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On two dimensional Schrödinger operators

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Abstract

Following the works of Berest [4], we compute explicitly Hadamard's coefficients for two dimensional Schrödinger operators. For this, we use some specific generating functions and compute their relative Wronskians.

1 Introduction

In 2008 Berest, Cramer and Eshmatov [4] studied the Schrödinger operator $\mathcal{L} = -\Delta + V$, with a C^∞ potential V in some open domain Ω , following several works on similar themes [2, 3]. The heat kernel of \mathcal{L} is the solution $\Phi_+(x, \zeta, t)$ in $\Omega \times \Omega \times \mathbf{R}^+$ of the initial value problem

$$(\partial_t + \mathcal{L})\Phi_+(x, \xi, t) = 0, \quad \lim_{t \rightarrow 0^+} \Phi_+(x, \xi, t) = \delta(x - \xi), \quad (1)$$

where $\delta(x - \xi)$ is the Dirac delta function on Ω with support at ξ . The function $\Phi_+(x, \xi, t)$ has an asymptotic expansion of the form [71]

$$\Phi_+(x, \xi, t) \sim \frac{e^{|x-\xi|^2/4t}}{(4\pi t)^{n/2}} \left(1 + \sum_{\nu=1}^{\infty} U_\nu(x, \xi) t^\nu \right) \text{ as } t \rightarrow 0^+. \quad (2)$$

The functions $U_\nu(x, \xi)$ are called Hadamard's coefficients of the Schrödinger operator $\mathcal{L} = -\Delta + V$.

In their paper, they consider the potential V defined by

$$V(x) = -\frac{2}{r^2} \frac{\partial^2}{\partial \varphi^2} \ln W(F_0, \dots, F_m), \quad (3)$$

where $W(F_0, \dots, F_m)$ is the wronskian of the functions F_0, \dots, F_m , and $k = (k_0, k_1, \dots, k_m)$ is a strictly increasing sequence of integers.

Considering the functions F_i defined by $F_i(x) = \cos(k_i x + x_i)$, where x is equal to φ or ϕ and x_i equal to φ_i or ϕ_i , they proved that the Hadamard's coefficients of $\mathcal{L} = -\Delta + V_k$ are given in terms of polar coordinates $x = (r \cos \varphi; r \sin \varphi)$, $\xi = (\rho \cos \phi; \rho \sin \phi)$ by

$$U_\nu(x, \xi) = \frac{(-2)^\nu}{(r\rho)^\nu} \sum_{i=0}^m c_i \Psi_i(\varphi) \Psi_i(\phi) T_{k_i}^{(\nu)}(\cos(\varphi - \phi)), \text{ for } \nu \geq 0 \quad (4)$$

where

$$\Psi_i := \frac{W(F_0, \dots, F_{i-1}, F_{i+1}, \dots, F_m)}{W(F_0, \dots, F_m)}, \quad (5)$$

$$c_i := \prod_{\substack{j=0, \\ j \neq i}}^m k_i^2 - k_j^2, \quad (6)$$

$T_N(z) := \cos N(\arccos z)$, with $T_N^{(\nu)}(z)$ being its derivative of order ν with respect to z .

In this statement we recognize the Chebychev polynomials defined by $T_N(z) = \cos N(\arccos z)$.

We set some classic notations for the text. We will denote the columns of order m of a matrix by C_m ; the classical binomial coefficient $\frac{n!}{p!(n-p)!}$ will be denoted by C_n^p .

In the following, we will compute $W(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m)$ and $W(f_1, \dots, f_m)$ for a certain type of functions f_i , following the ideas of [10, 7, 16]. Then we will be able to evaluate explicitly the Hadamard's coefficients and solutions to the two-dimensional Schrödinger equations with Huygens' potentials.

2 Computation of Wronskians

2.1 First type of formulas

Let m be some non negative integer, α and x_0 be some real parameters, and f_j the functions defined as follows :

$$f_j = \sin \alpha j(x - x_0), \quad 1 \leq j \leq m. \quad (7)$$

We use below the standard notation,

$$W(f_1, f_2, \dots, f_m) := \det \partial_x^{j-1} f_k(x); \quad j, k = 1, \dots, m, \quad (8)$$

for the Wronskian determinant of any m functions and $[x]$ the largest integer less than or equal to x .

We will denote W_{m-1}^i the determinant $W(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m)$.

We will use the following result [5, 6] :

Proposition 2.1 *For any smooth functions $f, \varphi_1, \dots, \varphi_n$, the Wronskian $W(f \cdot \varphi_1, \dots, f \cdot \varphi_n)$ verify*

$$W(f \cdot \varphi_1, \dots, f \cdot \varphi_n) = (f(x))^n W(\varphi_1, \dots, \varphi_n), \quad (9)$$

and the formula of change of independent variable in a Wronskian :

Proposition 2.2 *For $t = \varphi(x)$ a new variable so that $f_j(x) = \tilde{f}_j(t)$,*

$$W(f_1, \dots, f_n) = (\varphi'(x))^{\frac{n(n-1)}{2}} \cdot W(\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_n). \quad (10)$$

With these notations we have the following formula

Theorem 2.1

$$W_{m-1}^i = i(2)^{\frac{m(m-1)}{2} - i + 1} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \prod_{k=1}^{m-1} k! \times \sin^{\frac{m(m-1)}{2}} \alpha(x - x_0) \\ \times \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k}}{(i+k)!k!} \frac{\cos^{m-i-2k} \alpha(x - x_0)}{(m-i-2k)!} \right), \quad 1 \leq i \leq m. \quad (11)$$

Proof:

We start with the cases where $i = m$ and $i = m - 1$.

The details of the justifications will be given in the general case.

1. For W_{m-1}^m , all functions involved in the determinant are of the form $\sin(px)$. We use the expansion

$$\sin px = \sin x \sum_{k=0}^{[\frac{p-1}{2}]} (-1)^k C_{p-k-1}^k \cdot (2 \cos x)^{p-1-2k} = \\ \sin x [(2 \cos x)^{p-1} - (p-2)(2 \cos x)^{p-3} + \dots]. \quad (12)$$

the combinations on the columns and the previous properties (9), (10) to get the series of equalities as already used in [10, 7, 16]

$$\begin{aligned}
W_{m-1}^m &= W(\sin \alpha(x - x_0), \sin 2\alpha(x - x_0), \dots, \sin(m-1)\alpha(x - x_0)) \\
&= 2^{\frac{(m-1)(m-2)}{2}} W(\sin \alpha(x - x_0), \sin \alpha(x - x_0) \cos \alpha(x - x_0), \dots, \\
&\quad \dots, \sin \alpha(x - x_0) \cos^{m-2} \alpha(x - x_0)) \\
&= 2^{\frac{(m-1)(m-2)}{2}} (\sin \alpha(x - x_0))^{m-1} W(1, \cos \alpha(x - x_0), \dots, \cos^{m-2} \alpha(x - x_0)) \\
&= 2^{\frac{(m-1)(m-2)}{2}} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \sin^{\frac{m(m-1)}{2}} \alpha(x - x_0) W(1, t, \dots, t^{m-2}) \\
&\quad \text{(by the change } \cos \alpha(x - x_0) = t); \text{ thus} \\
W_{m-1}^m &= 2^{\frac{(m-1)(m-2)}{2}} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \prod_{j=1}^{m-2} j! \sin^{\frac{m(m-1)}{2}} \alpha(x - x_0).
\end{aligned}$$

So we get (11) for the case $i = m$.

2. For W_{m-1}^{m-1} , we use the same ideas that these given in the case $i = m$ and previous works [10, 7, 16]. We use the expansion 12, the combinations on the columns and the previous properties (9), (10) to get the following equalities :

$$\begin{aligned}
W_{m-1}^{m-1} &= W(\sin \alpha(x - x_0), \sin 2\alpha(x - x_0), \dots, \sin(m-2)\alpha(x - x_0), \sin m\alpha(x - x_0)), \\
&= 2^{\frac{m(m-1)}{2} - (m-2)} W(\sin \alpha(x - x_0), \sin \alpha(x - x_0) \cos \alpha(x - x_0), \dots, \\
&\quad \dots, \sin \alpha(x - x_0) \cos^{m-3} \alpha(x - x_0), \sin \alpha(x - x_0) \cos^{m-1} \alpha(x - x_0)) \\
&= 2^{\frac{m(m-1)}{2} - (m-2)} \sin^{m-1} \alpha(x - x_0) W(1, \cos \alpha(x - x_0), \dots, \\
&\quad \dots, \cos^{m-3} \alpha(x - x_0), \cos^{m-1} \alpha(x - x_0)). \\
&= 2^{\frac{m(m-1)}{2} - (m-2)} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \sin^{\frac{m(m-1)}{2}} (\alpha(x - x_0) W(1, t, \dots, t^{m-3}, t^{m-1}) \\
&\quad \text{(by the change } \cos \alpha(x - x_0) = t); \text{ thus} \\
W_{m-1}^{m-1} &= 2^{\frac{m(m-1)}{2} - (m-2)} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \prod_{j=1, j \neq m-2}^{m-1} j! \sin^{\frac{m(m-1)}{2}} \alpha(x - x_0) \cos \alpha(x - x_0).
\end{aligned}$$

This gives exactly formula (11) for $i = m - 1$.

3. Let i be an integer such that $1 \leq i \leq m - 2$.

First we use the expansion

$$\begin{aligned}
\sin px &= \sin x \sum_{k=0}^{\lfloor \frac{p-1}{2} \rfloor} (-1)^k C_{p-k-1}^k \cdot (2 \cos x)^{p-1-2k} = \\
\sin x [(2 \cos x)^{p-1} - (p-2)(2 \cos x)^{p-3} + \dots].
\end{aligned} \tag{13}$$

Therefore

$$\begin{aligned}
W_{m-1}^i &= W(\sin \alpha(x - x_0), \sin 2\alpha(x - x_0), \dots, \\
&\quad \sin(i-1)\alpha(x - x_0), \sin(i+1)\alpha(x - x_0), \dots, \sin m\alpha(x - x_0)),
\end{aligned}$$

can be reduced by replacing each term in the expansion of $\sin p\alpha(x - x_0)$, formula (13) by their first terms for $1 \leq p \leq i$, and p of the form $i + 2k$,

$1 \leq k \leq [(m-i)/2]$.

We can factorize in each column p the term 2^{p-1} for $1 \leq p \leq i-1$.

So we get

$$\begin{aligned} W_{m-1}^i &= 2^{\frac{(i-1)(i-2)}{2}} W(\sin \alpha(x-x_0), \sin \alpha(x-x_0) \cos \alpha(x-x_0), \dots, \\ &\sin \alpha(x-x_0) \cos^{i-2} \alpha(x-x_0), 2^i \sin \alpha(x-x_0) \cos^i \alpha(x-x_0), \\ &2^{i+1} \sin \alpha(x-x_0) \cos^{i+1} \alpha(x-x_0) - i2^{i-1} \sin \alpha(x-x_0) \cos^{i-1} \alpha(x-x_0), \\ &\dots, 2^{m-1} \sin \alpha(x-x_0) \cos^{m-1} \alpha(x-x_0) + \dots). \end{aligned} \quad (14)$$

Using the previous propositions 10, 10, then W_{m-1}^i can be written

$$\begin{aligned} W_{m-1}^i &= 2^{\frac{(i-1)(i-2)}{2}} (\sin \alpha(x-x_0))^{m-1} W(1, \cos \alpha(x-x_0), \dots, \\ &\cos^{i-2} \alpha(x-x_0), 2^i \cos^i \alpha(x-x_0), 2^{i+1} \cos^{i+1} \alpha(x-x_0) - i2^{i-1} \cos^{i-1} \alpha(x-x_0), (15) \\ &\dots, 2^{m-1} \cos^{m-1} \alpha(x-x_0) + \dots). \end{aligned}$$

Then, we make the following variable change defined by $t = \cos \alpha(x-x_0)$ is made and we can factorize $(-\alpha \sin \alpha(x-x_0))^{\frac{(m-1)(m-2)}{2}}$.

Therefore the Wronskian W_{m-1}^i can be written as

$$\begin{aligned} W_{m-1}^i &= 2^{\frac{(i-1)(i-2)}{2}} (-\alpha)^{\frac{(m-1)(m-2)}{2}} (\sin \alpha x)^{\frac{m(m-1)}{2}} \\ &\cdot \tilde{W}_{m-1}^i(1, t, \dots, t^{i-2}, 2^i t^i, 2^{i+1} t^{i+1} - i2^{i-1} t^{i-1}, 2^{i+2} t^{i+2}, \\ &2^{i+3} t^{i+3} - (i+2)2^{i+1} t^{i+1} + \frac{i(i+1)}{2} 2^{i-1} t^{i-1}, \\ &\dots, 2^{m-1} t^{m-1} - (m-2)2^{m-3} t^{m-3} + \dots + (-1)^{\frac{m-1}{2}} 2^{i-1} C_{(m+i)/2-1}^{(m-i)/2} t^{i-1}). \end{aligned} \quad (16)$$

If $m-i-1 = 2p$, we can factorize $2^i 2^{i+2} \dots 2^{m-1} (2^{i-1})^{\frac{(m-i-1)}{2}}$ which is equal to $2^{\frac{(m-i+1)(m+i-1)}{4} + \frac{(m-i-1)(i-1)}{2}}$.

If $m-i-1 = 2p+1$, we can factorize $2^i 2^{i+2} \dots 2^{m-2} (2^{i-1})^{\frac{(m-i)}{2}}$ which is equal to $2^{\frac{(m-i)(m+i-2)}{4} + \frac{(m-i)(i-1)}{2}}$.

So, the Wronskian $\tilde{W}_{m-1}^i := W(1, t, \dots, t^{i-2}, 2^i t^i, 2^{i+1} t^{i+1} - i2^{i-1} t^{i-1}, \dots, 2^{m-1} t^{m-1} - (m-2)2^{m-3} t^{m-3} + \dots)$ can be written as

$$\begin{aligned} \tilde{W}_{m-1}^i &= \prod_{j=1}^{i-2} j! 2^{\frac{(m-i+1)(m+i-1)}{4} + \frac{(m-i-1)(i-1)}{2}} \tilde{w}_{m-1,e}^i, \text{ if } m-i-1 = 2p \text{ is even,} \\ \tilde{W}_{m-1}^i &= \prod_{j=1}^{i-2} j! 2^{\frac{(m-i)(m+i-2)}{4} + \frac{(m-i)(i-1)}{2}} \tilde{w}_{m-1,o}^i, \text{ if } m-i-1 = 2p+1 \text{ is odd} \end{aligned} \quad (17)$$

where $\tilde{w}_{m-1,o}^i$ is the determinant if $m - i - 1 = 2p + 1$ is odd defined by,

$$\begin{vmatrix} i!t & 2^2(i+1)!t^2/2! - i(i-1)! & \dots & (m-2)! \frac{t^{m-i-1}}{(m-i-1)!} & 2^{m-i}(m-1)! \frac{t^{m-i}}{(m-i)!} + \dots + (-1)^{(m-i)/2} C_{(m+i)/2-1}^{(m-i)/2} 2^{i-1}(i-1)! \\ i! & 2^2(i+1)!t & \dots & (m-2)! \frac{t^{m-i-2}}{(m-i-2)!} & 2^{m-i}(m-1)! \frac{t^{m-i-1}}{(m-i-1)!} + \dots + (-1)^{(m-i)/2+1} C_{(m+i)/2}^{(m-i)/2+1} 2^{i+1}(i+1)!t \\ 0 & 2^2(i+1)! & \dots & (m-2)! \frac{t^{m-i-3}}{(m-i-3)!} & 2^{m-i}(m-1)! \frac{t^{m-i-2}}{(m-i-2)!} + \dots + (-1)^{(m-i)/2+1} C_{(m+i)/2}^{(m-i)/2+1} 2^{i+1}(i+1)! \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & (m-2)! & 2^{m-i}(m-1)!t \end{vmatrix} \quad (18)$$

and $\tilde{w}_{m-1,e}^i$ is the determinant if $m - i - 1 = 2p$ is even, defined by,

$$\begin{vmatrix} i!t & 2^2(i+1)!t^2/2! - i(i-1)! & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-1}}{(m-i-1)!} + \dots + (-1)^{(m-i-1)/2} C_{(m+i-1)/2}^{(m-i-1)/2} 2^{i-1}(i-1)! & (m-1)! \frac{t^{m-i}}{(m-i)!} \\ i! & 2^2(i+1)!t & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-2}}{(m-i-2)!} + \dots + (-1)^{(m-i-1)/2+1} C_{(m+i-1)/2+1}^{(m-i-1)/2+1} 2^{i+1}(i+1)!t & (m-1)! \frac{t^{m-i-1}}{(m-i-1)!} \\ 0 & 2^2(i+1)! & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-3}}{(m-i-3)!} + \dots + (-1)^{(m-i-1)/2+1} C_{(m+i-1)/2+1}^{(m-i-1)/2+1} 2^{i+1}(i+1)! & (m-1)! \frac{t^{m-i-2}}{(m-i-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2^{m-i-1}(m-2)! & (m-1)!t \end{vmatrix} \quad (19)$$

In the following, we calculate $\tilde{w}_{m-1,o}^i$ and $\tilde{w}_{m-1,e}^i$.

All the columns of the determinant $\tilde{w}_{m-1,o}^i$ and $\tilde{w}_{m-1,e}^i$ of the form C_{i+2k} can be written as

$$C_{i+2k}^* = {}^t((i+2k)! \frac{t^{2k+1}}{(2k+1)!}; (i+2k)! \frac{t^{2k}}{(2k)!}; \dots; (i+2k)!; 0; \dots; 0) \quad (20)$$

It remains to reduce the columns of the determinant $\tilde{w}_{m-1,o}^i$ and $\tilde{w}_{m-1,e}^i$ of the form C_{i+2k+1} .

a. We begin with the case $m - i - 1 = 2p$.

We consider the column

$$C_{i+1}^* = {}^t(2^2(i+1)! \frac{t^2}{2!} - i!; 2^2(i+1)!t; 2^2(i+1)!; 0; \dots; 0)$$

On each column C_{i+2k+1} of \tilde{w}_{m-1}^i , $1 \leq k \leq (m - i - 1)/2$, we make the following transformations

$$\begin{aligned} C_{i+3} &\rightarrow C_{i+3} + (i+2)C_{i+1}^* = C_{i+3}^* \\ &= {}^t(2^4(i+3)! \frac{t^4}{4!} - \frac{i!(i+3)}{2!}; 2^4(i+3)! \frac{t^3}{3!}; 2^4(i+3)! \frac{t^2}{2!}; 2^4(i+3)!t; 2^4(i+3)!; 0; \\ &\dots; 0) \end{aligned} \quad (21)$$

$$\begin{aligned}
C_{i+5} &\rightarrow C_{i+5} + (i+4)C_{i+3}^* - \frac{(i+3)(i+2)}{2}C_{i+1}^* = C_{i+5}^* \\
&= {}^t(2^6(i+5)! \frac{t^6}{6!} - \frac{i!(i+4)(i+5)}{3!}; 2^6(i+5)! \frac{t^5}{5!}; 2^6(i+5)! \frac{t^4}{4!}; 2^6(i+5)! \frac{t^3}{3!} \\
&\quad ; 2^6(i+5)! \frac{t^2}{2!}; 2^6(i+5)!t; 2^6(i+5)!; 0; \dots; 0)
\end{aligned} \tag{22}$$

until

$$\begin{aligned}
C_{i+2k+1} &\rightarrow C_{i+2k+1} + (i+2k)C_{i+2k-1}^* - \frac{(i+2k-1)(i+2k-2)}{2!}C_{i+2k-3}^* + \dots \\
&\dots + (-1)^{k+1} \frac{(i+k+1)\dots(i+2)}{k!}C_{i+1}^* = C_{i+2k+1}^*
\end{aligned} \tag{23}$$

We claim that

$$\begin{aligned}
C_{i+2k+1}^* &= {}^t(2^{2k+2}(i+2k+1)! \frac{t^{2k+2}}{(2k+2)!} - \frac{i!(i+2k+1)\dots(i+k+2)}{(k+1)!}; \\
&2^{2k+2}(i+2k+1)! \frac{t^{2k+1}}{(2k+1)!}; \dots; 2^{2k+2}(i+2k+1)!; 0; \dots; 0).
\end{aligned} \tag{24}$$

Suppose that the column C_{i+2i+1}^* can be written as in (24) for $i \leq k$. We have clearly the relation for $k=1$ (21) and $k=2$ (22).

Then we make the following transformation on the column C_{i+2k+3} :

$$\begin{aligned}
C_{i+2k+3} &\rightarrow C_{i+2k+3} + (i+2k+2)C_{i+2k+1}^* - \frac{(i+2k+1)(i+2k)}{2!}C_{i+2k-1}^* \\
&+ \dots + (-1)^{k+2} \frac{(i+k+2)\dots(i+2)}{(k+1)!}C_{i+1}^* = C_{i+2k+3}^*.
\end{aligned}$$

Clearly, all the rows from 2 to $m-1$ of the column C_{i+2k+3}^* take the form

$$(2^{k+4}(i+2k+3)! \frac{t^{2k+3}}{(2k+3)!}, 2^{k+4}(i+2k+3)! \frac{t^{2k+2}}{(2k+2)!}, \dots, 2^{k+4}(i+2k+3)!, 0, \dots, 0). \tag{25}$$

It remains to evaluate the first row of the column C_{i+2k+3} .

In a shortened way, the first row can be written as

$$(C_{i+2k+3}^*)_1 = -i! \sum_{j=1}^{k+2} \frac{(-1)^{j+1}}{i+k-2-j} C_{i+2k+3-j}^j C_{i+2k+3-2j}^{k+2-j}.$$

By reduction, it can be written as

$$(C_{i+2k+3}^*)_1 = -i! \frac{C_{i+2k+3}^{k+2}}{i+k+2} = -i! \frac{C_{i+2k+3}^{k+1}}{k+2} = -i! \frac{\prod_{j=0}^k (i+2k+3-j)}{(k+2)!}. \tag{26}$$

The relations (25) and (26) gives exactly the relation (24) at the rank $k+1$.

Then we can write the determinant $\tilde{w}_{m-1,e}^i$ defined in (18) as the following sum $\tilde{w}_{m-1,e}^{i,*} + \tilde{w}_{m-1,e}^{i,**}$ where $\tilde{w}_{m-1,e}^{i,*}$ is defined by

$$\tilde{w}_{m-1,e}^{i,*} = \begin{vmatrix} i!t & 2^2(i+1)! \frac{t^2}{2!} & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-1}}{(m-i-1)!} & (m-1)! \frac{t^{m-i}}{(m-i)!} \\ i! & 2^2(i+1)!t & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-2}}{(m-i-2)!} & (m-1)! \frac{t^{m-i-1}}{(m-i-1)!} \\ 0 & 2^2(i+1)! & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-3}}{(m-i-3)!} & (m-1)! \frac{t^{m-i-2}}{(m-i-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2^{m-i-1}(m-2)! & (m-1)!t \end{vmatrix}.$$

$\tilde{w}_{m-1,e}^{i,**}$ is equal to

$$\tilde{w}_{m-1,e}^{i,**} = \begin{vmatrix} 0 & -(i)! & 0 & \dots & 0 & -(i)! \frac{C_{i+2k+1}^k}{(k+1)} & 0 & -(i)! \frac{C_{m-2}^{(m-i-3)/2}}{(m-i-1)/2} & 0 \\ i! & 2^2(i+1)!t & (i+2)! \frac{t^2}{2!} & \dots & \dots & 2^{2k+2}(i+2k+1)! \frac{t^{2k+1}}{(2k+1)!} & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-2}}{(m-i-2)!} & (m-1)! \frac{t^{m-i-1}}{(m-i-1)!} \\ 0 & 2^2(i+1)! & (i+2)!t & \dots & \dots & 2^{2k+2}(i+2k+1)! \frac{t^{2k}}{(2k)!} & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-3}}{(m-i-3)!} & (m-1)! \frac{t^{m-i-2}}{(m-i-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2^{m-i-1}(m-2)! & (m-1)!t \end{vmatrix} \quad (27)$$

Clearly, we have

$$\tilde{w}_{m-1,e}^{i,*} = \prod_{j=1}^{m-1} j! \frac{t^{m-i}}{(m-i)!} 2^{\frac{(m-i-1)(m-i+1)}{4}}. \quad (29)$$

For $\tilde{w}_{m-1,e}^{i,**}$, we have

$$\tilde{w}_{m-1,e}^{i,**} = i! \sum_{k=0}^{(m-i-3)/2} \frac{C_{i+2k+1}^k}{(k+i)} \delta_{i+2k+1}.$$

where δ_{i+2k+1} is the determinant defined by

$$\delta_{i+2k+1} = \begin{vmatrix} i! & 2^2(i+1)!t & \dots & (i+2k)! \frac{t^{2k}}{(2k)!} & (i+2k+2)! \frac{t^{2k+2}}{(2k+2)!} & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-2}}{(m-i-2)!} & (m-1)! \frac{t^{m-i-1}}{(m-i-1)!} \\ 0 & 2^2(i+1)! & \dots & (i+2k)! \frac{t^{2k-1}}{(2k-1)!} & (i+2k+2)! \frac{t^{2k+1}}{(2k+1)!} & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-3}}{(m-i-3)!} & (m-1)! \frac{t^{m-i-2}}{(m-i-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2^{m-i-1}(m-2)! & (m-1)!t \end{vmatrix}.$$

δ_{i+2k+1} is equal to

$$\delta_{i+2k+1} = 2^{\frac{(m-i+1)(m-i-1)}{4} - (2k+2)} \prod_{j=i, j \neq i+2k+1}^{m-1} j! \frac{t^{m-i-2k-2}}{(m-i-2k-2)!}.$$

So $\tilde{w}_{m-1,e}^{i,**}$ can be written as

$$\tilde{w}_{m-1,e}^{i,**} = i! \sum_{k=0}^{(m-i-3)/2} \frac{C_{i+2k+1}^k}{(k+1)} 2^{\frac{(m-i+1)(m-i-1)}{4} - (2k+2)} \prod_{j=i, j \neq i+2k+1}^{m-1} j! \frac{t^{m-i-2k-2}}{(m-i-2k-2)!}. \quad (30)$$

Taking into account (29) and (30), with (17) we have for W

$$\begin{aligned} \tilde{W}_{m-1}^i &= \prod_{j=1}^{i-2} j! 2^{\frac{(m-i+1)(m+i-1)}{4} + \frac{(m-i-1)(i-1)}{2}} \left(\prod_{j=i}^{m-1} j! \frac{t^{m-i}}{(m-i)!} 2^{\frac{(m-i-1)(m-i+1)}{4}} \right. \\ &\quad \left. + i! \sum_{k=0}^{(m-i-3)/2} \frac{C_{i+2k+1}^k}{(k+1)} 2^{\frac{(m-i+1)(m-i-1)}{4} - (2k+2)} \prod_{j=i, j \neq 2k+1}^{m-1} j! \frac{t^{m-i-2k-2}}{(m-i-2k-2)!} \right) \end{aligned} \quad (31)$$

Thus, we have

if $m-i-1 = 2p$,

$$\begin{aligned} W_{m-1}^i &= (2)^{\frac{(m-1)(m-i+1)}{2} + \frac{(i-1)(m-3)}{2}} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \sin \frac{m(m-1)}{2} \alpha(x-x_0) \prod_{j=0}^{i-2} j! \\ &\cdot \left(\prod_{j=i}^{m-1} j! \frac{\cos^{m-i} \alpha(x-x_0)}{(m-i)!} + i! \cdot \sum_{k=0}^{(m-i-3)/2} \frac{C_{i+2k+1}^k}{k+1} 2^{-2k-2} \cdot \prod_{j=i, j \neq i+2k+1}^{m-1} j! \frac{\cos^{m-i-2k-2} \alpha(x-x_0)}{(m-i-2k-2)!} \right), \end{aligned} \quad (32)$$

Taking into account that $\frac{m(m-1)}{2} + \frac{(i-1)(m-2)}{2} = \frac{m(m-1)}{2} - (i-1)$, and $\frac{m-i-3}{2} = \lfloor \frac{m-i-2}{2} \rfloor$, we have :

$$\begin{aligned} W_{m-1}^i &= (2)^{\frac{m(m-1)}{2} - (i-1)} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \sin \frac{m(m-1)}{2} \alpha(x-x_0) \\ &\cdot i \prod_{j=1}^{m-1} j! \left(\frac{\cos^{m-i} \alpha(x-x_0)}{i!(m-i)!} + \sum_{k=0}^{\lfloor m-i-2 \rfloor / 2} \frac{2^{-2(k+1)}}{(i+k+1)!(k+1)!} \frac{\cos^{m-i-2k-2} \alpha(x-x_0)}{(m-i-2k-2)!} \right), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} W_{m-1}^i &= (2)^{\frac{m(m-1)}{2} - (i-1)} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \sin \frac{m(m-1)}{2} \alpha(x-x_0) \\ &\cdot i \prod_{j=1}^{m-1} j! \left(\frac{\cos^{m-i} \alpha(x-x_0)}{i!(m-i)!} + \sum_{k=1}^{\lfloor m-i \rfloor / 2} \frac{2^{-2k}}{(i+k)!(k)!} \frac{\cos^{m-i-2k} \alpha(x-x_0)}{(m-i-2k)!} \right). \end{aligned}$$

It is easy to see that we get exactly (11).

b. We consider now the case $m-i-1 = 2p+1$.

The ideas are the same, but the structure of $\tilde{w}_{m-1,o}^i$ is different (the two last columns are different).

On each column C_{i+2k+1} of $\tilde{w}_{m-1,o}^i$, $1 \leq k \leq m-i-1$, we make the same transformations as defined in the case $m-i-1 = 2p+1$, i.e. (21), (22) and (23) for the column C_{i+2k+1} .

We have the same relation (24) whose proof is the same. Clearly, all the rows

from 1 to $m - 1$ of the column C_{i+2k+3}^* take the same form as in (25) and (26). Then we can write the determinant $\tilde{w}_{m-1,o}^i$ defined in (19) as the following sum $\tilde{w}_{m-1,o}^{i,*} + \tilde{w}_{m-1,o}^{i,**}$ where $\tilde{w}_{m-1,o}^{i,*}$ is defined by

$$\tilde{w}_{m-1,o}^{i,*} = \begin{vmatrix} i!t & 2^2(i+1)! \frac{t^2}{2!} & \dots & (m-2)! \frac{t^{m-i-1}}{(m-i-1)!} & 2^{m-i}(m-1)! \frac{t^{m-i}}{(m-i)!} \\ i! & 2^2(i+1)!t & \dots & (m-2)! \frac{t^{m-i-2}}{(m-i-2)!} & 2^{m-i}(m-1)! \frac{t^{m-i-1}}{(m-i-1)!} \\ 0 & 2^2(i+1)! & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-3}}{(m-i-3)!} & (m-1)! \frac{t^{m-i-2}}{(m-i-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & (m-2)! & 2^{m-i}(m-1)!t \end{vmatrix}. \quad (33)$$

$\tilde{w}_{m-1,o}^{i,**}$ is equal to

$$\tilde{w}_{m-1,o}^{i,**} = \begin{vmatrix} 0 & -(i)! & 0 & \dots & 0 & -(i)! \frac{C_{i+2k+1}^k}{(k+1)} & 0 & 0 & -(i)! \frac{C_{m-1}^{(m-i-2)/2}}{(m-i)/2} \\ i! & 2^2(i+1)!t & (i+2)! \frac{t^2}{2!} & \dots & \dots & 2^{2k+2}(i+2k+1)! \frac{t^{2k+1}}{(2k+1)!} & \dots & (m-2)! \frac{t^{m-i-2}}{(m-i-2)!} & 2^{m-i}(m-1)! \frac{t^{m-i-1}}{(m-i-1)!} \\ 0 & 2^2(i+1)! & (i+2)!t & \dots & \dots & 2^{2k+2}(i+2k+1)! \frac{t^{2k}}{(2k)!} & \dots & (m-2)! \frac{t^{m-i-3}}{(m-i-3)!} & 2^{m-i}(m-1)! \frac{t^{m-i-2}}{(m-i-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & (m-2)! & 2^{m-i}(m-1)!t \end{vmatrix} \quad (34)$$

We have

$$\tilde{w}_{m-1,o}^{i,*} = \prod_{j=1}^{m-1} j! \frac{t^{m-i}}{(m-i)!} 2^{\frac{(m-i)(m-i+2)}{4}}. \quad (35)$$

$\tilde{w}_{m-1,o}^{i,**}$ can be written as

$$\tilde{w}_{m-1,o}^{i,**} = i! \sum_{k=0}^{(m-i-2)/2} \frac{C_{i+2k+1}^k}{(k+1)} 2^{\frac{(m-i+2)(m-i)}{4} - (2k+2)} \prod_{j=i, j \neq i+2k+1}^{m-1} j! \frac{t^{m-i-2k-2}}{(m-i-2k-2)!}. \quad (36)$$

Taking into account (35) and (36), with (17) we have for W

$$W = \prod_{j=1}^{i-2} j! 2^{\frac{(m-i)(m+i-2)}{4} + \frac{(m-i)(i-1)}{2}} \left(\prod_{j=i}^{m-1} j! \frac{t^{m-i}}{(m-i)!} 2^{\frac{(m-i)(m-i+2)}{4}} + i! \sum_{k=0}^{(m-i-2)/2} \frac{C_{i+2k+1}^k}{(k+1)} 2^{\frac{(m-i+2)(m-i)}{4} - (2k+2)} \prod_{j=i, j \neq i+2k+1}^{m-1} j! \frac{t^{m-i-2k-2}}{(m-i-2k-2)!} \right) \quad (37)$$

Thus we are able to give the expression of W_i^{m-1}

$$\begin{aligned}
& \text{if } m - i - 1 = 2p + 1, \\
W_{m-1}^i &= (2)^{\frac{m(m-i)}{2} + \frac{(i-1)(m-2)}{2}} (-\alpha)^{\frac{(m-1)(m-2)}{2}} \sin^{\frac{m(m-1)}{2}} \alpha(x - x_0) \prod_{j=1}^{i-2} j! \\
& \cdot \left(\prod_{j=i}^{m-1} j! \frac{\cos^{m-i} \alpha(x-x_0)}{(m-i)!} + i! \cdot \sum_{k=0}^{(m-i-2)/2} \frac{C_{i+2k+1}^k}{k+1} 2^{-2k-2} \cdot \prod_{j=i, j \neq i+2k+1}^{m-1} j! \frac{\cos^{m-i-2k-2} \alpha(x-x_0)}{(m-i-2k-2)!} \right),
\end{aligned} \tag{38}$$

Taking into account that $\frac{m-i-2}{2} = [\frac{m-i-2}{2}]$, and after some simplifications like these done in the part a., we have the expression of W_i^{m-1} according (11).

□

Remark 2.1 *The possible extension of the same calculations to the sequence of the hyperbolic functions can be easily made by the change $\alpha \rightarrow i\alpha$.*

2.2 Second type of formulas

Let m some non negative integer, α and x_0 be some real parameters, and g_j the functions defined as follows :

$$g_j = \cos \alpha j(x - x_0), \quad 1 \leq j \leq m. \tag{39}$$

We use the standard notation (8) for the Wronskian determinant of any m functions.

We will denote \overline{W} the determinant $W(g_1, \dots, g_m)$.

With these notations we have the following formulas

Theorem 2.2

$$\overline{W} = 2^{\frac{m(m-1)}{2}} \prod_{k=1}^m k! (-\alpha)^{\frac{m(m-1)}{2}} \sin^{\frac{m(m-1)}{2}} \alpha(x - x_0) \times \left(\sum_{k=0}^{[m/2]} \frac{2^{-2k} \cos^{m-2k} \alpha(x - x_0)}{k!^2 (m-2k)!} \right). \tag{40}$$

Proof

First we use the expansion

$$\begin{aligned}
\cos px &= 2^{p-1} \cos^p x + \sum_{k=1}^{[\frac{p}{2}]} \frac{p}{k} (-1)^k C_{p-1-k}^{k-1} \cdot 2^{p-1-2k} \cos^{p-1-2k} x = \\
& 2^{p-1} \cos^p x - p 2^{p-3} \cos^{p-2} x + \dots
\end{aligned} \tag{41}$$

We replace each term $\cos p\alpha(x - x_0)$ by its expansion (formula 41).

Then we use the formula of change of independent variable (10) defined by

$t = \cos \alpha(x - x_0)$ and we can factorize $(-\alpha \sin \alpha(x - x_0))^{\frac{(m)(m-1)}{2}}$.
Then \overline{W} can be written

$$\overline{W} = (-\alpha \sin \alpha(x - x_0))^{\frac{m(m-1)}{2}} W(t, 2t^2 - 1, \dots, 2^{m-1}t^m - m 2^{m-3}t^{m-2} + \dots). \quad (42)$$

We denote in the following w the Wronskian

$$w := W(t, 2t^2 - 1, \dots, 2^{m-1}t^m - m 2^{m-3}t^{m-2} + \dots). \quad (43)$$

In the following, we calculate w .

We are going to define the succession of transformations on the columns in the following .

All the odd columns of the form C_{2k+1} can be easily reduced to

$$C_{2k+1}^* = {}^t (2^{2k}(2k+1)! \frac{t^{2k+1}}{(2k+1)!}; 2^{2k}(2k+1)! \frac{t^{2k}}{(2k)!}; \dots; 2^{2k}(2k+1)!; 0; \dots; 0) \quad (44)$$

It remains to reduce the even columns of the form C_{2k+2} .

We consider the second column

$$C_2^* = {}^t (2 \cdot 2! \frac{t^2}{2!} - 1; 2 \cdot 2!t; 2 \cdot 2!; 0; \dots; 0). \quad (45)$$

On each column C_{2k} of w , $k \geq 2$, we make the following transformations

$$\begin{aligned} C_4 &\rightarrow C_4 + 4C_2^* = C_4^* \\ &= {}^t (2^3 \cdot 4! \frac{t^4}{4!} - 3; 2^3 \cdot 4! \frac{t^3}{3!}; 2^3 \cdot 4! \frac{t^2}{2!} 2^3 \cdot 4!t; 2^3 \cdot 4!; 0; \dots; 0) \end{aligned} \quad (46)$$

$$\begin{aligned} C_6 &\rightarrow C_6 + 6C_4^* - \frac{6 \times 3}{2} C_2^* = C_6^* \\ &= {}^t (2^5 \cdot 6! \frac{t^6}{6!} - 10; 2^5 \cdot 6! \frac{t^5}{5!}; 2^5 \cdot 6! \frac{t^4}{4!} 2^5 \cdot 6! \frac{t^3}{3} \\ &\quad ; 2^5 \cdot 6! \frac{t^2}{2!}; 2^5 \cdot 6!t; 2^5 \cdot 6!; 0; \dots; 0) \end{aligned} \quad (47)$$

until

$$\begin{aligned} C_{2k} &\rightarrow C_{2k} + (2k)C_{2k-2}^* - \frac{(2k)(2k-3)}{2!} C_{2k-4}^* + \dots \\ &\dots + (-1)^k \frac{2k}{k-1} \frac{(2k-1-(k-1)) \dots (2k-1-(k-1+k-1-2))}{(k-2)!} C_2^* = C_{2k}^* \end{aligned} \quad (48)$$

which can be rewritten as

$$\begin{aligned} C_{2k} &\rightarrow C_{2k} + (2k)C_{2k-2}^* - \frac{(2k)(2k-3)}{2!} C_{2k-4}^* + \dots \\ &\dots + (-1)^k k^2 C_2^* = C_{2k}^*. \end{aligned} \quad (49)$$

We claim that

$$\begin{aligned} C_{2k}^* &= {}^t (2^{2k-1}(2k)! \frac{t^{2k}}{(2k)!} - C_{2k-1}^{k-1}; \\ &2^{2k-1}(2k)! \frac{t^{2k-1}}{(2k-1)!}; \dots; 2^{2k-1}(2k)!; 0; \dots; 0). \end{aligned} \quad (50)$$

Suppose that the column C_{2i}^* can be written as in (24) for $i \leq k$.

We have clearly the relation for $k = 1$ (45), $k = 2$ (46) and $k = 3$ (47).

Then we make the following transformation on the column C_{2k+2} :

$$\begin{aligned} C_{2k+2} &\rightarrow C_{2k+2} + (2k+2)C_{2k}^* - \frac{(2k+2)(2k-1)}{2!}C_{2k-2}^* \\ &+ \dots + (-1)^{k+1}(k+1)^2C_2^* = C_{2k+2}^*. \end{aligned}$$

Clearly, all the rows from 2 to m of the column C_{2k+2}^* take the form

$$(2^{2k+1}(2k+2)! \frac{t^{2k+1}}{(2k+1)!}, (2^{2k+1}(2k+2)! \frac{t^{2k}}{(2k)!}, \dots, (2^{2k+1}(2k+2)!; 0, \dots, 0). \quad (51)$$

It remains to evaluate the first row of the column C_{2k+2}^* .

In a shortened way, the first row can be written as

$$(C_{2k+2}^*)_1 = (-1)^{k+1} + (2k+2) \sum_{j=1}^k \frac{(-1)^j}{j} C_{2k+1-j}^{j-1} C_{2k+1-2j}^{k-j}.$$

By reduction, it can be written as

$$(C_{2k+2}^*)_1 = -C_{2k+1}^k = \prod_{j=0}^{k-1} \frac{(2k+1-l)}{(k-l)}. \quad (52)$$

The relations (51) and (52) gives exactly the relation (50) at the rank $k+1$.

Then we can write the determinant w defined in (43) as the following sum $w_1 + w_2$ where w_1 is defined by depending on m :

a. If $m = 2q$ we have for w_1

$$w_1 = \begin{vmatrix} t & 2 \cdot 2! \frac{t^2}{2!} & \dots & 2^{m-1} m! \frac{t^m}{m!} \\ 1 & 2 \cdot t & \dots & 2^{m-1} m! \frac{t^{m-1}}{(m-1)!} \\ 1 & 2 & \dots & 2^{m-1} m! \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2^{m-1} m! t \end{vmatrix}.$$

w_2 is equal to

$$(53)$$

$$w_2 = \begin{vmatrix} 0 & -1 & 0 & \dots & 0 & -C_{2k-1}^k & 0 & \dots & -C_{m-1}^{m/2} \\ 1 & 2 \cdot t & 2^2(2)! \frac{t^2}{2!} & \dots & \dots & 2^{2j-1}(2j)! \frac{t^{2j-1}}{(2j-1)!} & \dots & \dots & 2^{m-1}(m)! \frac{t^{m-1}}{(m-1)!} \\ 0 & 2 & 2^2 t & \dots & \dots & 2^{2k+2}(i+2k+1)! \frac{t^{2k}}{(2k)!} & \dots & 2^{m-i-1}(m-2)! \frac{t^{m-i-3}}{(m-i-3)!} & (m-1)! \frac{t^{m-i-2}}{(m-i-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2^{m-2}(m-1)! & 2^{m-1} m! t \end{vmatrix} \quad (54)$$

Then we have

$$w_1 = \prod_{j=1}^m j! \frac{t^m}{m!} 2^{\frac{m(m-1)}{2}}. \quad (55)$$

For w_2 , we have

$$w_2 = \sum_{j=1}^{m/2} C_{2j-1}^{j-1} \delta_{2j},$$

where δ_{2j} is the determinant defined by

$$\delta_{2j} = \begin{vmatrix} 1 & 2t & 2^2 \cdot 3! \frac{t^2}{2!} & \dots & 2^{2j-2}(2j-1)! \frac{t^{2j-2}}{(2j-2)!} & 2^{2j}(2j+1)! \frac{t^{2j}}{(2j)!} & \dots & 2^{m-1} m! \frac{t^{m-1}}{(m-1)!} \\ 0 & 2 & 2^2 \cdot 3! t & \dots & 2^{2j-2}(2j-1)! \frac{t^{2j-3}}{(2j-3)!} & 2^{2j}(2j+1)! \frac{t^{2j-1}}{(2j-1)!} & \dots & 2^{m-1} m! \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2^{m-2}(m-1)! & 2^{m-1} m! t \end{vmatrix}.$$

δ_{2j} is equal to

$$\delta_{2j} = 2^{\frac{m(m-1)}{2} - 2j + 1} \prod_{k=1, k \neq 2j}^m k! \frac{t^{m-2j}}{(m-2j)!}.$$

So w_2 can be written as

$$w_2 = \sum_{j=1}^{m/2} C_{2j-1}^{j-1} 2^{\frac{m(m-1)}{2} - 2j + 1} \prod_{k=1, k \neq 2j}^m k! \frac{t^{m-2j}}{(m-2j)!}. \quad (56)$$

Taking into account (55), (56) and $[m/2] = m/2$, with (42) we have :

$$\overline{W} = 2^{\frac{m(m-1)}{2}} \prod_{k=i}^m k! (-\alpha)^{\frac{m(m-1)}{2}} \sin^{\frac{m(m-1)}{2}} \alpha(x-x_0) \cdot \left(\frac{\cos^m \alpha(x-x_0)}{m!} + \sum_{j=1}^{[m/2]} \frac{(2j-1)! 2^{-2j+1} \cos^{m-2j} \alpha(x-x_0)}{(j-1)! j! (2j)! (m-2j)!} \right),$$

which can be rewritten as

$$\overline{W} = 2^{\frac{m(m-1)}{2}} \prod_{k=i}^m k! (-\alpha)^{\frac{m(m-1)}{2}} \sin^{\frac{m(m-1)}{2}} \alpha(x-x_0) \cdot \left(\frac{\cos^m \alpha(x-x_0)}{m!} + \sum_{j=1}^{[m/2]} \frac{2^{-2j} \cos^{m-2j} \alpha(x-x_0)}{j!^2 (m-2j)!} \right)$$

This gives for \overline{W} exactly the formula (40).

b. We consider now the case $m = 2q + 1$.

The ideas are the same, but the structure of w is different (the last column is different).

We can write the determinant w defined in (43) as the following sum $w_1 + w_2$ where w_1 is defined by

$$w_1 = \begin{vmatrix} t & 2 \cdot 2! \frac{t^2}{2!} & \dots & 2^{m-1} m! \frac{t^m}{m!} \\ 1 & 2 \cdot t & \dots & 2^{m-1} m! \frac{t^{m-1}}{(m-1)!} \\ 1 & 2 & \dots & 2^{m-1} m! \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2^{m-1} m! t \end{vmatrix}.$$

w_2 is equal to

$$w_2 = \begin{vmatrix} 0 & -1 & 0 & \dots & 0 & -C_{2k-1}^{m-2} & \dots & -C_{m-2}^{(m-1)/2} & 0 \\ 1 & 2 \cdot t & 2^2 (2)! \frac{t^2}{2!} & \dots & \dots & 2^{2j-1} (2j)! \frac{t^{2j-1}}{(2j-1)!} & \dots & \dots & 2^{m-1} (m)! \frac{t^{m-1}}{(m-1)!} \\ 0 & 2 & 2^2 t & \dots & \dots & 2^{2k+2} (i+2k+1)! \frac{t^{2k}}{(2k)!} & \dots & 2^{m-i-1} (m-2)! \frac{t^{m-i-3}}{(m-i-3)!} & (m-1)! \frac{t^{m-i-2}}{(m-i-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2^{m-2} (m-1)! & 2^{m-1} m! t \end{vmatrix} \quad (57)$$

Then we have

$$w_1 = \prod_{j=1}^m j! \frac{t^m}{m!} 2^{\frac{m(m-1)}{2}}. \quad (59)$$

For w_2 , we have

$$w_2 = \sum_{j=1}^{(m-1)/2} C_{2j-1}^{j-1} \delta_{2j},$$

where δ_{2j} is the determinant defined by

$$\delta_{2j} = \begin{vmatrix} 1 & 2t & 2^2 \cdot 3! \frac{t^2}{2!} & \dots & 2^{2j-2} (2j-1)! \frac{t^{2j-2}}{(2j-2)!} & 2^{2j} (2j+1)! \frac{t^{2j}}{(2j)!} & \dots & 2^{m-1} m! \frac{t^{m-1}}{(m-1)!} \\ 0 & 2 & 2^2 \cdot 3! t & \dots & 2^{2j-2} (2j-1)! \frac{t^{2j-3}}{(2j-3)!} & 2^{2j} (2j+1)! \frac{t^{2j-1}}{(2j-1)!} & \dots & 2^{m-1} m! \frac{t^{m-2}}{(m-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2^{m-2} (m-1)! & 2^{m-1} m! t \end{vmatrix}.$$

δ_{2j} is equal to

$$\delta_{2j} = 2^{\frac{m(m-1)}{2}-2j+1} \prod_{k=1, k \neq 2j}^m k! \frac{t^{m-2j}}{(m-2j)!}.$$

So w_2 can be written as

$$w_2 = \sum_{j=1}^{(m-1)/2} C_{2j-1}^{j-1} 2^{\frac{m(m-1)}{2}-2j+1} \prod_{k=1, k \neq 2j}^m k! \frac{t^{m-2j}}{(m-2j)!}. \quad (60)$$

Taking into account (59), (60), $[m/2] = (m-1)/2$, with (42) and some simplifications as these done in the part a., we have for \overline{W} exactly the formula (40).

So we get the result.

□

Remark 2.2 *As in the previous section, the possible extension of the same calculations to the sequence of the hyperbolic functions can be easily made by the change $\alpha \rightarrow i\alpha$.*

3 Hadamard's coefficients and solutions to the 2D-Schrödinger equation with Huygens' potentials

3.1 Explicit expressions of Hadamard's coefficients

We want to give a reduction of Hadamard's coefficients, given in terms of polar coordinates $x = (r \cos \varphi; r \sin \varphi)$, $\xi = (\rho \cos \phi; \rho \sin \phi)$ by (4)

$$U_\nu(x, \xi) = \frac{(-2)^\nu}{(r\rho)^\nu} \sum_{i=0}^m c_i \Psi_i(\varphi) \Psi_i(\phi) T_{k_i}^{(\nu)}(\cos(\varphi - \phi)), \text{ for } \nu \geq 0,$$

where

$$\Psi_i = \frac{W(F_0, \dots, F_{i-1}, F_{i+1}, \dots, F_m)}{W(F_0, \dots, F_m)},$$

$$c_i = \prod_{j=0, j \neq i}^m k_i^2 - k_j^2,$$

and $T_N(x) := \cos N(\arccos x)$.

Let F_i the functions defined by

$$\text{for } x = \varphi, \text{ or } x = \phi, \quad F_i(x) = \cos k_i(x), \text{ with } k_i = i, \text{ for } 0 \leq i \leq n \quad (61)$$

Then, using the same notations as before, we have the obvious relations, where the functions f_i are defined in (7) with $\alpha = 1$ and $x_0 = 0$.

$$\begin{aligned} W(F_0, F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_m) &= (-1)^{m-1} \frac{m!}{i} W(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m) \\ &= (-1)^{\frac{(m)(m-1)}{2}} \times 2^{\frac{m(m-1)}{2} - i + 1} \prod_{k=1}^m k! \times \sin^{\frac{m(m-1)}{2}} x \times \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k}}{(i+k)!k!} \frac{\cos^{m-i-2k} x}{(m-i-2k)!} \right). \end{aligned}$$

We have also,

$$\begin{aligned} W(F_1, \dots, F_m) &= W(g_1, \dots, g_m) = \overline{W} \\ &= 2^{\frac{m(m-1)}{2}} \prod_{k=1}^m k! (-1)^{\frac{m(m-1)}{2}} \sin^{\frac{m(m-1)}{2}} x \times \left(\sum_{k=0}^{[m/2]} \frac{2^{-2k}}{k!^2} \frac{\cos^{m-2k} x}{(m-2k)!} \right) \end{aligned}$$

Moreover,

$$W(F_0, \dots, F_m) = (-1)^m m! W(f_1, \dots, f_m) = (-1)^m m! W_m.$$

But, we have from [16],

$$W_m = W(f_1, \dots, f_m) = (-2)^{\frac{m(m-1)}{2}} \prod_{k=1}^{m-1} k! \sin^{\frac{m(m+1)}{2}} x.$$

This gives the equality

$$W(F_0, \dots, F_m) = (-1)^m (-2)^{\frac{m(m-1)}{2}} \prod_{k=1}^m k! \sin^{\frac{m(m+1)}{2}} x.$$

Then for $1 \leq i \leq m$, we have for Ψ_i :

$$\Psi_i = -\frac{W_{m-1}^i}{iW_m} = (-1)^m (2)^{-i+1} \sin^{-m} x \cdot \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k}}{(i+k)!k!} \frac{\cos^{m-i-2k} x}{(m-i-2k)!} \right) \quad (62)$$

For Ψ_0 we get :

$$\Psi_0 = -\frac{\overline{W}}{(-1)^m m! W m} = (-1)^m \sin^{-m} x \cdot \left(\sum_{k=0}^{[m/2]} \frac{2^{-2k} \cos^{m-2k} x}{k!^2 (m-2k)!} \right). \quad (63)$$

We use also the following relations:

$$c_i = \frac{(-1)^{m-i} (m-i)! (m+i)!}{2},$$

It is obvious that for $0 \leq \nu \leq i$

$$\begin{aligned} T_{k_i}^{(\nu)}(\cos(\varphi - \phi)) &= T_i^{(\nu)}(\cos(\varphi - \phi)) \\ &= \sum_{p=0}^{[i/2]} \sum_{j=0, i-2p+2j-\nu \geq 0}^p (-1)^{p+j} \binom{i}{2p} \binom{p}{j} \prod_{l=0}^{\nu-1} (i-2p+2j-l) \cos^{i-2p+2j-\nu}(\varphi - \phi). \end{aligned}$$

$$T_{k_i}^{(\nu)} = 0 \text{ for } \nu > i, \text{ and } T_{k_0}^{(\nu)}(\cos(\varphi - \phi)) = T_0^{(\nu)}(\cos(\varphi - \phi)) = 1.$$

At this step, we can evaluate Hadamard's coefficients given in (4) :

Proposition 3.1 *With $x = (x_1, x_2) = (r \cos(\varphi), r \cos(\varphi))$, $\xi = (\xi_1, \xi_2) = (\rho \cos(\phi), \rho \cos(\phi))$, the Hadamard's coefficients can be written as*

$$\begin{aligned} U_0(x, \xi) &= \left\{ (-1)^m 2^{-1} (m!)^2 \sin^{-m} \varphi \sin^{-m} \phi \times \left(\sum_{k=0}^{[m/2]} \frac{2^{-2k} \cos^{m-2k} \varphi}{k!^2 (m-2k)!} \right) \right. \\ &\times \left(\sum_{k=0}^{[m/2]} \frac{2^{-2k} \cos^{m-2k} \phi}{k!^2 (m-2k)!} \right) + \sum_{i=1}^m (-1)^{m-i} 2^{-2i+1} (m-i)! (m+i)! \sin^{-m} \varphi \sin^{-m} \phi \\ &\times \sum_{p=0}^{[i/2]} \sum_{j=0, i-2p+2j-\nu \geq 0}^p (-1)^{p+j} \binom{i}{2p} \binom{p}{j} \cos^{i-2p+2j}(\varphi - \phi) \\ &\left. \times \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k} \cos^{m-i-2k} \varphi}{(i+k)! k! (m-i-2k)!} \right) \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k} \cos^{m-i-2k} \phi}{(i+k)! k! (m-i-2k)!} \right) \right\}. \end{aligned}$$

$$\begin{aligned}
U_\nu(x, \xi) &= \frac{(-2)^\nu}{(r\rho)^\nu} \left\{ \sum_{i=1}^m (-1)^{m-i} 2^{-2i+1} (m-i)! (m+i)! \sin^{-m} \varphi \sin^{-m} \phi \right. \\
&\times \sum_{p=0}^{\lfloor i/2 \rfloor} \sum_{j=0, i-2p+2j-\nu \geq 0}^p (-1)^{p+j} \binom{i}{2p} \binom{p}{j} \prod_{l=0}^{\nu-1} (i-2p+2j-l) \cos^{i-2p+2j-\nu}(\varphi - \phi) \\
&\times \left. \left(\sum_{k=0}^{\lfloor (m-i)/2 \rfloor} \frac{2^{-2k}}{(i+k)! k!} \frac{\cos^{m-i-2k} \varphi}{(m-i-2k)!} \right) \left(\sum_{k=0}^{\lfloor (m-i)/2 \rfloor} \frac{2^{-2k}}{(i+k)! k!} \frac{\cos^{m-i-2k} \phi}{(m-i-2k)!} \right) \right\}, \text{ for } \nu \geq 1.
\end{aligned}$$

3.2 Expression of the eigenfunctions of the 2-D Schrödinger operator with Huygens' potentials.

Following the results of Berest [4], the expressions of the eigenfunctions of the two dimensional Schrödinger operator with $x = (x_1, x_2) = (r \cos(\varphi), r \sin(\varphi))$, $\xi = (\xi_1, \xi_2) = (\rho \cos(\phi), \rho \sin(\phi))$ are given by the following result

Proposition 3.2 *The eigenfunctions to the two dimensional Schrödinger operator*

$$-\Delta + V$$

with

$$V = -\frac{2}{r^2} \partial_x^2 \ln(W(\psi_0, \dots, \psi_m))$$

and ψ defined by (61) can be written as

$$\Psi(x, \xi) = \left(\sum_{\nu=0}^m \frac{1}{2^\nu} U_\nu(x, \xi) \right) e^{r\rho \cos \varphi \cos \phi + r\rho \sin \varphi \sin \phi}$$

$$\begin{aligned}
&= e^{r\rho \cos \varphi \cos \phi + r\rho \sin \varphi \sin \phi} \left\{ (-1)^m 2^{-1} (m!)^2 \sin^{-m} \varphi \sin^{-m} \phi \times \left(\sum_{k=0}^{[m/2]} \frac{2^{-2k} \cos^{m-2k} \varphi}{k!^2 (m-2k)!} \right) \right. \\
&\quad \times \left(\sum_{k=0}^{[m/2]} \frac{2^{-2k} \cos^{m-2k} \phi}{k!^2 (m-2k)!} \right) + \sum_{i=1}^m (-1)^{m-i} 2^{-2i+1} (m-i)! (m+i)! \sin^{-m} \varphi \sin^{-m} \phi \\
&\quad \quad \quad \times \sum_{p=0}^{[i/2]} \sum_{j=0, i-2p+2j-\nu \geq 0}^p (-1)^{p+j} \binom{i}{2p} \binom{p}{j} \cos^{i-2p+2j}(\varphi - \phi) \\
&\quad \quad \quad \times \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k} \cos^{m-i-2k} \varphi}{(i+k)! k! (m-i-2k)!} \right) \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k} \cos^{m-i-2k} \phi}{(i+k)! k! (m-i-2k)!} \right) \\
&\quad \quad \quad + \sum_{\nu=1}^m \frac{(-1)^\nu}{(r\rho)^\nu} \sum_{i=1}^m (-1)^{m-i} 2^{-2i+1} (m-i)! (m+i)! \sin^{-m} \varphi \sin^{-m} \phi \\
&\quad \times \sum_{p=0}^{[i/2]} \sum_{j=0, i-2p+2j-\nu \geq 0}^p (-1)^{p+j} \binom{i}{2p} \binom{p}{j} \prod_{l=0}^{\nu-1} (i-2p+2j-l) \cos^{i-2p+2j-\nu}(\varphi - \phi) \\
&\quad \quad \quad \times \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k} \cos^{m-i-2k} \varphi}{(i+k)! k! (m-i-2k)!} \right) \left(\sum_{k=0}^{[(m-i)/2]} \frac{2^{-2k} \cos^{m-i-2k} \phi}{(i+k)! k! (m-i-2k)!} \right) \left. \right\}
\end{aligned}$$

4 Conclusion

We have constructed in this paper explicit expressions of Hadamard's coefficients by means of explicit expressions of some Wronskians.

Expressions of the Hadamard's coefficients for the Schrödinger operator with a matrix potential were given by algorithms in [1].

Other expressions of these coefficients were given in [70] in function of the tau functions of the KdV hierarchy. But these expressions are not explicit and practically unusable because they depend on a product of the sum of Schur polynomials by tau functions of the KdV equation which are more difficult to calculate.

As a byproduct, we give explicit expressions of the eigenfunctions of the two dimensional Schrödinger operator with Huygens' potentials.

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