad (74)$$

which is better than the estimate in Lemma 28.

A.2 Computation of the system with new coordinates

Let us look for the system (8) written in the new coordinates, first in forgetting quadratic and higher orders terms

$$\begin{aligned}B_0 x'_1 &= \frac{(\lambda_r^2 + \lambda_i^2)}{4\lambda_r} \left(A_1 + \frac{(1 + \delta^2)B_0 B_1}{\widetilde{A}_*} \right) + \frac{3\lambda_r^2 - \lambda_i^2}{4\lambda_r(\lambda_r^2 + \lambda_i^2)} A_3 \\ &\quad + \frac{A_2}{2} + \frac{(1 + \delta^2)}{2\widetilde{A}_*} B_0^2 \varepsilon^2 \left(\delta^2 (\widetilde{A}_*^2 - B_0^2) + 2(1 + \delta^2) \widetilde{A}_* \widetilde{A}_0 \right) - (\lambda_r^2 - \lambda_i^2) \widetilde{A}_0 \\ &= B_0 f_1 + \frac{(\lambda_r^2 + \lambda_i^2)}{4\lambda_r} B_0 (x_1 + y_1) + \frac{A_2}{2} + \frac{1}{4\lambda_r} A_3,\end{aligned}$$

$$\begin{aligned}
\lambda_i B_0 x'_2 &= -\frac{(\lambda_r^2 + \lambda_i^2)}{4} \left(A_1 + \frac{(1 + \delta^2) B_0 B_1}{\widetilde{A}_*} \right) - \frac{\lambda_r^2 - 3\lambda_i^2}{4(\lambda_r^2 - \alpha)} A_3 \\
&\quad - \frac{(\lambda_r^2 - \lambda_i^2)}{4\lambda_r} \left(A_2 + \frac{(1 + \delta^2) B_0^2 \varepsilon^2}{\widetilde{A}_*} \delta^2 (\widetilde{A}_*^2 - B_0^2) \right) \\
&\quad - \frac{1}{4\lambda_r} (\lambda_r^2 + \lambda_i^2)^2 \widetilde{A}_0 \\
&= \lambda_i B_0 f_2 - \frac{(\lambda_r^2 + \lambda_i^2)}{4} B_0 (x_1 + y_1) - \frac{(\lambda_r^2 - \lambda_i^2)}{4\lambda_r} A_2 \\
&\quad + \frac{1}{4} A_3 - \frac{1}{4\lambda_r} (\lambda_r^2 + \lambda_i^2)^2 \widetilde{A}_0,
\end{aligned}$$

with

$$\begin{aligned}
f_1 &= \frac{\varepsilon^2 \delta^2 B_0 (1 + \delta^2) (\widetilde{A}_*^2 - B_0^2)}{2\widetilde{A}_*}, \\
f_2 &= -\frac{\varepsilon^2 \delta^2 B_0 (1 + \delta^2) (\lambda_r^2 - \lambda_i^2) (\widetilde{A}_*^2 - B_0^2)}{4\lambda_r \lambda_i \widetilde{A}_*},
\end{aligned}$$

hence

$$\begin{aligned}
x'_1 &= f_1 + \lambda_r x_1 + \lambda_i x_2, \\
x'_2 &= f_2 - \lambda_i x_1 + \lambda_r x_2,
\end{aligned} \tag{75}$$

and in the same way

$$\begin{aligned}
y'_1 &= f_1 - \lambda_r y_1 + \lambda_i y_2, \\
y'_2 &= -f_2 - \lambda_i y_1 - \lambda_r y_2, \\
z'_1 &= \frac{2\varepsilon^2 \delta^2 (\widetilde{A}_*^2 - B_0^2)}{\widetilde{A}_*} = \frac{2f_1}{(1 + \delta^2) B_0}, \\
B'_* &= -\frac{(\lambda_r^2 - \lambda_i^2)}{(1 + \delta^2) B_0 \widetilde{A}_*} A_3 + \widetilde{A}_* B_0 z_1.
\end{aligned} \tag{76}$$

We notice that the following estimates hold

$$\begin{aligned}
|f_1| &\leq \frac{B_0 \varepsilon^2 \delta^2}{\widetilde{A}_*} \leq \frac{B_0 \varepsilon^2 \delta}{\alpha}, \\
|f_2| &\leq \frac{B_0 \varepsilon^4 \delta^2}{\widetilde{A}_*^2} \leq B_0 \varepsilon^2 \delta.
\end{aligned} \tag{77}$$

A.2.1 Full system in new coordinates

We intend to derive the full system (1) with coordinates $(x_1, x_2, y_1, y_2, B_0, z_1)$. Differentiating (17) and (18) we see that we respectively need to add to the

previous expressions (75) for x'_1 and x'_2

$$\begin{aligned} & \frac{1}{B_0} \left\{ \left(\frac{\widetilde{A}_*}{2\sqrt{2}\lambda_r} \right)' \widetilde{A}_0 + \left(\frac{(3\lambda_r^2 - \lambda_i^2)}{4\sqrt{2}\lambda_r\widetilde{A}_*} \right)' A_2 + \varepsilon^2 \left(\frac{(1+\delta^2)^2 B_0^2}{2\widetilde{A}_*^2} \right)' A_3 + \left(\frac{(1+\delta^2)B_0}{2\widetilde{A}_*} \right)' B_1 \right\} \\ & - \varepsilon^2 \frac{(1+\delta^2)^2 B_0}{2\widetilde{A}_*^2} [3\widetilde{A}_*\widetilde{A}_0^2 + \widetilde{A}_0^3] + \frac{B_0\varepsilon^2(1+\delta^2)^2\widetilde{A}_0^2}{2\widetilde{A}_*} - \frac{B_1}{B_0}x_1. \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{B_0} \left\{ - \left(\frac{\widetilde{A}_*}{2\sqrt{2}\lambda_i} \right)' \widetilde{A}_0 - \left(\frac{(\lambda_r^2 - \lambda_i^2)}{4\lambda_r\lambda_i} \right)' A_1 - \left(\frac{(\lambda_r^2 - 3\lambda_i^2)}{4\sqrt{2}\lambda_i\widetilde{A}_*} \right)' A_2 + \left(\frac{\varepsilon^2(1+\delta^2)^3 B_0^3}{4\lambda_r\lambda_i\widetilde{A}_*} \right)' B_1 \right\} \\ & + \frac{1}{B_0} \left(\frac{1}{4\lambda_r\lambda_i} \left[1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A}_*^2} \right] \right)' A_3 - \frac{1}{4\lambda_r\lambda_i B_0} \left(1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A}_*^2} \right) [3\widetilde{A}_*\widetilde{A}_0^2 + \widetilde{A}_0^3] \\ & - \frac{\varepsilon^4 B_0^3 (1+\delta^2)^4}{4\lambda_r\lambda_i\widetilde{A}_*} \widetilde{A}_0^2 - \frac{B_1}{B_0}x_2. \end{aligned}$$

We then arrive to the system (22,23,24,25).

We observe that (using (11))

$$\widetilde{A}_*' = - \frac{(1+\delta^2)B_0}{\widetilde{A}_*} B_1$$

$$(\lambda_r^2)' = - \frac{(1+\delta^2)B_0 B_1}{\sqrt{2}\widetilde{A}_*} (1 - \varepsilon^2\sqrt{2}(1+\delta^2)\widetilde{A}_*)$$

$$(\lambda_i^2)' = - \frac{(1+\delta^2)B_0 B_1}{\sqrt{2}\widetilde{A}_*} (1 + \varepsilon^2\sqrt{2}(1+\delta^2)\widetilde{A}_*)$$

$$\left(\frac{\widetilde{A}_*}{2\sqrt{2}\lambda_r} \right)' = a_1 B_0 B_1, \quad |a_1| \leq \frac{c}{\widetilde{A}_*^{3/2}}, \quad (78)$$

$$\left(\frac{\widetilde{A}_*}{2\sqrt{2}\lambda_i} \right)' = a_2 B_0 B_1, \quad |a_2| \leq \frac{c}{\widetilde{A}_*^{3/2}}, \quad (79)$$

$$\left(- \frac{(\lambda_r^2 - \lambda_i^2)}{4\lambda_r\lambda_i} \right)' = b_2 B_0 B_1, \quad |b_2| \leq \frac{c\varepsilon^2}{\widetilde{A}_*^3}, \quad (80)$$

$$\left(\frac{(3\lambda_r^2 - \lambda_i^2)}{4\sqrt{2}\lambda_r\widetilde{A}_*} \right)' = c_1 B_0 B_1, \quad |c_1| \leq \frac{c}{\widetilde{A}_*^{5/2}}, \quad (81)$$

$$\left(- \frac{(\lambda_r^2 - 3\lambda_i^2)}{4\sqrt{2}\lambda_i\widetilde{A}_*} \right)' = c_2 B_0 B_1, \quad |c_2| \leq \frac{c}{\widetilde{A}_*^{5/2}}, \quad (82)$$

$$\varepsilon^2 \left(\frac{(1 + \delta^2)^2 B_0^2}{2\widetilde{A}_*^2} \right)' = d_1 B_0 B_1, \quad |d_1| \leq \frac{c}{\widetilde{A}_*^3}, \quad (83)$$

$$\left(\frac{1}{4\lambda_r \lambda_i} \left[1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A}_*^2} \right] \right)' = d_2 B_0 B_1, \quad |d_2| \leq \frac{c}{\widetilde{A}_*^3}, \quad (84)$$

$$\left(\frac{(1 + \delta^2) B_0}{2\widetilde{A}_*} \right)' = e_1 B_1, \quad |e_1| \leq \frac{c}{\widetilde{A}_*^3} \quad (85)$$

$$\left(\frac{\varepsilon^2 (1 + \delta^2)^3 B_0^2}{4\lambda_r \lambda_i \widetilde{A}_*} \right)' = e_2 B_0 B_1, \quad |e_2| \leq \frac{c}{\widetilde{A}_*^3}, \quad (86)$$

with c independent of ε and $\delta \in [\delta_0, \delta_1]$.

A.3 System after elimination of z_1

A.3.1 System after scaling

After the scaling (27) our system (22,23,24,25) takes the form

$$\begin{aligned} \overline{X}' &= \mathbf{L}_0 \overline{X} + B_0 \overline{F}_0 + \mathbf{B}_{01}(\overline{X}, \overline{Y}) + \overline{z}_1 \mathbf{M}_{01}(\overline{X}, \overline{Y}) \\ &\quad + \overline{z}_1^2 B_0 \mathbf{n}_0 + \mathbf{C}_{01}(\overline{X}, \overline{Y}), \\ \overline{Y}' &= \mathbf{L}_1 \overline{Y} + B_0 \overline{F}_1 + \mathbf{B}_{11}(\overline{X}, \overline{Y}) + \overline{z}_1 \mathbf{M}_{11}(\overline{X}, \overline{Y}) \\ &\quad + \overline{z}_1^2 B_0 \mathbf{n}_1 + \mathbf{C}_{11}(\overline{X}, \overline{Y}), \end{aligned}$$

where $\overline{F}_0, \overline{F}_1, \mathbf{n}_0, \mathbf{n}_1$ are two-dimensional vectors $\mathbf{M}_{01}, \mathbf{M}_{11}$ are linear operators in $(\overline{X}, \overline{Y})$, $\mathbf{B}_{01}, \mathbf{B}_{11}$ are quadratic and $\mathbf{C}_{01}, \mathbf{C}_{11}$ are cubic in $(\overline{X}, \overline{Y})$, all functions of B_0 . More precisely we have

$$\overline{F}_0 = \begin{pmatrix} \frac{f_1}{\varepsilon \delta B_0} \\ \frac{f_2}{\varepsilon \delta B_0} \end{pmatrix}, \quad \overline{F}_1 = \begin{pmatrix} \frac{f_1}{\varepsilon \delta B_0} \\ -\frac{f_2}{\varepsilon \delta B_0} \end{pmatrix}, \quad |\overline{F}_j| \leq c \frac{\varepsilon}{\alpha},$$

$$\mathbf{n}_0 = \varepsilon \delta \begin{pmatrix} e_1 \widetilde{A}_*^2 \\ e_2 \widetilde{A}_*^2 B_0 - b_2 (1 + \delta^2) \widetilde{A}_* B_0^2 \end{pmatrix},$$

$$\mathbf{M}_{01}(\overline{X}, \overline{Y}) = \varepsilon \delta \begin{pmatrix} m_{01}(\overline{X}, \overline{Y}) \\ m_{02}(\overline{X}, \overline{Y}) \end{pmatrix},$$

$$m_{01}(\overline{X}, \overline{Y}) = \widetilde{A}_* B_0 \left(a_1 \overline{A}_0 + c_1 \overline{A}_2 + (d_1 - 2e_1(1 + \delta^2)) \varepsilon^2 \frac{B_0}{\widetilde{A}_*} \overline{A}_3 - \frac{\overline{x}_1}{B_0} \right),$$

$$\begin{aligned} m_{02}(\overline{X}, \overline{Y}) &= \widetilde{A}_* B_0 \left(-a_2 \overline{A}_0 + c_2 \overline{A}_2 + (d_2 - 2e_2(1 + \delta^2)) \varepsilon^2 \frac{B_0^2}{\widetilde{A}_*} \overline{A}_3 - \frac{\overline{x}_2}{B_0} \right) \\ &\quad + \widetilde{A}_* B_0^2 b_2 (\overline{x}_1 + \overline{y}_1) + (1 + \delta^2)^2 \varepsilon^2 \frac{B_0^3}{\widetilde{A}_*} b_2 \overline{A}_3, \end{aligned}$$

$$\mathbf{B}_{01}(\bar{X}, \bar{Y}) = \varepsilon \delta \begin{pmatrix} b_{01}(\bar{X}, \bar{Y}) \\ b_{02}(\bar{X}, \bar{Y}) \end{pmatrix},$$

$$\begin{aligned} b_{01}(\bar{X}, \bar{Y}) &= -\varepsilon^2 \frac{(1+\delta^2)(2-\delta^2)B_0}{2\widetilde{A}_*} \widetilde{A}_0^{-2} + e_1 \frac{\varepsilon^4(1+\delta^2)^2 B_0}{\widetilde{A}_*^2} \widetilde{A}_3^{-2} \\ &\quad -\varepsilon^2 \frac{(1+\delta^2)B_0}{\widetilde{A}_*} \widetilde{A}_3 [a_1 \widetilde{A}_0 + c_1 \widetilde{A}_2 + d_1 \widetilde{A}_3 - \frac{\bar{x}_1}{B_0}], \end{aligned}$$

$$\begin{aligned} b_{02}(\bar{X}, \bar{Y}) &= -\frac{1}{4\lambda_r \lambda_i \widetilde{A}_* B_0} \left(3\widetilde{A}_*^2 - 2\varepsilon^4 B_0^4 (1+\delta^2)^4 \right) \widetilde{A}_0^{-2} + e_2 \frac{\varepsilon^4(1+\delta^2) B_0^2}{\widetilde{A}_*^2} \widetilde{A}_3^{-2} \\ &\quad -\varepsilon^2 \frac{(1+\delta^2)B_0}{\widetilde{A}_*} \widetilde{A}_3 [-a_2 \widetilde{A}_0 + b_2 B_0 (\bar{x}_1 + \bar{y}_1) + c_2 \widetilde{A}_2 + d_2 \widetilde{A}_3 - \frac{\bar{x}_2}{B_0}], \end{aligned}$$

$$\mathbf{C}_{01}(\bar{X}, \bar{Y}) = \varepsilon^2 \delta^2 \widetilde{A}_0^{-3} \begin{pmatrix} -\varepsilon^2 \frac{(1+\delta^2)B_0}{\widetilde{A}_*^2} \\ -\frac{1}{4\lambda_r \lambda_i B_0} \left(1 - \frac{\varepsilon^4 B_0^4 (1+\delta^2)^4}{\widetilde{A}_*^2} \right) \end{pmatrix}.$$

$\mathbf{n}_1, \mathbf{M}_{11}, \mathbf{B}_{11}, \mathbf{C}_{11}$ are deduced respectively from $\mathbf{n}_0, \mathbf{M}_{01}, \mathbf{B}_{01}, \mathbf{C}_{01}$ in changing $(a_1, c_1, b_2, d_2, e_2)$ into their opposite.

A.3.2 System after elimination of z_1

Let us replace \bar{z}_1 by $\bar{z}_{10} + \mathcal{Z}(\bar{X}, \bar{Y}, B_0)$ in the differential system for (\bar{X}, \bar{Y}) . The new system becomes (notice that B_0 is in factor of the "constant" terms)

$$\begin{aligned} \bar{X}' &= \mathbf{L}_0 \bar{X} + B_0 \mathcal{F}_0 + \mathcal{L}_{01}(\bar{X}, \bar{Y}) + \mathcal{B}_{01}(\bar{X}, \bar{Y}), \\ \bar{Y}' &= \mathbf{L}_1 \bar{Y} + B_0 \mathcal{F}_1 + \mathcal{L}_{11}(\bar{X}, \bar{Y}) + \mathcal{B}_{11}(\bar{X}, \bar{Y}), \end{aligned}$$

which is (30) with

$$\begin{aligned} \mathcal{F}_0 &= \bar{F}_0 + \bar{z}_{10}^{-2} \mathbf{n}_0, \\ \mathcal{L}_{01}(\bar{X}, \bar{Y}) &= \bar{z}_{10} \mathbf{M}_{01}(\bar{X}, \bar{Y}), \end{aligned}$$

$$\begin{aligned} \mathcal{B}_{01}(\bar{X}, \bar{Y}) &= \mathbf{B}_{01}(\bar{X}, \bar{Y}) + \mathcal{Z}(\bar{X}, \bar{Y}) \mathbf{M}_{01}(\bar{X}, \bar{Y}) + \mathbf{C}_{01}(\bar{X}, \bar{Y}) \\ &\quad + 2\bar{z}_{10} \mathcal{Z}(\bar{X}, \bar{Y}) B_0 \mathbf{n}_0 + \mathcal{Z}(\bar{X}, \bar{Y})^2 B_0 \mathbf{n}_0. \end{aligned}$$

In using estimates (21), (78) to (86), it is straightforward to check that

$$|\mathcal{F}_0| + |\mathcal{F}_1| \leq \frac{c\varepsilon}{\alpha^3},$$

$$|\mathbf{M}_{01}(\bar{X}, \bar{Y})| \leq c \frac{\varepsilon \delta}{A_*} (|\bar{X}| + |\bar{Y}|),$$

hence

$$|\mathcal{L}_{01}(\bar{X}, \bar{Y})| + |\mathcal{L}_{11}(\bar{X}, \bar{Y})| \leq c \frac{\varepsilon}{\alpha^2} (|\bar{X}| + |\bar{Y}|).$$

For higher order terms we have

$$\begin{aligned}
|\mathbf{B}_{01}(\bar{X}, \bar{Y})| &\leq c\varepsilon(|\bar{X}| + |\bar{Y}|)^2, \\
|2\bar{z}_{10}\mathcal{Z}(\bar{X}, \bar{Y})\mathbf{n}_0| &\leq c\frac{\varepsilon\widetilde{A}_*}{\delta}(|\bar{X}| + |\bar{Y}|)^2, \\
|\mathcal{Z}(\bar{X}, \bar{Y})\mathbf{M}_{01}(\bar{X}, \bar{Y})| &\leq c\frac{\varepsilon^2\widetilde{A}_*}{\delta}(|\bar{X}| + |\bar{Y}|)^3, \\
|\mathcal{Z}(\bar{X}, \bar{Y})^2\mathbf{n}_0| &\leq c\frac{\varepsilon^3\widetilde{A}_*^3}{\delta^3}(|\bar{X}| + |\bar{Y}|)^4, \\
|\mathbf{C}_{01}(\bar{X}, \bar{Y})| &\leq c\frac{\varepsilon}{\alpha^{1/2}}(|\bar{X}| + |\bar{Y}|)^3,
\end{aligned}$$

hence, choosing α small enough and for

$$|\bar{X}| + |\bar{Y}| \leq \rho, \tag{87}$$

we obtain

$$|\mathcal{B}_{01}(\bar{X}, \bar{Y})| + |\mathcal{B}_{11}(\bar{X}, \bar{Y})| \leq \frac{c\varepsilon}{\alpha^{1/2}}(|\bar{X}| + |\bar{Y}|)^2.$$

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