ON A NON-HOMOGENEOUS PANTOGRAPH FUNCTIONAL DIFFERENTIAL EQUATION

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Abstract

This paper is devoted to the analytical study of the non-homogeneous functional differential equation $\boldsymbol{y'(x) = ay(qx) + by(x) + \frac{1}{x}}$, where $\boldsymbol{q}$ is a constant in $\boldsymbol{(0,1)}$ and where $\boldsymbol{a}$ and $\boldsymbol{b}$ are two non-zero real or complex numbers. This equation represents a $\boldsymbol{q}$-perturbation of the first-order ODE $\boldsymbol{y'(x) = by(x) + \frac{1}{x}}$, that admits an irregular singular point at infinity in the complex plane. First, combing hypergeometric-type and $\boldsymbol{q}$-hypergeometric-type power series yields solutions at zero and infinity. Next, using the Laplace transform allows us to convert this equation into a first-order linear $\boldsymbol{q}$-difference equation. In this way, we express the above-mentioned power series-type solutions in terms of Laplace integrals involving the Jacobi theta functions. Finally, we get the connection formula between solutions at zero and infinity, which plays a crucial role in determining the asymptotic behavior of solutions at infinity.

Keywords: differential $\boldsymbol{q}$-difference equation, Laplace transform, Jacobi theta function, connection formula, asymptotic behavior.

1 Introduction

The functional differential equation

$$y'(x) = ay(qx) + by(x) \tag{1.1}$$

represents an idealized mathematical model derived from an industrial problem concerning the wave motion of the overhead supply line in an electrified railway system [1], where a, b are non-zero real numbers, and y is a real-valued function. Equation (1.1) is often called the pantograph equation, as mentioned in [2], the term "pantograph" comes from Ockendon and Tayler [3] and Iserles [4]. It appears in applications in many areas: analytic number theory [5], nonlinear dynamical systems [6], etc.

Several extensions of equation (1.1) had been widely studied [7–12]. For instance, Carr and Dyson [9] analyzed the asymptotic behavior of solutions of equation (1.1) with $\mathrm{Re}(b) = 0$ but $b \neq 0$. Kato and Mcleod [13] studied the existence and uniqueness of solutions of (1.1) with $y(0) = 1$ and obtained the asymptotic form of solutions as $x \to \infty$, where a is a possibly complex constant. Shaldanbayev *et al.* [11] discussed the spectral properties of the associated Cauchy problem to (1.1) with $b = 0$, they obtained that in the case of $0 < q < 1$, the Cauchy problem is always a Volterra problem; in the case of $q > 1$, the problem is not solvable for all values of a. The series solutions at zero of (1.1) belong to a family of hypergeometric-type and q-hypergeometric-type series. See [14, p. 4, formula (1.2.22)] and [15, 16]. Note that Ismail and Zhang applied a technique of integral representations for certain entire q-functions to obtain the convergent asymptotic series expansion of the large zeros of the Ramanujan entire function [17].

In addition to the research on the homogeneous equation (1.1), several studies have also been devoted to investigating its corresponding non-homogeneous form. For example, Lim [18] studied the asymptotic bounds of solutions of the non-homogeneous functional differential equation (for $0 < q < 1$),

$$y'(x) = ay(qx) + by(x) + g(x), \tag{1.2}$$

where g is a continuous function defined on $[0, \infty)$ such that $g(x) = O(x^\alpha)$ for $x \to \infty$.

In this paper, we shall study (1.2) from the point of view of analytic approaches so that g will be some analytic function. To simplify, we suppose that g is a rational function, leading to the classification of three distinct cases:

Case 1: g is a polynomial;
Case 2: g is of the form $\frac{1}{x^m}$, $m > 0$;
Case 3: g is of the form $\frac{1}{(x-c)^m}$, $m > 0$ and $c \neq 0$.

For Case 1, if g is a polynomial, then equation (1.2) has a unique polynomial solution of the same degree as g. The treatment of Case 3, being more complex, is left for future work. In this paper, we only treat Case 2. Initially, a unique polynomial transformation exists, allowing us to convert equation (1.2) with $g(x) = \frac{1}{x^m}$ into an equation with $g(x) = \frac{1}{x}$ (see Proposition 2.1). Therefore, we are led to the study of a non-homogeneous functional differential equation

$$y'(x) = ay(qx) + by(x) + \frac{1}{x}, \tag{1.3}$$

where a, b are non-zero complex numbers. The boundary-value problem with $y(0) = 1$ is "well-posed" if $0 < q < 1$, but not if $q > 1$ (see [13]). Consequently, this paper will only deal with the case $0 < q < 1$.

Furthermore, for equation (1.3), it suffices only to consider the case where $b = -1$ or $b = 0$, as we will always assume that x belongs to $\mathbb{C}^*$ or to the Riemann surface $\widetilde{\mathbb{C}^*}$ of the logarithm function. Indeed, if $b \neq 0$, by making $x = -\frac{t}{b}$ and $u(t) = y(x) = y(-\frac{t}{b})$, one gets $u'(t) = -\frac{1}{b}y'(-\frac{t}{b})$. Letting $x = -\frac{t}{b}$ into (1.3) gives that

$$u'(t) = \alpha u(qt) - u(t) + f(t),$$

where $\alpha = -\frac{a}{b} \neq 0$ and $f(t) = -\frac{1}{b}g(-\frac{t}{b}) = \frac{1}{t}$. This is to say, we shall consider the equation given by the following:

$$y'(x) = \alpha y(qx) - y(x) + \frac{1}{x}. \tag{1.4}$$

On the one hand, the corresponding homogeneous equation associated with (1.4) is:

$$y'(x) = \alpha y(qx) - y(x). \tag{1.5}$$

Zhang [19] proved that if $y(0) = 1$, the unique power series solution can be expressed by a linear combination of all the elements of a system of fundamental solutions at ∞ in the complex plane. Termed as the connection formula between solutions at 0 and ∞, this formula plays an important role in determining the asymptotic behavior of the solution of this Cauchy problem. Note that the difference between two solutions of equation (1.4) is a solution of equation (1.5). This simple observation will play a key role in our study.

On the other hand, if $\alpha = 0$ in equation (1.4), the corresponding non-homogeneous ODE is the well-known Euler differential equation, whose power series solution at ∞ is simply the divergent series $\sum_{n\geq 0} n!x^{-n-1}$ (see Malgrange [20] and Ramis [21]). Even in this simple case, solutions of equation (1.4) include some complex properties!

The present paper aims to study the asymptotic behavior of solutions of (1.4). Hence, we first study the connection formula between solutions of equation (1.4) at 0 and ∞. Because the theory of elliptic functions is closely related to the analytic theory of linear functional q-difference equations [22–26], we use the Jacobi theta function to express solutions of (1.4).

The paper is organized as follows. In Section 2, we will construct solutions at 0 and ∞ by means of convergent power series and the logarithm function. In Section 3, equation (1.4) is transformed into a linear q-difference equation by means of the Laplace transform. This gives two integral solutions, whose relationship is obtained by using the associated q-difference equation. In Section 4, it is shown that one of these two integral solutions can be identified with a distinguished solution whose behavior near zero depends of the so-called critical value "γ_0", the other one being the power series solution at infinity. In Section 5, the connection formula between solutions near zero and infinity is obtained by combining the results from Sections 3 and 4. In this way, we obtain

the asymptotic behaviors at infinity over the right half-plane for all the solutions of (1.4) that can be expressed in terms of power series and the logarithm function. Notice that this is done using the above-mentioned critical value γ_0, which depends on the Euler's constant. Finally, we will note how to interpret our results by means of that obtained in the paper [18].

We first introduce some notations. Denote $\widetilde{\mathbb{C}}^* = \mathbb{C}^* \cup \{\infty\}$ as the Riemann surface of the logarithm function, and log as the principal branch value of the logarithm function defined over $\widetilde{\mathbb{C}}^*$. Thus, $x^a = e^{a \log x}$ for all $a \in \mathbb{C}$ and $x \in \widetilde{\mathbb{C}}^*$.

For any $\alpha \in \mathbb{C}$ and $n \geq 1$, we define $(\alpha)_n$ and $(\alpha; q)_n$ in the following form:

$$(\alpha)_n = \prod_{j=0}^{n-1} (\alpha + j) \quad \text{and} \quad (\alpha; q)_n = \prod_{j=0}^{n-1} (1 - \alpha q^j),$$

and we denote usually $(\alpha; q)_0 = (\alpha)_0 = 1$. One can easily get that $(\alpha; q)_n$ and $(\alpha)_n$ can be extended to $(\alpha; q)_\infty$ and $(\alpha)_\infty$ respectively, as $n \to \infty$. It is obvious to see that the following relation holds:

$$(\alpha q^{-n};\ q)_n = (-1)^n \big(\frac{q}{\alpha}; q\big)_n \big(\frac{\alpha}{q}\big)^n q^{-\frac{n(n-1)}{2}}. \tag{1.6}$$

The Jacobi theta function $\theta(q; x)$ is defined by

$$\theta(q; x) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n;$$

we write it as $\theta(x)$ in the absence of ambiguity. It verifies the following functional equation:

$$\theta(q^n x) = q^{-\frac{n(n-1)}{2}} x^{-n} \theta(x), \text{ for } n \in \mathbb{Z}. \tag{1.7}$$

Following [19], we define

$$F_0(\alpha; q, x) = \sum_{n \geq 0} \frac{(-1)^n (\alpha; q)_n}{n!} x^n, \tag{1.8}$$

and we recall that $F_0(\alpha; q, x)$ is the unique solution of homogeneous equation (1.5) with the initial condition $y(0) = 1$. The following identities can be found in [15, p. 490, Corollary 10.2.2 (a)-(b)], which will be used in this paper:

$$\sum_{n \geq 0} \frac{x^n}{(q; q)_n} = \frac{1}{(x; q)_\infty}, \qquad |x| < 1,\ |q| < 1, \tag{1.9}$$

and

$$\sum_{n\geq 0}\frac{(-1)^n q^{\frac{n(n-1)}{2}}x^n}{(q;\,q)_n}=(x;\,q)_\infty,\qquad |q|<1. \tag{1.10}$$

2 Solutions expressed in terms of power series

As stated in the Introduction, this paper is devoted to Case 2, where equation (1.2) has the form $g(x)=\frac{1}{x^m}$. Notice that this can be converted into equation (1.4) by a simple transformation using the following proposition. Therefore, we will only consider equation (1.4) in the rest of this paper.

Proposition 2.1. *Let m be an integer with $m>1$. If $\alpha\notin\{q,q^2,\cdots,q^{m-1}\}$, then there is a unique transformation of the form $y(x)=\sum\limits_{j=1}^{m-1}a_jx^{-j}+dz(x)$, where $a_j,d\in\mathbb{C}^*$, such that the equation $y'(x)=\alpha y(qx)-y(x)+\frac{1}{x^m}$ is converted to $z'(x)=\alpha z(qx)-z(x)+\frac{1}{x}$.*

Proof By direct computation, one gets that, for $j=1,\,2,\,\cdots,\,m-1$,

$$a_j=-(\alpha q^{-m+1};\,q)_{m-j-1}\frac{(j-1)!}{(m-1)!},\quad d=\frac{(\alpha q^{-m+1};\,q)_{m-1}}{(m-1)!}.$$

□

Remark 2.1. If $\alpha=q^k$ for some $k\in\{1,2,\cdots,m-1\}$, then there is a transformation of the form $y(x)=\sum\limits_{j=k}^{m-1}a_jx^{-j}+z(x)$, such that equation $y'(x)=\alpha y(qx)-y(x)+\frac{1}{x^m}$ can be converted to the homogeneous equation $z'(x)=\alpha z(qx)-z(x)$.

We now study solutions of equation (1.4) by means of power series at zero. For this purpose, we need the following lemma.

Lemma 2.1. *Let $F_0(\alpha;\,q,x)$ be as in (1.8), and let u,v be two convergent power series. The function*

$$y(x)=u(x)\log x+v(x)$$

is a solution of equation (1.4) iff the two functions $u(x)$ and $v(x)$ satisfy the following equations:

$$\begin{aligned}&u'(x)=\alpha u(qx)-u(x),\quad u(0)=1,\\&v'(x)=\alpha v(qx)-v(x)+\alpha\ln q\cdot u(qx)-\tfrac{u(x)}{x}+\tfrac{1}{x}.\end{aligned} \tag{2.1}$$

Hence, $u(x)=F_0(\alpha;\,q,x)$.

Proof By substituting $y(x) = u(x) \cdot \log x + v(x)$ into (1.4), we obtain that

$$\begin{aligned} &\log x[u'(x) - \alpha u(qx) + u(x)] = 0, \\ &v'(x) = \alpha v(qx) - v(x) + \alpha \ln q \cdot u(qx) - \tfrac{u(x)}{x} + \tfrac{1}{x}, \end{aligned} \tag{2.2}$$

The first equation of (2.2) is equivalent to equation (1.5), hence $u(x) = u_0 F_0(\alpha; q, x)$, where $u_0 = u(0)$. Furthermore, the second equation of (2.2) admits a power series solution v only when $u(0) = 1$. So, we find the system given in (2.1). □

Lemma 2.2. *Let $y(x)$ be an analytic solution of equation (1.5) on an interval $(0, a)$ $(a > 0)$. Then, $y(x) = 0$ for all $x \in (0, a)$ iff $\lim\limits_{x\to 0} y(x) = 0$.*

Proof Set $z(x) = y(x)e^x$ for $x \in (0, a)$. Then

$$z'(x) = (\alpha y(qx) - y(x))e^x + y(x)e^x = \alpha e^x y(qx) = \alpha e^{(1-q)x} z(qx).$$

If $0 < x_0 < x < x_1 < a$, then, one has

$$z(x) = z(x_0) + \alpha \int_{x_0}^{x} e^{(1-q)t} z(qt) \mathrm{d}t.$$

Hence, by taking $x_0 \to 0$, we obtain

$$z(x) = \alpha \int_{0}^{x} e^{(1-q)t} z(qt) \mathrm{d}t,$$

which implies that, for all $x \in (0, x_1]$,

$$|z(x)| \le |\alpha| x_1 e^{(1-q)x_1} \max_{0<t\le qx_1} |z(t)|.$$

Consequently, it follows that

$$|y(x)| \le |\alpha| x_1 e^{-x} e^{(1-q)x_1} \max_{0<t\le qx_1} |e^t y(t)| \le C x_1 \max_{0<t\le qx_1} |y(t)|,$$

where $C = |\alpha| e^{x_1}$. Therefore,

$$\max_{x\in(0,x_1]} |y(x)| \le C x_1 \max_{0<t\le qx_1} |y(t)|;$$

and by iteration, we obtain

$$\max_{x\in(0,x_1]} |y(x)| \le C^n q^{n(n-1)/2} x_1^n \max_{0<t\le q^n x_1} |y(t)|.$$

Finally, taking $n \to +\infty$, one gets that $\max\limits_{0<x\le x_1} |y(x)| = 0$, which completes the proof. □

As usual, we denote by $H_n = \sum\limits_{k=1}^{n} \dfrac{1}{k}$, the n-th harmonic number.

Theorem 2.1. *Let $\alpha \notin q^{\mathbb{Z}_{\le 0}}$. For any constant c_0, we define*

$$F(\alpha; q, c_0, x) = (c_0 + \log x) F_0(\alpha; q, x) + F_1(\alpha; q, x), \tag{2.3}$$

where $F_0(\alpha; q, x)$ *is as in* (1.8) *and where*

$$F_1(\alpha; q, x) = \sum_{n\geq 1} \Big(H_n + \sum_{k=0}^{n-1} \frac{\alpha q^k \ln q}{1-\alpha q^k}\Big) \frac{(-1)^{n-1}(\alpha; q)_n}{n!} x^n. \tag{2.4}$$

Then the function $F(\alpha; q, c_0, x)$ *is a unique analytic solution on* $\widetilde{\mathbb{C}^*}$ *of equation* (1.4) *with the initial asymptotic condition* $y(x) = \log x + c_0 + o(1)$ *as* $x \to 0$ *along a certain direction.*

Proof Since $F_0(\alpha; q, x)$ is an analytic solution of equation (1.5), substituting $F(\alpha; q, c_0, x)$ into equation (1.4), we see that the first equation in (2.1) is satisfied. By taking $v(x) = F_1(\alpha; q, x)$, the second equation in (2.1) is satisfied.

Since the sequence $|1 + \alpha q^{k-1}(k \ln q - 1)]|$ is convergent as $k \to \infty$, then there exists a constant C_1 such that $|1 + \alpha q^{k-1}(k \ln q - 1)]| \leq C_1$, for all $k \geq 1$. Because $\alpha \notin q^{-\mathbb{N}}$, there exists a constant C_2 such that $|1 - \alpha q^{k-1}| \geq C_2$ for all $k \geq 1$. Therefore, for $n \geq 1$,

$$\Big|\sum_{k=1}^{n} \frac{(-1)^{n-1}(\alpha; q)_n [1 + \alpha q^{k-1}(k \ln q - 1)]}{n! k (1 - \alpha q^{k-1})}\Big| \leq \frac{C_1 |(\alpha; q)_n|}{C_2 (n-1)!}.$$

Hence, the radius of convergence of $F_1(\alpha; q, x)$ is ∞.

Furthermore, it is trivial that $F(\alpha; q, c_0, x) = \log x + c_0 + o(1)$ as $x \to 0$ along any direction. From the analysis above, the function $F(\alpha; q, c_0, x)$ is a solution of equation (1.4) as described in the theorem.

For uniqueness, we can easily see that using a suitable change of variable, the difference of two solutions with the given asymptotic initial condition satisfies the conditions of Lemma 2.2. □

Next, we look for a power series solution at ∞.

Theorem 2.2. *If* $\alpha \notin q^{\mathbb{Z}_{>0}}$, *then the following power series*

$$G(\alpha; q, x) = x^{-1} \sum_{n\geq 0} \frac{(-1)^{n+1} \alpha^{-(n+1)} q^{(n+1)(n+2)/2} n!}{(\alpha^{-1} q; q)_{n+1}} x^{-n} \tag{2.5}$$

represents the unique analytic solution of equation (1.4) *in* $\mathbb{C}^* \cup \{\infty\}$ *which vanishes at infinity.*

Proof We suppose $y = \sum_{n\geq 0} a_n x^{-n-1}$ and substitute it into equation (1.4), then

$$-\sum_{n\geq 0} (n+1) a_n x^{-n-2} = \sum_{n\geq 0} (\alpha q^{-n-1} - 1) a_n x^{-n-1} + x^{-1}.$$

Comparing the coefficients of x^{-1}, one obtains $a_0 = \frac{1}{1-\alpha q^{-1}}$ (for $\alpha \neq q$). By comparing the coefficients of $x^{-2}, x^{-3}, \cdots$, we have $(\alpha q^{-n-1} - 1) a_n = -n a_{n-1}$ for $n \geq 1$. Therefore,

$$a_n = \frac{n}{1 - \alpha q^{-n-1}} a_{n-1} = \cdots = \frac{(-1)^n \alpha^{-n} q^{n(n+3)/2} n!}{(\alpha^{-1} q^2; q)_n} a_0$$

for $\alpha \notin q^{\mathbb{Z}_{>0}}$. Thus, the expression for a_n is unique. Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,q^{n+2}}{|\alpha - q^n|} = 0\,,$$

the power series (in $\frac{1}{x}$) $G(\alpha; q, x)$ defined by (2.5) is the unique analytic solution of equation (1.4) in Riemann sphere $\mathbb{C}^* \cup \{\infty\}$, such that $\lim_{x\to\infty} G(\alpha; q, x) = 0$. □

Remark 2.2. When $\alpha \neq 1$, any analytic solution at infinity is null there. In this case, $G(\alpha; q, x)$ is the unique analytic solution at infinity.

This comes from the fact that $\lim_{x\to\infty}(y'(x) - \frac{1}{x}) = 0$, which implies that $(\alpha - 1)\lim_{x\to\infty} y(x) = 0$.

The connection formula between $F(\alpha; q, c_0, x)$ and $G(\alpha; q, x)$ will be given in Section 5. In formula (2.5), one can notice that, for any $\alpha \notin q^{\mathbb{Z}_{>0}}$,

$$\left|\frac{\alpha^{-(n+1)}}{(\alpha^{-1}q;\, q)_{n+1}}\right| = \frac{1}{|(\alpha - q)(\alpha - q^2)\cdots(\alpha - q^{n+1})|} \leq \frac{1}{\epsilon^{n+1}}, \tag{2.6}$$

where $\epsilon = \min_{k\in\mathbb{Z}_{>0}} |\alpha - q^k|$. Furthermore, we have the following Remark.

Remark 2.3. The function $F(\alpha; q, c_0, x)$ of Theorem 2.1 is analytic for $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$, while the function $G(\alpha; q, x)$ of Theorem 2.2 is analytic for $\alpha \notin q^{\mathbb{Z}_{>0}}$.

Indeed, by using the uniform convergence theorem, one can see that $F_0(\alpha; q, x)$ and $F_1(\alpha; q, x)$ are respectively analytic for $(\alpha, x) \in \mathbb{C}\times\mathbb{C}$ and for all $(\alpha, x) \in \mathbb{C}\times\mathbb{C}$ such that $\alpha q^k \neq 1$ for any $k \in \mathbb{Z}_{\geq 0}$. This implies the analyticity of $F(\alpha; q, c_0, x)$ for $(\alpha, x) \in \left(\mathbb{C} \setminus q^{\mathbb{Z}_{\leq 0}}\right) \times \mathbb{C}$. In the same way, we get the analyticity of $G(\alpha; q, x)$, by using (2.6).

3 Laplace integral solutions and the relationship between them

In this section, we will look for solutions of (1.4) in the form of Laplace integral. For the convergence of Laplace integral, we refer to [20, p. 216, Definition A.2.3], whose main content is: Let $f \in C^2([0, +\infty))$. Suppose there are $A, B > 0$ such that $|f(x)| \leq Ae^{Bx}$, then the function $\xi \mapsto \int_0^{+\infty} f(x)e^{-x\xi}\mathrm{d}x$ converges for $\mathrm{Re}(\xi) > B$. It is denoted Lf, Laplace transform of f, which holomorphic for $\mathrm{Re}(\xi) > B$. We first give an elementary lemma as follows.

Lemma 3.1. *Assume that $d \in \mathbb{R}$, $\epsilon > 0$, and h is an analytic function in the open sector $\{t \in \widetilde{\mathbb{C}}^* \,|\, |\arg(t) - d| < \epsilon\}$, verifying $h(t) = O(t)$ for $t \to 0$, and $h(t) = O(e^{\lambda|t|})$ for $t \to \infty$ $(\lambda \geq 0)$. An integral of the form*

$$y(x) = \int_0^{\infty e^{id}} h(t)e^{-tx}\frac{\mathrm{d}t}{t}$$

is a solution of equation (1.4) *in the domain*

$$S^d_{\epsilon,\lambda} = \bigcup_{\delta\in(d-\epsilon,d+\epsilon)} \{x \in \widetilde{\mathbb{C}}^* \big| \mathrm{Re}(xe^{i\delta}) > \lambda\}$$

if and only if the function $h(t)$ *satisfies*

$$\alpha h(t) + (qt-1)h(qt) = -qt. \tag{3.1}$$

Proof One has $\frac{1}{x} = \int_0^{\infty e^{id}} te^{-tx}\frac{\mathrm{d}t}{t}$ for $\mathrm{Re}(xe^{\mathrm{i}d}) > 0$, and $y(x)$ can be extended into an analytic function in $S^d_{\epsilon,\lambda}$ under the given conditions. Thus, we can obtain the result by substituting $y(x)$ in equation (1.4). □

3.1 Two types of Laplace integral solutions

Suppose $\phi(t) = \sum_{n\geq 1}\phi_n t^n$ and substisute it into equation (3.1). Comparing the coefficients of t^n, we obtain

$$\phi_1 = \frac{1}{1-\alpha q^{-1}}, \text{ and for } n \geq 2, \phi_n = \frac{1}{1-\alpha q^{-n}}\phi_{n-1}. \tag{3.2}$$

Next, we use it to get an integral solution of equation (1.4), as shown in the following lemma.

Lemma 3.2. *Let* $\alpha \notin q^{\mathbb{Z}_{>0}}$, $d \in [0, 2\pi]$, *and*

$$\phi(t) = \sum_{n\geq 0}\phi_{n+1}t^{n+1} = \sum_{n\geq 0}\frac{t^{n+1}}{(\alpha q^{-n-1};\, q)_{n+1}}. \tag{3.3}$$

Then the integral $L_1^{[d]}(\alpha; q, x) = \int_0^{\infty e^{id}} \phi(t)e^{-tx}\frac{\mathrm{d}t}{t}$ *is an analytic solution of equation* (1.4) *for* x *in the Riemann surface* $\widetilde{\mathbb{C}}^*$ *such that* $\arg(x) \in (-d - \frac{\pi}{2}, -d + \frac{\pi}{2})$.

Consequently, glueing all the functions $L_1^{[d]}(\alpha; q, x)$ $(d \in [0, 2\pi])$ *gives rise to an analytic solution of equation* (1.4) *in* $\mathbb{C}^* \cup \{\infty\}$, *which will be denoted by* $L_1(\alpha; q, x)$.

Proof First, from (3.2) and (1.6), we have $\phi_n = O(\alpha^{-n}q^{n(n+1)/2})$, we get that $\phi(t)$ is an entire function satisfying (3.1) and $\phi(0) = 0$. According to Lemma 3.1, it suffices to prove that $\phi(t)$ has at most exponential growth at ∞.

From [27, p. 59, Proposition 2.1], we obtain that

$$\left|\frac{\phi(t)}{t}\right| \leq Ce^{-\frac{(\log|t/\alpha\sqrt{q}|)^2}{2\ln q}}, \quad \text{as } |t| \to +\infty.$$

Therefore, as $t \to \infty$ in $\mathbb{C}$, $\phi(t) = O(e^{\lambda|t|})$ for any $\lambda > 0$. Hence, the integral $L_1^{[d]}(\alpha; q, x)$ is well-defined in any direction $d \in [0, 2\pi]$. This is to say, there is no singular direction. Thus, one obtains an analytic function on $\mathbb{C}^* \cup \{\infty\}$. □

Note that the proof above for Lemma 3.2 is based on a basic idea of the Borel-Laplace summation theory (see [20, p. 216, Proposition A.2.3]).

Theorem 3.1. *If $\alpha \notin q^{\mathbb{Z}_{>0}}$, then the relation*

$$G(\alpha; q, x) = L_1(\alpha; q, x) \tag{3.4}$$

holds for $x \in \mathbb{C}^* \cup \{\infty\}$.

Proof By the definition in (2.5), one can get that

$$G(\alpha; q, x) = \sum_{n\geq 0} \frac{n!}{(\alpha q^{-n-1}; q)_{n+1}} x^{-n-1}.$$

Given any direction d and x such that $\mathrm{Re}(xe^{id}) > 0$, by using dominated convergence theorem, we have

$$L_1^{[d]}(\alpha; q, x) = \int_0^{\infty e^{id}} e^{-tx}\phi(t)\frac{\mathrm{d}t}{t} = \sum_{n\geq 0} \frac{n!}{(\alpha q^{-n-1}; q)_{n+1}} x^{-n-1}.$$

Then $G(\alpha; q, x) = L_1^{[d]}(\alpha; q, x)$. By the analytic continuation process, relation (3.4) holds for all $x \in \mathbb{C}^* \cup \{\infty\}$. □

In the following, we will introduce another kind of integral solution.

Lemma 3.3. *Let $|\alpha| < 1$, $d \in (0, 2\pi)$, and*

$$\psi(t) = (\alpha; q)_\infty \sum_{n\geq 0} \frac{\alpha^n t}{(q; q)_n (q^n - t)}. \tag{3.5}$$

The integral

$$L_2^{[d]}(\alpha; q, x) = \int_0^{\infty e^{id}} \psi(t) e^{-tx} \frac{\mathrm{d}t}{t} \tag{3.6}$$

is an analytic solution of equation (1.4) for x in the Riemann surface $\widetilde{\mathbb{C}}^*$ such that $\arg(x) \in (-d - \frac{\pi}{2}, -d + \frac{\pi}{2})$.

Consequently, glueing all the functions $L_2^{[d]}(\alpha; q, x)$ $(d \in (0, 2\pi))$ gives rise to an analytic solution of equation (1.4) for $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$, which will be denoted by $L_2(\alpha; q, x)$.

Proof To see that $L_2^{[d]}(\alpha; q, x)$ is a solution of (1.4), we first prove that $\psi(t)$ is a solution of (3.1) (Lemma 3.1). Letting $g(t) = \psi(t)/t$, we only need to prove that $g(t)$ is a solution of

$$(1-t)g(t) - \frac{\alpha}{q} g(\frac{t}{q}) = 1. \tag{3.7}$$

It can be proved by direct computation with the help of the following identity:

$$(\alpha; q)_\infty \sum_{n\geq 0} \frac{\alpha^n}{(q; q)_n} = 1.$$

Then $\psi(t)$ is a solution of equation (3.1).

Next, we consider the growth of $\psi(t)$ at infinity to prove the convergence of the integral. Since

$$\begin{aligned} |\psi(t)| \leq & |(\alpha; q)_\infty| \sum_{n\geq 0} \frac{|\alpha|^n(|q^n| + |t - q^n|)}{(q; q)_n |q^n - t|} \\ = & |(\alpha; q)_\infty| \sum_{n\geq 0} \frac{|\alpha|^n q^n}{(q; q)_n |q^n - t|} + |(\alpha; q)_\infty| \sum_{n\geq 0} \frac{|\alpha|^n}{(q; q)_n}, \end{aligned} \tag{3.8}$$

the function $\psi(t)$ is bounded as $t \to \infty$ in any direction $d \in (0, 2\pi)$. The result follows by applying Lemma 3.1 ($\lambda = 0, d = \epsilon = \pi$). □

The following theorem describes the Stokes phenomenon of $L_2(\alpha; q, x)$.

Theorem 3.2. *Let* $|\alpha| < 1$ *and* $\arg(x) \in (-\frac{5\pi}{2}, -\frac{3\pi}{2})$, *the following formula holds:*

$$L_2(\alpha; q, xe^{2\pi i}) - L_2(\alpha; q, x) = 2\pi i(\alpha; q)_\infty \sum_{n\geq 0} \frac{\alpha^n e^{-q^n x}}{(q; q)_n}, \tag{3.9}$$

the right-hand side of (3.9) *is a solution of the homogenous equation* (1.5).

Proof Let L be any smooth and anti-clockwise curve whose interior contains the set $\{1, q, q^2 \cdots\}$. By the residue theorem,

$$\begin{aligned} L_2(\alpha; q, xe^{2\pi i}) - L_2(\alpha; q, x) &= \int_L (\alpha; q)_\infty \sum_{n\geq 0} \frac{\alpha^n}{(q; q)_n (t - q^n)} e^{-tx} \mathrm{d}t \\ &= 2\pi i(\alpha; q)_\infty \sum_{n\geq 0} \frac{\alpha^n e^{-q^n x}}{(q; q)_n}, \end{aligned}$$

which is (3.9). Since $|\alpha| < 1$, the series in (3.9) is convergent for $\mathrm{Re}(x) > 0$, which is satisfied when $\arg(x) \in (-\frac{5\pi}{2}, -\frac{3\pi}{2})$. By direct calculation, one can prove that the right-hand side of (3.9) is a solution of equation (1.5). □

3.2 Relationship between the two Laplace integrals

We have established that $L_1(\alpha; q, x)$ and $L_2(\alpha; q, x)$ are solutions of equation (1.4). Then $L_1(\alpha; q, x) - L_2(\alpha; q, x)$ is a solution of the homogeneous equation (1.5). We now study the link formula between $L_1(\alpha; q, x)$ and $L_2(\alpha; q, x)$, by giving an explicit expression of $L_1(\alpha; q, x) - L_2(\alpha; q, x)$. For doing this, we should study the relation between the functions ϕ and ψ.

Lemma 3.4. *Let $\psi(t)$ be as in Lemma 3.3. For $|t| > 1$, the following equation holds:*

$$\psi(t) = -\sum_{n\geq 0}(\alpha;\, q)_n \big(\frac{1}{t}\big)^n.$$

Proof By using $\frac{1}{1-\frac{q^n}{t}} = \sum_{m\geq 0}(\frac{q^n}{t})^m$ for every fixed n, equation (3.5) becomes

$$\psi(t) = -(\alpha;\, q)_\infty \sum_{n\geq 0, m\geq 0} \frac{(\alpha q^m)^n}{(q;\, q)_n}\Big(\frac{1}{t}\Big)^m. \tag{3.10}$$

For every fixed m, by applying equation (1.9), we obtain

$$(\alpha;\, q)_\infty \sum_{n\geq 0} \frac{(\alpha q^m)^n}{(q;\, q)_n} = \frac{(\alpha;\, q)_\infty}{(\alpha q^m;\, q)_\infty} = (\alpha;\, q)_m. \tag{3.11}$$

Putting (3.11) into (3.10) allows us to complete the proof. □

The functions ϕ and ψ are solutions of (3.1), and ϕ is an entire function while ψ has simple poles on the set $q^{\mathbb{N}}$. The homogeneous equation of (3.1) has special solutions, for example,

$$H(t) = \frac{\theta(-\frac{q}{\alpha}t)}{(\frac{q}{\alpha};\, q)_\infty(\frac{1}{t};\, q)_\infty}, \tag{3.12}$$

who has the same poles as ψ. We will obtain the relationship between $\phi(t), \psi(t)$, and $H(t)$, as shown in Lemma 3.5. From (1.6), we have

$$\phi(t) = \sum_{n\geq 0} \frac{t^{n+1}}{(\alpha q^{-n-1};\, q)_{n+1}} = \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(\frac{q}{\alpha};\, q)_{n+1}}\Big(-\frac{qt}{\alpha}\Big)^{n+1}. \tag{3.13}$$

According to the definition of basic hypergeometric series in [14, p. 4, formula (1.2.22)]:

$$\begin{aligned} &{}_r\phi_s(a_1, a_2, \cdots, a_r;\, b_1, b_2, \cdots, b_s;\, q, z) \\ &= \sum_{n\geq 0} \frac{(a_1;\, q)_n (a_2;\, q)_n \cdots (a_r;\, q)_n}{(q;\, q)_n (b_1;\, q)_n \cdots (b_s;\, q)_n}\Big[(-1)^n q^{\frac{n(n-1)}{2}}\Big]^{1+s-r} z^n, \end{aligned}$$

the right-hand side of the series in (3.13) can be viewed as a basic hypergeometic series, written as ${}_1\phi_1(q;\, \frac{q}{\alpha};\, q, \frac{qt}{\alpha}) - 1$. The connection formula for ${}_2\phi_1$ is proposed in [14, p. 117, formula (4.3.2)], and the following relationship can also be regarded as the connection formula for ${}_1\phi_1$.

Lemma 3.5. *Let $\phi(t)$ and $\psi(t)$ be as in (3.3) and (3.5). For $|t| > 1$, the following equation holds:*

$$\phi(t) = H(t) + \psi(t). \tag{3.14}$$

Proof Let $c=-\frac{q}{\alpha}$, equation (3.12) can be rewritten as $H(t)=\frac{\theta(ct)}{(-c;\,q)_\infty(\frac{1}{t};\,q)_\infty}$. From formula (1.9), we have, for $|t|>1$,

$$\frac{1}{(\frac{1}{t};\,q)_\infty}=\sum_{k\geq 0}\frac{1}{(q;\,q)_k}\big(\frac{1}{t}\big)^k.$$

Multiplying with the series $\theta(ct)=\sum_{l\in\mathbb{Z}}q^{\frac{l(l-1)}{2}}(ct)^l$, we obtain

$$H(t)=\frac{1}{(-c;\,q)_\infty}\sum_{k\geq 0}\frac{1}{(q;\,q)_k}\big(\frac{1}{t}\big)^k\sum_{l\in\mathbb{Z}}q^{\frac{l(l-1)}{2}}(ct)^l=H^+(t)+H^-(t),$$

where

$$H^+(t)=\sum_{k\geq 0}\sum_{l>k}\frac{c^lq^{\frac{l(l-1)}{2}}}{(-c;\,q)_\infty(q;\,q)_k}t^{l-k},\quad H^-(t)=\sum_{k\geq 0}\sum_{l\leq k}\frac{c^lq^{\frac{l(l-1)}{2}}}{(-c;\,q)_\infty(q;\,q)_k}\big(\frac{1}{t}\big)^{k-l}.$$

Letting $m=l-k$ and replacing l with $k+m$, it follows that

$$H^+(t)=\sum_{m>0,k\geq 0}\frac{(cq^m)^kq^{\frac{k(k-1)}{2}}q^{\frac{m(m-1)}{2}}}{(-c;\,q)_\infty(q;\,q)_k}(ct)^m.$$

By applying equation (1.10), for fixed m, we have $\sum_{k\geq 0}\frac{(cq^m)^kq^{\frac{k(k-1)}{2}}}{(q;\,q)_k}=(-cq^m;\,q)_\infty$.

Therefore,

$$H^+(t)=\sum_{m>0}\frac{(-cq^m;\,q)_\infty}{(-c;\,q)_\infty}q^{\frac{m(m-1)}{2}}(ct)^m=\sum_{m>0}\frac{q^{\frac{m(m-1)}{2}}}{(\frac{q}{\alpha};\,q)_m}(-\frac{q}{\alpha}t)^m=\phi(t).$$

Next, letting $n=k-l$ and replacing l with $k-n$, we have

$$H^-(t)=\sum_{n\geq 0,k\geq 0}\frac{c^{k-n}q^{\frac{(k-n)(k-n-1)}{2}}}{(-c;\,q)_\infty(q;\,q)_k}\big(\frac{1}{t}\big)^n.$$

Since $\frac{(k-n)(k-n-1)}{2}=\frac{k(k-1)}{2}+\frac{n(n+1)}{2}-kn$, by summing on k and using equation (1.10), we obtain

$$H^-(t)=\sum_{n\geq 0}c_n\big(\frac{1}{t}\big)^n,$$

where

$$c_n=\frac{(-\frac{c}{q^n};\,q)_\infty}{(-c;\,q)_\infty}c^{-n}q^{\frac{n(n+1)}{2}}=(-\frac{c}{q^n};\,q)_nc^{-n}q^{\frac{n(n+1)}{2}}.$$

By using formula (1.6), we have $c_n=(\alpha;\,q)_n$. From Lemma 3.4, we can obtain that $H^-(t)=-\psi(t)$, which completes the proof. □

Remark 3.1. We note that equation (3.14) is still valid for all $t\in\mathbb{C}^*\backslash q^{\mathbb{N}}$. This property will be used in the sequel.

We obtain the link formula between the two integral solutions as follows.

Theorem 3.3. *Let $|\alpha|<1$ and $\alpha\notin q^{\mathbb{Z}_{>0}}$. For any x verifying $\arg(x)\in\big(-\frac{5\pi}{2},\frac{\pi}{2}\big)$, there exists a direction $d\in(0,2\pi)$ such that the following relation holds:*

$$L_1(\alpha;\,q,x)=L_2(\alpha;\,q,x)+\int_0^{\infty e^{\mathrm{i}d}}e^{-tx}H(t)\frac{\mathrm{d}t}{t}.\tag{3.15}$$

Proof For any given such x, one can find a direction $d \in (0, 2\pi)$ such that $\arg(x) \in \left(-d - \frac{\pi}{2}, -d + \frac{\pi}{2}\right)$, the theorem follows by using Lemma 3.5, Remark 3.1 and integration. □

4 Expression of integral solutions in terms of series

This section focuses on exploring the relationship between series and integral solutions, which is structured into the following aspects. By applying perturbations of the equation, we derive an alternative expression for $L_2(\alpha; q, x)$, which is represented in terms of series expansions at zero. Finally, we establish the link formula between $F(\alpha; q, c_0, x)$ and $L_2(\alpha; q, x)$ by using Theorem 3.1.

4.1 Perturbations of the parameter α

The idea of perturbations of the equation gives another representation of $L_2(\alpha; q, x)$. In the following, we consider α as a parameter close to zero.

If $\alpha = 0$, then equation (1.4) becomes

$$y'(x) = -y(x) + \frac{1}{x}, \tag{4.1}$$

which has the divergent series solution $\sum_{n\geq 0} n!(\frac{1}{x})^{n+1}$. From the Definition 1.3.1.2 in [20]: We place ourselves in the plane of $\tau = \frac{1}{x}$. $\sum \frac{a_n}{\tau^n}$ is Borel-summable in $U = \{\mathrm{Re}(\tau) > 0\}$, of sum f, if:

$$\exists C > 0,\ \forall n \in \mathbb{N},\ \forall \tau \in U, \quad \left|f(\tau) - \sum_{k<n} \frac{a_k}{\tau^k}\right| \leq \frac{C^n \cdot n!}{|\tau|^n}.$$

We obtain that, given any $d \in (0, 2\pi)$, the corresponding Borel-sum in the direction d is

$$E^{[d]}(x) = \int_0^{\infty e^{\mathrm{i}d}} e^{-xt} \frac{\mathrm{d}t}{1-t},$$

for $\mathrm{Re}(xe^{\mathrm{i}d}) > 0$, *i.e.*, $\arg(x) \in (-d - \frac{\pi}{2}, -d + \frac{\pi}{2})$. By the analytic continuation process, it yields a function $E(x)$, which is a solution of (4.1) for $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$ on $\widetilde{\mathbb{C}}^*$, for more details, see Ramis [21, p. 183, Definition 3.1] and Malgrange [20, p. 217, Proposition A.2.4].

Since $L_2(\alpha; q, x)$ is a solution of (1.4) with $\alpha \neq 0$ and $E(x)$ is a solution of (1.4) with $\alpha = 0$, equation (1.4) can be seen as the perturbation of equation (4.1). We expect to expand $L_2(\alpha; q, x)$ as a perturbation series related to $E(x)$.

Proposition 4.1. *Let $\alpha \neq 0$ and $|\alpha| < 1$. For $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$ on $\widetilde{\mathbb{C}}^*$, the following relation holds*

$$L_2(\alpha; q, x) = (\alpha; q)_\infty \sum_{n\geq 0} \frac{\alpha^n}{(q; q)_n} E(q^n x). \tag{4.2}$$

Consequently, $L_2(\alpha; q, x) \to E(x)$ as $\alpha \to 0$.

Proof For any given x with $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$, one can find a direction $d \in (0, 2\pi)$ such that $\arg(x) \in \left(-d - \frac{\pi}{2}, -d + \frac{\pi}{2}\right)$. By the definition of $L_2(\alpha; q, x)$, we have

$$L_2^{[d]}(\alpha; q, x) = (\alpha; q)_\infty \int_0^{\infty e^{id}} \sum_{n\geq 0} \frac{\alpha^n\, t}{(q; q)_n (q^n - t)} e^{-tx} \frac{\mathrm{d}t}{t},$$

where the series under the integral verifies (3.8). Since

$$\int_0^{\infty e^{id}} \frac{t}{q^n - t} e^{-tx} \frac{\mathrm{d}t}{t} = E^{[d]}(q^n x),$$

we can apply the dominated convergence theorem to obtain (4.2). □

Let $\alpha \neq 0$ and close to 0. Note that a function in the form $y(x) = \sum_{n\geq 0} y_n(x)\alpha^n$ is a solution of equation (1.4) iff the functions $y_n(x)$ satisfy the following system of differential equations:

$$\begin{aligned} &y_0'(x) + y_0(x) = \tfrac{1}{x}, \\ &y_{n+1}'(x) + y_{n+1}(x) = y_n(qx), \quad n \geq 0. \end{aligned} \tag{4.3}$$

By using equations (1.10) and (4.2), we obtain the following corollary, which gives a particular solution to the above system.

Corollary 4.1. *For $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$, a particular solution of system (4.3) is given by the following*

$$y_n(x) = \sum_{k=0}^{n} \frac{(-1)^k q^{k(k-1)/2}}{(q; q)_k (q; q)_{n-k}} E(q^{n-k} x), \text{ for } n \geq 0.$$

4.2 Relationship between $L_2(\alpha; q, x)$ and $F(\alpha; q, c_0, x)$

To derive the link formula between $L_2(\alpha; q, x)$ and $F(\alpha; q, c_0, x)$, we will express $L_2(\alpha; q, x)$ in terms of power series at 0. This is done by establishing the relationship between $E(x)$ and the series solution at 0 of equation (4.1), because formula (4.2) indicates that $L_2(\alpha; q, x)$ can be expressed in terms of $E(x)$, and $E(x)$ is a solution of (4.1).

Lemma 4.1. *Let* $\gamma = \Gamma'(1)$ *be Euler's constant ([28, p. 185, formula (10.8.1)]) and*

$$w_\gamma(x) = \gamma e^{-x} + e^{-x}\log(e^{i\pi}x) + \sum_{n\geq 1}\sum_{k=0}^{n-1}\frac{(-1)^k}{(n-k)(n-k)!\,k!}x^n. \tag{4.4}$$

Then, for $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$, *we have* $E(x) = w_\gamma(x)$.

Proof Choose $d = \pi$ for $E^{[d]}(x)$. Given x such that $\arg(x) \in (-\frac{3\pi}{2}, -\frac{\pi}{2})$, we have $\mathrm{Re}(x) < 0$, and

$$E^{[\pi]}(x) = \int_0^{\infty e^{i\pi}} \frac{e^{-tx}}{1-t}\mathrm{d}t = -\int_0^{+\infty}\frac{e^{tx}}{1+t}\mathrm{d}t.$$

From [28, p. 276, formula (14.1.10)], we have the exponential integral

$$E_1(z) := \int_1^\infty \frac{e^{-zt}}{t}\mathrm{d}t = -\gamma - \mathrm{Ln}(z) - \sum_{k\geq 1}\frac{(-1)^k z^k}{k!k},$$

where $\mathrm{Re}(z) > 0$. If $z = e^{i\pi}x$, then we have

$$E^{[\pi]}(x) = -e^{-x}E_1(e^{i\pi}x) = \gamma e^{-x} + e^{-x}\log(e^{i\pi}x) + e^{-x}\sum_{k\geq 1}\frac{x^k}{k!k},$$

One completes the proof by direct computation and analytic continuation. □

Remark 4.1. The series in (4.4) has another expression as follows

$$w_\gamma(x) = \gamma e^{-x} + e^{-x}\log(e^{i\pi}x) + \sum_{n\geq 1}\frac{(-1)^{n-1}H_n}{n!}x^n. \tag{4.5}$$

By utilizing formula (4.2) and Lemma 4.1, we have the following corollary.

Corollary 4.2. *Let* $|\alpha| < 1$ *and* $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$. *The function* $L_2(\alpha; q, x)$ *can be expressed in terms of* $w_\gamma(x)$ *as follows:*

$$L_2(\alpha; q, x) = (\alpha; q)_\infty \sum_{m\geq 0}\frac{\alpha^m}{(q; q)_m}w_\gamma(q^m x). \tag{4.6}$$

The following lemma presents a relation that will be used to study the relationship between $L_2(\alpha; q, x)$ and solutions at 0 of equation (1.4).

Lemma 4.2. *For any* $|x| < 1$, *the following equation holds*

$$\sum_{m\geq 0}\frac{mx^m}{(q; q)_m} = \frac{1}{(x; q)_\infty}\sum_{k\geq 0}\frac{xq^k}{1-xq^k}. \tag{4.7}$$

Proof For any $|x| < 1$, by using (1.9), we have

$$\Big[\frac{x}{(x;\,q)_\infty}\Big]' = \Big[\sum_{m\geq 0}\frac{x^{m+1}}{(q;\,q)_m}\Big]' = \sum_{m\geq 0}\frac{(m+1)x^m}{(q;\,q)_m},$$

then

$$\begin{aligned}\sum_{m\geq 0}\frac{mx^m}{(q;\,q)_m} &= \Big[\frac{x}{(x;\,q)_\infty}\Big]' - \sum_{m\geq 0}\frac{x^m}{(q;\,q)_m}\\ &= \frac{(x;\,q)_\infty - x[(x;\,q)_\infty]'}{(x;\,q)_\infty^2} - \sum_{m\geq 0}\frac{x^m}{(q;\,q)_m} = \frac{-x[(x;\,q)_\infty]'}{(x;\,q)_\infty^2}.\end{aligned}$$

Since $[(x;\,q)_\infty]' = (x;\,q)_\infty\sum_{k\geq 0}\frac{-q^k}{1-xq^k}$, equation (4.7) can be easily proved. □

We now examine the relationship between $L_2(\alpha;\,q,x)$ and $F(\alpha;\,q,c_0,x)$ where the function $F(\alpha;\,q,c_0,x)$ is defined in Theorem 2.1. For a special value of c_0, we have the following result.

Theorem 4.1. *Let $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$ and $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$. The following relation holds:*

$$L_2(\alpha;\,q,x) = F(\alpha;\,q,\gamma_0,x), \tag{4.8}$$

where

$$\gamma_0 = \sum_{k\geq 0}\frac{\alpha q^k\ln q}{1-\alpha q^k} + \gamma + i\pi. \tag{4.9}$$

Proof According to Remark 2.3, the function $F(\alpha;\,q,\gamma_0,x)$ is analytic for $\alpha \notin q^{\mathbb{Z}_{\leq 0}}$. Thus, we need only to establish the validity of (4.8) for $|\alpha| < 1$.

First, we have (see [19, p. 7, Proposition 2.1])

$$F_0(\alpha;\,q,x) = (\alpha;\,q)_\infty\sum_{m\geq 0}\frac{\alpha^m e^{-q^m x}}{(q;\,q)_m} = \sum_{m\geq 0}\frac{(-1)^m(\alpha;\,q)_m}{m!}x^m.$$

Then, from (4.5) and (4.6), we obtain

$$L_2(\alpha;\,q,x) = (\log(e^{i\pi}x) + \gamma)F_0(\alpha;\,q,x) + A_1 + A_2,$$

where

$$A_1 = (\alpha;\,q)_\infty\sum_{m\geq 0}\frac{\alpha^m}{(q;\,q)_m}(\log(q^m))e^{-q^m x}$$

and

$$A_2 = (\alpha;\,q)_\infty\sum_{m\geq 0}\frac{\alpha^m}{(q;\,q)_m}\sum_{n\geq 1}\sum_{k=1}^{n}\frac{(-1)^{n-1}}{n!k}q^{mn}x^n.$$

By using the exponential series and inversion of the summation order of convergent power series, we get

$$\begin{aligned}A_1 &= (\alpha;\,q)_\infty(\ln q)\sum_{m\geq 0}\frac{m\alpha^m}{(q;\,q)_m}\sum_{k\geq 0}\frac{(-q^m x)^k}{k!}\\ &= (\alpha;\,q)_\infty(\ln q)\sum_{k\geq 0}\frac{(-x)^k}{k!}\sum_{m\geq 0}\frac{m(\alpha q^k)^m}{(q;\,q)_m}.\end{aligned}$$

By applying (3.11) and Lemma 4.2, we further have

$$\begin{aligned} A_1 &= (\alpha;q)_\infty(\ln q)\sum_{k\geq 0}\frac{(-x)^k}{k!(\alpha q^k;q)_\infty}\sum_{m\geq 0}\frac{\alpha q^{k+m}}{1-\alpha q^{k+m}} \\ &= \sum_{n\geq 0}\sum_{k\geq n}\frac{\alpha q^k\ln q}{1-\alpha q^k}\frac{(-1)^n(\alpha;q)_n}{n!}x^n. \end{aligned}$$

Together with (1.9), we get

$$A_2 = (\alpha;q)_\infty\sum_{n\geq 1}\sum_{k=1}^{n}\Big(\sum_{m\geq 0}\frac{\alpha^m q^{mn}}{(q;q)_m}\Big)\frac{(-1)^{n-1}}{n!}\frac{x^n}{k} = \sum_{n\geq 1}\frac{(-1)^{n-1}H_n}{n!}(\alpha;q)_n x^n.$$

Hence,

$$A_1 + A_2 = F_1(\alpha;q,x) + \sum_{k\geq 0}\frac{\alpha q^k\ln q}{1-\alpha q^k}F_0(\alpha;q,x)$$

Thus, the proof is complete. □

5 Connection formula and asymptotic behaviors

We recall that $F(\alpha;q,c_0,x)$ and $G(\alpha;q,x)$ are as in (2.3) and (2.5). In this section, we first introduce some lemmas about q-periodic functions. Then, we present the connection formula between $F(\alpha;q,c_0,x)$ and $G(\alpha;q,x)$ for a special case $c_0=\gamma_0$. The asymptotic behaviors of $F(\alpha;q,\gamma_0,x)$ are obtained using the connection formula. Finally, we draw conclusions about the connection formula and the asymptotic behaviors at ∞ of solutions around zero for the general case.

5.1 Two families of q-periodic functions

Let μ be a fixed complex number such that $\alpha = q^\mu$ and $-\frac{\pi}{|\ln q|} < \mathrm{Im}(\mu) \leq \frac{\pi}{|\ln q|}$. There are infinity numbers of $\mu_l = \mu + i\kappa l$ ($l\in\mathbb{Z}$ and $\kappa = -\frac{2\pi}{\ln q}$), such that $q^{\mu_l}=\alpha$. All values of μ_l form a set, which we call $\Lambda_\alpha = \{\mu_l\in\mathbb{C} : q^{\mu_l}=\alpha\}$.

Lemma 5.1. *Let $\alpha\notin q^{\mathbb{Z}}$. For $\arg(x)\in(-\frac{5\pi}{2},\frac{\pi}{2})$, the function*

$$g_n(\alpha;q,x) = \sum_{\mu_l\in\Lambda_\alpha}\frac{\Gamma(n+\mu_l)x^{-\mu_l}}{1-e^{2\pi i\mu_l}},\ n\geq 0 \tag{5.1}$$

is an analytic solution of the equation $y(x)=\alpha y(qx)$.

Proof For $\mu_l\in\Lambda_\alpha$, we have

$$|x^{-\mu_l}| = |e^{-(\mathrm{Re}(\mu_l)+i\mathrm{Im}(\mu_l))(\log|x|+i\arg(x))}| = e^{\mathrm{Im}(\mu_l)\arg(x)}|x|^{-\mathrm{Re}(\mu)}. \tag{5.2}$$

Therefore,

$$|1-e^{2\pi i\mu_l}| \geq |1-|e^{2\pi i\mu_l}|| = |1-e^{-2\pi\mathrm{Im}\mu}e^{-2\pi\kappa l}|. \tag{5.3}$$

Since $\alpha \notin q^{\mathbb{Z}}$, we have $\mu_l \notin \mathbb{Z} \oplus i\kappa\mathbb{Z}$. By using Stirling's formula for the Gamma function (see [15, p. 21, Corollary 1.4.4]), it yields that

$$\Gamma(n+\mu_l) = \sqrt{2\pi}|\mathrm{Im}(\mu_l)|^{n+\mathrm{Re}(\mu_l)-\frac{1}{2}} e^{-\frac{\pi|\mathrm{Im}(\mu_l)|}{2}} \Big(1 + O\big(\frac{1}{|\mathrm{Im}(\mu_l)|}\big)\Big) \tag{5.4}$$

as $|\mathrm{Im}(\mu_l)| \to +\infty$.

(i) If $\mathrm{Im}(\mu_l) > 0$ $(l > 0)$, then for $\mathrm{Im}(\mu_l) = \mathrm{Im}(\mu) + \kappa l$, we have

$$\Gamma(n+\mu_l) = O\big(l^{n+\mathrm{Re}(\mu)-\frac{1}{2}} e^{-\frac{\pi\kappa l}{2}}\big), \quad \text{as } l \to +\infty.$$

Together with equations (5.2) and (5.3), we obtain that

$$\begin{aligned} &\Big|\sum_{\mathrm{Im}(\mu_l)>0} \frac{\Gamma(n+\mu_l)x^{-\mu_l}}{1-e^{2\pi i\mu_l}}\Big| \\ &\le Ce^{\mathrm{Im}(\mu)\arg(x)} \sum_{l\in\mathbb{Z}_{>0}} \frac{l^{n+\mathrm{Re}(\mu)-\frac{1}{2}} e^{\kappa l(\arg(x)-\frac{\pi}{2})}}{1-e^{-2\pi\mathrm{Im}(\mu)-2\pi\kappa l}} |x|^{-\mathrm{Re}(\mu)}. \end{aligned} \tag{5.5}$$

The series of the right-hand side of (5.5) is convergent if the ratio of two consecutive terms tends to a limit smaller than 1, that is to say:

$$\lim_{l\to+\infty} \Big|\big(1+\frac{1}{l}\big)^{n+\mathrm{Re}(\mu)-\frac{1}{2}} \frac{(1-e^{-2\pi\mathrm{Im}\mu-2\pi\kappa l})e^{\kappa(\arg(x)-\frac{\pi}{2})}}{1-e^{-2\pi\mathrm{Im}\mu-2\pi\kappa(l+1)}}\Big| = e^{\kappa(\arg(x)-\frac{\pi}{2})} < 1,$$

i.e. $\arg(x) < \frac{\pi}{2}$.

(ii) If $\mathrm{Im}(\mu_l) < 0$ $(l < 0)$, then equation (5.4) becomes

$$\Gamma(n+\mu_l) = O\big((-\mathrm{Im}(\mu_l))^{n+\mathrm{Re}(\mu)-\frac{1}{2}} e^{\frac{\pi\mathrm{Im}(\mu_l)}{2}}\big), \quad \text{as } \mathrm{Im}(\mu_l) \to -\infty.$$

By taking $m = -l$, we have

$$\Gamma(n-\mu_m) = O\big((-m)^{n+\mathrm{Re}(\mu)-\frac{1}{2}} e^{-\frac{\pi\kappa m}{2}}\big), \quad \text{as } m \to +\infty.$$

Equation (5.2) becomes $\big|x^{-\mu_l}\big| = e^{(\mathrm{Im}(\mu)-\kappa m)\arg(x)}|x|^{-\mathrm{Re}(\mu)}$. Therefore, we have

$$\begin{aligned} &\Big|\sum_{\mathrm{Im}(\mu_l)<0} \frac{\Gamma(n+\mu_l)x^{-\mu_l}}{1-e^{2\pi i\mu_l}}\Big| \\ &\le Ce^{\mathrm{Im}(\mu)\arg(x)} \sum_{m\in\mathbb{Z}_{>0}} \frac{m^{n+\mathrm{Re}(\mu)-\frac{1}{2}} e^{-\kappa m(\arg(x)+\frac{\pi}{2})}}{e^{-2\pi\mathrm{Im}(\mu)+2\pi\kappa m}-1} |x|^{-\mathrm{Re}(\mu)}. \end{aligned} \tag{5.6}$$

The series of the right-hand side of (5.6) is convergent if

$$\begin{aligned} &\lim_{m\to+\infty} \Big|\big(1+\frac{1}{m}\big)^{n+\mathrm{Re}(\mu)-\frac{1}{2}} \frac{(1-e^{-2\pi\mathrm{Im}\mu+2\pi\kappa m})e^{-\kappa(\arg(x)+\frac{\pi}{2})}}{1-e^{-2\pi\mathrm{Im}\mu+2\pi\kappa(m+1)}}\Big| \\ &\quad = e^{-\kappa(\arg(x)+\frac{5\pi}{2})} < 1, \end{aligned}$$

i.e. $\arg(x) > -\frac{5\pi}{2}$.

To sum up, the Laurent series in (5.1) is convergent if the series of the right-hand side of (5.5) and (5.6) are convergent. Therefore, the function $g_n(\alpha; q, x)$ is well-defined and analytic for $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$.

Finally, it is obvious that $g_n(\alpha; q, x)$ satisfies $\alpha y(qx) = y(x)$ by direct computation. □

Remark 5.1. For $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, let

$$h_n(\alpha; q, x) = g_n(\alpha; q, x) - g_n(\alpha; q, xe^{-2\pi i}).$$

Then, we have $h_n(\alpha; q, x) = \sum_{\mu_l \in \Lambda_\alpha} \Gamma(n + \mu_l) x^{-\mu_l}$.

Lemma 5.2. *For any fixed n, the functions $\hat{g}_n(\alpha; q, x) = x^\mu g_n(\alpha; q, x)$ and $\hat{h}_n(\alpha; q, x) = x^\mu h_n(\alpha; q, x)$ are q-periodic functions and bounded as $x \to \infty$ in any direction $d \in (-\frac{5\pi}{2}, \frac{\pi}{2})$ and $d \in (-\frac{\pi}{2}, \frac{\pi}{2})$ respectively.*

Proof Both $g_n(\alpha; q, x)$ and $h_n(\alpha; q, x)$ are solutions of equation $y(x) = \alpha y(qx)$, it is easy to verify that $\hat{g}_n$ and $\hat{h}_n$ are q-periodic functions, *i.e.*, satisfying $\hat{y}(qx) = \hat{y}(x)$. We only need to prove that $\hat{g}_n$ satisfies $|\hat{g}_n(\alpha; q, x)| \le C$ as $x \to \infty$ in any direction $\arg(x) \in (-\frac{5\pi}{2}, \frac{\pi}{2})$, for $\hat{h}_n$, we change the direction to $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

For any $d \in (-\frac{5\pi}{2}, \frac{\pi}{2})$ and $x \in [\frac{1}{q}e^{id}, \frac{1}{q^2}e^{id}]$, $\hat{y}(x) = \hat{y}(qx)$, where $qx \in [e^{id}, \frac{1}{q}e^{id}]$. Therefore, using the continuity and taking $C = \max_{x \in [e^{id}, \frac{1}{q}e^{id}]} |\hat{y}(x)|$, we have $|\hat{y}(x)| \le C$ for all $x = te^{id}$ with $t \ge 1$. □

5.2 The connection formula for the critical value $c_0 = \gamma_0$

In this section, we will present the connection formula between $F(\alpha; q, \gamma_0, x)$ and $G(\alpha; q, x)$. We first introduce a lemma that will be used later.

Define $e(q; x) = e^{-\frac{\log^2 \frac{x}{\sqrt{q}}}{2 \ln q}}$. One can get that both $e(q; x)$ and $\theta(q; x)$ satisfy equation $xy(qx) = y(x)$. Let $q^* = e^{-2\pi\kappa}$ and $x^* = x^{-i\kappa}$. From [15, p. 498, (10.4.2)] and [19, p. 12, (4.2)], we have

$$\theta(q; e^{i\pi}x) = \sqrt{\kappa}\, e(q; e^{i\pi}x)\theta(q^*; e^{i\pi}x^*), \tag{5.7}$$

and from [19, p. 12, Lemma 4.1], we have

$$\frac{\theta(q; -q^\mu x)}{\theta(q; -x)} = q^{-\mu(\mu-1)/2}(e^{i\pi})^{-\mu}\frac{\theta(q^*; -e^{2\pi i\mu}x^*)}{\theta(q^*; -x^*)}. \tag{5.8}$$

Lemma 5.3. *Let $\mu \in \mathbb{C} \setminus \mathbb{Z}$. The following relation holds:*

$$\frac{\theta(q; -\frac{q^\mu}{t})}{(\frac{1}{t}; q)_\infty} = \frac{\kappa(q^\mu, q^{1-\mu}; q)_\infty}{i(q; q)_\infty} \sum_{n \ge 0} \sum_{l \in \mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2} t^{n+\mu+i\kappa l}}{(q; q)_n (1 - e^{2\pi i(\mu + i\kappa l)})} \tag{5.9}$$

for $\arg(t) \in (0, 2\pi)$.

Proof Let $x = \frac{1}{t}$. Since $\arg(t) \in (0, 2\pi)$, we have $\arg(x) \in (-2\pi, 0)$. Then

$$|x^*| = e^{\kappa \arg(x)} \in (e^{-2\kappa\pi}, 1) = (q^*, 1).$$

From [15, p. 502, (10.5.3)]: for $|q|<1$ and $|ba^{-1}|<1$,

$$\sum_{l\in\mathbb{Z}}\frac{(a;\,q)_l}{(b;\,q)_l}x^l=\frac{(ax;\,q)_\infty(q/ax;\,q)_\infty(q;\,q)_\infty(b/a;\,q)_\infty}{(x;\,q)_\infty(b/ax;\,q)_\infty(b;\,q)_\infty(q/a;\,q)_\infty},$$

letting $q=q^*$, $x=x^*$, $a=e^{2\pi i\mu}$ and $b=q^*e^{2\pi i\mu}$ yields that

$$\frac{\theta(q^*;\,-e^{2\pi i\mu}x^*)}{\theta(q^*;\,-x^*)}=\frac{\theta(q^*;\,-e^{2\pi i\mu})}{(q^*,q^*)_\infty^3}\sum_{l\in\mathbb{Z}}\frac{x^{*l}}{1-e^{2\pi i\mu}q^{*l}}. \tag{5.10}$$

From (5.8) and (5.10), we have

$$\frac{\theta(q;\,-q^\mu x)}{\theta(q;\,-x)}=q^{-\mu(\mu-1)/2}(e^{i\pi})^{-\mu}\frac{\theta(q^*;\,-e^{2\pi i\mu})}{(q^*,q^*)_\infty^3}\sum_{l\in\mathbb{Z}}\frac{x^{*l}}{1-e^{2\pi i\mu}q^{*l}}.$$

By using (5.7) to $\theta(q^*;\,-e^{2\pi i\mu})$ and $(q^*;\,q^*)_\infty=\frac{q^{1/24}}{\sqrt{\kappa}}e^{\kappa\pi/12}(q;\,q)_\infty$, it yields that

$$\begin{aligned}\frac{\theta(q;\,-q^\mu x)}{(x;\,q)_\infty}=&\frac{\kappa(q^\mu,q^{1-\mu};\,q)_\infty}{i(q;\,q)_\infty}(\frac{q}{x};\,q)_\infty\sum_{l\in\mathbb{Z}}\frac{x^{-(\mu+i\kappa l)}}{1-e^{2\pi i(\mu+i\kappa l)}}\\=&\frac{\kappa(q^\mu,q^{1-\mu};\,q)_\infty}{i(q;\,q)_\infty}\sum_{n\geq0}\frac{(-1)^nq^{n(n+1)/2}x^{-n}}{(q;\,q)_n}\sum_{l\in\mathbb{Z}}\frac{x^{-(\mu+i\kappa l)}}{1-e^{2\pi i(\mu+i\kappa l)}},\end{aligned}$$

the series in the right-hand side is normally convergent on any compact of $\{x|\,\arg(x)\in(-2\pi,0)\}$ ($|x^*|<1$). The proof is completed by replacing x with $\frac{1}{t}$. □

Theorem 5.1. *Let $\alpha\notin q^{\mathbb{Z}}$ and γ_0 be as in (4.9). The following relation holds for any $x\in\widetilde{\mathbb{C}^*}$ with $\arg(x)\in(-\frac{5\pi}{2},\frac{\pi}{2})$:*

$$F(\alpha;\,q,\gamma_0,x)=G(\alpha;\,q,x)+\frac{i\kappa(\alpha;\,q)_\infty}{(q;\,q)_\infty}\sum_{n\geq0}\frac{(-1)^nq^{n(n+1)/2}}{(q;\,q)_n}g_n(\alpha;\,q,x)x^{-n}.$$

Proof From equations (3.4), (3.15) and (4.8), letting $d=\pi$, we have, for $\arg(x)\in(-\frac{3\pi}{2},\frac{\pi}{2})$,

$$F(\alpha;\,q,\gamma_0,x)=G(\alpha;\,q,x)-\int_0^{\infty e^{i\pi}}\frac{e^{-tx}\,\theta\left(-\frac{\alpha}{t}\right)}{\left(\frac{q}{\alpha};\,q\right)_\infty\left(\frac{1}{t};\,q\right)_\infty}\frac{\mathrm{d}t}{t}.$$

Assuming that $|\alpha|<1$, then we have $\mathrm{Re}(\mu)>0$. From $\alpha\notin q^{\mathbb{Z}}$, we have $\mu+i\kappa l\notin\mathbb{Z}$. By using equation (5.9), Lebesgue's dominated convergence Theorem and

$$\int_0^\infty e^{-t}t^{n+\mu+i\kappa l}\frac{\mathrm{d}t}{t}=\Gamma(n+\mu+i\kappa l),$$

we obtain the equation give in the theorem, where $g_n(\alpha;\,q,x)$ is shown in Lemma 5.1. By applying Remark 2.3 and analytic continuation, the equation in the theorem holds for any value of α and $\arg(x)\in(-\frac{5\pi}{2},\frac{\pi}{2})$. □

Then, we obtain the asymptotic form of solution at zero for the critical value $c_0=\gamma_0$.

Theorem 5.2. *Let $\alpha \notin q^{\mathbb{Z}}$ and γ_0 be as in (4.9). Then the following relation holds:*

(i) If $|\alpha| < q$, then

$$F(\alpha; q, \gamma_0, x) = \frac{1}{1-\alpha q^{-1}} x^{-1} + o(x^{-1})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) If $|\alpha| > q$, then

$$F(\alpha; q, \gamma_0, x) = \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} \hat{g}_0(\alpha; q, x) x^{-\mu} + o(x^{-\mu})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where $\hat{g}_0(\alpha; q, x)$ is a bounded q-periodic function shown in Lemma 5.2 with $n = 0$.

(iii) If $|\alpha| = q$, then

$$F(\alpha; q, \gamma_0, x) = \frac{1}{1-\alpha q^{-1}} x^{-1} + \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} \hat{g}_0(\alpha; q, x) x^{-\mu} + o(x^{-1})$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof In order to study the asymptotic behavior of $F(\alpha; q, c_0, x)$, we distinguish two cases:

(i) If $|\alpha| < q$, then $-\mathrm{Re}(\mu) = -\ln|\alpha|/\ln q < -1$. From Theorem 5.1, we have

$$\begin{aligned} F(\alpha; q, \gamma_0, x) &= x^{-1}\big[\tfrac{1}{1-\alpha q^{-1}} + \sum_{n\geq 1} \frac{\alpha^{-(n+1)} q^{(n+1)(n+2)/2} n!}{(\alpha^{-1} q; q)_{n+1}} x^{-n} \\ &\quad + O(x^{-\mathrm{Re}(\mu)+1})\big] \\ &= \tfrac{1}{1-\alpha q^{-1}} x^{-1}\big[1 + O(x^{-1}) + O(x^{-\mathrm{Re}(\mu)+1})\big] \end{aligned}$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) If $|\alpha| > q$, then $\mathrm{Re}(\mu) - 1 < 0$. By using Lemma 5.2 and Theorem 5.1, it follows that

$$F(\alpha; q, \gamma_0, x) = \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} g_0(\alpha; q, x) + x^{-\mu}\big[O(x^{-1}) + O(x^{\mathrm{Re}(\mu)-1})\big]$$

as $x \to \infty$ in the direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(iii) If $|\alpha| = q$, then we assume that $\mu = 1 + \frac{ia}{\ln q}$ ($a \notin 2\pi\mathbb{Z}$). From Theorem 5.1, we have

$$F(\alpha; q, \gamma_0, x) = \frac{1}{1-\alpha q^{-1}} x^{-1}\big[1 + O(x^{-1})\big] + \hat{g}_0(\alpha; q, x) x^{-\mu}\big[1 + O(x^{-1})\big],$$

the proof is thus completed. □

5.3 Concluding results for general c_0

From the above analysis, we draw conclusions about the connection formula and asymptotic behaviors in the general case.

Corollary 5.1. *Let $\alpha \notin q^{\mathbb{Z}}$. The following relation holds for any $c_0 \in \mathbb{C}^*$ and $x \in \widetilde{\mathbb{C}^*}$ with $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$:*

$$F(\alpha; q, c_0, x) = G(\alpha; q, x) + \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} \sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_n} \left[g_n(\alpha; q, x) + \frac{c_0 - \gamma_0}{2\pi i} h_n(\alpha; q, x) \right] x^{-n}.$$

Proof The proof is completed, by using Theorem 5.1,

$$F(\alpha; q, c_0, x) = F(\alpha; q, \gamma_0, x) + (c_0 - \gamma_0) F_0(\alpha; q, x),$$

and the connection formula

$$F_0(\alpha; q, x) = \frac{\kappa(\alpha; q)_\infty}{2\pi (q; q)_\infty} \sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)/2}}{(q; q)_n} h_n(\alpha; q, x) x^{-n},$$

for $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ in [19, p. 6, Theorem 1.2]. □

Therefore, we obtain the asymptotic form at ∞ of solutions around zero.

Theorem 5.3. *Let $\alpha \notin q^{\mathbb{Z}}$.*

(i) If $|\alpha| < q$, then

$$F(\alpha; q, c_0, x) = \frac{x^{-1}}{1 - \alpha q^{-1}} + o(x^{-1}),$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) If $|\alpha| > q$, then

$$F(\alpha; q, c_0, x) = \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} \left(\hat{g}_0(\alpha; q, x) + \frac{c_0 - \gamma_0}{2\pi i} \hat{h}_0(\alpha; q, x) \right) x^{-\mu} + o(x^{-\mu}),$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where the functions $\hat{g}_0(\alpha; q, x)$ and $\hat{h}_0(\alpha; q, x)$ are bounded q-period functions in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, as shown in Lemma 5.2 with $n = 0$.

(iii) If $|\alpha| = q$, then

$$F(\alpha; q, c_0, x) = \frac{x^{-1}}{1 - \alpha q^{-1}} + \frac{i\kappa(\alpha; q)_\infty}{(q; q)_\infty} \left(\hat{g}_0(\alpha; q, x) + \frac{c_0 - \gamma_0}{2\pi i} \hat{h}_0(\alpha; q, x) \right) x^{-\mu} + o(x^{-1}),$$

as $x \to \infty$ in any direction $\arg(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proof Similar to the proof of Theorem 5.2, it can be easily obtained by the above corollary. □

From the above results, we remark that for small $|\alpha|$, equation (1.4) behaves like a differential equation. In contrast, for large $|\alpha|$, the q-difference operator plays a more central role than the differential operator.

As mentioned in the Introduction, Lim obtained the asymptotic boundaries for all solutions of equation (1.2). We now compare the asymptotic form obtained in this paper with the results in [18]. In this paper, the non-homogeneous term $g(x) = \frac{1}{x}$ has a sigularity at 0, which is defined on $(0, \infty)$, does not satisfy the condition in [18] that g is defined on $[0, \infty)$, but it satisfies other conditions such as $g = O(x^{-1})$ and $g' = O(x^{-2})$ (equivalent to the case where $\alpha = -1$ in [18]). Recall Lim's results in [18, Theorem 1] (let's make $\alpha = -1$ for comparison):

Let $b < 0$. Assume that g' exists. Let $g(x) = O(x^{-1})$ and $g'(x) = O(x^{-2})$. Then:

(i) If $\frac{\ln|b/a|}{\ln q} > -1$, every solution of (1.2) is $O(x^{\frac{\ln|b/a|}{\ln q}})$ as $x \to \infty$.

(ii) If $\frac{\ln|b/a|}{\ln q} = -1$, every solution of (1.2) is $O(x^{\frac{\ln|b/a|}{\ln q}} \ln x)$ as $x \to \infty$.

(iii) If $\frac{\ln|b/a|}{\ln q} < -1$, every solution of (1.2) is $O(x^{-1})$ as $x \to \infty$.

From the tranform shown in the Introduction, we know that $\alpha = -\frac{a}{b}$. Therefore, the above case (i) is equivalent to: if $|\alpha| > q$, every solution of (1.2) is $O(x^{-\frac{\ln|\alpha|}{\ln q}})$ as $x \to \infty$, which is consistent with the case (ii) in Theorem 5.3 (because $\frac{\ln|\alpha|}{\ln q} = \mathrm{Re}(\mu)$ and every solution is $O(x^{-\mathrm{Re}(\mu)})$ as $x \to \infty$). The above case (iii) is equivalent to: if $|\alpha| < q$, every solution of (1.2) is $O(x^{-1})$ as $x \to \infty$, which is also consistent with the case (i) in Theorem 5.3.

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