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1 **Phonon transmission through a nonlocal metamaterial slab**

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12

13 **Abstract:**

14 Previous theory and experiment has shown that introducing strong (nonlocal) beyond-nearest-neighbor
15 interactions in addition to (local) nearest-neighbor interactions into rationally designed periodic lattices
16 called metamaterials can lead to unusual wave dispersion relations of the lowest band. For roton-like
17 dispersions, this especially includes the possibility of multiple solutions for the wavenumber at a given
18 frequency. Here, we study the one-dimensional frequency-dependent acoustical phonon transmission of
19 a slab of such nonlocal metamaterial in a local surrounding. In addition to the usual Fabry-Perot
20 resonances, we find a series of bound states in the continuum. In their vicinity, sharp Fano-type
21 transmission resonances occur, with sharp zero-transmission minima next to sharp transmission maxima.
22 Our theoretical discussion starts with a discrete mass-and-spring model. We compare these results with
23 solutions of a generalized wave equation for heterogeneous nonlocal effective media. We validate our
24 findings by numerical calculations on three-dimensional metamaterial microstructures for one-
25 dimensional acoustical wave propagation.

26

27 Introduction

28 The wave properties of ordinary crystals are determined by the atoms forming the crystal as well as by
29 their interactions. Likewise, the wave properties of rationally designed artificial periodic lattices called
30 metamaterials¹⁻³ are determined by the interior of the metamaterial unit cells as well as by the
31 interactions among the unit cells⁴⁻⁶. A bulk of literature has used the approximation of considering
32 interactions among only the nearest neighbors⁷⁻⁹. Interactions beyond the nearest neighbors have been
33 considered to test the validity of this approximation¹⁰. However, in metamaterials, the interactions
34 beyond the nearest neighbors can be designed rationally and can be made strong¹⁰⁻¹⁴. This additional
35 design freedom has lately been used to realize unusual dispersion relations of the lowest acoustic or
36 elastic metamaterial band¹⁵⁻¹⁸. For example, the latter can resemble the unusual dispersion relation,
37 $\omega(k)$, of sound waves in superfluid helium^{19, 20} that starts with an angular frequency of the wave, ω ,
38 proportional to its wavenumber, k , followed by a maximum (the “maxon”) and a minimum (the “roton”)
39 versus k ^{21, 22}. Such unusual phonon dispersion relations have been observed experimentally using three-
40 dimensional macroscopic metamaterials for airborne sound at audible frequencies^{17, 18} and using three-
41 dimensional microstructured metamaterials for elastic waves at ultrasound frequencies¹⁷.

42 However, structures and devices in applications usually exploit multiple dissimilar materials and the
43 interplay between them and their interfaces. A paradigmatic textbook heterostructure geometry is a slab
44 with thickness L of material A clad between two semi-infinite half spaces of material B. For usual local
45 materials A and B, it is well-known that this setting leads to Fabry-Perot resonances connected to unity
46 wave transmission, $|T(\omega)| = 1$, through the slab at particular angular frequencies $\omega = \omega_i$ of the incident
47 wave²³. At these particular frequencies, the phase that the wave accumulates in one round trip through
48 the slab is an integer multiple of 2π . For a slab with a sufficiently large number of unit cells within, this
49 condition translates into $2kL = n_i 2\pi$, where $k = k(\omega_i)$ is the single wavenumber in material A at the
50 angular frequency ω_i and n_i is an integer. Fabry-Perot resonances with high quality factors have
51 numerous applications, e.g., as optical filters or interferometry²⁴.

52 Here, we discuss the case that material A in the slab is replaced by a nonlocal metamaterial. At a given
53 angular frequency ω , such medium generally supports more than a single wave mode with single
54 wavenumber k . For different wavenumbers $k_j(\omega)$, with $j = 1, 2, \dots, N$, at a given angular frequency ω , the
55 behavior is richer than for local material slabs. We start by discussing the problem using a previously
56 introduced simple discrete one-dimensional (1D) mass-and-spring model¹⁵. Apart from the nearest-
57 neighbor interactions via Hooke’s springs, it contains N -th nearest-neighbor interactions with integer $N \geq$

58 2. Here, we emphasize the example of $N = 3$, which is the smallest N for which the roton-like minimum
59 fully lies inside of the first Brillouin zone (for $N = 2$ it lies right at the Brillouin zone border). We find a
60 series of sharp Fano-type resonances in the frequency-dependent transmission $|T(\omega)|$ in the frequency
61 region for which multiple solutions $k_j(\omega)$ for the wavenumber exist. We show that the linewidth of the
62 Fano-type resonances tends to zero towards special points in material-parameter space corresponding to
63 bound states in the continuum (BIC)²⁵. BIC physics in general, not related to beyond-nearest-neighbor
64 interactions in periodic lattices, has a long history in acoustics²⁶, elasticity^{27, 28}, as well as optics^{29, 30}, and
65 has recently attracted renewed attention in the metamaterials community³¹. We refer the reader to the
66 review articles^{25, 29} for an introduction to and comprehensive reviews of the BIC field. Next, we discuss the
67 nonlocal slab transmission on the level of a 1D effective-medium approximation for the displacement field
68 of the heterogeneous 1D mass-and-spring model, which leads to a phenomenological generalized wave
69 equation containing spatial derivatives up to order $2N$. Finally, we present numerical calculations for
70 three-dimensional nonlocal metamaterial microstructures for wave propagation along one direction,
71 again showing BIC behavior.

72

73 **Results and Discussion**

74 **Mass-and-spring model.** Figure 1(a) illustrates the infinite one-dimensional mass-and-spring toy model
75 that we have discussed previously¹⁵. Herein, identical masses m , periodically arranged with period or
76 lattice constant a , are connected to their immediate neighbors along the x -axis on the left and on the
77 right by linear elastic Hooke' springs with spring constant K_1 . In this form (i.e., for $K_N = 0$), Fig. 1(a)
78 corresponds to the paradigmatic one-dimensional model for acoustical phonons in usual local media as
79 described in any solid-state-physics textbook³². For the nonlocal case, the masses in Fig. 1(a) are
80 additionally connected to their N -th nearest neighbor on the left and on the right by Hooke's springs with
81 spring constant K_N . Shown is the example of $N = 3$. This is the lowest integer for which roton-like
82 dispersion relations¹⁵ can occur within the first Brillouin zone of the model. For $N = 2$, the roton-like
83 minimum is right at the boundary of the first Brillouin zone. Clearly, the model can be extended to contain
84 multiple orders of beyond-nearest-neighbor interactions³³. Here, for simplicity, we only consider nearest
85 neighbors plus neighbors with $N = 3$. We will see that the resulting behavior of slabs is extremely rich
86 and complex already. The beyond-nearest-neighbor springs in Fig. 1 are meant symbolically, an actual
87 feasible realization is discussed in Section V.

88 As an example for $N = 3$, Fig. 1(b) shows a slab of relative thickness $L/a = 6$ of such nonlocal material
 89 clad between a local mass-and-spring model. The thinnest possible slab corresponds to $L/a = N$, for
 90 which only a single N -th nearest-neighbor spring is left. For simplicity and clarity, we depict and study in
 91 what follows the case that the lattice constant a , the masses m , and the spring constants K_1 are constant
 92 throughout the entire structure considered. We notice that, for a given well-defined integer ratio L/a ,
 93 the left and right boundaries of the nonlocal slab in Fig. 1(b) cannot be defined unambiguously anymore.
 94 Six of the seven masses in the slab have only third-nearest-neighbor springs to one side. Further inside of
 95 the nonlocal slab (in Fig. 1(b) only the middle mass), the masses have long-range interactions to their left
 96 and to their right-hand side. For the phenomenological effective-medium description to be discussed
 97 below, this obvious fact means that the boundaries between the local and the nonlocal medium cannot
 98 be considered as being sharp or discontinuous anymore. The boundaries are rather smeared out, which
 99 is a direct consequence of the nonlocality of the slab. This simple observation will become important for
 100 an intuitive interpretation of our results and for the effective-medium description described below.

101 Before discussing the nonlocal slab, let us briefly recapitulate the expected transmission, $T(\omega)$, of a slab
 102 of a local material embedded in a different local material, at the real-valued angular frequency ω . We
 103 define the complex-valued transmission as the ratio of the transmitted displacement amplitude or output,
 104 u_{out} , and the displacement amplitude incident onto the slab, u_{in} , i.e.,

$$105 \quad T(\omega) = \frac{u_{\text{out}}}{u_{\text{in}}}. \quad (1)$$

106 The phase of $T(\omega)$ clearly depends on at which lattice site exactly we take the incident and the
 107 transmitted displacement, respectively. This dependence drops out when considering the modulus,
 108 i.e., $|T(\omega)|$. Therefore, we consider $|T(\omega)|$ in what follows. As pointed out in the introduction, for a local
 109 slab in a local surrounding, $|T(\omega)|$ generally exhibits Fabry-Perot resonances with $|T(\omega_i)| = 1$ at
 110 particular angular frequencies ω_i which fulfill the standing-wave condition²³

$$111 \quad k(\omega_i)L = n_i\pi, \quad (2)$$

112 with integer n_i . Clearly, this reasoning implies that the slab contains sufficiently many unit cells, such that
 113 the wavenumber k can assume nearly any value. For these particular angular frequencies, the wave
 114 accumulates a phase in one round trip within the slab that is an integer multiple of 2π . For the special
 115 case that the impedances between the two materials are matched, we have $|T(\omega)| = 1$ for all angular
 116 frequencies.

117 Let us apply this intuitive reasoning to a nonlocal slab with sufficiently many unit cells inside. As we have
 118 shown previously¹⁵ and as can be seen from roton-like dispersion relation shown in Fig. 2(a), one generally
 119 has three solutions (for $N = 3$) for each direction (left/right or $+k/-k$) for the (real part of the)
 120 wavenumber at a given angular frequency, i.e., $k(\omega_i) \rightarrow k_j(\omega_i)$ and $n_i \rightarrow n_{ij}$ with $j = 1,2,3$. Intuitively,
 121 a standing-wave condition Eq. (2) has to be fulfilled for each one of them simultaneously to obtain a
 122 “special” behavior of $|T(\omega)|$ at certain angular frequencies ω_i . Below, we will connect this “special”
 123 behavior to bound states in the continuum (BIC). For arbitrary parameter choices of m , K_1 , K_3 , and a , and
 124 hence arbitrary dispersion relations $\omega(k)$, it is unlikely that the condition Eq. (2) can be fulfilled three
 125 times simultaneously for any one angular frequency ω_i . However, as pointed out above (see Fig. 1(b)),
 126 the boundaries of the nonlocal slab are not sharp (see above discussion on Fig. 1(a)), and, hence, the
 127 effective slab thickness, L_j^{eff} , may be different from L in Eq. (2), i.e., we have to replace $L \rightarrow L_j^{\text{eff}}$ in Eq. (2).
 128 Together, we obtain

$$129 \quad k_j(\omega_i)L_j^{\text{eff}} = n_{ij}\pi. \quad (3)$$

130 Unfortunately, there is no obvious and unambiguous way to calculate the effective slab thicknesses L_j^{eff}
 131 and thereby the special frequencies ω_i from Eq. (3) and the given dispersion relation $k(\omega)$. Nevertheless,
 132 this simple reasoning connects the textbook treatment of Fabry-Perot resonances for ordinary local slabs
 133 to the more unusual resonances in nonlocal slabs discussed in this paper.

134 Before we discuss the problem more rigorously, especially including the possibility of only a small number
 135 of unit cells within the slab, let us address a subtlety of the dispersion relation connected to the finite-
 136 thickness slab that turns out to be important for an intuitive interpretation of our results. For the infinitely
 137 extended periodic nonlocal mass-and-spring model (see Fig. 1(a)), Newton’s law for the displacement u_l
 138 of the mass m at site l along the x -axis reads

$$139 \quad m \frac{\partial^2 u_l}{\partial t^2} = K_1(u_{l+1} - 2u_l + u_{l-1}) + K_N(u_{l+N} - 2u_l + u_{l-N}), l = -\infty, \dots, 0, \dots + \infty. \quad (4)$$

140 Without further assumptions or approximations, the plane-wave ansatz $u_l = \tilde{u} \exp(i(kx - \omega t))$, with
 141 $x = la$ and constant prefactor \tilde{u} , leads to the phonon dispersion relation $\omega(k)$ given by¹⁵

$$142 \quad \omega^2(k) = \frac{4}{m} \left(K_1 \sin^2 \left(\frac{ka}{2} \right) + K_N \sin^2 \left(\frac{Nka}{2} \right) \right). \quad (5)$$

143 Clearly, when taking the square root on both sides of Eq. (5), we obtain two signs for ω . As usual, we
 144 follow the convention to consider positive (real parts of the) angular frequencies. For an infinite non-
 145 dissipative nonlocal medium, according to Bloch’s theorem³², the wavenumber must be real. However,

146 for a finite-thickness nonlocal slab, the wavenumber is not necessarily real because evanescent modes
147 may appear. For the considered transmission *Gedankenexperiment*, the angular frequency is purely real
148 (by definition) and positive by convention. Nevertheless, we plot in Fig. 2 all mathematical solutions of Eq.
149 (5) for the most general case of complex-valued k and complex-valued ω . Panel (a) is for a parameter set
150 (see caption) for which $\text{Re}(\omega)$ versus $\text{Re}(k)$ shows a roton-like dispersion relation with a pronounced
151 maximum and a pronounced minimum. Panel (b) is for a parameter set (see caption) for which $\text{Re}(\omega)$
152 versus $\text{Re}(k)$ shows no roton minimum in the phonon dispersion relation. Nevertheless, in Fig. 2(b), we
153 still obtain three solutions (three modes) for the complex-valued wavenumber k for real and positive ω
154 in the range of $\text{Re}(k) > 0$. We repeat that $\text{Im}(k) \neq 0$ indicates evanescent modes that drop out for an
155 infinite medium, but that we have to consider for a finite-thickness nonlocal slab. For a local medium
156 with $K_N = 0$, be it finite or infinite in thickness, this subtlety does not apply because $\text{Im}(k) = 0$ holds
157 true for any real-valued $\omega > 0$.

158 We note in passing that the behavior shown in Fig. 2 can be understood in terms of the roton minimum
159 being an exceptional point³⁴⁻³⁶. In fact, any k -position of a minimum or maximum of $\omega(k)$ in the first
160 Brillouin zone of any type of wave in any kind of lossless system is an exceptional point in the sense that
161 two eigenmodes coalesce in both eigenvalues and eigenvectors for the angular eigenfrequency ω at the
162 k -position of the maximum or minimum. At the position of a saddle point (see Fig. 2(b)), even three
163 eigenmodes coalesce. This exceptional degeneracy is lifted as soon as one introduces a perturbation. It is
164 also lifted as soon as one considers finite imaginary parts of k (i.e., evanescent waves). As a result, one
165 black line emerges from the roton minimum for increasing imaginary part of the wavenumber in Fig. 2(a).
166 In Fig. 2(b), two black lines emerge from the saddle point for $\text{Im}(k) > 0$.

167 Next, we discuss solutions for $|T(\omega)|$ of the nonlocal slab. As our model contains no losses, the sum of
168 kinetic and potential energy is conserved, and the reflectivity spectrum, $|R(\omega)|$, is directly connected to
169 the transmission spectrum by the relation

$$170 \quad |R(\omega)|^2 + |T(\omega)|^2 = 1. \quad (6)$$

171 This expression is only meaningful and valid for a local surrounding that supports only a single relevant
172 mode (in either direction). This condition is automatically fulfilled for the discrete mass-and-spring model
173 (cf. Fig. 1), but has to be taken with caution for the below approximate effective-medium description in
174 which a very small but finite nonlocality needs to be added to the surrounding of the slab. We will come
175 back to this point below.

176 To mathematically compute the transmission spectrum for the discrete model (see Fig. 1(b)), we proceed
 177 as follows. An incident wave with angular frequency ω impinges onto the slab from the left-hand side. We
 178 aim at computing the frequency-dependent reflection and transmission coefficients. We write the
 179 displacements corresponding to masses with label $l \leq 0$ (see Fig. 1(b)) as

$$180 \quad u_l = u_{\text{in}} \exp(ikla) + u_{\text{ref}} \exp(-ikla), \quad (7)$$

181 where u_{in} indicates the complex-valued amplitude of the incident wave, k is the wavenumber, and u_{ref}
 182 represents the unknown amplitude of the reflected wave, respectively. To ease readability, the time
 183 harmonic factor $\exp(-i\omega t)$ is omitted here and throughout the following. It can be shown that the
 184 displacements of the masses with label $l \leq -1$ satisfy their balance equations automatically. Likewise,
 185 we represent the displacements of the masses with label $l \geq L/a$ by,

$$186 \quad u_l = u_{\text{out}} \exp(ik(l - L/a)a), l \geq L/a \quad (8)$$

187 Here, u_{out} indicates the unknown amplitude of the transmitted wave. In total, we have $L/a + 1$
 188 unknowns, including u_{ref} , u_{out} , and the displacements, u_l , with $l = 1, 2 \dots (L/a - 1)$. These unknowns
 189 are obtained from $L/a + 1$ equilibrium equations for the masses with labels $l = 0, 2 \dots L/a$. As defined
 190 above, the transmission coefficient is obtained via $T(\omega) = u_{\text{out}}/u_{\text{in}}$.

191 For example, for $L/a = 4$, we obtain the transmission spectrum

$$192 \quad T(\omega) = \frac{2i \sin(ka) K_1^2 (K_1(K_1 + K_3)(K_1 + 5K_3) - 2K_3(4K_1 + K_3)m\omega^2 + 2K_3m^2\omega^4)}{F_1(\omega)F_2(\omega)}. \quad (9)$$

193 Herein,

$$194 \quad F_1(\omega) = \exp(ika) K_1(2K_1 + K_3 - m\omega^2) - (3K_1 - m\omega^2)(K_1 + 2K_3 - m\omega^2), \quad (10)$$

195 and

$$196 \quad F_2(\omega) = 2(\exp(ika) - 1)K_1^3 + m^2\omega^4(M\omega^2 - 2K_3) + (\exp(ika) - 6)K_1m\omega^2(M\omega^2 - K_3) + \\ 197 \quad K_1^2(9m\omega^2 - 2K_3 + 2\exp(ika)(K_3 - 2m\omega^2)). \quad (11)$$

198 The corresponding explicit expressions become very lengthy for slab length $L/a \geq 5$, and are hence not
 199 provided here.

200 Fig. 3(a) depicts an example of the calculated transmission $|T(\omega)|$ (gray scale) of the nonlocal slab (see
 201 Fig. 1(b)) versus ω and versus the spring-constant ratio K_3/K_1 . For simplicity, all other model parameters
 202 are fixed (see caption). For reference, panel (b) shows the phonon dispersion relation for the slab for
 203 selected values of K_3/K_1 (see dashed lines). We find a complex behavior. In Fig. 3(a), for low frequencies,

204 transmission peaks occur that follow the expectation for ordinary Fabry-Perot resonances (labelled “FP”
205 in Fig. 3). At higher frequencies, near specific special frequencies (see arrows in Fig. 3(a)), the resonances
206 in transmission become more and more narrow. Exactly at these special frequencies and spring constant
207 ratio K_3/K_1 , the resonances disappear. We interpret these special frequencies as being due to bound
208 states in the continuum (BIC).

209 To test this interpretation, we have performed additional numerical calculations of the eigenfrequencies
210 and eigenmodes of the slab alone, i.e., without the surrounding (not shown). We find eigenfrequencies,
211 ω_{BIC} , for which the corresponding eigenmodes exhibit strictly zero displacement amplitude at the left and
212 right end of the slab for all times t . Obviously, an incident plane wave with non-zero amplitude impinging
213 from the surrounding cannot couple to such an eigenmode. Correspondingly, the lifetime of this mode is
214 infinitely long – provided that friction plays no role, as implied in our model, see Fig. 1 or Eq. (4). This
215 means that the special frequencies of BIC resonances only depend on the slab properties, but not on the
216 properties of the surrounding. The same holds true for usual Fabry-Perot resonances.

217 To connect to our above intuitive discussion for sufficiently many unit cells within the slab, we can
218 decompose the BIC modes of the slab corresponding to the BIC angular frequencies ω_i into the three ($j =$
219 $1, 2, 3$) eigenmodes with wavenumbers $k_j(\omega_i)$ of the nonlocal dispersion relation according to Eq. (5) to
220 fulfill the three standing-wave conditions Eq. (3) simultaneously. However, the reverse is not true. Just
221 any arbitrary linear combination of the three standing-wave solutions fulfilling Eq. (3) will generally not
222 lead to a BIC mode as the displacement of the masses at the two ends of the slab is not necessarily strictly
223 zero.

224 For special (small) integer values of the relative slab thickness L/a , the BIC resonance frequencies ω_{BIC}
225 can be obtained analytically. We consider those $(1 + L/a)$ eigenfrequencies of the $(1 + L/a)$ coupled
226 masses in the slab in Fig. 1(b) for which the corresponding eigenmode is such that the mass on the left-
227 hand side and the right-hand side of the slab have strictly zero displacement amplitude at all times t (but
228 the masses in between have nonzero amplitude). Such solutions occur only for special combinations of
229 the three slab parameters m , K_1 , and K_N . For any N and $K_N = 0$, BIC solutions do not occur for any value
230 of L/a . For $K_N \neq 0$, $N = 3$ and $L/a = 3$ (i.e., only a single third-nearest-neighbor spring), a BIC does not
231 occur either. The simplest non-trivial case is $N = 3$ and $L/a = 4$, for which we have only two third-
232 nearest-neighbor springs in the slab. It is straightforward to obtain the eigenstates for this system
233 composed of five coupled masses. By demanding that the displacements of the two masses on the left
234 end and on the right end of this chain are zero for all times (see Supplementary Note 1), we obtain

$$(K_3/K_1)_{\text{BIC}} = 1; \omega_{\text{BIC}} = \sqrt{\frac{3K_3}{m}}. \quad (12)$$

For large relative slab thicknesses L/a we find BIC modes numerically as it seems hard to obtain closed analytical solutions.

For frequencies and parameters near but not identical to these BIC conditions, an incident propagating plane wave can couple to the resonance mode localized within the slab. The interference of a continuum of propagating modes and a spectrally-sharp localized mode is well known to give rise to Fano-type line shapes³⁷, the detailed form of which depends on the Fano coupling parameter. In Fig. 4(a), we show a zoomed-in view of one BIC point highlighted by the yellow box in Fig. 3(a). For K_3/K_1 values below the BIC points and with increasing angular frequency ω , we find a transmission dip (zero transmission) followed by a transmission peak (complete transmission), whereas for K_3/K_1 ratios above the BIC frequency, the sequence flips and we find a transmission peak followed by a transmission dip with increasing frequency. This behavior is more clearly seen from the selected cuts shown in Fig. 4(b) corresponding to three different K_3/K_1 values.

Further examples for other L/a , represented likewise in Fig. 3(a), are shown in Supplementary Figure 1. We find BIC modes even if only two third-nearest neighbor Hooke's springs are kept ($L/a = 4$ and $N = 3$, see Eq. (6)). In the opposite limit of a thick slab, $L/a \gg 1$ (see Fig. 1(b)), in which we expect that we can consider the slab as an effective medium, the BIC resonances survive as well. This brings us to a possible effective-medium description.

Effective-medium description. For an infinitely periodic nonlocal mass-and spring model and for $N = 3$, we have previously argued¹⁷ that one gets the following general form for the displacement field $u = u(x, t)$ within the long-wavelength limit ($ka \rightarrow 0$)

$$m \frac{\partial^2 u}{\partial t^2} = A_2 \frac{\partial^2 u}{\partial x^2} + A_4 \frac{\partial^4 u}{\partial x^4} + A_6 \frac{\partial^6 u}{\partial x^6}. \quad (13)$$

In a previous study¹⁷, we have derived explicit expressions for the parameters A_2 , A_4 , and A_6 . However, it should be noted that one gets different explicit expressions for A_2 , A_4 , and A_6 depending on which terms of the expansion one keeps. For example, even for the nearest-neighbor interactions alone (i.e., for $K_1 \neq 0$ and $K_3 = 0$) one can obtain finite terms for all three coefficients A_2 , A_4 , and A_6 in Eq. (13). Unless $K_1 \ll K_3$ (which does not hold true for the parameters considered in this paper), these terms are not negligible compared to the ones originating from the third-nearest-neighbor interactions. Therefore, we

264 have assumed a phenomenological spirit and have considered the parameters A_2 , A_4 , and A_6 in the
 265 general form Eq. (13) as fit parameters when plotting the phonon dispersion relations as gray curves in
 266 Fig. 4B and 4D in Martínez et al.¹⁷. Further examples are given in Wang et al.¹⁶.

267 If one wants to go beyond this phenomenological treatment, one would have to expand the finite
 268 differences on the right-hand side of Eq. (4) to yet much-higher orders of spatial derivatives than in Eq.
 269 (13) in order to quantitatively reproduce the results of the discrete mass-and-spring model. However, in
 270 this case, nothing is gained because the point of a meaningful effective-medium description is that it
 271 should be simpler than the underlying discrete model (or microstructure or atomic structure). Otherwise,
 272 one could rather continue working with the more complete discrete model.

273 We assume the same phenomenological spirit here. However, importantly, for the slab geometry of
 274 interest in this paper, the coefficients A_2 , A_4 , and A_6 are no longer constant versus the x -coordinate (see
 275 Fig. 1(b)). For this case of a heterogeneous nonlocal medium, it is straightforward to derive, starting from
 276 Eq. (4), the more general form

$$277 \quad m \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left(A_2(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2}{\partial x^2} \left(A_4(x) \frac{\partial^2 u}{\partial x^2} \right) + \frac{\partial^3}{\partial x^3} \left(A_6(x) \frac{\partial^3 u}{\partial x^3} \right), \quad (14)$$

278 in the limit of $a \rightarrow 0$. The coefficients $A_2(x)$, $A_4(x)$, and $A_6(x)$ can be expressed by the model
 279 parameters $K_1(x)$, $K_N(x)$ and spatial derivatives up to third order thereof (see Supplementary Materials
 280 of Martínez et al.¹⁷ for the case of constant coefficients). However, again, the expressions for $A_2(x)$,
 281 $A_4(x)$, and $A_6(x)$ depend on which terms of the expansion one keeps. If one considers the
 282 mathematically strict limit of $a \rightarrow 0$, one gets discontinuous steps of the coefficients $A_2(x)$, $A_4(x)$, and
 283 $A_6(x)$ at the interfaces of the slab, leading to diverging derivatives on the right-hand side of Eq. (14). One
 284 possible strategy to solve Eq. (14) with such discontinuous jumps of parameters is to introduce additional
 285 continuity conditions (as described for low-order differential equations in many textbooks³⁸) or to treat
 286 the derivatives in a distributional sense³⁹. However, in the current paper, we rather assume continuous
 287 coefficients as detailed below.

288 We rather make a second phenomenological assumption: We search for reasonable coefficients $A_2(x)$,
 289 $A_4(x)$, and $A_6(x)$ that lead to a behavior of the transmission $|T(\omega)|$ of the nonlocal slab that at least
 290 roughly qualitatively resembles the behavior we have found for the discrete mass-and-spring model
 291 shown in Fig. 3 or Fig. 4. By “reasonable”, we mean that the dependencies $A_2(x)$, $A_4(x)$, and $A_6(x)$ must
 292 assume constant values far away from the interfaces. However, we must assume phenomenological
 293 shapes of the transition in the smeared-out interface regions (see above discussion on Fig. 1(b)).

294 Intuitively, the smearing out extends over a length scale Na given by the nonlocal interaction of order N .
 295 Furthermore, the coefficients $A_4(x)$ and $A_6(x)$ must be extremely small in the local surrounding.
 296 Conceptually, they should be zero in a local medium. However, mathematically, they cannot be strictly
 297 zero there, because this would again lead to discontinuous jumps and hence divergences of spatial
 298 derivatives when attempting to solve Eq. (14).

299 We do *not* expect a quantitative agreement with our results for the discrete heterogeneous mass-and-
 300 spring model (see, e.g., Fig. 3) because this form of a phenomenological effective-medium description
 301 does not even capture the dispersion relation of the nonlocal model quantitatively (see Figs. 4B and 4D in
 302 a previous study¹⁷). The asymptotics for $|k| \rightarrow \pi/a$ is incorrect, too⁴⁰. Our effective-medium description
 303 can only capture roughly and qualitatively the fact that there is a roton minimum at a finite wavenumber
 304 within the first Brillouin zone. Nevertheless, we feel that it is interesting and relevant to identify a simple
 305 effective-medium description that can at least capture the existence of BIC behavior for nonlocal slabs.

306 To compute the transmission spectrum of a nonlocal slab according to Eq. (14) within the effective-
 307 medium description numerically, we proceed as follows. Figure 6(a) and (b) illustrate the discrete model
 308 and the corresponding continuum model. Here, $L = 9a$ serves as an example. In the discrete model (see
 309 Fig. 5(a)), all springs connecting two neighboring masses are the same. Therefore, we can naturally set
 310 $K_1(x) = 1$ in the continuum model. The spatial dependence of the non-local spring constant $K_3(x)$ needs
 311 to be manually constructed. We assume a smooth function for $K_3(x)$ in the region of $0 < x < 3a$, roughly
 312 corresponding to the boundary length scale of the discrete slab (compare Fig. 5(a) and (b)). Due to mirror
 313 symmetry of the discrete system, $K_3(x)$ for $L - 3a < x < L$ is obtained by symmetry. For the central part
 314 of the slab, i.e., $3a < x < L - 3a$, and the two surroundings to the left and right of the slab, i.e., $x < 0$
 315 and $x > L$, $K_3(x)$ becomes constant. This constant is determined by the value of the graded profiles at
 316 $x = 0$ and $x = L$. The effective coefficients, $A_2(x)$, $A_4(x)$, and $A_6(x)$ of the continuum model are
 317 chosen phenomenologically as described above.

318 Now, we consider a plane wave with angular frequency ω incident onto the left interface of the slab. Since
 319 the surrounding has small but non-zero coefficients A_4 and A_6 , three reflected modes exist, one with a
 320 real wavenumber, corresponding to a propagating mode, and two with complex wavenumbers, denoting
 321 evanescent modes that exponentially decay away from the interface. The total displacement field can be
 322 written as

$$323 \quad u(x) = u_{\text{in}} \exp(ikx) + R_1 \exp(-ik_1x) + R_2 \exp(-ik_2x) + R_3 \exp(-ik_3x), x \leq 0. \quad (15)$$

324 The wavenumber k_1 is purely real, while k_2 and k_3 should have positive imaginary parts to ensure
 325 exponential decay for $x < 0$. The three wavenumbers all satisfy the dispersion relation

$$326 \quad \omega^2 = A_2 k_i^2 - A_4 k_i^4 + A_6 k_i^6 \quad (16)$$

327 for $i = 1, 2, 3$. In the transmission region, we start from the displacement field

$$328 \quad u(x) = T_1 \exp(ik_1 x) + T_2 \exp(ik_2 x) + T_3 \exp(ik_3 x), x \geq La. \quad (17)$$

329 Here, the three wavenumbers $k_i, i = 1, 2, 3$ are the same as in Eq. (15). In the above two expressions, R_i
 330 and $T_i, i = 1, 2, 3$, are the corresponding unknown reflection and transmission coefficients for the three
 331 modes.

332 To solve the six unknown coefficients, R_i and $T_i, i = 1, 2, 3$, wave propagation inside the non-local slab
 333 must be considered. However, due to inhomogeneous material parameters, the displacement fields
 334 cannot be constructed analytically. Here, we implement a state-space approach for solving the high-order
 335 ordinary differential equation⁴¹.

336 We first re-write the above sixth-order ordinary differential equation (14) into the following matrix form,

$$337 \quad \frac{d\mathbf{S}(x)}{dx} = \mathbf{P}(x) \cdot \mathbf{S}(x), \quad (18)$$

$$338 \quad \mathbf{S}(x) = \begin{pmatrix} u(x) \\ A_2(x)u'(x) \\ A_4(x)u''(x) \\ A_6(x)u'''(x) \\ (A_6(x)u'''(x))' + A_4(x)u''(x) \\ (A_6(x)u'''(x))'' + (A_4(x)u''(x))' + A_2(x)u'(x) \end{pmatrix}, \quad (19)$$

$$339 \quad \mathbf{P}(x) = \begin{pmatrix} 0 & \frac{1}{A_2(x)} & 0 & 0 & 0 & 0 \\ 0 & \frac{A_2'(x)}{A_2(x)} & \frac{A_2(x)}{A_4(x)} & 0 & 0 & 0 \\ 0 & 0 & \frac{A_4'(x)}{A_4(x)} & \frac{A_4(x)}{A_6(x)} & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -m\omega^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (20)$$

340 Here, the prime symbol $'$ represents the spatial derivative with respect to the coordinate x and $\mathbf{S}(x)$ is
 341 called the state-space vector.

342 Next, the slab is discretized into many thin layers. The left location and right location of the j^{th} layer are
 343 denoted as x_{j-1} and x_j , respectively. Each layer is assumed to be homogeneous with its material

344 parameters being evaluated at its middle, i.e., $A_2 \left((x_{j-1} + x_j)/2 \right)$, $A_4 \left((x_{j-1} + x_j)/2 \right)$, and $A_6 \left((x_{j-1} + \right.$
345 $x_j)/2 \left. \right)$, respectively. The discretized problem will converge to the original problem with graded material
346 parameter distribution if the discretized layers are sufficiently thin.

347 Within the j^{th} layer, the matrix $\mathbf{P}(x)$ becomes a constant matrix and the Eq. (19) has an exponential
348 solution⁴¹. Furthermore, the two state-space vectors at both ends of the thin layer have the following
349 transfer relation,

$$350 \quad \mathbf{S}(x_j) = \mathbf{t}(x_j) \cdot \mathbf{S}(x_{j-1}), j = 1, 2 \dots, \quad (21)$$

351 with

$$352 \quad \mathbf{t}(x_j) = \exp \left((x_j - x_{j-1}) \mathbf{P} \left(\frac{x_{j-1} + x_j}{2} \right) \right). \quad (22)$$

353 Note that the state-space vector is continuous across the interface between two adjacent thin layers.
354 Therefore, we can apply the transfer relation Eq. (21) sequentially to obtain the transfer relation between
355 the two state space vectors at both ends, i.e., $x = 0$ and $x = La$, of the slab region,

$$356 \quad \mathbf{S}(La) = \mathbf{T} \cdot \mathbf{S}(0), \mathbf{T} = \prod_j \mathbf{t}(x_j). \quad (23)$$

357 The two state-space vectors $\mathbf{S}(La)$ and $\mathbf{S}(0)$ are also obtained from the derived displacement fields Eqs.
358 (21) - (22) for the incidence region and transmission region. Together with the transfer relation Eq. (23),
359 the six unknown coefficients, R_i and T_i , $i = 1, 2, 3$ can be obtained.

360 In Fig. 6(a), we show the numerically calculated transmission results by using the above effective-medium
361 model for a slab with relative length $L/a = 9$. The other chosen parameters are given in the figure caption.
362 By comparing Fig. 3(a) and Fig. 6(a), we see that the effective model can capture the BIC behavior as well
363 as the usual Fabry-Perot resonance qualitatively well. The BIC behavior also occurs in the frequency range
364 where multiple eigenstates coexist (the roton part of the dispersion relation). The agreement with respect
365 to the discrete model cannot be quantitative because the dispersion relations for the discrete model and
366 the effective-medium model do not match exactly (Fig. 6(b)). As for previous discrete model (see Fig. 4),
367 Figure 7 shows an enlarged view of the BIC point enclosed by the yellow box in Fig. 6(a). While the BIC
368 point appears in both, Fig. 4(b) and Fig. 7(b), the transmission line shapes are qualitatively different.
369 Results for different relative slab thicknesses in the effective-medium model are shown in Supplementary
370 Figure 2. There, one can again see the trend that, as the slab thickness increases, more and more BIC
371 points appear.

372

373 **Metamaterial microstructures.** So far, we have only considered a conceptual discrete mass-and-spring
 374 toy model and an effective-medium simplification thereof. This model itself can hardly be called a
 375 metamaterial. We have previously discussed that acoustic metamaterials for airborne sound can be
 376 described approximately by the mass-and-spring toy model¹⁶. Therefore, in this section, we perform
 377 numerical calculations for a slab of a specific acoustic metamaterial.

378 Figure 8(a) illustrates the considered metamaterial for airborne sound. The metamaterial is composed of
 379 acoustical cavities (blue cylinders) and acoustical tubes (green and red pipes). Based on our previous
 380 theoretical and numerical studies¹⁶, the acoustical cavities can be treated as masses in the discrete model
 381 in Fig. 1(a), and the green (red) acoustical tubes correspond to nearest-neighbor (third-nearest-neighbor)
 382 springs. The ratio between the strength of the third-nearest-neighbor interactions and that of the nearest-
 383 neighbor interactions can be tuned through the geometry parameter R_3/R_1 . The metamaterial structure
 384 shown in Fig. 10(a) forms the basis for the following calculations. Figure 8(b) exhibits a specific realization
 385 of the discrete model in Fig. 1(b) by using the illustrated nonlocal metamaterial in Fig. 8(a). The length of
 386 the metamaterial structure in Fig. 8(b) is about $L = 5a$. The surrounding tubes have no cut-off frequency,
 387 which is similar to the continuum model in the preceding section. Viscosity of air usually leads to losses in
 388 acoustic systems⁴² and can influence the high-quality-factor resonances near the expected BIC points.
 389 Therefore, in what follows, we will show and discuss numerical results with and without losses.

390 We simulate the sound wave propagation in the metamaterial shown in Fig. 8(b) by using the commercial
 391 software COMSOL Multiphysics. A plane-wave radiation condition is applied at the bottom of the model
 392 to mimic an incident plane wave. A perfectly matched layer is employed at the top to mimic a semi-infinite
 393 transmission region with no reflections⁴³. All other boundaries are treated as acoustic rigid boundaries²³.
 394 The linear acoustic equation in frequency,

$$395 \quad \nabla \cdot (\nabla p_\omega(\mathbf{r})) = -\frac{\omega^2}{c_{\text{air}}^2} p_\omega(\mathbf{r}) \quad (24)$$

396 is solved with the above specified boundary conditions. ω again represents the excitation angular
 397 frequency, $p_\omega(\mathbf{r})$ is the corresponding pressure field, and c_{air} is the speed of sound wave in air. The
 398 transmission coefficient T is extracted from the pressure field in the transmission region.

399 Results for the transmission behavior of the lossless microstructured slab are given in Fig. 9. Panel (a)
 400 depicts the transmission amplitude $|T|$ versus the wave frequency ω and the geometry parameters
 401 R_3/R_1 . Panel (b) exhibits the calculated lowest phonon band for the periodic metamaterial in Fig. 9(a) for

402 different ratios R_3/R_1 . In the numerical simulations, we fix the radius $R_1 = 0.1a$ and vary the parameter
403 R_3 . In analogy to the above mass-and-spring model and continuum model, Fabry-Perot resonances are
404 observed in Fig. 9(a). Furthermore, a BIC behavior is clearly identified within that frequency range, for
405 which multiple Bloch wave modes coexist. Near by the BIC point, very sharp Fano resonances appear – as
406 for the discrete model as well as for the effective-medium model (see above).

407 In Fig. 10, we show the calculated transmission amplitude $|T|$ as in Fig. 9(a), but with losses accounted for.
408 Here, viscous damping in the acoustic pipes is treated *via* the “narrow region acoustics” in COMSOL
409 Multiphysics. A quasi-BIC behavior is still observed from the plot. Here, the resonances near the BIC points
410 have much smaller quality factors compared to the lossless case in Fig. 10(a). Nevertheless, the behavior
411 is qualitatively similar to that of the discrete model and that of the effective-medium model, respectively.

412 We expect that our findings for nonlocal elastic slabs can be translated to other systems. For example, a
413 thin film of superfluid helium, for which rotons were originally discovered, in a local surrounding should
414 show a similar overall transmission behavior according to our intuitive interpretation. The detailed
415 mathematical description might be quite different though. Furthermore, the nonlocal discrete mass-and-
416 spring model discussed here can be exactly mapped onto an electrical circuit composed of lumped
417 capacitors and inductors, where the capacitors correspond to the masses and two types of inductors to
418 the nearest-neighbor and beyond-nearest-neighbor Hooke’s springs, respectively.

419 Finally, we note again that the minimum in the roton-like dispersion relation corresponds to an
420 exceptional point. Furthermore, we have shown that the roton-like dispersion relation leads to BIC for a
421 nonlocal metamaterial slab. This BIC behavior has already occurred at frequencies near to those of the
422 roton minimum of the slab. We speculate that further interesting behavior might occur if one tunes the
423 system parameters such that the BIC frequency coincides with that of the roton exceptional point.

424

425 **Data availability**

426 The data that support the plots within this paper and other findings of this study are published on the
427 open access data repository of the Karlsruhe Institute of Technology (<https://doi.org/10.35097/860>).

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431 **Code availability**

432 The numerical simulations in this work for the mass-and-spring model have been performed by using the
433 commercial software MATLAB. Numerical simulations for the elastic metamaterials are performed using
434 the commercial software COMSOL Multiphysics. The code and models are published on the open access
435 data repository of the Karlsruhe Institute of Technology (<https://doi.org/10.35097/860>).

436

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443

444 **Author contributions**

445 Y.C. and K.W. performed the numerical simulations. K.W. and M.K. designed the metamaterials. Y.C. and
446 S.G. developed the theory. M.W. wrote the first draft. C.W. and M.W. supervised the effort. All authors
447 discussed the results and contributed to the writing and reviewing of the manuscript.

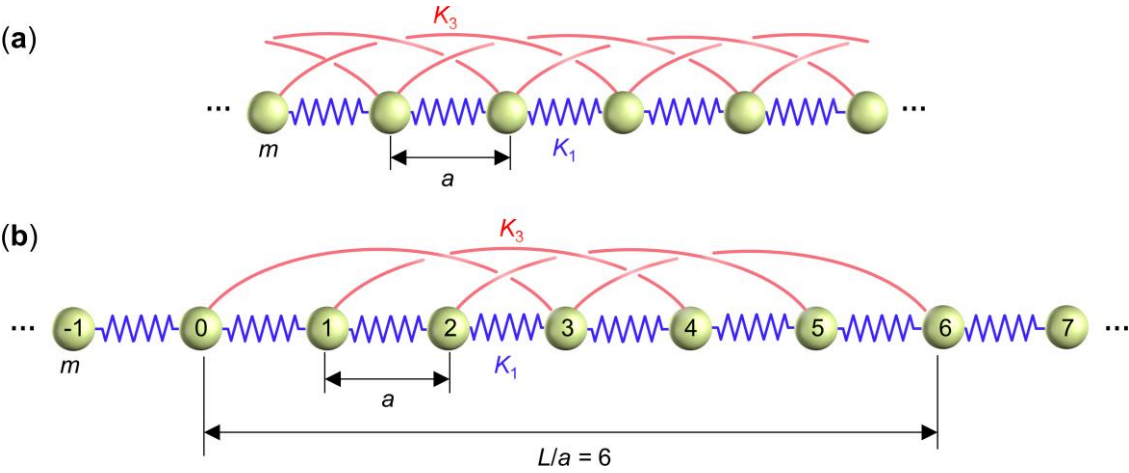
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449 **Competing interests**

450 All authors declare no competing interests.

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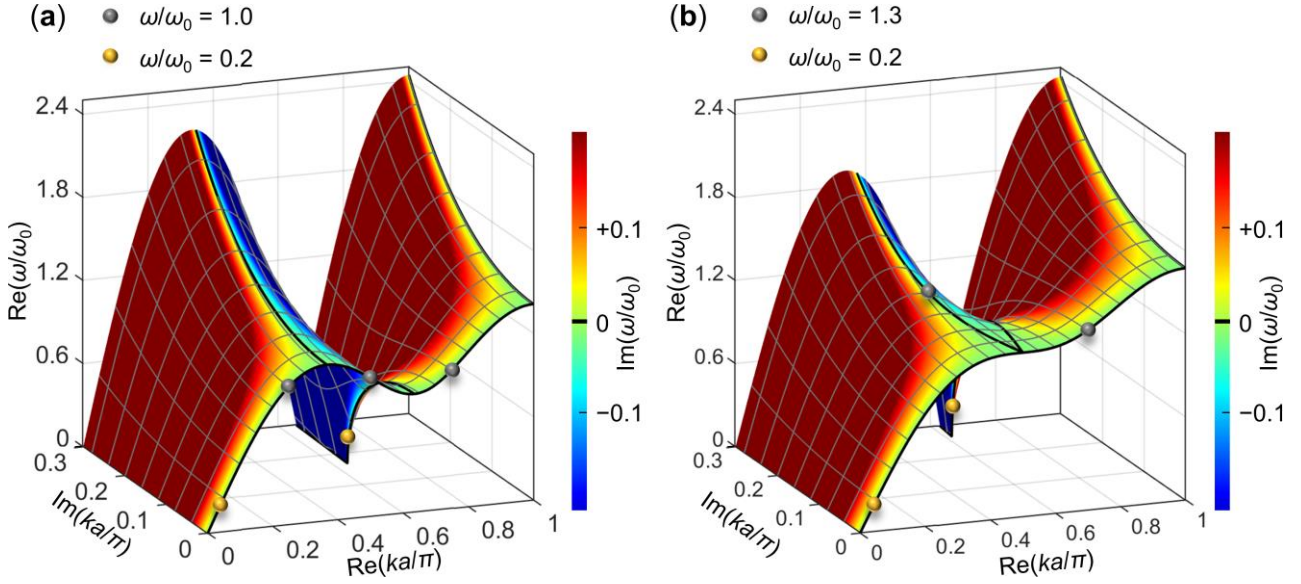
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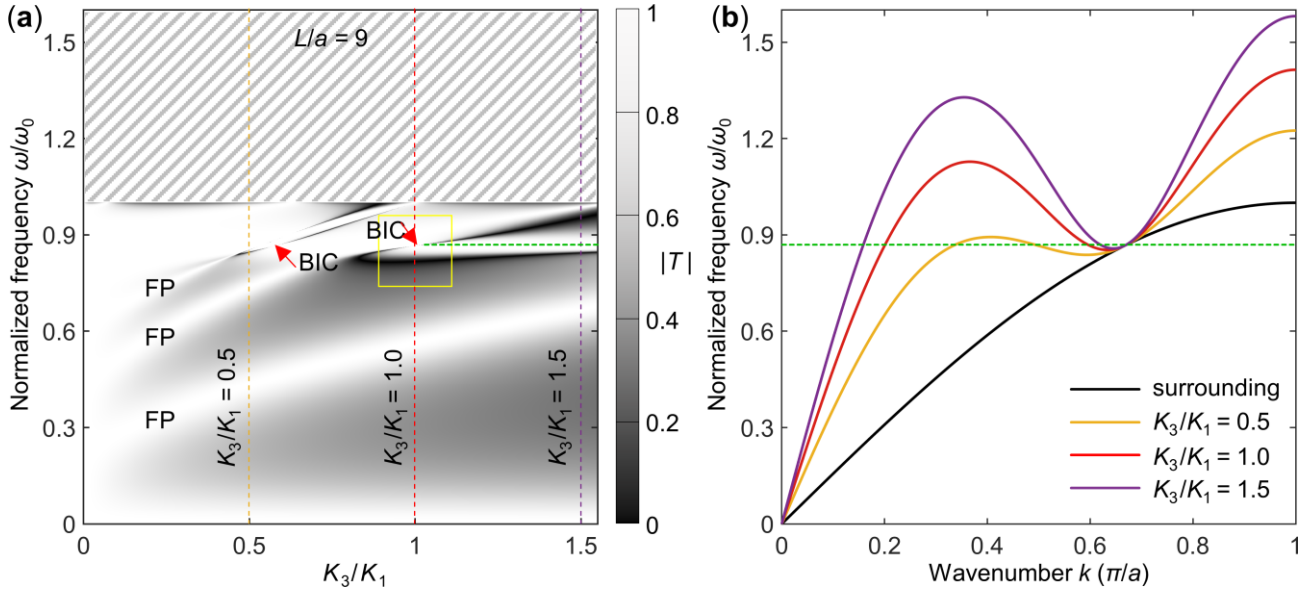
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455 **Figure 1. Illustration of mass-and-spring model.** (a) An infinite periodic one-dimensional mass-and-spring
 456 model composed of masses (light yellow), m , connected to their nearest neighbors by Hooke's springs
 457 (blue) with spring constant K_1 and additionally connected to their N -th nearest neighbors by Hooke's
 458 springs (red) with spring constant K_N . Shown is the example of $N = 3$, which we emphasize in this paper
 459 because it is the smallest integer for which one obtains a roton-like minimum inside of the first Brillouin
 460 zone. The lattice constant is a . (b) A slab of such nonlocal material cladded between half spaces of an ordinary
 461 local mass-and-spring model with only nearest-neighbor interactions. The slab thickness is defined by the
 462 integer ratio L/a . Shown is the example of $L/a = 6$ and $N = 3$. Note that the boundaries of the slab are
 463 smeared out in the sense that only the center mass out of the $(1 + L/a)$ masses in the slab has two third-
 464 nearest-neighbor connections. The remaining six masses have only one such connection. This smearing-
 465 out is an immediate consequence of the nonlocality.

466



467
 468 **Figure 2. Dispersion relations of the mass-and-spring model.** (a) Surface plot of real component of
 469 frequency ω versus the real and the imaginary components of the wavenumber k following Eq. (5). For
 470 the conditions discussed in this paper, the angular frequency ω is purely real. The wavenumber k is also
 471 purely real for a Bloch-periodic solution of an infinite periodic model. For a finite-thickness slab (see Fig.
 472 1(b)), evanescent modes can play a role and the imaginary part of k is generally not zero. The imaginary
 473 part of the complex-valued angular frequency ω is shown by the false-color scale. Only the positive parts
 474 of the real and imaginary components of the wavenumber are shown here as the corresponding negative
 475 parts can be obtained by mirror symmetry. The four highlighted black lines on the surface lead to purely
 476 real angular frequency ω . Among them, one corresponds to purely real wavenumber and the other three
 477 correspond to complex wavenumbers in the range of $\text{Re}(k) > 0$. For a normalized frequency of $\omega/\omega_0 =$
 478 1.0, in between the local maximum and roton minimum, three real wavenumbers (see the three gray dots)
 479 can be obtained from the dispersion relation. For $\omega/\omega_0 = 0.2$ below the roton minimum, a real
 480 wavenumber and a pair of complex conjugate wavenumber are obtained (see two yellow dots).
 481 Parameters are $K_3/K_1 = 1.0$ and the normalization frequency is $\omega_0 = \sqrt{4K_1/m}$. (b) Parameters
 482 corresponding to the critical case without roton minimum in the dispersion relation, i.e., $m = 1$ and
 483 $K_3/K_1 = 1/3$. Note that still three solutions for the complex-valued k in the range of $\text{Re}(k) > 0$ occur at
 484 a given angular frequency ω .



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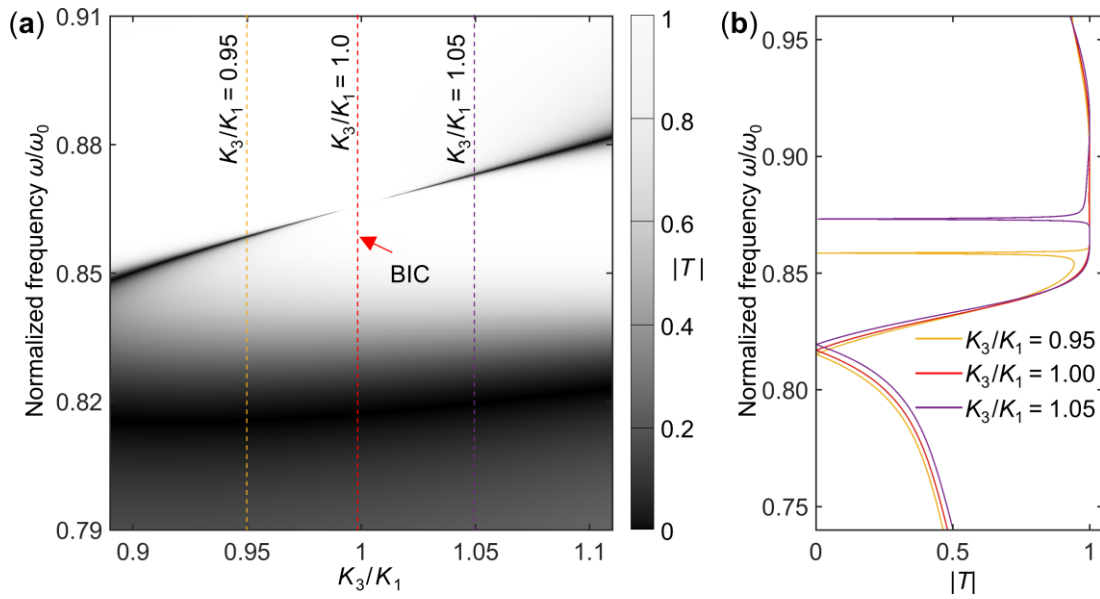
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Figure 3. Phonon transmission results of the mass-and-spring model. (a) Calculated transmission amplitude $|T(\omega)|$ of the discrete nonlocal mass-and-spring-model slab (see Fig. 1(b)) shown on a gray scale versus ω and K_3/K_1 . In the hatched region above the cut-off frequency $\omega/\omega_0 = 1.0$, waves cannot propagate in the surrounding medium. “FP” denotes Fabry-Perot resonances, “BIC” bound-states-in-the-continuum points. Note the Fano-type line shapes of $|T(\omega)|$ near the BIC points. Two “BIC” points are indicated. A zoom into one of them (see yellow box) is shown in Fig. 4(a). The normalization frequency is $\omega_0 = \sqrt{4K_1/m}$. Parameters are: $m = 1$, $L/a = 9$. (b) Illustration of the corresponding dispersion relations of the slab region for purely real ω and purely real k for different ratios of K_3/K_1 .

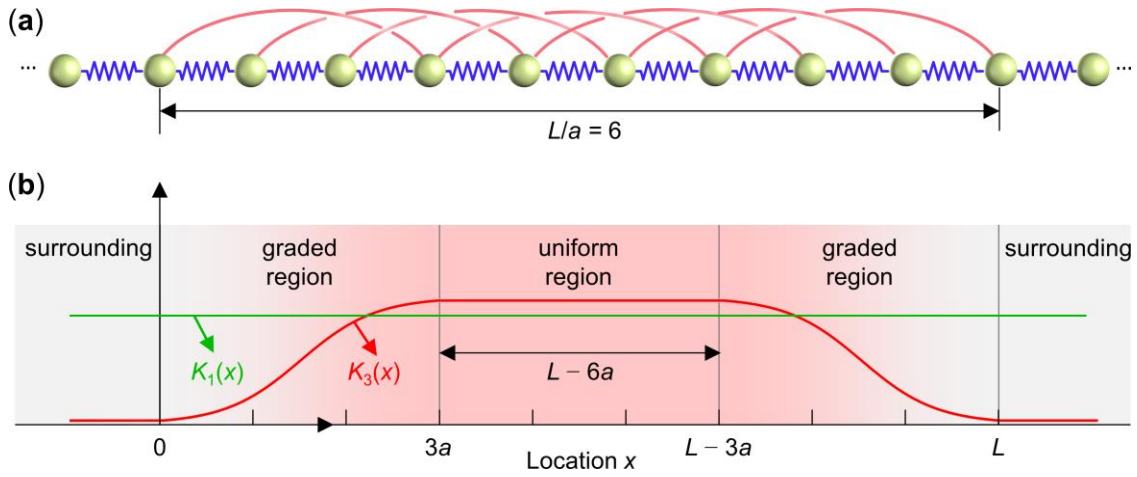


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499 **Figure 4. Zoomed-in view of the BIC and sharp resonances.** (a) Zoomed-in view of the bound-states-in-
 500 the-continuum (BIC point) highlighted by the yellow box in Fig. 3(a). (b) Cuts through the data in panel (a)
 501 at three selected ratios K_3/K_1 (see dashed vertical lines in (a)). Extremely sharp resonance versus
 502 frequency ω occur for parameters close to the BIC point (yellow and purple curves). At the BIC point (red
 503 curve), the sharp resonance disappears as incident waves strictly do not couple to the BIC.

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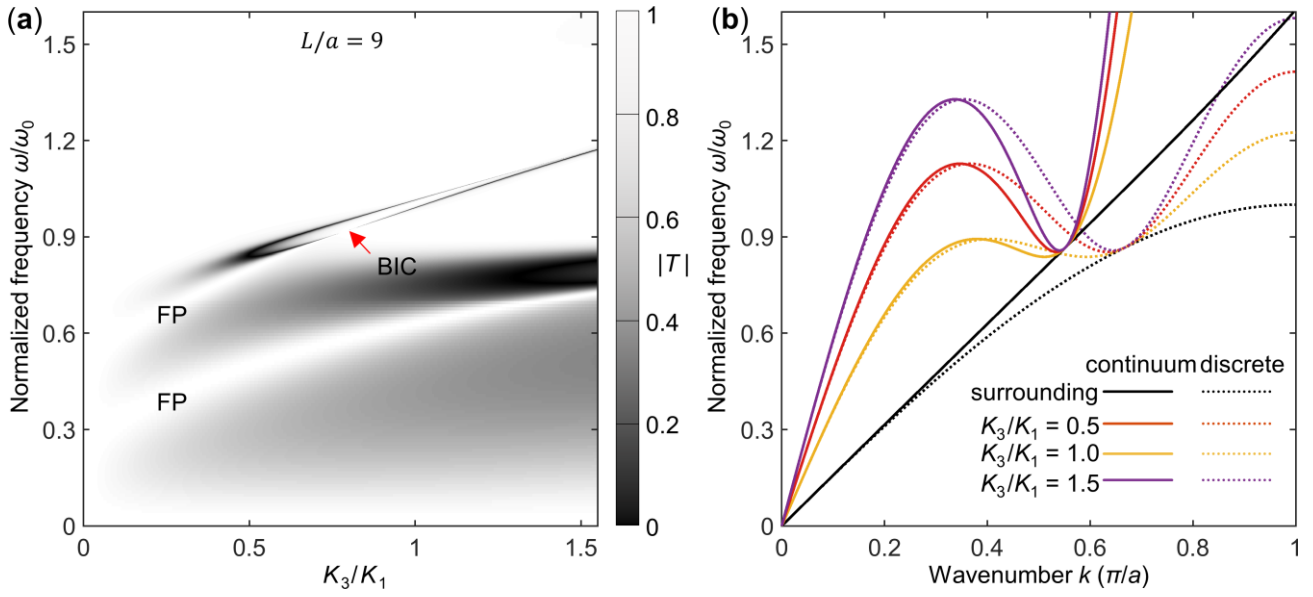
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Figure 5. Illustration of the effective-medium model. (a) Discrete system with slab thickness L . $L = 9a$ is used as an example. (b) Scheme of the continuum model composed of the slab region and the two semi-infinite surroundings. The slab is further decomposed into a central uniform region, i.e., $3a < x < L - 3a$, and two graded regions, i.e., $0 < x < 3a$, and $L - 3a < x < L$. The graded regions represent smooth transitions of the effective parameters to those in the two surroundings. In the region of $0 < x < 3a$, a function that increases smoothly from an extremely small value to a finite value is assumed for $K_3(x)$, indicating the third-nearest-neighbor constants. $K_3(x)$ for $L - 3a < x < L$ is obtained from mirror symmetry of the system. For the central uniform part of the slab, i.e., $3a < x < L - 3a$, and the two surroundings, i.e., $x < 0$ and $x > L$, $K_3(x)$ becomes constant and is obtained from continuity. The parameter $K_1(x)$ is assumed to be constant throughout the 1D system, $K_1(x) = 1$. The effective parameters of the continuum model are constructed from the two spring constants $K_1(x)$ and $K_3(x)$, i.e., from $A_2(x) = K_1(x) + 9K_3(x)$, $A_4(x) = 6K_3(x)$, and $A_6(x) = K_3(x)$, respectively.

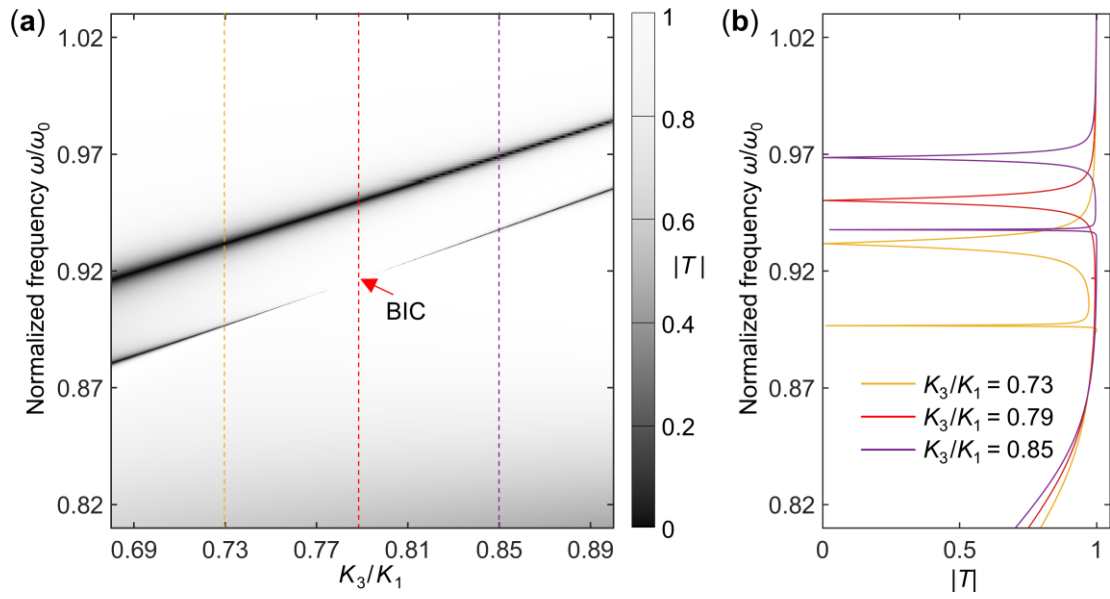


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521 **Figure 6. Phonon transmission results for the phenomenological effective-medium model rather than**
 522 **the discrete mass-and-spring model.** (a) Phonon transmission results. Parameters are $L = 9a$ and
 523 $K_3(x)/K_1 = 1 - 1/(1 + \exp(2(x - 3a/2)))$ for $0 < x < 3a$. The assumed $K_3(x)$ ensures that the two
 524 surrounding regions exhibit extremely small non-local stiffness parameters (about two orders of
 525 magnitude smaller than for the slab). (b) Dispersion relations for the effective-medium model (solid
 526 curves). The dashed curves correspond to the data in Fig. 3(b) for the discrete mass-and-spring model and
 527 can be compared directly to the effective-medium model.

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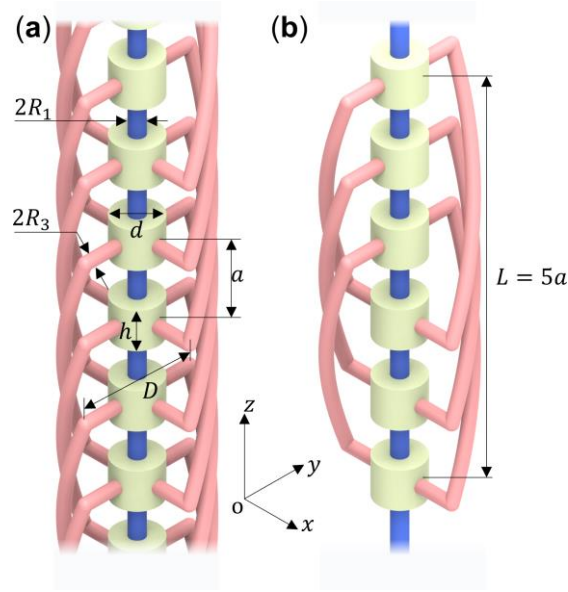


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531 **Figure 7. Zoomed-in view of the BIC and sharp resonances.** (a) Zoomed-in view of the bound-states-in-
 532 the-continuum (BIC) point highlighted by the yellow box in Fig. 6(a). (b) Cuts through the data in panel (a)
 533 at three selected ratios K_3/K_1 (see dashed vertical lines in (a)).

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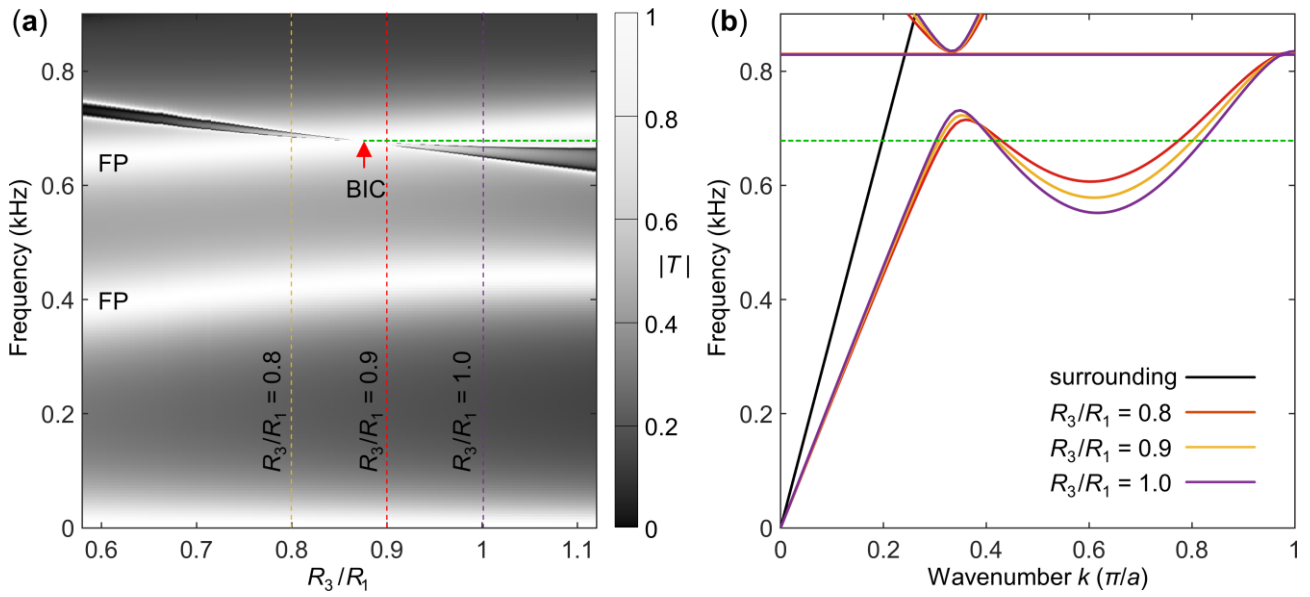
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537 **Figure 8. Illustration of the considered 3D acoustical metamaterial for airborne sound.** (a) Infinite
 538 periodic metamaterial with non-local interactions. The metamaterial is composed of acoustical cavities
 539 (yellow cylinders) and acoustical channels (blue and red pipes). Colors are for illustration only, all parts
 540 represent voids for air. The yellow cylinders, with height h and diameter d , correspond to masses in the
 541 discrete mass-and-spring model, and the blue (red) pipes, with diameter $2R_1$ ($2R_3$), represent the
 542 nearest-neighbor interactions (third-nearest-neighbor interactions). The helix part of the red pipes has a
 543 major radius, $D/2$. (b) A specific realization of the discrete model in Fig. 1(b) by using the metamaterial
 544 structure in (a). The two semi-infinite pipes at both ends represent the surrounding. Therefore, the
 545 surrounding medium has no cut-off frequency, analogous to the effective-medium model shown in Fig. 6.
 546 Geometry parameters are: $h = 0.5a$, $d = 0.6a$, $D = 1.5a$, $R_1 = 0.1a$, and $a = 0.1$ m, respectively. For
 547 air, we choose the sound velocity $c_{\text{air}} = 343$ m s⁻¹ and the mass density $\rho_{\text{air}} = 1.29$ kg m⁻³.

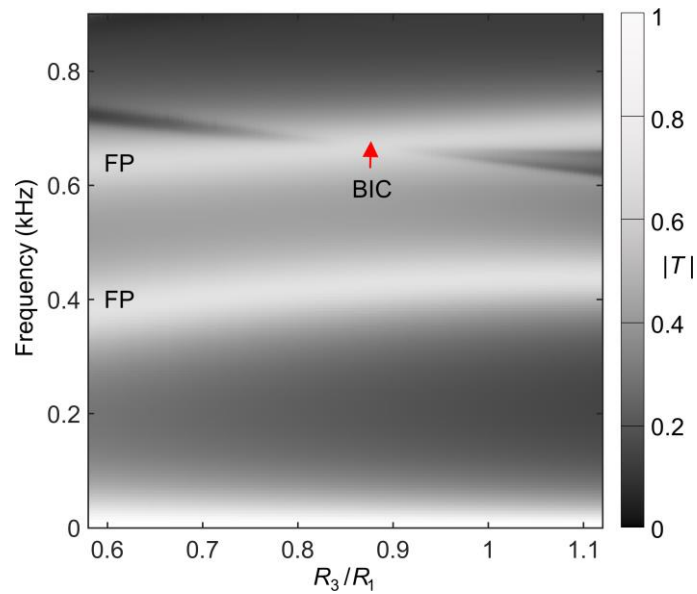
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550 **Figure 9. Phonon transmission results of the designed metamaterial.** (a) Numerically obtained
 551 transmission spectrum $|T|$ for the metamaterial structure shown in Fig. 8(b) versus exciting frequency
 552 $\omega/(2\pi)$ and versus the ratio R_3/R_1 . Damping is neglected. The bound-states-in-the-continuum (BIC)
 553 point is marked by the red arrow. (b) Calculated phonon dispersion relation for three selected ratios
 554 R_3/R_1 (see legend). The lowest acoustic band exhibits a pronounced roton-like behavior. For comparison,
 555 the dispersion relation of the surrounding (a straight line) is depicted by the black solid curve.

556



557
 558 **Figure 10. Numerically obtained transmission spectrum $|T|$ with viscous damping in the acoustic pipes**
 559 **is accounted for.** As a result, the resonances around the BIC point are smeared out, but the overall
 560 qualitative behavior remains unchanged.

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 563 **References:**

- 564 1. Liu, Z., et al. Locally resonant sonic materials. *Science* **289**, 1734-1736 (2000).
- 565 2. Kadic, M., Milton G. W., van Hecke M. & Wegener M. 3D metamaterials. *Nat. Rev. Phys.* **1**, 198-210
 566 (2019).
- 567 3. Frenzel, T., Kadic M. & Wegener M. Three-dimensional mechanical metamaterials with a twist. *Science*
 568 **358**, 1072-1074 (2017).
- 569 4. Pendry, J. B., Holden A. J., Stewart W. J. & Youngs I. Extremely low frequency plasmons in metallic
 570 mesostructures. *Phys. Rev. Lett.* **76**, 4773 (1996).
- 571 5. Smith, D. R., Pendry J. B. & Wiltshire M. Metamaterials and negative refractive index. *Science* **305**, 788-
 572 792 (2004).
- 573 6. Zhu, R., Liu X. N., Hu G. K., Sun C. T. & Huang G. L. Negative refraction of elastic waves at the deep-
 574 subwavelength scale in a single-phase metamaterial. *Nat. Commun.* **5**, 5510 (2014).

575 7. Liu, X. N., Huang G. L. & Hu G. K. Chiral effect in plane isotropic micropolar elasticity and its application
576 to chiral lattices. *J. Mech. Phys. Solids* **60**, 1907-1921 (2012).

577 8. Chen, Y., Frenzel T., Guenneau S., Kadic M. & Wegener M. Mapping acoustical activity in 3D chiral
578 mechanical metamaterials onto micropolar continuum elasticity. *J. Mech. Phys. Solids* **137**, 103877 (2020).

579 9. Xu, X., et al. Physical Realization of Elastic Cloaking with a Polar Material. *Phys. Rev. Lett.* **124**, 114301
580 (2020).

581 10. Farzbod, F. & Scott-Emuakpor O. E. Interactions beyond nearest neighbors in a periodic structure:
582 Force analysis. *Int. J. Solids Struct.* **199**, 203-211 (2020).

583 11. Di Paola, M. & Zingales M. Long-range cohesive interactions of non-local continuum faced by fractional
584 calculus. *Int. J. Solids Struct.* **45**, 5642-5659 (2008).

585 12. Chaplain, G. J., Hooper I. R., Hibbins A. P. & Starkey T. A. Reconfigurable Elastic Metamaterials:
586 Engineering Dispersion with Meccano™. *arXiv preprint arXiv:2206.10487*, (2022).

587 13. Kutsenko, A. A., Shuvalov, A. L., Poncelet, O. & Darinskii, A. N. Quasistatic stopband and other unusual
588 features of the spectrum of a one-dimensional piezoelectric phononic crystal controlled by negative
589 capacitance. *C. R. Mecanique* **343**, 680-688 (2015).

590 14. Kutsenko, A. A., Shuvalov, A. L. & Poncelet, O. Dispersion spectrum of acoustoelectric waves in 1D
591 piezoelectric crystal coupled with 2D infinite network of capacitors. *J. Appl. Phys.* **123**, 044902 (2018).

592 15. Chen, Y., Kadic M. & Wegener M. Roton-like acoustical dispersion relations in 3D metamaterials. *Nat.*
593 *Commun.* **12**, 3278 (2021).

594 16. Wang, K., Chen Y., Kadic M., Wang C. & Wegener M. Nonlocal interaction engineering of 2D roton-like
595 dispersion relations in acoustic and mechanical metamaterials. *Commun. Mat.* **3**, 35 (2022).

596 17. Martínez, J. A. I., et al. Experimental Observation of Roton-Like Dispersion Relations in Metamaterials.
597 *Sci. Adv.* **7**, m2189 (2021).

598 18. Zhu, Z., et al. Observation of multiple rotons and multidirectional roton-like dispersion relations in

599 acoustic metamaterials. *New J. Phys.* **24**, 123019 (2022).

600 19. Landau, L. Theory of the Superfluidity of Helium II. *Phys. Rev.* **60**, 356 (1941).

601 20. Feynman, R. P. & Cohen M. Energy spectrum of the excitations in liquid helium. *Phys. Rev.* **102**, 1189
602 (1956).

603 21. Godfrin, H., et al. Dispersion relation of Landau elementary excitations and thermodynamic properties
604 of superfluid He 4. *Phys. Rev. B* **103**, 104516 (2021).

605 22. Dietrich, O. W., Graf E. H., Huang C. H. & Passell L. Neutron scattering by rotons in liquid helium. *Phys.*
606 *Rev. A* **5**, 1377 (1972).

607 23. Kinsler, L. E., Frey A. R., Coppens A. B. & Sanders J. V. *Fundamentals of acoustics* (Wiley, 1999).

608 24. Hernández, G. *Fabry-perot interferometers* (Cambridge University Press, 1988).

609 25. Hsu, C. W., Zhen B., Stone A. D., Joannopoulos J. D. & Soljačić M. Bound states in the continuum. *Nat.*
610 *Rev. Mat.* **1**, 16048 (2016).

611 26. Parker, R. & Griffiths W. M. Low frequency resonance effects in wake shedding from parallel plates. *J.*
612 *Sound Vib.* **7**, 371-379 (1968).

613 27. Chen, Y., Liu X. & Hu G. Influences of imperfectness and inner constraints on an acoustic cloak with
614 unideal pentamode materials. *J. Sound Vib.* **458**, 62-73 (2019).

615 28. Haq, O. & Shabanov S. Bound states in the continuum in elasticity. *Wave Motion* **103**, 102718 (2021).

616 29. Limonov, M. F., Rybin M. V., Poddubny A. N. & Kivshar Y. S. Fano resonances in photonics. *Nat.*
617 *Photonics* **11**, 543-554 (2017).

618 30. Marinica, D. C., Borisov A. G. & Shabanov S. V. Bound States in the continuum in photonics. *Phys. Rev.*
619 *Lett.* **100**, 183902 (2008).

620 31. Zhou, Q., et al. Geometry symmetry-free and higher-order optical bound states in the continuum. *Nat.*
621 *Commun.* **12**, 4390 (2021).

622 32. Kittel, C. *Introduction to solid state physics* (Wiley, 2005).

623 33. Fleury, R. Non-local oddities. *Nat. Phys.* **17**, 766-767 (2021).

624 34. Heiss, W. D. The physics of exceptional points. *Journal of physics. A, Mathematical and theoretical* **45**,
625 444011-444016 (2012).

626 35. Zhen, B., et al. Spawning rings of exceptional points out of Dirac cones. *Nature* **525**, 354-358 (2015).

627 36. Rosa, M. I. N., Mazzotti M. & Ruzzene M. Exceptional points and enhanced sensitivity in PT-symmetric
628 continuous elastic media. *J. Mech. Phys. Solids* **149**, 104325 (2021).

629 37. Hein, S., Koch W. & Nannen L. Fano resonances in acoustics. *J. Fluid Mech.* **664**, 238-264 (2010).

630 38. Graff, K. F. Wave motion in elastic solids (Courier Corporation, 2012).

631 39. Kanwal, R. P. *Generalized functions theory and technique: Theory and technique* (Springer Science &
632 Business Media, 1998).

633 40. Norris, A. N., Shuvalov A. L. & Kutsenko A. A. Analytical formulation of three-dimensional dynamic
634 homogenization for periodic elastic systems. *Proc. R. Soc. A* **468**, 1629-1651 (2012).

635 41. Chen, W. Q., Bian Z. G. & Ding H. J. Three-dimensional vibration analysis of fluid-filled orthotropic FGM
636 cylindrical shells. *Int. J. Mech. Sci.* **46**, 159-171 (2004).

637 42. Frenzel, T., David Brehm J., Bückmann T., Schittny R., Kadic M. & Wegener M. Three-dimensional
638 labyrinthine acoustic metamaterials. *Appl. Phys. Lett.* **103**, 61907 (2013).

639 43. Berenger, J. A perfectly matched layer for the absorption of electromagnetic waves. *J. Comput. Phys.*
640 **114**, 185-200 (1994).

641