

# Time spent close to an unstable point

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We consider the one dimensional dynamics of a particle in a potential  $V$  which has at least one unstable equilibrium position at  $x = x_c$ . The particle starts from a given position with an initial velocity which depends on a control parameter  $p$ . The potential shape also depends on  $p$ . For a critical value  $p_c$  of  $p$ , the particle reaches the unstable position of the potential. We show that the time spent close to the unstable point scales as  $-\log(|p - p_c|)$  when  $p \rightarrow p_c$  both in the stable case when the particle does not reach  $x_c$  and in the unstable case where the particle crosses  $x_c$ .

## I. NOTATIONS AND PHYSICAL CONFIGURATION

Let consider the following conservative dynamics:

$$\ddot{x} = f(x, p) = -\partial_x V(x, p) \quad (1)$$

where  $p$  is some control parameter. Dots represent time derivatives.  $V(x, p)$  is a potential which for some  $p$  has at least one unstable position in  $x_c$ . In the simplest case, this means that  $f(x_c, p) = 0$  and  $\partial_x f(x_c, p) > 0$ . The particle starts from  $x_0 < x_c$ , with a positive initial velocity which depends on  $p$ . When  $p < p_c$  the particle does not reach  $x_c$  and turns back at  $x = x_{\max}(p)$ . For  $p > p_c$ , the particle crosses  $x_c$ . We want to find how the time spent in the vicinity of  $x_c$  depends on  $p - p_c$ .

We set  $u = x_c - x$ ,  $\epsilon = p_c - p$ . At lowest order in  $u$  and  $\epsilon$ , equation (1) writes:

$$\ddot{u} = u\partial_x f|_c + \epsilon\partial_p f|_c \quad (2)$$

Since  $\partial_x f(x_c, p_c) = \partial_x f|_c > 0$ , in the following we write:  $\gamma = \sqrt{\partial_x f|_c}$ .

## II. STABLE CASE, $p < p_c$

Let first consider the case when the particle does not reach  $x_c$ . In this case, we know that  $x_c - x_{\max} = u_{\min} \propto \sqrt{p_c - p}$  (see the main text).

### A. Forcing independent of $p$

When  $f$  is independent of  $p$ , equation (2) simplifies to:

$$\ddot{u} = u\partial_x f|_c \quad (3)$$

whose solutions are of the form:

$$u(t) = u_0 e^{\gamma t} + u_1 e^{-\gamma t}$$

where  $u_0$  and  $u_1$  are some constants that depend on the initial conditions. Since we are considering a stable trajectory, there exists a point at which the particle turns around and moves away from  $x_c$ . To simplify notations, this time is taken as the origin of times. Then, we have

$$\begin{aligned} u(0) &= u_{\min} \\ \dot{u}(0) &= 0 \end{aligned}$$

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Which gives the solution of (3):

$$u(t) = \frac{u_{\min}}{2}(e^{\gamma t} + e^{-\gamma t}) = u_{\min} \cosh(\gamma t)$$

Let  $U$  be an arbitrary distance from  $x_c$  reached at  $t = T$ .  $U$  reads:

$$U = u_{\min} \cosh(\gamma T)$$

When  $p \rightarrow p_c$ , since  $u_{\min} \propto \sqrt{p_c - p}$  (showed in the main text),  $u_{\min} \rightarrow 0$  and so  $\cosh(\gamma T) \rightarrow \infty$ . In this limit:  $\cosh(\gamma T) \sim e^{\gamma T}$ . And so:

$$\gamma T \propto -\log(u_{\min}) \propto -\log(p_c - p) \quad (4)$$

As  $p$  gets closer and closer to  $p_c$ , the time spent close to  $x_c$  diverges as  $-\log(p_c - p)$ . Let's now consider the case where the potential also depends on the control parameter  $p$ .

### B. General expression

If now the potential also depends on  $p$ , we have to introduce a new variable  $v$  such that  $v = u + \epsilon \frac{\partial_p f|_c}{\partial_x f|_c} = u + \epsilon C$ .  $v$  is a solution of equation (3):

$$\ddot{v} = v \partial_x f|_c \quad (5)$$

with the following initial conditions:

$$\begin{aligned} v(0) &= v_{\min} = u_{\min} + \epsilon C \\ \dot{v}(0) &= 0 \end{aligned}$$

As previously, the solution reads:

$$v(t, p) = v_{\min} \cosh(\gamma t)$$

Using the definition of  $v$  and the fact that  $v_{\min} = u_{\min} + \epsilon C$  we get:

$$u(t, p) = u_{\min} \cosh(\gamma t) + \epsilon C \cosh(\gamma t) - \epsilon C$$

As before, we fix  $U$ , an arbitrary distance from  $x_c$ , reached at  $T$ .

$$U = u_{\min} \cosh(\gamma T) + \epsilon C \cosh(\gamma T) - \epsilon C$$

Since we still have  $u_{\min} \propto \sqrt{\epsilon}$  and so, at lowest order in  $\epsilon$ , when  $p \rightarrow p_c$ , since  $\gamma T \rightarrow \infty$ :

$$\gamma T \propto -\log u_{\min}$$

We get again the same divergence of  $T$ :  $\gamma T \propto -\log(p_c - p)$ .

### III. UNSTABLE CASE, $p > p_c$

When  $p > p_c$ , the particle crosses  $x_c$ . We then choose as the origin of times the moment where  $x = x_c$ , which corresponds to  $u = 0$  and to the minimum of  $\dot{u}$ , written  $\dot{u}_{\min}$ . We first look for the scaling of  $\dot{u}_{\min}$  with  $\epsilon$ .

By conservation of energy we get:

$$\frac{1}{2} \dot{x}^2 + V(x, p) = \frac{1}{2} \dot{x}_0^2$$

where  $V$  is redefined so that  $V(x_0) = 0$ . Let define  $g(p)$  such that  $\dot{x}_0 = \sqrt{2g(p)}$ . The previous equation rewrites:

$$\frac{1}{2} \dot{x}^2 = g(p) - V(x, p)$$

Since  $x_c$  is a maximum of  $V$ ,  $\dot{x}$  is minimum at  $x = x_c$  and reads:

$$\frac{1}{2}\dot{x}_{\min}^2 = g(p) - V(x_c, p)$$

We develop the right hand side at lowest order in  $\epsilon$  taking advantage of the fact that  $g(p_c) = V(x_c, p_c)$ . We get:

$$\frac{1}{2}\dot{x}_{\min}^2 = -\epsilon(g'(p_c) - \partial_p V(x_c, p_c))$$

And so:

$$\dot{x}_{\min} = \dot{u}_{\min} = \sqrt{-2\epsilon(g'(p_c) - \partial_p V(x_c, p_c))} \propto \sqrt{p - p_c} \quad (6)$$

Now, equation (5) still holds in the unstable case with the new initial conditions

$$\begin{aligned} v(0) &= u(0) + \epsilon C = \epsilon C \\ \dot{v}(0) &= \dot{u}(0) = \dot{u}_{\min} \end{aligned}$$

Using the same trick as in the previous section we obtain the general solution in this case:

$$u(t, p) = \epsilon C \cosh(\gamma t) + \frac{\dot{u}_{\min}}{\gamma} \sinh(\gamma t) - \epsilon C \quad (7)$$

As before, let's take  $U$ , an arbitrary distance from the critical point which is reached at time  $T$ .

$$U = \epsilon C \cosh(\gamma T) + \frac{\dot{u}_{\min}}{\gamma} \sinh(\gamma T) - \epsilon C \quad (8)$$

When  $\epsilon \rightarrow 0$ , since  $\dot{u}_{\min} \propto \sqrt{-\epsilon}$ , as previously, we get:

$$U \propto \frac{\dot{u}_{\min}}{\gamma} e^{\gamma T} \quad (9)$$

and we recover:

$$\gamma T \propto -\log(p - p_c) \quad (10)$$