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On the Admissibility and Stability of Multi-Agent Nonlinear Interconnected Positive Systems with Heterogeneous Delays

Abdelhafid Zenati, Nabil Aouf, Mohamed Tadjine, and Taous-Meriem Laleg-Kirati.

Abstract—Many multi-agent interconnected systems include typical nonlinearities which are highly sensitive to inevitable communication delays. This makes their analysis challenging and the generalization of results from linear interconnected systems theory to those nonlinear interconnected systems very limited. This paper deals with the analysis of Multi-Agent Nonlinear Interconnected Positive Systems (MANIPS). The main contributions of this work are two fold. Based on Perron-Frobenius theorem, we first study the "admissibility" property for MANIPS, and show that it is a fundamental requirement for this category of systems. Then, using admissibility/positivity properties and sequences of functions theory, we propose a suitable Lyapunov function candidate to conduct the analysis of the dynamical behavior of such systems. We show that the stability of MANIPS is reduced to the positiveness property (i.e. negative or positive definiteness) of a new specific matrix-valued function (\mathcal{Z}) that we derive in this paper. Furthermore, the obtained results generalize the existing theory. The quality of the results achieved are demonstrated through the applications of the developed theory on cells with multi-stage maturation process dynamical models.

Index Terms—Admissibility, Delays, Interconnected System, Nonlinear Systems, Lyapunov Function, Positive Systems, Stability.

I. INTRODUCTION

In this work, we first characterize the admissibility property required for the stability of interconnected systems using Perron-Frobenius theorem. Then, we prove that the admissibility defined in [1] for the linear case remains valid for the nonlinear case. We also show that admissibility is a sufficient condition for the system to be positive and well-posed. No systematic method to analyze the stability of this class of systems is available. The Lyapunov method, for example, presents a real challenge to find suitable Lyapunov function candidates. Therefore, by combining positiveness/admissibility criteria and time functions sequence theory, we construct a suitable non-quadratic Lyapunov functional. We prove the uniform convergence property of this functional to guarantee that converges to a time function and the time derivative of this

later (i.e. of the time function) is the limit of the time derivative of the suitable non-quadratic Lyapunov functional. Then, we exploit the time function and its time derivative to develop the global asymptotic stability and instability conditions for the multi-agent nonlinear positive interconnected systems with constant and distributed delays. We illustrate the results of the proposed method on a biological problem proposed recently in [2] and which describes stem cells dynamics in hematopoiesis with multi-stage of maturation.

The paper is organized as follows. In Section II, we formulate what we call a multi-agent nonlinear positive interconnected system with heterogeneous delays and discuss the importance of stability analysis for this class of systems. We present its admissibility and stability conditions through constructing an appropriate Lyapunov functional in Section III. The effectiveness of our findings is shown through the results obtained on an application problem as presented in Section IV. We conclude with some remarks in Section V.

Notations and Definitions: We use the following notations: \mathbb{R}_+ denotes the positive real, \mathbb{N} denotes the set of all non-negative integers. We define $\mathbb{R}_{\geq 0}^N = \{\forall x \in \mathbb{R}^N \mid x \geq 0\}$, Matrix $\mathcal{M}^N \in \mathbb{R}^{N \times N}$ is said to be Metzler or cooperative provided that all of its off-diagonal entries are non-negative, i.e. $\mathcal{M}_{i,j} \geq 0$ for $i \neq j$. A matrix \mathbf{M} is said to be non-negative provided every entry $\mathbf{M}_{i,j}$ of \mathbf{M} satisfies $\mathbf{M}_{i,j} \geq 0$ and we write $\mathbf{M} \in \mathbb{R}_{\geq 0}^{N \times N}$. For $M \in \mathbb{R}^{N \times N}$, the spectral radius of matrix M is denoted by $\rho(M)$ where for all $1 \leq i \leq s$, λ_i be the eigenvalues of the matrix M . The spectral radius $\rho(M)$ is defined as: $\max_{1 \leq i \leq N} \{\|\lambda_i\|\}$. We denote $\varphi_x \in \mathcal{D}([-\tau, 0], \mathbb{R}_{\geq 0}^N)$ the set of all continuous $\mathbb{R}_{\geq 0}^N$ -valued functions defined on $[-\tau, 0]$. The absolute value is denoted by $\|\cdot\|$. For all $t \in [0, +\infty)$, $\{V_{q,n,m}(t), q = 1, 2, \dots\}$ is a positive sequence of time function.

II. MULTI-AGENT NONLINEAR INTERCONNECTED POSITIVE SYSTEM

Let consider dynamic of an agent i represented by a positive sub-systems $\Sigma_i (i = 1, \dots, N)$ as follow:

$$\Sigma_i := \begin{cases} \dot{x}_i(t) &= \Psi_i(x_i, t)x_i(t) + \mathbf{H}_i w_i(t), \\ z_i(t) &= \xi_i(x_i, t)x_i(t) + \mathbf{K}_i w_i(t), \end{cases} \quad (1)$$

where $x_i(t)$ is the state vector, $w_i(t)$ is the interconnection signal with the other agents and the measured signal $z_i(t)$

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and its delayed versions due to heterogeneous communication delays, and

$$\Psi_i(\cdot, \cdot) \in \mathbb{R}^{N_{x_i} \times N_{x_i}}, \quad H_i \in \mathbb{R}^{N_{x_i} \times N_{w_i}}, \\ K_i \in \mathbb{R}^{N_{z_i} \times N_{w_i}}, \quad \xi_i(\cdot, \cdot) \in \mathbb{R}^{N_{z_i} \times N_{x_i}}.$$

In the subsection III-A, we will prove that the sub-system (1) is positive when it is associated with appropriate positive initial conditions if and only if:

$$\Psi_i(\cdot, \cdot) \in \mathcal{M}^{N_{x_i}}, \quad H_i \in \mathbb{R}_{\geq 0}^{N_{x_i} \times N_{w_i}}, \\ \xi_i(\cdot, \cdot) \in \mathbb{R}_{\geq 0}^{N_{z_i} \times N_{x_i}}, \quad K_i \in \mathbb{R}_{\geq 0}^{N_{z_i} \times N_{w_i}}. \quad (2)$$

With these positive sub-systems, let us define an augmented positive system Σ of N agents by

$$\Sigma := \text{diag}(\Sigma_{System}^1, \dots, \Sigma_{System}^N). \quad (3)$$

The state space realization of Σ is given by

$$\Sigma := \begin{cases} \dot{x}(t) &= \Psi(x, t)x(t) + Hw(t), \\ z(t) &= \xi(x, t)x(t) + Kw(t), \end{cases} \quad (4)$$

where

$$\Psi := \text{diag}(\Psi_1, \dots, \Psi_N), \quad \xi := \text{diag}(\xi_1, \dots, \xi_N),$$

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{\sum_{i=1}^N N_{x_i}}, \quad z := \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \in \mathbb{R}^{\sum_{i=1}^N N_{z_i}}, \\ w := \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \in \mathbb{R}^{\sum_{i=1}^N N_{w_i}}. \quad (5)$$

Also, the interconnection among the fleet of the agents is established via the matrices K and H where :

$$K_{ii} := K_i, \quad K_{ij} \in \mathbb{R}_{\geq 0}^{N_{z_j} \times N_{w_i}}, \quad H_{ii} := H_i, \quad H_{ij} \in \mathbb{R}_{\geq 0}^{N_{x_j} \times N_{w_i}}. \quad (6)$$

In [1], [3]–[8], the authors considered only the linear case of positive systems which limits the application of their results for real challenging problems. Authors of [1] considered interconnected systems, without delays, built from positive subsystems. However, ignoring communication delays for the interconnected system often cause instability [9], [10]. Thus, these delays should be taken into consideration. Regarding the nonlinear interconnected systems, it is true that their stability issue was investigated in [11] through a comparison principle. Some interesting results have been developed towards control design in [12]. The comparison principle is based on finding a suitable compared system whose stability can be analyzed. Unfortunately, building such a system is still a real challenge and difficult. In this paper, we will develop and generalize stability conditions that can be tested on any nonlinear positive interconnected class of systems. Moreover, in this work, we take into consideration heterogeneous communications delays (constant and distributed). The distributed delay is an inherent characteristic of various physical and biological processes because. In real world, several events related to the delay have a distributed density that is not a delta function. Via an integral

action, the delay operator has accumulative effect over the past values of the dynamics. For this reason, distributed delay systems exist in a wide range of applications [2]. Thus, if the interconnected system is modelled including heterogeneous communication delays ($n \in \mathbb{N}$ constant delays and $m \in \mathbb{N}$ distributed delays) as follow:

$$w(t) = \sum_{k=0}^m \int_{t-\tau_k}^t L_k(t-a)z(a) da + \sum_{i=0}^n D_i z(t-h_i), \quad (7)$$

where

$$L_k(t) \in \mathbb{R}_{\geq 0}^{\left(\sum_{i=1}^N N_{w_i}\right) \times \left(\sum_{i=1}^N N_{z_i}\right)}, \quad D_i \in \mathbb{R}_{\geq 0}^{\left(\sum_{i=1}^N N_{w_i}\right) \times \left(\sum_{i=1}^N N_{z_i}\right)}. \quad (8)$$

This makes (4) as

$$\Sigma_{System} := \begin{cases} \dot{x}(t) &= \Psi(x, t)x(t) + \sum_{i=0}^n B_i z(t-h_i) \\ &+ \sum_{k=0}^m \int_{t-\tau_k}^t g_k(t-a)z(a) da, \\ z(t) &= \xi(x, t)x(t) + \sum_{i=0}^n C_i z(t-h_i) \\ &+ \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da, \end{cases} \quad (9)$$

where

$$f_k(t) = KL_k(t), \quad g_k(t) = HL_k(t), \quad B_i = HD_i, \quad C_i = KD_i.$$

We will prove in the next section that system (9) is positive (i.e. $x \in \mathbb{R}_{\geq 0}^{\sum_{i=1}^N N_{x_i}}$ and $z \in \mathbb{R}_{\geq 0}^{\sum_{i=1}^N N_{z_i}}$) when it is associated with appropriate initial conditions φ_x and φ_z such that

$$\varphi_x \in \mathbb{C} \left(\left[-\max_{0 \leq i \leq n, 0 \leq k \leq m} \{h_i, \tau_k\}, 0 \right], \mathbb{R}_{\geq 0}^{\sum_{i=1}^N N_{x_i}} \right), \quad (10)$$

$$\varphi_z \in \mathbb{C} \left(\left[-\max_{0 \leq i \leq n, 0 \leq k \leq m} \{h_i, \tau_k\}, 0 \right], \mathbb{R}_{\geq 0}^{\sum_{i=1}^N N_{z_i}} \right).$$

The mathematical form of (9) is called positive coupled differential-difference equation and is used in different control applications [8], [13]–[15]. Also, it covers many important classes of dynamical systems, such as neutral systems, systems with multiple commensurate delays and some singular systems [16]–[19].

III. ADMISSIBILITY AND GLOBAL ASYMPTOTIC STABILITY ANALYSIS

Consider the interconnected system (9). One must fulfill two fundamental requirements in order to get well defined states :

- 1) *Well posedness* : that is the interconnection does not bring singularity of solutions.
- 2) *Positivity* : that is the interconnection maintains the positivity property of the signals under suitable initial conditions.

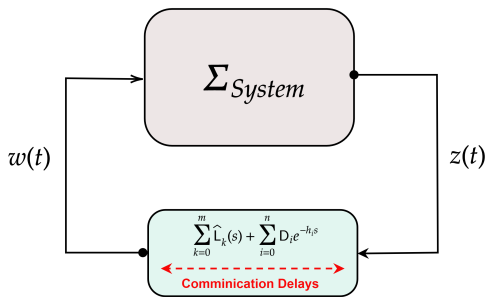


Fig. 1. Multi-agent nonlinear interconnected system with heterogeneous n constant and m distributed communication delays, where $\hat{L}_k(s)$ and $e^{-h_i s}$ are Laplace transform of $L_k(t)$ the distributed density in $[t - \tau_k, t]$ and the delta function $\Delta_t(t - h_i)$, respectively.

In [1], authors combine these two facts into a new notion called admissibility for a certain class of linear interconnected positive systems. Furthermore, they characterize the admissibility notion by simple algebraic test involving a particular Metzler matrix. In the sequel, we generalize the admissibility notion for the nonlinear case.

A. Positivity Property

Definition 1 (Positiveness): System (9) is said to be positive if it has a piece-wise continuous positive solutions $x(t)$ and $z(t)$ for all $t > 0$, when the initial conditions φ_x and φ_z are positive and satisfy (10).

Proposition 1 (Positiveness): Let φ_x and φ_z be the initial conditions satisfying (10), the solutions of system (9) are positive for all $t > 0$.

Proof: For the proof, please see the reference [21]. ■

B. Admissibility Analysis

Definition 2 (Admissible System [1]): The interconnected positive system (9) is said to be admissible if the Metzler matrix $\Omega - I$ is Hurwitz, where

$$\Omega = \sum_{i=0}^n C_i + \sum_{k=0}^m F_k, \quad F_k = \int_{t-\tau_k}^t f_k(t-a) da. \quad (11)$$

The admissibility as a sufficient condition for the positivity and well-posedness presents an issue to build interconnected systems. These systems when not inadmissible are unstable in the presence of arbitrarily small communication delays. From a control theory point of view, the admissibility of nonlinear interconnected positive systems is an essential requirement to build the appropriate Lyapunov function. As we can see later during the design of the appropriate Lyapunov Functional, we utilize the property of the admissibility to build the proposed functional and also to verify its uniform convergence as we see in the lemma 2.

We use the Perron-Frobenius theorem [20] to characterize the admissibility condition for the system in (9). We start by the following theorem:

Theorem 1 (Admissibility): Let $\rho(\Omega)$ be the Perron-Frobenius eigenvalue of the non-negative matrix Ω , the system (9) is admissible if and only if

$$\rho(\Omega) < 1. \quad (12)$$

Moreover, if (9) is not admissible then it has an unbounded solution.

Proof:

According to the definition of the admissibility, the system (9) is admissible if and only if the Metzler matrix $\Omega - I$ is Hurwitz. Then, the eigenvalues of the Metzler matrix $\Omega - I$ is the solution of

$$\begin{aligned} \det[\lambda I - (\Omega - I)] &= \det[(\lambda + 1)I - \Omega] \\ &= \det[\tilde{\lambda}I - \Omega] = 0, \end{aligned} \quad (13)$$

where $\tilde{\lambda} = \lambda + 1$, knowing that Ω is non-negative matrix, so, according to Perron-Frobenius theorem

$$\rho(\Omega) = \max_{1 \leq i \leq N} |\tilde{\lambda}_i|. \quad (14)$$

Since that $\tilde{\lambda} = \lambda + 1$ then the Metzler matrix $\Omega - I$ is Hurwitz or the system (9) is admissible if and only if

$$\rho(\Omega) < 1. \quad (15)$$

On the other hand, assuming that system (9) is not admissible, we have $z(t)$ from the second equation of (9) is given by

$$z(t) = \xi(x, t)x(t) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da + \sum_{i=0}^n C_i z(t-h_i). \quad (16)$$

Subtracting the positive vector $\Omega z(t)$ from then multiplying each side of the equation and multiplying it by positive left eigenvector ν of Ω [20], we get

$$\begin{aligned} \nu^t (I - \Omega) z(t) &= \nu^t \left[\xi(x, t)x(t) + \sum_{i=0}^n C_i \Delta_t(t-h_i) \right] \\ &+ \nu^t \left[\sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a) \Delta_t(a) da \right], \end{aligned} \quad (17)$$

where $\Delta_t(\phi) = z(\phi) - z(t)$.

According to [20], we have $\nu^t \Omega = \rho(\Omega) \nu^t$, this gives

$$\begin{aligned} (1 - \rho(\Omega)) \nu^t z(t) &= \nu^t \left[\xi(x, t)x(t) + \sum_{i=0}^n C_i \Delta_t(t-h_i) \right] \\ &+ \nu^t \left[\sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a) \Delta_t(a) da \right], \end{aligned}$$

Knowing that $\xi(x, t)x(t) \geq 0$ and $\nu_i > 0, i = 1, \dots, N_{z_i}$, it is clear that if the system (9) is not admissible (i.e. $\rho(\Omega) \geq 1$) then $\forall t > 0$

$$\nu^t \left[\sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a) \Delta_t(a) da + \sum_{i=0}^n C_i \Delta_t(t-h_i) \right] < 0.$$

This implies that the magnitude $|z(t)|$ is an increasing function with time, which means that (For more details see [21]):

$$\lim_{t \rightarrow +\infty} |z(t)| = +\infty \implies \lim_{t \rightarrow +\infty} |x(t)| = +\infty. \quad (18)$$

■

C. Boundedness

In this subsection, the sufficient conditions for boundedness of solutions of system (9) are derived.

Proposition 2 (Boundedness): Let φ_x and φ_z be the initial conditions satisfying (10), if system (9) is admissible and $\xi(x, t)x(t)$ is bounded, then, the difference operator $z(t)$ has a bounded solution.

Proof: Consider system (9) is admissible (i.e. $\rho(\Omega) < 1$), from (9), we have

$$z(t) = \xi(x, t)x(t) + \sum_{i=0}^n C_i z(t-h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da. \quad (19)$$

Knowing that $\xi(x, t)x(t)$ is bounded, there exists $\Upsilon \in \mathbb{R}_{\geq 0}^{\sum_{i=1}^N N_{z_i}}$ such as

$$\xi(x, t)x(t) \leq \Upsilon < +\infty, \quad \forall(x, t) \in \mathbb{R}_{\geq 0}^{\sum_{i=1}^N N_{z_i}} \times [-\tau, +\infty[. \quad (20)$$

Thus,

$$z(t) \leq \Upsilon + \sum_{i=0}^n C_i z(t-h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da. \quad (21)$$

Let's $Z(s)$ be the Laplace transform for $z(t)$, (21) implies ¹

$$Z(s) \leq \frac{1}{s}\Upsilon + \sum_{i=0}^n C_i e^{-h_i s} Z(s) + \sum_{k=0}^m \hat{\mathbf{F}}_k(s) Z(s). \quad (22)$$

Therefore,

$$\left[I - \sum_{i=0}^n C_i e^{-h_i s} + \sum_{k=0}^m \hat{\mathbf{F}}_k(s) \right] Z(s) \leq \frac{1}{s}\Upsilon. \quad (23)$$

According to the final values theorem, we have

$$\lim_{t \rightarrow +\infty} z(t) = \lim_{s \rightarrow 0} sZ(s). \quad (24)$$

Therefore, the equations (22) and (24) follow that

$$\begin{aligned} \Upsilon &\geq \lim_{s \rightarrow 0} \left[I - \sum_{i=0}^n C_i e^{-h_i s} + \sum_{k=0}^m \hat{\mathbf{F}}_k(s) \right] sZ(s) \\ &= \lim_{s \rightarrow 0} \left[I - \sum_{i=0}^n C_i e^{-h_i s} + \sum_{k=0}^m \hat{\mathbf{F}}_k(s) \right] \times \lim_{s \rightarrow 0} s\tilde{Z}(s) \quad (25) \\ &= \lim_{s \rightarrow 0} \left[I - \sum_{i=0}^n C_i e^{-h_i s} + \sum_{k=0}^m \hat{\mathbf{F}}_k(s) \right] \times \lim_{t \rightarrow \infty} z(t) \end{aligned}$$

Then,

$$\lim_{s \rightarrow 0} \left[I - \sum_{i=0}^n C_i e^{-h_i s} + \sum_{k=0}^m \hat{\mathbf{F}}_k(s) \right] \times \lim_{t \rightarrow \infty} z(t) \leq \Upsilon \quad (26)$$

Also, having $\lim_{s \rightarrow 0} e^{-h_i s} = 1$, $\lim_{s \rightarrow 0} e^{-ts} = 1$ and

$$\lim_{s \rightarrow 0} \hat{\mathbf{F}}_k(s) = \lim_{s \rightarrow 0} \int_{-\tau_k}^0 f_k(t)e^{-ts} dt = \int_{-\tau_k}^0 f_k(t) dt \quad (27)$$

¹The inequality is still valid for the complex numbers as well.

Therefore, $\lim_{s \rightarrow 0} \hat{\mathbf{F}}_k(s) = F_k$ where F_k is given in (11), this yields

$$\begin{aligned} &\lim_{s \rightarrow 0} \left[I - \sum_{i=0}^n C_i e^{-h_i s} + \sum_{k=0}^m \hat{\mathbf{F}}_k(s) \right] \\ &= \left[I - \sum_{i=0}^n C_i + \sum_{k=0}^m F_k \right] = [I - \Omega]. \end{aligned} \quad (28)$$

where Ω is given in (11). Knowing that the system is admissible, the matrix $[I - \Omega]$ is invertible, then, the equation (26) implies

$$\lim_{t \rightarrow +\infty} z(t) = \Upsilon [I - \Omega]^{-1} < +\infty, \quad (29)$$

this allows us conclude that $z(t) < +\infty$. Hence $z(t)$ is bounded if system (9) is admissible and $\xi(x, t)x(t)$ is bounded. ■

We take into consideration the result presented above to build the Lyapunov function candidate in the following subsection by assuming that $\rho(\Omega) < 1$.

D. Design a Lyapunov Functional

An appropriate Lyapunov functional candidate is required to solve the stability of the system described in (9). We are aiming at providing sufficient and necessary conditions for global asymptotic stability. Lemma 1 and lemma 2 are related to the construction and time derivation of a positive function, which permits to characterize the dynamics of (9). Introducing for later use the following:

$$\begin{aligned} \vartheta_n(t) &= \sum_{i=0}^n B_i \int_{t-h_i}^t z(a) da, \\ \omega_n(t) &= \sum_{i=0}^n C_i \int_{t-h_i}^t z(a) da, \\ \varrho_m(t) &= \sum_{k=0}^m \int_{t-\tau_k}^t \int_{\sigma}^t g_k(\sigma - a + \tau_k) z(a) da d\sigma, \\ \eta_m(t) &= \sum_{k=0}^m \int_{t-\tau_k}^t \int_{\sigma}^t f_k(\sigma - a + \tau_k) z(a) da d\sigma. \end{aligned} \quad (30)$$

Lemma 1 (Appropriate Lyapunov Functional): Let $V_{q,n,m}(t)$ be a differentiable positive sequence of time function given by:

$$V_{q,n,m}(t) = x(t) + \vartheta_n(t) + \varrho_m(t) + \Phi \sum_{j=0}^{q-1} (\Omega)^j \psi(t). \quad (31)$$

Then, its time derivative along (9) satisfies $\dot{V}_{q,n,m}(t) =$

$$\begin{aligned} &\left[\Psi(x, t) + \Phi \sum_{j=0}^{q-1} (\Omega)^j \xi(x, t) \right] x(t) + \Phi (\Omega)^q \\ &\times \left[\sum_{i=0}^n C_i z(t-h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da \right], \end{aligned} \quad (32)$$

where $\psi(t) = \omega_n(t) + \eta_m(t)$ and $\Phi = \sum_{k=0}^m (F_k + G_k)$, such that

$$F_k = \int_{t-\tau_k}^t f_k(t-a) da, \quad G_k = \int_{t-\tau_k}^t g_k(t-a) da. \quad (33)$$

Proof: We start by computing the time derivative of $\vartheta_n(t)$, $\omega_n(t)$, $\eta_m(t)$ and $\varrho_m(t)$.

Using Leibniz's rule for differentiation under the integral sign [22], we get

$$\begin{cases} \dot{\vartheta}_n(t) = \sum_{i=0}^n B_i z(t) - \sum_{i=0}^n B_i z(t - h_i), \\ \dot{\omega}_n(t) = \sum_{i=0}^n C_i z(t) - \sum_{i=0}^n C_i z(t - h_i), \\ \dot{\varrho}_m(t) = \sum_{k=0}^m \mathbf{G}_k z(t) - \sum_{k=0}^m \int_{t-\tau_k}^t g_k(t-a)z(a) da, \\ \dot{\eta}_m(t) = \sum_{k=0}^m F_k z(t) - \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da. \end{cases} \quad (34)$$

Using (34) and applying mathematical induction, we prove lemma 1.

1. Based case: For $q = 1$ and from (31), we have

$$V_{1,n,m}(t) = x(t) + \vartheta_n(t) + \varrho_m(t) + \Phi \sum_{j=0}^0 (\Omega)^j \psi(t). \quad (35)$$

Which is equivalent to

$$V_{1,n,m}(t) = x(t) + \vartheta_n(t) + \varrho_m(t) + \Phi [\omega_n(t) + \eta_m(t)]. \quad (36)$$

Its time derivative is given by

$$\dot{V}_{1,n,m}(t) = \dot{x}(t) + \dot{\vartheta}_n(t) + \dot{\varrho}_m(t) + \Phi [\dot{\omega}_n(t) + \dot{\eta}_m(t)]. \quad (37)$$

By replacing $\dot{x}(t)$, $\dot{\vartheta}_n(t)$, $\dot{\varrho}_m(t)$, $\dot{\omega}_n(t)$ and $\dot{\eta}_m(t)$ with their expressions from (9), (34), respectively, we obtain

$$\begin{aligned} \dot{V}_{1,n,m}(t) &= \Psi(x, t)x(t) + \Phi [I + \Omega]z(t) - \Phi \\ &\times \left[\sum_{i=0}^n C_i z(t - h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da \right]. \end{aligned} \quad (38)$$

Now, we replace $z(t)$ with its expression from equation two of (9), we then have

$$\begin{aligned} \dot{V}_{1,n,m}(t) &= \Psi(x, t)(t) + \Phi [I + \Omega] \xi(x(t))x(t) - \Phi \Omega \\ &\times \left[\sum_{i=0}^n C_i z(t - h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da \right]. \end{aligned} \quad (39)$$

Which is

$$\begin{aligned} \dot{V}_{1,n,m}(t) &= \left[\Psi(x, t) + \Phi \sum_{j=0}^1 (\Omega)^j \xi(x, t) \right] x(t) + \Phi \Omega \\ &\times \left[\sum_{i=0}^n C_i x(t - h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da \right]. \end{aligned} \quad (40)$$

Therefore, we deduce that (32) is time derivative of (31) for $q = 1$.

2. Induction Step: Now, we assume that (31) and (32) hold and lemma 1 is true for q (Induction hypothesis), and we show that it is also true for $q+1$. For $q+1$, the sequence $V_{q+1,n,m}(t)$ of time function is

$$\begin{aligned} V_{q+1,n,m}(t) &= x(t) + \vartheta_n(t) + \varrho_m(t) + \Phi \sum_{j=0}^q (\Omega)^j \psi(t) \\ &= x(t) + \vartheta_n(t) + \varrho_m(t) + \Phi \sum_{j=0}^{q-1} (\Omega)^j \psi(t) + \Phi \Omega^q \psi(t) \\ &= V_{q,n,m}(t) + \Phi (\Omega)^q \psi(t). \end{aligned} \quad (41)$$

Then, its time derivative along (9) is given by

$$\dot{V}_{q+1,n,m}(t) = \dot{V}_{q,n,m}(t) + \Phi (\Omega)^q [\dot{\omega}_n(t) + \dot{\eta}_m(t)]. \quad (42)$$

Replacing $\dot{V}_{q,n,m}(t)$ which is given in (32) (Induction hypothesis) and $\dot{\omega}_n(t)$ and $\dot{\eta}_m(t)$ which are given in (34), we then get

$$\begin{aligned} \dot{V}_{q+1,n,m}(t) &= \left[\Psi(x, t) + \Phi \sum_{j=0}^q (\Omega)^j \xi(x, t) \right] x(t) + \Phi (\Omega)^{q+1} z(t) \\ &+ \Phi (\Omega)^q \left[\sum_{i=0}^n C_i x(t - h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da \right] \\ &- \Phi (\Omega)^q \left[\sum_{i=0}^n C_i x(t - h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da \right] \\ &= \Phi (\Omega)^{q+1} z(t) + \left[\Psi(x, t) + \Phi \sum_{j=0}^q (\Omega)^j \xi(x, t) \right] x(t). \end{aligned}$$

Now, we replace $z(t)$ with its expression which is given in equation two of (9), we get

$$\begin{aligned} \dot{V}_{q+1,n,m}(t) &= \left[\Psi(x, t) + \Phi \sum_{j=0}^{q+1} (\Omega)^j \xi(x, t) \right] x(t) + \Phi \\ &\times \Omega^{q+1} \left[\sum_{i=0}^n C_i x(t - h_i) + \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)z(a) da \right]. \end{aligned}$$

This allow us to conclude that for all $q \in \mathbb{N}$ we have (32) is time derivative of (31). ■

In the next lemma, we will check the uniform convergence to ensure that

$$\frac{d}{dt} \left[\lim_{q \rightarrow +\infty} V_{q,n,m}(t) \right] = \lim_{q \rightarrow +\infty} \dot{V}_{q,n,m}(t). \quad (43)$$

Lemma 2 (Uniform Convergence): Let the non-negative matrix Ω stratifying $\rho(\Omega) < 1$, the sequences $V_{q,n,m}(t)$ and $\dot{V}_{q,n,m}(t)$ converge uniformly to $V_{n,m}(t)$ and $\dot{V}_{n,m}(t)$, respectively. Such that

$$\lim_{q \rightarrow +\infty} V_{q,n,m}(t) = x(t) + \vartheta_n(t) + \varrho_m(t) + \Phi \bar{\Omega}^{-1} \psi(t), \quad (44)$$

and

$$\lim_{q \rightarrow +\infty} \dot{V}_{q,n,m}(t) = [\Psi(x, t) + \Phi \bar{\Omega}^{-1} \xi(x, t)] x(t). \quad (45)$$

where $\psi(t) = \omega_n(t) + \eta_m(t)$ and $\bar{\Omega} = I - \Omega$.

Proof: To prove this lemma, we replace in (31) and (32) the sum of the geometric matrix sequence $\sum_{j=0}^{\infty} (\Omega)^j$ and $\lim_{q \rightarrow +\infty} \Omega^q$. Then, we verify the uniform convergence i.e.

$$\lim_{q \rightarrow +\infty} \left\| \begin{aligned} &V_{q,n,m}(t) - V_{n,m}(t) \\ &\dot{V}_{q,n,m}(t) - \dot{V}_{q,n,m}(t) \end{aligned} \right\| = 0. \quad (46)$$

Having $\rho(\Omega) < 1$, we demonstrate in the next that

$$\lim_{q \rightarrow +\infty} \Omega^q = 0, \text{ and } \sum_{k=0}^{\infty} (\Omega)^k = (I - \Omega)^{-1} = \bar{\Omega}^{-1}. \quad (47)$$

We start by calculations of Ω^q . Using eigendecomposition, Ω can therefore be decomposed into a matrix composed of its eigenvectors P , a diagonal matrix Ξ with its eigenvalues along the diagonal, and the inverse of the matrix of eigenvectors P^{-1} , such as

$$\Omega = \underbrace{[v_1, \dots, v_N]}_P \times \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}}_{\Xi} \times \underbrace{[v_1, \dots, v_N]^{-1}}_{P^{-1}}. \quad (48)$$

This implies that

$$\begin{aligned} \Omega^j &= [P \Xi P^{-1}]^j = \underbrace{P \Xi P^{-1} \times P \Xi P^{-1} \times \dots \times P \Xi P^{-1}}_{j \text{ times}} \\ &= \underbrace{P \Xi \times \dots \times \Xi}_{j \text{ times}} P^{-1} = P \Xi^j P^{-1}. \end{aligned} \quad (49)$$

Therefore

$$\Omega^j = [v_1, \dots, v_N] \times \begin{pmatrix} \lambda_1^j & & \\ & \ddots & \\ & & \lambda_N^j \end{pmatrix} \times [v_1, \dots, v_N]^{-1}. \quad (50)$$

Knowing that $|\lambda_\kappa| < \rho(\Omega) < 1$ for all $\kappa = \{1, 2, \dots, N\}$, it follows that $\lim_{q \rightarrow +\infty} \lambda_\kappa^q = 0$. Thus

$$\lim_{q \rightarrow +\infty} \Omega^q = 0. \quad (51)$$

Then, we can write

$$\begin{cases} V_{n,m}(t) = x(t) + \vartheta_n(t) + \varrho_m(t) + \Phi \sum_{j=0}^{\infty} (\Omega)^j \omega_n(t), \\ \dot{V}_{n,m}(t) = [\Psi(x, t) + \Phi \sum_{j=0}^{\infty} (\Omega)^j \xi(x, t)] x(t). \end{cases} \quad (52)$$

Now, we calculate the geometric sum of matrix $\sum_{j=0}^{\infty} (\Omega)^j$.

According (49), we have

$$\sum_{j=0}^{\infty} (\Omega)^j = \sum_{j=0}^{\infty} (P \Xi P^{-1})^j = \sum_{j=0}^{\infty} P \Xi^j P^{-1} = P \sum_{j=0}^{\infty} \Xi^j P^{-1},$$

and it follows

$$\begin{aligned} \sum_{j=0}^{\infty} (\Omega)^j &= P \times \begin{pmatrix} \sum_{j=0}^{\infty} \lambda_1^j & & \\ & \ddots & \\ & & \sum_{j=0}^{\infty} \lambda_N^j \end{pmatrix} \times P^{-1} \\ &= P \times \begin{pmatrix} \frac{1}{1-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{1-\lambda_r} \end{pmatrix} \times P^{-1}. \end{aligned} \quad (53)$$

Knowing that inverse of matrix product $(AB)^{-1} = B^{-1}A^{-1}$, therefore

$$\begin{aligned} \sum_{j=0}^{\infty} (\Omega)^j &= P \times \begin{pmatrix} \frac{1}{1-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{1-\lambda_r} \end{pmatrix} \times P^{-1} \\ &= \left[P \times \begin{pmatrix} 1-\lambda_1 & & \\ & \ddots & \\ & & 1-\lambda_N \end{pmatrix} \times P^{-1} \right]^{-1} \end{aligned} \quad (54)$$

This implies that

$$\sum_{j=0}^{\infty} (\Omega)^j = [P(I - \Xi)P^{-1}]^{-1} = (I - \Omega)^{-1} = \bar{\Omega}^{-1}. \quad (55)$$

By replacing this result in (52), we get

$$\begin{cases} V_{n,m}(t) = x(t) + \vartheta_n(t) + \varrho_m(t) + \Phi \bar{\Omega}^{-1} \psi(t), \\ \dot{V}_{n,m}(t) = [\Psi(x, t) + \Phi \bar{\Omega}^{-1} \xi(x, t)] x(t). \end{cases} \quad (56)$$

Remarks: In the case where the matrix Ω is not diagonalizable, the above results remain correct because there exists an invertible matrix P such that $\Omega = M J M^{-1}$. J is called the Jordan normal form of Ω .

E. Stability Analysis

Theorem 2 (Stability): Let $\varrho(\Omega) < 1$ and the matrix \mathcal{Z} given by:

$$\mathcal{Z}(x, t) = [\Psi(x, t) + \Phi[I - \Omega]^{-1} \xi(x, t)], \quad (57)$$

(i) if $\mathcal{Z}(x, t)$ is negative semi-definite, negative definite on $\mathfrak{G} \times [-\max_{0 \leq i \leq n, 0 \leq k \leq m} \{h_i, \tau_k\}, +\infty[$, respectively, then, the origin of the system (9) is (globally if $\mathfrak{G} = \mathbb{R}^N$) uniformly stable, uniformly asymptotically stable, respectively.

(ii) if there exists a region \mathfrak{D} around the origin where $\mathcal{Z}(x, t) \succ 0$ for all $x(t) \in \mathfrak{D}$ then (9) is unstable. where

$$\mathfrak{D} = \{\exists \varepsilon > 0, x \in \mathbb{R}_+^n \mid x \leq \varepsilon, \mathcal{Z}(x, t) \succ 0\}. \quad (58)$$

Proof: We start by proving the stability of case (i) where the system (9) is claimed globally asymptotically stable.

a) *First case (i):* Let assume that we have the following positive quadratic functional:

$$V(t) = \frac{1}{2} V_{n,m}^T(t) \cdot V_{n,m}(t) = \frac{1}{2} [V_{n,m}^1, \dots, V_{n,m}^N] \begin{bmatrix} V_{n,m}^1 \\ \vdots \\ V_{n,m}^N \end{bmatrix}. \quad (59)$$

This follows:

$$\dot{V}(t) = V_{n,m}^T(t) \cdot \dot{V}_{n,m}(t). \quad (60)$$

From (44), we have

$$V_{n,m}(t) = x(t) + \underbrace{\vartheta_n(t) + \varrho_m(t) + \Phi \bar{\Omega}^{-1} \psi(t)}_{\geq 0} \geq x(t), \quad (61)$$

Also, we get from (45) the time derivative of the positive linear vector functional $\dot{V}_{n,m}(t)$ satisfies:

$$\dot{V}_{n,m}(t) = [\dot{V}_{n,m}^1 \cdots \dot{V}_{n,m}^N]^T = \mathcal{Z}(x, t)x(t). \quad (62)$$

This implies

$$\begin{aligned} \dot{V}(t) &= V_{n,m}^T(t)\mathcal{Z}(x, t)x(t) \\ &= [V_{n,m}^T(t) - x(t) + x(t)]^T \mathcal{Z}(x, t)x(t) \\ &= [V_{n,m}^T(t) - x(t)]^T \mathcal{Z}(x, t)x(t) + x^T(t)\mathcal{Z}(x, t)x(t). \end{aligned} \quad (63)$$

It is clear that as $x \geq 0$ (i.e. $\|x\| = x$) and $V_{n,m} - x(t) \geq 0$, thus, if $\mathcal{Z}(x, t)$ is negative semi-definite or negative definite, respectively, on $\mathfrak{G} \times [-\max_{0 \leq i \leq n, 0 \leq k \leq m} \{h_i, \tau_k\}, +\infty[$ we get

$$\dot{V}(t) \leq x^T(t)\mathcal{Z}(x, t)x(t) = \psi^T(t)\mathcal{Z}(x, t)\psi(t), \quad (64)$$

where $\psi = [x_1, \dots, x_N]$.

According to [23], the origin of the system (9) is uniformly stable or uniformly asymptotically stable. Obviously, the origin of the system (9) is globally uniformly stable or globally uniformly asymptotically stable if $\mathcal{Z}(x, t)$ is negative semi-definite or negative definite, respectively, on $\mathbb{R}^N \times [-\max_{0 \leq i \leq n, 0 \leq k \leq m} \{h_i, \tau_k\}, +\infty[$. Regarding $z(t)$, we have:

$$\begin{aligned} (I - \Omega)z(t) &= \xi(x, t)x(t) + \sum_{i=0}^n C_i \Delta_t(t - h_i) \\ &+ \sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)\Delta_t(a) da \end{aligned} \quad (65)$$

Knowing that $\lim_{t \rightarrow +\infty} x(t) = 0$ and $(I - \Omega) > 0$, it follows that:

$$\sum_{k=0}^m \int_{t-\tau_k}^t f_k(t-a)\Delta_t(a) da + \sum_{i=0}^n C_i \Delta_t(t - h_i) > 0. \quad (66)$$

This implies that $z(t) > 0$ is decreasing with time then

$$\lim_{t \rightarrow +\infty} z(t) = 0. \quad (67)$$

b) Second case (ii): We demonstrate this result of (ii) by contradiction. Let us assume that there exists $\mathfrak{D} \in R_+^N$ where $0 \in \mathfrak{D}$ and $\forall x(t) \in \mathfrak{D}$ we have $\mathcal{Z}(x, t)$ is positive definite and the system (9) is stable. From (63), we have:

$$\begin{aligned} \dot{V}(t) &= V_{n,m}^T(t)\mathcal{Z}(x, t)x(t) \\ &= [V_{n,m}^T(t) - x(t) + x(t)]^T \mathcal{Z}(x, t)x(t) \\ &= [V_{n,m}^T(t) - x(t)]^T \mathcal{Z}(x, t)x(t) + x^T(t)\mathcal{Z}(x, t)x(t). \end{aligned} \quad (68)$$

As $V_{n,m}^T(t) - x(t) \geq 0$ and $\mathcal{Z}(x, t)$ is positive definite, thus, $[V_{n,m}^T(t) - x(t)]^T \mathcal{Z}(x, t)x(t) \geq 0$. Therefore

$$\dot{V}(t) \geq x^T(t)\mathcal{Z}(x, t)x(t) > 0. \quad (69)$$

When $x(t_s) \in \mathfrak{D}$, $\dot{V}(t) > 0$. It follows that for $t > t_s$, $V(t) > V(t = t_s)$. Therefore, $V(t)$ does not converge to zero. However and as we considered a solution $x(t)$ that converges to the origin implying that $x(t)$ converges to zero, this yields the contradiction. ■

F. Discussion on the Obtained Results

- 1) Of particular interest, we showed that the stability of Multi-Agent Nonlinear Interconnected Positive System is reduced to the sign (i.e. negative or positive definite) of the novel specific matrix \mathcal{Z} .
- 2) The results achieved are more general and concise with respect to the previously published results. For example: the linear interconnected positive system which is a particular case when $\Psi(x, t) = A$ and $\xi(x, t) = C$ are a constant matrix, the matrix \mathcal{Z} becomes:
$$\mathcal{Z}(x, t) = [A + \Phi(I - \Omega)^{-1}C]. \quad (70)$$
This linear interconnected system is admissible and stable if and only if $\mathcal{Z}(x, t)$ is Hurwitz and $(I - \Omega) < 0$. These results are similar to those given in ([1], [3]).
- 3) The results proved that the communication topology or the interconnection structure represented by the matrix Ω is the most important part in admissibility. It has a significant and direct impact on the admissibility and therefore on the stability. Moreover, delayed communication can be a source of non-admissibility and instability.

IV. SIMULATION VALIDATION

Let consider cells dynamics with multi-stage maturation process [2], the process dynamics are given by the following distributed delay system:

For all $i \in 1 \cdots N$ and $t > 0$

$$\begin{aligned} \dot{x}_i(t) &= -[\delta_i + \beta_i(x_i(t))]x_i(t) + L_i \int_0^{\tau_i} \omega_i(a)u_i(t-a) da \\ &+ 2K_{i-1} \int_0^{\tau_{i-1}} \omega_{i-1}(a)u_{i-1}(t-a) da \\ u_i(t) &= \beta_i(x_i(t))x_i(t) + \tilde{L}_i \int_0^{\tau_i} \omega_i(a)u_i(t-a) da \end{aligned} \quad (71)$$

We invite the reader to refer [2] to get a detailed presentation of the biological system model in (71). The parameters and functions of this model are given in [24]–[27]). Therefore, we can write the system (71) under the form (9) as follow

$$\begin{aligned} \Psi(x, t) &= \text{diag}(-\delta_1 - \beta_1(x_1(t)), \dots, -\delta_N - \beta_N(x_N(t))), \\ \xi(x, t) &= \text{diag}(\beta_1(x_1(t)), \dots, \beta_N(x_N(t))), \\ f(a) &= \text{diag}(\tilde{L}_1 \omega_1(a), \dots, \tilde{L}_N \omega_N(a)), \end{aligned}$$

$$g(a) = \begin{pmatrix} a_1(a) & 0 & 0 & 0 \\ b_1(a) & a_2(a) & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & b_{N-1}(a) & a_N(a) \end{pmatrix},$$

where $a_i(a) = L_i \omega_i(a)$, $i = 1, \dots, N$ and $b_j(a) = 2K_j \omega_j(a)$, $j = 1, \dots, N - 1$. Therefore, the admissibility matrix Ω

$$\Omega = \text{diag} \left(\tilde{L}_1 \int_0^{\tau_1} \omega_1(a) da, \dots, \tilde{L}_N \int_0^{\tau_N} \omega_N(a) da \right), \quad (72)$$

Then, the system (71) is admissible if and only if:

$$1 - \tilde{L}_i \int_0^{\tau_i} \omega_i(a) da > 0, \quad \forall i = 1, \dots, N. \quad (73)$$

Moreover, we have the stability matrix $\mathcal{Z}(x, t)$

$$\mathcal{Z}(x, t) = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ B_1 & A_2 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & BN - 1 & A_N \end{pmatrix}, \quad (74)$$

where

$$A_i = -\delta_i - \frac{1 - (\tilde{L}_i + L_i) \int_0^{\tau_i} \omega_i(a) da}{1 - \tilde{L}_i \int_0^{\tau_i} \omega_i(a) da} \beta_i(x_i(t)),$$

$$B_j = \frac{2K_j \int_0^{\tau_j} \omega_j(a) da}{1 - \tilde{L}_j \int_0^{\tau_j} \omega_j(a) da} \beta_j(x_j(t)), \quad (75)$$

where $i = 1, \dots, N$ and $j = 1, \dots, N - 1$. It is clear that the eigenvalues of $\mathcal{Z}(x, t)$ are $\lambda_i = A_i$. This allow us to conclude that $\mathcal{Z}(x, t)$ is semi-negative define (i.e. (71) is stable) if and only $A_i \leq 0$. This result is exactly the same one given [2].

V. CONCLUSION

In this paper, the admissibility and stability were investigated for nonlinear interconnected positive system. We showed that the obtained results are more general and con-size with respect the previously published results. As an interesting outcome, we reduced the complexity analysis of MANIPS to the sign (i.e. negative or positive definite) of the novel specific matrix \mathcal{Z} . The results clearly illustrated that the communication topology or the interconnection structure has a significant influence on the admissibility and therefore on the stability. The study of recent model in biology and its simulations were performed to illustrate the effectiveness of the proposed approach.

The future work is to use the matrix \mathcal{Z} in the control design for MANIPS and also in the analysis of its persistence property. This later is characterized the system state converges to a unique strictly positive vector for any strict positive initial state.

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