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NONLOCAL DEGENERATE PARABOLIC HYPERBOLIC EQUATIONS ON BOUNDED DOMAINS

NATHAËL ALIBAUD, JØRGEN ENDAL, ESPEN R. JAKOBSEN, AND OLA MÆHLEN

ABSTRACT. We study well-posedness of degenerate mixed-type parabolic-hyperbolic equations

$$\partial_t u + \operatorname{div}(f(u)) = \mathcal{L}[b(u)]$$

on bounded domains with general Dirichlet boundary/exterior conditions. The nonlocal diffusion operator \mathcal{L} can be any symmetric Lévy operator (e.g. fractional Laplacians) and b is nondecreasing and allowed to have degenerate regions ($b' = 0$). We propose an entropy solution formulation for the problem and show uniqueness and existence of bounded entropy solutions under general assumptions.

The uniqueness proof is based on the Kružkov doubling of variables technique and incorporates several a priori results derived from our entropy formulation: an L^∞ -bound, an energy estimate, strong initial trace, weak boundary traces, and a *nonlocal* boundary condition. The existence proof is based on fixed point iteration for zero-order operators \mathcal{L} , and then extended to more general operators through approximations, weak- \star compactness of approximate solutions u_n , and *strong* compactness of $b(u_n)$. Strong compactness follows from energy estimates and arguments we introduce to transfer weak regularity from $\partial_t u_n$ to $\partial_t b(u_n)$. Our work can be seen as both extending nonlocal theories from the whole space to domains and local theories on domains to the nonlocal case. Unlike local theories our formulation does not assume energy estimates. They are now a consequence of the formulation, and as opposed to previous nonlocal theories, play an essential role in our arguments. Several results of independent interest are established, including a characterization of the \mathcal{L} 's for which the corresponding energy/Sobolev-space compactly embeds into L^2 .

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1. INTRODUCTION

In this paper we prove existence and uniqueness results on bounded domains $\Omega \subset \mathbb{R}^d$ for mixed type hyperbolic-parabolic equations with nonlinear and nonlocal diffusion:

$$(1.1) \quad \begin{cases} \partial_t u + \operatorname{div}(f(u)) = \mathcal{L}[b(u)] & \text{in } Q := (0, T) \times \Omega, \\ u = u^c & \text{in } Q^c := (0, T) \times \Omega^c, \\ u(0, \cdot) = u_0 & \text{on } \Omega, \end{cases}$$

where $T > 0$, the initial/boundary (exterior) data u_0, u^c are bounded, the ‘fluxes’ f, b are (at least) locally Lipschitz and b is nondecreasing (possibly degenerate),¹ ‘div’ is the x -divergence, and the nonlocal diffusion operator \mathcal{L} is defined for $\phi \in C_c^\infty(\mathbb{R}^d)$ as a singular integral:

$$(1.2) \quad \mathcal{L}[\phi](x) := P.V. \int_{|z|>0} (\phi(x+z) - \phi(x)) \, d\mu(z) := \lim_{\epsilon \rightarrow 0} \int_{|z|>\epsilon} (\phi(x+z) - \phi(x)) \, d\mu(z),$$

where the (Lévy) measure $\mu \geq 0$, symmetric, and $\int_{|z|>0} |z|^2 \wedge 1 \, d\mu(z) < \infty$.² This class of anomalous diffusion operators coincides with the generators of the symmetric pure-jump Lévy processes [19, 101, 102, 14], including α -stable, tempered, relativistic, and compound Poisson processes. The corresponding generators include the fractional Laplacians [14, 101]

$$\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}, \quad \alpha \in (0, 2), \quad \text{where} \quad d\mu_\alpha(z) := C_{d,\alpha} |z|^{-(d+\alpha)} \, dz,$$

anisotropic operators $-\sum_{i=1}^N (-\partial_{x_i x_i}^2)^{\frac{\alpha_i}{2}}$, $\alpha_i \in (0, 2)$, relativistic Schrödinger operators $m^\alpha I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$, degenerate and 0-order operators, and numerical discretizations [49, 50] of these. Since the operators \mathcal{L} are naturally defined on the whole space, they require Dirichlet data on Ω^c for (1.1) to be well-defined. In addition, inflow conditions are needed in hyperbolic regions. We will discuss this in more detail below.

Equation (1.1) is a possibly degenerate nonlinear convection-diffusion equation. Convection and diffusion phenomena are ubiquitous in sciences, economics, and engineering, their history dating back centuries. The study of nonlocal models is a strong trend across disciplines, motivated by applications [88, 114, 44], but also by deep connections between different fields and theoretical breakthroughs within mathematics [34, 33, 24, 109, 82]. Of the many applications of equations like (1.1), we mention reservoir simulation [59], sedimentation processes [32], and traffic flow [113] in the local case (e.g. $\mathcal{L} = \Delta$); detonation in gases [42], radiation hydrodynamics [97, 95], and semiconductor growth [114] in the nonlocal case; and porous media flow [90, 108, 47] and mathematical finance [44, 93] in both cases.

¹Both f and b may be strongly degenerate in the sense that f' or b' is 0 on some interval.

²Hence $-\mathcal{L}$ is a nonnegative, self-adjoint (on L^2), pseudo-differential operator of order $\alpha \in (0, 2)$.

The solution structure of equation (1.1) is very rich. Special cases include scalar conservation laws ($b = 0$) [45, 64, 86], fractional conservation laws ($b(u) = u$) [21, 55, 1], nonlinear fractional diffusion equations ($f = 0$), degenerate porous medium type equations [46, 47, 48, 110], strongly degenerate Stefan type problems [51, 52], and problems of mixed hyperbolic-parabolic type [40, 5, 57]. Solutions may have fundamentally different behaviour in different regions: The problem is hyperbolic with possible discontinuous shock solutions where the diffusion degenerates ($b'(u) = 0$). In diffusive regions ($b'(u) > 0$), the problem is ‘parabolic’ and the behaviour of solutions is determined by the order α of \mathcal{L} : Smooth solutions in the subcritical/diffusion dominated case $\alpha \in (1, 2)$ [55], possibly discontinuous solutions in the supercritical/convection dominated case $\alpha \in [0, 1)$ [1, 2]. The critical case $\alpha = 1$ is delicate, but can sometimes give regular solutions [43, 74]. In subcritical diffusive regions boundary conditions are expected to be satisfied pointwise, while in other regions, there may be sharp boundary layers and loss of boundary conditions (at ‘outflow’).

To accommodate non-smooth solutions, a weak/distributional solutions concept is needed, and then solutions are non-unique unless additional (entropy) conditions are imposed [45, 2]. In [78] Kruřkov developed an entropy solution theory and well-posedness for the Cauchy problem for scalar conservation laws (1.1) with $b = 0$. After a long time and attempts by different authors, Carrillo came in [35] up with a highly non-trivial extension to local (degenerate) convection-diffusion equations (1.1) with $\mathcal{L} = \Delta$. Inspired by ideas from nonlocal viscosity solution theories (see e.g. [68, 16]), Alibaud in [1] introduced an entropy solution theory and well-posedness for the Cauchy problem for (1.1) with linear diffusion $b(u) = u$ and $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$. Cifani and Jakobsen in [40] then extended these ideas to cover the Cauchy problem for (1.1) in full generality, including nonlinear possibly degenerate diffusions and arbitrary pure-jump Levy diffusion operators. In nonlocal entropy solution theories, a key point is to split the nonlocal operator in a singular and a nonsingular part, where the singular part vanishes after taking limits, and the nonsingular part leads to beneficial cancellations in the Kruřkov doubling of variables argument. This idea has no local analogue, and the theory is quite different from local theories [35, 71, 39]. We refer to [4, 5, 73, 57, 41, 66, 20, 3] and references therein for more results on Cauchy problems and recent extensions.

Hyperbolic and mixed type boundary value problems. In hyperbolic problems, boundary conditions can be imposed only at inflow, where characteristics go into the domain. When $b = 0$ and (1.1) is a nonlinear scalar conservation law,

$$(1.3) \quad \begin{cases} \partial_t u + \operatorname{div}(f(u)) = 0 & \text{in } Q, \\ u = u^c & \text{on } \Gamma := (0, T) \times \partial\Omega, \\ u(0, \cdot) = u_0 & \text{on } \Omega, \end{cases}$$

characteristics depend on u itself and the correct way to impose boundary conditions only at inflow is

$$(1.4) \quad \operatorname{sgn}(u - u^c)(f(u) - f(k)) \cdot \hat{n} \geq 0 \quad \forall k : u \wedge u^c \leq k \leq u \vee u^c \quad \text{on } (0, T) \times \partial\Omega,$$

where \hat{n} is the outward unit normal. This strong formulation of Bardos, Le Roux, and Nedelec [15] (the BLN condition) requires BV regularity and strong boundary traces for u . Otto introduced in [91] (see also [86]) an L^∞ -theory using weak boundary traces and a weak entropy formulation on the boundary, and then this theory was simplified by the semi-Kruřkov boundary entropy formulation of Vovelle [111] – see also [60]. In [38] divergence-measure fields were analyzed to obtain weak traces and applied to (1.3). The divergence structure of the equation makes entropy solutions regular at the boundaries, and L^∞ -solutions of (1.3) are shown to have strong traces in [107, 92, 81]. The weak trace approach of Otto is in a sense bypassed, but the case of general fluxes $f = f(t, x, u)$ is not well-understood. In the BV setting, an extensive overview of well-posedness for (1.3) is given in [98], see also [103, 45].

Measure-valued solutions are considered in [104, 105], renormalized solutions in [37, 94, 8], kinetic solutions in [67], saturated solutions in [84], and Robin and Neumann conditions in [11].

When both convection and nonlinear local diffusion ($\mathcal{L} = \Delta$) are present in (1.1), the problem is of mixed type and will be hyperbolic in regions where the diffusion degenerates. A correct formulation taking into account inflow conditions are needed, and the first result was given in [35] for homogeneous Dirichlet conditions. The case of $u^c \neq 0$ is more delicate in the sense that the question of traces becomes more visible. Well-posedness is proved by Mascia, Poretta, and Terracina [87] using a careful parabolic extension of the Otto-formulation. Michel and Vovelle [89] recasted the problem as a single integral inequality using semi-Kruřkov entropies, and also proved existence and convergence of finite volume schemes. Further improvements can be found in [106]. Strong traces exist in some cases [80, 62], strong entropy solutions can be found in [99, 10] and kinetic solutions in [75, 76]. We also mention work on related boundary conditions [12, 31], on anisotropic diffusion [77, 83, 112], and on doubly and triply nonlinear degenerate parabolic equations [18, 22, 36, 7, 23, 17, 13, 9, 6, 54].

Nonlocal boundary value problems. Our nonlocal operators \mathcal{L} are naturally defined on the whole space. For problem (1.1) to be well-defined, we need to restrict \mathcal{L} to the bounded domain Ω . Likely the most classical way of doing this is to impose Dirichlet data on the complement Ω^c as in (1.1). In probability theory this corresponds to stopping or killing the underlying (particle) processes the first time it exits from Ω [56]. This approach is by far the most studied model in the PDE setting, with a large literature including important contributions of Caffarelli and Silvestre [34, 33, 109, 96, 30]. As opposed to the local case, there are many other ways to restrict \mathcal{L} to a bounded domain and impose Dirichlet conditions. Equivalent definitions in \mathbb{R}^d [79] via PDEs, probability, calculus of variations, spectral theory etc., may differ on domains and lead to different interpretations of Dirichlet conditions. Popular choices are censoring and spectral definitions [24, 63] where boundary data is only required on $\partial\Omega$. But the resulting operators \mathcal{L} are no longer translation invariant, and their integral representations depend explicitly on Ω . See e.g. [63] for a comparison of the three main models and [26, 25] for a discussion in a nonlinear PDE setting.

Problem (1.1) is a nonlocal problem of mixed type, and any formulation of it needs to take account inflow conditions in hyperbolic regions. Up to now the literature on such problems is very limited. For purely parabolic equations ($f \equiv 0$), boundary value problems have been studied in e.g. [27, 28, 26, 25]. Here there are no hyperbolic regions and inflow conditions are not needed. When the diffusion is linear and non-degenerate ($b = \text{Id}$) and $u^c \equiv 0$, there are results in [85] for $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$ (and stochastic perturbations) and in [29, 69, 70] when \mathcal{L} is the spectral fractional Laplacian. In the latter case the boundary condition is satisfied in a pointwise sense when $\alpha \in (1, 2)$. Recently, when \mathcal{L} is the regional/censored fractional Laplacian, a priori estimates and existence for L^∞ solutions of (1.1) was proved in [65] using the vanishing viscosity method. There is no result on uniqueness, and the nonlocal operator is different from the ones we consider here.

Main contributions of the paper:

1. An appropriate entropy solution formulation for problem (1.1).
2. A uniqueness result for entropy solutions of (1.1).
3. Existence of entropy solutions of (1.1) under mild additional assumptions on \mathcal{L} .

Entropy solutions of (1.1) are defined to be L^∞ functions satisfying pointwise exterior conditions and a family of weak semi-Kruřkov entropy inequalities incorporating both the equation in the interior and inflow conditions in hyperbolic parts of the boundary. The definition includes splitting the nonlocal operator into singular and nonsingular parts as in [1, 40] and a version of the entropy formulation on

the boundary given in [87, 89]. See Definition 2.2 for the precise statement. The entropy inequalities include initial and boundary terms that impose strong initial traces (Lemma 2.5), hyperbolic weak traces on the boundary (Proposition 4.8) and a boundary condition (Proposition 4.10) that coincides with the BLN-condition (1.4) whenever strong traces exist [111].

The most important contribution of the paper is the uniqueness result for our class of entropy solutions (Theorem 5.1). As in the local case,³ it takes the form of a partial L^1 -contraction estimate, $\|u(t, \cdot) - v(t, \cdot)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}$ for solutions u and v coinciding on Ω^c . This result seems to be the first uniqueness result for degenerate parabolic hyperbolic equations that are nonlocal. The proof is based on an adaptation of the Kruřkov doubling of variables technique and combine nonlocal elements from [1, 40] with arguments to handle boundary conditions from [89]. Such proofs require a lot of ingredients to work and are more technically involved on bounded domains than in the whole space.

On bounded domains, a key part of any uniqueness proof is making sense of and controlling fluxes at the boundary. For example, in the local case ($\mathcal{L} = \Delta$) the in-flux of $|u - \bar{u}|$ at the boundary is formally given as the measure

$$\left(\operatorname{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u})) - \nabla |b(u) - b(\bar{u})| \right) \cdot \nabla \chi_\Omega \quad \text{on} \quad \Gamma = (0, T) \times \partial\Omega.$$

The existence and control of such a (weak) flux-trace is part of the solution-concept in [87], but not in [89] where it is instead proved from the entropy formulation and a finite energy assumption. In this paper we work with a nonlocal version of the weak trace given by

$$\operatorname{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c)) \cdot \nabla \chi_\Omega - \mathcal{B}(|b(u) - b(\bar{u}^c)|, \chi_\Omega) \quad \text{in} \quad M := (0, T) \times \mathbb{R}^d,$$

where \mathcal{B} is the bilinear operator whose integral is the bilinear form associated to \mathcal{L} . The latter measure has a singular part on Γ and an absolutely continuous part in $M \setminus \Gamma$ which is due to the nonlocal \mathcal{B} -term. Like in [89], we also prove the necessary properties of this trace (Proposition 4.8) from the entropy formulation using a nonlocal energy estimate. Contrary to the local case, this energy estimate is not a part of the formulation, but follows as a consequence of it (see Proposition 4.1).⁴

The strategy employed when proving these a priori results, including the aforementioned weak trace, boundary condition, and nonlocal finite energy, follows a general procedure of doubling of variables arguments using carefully chosen boundary layer sequences and passing to the limit (i) to undo the doubling and (ii) to see the boundary. Here we deviate from the local case [89] where nice and rather explicit ‘inner’ boundary layer sequences can be constructed via local Poisson problems. Such arguments are difficult to generalize to our family of nonlocal operators which may not be uniformly elliptic and have integral representations that are more difficult to handle. We avoid this problem by taking a more crude ‘outer’ boundary layer sequences and reverse the order of the limits (i) and (ii) mentioned above. The price for this approach is more regularity requirements on b and \bar{u}^c in problem (1.1), see Remark 4.2 for more details.

The second most important contribution of the paper is an existence result for entropy solutions of (1.1) under mild additional assumptions on the operators \mathcal{L} (Theorem 2.9). Our proof is based on a series of approximations of (1.1), combined with compactness and stability arguments. For the Cauchy problem in the whole space, see e.g. [1, 40, 57, 50], such arguments use strong compactness and rely on L^1 -contraction estimates. Translation invariance then yield translation estimates in L^1 , and the approximations have a limit in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^d))$ by standard strong compactness results. Stability results are used to conclude that this limit is an entropy solution. In view of the L^1 -contraction results of e.g. [8, Corollary 2.6], similar arguments work on bounded domains when $b \equiv 0$, see e.g. [86, 8]. In

³Problem (1.1) with $\mathcal{L} = \Delta$, see e.g. [35, 87, 89] for uniqueness results.

⁴Energy estimates play no role in nonlocal problems on the whole space [40], but are still necessary for local ones [72].

the presence of diffusion ($b \neq 0$), the only L^1 -contraction results known to us are the time-averaged results of [75, 76] for the local case ($\mathcal{L} = \Delta$). These bounds do not imply translation bounds, and an approach based on weak compactness arguments is needed. Approximate solutions u_n to (1.1) typically satisfy uniform L^∞ -bounds, and hence are weak- \star precompact in L^∞ . An approach based on nonlinear weak- \star compactness and measure valued solutions has been developed for hyperbolic conservation laws [60]⁵ and later extended to mixed hyperbolic-parabolic problems [89]. In [89] additional energy estimates are needed to show strong L^2 -compactness of $b(u_n)$ which is used to conclude that the limit is a solution (stability).

We adapt the entropy-process solution concept and nonlinear weak- \star approach of [89] to our nonlocal setting. Our approximation procedure is different from [89] and adapted to nonlocal operators. The existence proof starts from the hyperbolic problem (1.3). Through a fixed point argument we obtain existence for (1.1) when \mathcal{L} is bounded. Then the proof proceeds via two successive approximations (approximation by bounded operators followed by vanishing viscosity) to the most general case where \mathcal{L} is any symmetric Lévy operator whose bilinear form induce compactness in L^2 . This very general class of operators include all fractional Laplacians, logarithmic Laplacians, and practically any (symmetric) nonnegative nonlocal operator seen in applications.

Key contributions are stability results for (1.1) with respect to change of \mathcal{L} (cf. Proposition 6.6) and strong compactness results for $b(u_n)$ in L^2 (see Section 6.4). Spatial translation estimates of $b(u_n)$ follow from the energy estimate when our nonlocal operator \mathcal{L} induces L^2 compactness, a property we characterize (in terms of sufficient/necessary conditions) in Appendix E. This result seems to be new in our setting. Getting translation estimates in time however, is not easy. From the equation we can only induce weak L^1 estimates for $\partial_t u$, and then L^1 time-continuity of regularized quantities like $b(u_n * \rho)$ where ρ is a mollifier. Time translation estimates follow if we can compare this regularized quantity with $b(u_n)$. To do this we introduce a ‘commutator estimate’ (Lemma 6.9) to interchange $b(\cdot)$ with $*\rho$, the proof of which relies on the novel Lemma F.1.

Outline of paper. Section 2 is reserved for assumptions, concept of solution, and main results. Some preliminary results regarding nonlocal operators and boundary layer sequences can be found in Section 3. We collect important properties of entropy solutions, like energy estimates, boundary conditions, and weak traces, in Section 4. These properties are then used to prove uniqueness in Section 5, and existence in Section 6. The Appendices contain technical results.

2. ASSUMPTIONS, CONCEPT OF SOLUTION, AND MAIN RESULTS

In this section we state the notation and assumptions, present the (entropy) solution concept we will use, and give the main results – including uniqueness, existence, a priori estimates, and boundary trace/condition results.

We start with some notation. Let $\phi \vee^+ \psi := \max\{\phi, \psi\} = \phi \vee \psi$, $\phi \vee^- \psi := \min\{\phi, \psi\} = \phi \wedge \psi$, and $\text{sgn}^\pm(a) = \pm 1$ if $\pm a > 0$ and zero otherwise. We define

$$Q = (0, T) \times \Omega, \quad \Gamma = (0, T) \times \partial\Omega, \quad \bar{\Omega} = \Omega \cup \partial\Omega,$$

$$Q^c = (0, T) \times \Omega^c, \quad \bar{\Gamma} = [0, T] \times \partial\Omega, \quad M = (0, T) \times \mathbb{R}^d.$$

For a set $\mathcal{S} \subset \mathbb{R}^d$ we define the signed distance by $d_{\mathcal{S}}(x) = \text{dist}(x, \partial\mathcal{S})$ if $x \in \mathbb{R}^d \setminus \mathcal{S}$ and $d_{\mathcal{S}}(x) = -\text{dist}(x, \partial\mathcal{S})$ otherwise. For $\varepsilon > 0$, the $\pm\varepsilon$ -neighborhoods of \mathcal{S} and $\partial\Omega$ are given by

$$\mathcal{S}_{\pm\varepsilon} := \{x \in \Omega^c : d_{\mathcal{S}}(x) \leq \pm\varepsilon\} \quad \text{and} \quad \partial\Omega_{\pm\varepsilon} := (\partial\Omega)_{\pm\varepsilon}.$$

⁵The context was finite volume schemes on irregular grids which satisfy no known translation estimate in L^1 .

The $(d-1)$ -dimensional Hausdorff measure on $\partial\Omega$ is denoted by $d\sigma(x)$. When extending \mathcal{L} to a larger class of functions, the limit in (1.2) is taken in $L^1_{\text{loc}}(\mathbb{R}^d)$ and the corresponding bilinear operator is

$$(2.1) \quad \mathcal{B}[\phi, \psi](x) := \lim_{\epsilon \rightarrow 0} \frac{1}{2} \int_{|z| > \epsilon} (\phi(x+z) - \phi(x)) (\psi(x+z) - \psi(x)) d\mu(z),$$

again with a limit in $L^1_{\text{loc}}(\mathbb{R}^d)$. We denote truncated operators by

$$(2.2) \quad \mathcal{L}^{\geq r}, \quad \mathcal{L}^{< r}, \quad \mathcal{L}^{r' > \dots \geq r}, \quad \mathcal{B}^{\geq r}, \quad \mathcal{B}^{< r}, \quad \mathcal{B}^{r' > \dots \geq r},$$

where the domains of integration are restricted to $\{|z| \geq r\}$, $\{|z| < r\}$, and $\{r' > |z| \geq r\}$, and

$$(2.3) \quad \mathcal{L}^{\geq r}_{x+y}[\phi(\cdot, \cdot)](x, y) := \int_{|z| \geq r} (\phi(x+z, y+z) - \phi(x, y)) d\mu(z).$$

Let $\Omega \subset \mathbb{R}^d$ be open and \mathcal{L} an L^2 -self-adjoint Lévy operator. We identify $L^2(\Omega)$ with a subspace of $L^2(\mathbb{R}^d)$ through zero extensions and define $H^{\mathcal{L}}_0(\Omega) \subset L^2(\Omega)$ to be the Hilbert space with norm

$$(2.4) \quad \|\phi\|_{H^{\mathcal{L}}_0(\Omega)}^2 := \|\phi\|_{L^2(\Omega)}^2 + \int_{\mathbb{R}^d} \mathcal{B}[\phi, \phi] dx.$$

The Lipschitz constant of a function g on a set K is $L_{g,K}$ or L_g if the set is clear from the context.

In this paper we will use the following assumptions:

- (A $_{\Omega}$) $\Omega \subset \mathbb{R}^d$ is open, bounded with C^2 -boundary $\partial\Omega$, and outward pointing normal \hat{n} .
- (A $_f$) $f = (f_1, f_2, \dots, f_d) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R}^d)$.
- (A $_b$) $b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}; \mathbb{R})$ is non-decreasing, and the weak derivative $b' \in TV_{\text{loc}}(\mathbb{R})$.
- (A $_{u^c}$) $u^c \in (C^2 \cap L^\infty)(Q^c)$ and has an extension $\bar{u}^c \in (C^2 \cap L^\infty)([0, T] \times \mathbb{R}^d)$.
- (A $_{u_0}$) $u_0 \in L^\infty(\Omega)$.
- (A $_{\mu}$) μ is a symmetric, nonnegative Radon measure such that $\int_{\mathbb{R}^d} (|z|^2 \wedge 1) d\mu(z) < \infty$.
- (A' $_{\mu}$) μ is such that the symbol $m(\xi) := \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z)) d\mu(z) \rightarrow \infty$ as $|\xi| \rightarrow \infty$.

Here m is the symbol, or Fourier multiplier, of $-\mathcal{L}$.

Remark 2.1.

- (a) In (A $_f$) and (A $_b$), we can assume without loss of generality that $f(0) = 0$ and $b(0) = 0$ (add constants to f and b), and that f and b are globally Lipschitz and b' has bounded total variation (solutions are uniformly bounded by Lemma 2.4).
- (b) In (A $_b$) the condition $b' \in TV_{\text{loc}}(\mathbb{R})$ implies $b \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$, the standard assumption for the Cauchy problem. Our stronger assumption still allows for classical power type and strongly degenerate nonlinearities $b(r) = r^m$ for $m > 1$ (porous medium) and $b(r) = \max\{r - L, 0\}$ (one-phase Stefan).
- (c) In proofs we will often use the notation, for some $\epsilon > 0$,

$$L_{b(\bar{u}^c)} = \|\partial_t b(\bar{u}^c)\|_{L^\infty(Q_\epsilon)} + \|\nabla b(\bar{u}^c)\|_{L^\infty(Q_\epsilon)} \leq L_b (\|\partial_t \bar{u}^c\|_{L^\infty(Q_\epsilon)} + \|\nabla \bar{u}^c\|_{L^\infty(Q_\epsilon)}) = L_b L_{\bar{u}^c}.$$

By our assumptions, $\mathcal{L}[b(\bar{u}^c)] \in L^1(Q)$ (Corollary 3.7) and $\partial_t \bar{u}^c + \text{div}_x(f(\bar{u}^c)) - \mathcal{L}[b(\bar{u}^c)] \in L^1(Q)$, which will be important in many calculations and results concerning $u - \bar{u}^c$.

- (d) Assumption (A' $_{\mu}$) is a necessary and sufficient condition for a family of functions bounded in the energy space $H^{\mathcal{L}}_0$ to be compact in L^2 , see Proposition 6.11. It is needed in the existence proof in Section 6 where it compensates for the lack of L^1 -contraction results.

For the definition of entropy solutions we introduce the semi-Kruřkov entropy-entropy flux pairs

$$(u - k)^\pm, \quad F^\pm(u, k) := \operatorname{sgn}^\pm(u - k)(f(u) - f(k)) \quad \text{for } u, k \in \mathbb{R},$$

where $(\cdot)^+ := \max\{\cdot, 0\}$ and $(\cdot)^- := (-\cdot)^+$, and the usual splitting of the nonlocal operator,

$$\mathcal{L}[\phi](x) = \mathcal{L}^{<r}[\phi](x) + \mathcal{L}^{\geq r}[\phi](x) \quad \text{for } \phi \in C_c^\infty(\mathbb{R}^d), \quad r > 0, \quad x \in \mathbb{R}^d,$$

where $\mathcal{L}^{<r}$ and $\mathcal{L}^{\geq r}$ are defined in (2.2). To motivate the definition (see also Appendix A), assume u is a smooth solution of (1.1), $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$, and $k \in \mathbb{R}$. Replace $(u, f(u), b(u))$ by $(u - k, f(u) - f(k), b(u) - b(k))$ in (1.1), multiply the resulting PDE by $\operatorname{sgn}^\pm(u - k)\varphi$, and integrate over Q . After several integrations by parts and chain rules, we get the equality

$$\begin{aligned} & - \int_Q \left((u - k)^\pm \partial_t \varphi + F^\pm(u, k) \cdot \nabla \varphi \right) dx dt \\ & - \int_M \mathcal{L}^{\geq r}[b(u) - b(k)] \operatorname{sgn}^\pm(u - k) \varphi \chi_\Omega dx dt - \int_M (b(u) - b(k))^\pm \mathcal{L}^{<r}[\varphi] dx dt \\ & = \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) dx + L_f \int_\Gamma (\bar{u}^c - k)^\pm \varphi d\sigma(x) dt - \int_\Gamma \varphi d\mu_c^\pm - \int_M \varphi d\mu_d^{\pm, r}, \end{aligned}$$

where the μ -measures are absolutely continuous with respect to $d\sigma dt$ and $dx dt$ with densities

$$F^\pm(u, k) \cdot \hat{n} + L_f(u - k)^\pm \quad \text{and} \quad \mathcal{L}^{<r}[b(u) - b(k)] \operatorname{sgn}^\pm(u - k) \chi_\Omega + \mathcal{L}^{<r}[(b(u) - b(k))^\pm].$$

For our definition we want these measures to be nonnegative. For the classical measure μ_c^\pm , this follows by definition of F^\pm and Lipschitz continuity of f . The new measure $\mu_d^{\pm, r}$ however, requires an additional assumption. Since $b' \geq 0$ and $\operatorname{sgn}^\pm(u - k)(b(u) - b(k)) = \operatorname{sgn}^\pm(b(u) - b(k))(b(u) - b(k))$, if

$$\operatorname{sgn}^\pm(b(u^c) - b(k))\varphi = 0 \quad \text{a.e. in } Q^c,$$

then we get

$$\begin{aligned} & \int_M \left(\mathcal{L}^{<r}[b(u) - b(k)] \operatorname{sgn}^\pm(b(u) - b(k)) \chi_\Omega - \mathcal{L}^{<r}[(b(u) - b(k))^\pm] \right) \varphi dx dt \\ & = \int_M \left(\mathcal{L}^{<r}[b(u) - b(k)] \operatorname{sgn}^\pm(b(u) - b(k)) - \mathcal{L}^{<r}[(b(u) - b(k))^\pm] \right) \varphi dx dt. \end{aligned}$$

The last integral is nonpositive by the convex inequality $\operatorname{sgn}^+(v) \mathcal{L}^{<r}[v] \leq \mathcal{L}^{<r}[v^+]$ (cf. Corollary C.2), and hence we can conclude that also $\mu_d^{\pm, r}$ is nonnegative.

Our definition is then:

Definition 2.2 (Entropy solution). A function $u \in L^\infty(M)$ is an *entropy solution* of (1.1) if:

(a) (Entropy inequalities in \bar{Q}) For all $r > 0$, and all $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ satisfying

$$(2.5) \quad (b(u^c) - b(k))^\pm \varphi = 0 \quad \text{a.e. in } Q^c,$$

the following inequality holds

$$\begin{aligned} & - \int_Q \left((u - k)^\pm \partial_t \varphi + F^\pm(u, k) \cdot \nabla \varphi \right) dx dt \\ (2.6) \quad & - \int_Q \mathcal{L}^{\geq r}[b(u)] \operatorname{sgn}^\pm(u - k) \varphi dx dt - \int_M (b(u) - b(k))^\pm \mathcal{L}^{<r}[\varphi] dx dt \\ & \leq \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) dx + L_f \int_\Gamma (\bar{u}^c - k)^\pm \varphi d\sigma(x) dt. \end{aligned}$$

(b) (Data in Q^c) $u = u^c$ a.e. in Q^c .

This definition is an extension of both [40, Definition 2.1] (see also [1]) for nonlocal problems in the whole space and [89, Definition 2.1] (see also [87]) for local problems on bounded domains. In the hyperbolic case ($b' \equiv 0$), it is equivalent [111] to the original definition of Otto [91]. By (A_Ω) – (A_μ) , all the integrals in (2.6) are well-defined.

Remark 2.3.

- (a) The condition $(b(u^c) - b(k))^+ \varphi = 0$ is weaker than $|b(u^c) - b(k)|\varphi = 0$. When $b(u^c) < b(k)$, the first condition holds for any φ while the second then implies that $\varphi = 0$. This gives a hint to why standard Kruřkov entropy-entropy flux pairs are too restrictive in this setting, see [111] for counterexamples to uniqueness in the hyperbolic case when $b = 0$.
- (b) Since u is a Lebesgue measurable function, it is not immediately clear that $b(u)$ is μ -measurable and $\mathcal{L}^{\geq r}[b(u) - b(k)]$ is well-defined pointwise. To avoid this problem we always consider Borel measurable representatives of u ; see also Remark 2.1 in [4] for a discussion.

A first consequence of our definition is the L^∞ a priori estimate for solutions.

Lemma 2.4 (L^∞ -bound). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\min \left\{ \operatorname{ess\,inf}_\Omega u_0, \operatorname{ess\,inf}_{Q^c} u^c \right\} \leq u(t, x) \leq \max \left\{ \operatorname{ess\,sup}_\Omega u_0, \operatorname{ess\,sup}_{Q^c} u^c \right\} \quad \text{for a.e. } (t, x) \in Q.$$

The weakly posed initial conditions u_0 are in fact assumed in the strong L^1 -sense.

Lemma 2.5 (Time continuity at $t = 0$). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\operatorname{ess\,lim}_{t \rightarrow 0^+} \|u(t, \cdot) - u_0\|_{L^1(\Omega)} = 0.$$

These two results are natural, and their rather standard proofs can be found in Appendix B. We continue by listing additional a priori estimates that are crucial for the uniqueness proof. The proofs are given in Section 4. Recall that \mathcal{B} is the bilinear operator associated with \mathcal{L} defined in (2.1), and define

$$F := F^+ + F^-.$$

Proposition 4.1 (Energy estimate). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$(4.1) \quad \begin{aligned} & \int_M \mathcal{B}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \, dx \, dt \\ & \leq \int_\Omega H(u_0, \bar{u}^c(0)) \, dx - \int_Q \left[(u - \bar{u}^c) \bar{u}_t^c + F(u, \bar{u}^c) \cdot \nabla \bar{u}^c \right] b'(\bar{u}^c) \, dx \, dt \\ & \quad + \int_Q \mathcal{L}[b(\bar{u}^c)](b(u) - b(\bar{u}^c)) \, dx \, dt, \end{aligned}$$

where $F(u, \bar{u}^c) = \operatorname{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c))$ and where H is the nonnegative function

$$H(u, k) := \int_k^u (b(\xi) - b(k)) \, d\xi.$$

Moreover, the right-hand side of (4.1) is finite.

Remark 2.6. This is a global energy estimate/regularity estimate for $b(u) - b(\bar{u}^c)$. When $b(\bar{u}^c)$ is smooth, it carries over (locally) to $b(u)$ as well. Indeed, by the inequality $x^2 \leq 2(x - y)^2 + 2y^2$ we have

$$\int_M \mathcal{B}[b(u), b(u)] \varphi \, dx \, dt \leq 2 \int_M \mathcal{B}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \varphi \, dx \, dt + 2 \int_M \mathcal{B}[b(\bar{u}^c), b(\bar{u}^c)] \varphi \, dx \, dt,$$

for any $0 \leq \varphi \in C_c^\infty(M)$. If $\int_M \mathcal{B}[b(\bar{u}^c), b(\bar{u}^c)] dx dt < \infty$ we can send $\varphi \rightarrow 1$ and get a global estimate.

To write down our results regarding boundary integrability, weak trace, and boundary condition, we need different sequences of test functions to approximate the (indicator function of the) domain. We say $(\zeta_\delta)_{\delta>0} \subset C_c^\infty(\mathbb{R}^d)$ is a *boundary layer sequence* if, for some Borel function $\zeta: \partial\Omega \rightarrow [0, 1]$,

$$(2.7) \quad 0 \leq \zeta_\delta(x) \leq 1, \quad x \in \mathbb{R}^d, \quad \forall \delta > 0, \quad \lim_{\delta \rightarrow 0} \zeta_\delta(x) = \begin{cases} 1, & x \in \Omega, \\ \zeta(x), & x \in \partial\Omega, \\ 0, & x \in \mathbb{R}^d \setminus \Omega. \end{cases}$$

Boundary layer sequences $(\bar{\zeta}_\delta)_{\delta>0}$ and $(\zeta_\delta)_{\delta>0}$ are called *outer* and *inner boundary layer sequences* if

$$(2.8) \quad \bar{\zeta}_\delta = 1 \quad \text{in} \quad \bar{\Omega} \quad \text{and} \quad \lim_{\delta \rightarrow 0} \bar{\zeta}_\delta(x) = \chi_{\bar{\Omega}}(x), \quad x \in \mathbb{R}^d,$$

$$(2.9) \quad \zeta_\delta = 0 \quad \text{in} \quad \Omega^c \quad \text{and} \quad \lim_{\delta \rightarrow 0} \zeta_\delta(x) = \chi_\Omega(x), \quad x \in \mathbb{R}^d.$$

Typically, we will require of these sequences to satisfy $\limsup_{\delta \rightarrow 0} \|\nabla \zeta_\delta\|_{L^1(\mathbb{R}^d)} < \infty$. For inner boundary layer sequences, $-\nabla \zeta_\delta \rightarrow \hat{n} \delta_{\partial\Omega}$ weakly in $H^1(\Omega; \mathbb{R}^d)$: For any $\psi \in H^1(\Omega; \mathbb{R}^d)$,

$$\lim_{\delta \rightarrow 0} \int_\Omega \psi(-\nabla \zeta_\delta) dx = \lim_{\delta \rightarrow 0} \int_\Omega \operatorname{div}(\psi) \zeta_\delta dx = \int_\Omega \operatorname{div}(\psi) dx = \int_{\partial\Omega} \psi \cdot \hat{n} d\sigma(x).$$

The next result restricts how much $b(u)$ can differ from $b(\bar{u}^c)$ close to the boundary Γ . It is a purely nonlocal result and prevents blow-up of certain integrals.

Proposition 4.6 (Boundary integrability). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$(4.27) \quad \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} |b(u(t, x)) - b(\bar{u}^c(t, x))| d\mu(z) dx dt \leq C,$$

where

$$(4.28) \quad C := |\Omega|T(1 + L_f)L_{\bar{u}^c} + |\Omega|(\|u_0\|_{L^\infty(\Omega)} + \|\bar{u}^c\|_{L^\infty(M)}) \\ + \left(\sup_c \|\mathcal{L}[b(\bar{u}^c \wedge c)]\|_{L^1(Q)} \right) \vee \left(\sup_c \|\mathcal{L}[b(\bar{u}^c \vee c)]\|_{L^1(Q)} \right).$$

Remark 2.7. The second line of (4.28) is finite by Corollary 3.7 with $h_c^\pm(\cdot) := b(\cdot \vee^\pm c)$ replacing h since $\|(h_c^\pm)'\|_{L^\infty(\mathbb{R})} \leq \|b'\|_{L^\infty(\mathbb{R})}$ and $|(h_c^\pm)'|_{TV(\mathbb{R})} \leq \|b'\|_{L^\infty(\mathbb{R})} + |b'|_{TV(\mathbb{R})}$.

The two next results have local analogues and are essential for the uniqueness proof.

Proposition 4.8 (Weak trace). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then there is a finite non-positive Borel measure ν on $\bar{\Gamma} = [0, T] \times \partial\Omega$ such that for every $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$, $r > 0$, and boundary layer sequence $(\zeta_\delta)_{\delta>0}$,*

$$(4.31) \quad \lim_{\delta \rightarrow 0} \left(\int_Q F(u, \bar{u}^c) \cdot (\nabla \zeta_\delta) \varphi dx dt - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta] \varphi dx dt \right) \\ = \int_{\bar{\Gamma}} (1 - \zeta) \varphi d\nu - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] \varphi dx dt,$$

where ζ is the limit of ζ_δ on $\partial\Omega$ and $F(u, \bar{u}^c) = \operatorname{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c))$.

Remark 2.8. The $\mathcal{B}^{<r}$ -term on the right-hand side is finite by boundary integrability Proposition 4.6.

The $\mathcal{B}^{<r}$ -terms are new and represent the contribution from the nonlocal diffusion, while the hyperbolic part (the F and ν terms) coincides with the weak trace of Otto⁶ for scalar conservation laws [111, 86]. The ‘flux density’ $F(u, \bar{u}^c) \cdot \nabla \chi_\Omega - \mathcal{B}[|b(u) - b(\bar{u}^c)|, \chi_\Omega]$ measures the inflow of $|u - \bar{u}^c|$ into Q . The convective part is a singular measure concentrated on Γ while the nonlocal diffusive part is absolutely continuous and supported on M . The term $-\mathcal{B}[|b(u) - b(\bar{u}^c)|, \chi_\Omega](t, x)$ (see (2.1)) represents the inflow density from $(t, x) \in Q^c$ into Q or minus the outflow density from $(t, x) \in Q$ to Q^c .

The weak trace becomes important in the uniqueness proof during the Kruřkov doubling of variables technique where we want to replace ‘difficult’ terms by ‘easy’ terms using the boundary condition below. Define the Kruřkov boundary entropy-entropy flux pairs as

$$\begin{aligned} \mathcal{F} &= \mathcal{F}(u, \bar{u}^c, k) := F(u, \bar{u}^c) + F(u, k) - F(\bar{u}^c, k), \\ \Sigma &= \Sigma(u, \bar{u}^c, k) := |b(u) - b(\bar{u}^c)| + |b(u) - b(k)| - |b(\bar{u}^c) - b(k)|. \end{aligned}$$

Proposition 4.10 (Boundary condition). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). For all inner boundary layer sequences ζ_δ , $k \in \mathbb{R}$, $r > 0$, and $0 \leq \varphi \in C^\infty([0, T] \times \mathbb{R}^d)$,*

$$(4.38) \quad \limsup_{\delta \rightarrow 0} \left(\int_Q (\mathcal{F} \cdot \nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r}[\Sigma, \zeta_\delta] \varphi \, dx \, dt \right) \leq 0.$$

At this point we could formulate an equivalent definition of entropy solutions in the spirit of Mascia, Porretta, and Terracina [87, Definition 1.2], by combining Lemma 2.5, Propositions 4.6, 4.8, 4.10, and the classical Kruřkov entropy inequalities in Q . However, the way the Definition 2.2 is formulated now, makes it easier to use in the existence proof where we take limits of approximate solutions.

We now state the uniqueness result, the most important contribution of this paper.

Theorem 5.1 (Uniqueness). *Assume (A_Ω) – (A_μ) . If u and v are entropy solutions of (1.1) with data u_0, v_0 , and $u^c = v^c$, then for a.e. $t \in (0, T)$*

$$\int_\Omega |u(t, x) - v(t, x)| \, dx \leq \int_\Omega |u_0(x) - v_0(x)| \, dx.$$

In particular, if also $u_0 = v_0$, then $u = v$ a.e. in $(0, T) \times \mathbb{R}^d$.

The second main contribution of this paper is an existence result for (1.1) under slightly stronger assumptions on \mathcal{L} (or μ).

Theorem 2.9 (Existence). *Assume (A_Ω) – (A_μ) and either (A'_μ) or $\mu(\mathbb{R}^d) < \infty$. Then there exists an entropy solution u of (1.1).*

This result is a combination of Theorem 6.20 and Theorem 6.17 in Section 6. In the proof of existence we will show several compactness and stability results for problem (1.1), see Proposition 6.6 and Sections 6.3 and 6.4. As a simple consequence of these, we obtain a convergence result for the fractional-nonlinear vanishing viscosity method,

$$(2.10) \quad \begin{cases} \partial_t u_n + \operatorname{div}(f(u_n)) = -\frac{1}{n}(-\Delta)^{\frac{\alpha}{2}}[b(u_n)] & \text{in } Q, \\ u_n = u^c & \text{on } Q^c, \\ u_n(0, \cdot) = u_0 & \text{on } \Omega. \end{cases}$$

We prove that solutions u_n converge as $n \rightarrow \infty$ to the solution u of the Dirichlet boundary value problem for the scalar conservation law (1.3).

⁶It is an equivalent restatement in terms of semi-Kruřkov entropy-entropy fluxes due to Vovelle [111] of the result of Otto (see Definition 7.2 and Lemma 7.34 in [86]).

Proposition 6.21 (Vanishing viscosity). *Assume (A_Ω) – (A_{u_0}) and u and u_n are entropy solutions of (1.3) and (2.10). Then*

$$u_n \rightarrow u \quad \text{as } n \rightarrow \infty \quad \text{in } L^p(Q) \quad \text{for all } p \in [1, \infty).$$

Similar results holds for a larger class of nonlocal operators \mathcal{L} , see Remark 6.22.

3. PRELIMINARIES ON THE LÉVY OPERATOR

This section is devoted to proving various identities and control estimates for the operators \mathcal{L} and \mathcal{B} . We will here say that the expressions $\mathcal{L}[\phi]$ and $\mathcal{B}[\psi, \varphi]$ are well-defined in $L^1(\mathbb{R}^d)$ when the limits $\lim_{r \searrow 0} \mathcal{L}^{\geq r}[\phi]$ and $\lim_{r \searrow 0} \mathcal{B}^{\geq r}[\psi, \varphi]$ exist in $L^1(\mathbb{R}^d)$.

3.1. Identities for \mathcal{L} and \mathcal{B} . We begin by establishing a connection between \mathcal{L} and \mathcal{B} and follow up with a product rule for \mathcal{B} . These results resemble well-known identities for Δ and ∇ , and the proofs boil down to truncations of the operators \mathcal{L} and \mathcal{B} , interchanging integrals, and change of variables. To ensure that Fubini's theorem can be applied, some 'absolute integrability' of the relevant components is required. For this purpose an assumption of compact support is included in the propositions (though various other assumptions would suffice) as we shall frequently encounter compactly supported functions.

Proposition 3.1 (Integration by parts formulas). *Assume (A_μ) and let either $\phi, \psi \in L^\infty(\mathbb{R}^d)$ have compact support. If both $\mathcal{L}[\phi]$ and $\mathcal{B}[\phi, \psi]$ are well-defined in $L^1(\mathbb{R}^d)$, then*

$$(3.1) \quad \int_{\mathbb{R}^d} \mathcal{L}[\phi]\psi \, dx = - \int_{\mathbb{R}^d} \mathcal{B}[\phi, \psi] \, dx.$$

If both $\mathcal{L}[\phi]$ and $\mathcal{L}[\psi]$ are well-defined in $L^1(\mathbb{R}^d)$, then

$$(3.2) \quad \int_{\mathbb{R}^d} \mathcal{L}[\phi]\psi \, dx = \int_{\mathbb{R}^d} \phi \mathcal{L}[\psi] \, dx.$$

Proof. We prove only (3.1) as the proof of (3.2) is similar. Fix $r > 0$, and note that $(z, x) \mapsto (\phi(x+z) - \phi(x))\psi(x)$ is absolutely integrable on the set $\{(z, x) \in \mathbb{R}^d \times \mathbb{R}^d : |z| \geq r\}$ with respect to $d\mu(z) \, dx$. We may then use Fubini's theorem to compute

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{L}^{\geq r}[\phi]\psi \, dx &= \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z) - \phi(x))\psi(x) \, d\mu(z) \, dx \\ &= \int_{|z| \geq r} \int_{\mathbb{R}^d} (\phi(x) - \phi(x-z))\psi(x-z) \, dx \, d\mu(z) \\ &= - \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z) - \phi(x))\psi(x+z) \, d\mu(z) \, dx, \end{aligned}$$

where we made the substitutions $x \mapsto x - z$ and then $z \mapsto -z$ exploiting the symmetry of μ . Thus

$$\int_{\mathbb{R}^d} \mathcal{L}^{\geq r}[\phi]\psi \, dx = \frac{1}{2} \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z) - \phi(x))(\psi(x) - \psi(x+z)) \, d\mu(z) \, dx = - \int_{\mathbb{R}^d} \mathcal{B}^{\geq r}[\phi, \psi] \, dx.$$

The result follows by letting $r \rightarrow 0$. □

Remark 3.2. If ψ in the previous proposition has compact support, the result holds when $\mathcal{L}[\phi]$ is (only) well-defined in the L^1_{loc} -sense; the proof is identical.

The next formula is a nonlocal analogue of the product rule $\nabla\phi \cdot \nabla(\psi\varphi) = (\nabla\phi \cdot \nabla\psi)\varphi + (\nabla\phi \cdot \nabla\varphi)\psi$.

Proposition 3.3 (Product rule). *Assume (A_μ) and either $\phi, \psi, \varphi \in L^\infty(\mathbb{R}^d)$ have compact support. Then*

$$(3.3) \quad \int_{\mathbb{R}^d} \mathcal{B}[\phi, \psi\varphi] \, dx = \int_{\mathbb{R}^d} \mathcal{B}[\phi, \psi]\varphi \, dx + \int_{\mathbb{R}^d} \mathcal{B}[\phi, \varphi]\psi \, dx,$$

provided the integrands are well-defined in $L^1(\mathbb{R}^d)$.

Proof. Fix $r > 0$. Similar to the previous proof, the compact support of one of ϕ, ψ, φ will guarantee that the following integrands are absolutely integrable with respect to $d\mu(z) \, dx$ on the given domain. By Fubini's theorem we compute

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathcal{B}^{\geq r}[\phi, \psi\varphi] - \mathcal{B}^{\geq r}[\phi, \psi]\varphi) \, dx &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z) - \phi(x))(\varphi(x+z) - \varphi(x))\psi(x+z) \, d\mu(z) \, dx \\ &= \frac{1}{2} \int_{|z| \geq r} \int_{\mathbb{R}^d} (\phi(x) - \phi(x-z))(\varphi(x) - \varphi(x-z))\psi(x) \, dx \, d\mu(z) \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z) - \phi(x))(\varphi(x+z) - \varphi(x))\psi(x) \, d\mu(z) \, dx \\ &= \int_{\mathbb{R}^d} \mathcal{B}^{\geq r}[\phi, \varphi]\psi \, dx, \end{aligned}$$

where we made the substitution $x \mapsto x - z$ followed by $z \mapsto -z$ and exploited the symmetry of $d\mu(z)$. The result follows by letting $r \rightarrow 0$. \square

3.2. Bounds for \mathcal{L} and \mathcal{B} . We here provide bounds and inequalities that will be used to make sense of the expression $\mathcal{L}[\phi]$ and $\mathcal{B}[\psi, \varphi]$ in non-trivial situations. We begin with a standard estimate for smooth functions.

Lemma 3.4. *Let $p \in [1, \infty]$, $\phi \in C_c^2(\mathbb{R}^d)$, and let $D^2\phi(x)$ denote the Hessian matrix of ϕ at x . Then*

$$(3.4) \quad \|\mathcal{L}[\phi]\|_{L^p(\mathbb{R}^d)} \leq \left[2\|\phi\|_{L^p(\mathbb{R}^d)} + \frac{1}{2}\|D^2\phi\|_{L^p(\mathbb{R}^d)} \right] \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z),$$

where $\|D^2\phi\|_{L^p(\mathbb{R}^d)}$ denotes the L^p norm of $x \mapsto |D^2\phi(x)|$, with $|D^2\phi|$ being the spectral radius of $D^2\phi$.

Up to the choice of constants, this result is proved e.g. in [58], see Lemma 2.1 and Remark 2.2.

When $\phi \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, the previous lemma guarantees that $\mathcal{L}[\phi]$ makes sense as a distribution if we canonically define $\langle \mathcal{L}[\phi], \varphi \rangle := \langle \phi, \mathcal{L}[\varphi] \rangle$ for all $\varphi \in C_c^2(\mathbb{R})$. Moreover, it follows that $\mathcal{L}^{\geq r}[\phi] \rightarrow \mathcal{L}[\phi]$ in the sense of distributions as $r \rightarrow 0$. However, a distributional notion of $\mathcal{L}[\phi]$ will often be too weak for our purpose, and so the next proposition is useful.

Proposition 3.5 (Lévy operator on compositions). *Assume $h : \mathbb{R} \rightarrow \mathbb{R}$ admits a weak derivative of bounded variation and $\phi \in L^\infty(\mathbb{R}^d)$ has compact support. If $\mathcal{L}[\phi]$ is well-defined in $L^1(\mathbb{R}^d)$, then so is $\mathcal{L}[h(\phi)]$, and moreover we have*

$$(3.5) \quad \|\mathcal{L}[h(\phi)]\|_{L^1(\mathbb{R}^d)} \leq \left[\|h'\|_{L^\infty(\mathbb{R})} + |h'|_{TV(\mathbb{R})} \right] \|\mathcal{L}[\phi]\|_{L^1(\mathbb{R}^d)}.$$

Remark 3.6. Note that $|h'|_{TV(\mathbb{R})} < \infty$ implies $\|h'\|_{L^\infty(\mathbb{R})} < \infty$ so that h is actually Lipschitz.

Proof of Proposition 3.5. As h' is of bounded variation, associate it with its left-continuous representation (so that the following pointwise estimates are unambiguous). We consider first the case when the Lévy measure is finite $\mu(\mathbb{R}^d) < \infty$, so that \mathcal{L} is a zero order operator; in particular we have

$\mathcal{L}[\psi] = \mu * \psi - \mu(\mathbb{R}^d)\psi$ for any $\psi \in L^\infty(\mathbb{R}^d)$. Due to its bounded variation, h' admits an essentially unique representation $h' = \beta_+ - \beta_-$ for two non-decreasing functions β_+, β_- satisfying

$$|h'|_{TV(\mathbb{R})} = |\beta_+|_{TV(\mathbb{R})} + |\beta_-|_{TV(\mathbb{R})} = \beta_+(\infty) + \beta_-(\infty) - \beta_+(-\infty) - \beta_-(-\infty).$$

Introducing then the monotone function $\beta(v) := \beta_+(v) + \beta_-(v) - \frac{1}{2}|h'|_{TV(\mathbb{R})}$, which has been shifted to satisfy $2\|\beta\|_{L^\infty(\mathbb{R})} = |h'|_{TV(\mathbb{R})}$, we observe that we have the inequality

$$(3.6) \quad |h'(v+w) - h'(v)| \leq \text{sgn}(w)(\beta(v+w) - \beta(v)),$$

for any $v, w \in \mathbb{R}$.

In the next computation, let $\delta := \phi(x+z) - \phi(x)$ for notational simplicity. Subtracting $h'(\phi)\mathcal{L}[\phi]$ from $\mathcal{L}[h(\phi)]$ we find for a.e. $x \in \mathbb{R}^d$

$$(3.7) \quad \begin{aligned} \left| \mathcal{L}[h(\phi)] - h'(\phi)\mathcal{L}[\phi] \right|(x) &= \left| \int_{\mathbb{R}^d} h(\phi(x) + \delta) - h(\phi(x)) - h'(\phi(x))\delta \, d\mu(z) \right| \\ &\leq \int_{\mathbb{R}^d} \int_0^1 |(h'(\phi(x) + s\delta) - h'(\phi(x)))\delta| \, ds \, d\mu(z) \\ &\leq \int_{\mathbb{R}^d} \int_0^1 (\beta(\phi(x) + s\delta) - \beta(\phi(x)))\delta \, ds \, d\mu(z), \end{aligned}$$

where we used (3.6). By monotonicity of β , we further have $(\beta(\phi + s\delta) - \beta(\phi))\delta \leq (\beta(\phi + \delta) - \beta(\phi))\delta$ whenever $s \in [0, 1]$. And so by (3.7) we infer

$$(3.8) \quad \|\mathcal{L}[h(\phi)] - h'(\phi)\mathcal{L}[\phi]\|_{L^1(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} 2\mathcal{B}[\beta(\phi), \phi] \, dx \leq 2\|\beta\|_{L^\infty(\mathbb{R}^d)}\|\mathcal{L}[\phi]\|_{L^1(\mathbb{R}^d)},$$

where we shifted \mathcal{B} over to \mathcal{L} using Proposition 3.1. Thus, (3.5) follows for zero order \mathcal{L} when combining the triangle inequality with (3.8) and the bound $2\|\beta\|_{L^\infty(\mathbb{R}^d)} = |h'|_{TV(\mathbb{R})}$.

For general \mathcal{L} , we first note that $\mathcal{L}[h(\phi)]$ is necessarily well-defined in $L^1(\mathbb{R}^d)$: Let $r' > r > 0$ and note that

$$\begin{aligned} \limsup_{r', r \rightarrow 0} \|\mathcal{L}^{\geq r'}[h(\phi)] - \mathcal{L}^{\geq r}[h(\phi)]\|_{L^1(\mathbb{R}^d)} &= \limsup_{r', r \rightarrow 0} \|\mathcal{L}^{r' > \dots \geq r}[h(\phi)]\|_{L^1(\mathbb{R}^d)} \\ &\lesssim \limsup_{r', r \rightarrow 0} \|\mathcal{L}^{r' > \dots \geq r}[\phi]\|_{L^1(\mathbb{R}^d)} = 0, \end{aligned}$$

where we used (3.5) on the zero order Lévy operator $\mathcal{L}^{r' > \dots \geq r}$ and the fact that $\mathcal{L}[\phi]$ is well-defined in $L^1(\mathbb{R}^d)$. The inequality (3.5) then follows for $\mathcal{L}[h(\phi)]$ by using the corresponding one for $\mathcal{L}^{\geq r}[h(\phi)]$ and letting $r \rightarrow 0$. \square

We combine the previous two results to compute a local L^1 bound for $\mathcal{L}[h(\phi)]$ with h as above and with ϕ bounded and locally C^2 .

Corollary 3.7. *Let h be as in Proposition 3.5, and let $\phi \in C_b^2(\mathbb{R}^d)$. Then for any bounded measurable set $U \subset \mathbb{R}^d$ and any smooth cut-off function $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfying $\psi(x) = 1$ whenever $\text{dist}(x, U) \leq 1$ we have the bound*

$$(3.9) \quad \begin{aligned} \|\mathcal{L}[h(\phi)]\|_{L^1(U)} &\leq \left[\|h'\|_{L^\infty(\mathbb{R})} + |h'|_{TV(\mathbb{R})} \right] \times \\ &\quad \times \left[2\|\phi\|_{L^\infty(\mathbb{R}^d)}|U| + 2\|\phi\psi\|_{L^1(\mathbb{R}^d)} + \frac{1}{2}\|D^2(\phi\psi)\|_{L^1(\mathbb{R}^d)} \right] \times \\ &\quad \times \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z), \end{aligned}$$

where $|U|$ denotes the Lebesgue measure of U while $\|D^2(\phi\psi)\|_{L^1(\mathbb{R}^d)}$ is as in Lemma 3.4. Thus $\mathcal{L}[h(\phi)] \in L^1_{\text{loc}}(\mathbb{R}^d)$, and in particular, we have

$$\lim_{r \rightarrow 0} \mathcal{L}^{\geq r}[h(\phi)] = \mathcal{L}[h(\phi)] \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^d).$$

Proof. To prove the bound (3.9), let us decompose $\mathcal{L} = \mathcal{L}^{\geq 1} + \mathcal{L}^{< 1}$. As $\mathcal{L}^{< 1}[h(\phi)] = \mathcal{L}^{< 1}[h(\phi\psi)]$ on U , we find

$$\begin{aligned} \|\mathcal{L}^{< 1}[h(\phi)]\|_{L^1(U)} &\leq \|\mathcal{L}^{< 1}[h(\phi\psi)]\|_{L^1(\mathbb{R}^d)} \\ &\leq \left[\|h'\|_{L^\infty(\mathbb{R})} + |h'|_{TV(\mathbb{R})} \right] \left[2\|\phi\psi\|_{L^1(\mathbb{R}^d)} + \frac{1}{2}\|D^2(\phi\psi)\|_{L^1(\mathbb{R}^d)} \right] \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z), \end{aligned}$$

where we used Proposition 3.5 and Lemma 3.4. As for $\mathcal{L}^{\geq 1}[h(\phi)]$, we compute

$$\begin{aligned} \|\mathcal{L}^{\geq 1}[h(\phi)]\|_{L^1(U)} &\leq \int_U \int_{|z| \geq 1} 2\|h'\|_{L^\infty(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R}^d)} \, d\mu(z) \, dx \\ &\leq 2 \left[\|h'\|_{L^\infty(\mathbb{R})} + |h'|_{TV(\mathbb{R})} \right] \|\phi\|_{L^\infty(\mathbb{R}^d)} |U| \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu(z), \end{aligned}$$

where the coarseness of the second inequality is so the bound factors nicely with that of $\|\mathcal{L}^{< 1}[h(\phi)]\|_{L^1(U)}$. Combining these two bounds we get (3.9). The fact that $\lim_{r \rightarrow 0} \mathcal{L}^{\geq r}[h(\phi)] = \mathcal{L}[h(\phi)]$ in $L^1_{\text{loc}}(\mathbb{R})$ follows if we can prove L^1 convergence on U as the latter set was arbitrary. This fact follows immediately from (3.9) with $\mathcal{L}^{< r}$ as the respective Lévy operator; we get

$$\|\mathcal{L}[h(\phi)] - \mathcal{L}^{\geq r}[h(\phi)]\|_{L^1(U)} = \|\mathcal{L}^{< r}[h(\phi)]\|_{L^1(U)} \lesssim \int_{|z| < r} (|z|^2 \wedge 1) \, d\mu(z),$$

which tends to zero as $r \rightarrow 0$. □

We end the section by listing a few sufficient conditions on $\psi, \varphi \in L^\infty(\mathbb{R}^d)$ for $\mathcal{B}[\psi, \varphi]$ to be well-defined. Here we shall say that ‘the integrand of $\mathcal{B}[\psi, \varphi]$ is absolutely integrable with respect to $d\mu(z) \, dx$ ’ to mean

$$(3.10) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} |\psi(x+z) - \psi(x)| |\varphi(x+z) - \varphi(x)| \, d\mu(z) \, dx < \infty.$$

This condition ameliorates the expression $\mathcal{B}[\psi, \varphi]$ and in particular ensures it to be well-defined in $L^1(\mathbb{R}^d)$; even the pointwise definition (2.1) is then necessarily meaningful almost everywhere.

Proposition 3.8 (List of \mathcal{B} -compatible pairs). *Let $\psi, \varphi \in L^\infty(\mathbb{R}^d)$. The integrand of $\mathcal{B}[\psi, \varphi]$ is absolutely integrable with respect to $d\mu(z) \, dx$ if either of the following conditions are satisfied:*

- (i) $\psi, \varphi \in H^{\mathcal{L}}(\mathbb{R}^d)$.
- (ii) $\psi \in H^{\mathcal{L}}(\mathbb{R}^d)$, φ is Lipschitz continuous, and ψ or φ has compact support.
- (iii) ψ is of bounded variation, φ is Lipschitz continuous, and ψ or φ has compact support.
- (iv) ψ or φ has compact support and the Lévy measure μ is finite.

Proof. (i). This follows from the definition of $H^{\mathcal{L}}(\mathbb{R}^d)$ and the Cauchy–Schwarz inequality.

(ii). Due to the compact support, there is some bounded measurable set $E \in \mathbb{R}^d$ such that

$$\begin{aligned} &\int_{\mathbb{R}^d \times \mathbb{R}^d} |\psi(x+z) - \psi(x)| |\varphi(x+z) - \varphi(x)| \, d\mu(z) \, dx \\ &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |\psi(x+z) - \psi(x)| |\varphi(x+z) - \varphi(x)| (1_E(x) + 1_E(x+z)) \, d\mu(z) \, dx. \end{aligned}$$

By splitting the integral on the right hand side in two, one integral featuring $1_E(x)$ and the other $1_E(x+z)$, one may perform the substitution of variable $x \mapsto x-z$ followed by $z \mapsto -z$ to see that the two coincide. Thus, we need only bound the first. By the Lipschitz continuity of φ and an application of Young's inequality we find

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} |\psi(x+z) - \psi(x)| |\varphi(x+z) - \varphi(x)| 1_E(x) \, d\mu(z) \, dx \\ & \leq \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(|\psi(x+z) - \psi(x)|^2 + \left((L|z|)^2 \wedge (2\|\varphi\|_{L^\infty(\mathbb{R})})^2 \right) 1_E(x) \right) d\mu(z) \, dx, \end{aligned}$$

where L is a Lipschitz constant for φ . The integral on the right-hand side is finite: Indeed, the first part of the integrand is integrable as $\psi \in H^{\mathcal{L}}(\mathbb{R}^d)$ while the second part is integrable as it has compact support in x and is bounded by $C(|z|^2 \wedge 1)$ for an appropriate constant C .

(iii). By the same trick as before, it will suffice to prove (3.10) for x restricted to some bounded measurable set E . By Lipschitz continuity of φ and ψ being of bounded variation we find

$$\begin{aligned} \int_E |\psi(x+z) - \psi(x)| |\varphi(x+z) - \varphi(x)| \, dx & \leq \int_E |\psi(x+z) - \psi(x)| \, dx \left(L|z| \wedge 2\|\varphi\|_{L^\infty(\mathbb{R}^d)} \right) \\ & \leq \left(|\psi|_{TV(\mathbb{R})} |z| \wedge 2|E| \|\psi\|_{L^\infty(\mathbb{R}^d)} \right) \left(L|z| \wedge 2\|\varphi\|_{L^\infty(\mathbb{R}^d)} \right) \\ & \leq C(|z|^2 \wedge 1), \end{aligned}$$

where $|E|$ denotes the d -dimensional Lebesgue measure of E and C is some large constant. Integrating over $z \in \mathbb{R}^d$ with respect to $d\mu(z)$ and we are done.

(iv). As before, we may restrict x to some bounded measurable set E , and the result then follows by replacing ψ, φ with their L^∞ norms. \square

One expression we will encounter frequently is $\mathcal{B}[\chi_\Omega, \varphi]$ where χ_Ω is the characteristic function on our bounded domain (with C^2 boundary) and where φ is bounded and Lipschitz continuous. It is a fact that $|\chi_\Omega|_{TV(\mathbb{R}^d)} = |\partial\Omega|_{\mathcal{H}^{d-1}} < \infty$, where $|\cdot|_{\mathcal{H}^{d-1}}$ is the $(d-1)$ -dimensional Hausdorff measure. This equality follows from the definition of $|\cdot|_{TV(\mathbb{R}^d)}$ and the Gauss divergence theorem. By the previous proposition we conclude that $\mathcal{B}[\chi_\Omega, \varphi]$ is well-defined in $L^1(\mathbb{R}^d)$.

4. PROPERTIES OF ENTROPY SOLUTIONS

In this section we establish some results and a priori estimates for entropy solutions of (1.1).

4.1. Finite energy. We will repeatedly need that entropy solutions have finite energy.

Proposition 4.1 (Energy estimate). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$\begin{aligned} & \int_M \mathcal{B}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \, dx \, dt \\ (4.1) \quad & \leq \int_\Omega H(u_0, \bar{u}^c(0)) \, dx - \int_Q \left[(u - \bar{u}^c) \bar{u}_t^c + F(u, \bar{u}^c) \cdot \nabla \bar{u}^c \right] b'(\bar{u}^c) \, dx \, dt \\ & \quad + \int_Q \mathcal{L}[b(\bar{u}^c)] (b(u) - b(\bar{u}^c)) \, dx \, dt, \end{aligned}$$

where $F(u, \bar{u}^c) = \text{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c))$ and where H is the following nonnegative function

$$H(u, k) := \int_k^u (b(\xi) - b(k)) \, d\xi.$$

Moreover, the right-hand side of (4.1) is finite.

Proof. The family of entropy inequalities (2.6) are those corresponding to the semi-Kružkov entropy pairs

$$u \mapsto ((u - k)^\pm, F^\pm(u, k)).$$

We shall now construct a second family of entropy inequalities corresponding to the entropy pairs $u \mapsto (H^\pm(u, k), G^\pm(u, k))$ defined by

$$(4.2) \quad \begin{aligned} H^+(u, k) &= \int_k^\infty (u - \xi)^+ b'(\xi) \, d\xi, & G^+(u, k) &= \int_k^\infty F^+(u, \xi) b'(\xi) \, d\xi, \\ H^-(u, k) &= \int_{-\infty}^k (u - \xi)^- b'(\xi) \, d\xi, & G^-(u, k) &= \int_{-\infty}^k F^-(u, \xi) b'(\xi) \, d\xi. \end{aligned}$$

We proceed by performing the calculations for the (+) and (-) case simultaneously, though we stress that one should consider them separate calculations.

Let $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty(M)$ be such that

$$(4.3) \quad (b(u^c) - b(k))^\pm \varphi = 0,$$

for all $(t, x) \in Q^c$. Observe that (4.3) is still valid when replacing k with ξ provided $\pm \xi \geq \pm k$. In particular, it follows that we for all $\pm \xi \geq \pm k$ have (2.6) with ξ replacing k , yielding

$$(4.4) \quad \begin{aligned} & - \int_Q ((u - \xi)^\pm \varphi_t + F^\pm(u, \xi) \cdot \nabla \varphi) \, dx \, dt \\ & - \int_Q \mathcal{L}^{\geq r}[b(u)] \operatorname{sgn}^\pm(u - \xi) \varphi \, dx \, dt \\ & - \int_M (b(u) - b(\xi))^\pm \mathcal{L}^{< r}[\varphi] \, dx \, dt \\ & \leq L_f \int_\Gamma (\bar{u}^c - \xi)^\pm \varphi \, d\sigma(x) \, dt, \end{aligned}$$

where there is no ‘initial term’ on the right-hand side as $\varphi(0, x) = 0$. Multiplying (4.4) by the nonnegative quantity $b'(\xi)$ and integrating over $\xi \in (k, \infty)$ in the (+) case and $\xi \in (-\infty, k)$ in the (-) case we get

$$(4.5) \quad \begin{aligned} & - \int_Q (H^\pm(u, k) \varphi_t + G^\pm(u, k) \cdot \nabla \varphi) \, dx \, dt \\ & \mp \int_M \mathcal{L}^{\geq r}[b(u)] (b(u) - b(k))^\pm \varphi \, dx \, dt \\ & - \frac{1}{2} \int_M ((b(u) - b(k))^\pm)^2 \mathcal{L}^{< r}[\varphi] \, dx \, dt \\ & \leq L_f \int_\Gamma H^\pm(\bar{u}^c, k) \varphi \, d\sigma(x) \, dt, \end{aligned}$$

where H^\pm and G^\pm are as in (4.2). We have for later convenience extended the domain of integration for the second integral in (4.5) from Q to M ; this is possible as $(b(u) - b(k))^\pm \varphi$ is zero a.e. in Q^c . To summarize, (4.5) is valid for all $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty(M)$ satisfying (4.3).

Next, we wish to set $k = \bar{u}^c(t, x)$ in (4.5), and while this cannot be done directly, we will accomplish this by a ‘doubling of variables’ argument. For parameters $(s, y) \in M$ and $\varepsilon \in (0, 1)$, we introduce the

pair $(\kappa^\pm, \psi) \in \mathbb{R} \times C_c^\infty(M)$ defined as follows

$$(4.6) \quad \begin{aligned} \kappa^\pm &= \kappa_{s,y,\varepsilon}^\pm = \bar{u}^c(s, y) \pm \varepsilon L_{\bar{u}^c}, \\ \psi &= \psi_{s,y,\varepsilon}(t, x) = \tilde{\varphi}(t, x) \rho_\varepsilon(t - s, x - y), \end{aligned}$$

where $0 \leq \tilde{\varphi} \in C_c^\infty(M)$ is arbitrary, ρ_ε is a standard mollifier with support in the centered ball of radius ε in $\mathbb{R} \times \mathbb{R}^d$ and $L_{\bar{u}^c}$ is a local Lipschitz constant for \bar{u}^c valid on $\{(t, x) \in M : \text{dist}((t, x), \text{supp } \tilde{\varphi}) < 1\}$. The support of ρ_ε implies that ψ is zero when $|(t - s, x - y)| \geq \varepsilon$, while $|(t - s, x - y)| < \varepsilon$ implies that $\pm \kappa^\pm \geq \pm \bar{u}^c(t, x)$. In particular, we have

$$(b(\bar{u}^c(t, x)) - b(\kappa_{s,y,\varepsilon}^\pm))^\pm \psi_{s,y,\varepsilon}(t, x) = 0,$$

for all t, s, x, y, ε , and we may thus use the pair (κ^\pm, ψ) as the constant and test function in (4.5). Doing so, and additionally writing φ instead of $\tilde{\varphi}$ for notional simplicity, we get after integrating over $(s, y) \in M$

$$(4.7) \quad \begin{aligned} & - \int_{Q_x \times M_y} \left[H^\pm(u, \kappa^\pm) \varphi_t + G^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi \right] \rho_\varepsilon \\ & \mp \int_{M_x \times M_y} \mathcal{L}_x^{\geq r} [b(u)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon \\ & - \frac{1}{2} \int_{M_x \times M_y} ((b(u) - b(\kappa^\pm))^\pm)^2 \mathcal{L}_x^{< r} [\varphi \rho_\varepsilon] \\ & \leq - \int_{Q_x \times M_y} \left[H^\pm(u, \kappa^\pm) \partial_s \rho_\varepsilon + G^\pm(u, \kappa^\pm) \cdot \nabla_y \rho_\varepsilon \right] \varphi. \end{aligned}$$

For brevity, we have here suppressed the measure $dt dx ds dy$ from the integrals and added sub-indexes to the operators (clarifying in what variables they act) and the domains of integration (clarifying which variables they are parameterized by); here Q_x and M_x are parameterized by (t, x) , while Q_y and M_y are parameterized by (s, y) . We also used the identities $\partial_t \rho_\varepsilon = -\partial_s \rho_\varepsilon$ and $\nabla_x \rho_\varepsilon = -\nabla_y \rho_\varepsilon$ and dropped the term on the right-hand side of (4.5) since

$$H^\pm(\bar{u}^c(t, x), \kappa^\pm) \varphi \rho_\varepsilon = 0,$$

for all t, x, s, y, ε .

Next, we integrate by parts in (s, y) on the right-hand side of (4.7) and add the integral of the quantity $\pm \mathcal{L}_y^{\geq r} [b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon$ to each side:

$$(4.8) \quad \begin{aligned} & - \int_{Q_x \times M_y} \left[H^\pm(u, \kappa^\pm) \varphi_t + G^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi \right] \rho_\varepsilon \\ & \mp \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon \\ & - \frac{1}{2} \int_{M_x \times M_y} ((b(u) - b(\kappa^\pm))^\pm)^2 \mathcal{L}_x^{< r} [\varphi \rho_\varepsilon] \\ & \leq \mp \int_{Q_x \times M_y} \left[(u - \kappa^\pm)^\pm \partial_s \kappa^\pm + F^\pm(u, \kappa^\pm) \cdot \nabla_y \kappa^\pm \right] b'(\kappa^\pm) \varphi \rho_\varepsilon \\ & \pm \int_{Q_x \times M_y} \mathcal{L}_y^{\geq r} [b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon. \end{aligned}$$

We here used that $\mathcal{L}_{x+y}^{\geq r}[w_1 - w_2](x, y) = \mathcal{L}_x^{\geq r}[w_1](x) - \mathcal{L}_y^{\geq r}[w_2](y)$ when w_1 and w_2 are functions in x and y respectively, and we used the formulas

$$\begin{aligned}\partial_s H^\pm(u, \kappa^\pm) &= \mp (u - \kappa^\pm)^\pm b'(\kappa^\pm) \partial_s \kappa^\pm, \\ \nabla_y G^\pm(u, \kappa^\pm) &= \mp F^\pm(u, \kappa^\pm) b'(\kappa^\pm) \nabla_y \kappa^\pm,\end{aligned}$$

which follow from (4.2). Fixing $r' > r$ we split up the operator $\mathcal{L}_{x+y}^{\geq r} = \mathcal{L}_{x+y}^{\geq r'} + \mathcal{L}_{x+y}^{r' > \dots \geq r}$. Looking at the first inequality of Corollary C.2 for the choice $\eta(\phi) = \frac{1}{2}(\phi^\pm)^2$, we get the pointwise inequality

$$\pm \mathcal{L}_{x+y}^{r' > \dots \geq r}b(u) - b(\kappa^\pm)^\pm \leq \frac{1}{2} \mathcal{L}_{x+y}^{r' > \dots \geq r} \left[((b(u) - b(\kappa^\pm))^\pm)^2 \right].$$

We thus conclude

$$\begin{aligned}& \mp \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r}b(u) - b(\kappa^\pm)^\pm \varphi \rho_\varepsilon \\ & \geq \mp \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r'}b(u) - b(\kappa^\pm)^\pm \varphi \rho_\varepsilon \\ & \quad - \frac{1}{2} \int_{M_x \times M_y} ((b(u) - b(\kappa^\pm))^\pm)^2 \mathcal{L}_x^{r' > \dots \geq r}[\varphi] \rho_\varepsilon,\end{aligned}$$

where we used the self-adjointness of $\mathcal{L}_x^{r' > \dots \geq r}$ (Proposition 3.1) to move it over to $\varphi \rho_\varepsilon$ on which the operator satisfies $\mathcal{L}_x^{r' > \dots \geq r}[\varphi \rho_\varepsilon] = \mathcal{L}_x^{r' > \dots \geq r}[\varphi] \rho_\varepsilon$. Exploiting this in (4.8), letting $r \rightarrow 0$ and writing for simplicity r instead of r' we get

$$\begin{aligned}(4.9) \quad & - \int_{Q_x \times M_y} \left[H^\pm(u, \kappa^\pm) \varphi_t + G^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi \right] \rho_\varepsilon \\ & \mp \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r}b(u) - b(\kappa^\pm)^\pm \varphi \rho_\varepsilon \\ & - \frac{1}{2} \int_{M_x \times M_y} ((b(u) - b(\kappa^\pm))^\pm)^2 \mathcal{L}_x^{< r}[\varphi] \rho_\varepsilon \\ & \leq \mp \int_{Q_x \times M_y} \left[(u - \kappa^\pm)^\pm \partial_s \kappa^\pm + F^\pm(u, \kappa^\pm) \cdot \nabla_y \kappa^\pm \right] b'(\kappa^\pm) \varphi \rho_\varepsilon \\ & \quad \pm \int_{Q_x \times M_y} \mathcal{L}_y^{\geq r}[b(\kappa^\pm)](b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon,\end{aligned}$$

which only differs from (4.8) in that ρ_ε is outside of the operator $\mathcal{L}_x^{< r}$.

We may now let $\varepsilon \rightarrow 0$ and by the regularity of \bar{u}^c , b , H^\pm and G^\pm we get through standard estimates

$$\begin{aligned}(4.10) \quad & - \int_Q \left[H^\pm(u, \bar{u}^c) \varphi_t + G^\pm(u, \bar{u}^c) \cdot \nabla \varphi \right] \\ & \mp \int_M \mathcal{L}^{\geq r}b(u) - b(\bar{u}^c)^\pm \varphi \\ & - \frac{1}{2} \int_M ((b(u) - b(\bar{u}^c))^\pm)^2 \mathcal{L}^{< r}[\varphi] \\ & \leq \mp \int_Q \left[(u - \bar{u}^c)^\pm \bar{u}_t^c + F^\pm(u, \bar{u}^c) \cdot \nabla \bar{u}^c \right] b'(\bar{u}^c) \varphi \\ & \quad \pm \int_Q \mathcal{L}^{\geq r}[b(\bar{u}^c)](b(u) - b(\bar{u}^c))^\pm \varphi,\end{aligned}$$

where \bar{u}^c is here a function in (t, x) ; now the only variables of integration.

Set $\varphi(t, x) = \theta(t)\phi(x)$, where $0 \leq \theta \in C_c^\infty((0, T))$ and $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$, and where $\phi(x) = 1$ whenever $\text{dist}(x, \Omega) \leq r$ so that both $\nabla\phi$ and $\mathcal{L}^{<r}[\phi]$ are zero in Q . Then (4.10) collapses to

$$\begin{aligned}
& - \int_Q H^\pm(u, \bar{u}^c)\theta' \\
& \mp \int_M \mathcal{L}^{\geq r}b(u) - b(\bar{u}^c)^\pm \theta \\
(4.11) \quad & \leq \mp \int_Q \left[(u - \bar{u}^c)^\pm \bar{u}_t^c + F^\pm(u, \bar{u}^c) \cdot \nabla \bar{u}^c \right] b'(\bar{u}^c)\theta \\
& \pm \int_Q \mathcal{L}^{\geq r}[b(\bar{u}^c)](b(u) - b(\bar{u}^c))^\pm \theta,
\end{aligned}$$

where we used that $b(u) - b(\bar{u}^c) = 0$ in $M \setminus Q$. Add the (+) case to the (-) case, and we further get

$$\begin{aligned}
& - \int_Q H(u, \bar{u}^c)\theta' \\
& + \int_M \mathcal{B}^{\geq r}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)]\theta \\
(4.12) \quad & \leq - \int_Q \left[(u - \bar{u}^c)\bar{u}_t^c + F(u, \bar{u}^c) \cdot \nabla \bar{u}^c \right] b'(\bar{u}^c)\theta \\
& + \int_Q \mathcal{L}^{\geq r}[b(\bar{u}^c)](b(u) - b(\bar{u}^c))\theta.
\end{aligned}$$

where $H = H^+ + H^-$ and we shifted $-\mathcal{L}^{\geq r}$ over to $\mathcal{B}^{\geq r}$ using Proposition 3.1.

Finally, for $\epsilon > 0$ let $\theta = \theta_\epsilon$ be such that $\theta_\epsilon(t) = 1$ for $t \in [\epsilon, T - \epsilon]$ and $\|\theta'_\epsilon\|_{L^1(0, T)} = 2$. Letting $r, \epsilon \rightarrow 0$ we may for the first term in (4.12) use the continuity of u at $t = 0$ (Lemma 2.5) and the positivity of H to conclude that

$$\liminf_{r, \epsilon \rightarrow 0} \left(- \int_Q H(u, \bar{u}^c)\theta'_\epsilon \right) \geq - \int_\Omega H(u_0, \bar{u}^c(0)),$$

where $\bar{u}^c(0) = \bar{u}^c(0, \cdot)$. For the second term, we apply Fatou's lemma to conclude

$$\liminf_{r, \epsilon \rightarrow 0} \int_M \mathcal{B}^{\geq r}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)]\theta_\epsilon \geq \int_M \mathcal{B}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)].$$

By Corollary 3.7, the last term in (4.12) converges to its canonical limit, and we conclude as $r, \epsilon \rightarrow 0$

$$\begin{aligned}
& - \int_\Omega H(u_0, \bar{u}^c(0)) \\
& + \int_M \mathcal{B}[b(u) - b(\bar{u}^c), b(u) - b(\bar{u}^c)] \\
(4.13) \quad & \leq - \int_Q \left[(u - \bar{u}^c)\bar{u}_t^c + F(u, \bar{u}^c) \cdot \nabla \bar{u}^c \right] b'(\bar{u}^c) \\
& + \int_Q \mathcal{L}[b(\bar{u}^c)](b(u) - b(\bar{u}^c)).
\end{aligned}$$

The proposition is proved. \square

4.2. Regularity results near the boundary. In Lemma 4.3 below we present a family of inequalities that follow from Definition 2.2. From this family, we shall derive appropriate boundary regularity of an entropy solution, which is vital for the uniqueness argument. The proof is similar to that of Proposition 4.1, but here we perform coarser computations resulting in a less explicit inequality (4.14) compared to (4.1). This is due to low-regularity terms that are difficult to handle when ‘going to the diagonal’ in the doubling of variables argument. Hence, these terms are replaced by upper bounds; while precise limits can be computed, we have instead chosen simpler and coarser calculations.

Remark 4.2. The coming lemma is analogous to Lemma 4.2 in [89], but the two proofs differ in that the main steps are interchanged. In [89] the (local analogue of the) below inequality is proved first for test functions compactly supported inside the domain, done through a doubling of variables argument. Afterwards, the result is extended up to the boundary using a boundary layer sequence satisfying both $\inf_{\delta>0} \inf_{x \in \Omega} \Delta \zeta_\delta(x) \geq 0$ and $\sup_{\delta>0} \|\nabla \zeta_\delta\|_{L^1(\Omega)} < \infty$. Both properties of ζ_δ are crucial for this extension argument, and it is not clear if an analogous boundary layer sequence always exists in our case (where \mathcal{L} replaces Δ). Thus, we instead prove the below inequality *directly* for test functions support up to the boundary. This approach bypasses the need for a special boundary layer sequence, but a new difficulty arises: The admissibility condition (2.5) outside the domain hinders a straight forward doubling of variables argument, and the data \bar{u}^c must be ‘lifted’ as done in the proof of Proposition 4.1. This lifting leads to expressions like $\mathcal{L}[b(\bar{u}^c \pm \varepsilon)]$, which we control by imposing regularity on \bar{u}^c and b separately rather than just on their composition $b(\bar{u}^c)$ as done in [89].

Lemma 4.3. *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then for $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and $k \in \mathbb{R}$,*

$$\begin{aligned}
(4.14) \quad & - \int_Q \left[(u - (\bar{u}^c \vee^\pm k))^\pm \varphi_t + F^\pm(u, \bar{u}^c \vee^\pm k) \cdot \nabla \varphi \right] dx dt \\
& + \int_M \mathcal{B}[(b(u) - b(\bar{u}^c \vee^\pm k))^\pm, \varphi] dx dt \\
& \leq \int_Q \left[(1 + L_f) L_{\bar{u}^c} + |\mathcal{L}[b((\bar{u}^c \vee^\pm k))]| \right] \varphi dx dt \\
& + \int_\Omega \left[\|u_0\|_{L^\infty(\Omega)} + \|\bar{u}^c\|_{L^\infty(M)} \right] \varphi(0, \cdot) dx,
\end{aligned}$$

where $F(u, v) := \text{sgn}(u - v)(f(u) - f(v))$.

Proof. We start from the definition of entropy solutions and the entropy inequality (2.6). Seeking to replace k by $\bar{u}^c \vee^\pm k$, we double the variables; the first part of this proof is similar to that of Proposition 4.1. In particular we also here carry out the proof of the (+) and (−) case simultaneously.

For parameters $(s, y) \in M$, $k \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, we introduce the pair $(\kappa^\pm, \psi) \in \mathbb{R} \times C_c^\infty(M)$ defined as follows

$$\begin{aligned}
(4.15) \quad & \kappa^\pm = \kappa_{s, y, \varepsilon, k}^\pm = (\bar{u}^c(s, y) \pm \varepsilon L_{\bar{u}^c}) \vee^\pm k, \\
& \psi = \psi_{s, y, \varepsilon}(t, x) = \varphi(t, x) \rho_\varepsilon(t - s, x - y),
\end{aligned}$$

where $0 \leq \varphi \in C_c^\infty(M)$ is arbitrary, ρ_ε is a standard mollifier with support in the centered ball of radius ε in $\mathbb{R} \times \mathbb{R}^d$ and $L_{\bar{u}^c}$ is a Lipschitz constant for \bar{u}^c valid on $\{(t, x) \in M : \text{dist}((t, x), \text{supp } \tilde{\varphi}) < 1\}$. The support of ρ_ε implies that ψ is zero when $|(t - s, x - y)| \geq \varepsilon$, while $|(t - s, x - y)| < \varepsilon$ implies that $\pm \kappa^\pm \geq \pm \bar{u}^c(t, x)$. In particular, we have

$$(4.16) \quad (b(\bar{u}^c(t, x)) - b(\kappa^\pm))^\pm \psi(t, x) = 0,$$

for all t, s, x, y, ε . Thus, the pair (κ^\pm, ψ) qualifies for use in the entropy inequality (2.6) as the constant and test function (which are denoted k, φ in said inequality, and should not to be confused with the same symbols used here). Inserting this pair in (2.6), we get after integrating over $(s, y) \in M$

$$\begin{aligned}
(4.17) \quad & - \int_{Q_x \times M_y} \left[(u - \kappa^\pm)^\pm \varphi_t + F^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi \right] \rho_\varepsilon \\
& - \int_{Q_x \times M_y} \mathcal{L}_x^{\geq r} [b(u)] \operatorname{sgn}(u - \kappa^\pm)^\pm \varphi \rho_\varepsilon \\
& - \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{< r} [\varphi \rho_\varepsilon] \\
& \leq - \int_{Q_x \times M_y} \left[(u - \kappa^\pm)^\pm \partial_s \rho_\varepsilon + F^\pm(u, \kappa^\pm) \cdot \nabla_y \rho_\varepsilon \right] \varphi,
\end{aligned}$$

where the ‘initial term’ from the right-hand-side of (2.6) is here dropped as $\varphi = 0$ for $t = 0$, while the ‘boundary term’ is omitted by the same argument that gave (4.16). We also used the integral/operator notation from the proof of Proposition 4.1 (see below (4.7)) and $\partial_t \rho_\varepsilon = -\partial_s \rho_\varepsilon$ and $\nabla_x \rho_\varepsilon = -\nabla_y \rho_\varepsilon$.

Adding the integral of $\mathcal{L}_y^{\geq r} [b(\kappa^\pm)] (b(u) - b(\kappa^\pm))^\pm \varphi \rho_\varepsilon$ over $Q_x \times M_y$ to each side we further get

$$\begin{aligned}
(4.18) \quad & - \int_{Q_x \times M_y} \left[(u - \kappa^\pm)^\pm \varphi_t + F^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi \right] \rho_\varepsilon \\
& - \int_{Q_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(\kappa^\pm)] \operatorname{sgn}(u - \kappa^\pm)^\pm \varphi \rho_\varepsilon \\
& - \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{< r} [\varphi \rho_\varepsilon] \\
& \leq \int_{Q_x \times M_y} \left[\partial_s ((u - \kappa^\pm)^\pm) + \operatorname{div}_y (F^\pm(u, \kappa^\pm)) \right] \varphi \rho_\varepsilon \\
& + \int_{Q_x \times M_y} \mathcal{L}_y^{\geq r} [b(\kappa^\pm)] \operatorname{sgn}(u - \kappa^\pm)^\pm \varphi \rho_\varepsilon.
\end{aligned}$$

We have here used that $\mathcal{L}_x^{\geq r} [w_1] - \mathcal{L}_y^{\geq r} [w_2] = \mathcal{L}_{x+y}^{\geq r} [w_1 - w_2]$ when w_1 and w_2 are functions in x and y respectively, and we have also integrated by parts in (s, y) on the right-hand side. By Lipschitz continuity, it is clear that the weak derivatives on the right-hand side of (4.18) satisfy

$$|\partial_s ((u - \kappa^\pm)^\pm)| + |\operatorname{div}_y (F^\pm(u, \kappa^\pm))| \leq L_{\bar{u}^c} + L_f L_{\bar{u}^c},$$

since $L_{\bar{u}^c}$ is a Lipschitz constant for κ .

Observe next that the domain of integration for the second integral on the left-hand side of (4.18) may be extended from $Q_x \times M_y$ to $M_x \times M_y$ as $\operatorname{sgn}^\pm(u - \kappa)^\pm \rho_\varepsilon = 0$ on $Q_x^c \times M_y$ (by the same argument used for (4.16)). Doing so, we may further infer

$$\begin{aligned}
\int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(\kappa^\pm)] \operatorname{sgn}(u - \kappa^\pm)^\pm \varphi \rho_\varepsilon & \leq \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [(b(u) - b(\kappa^\pm))^\pm] \varphi \rho_\varepsilon \\
& = \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{\geq r} [\varphi] \rho_\varepsilon,
\end{aligned}$$

where we used the second inequality of Corollary C.2, the self-adjointness of $\mathcal{L}_{x+y}^{\geq r}$ (Proposition 3.1) and the identity $\mathcal{L}_{x+y}^{\geq r}[\varphi\rho_\varepsilon] = \rho_\varepsilon\mathcal{L}_x^{\geq r}[\varphi]$. Exploiting these bounds in (4.18), we find that

$$(4.19) \quad \begin{aligned} & - \int_{Q_x \times M_y} \left[(u - \kappa^\pm)^\pm \varphi_t + F^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi \right] \rho_\varepsilon \\ & - \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{\geq r}[\varphi] \rho_\varepsilon \\ & - \int_{Q_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x^{< r}[\varphi\rho_\varepsilon] \\ & \leq \int_{Q_x \times M_y} \left[(1 + L_f)L_{\bar{u}^c} + |\mathcal{L}_y^{\geq r}[b(\kappa^\pm)]| \right] \varphi\rho_\varepsilon. \end{aligned}$$

Next, we let $r \rightarrow 0$. The limits on the left-hand side of (4.19) are straight forward by Lemma 3.4 and so we turn our attention to the right-hand side. It is easily verified that the function $h_\varepsilon^\pm(\cdot) := b((\cdot \pm \varepsilon L_{\bar{u}^c}) \vee^\pm k)$ satisfies

$$(4.20) \quad \|(h_\varepsilon^\pm)'\|_{L^\infty(\mathbb{R})} \leq \|b'\|_{L^\infty(\mathbb{R})} \quad \text{and} \quad |(h_\varepsilon^\pm)'|_{TV(\mathbb{R})} \leq |b'|_{TV(\mathbb{R})} + \|b'\|_{L^\infty(\mathbb{R})}$$

(the regularity of b is by Remark 2.1), and that we have the identity

$$(4.21) \quad h_\varepsilon^\pm(\bar{u}^c) = b(\kappa^\pm)$$

when using the definition (4.15) of κ^\pm . By Corollary 3.7 we get $\lim_{r \rightarrow 0} \mathcal{L}_y^{\geq r}[b(\kappa)] = \mathcal{L}_y[b(\kappa)]$ in $L^1_{\text{loc}}(M_y)$, and so we conclude from (4.19) that

$$(4.22) \quad \begin{aligned} & - \int_{Q_x \times M_y} \left[(u - \kappa^\pm)^\pm \varphi_t + F^\pm(u, \kappa^\pm) \cdot \nabla_x \varphi \right] \rho_\varepsilon \\ & - \int_{M_x \times M_y} (b(u) - b(\kappa^\pm))^\pm \mathcal{L}_x[\varphi] \rho_\varepsilon \\ & \leq \int_{Q_x \times M_y} \left[(1 + L_f)L_{\bar{u}^c} + |\mathcal{L}_y[b(\kappa^\pm)]| \right] \varphi\rho_\varepsilon. \end{aligned}$$

Finally, we wish to send $\varepsilon \rightarrow 0$. The left hand can be dealt with using classical arguments (cf. e.g. [40]) and so we again focus on the right-hand side. We shall first establish the limit

$$(4.23) \quad \lim_{r \rightarrow 0} \sup_{\varepsilon \in (0,1)} \int_{Q_x \times M_y} |\mathcal{L}_y^{< r}[b(\kappa^\pm)]| \varphi\rho_\varepsilon = 0.$$

Letting U_y denote the set of points in M_y such that $\text{dist}(y, \Omega_y) \leq 1$, we see that the support of $(s, y) \mapsto \rho_\varepsilon(t - s, x - y)$ lies in U_y for all $(t, x) \in Q_x$ and $\varepsilon \in (0, 1)$. Thus, for $\varepsilon \in (0, 1)$

$$\int_{Q_x \times M_y} |\mathcal{L}_y^{< r}[b(\kappa^\pm)]| \varphi\rho_\varepsilon \leq \|\varphi\|_{L^\infty(M)} \int_{U_y} |\mathcal{L}_y^{< r}[h_\varepsilon^\pm(\bar{u}^c)]|,$$

where we use the notation from (4.21). By the explicit bound (3.9) of Corollary 3.7, it follows that $\mathcal{L}_y^{< r}[h_\varepsilon^\pm(\bar{u}^c)] \rightarrow 0$ in $L^1(U_y)$ as $r \rightarrow 0$, and that the limits holds uniformly in ε due to the uniform bounds (4.20). Thus we attain (4.23), and exploiting it, we may compute

$$\lim_{\varepsilon \rightarrow 0} \int_{Q_x \times M_y} |\mathcal{L}_y[b(\kappa^\pm)]| \varphi\rho_\varepsilon = \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{Q_x \times M_y} |\mathcal{L}_y^{\geq r}[b(\kappa^\pm)]| \varphi\rho_\varepsilon = \int_Q |\mathcal{L}[b(\bar{u}^c \vee^\pm k)]| \varphi \, dx \, dt,$$

where \bar{u}^c on the right-most side is a function in (t, x) . Here we approximated \mathcal{L}_y by $\mathcal{L}_y^{\geq r}$ uniformly in ε by (4.23), used that $\mathcal{L}_y^{\geq r}$ is a zero order operator to pass to the limit in ε , and then concluded by the limit definition of \mathcal{L} . Sending $\varepsilon \rightarrow 0$ in (4.22), we then find that

$$(4.24) \quad \begin{aligned} & - \int_Q \left[(u - (\bar{u}^c \vee^\pm k))^\pm \varphi_t + F^\pm(u, (\bar{u}^c \vee^\pm k)) \cdot \nabla \varphi \right] dx dt \\ & - \int_M (b(u) - b(\bar{u}^c \vee^\pm k))^\pm \mathcal{L}[\varphi] dx dt \\ & \leq \int_Q \left[(1 + L_f) L_{\bar{u}^c} + |\mathcal{L}[b(\bar{u}^c \vee^\pm k))]| \right] \varphi dx dt, \end{aligned}$$

where \bar{u}^c now is a function in (t, x) . Finally, we want to extend the support of φ to include $t = 0$ and $t = T$ as we have so far assumed it to have compact support in $M = (0, T) \times \mathbb{R}^d$. In (4.24) we make the substitution $\varphi \mapsto \theta_\varepsilon \varphi$, where $(\theta_\varepsilon)_{\varepsilon > 0} \subset C_c^\infty((0, T))$ is a family of nonnegative functions satisfying $\lim_{\varepsilon \rightarrow 0} \theta_\varepsilon(t) = 1$ for $t \in (0, T)$ and $\|\theta'_\varepsilon\|_{L^1((0, T))} = 2$ for all ε while the new φ is like previous one. Exploiting the time continuity of u at $t = 0$, we get the inequality

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_Q (u - (\bar{u}^c \vee^\pm k))^\pm \varphi \theta'_\varepsilon dx dt & \leq \int_{\{0\} \times \Omega} (u_0 - (\bar{u}^c \vee^\pm k))^\pm \varphi dx \\ & \leq \int_{\{0\} \times \Omega} (\|u_0\|_{L^\infty(\Omega)} + \|\bar{u}^c\|_{L^\infty(M)}) \varphi dx. \end{aligned}$$

Using this, after inserting for φ in (4.24) and letting $\varepsilon \rightarrow 0$, we get (4.14), except that we have to go from \mathcal{L} over to \mathcal{B} . This is justified by Proposition 3.1 and the fact that $(b(u) - b(\bar{u}^c \vee^\pm k))^\pm$ is of finite energy (it is dominated by the energy of $b(u) - (\bar{u}^c)$). \square

Remark 4.4. One should note that the right hand side of (4.14) is bounded by $C\|\varphi\|_{L^\infty([0, T] \times \mathbb{R}^d)}$ where C is given by (4.28). While C depends on $\Omega, T, f, b, \mu, \bar{u}^c$ and u_0 it does not depend on our choice of $k \in \mathbb{R}$ in (4.14); this uniformity in k will be important for the uniqueness proof of Section 5 where k will depend on a second set of variables.

Corollary 4.5. *Under the assumptions of Lemma 4.3, there exists a finite signed Borel measure ν_k^\pm on $[0, T] \times \mathbb{R}^d$, which integrates to zero and is non-positive on $[0, T] \times (\mathbb{R}^d \setminus \Omega)$, and such that*

$$(4.25) \quad \begin{aligned} \int_{[0, T] \times \mathbb{R}^d} \varphi d\nu_k^\pm & = - \int_Q \left[(u - (\bar{u}^c \vee^\pm k))^\pm \varphi_t + F^\pm(u, \bar{u}^c \vee^\pm k) \cdot \nabla \varphi \right] dx dt \\ & + \int_M \mathcal{B}[(b(u) - b(\bar{u}^c \vee^\pm k))^\pm, \varphi] dx dt, \end{aligned}$$

for any $\varphi \in C_b^\infty([0, T] \times \mathbb{R}^d)$. Moreover, the total variation norm of ν_k^\pm admits the bound $\|\nu_k^\pm\| \leq 2C$, where C is given by (4.28).

Proof. For brevity, let $T_k^\pm: C_b^\infty([0, T] \times \mathbb{R}^d) \rightarrow \mathbb{R}$ denote the operator such that $T_k^\pm(\varphi)$ coincides with the right-hand side of (4.25) or, equivalently, the left-hand side of (4.14). Pick $0 \leq \psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi = 1$ for $|x| \leq 1$ and set $\psi_n(x) = \psi(x/n)$. Note that $\lim_{n \rightarrow \infty} T_k^\pm(\psi_n) = 0$. For an arbitrary $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ satisfying $|\varphi| \leq 1$, we can exploit the linearity of T_k^\pm to compute

$$T_k^\pm(\varphi) = \lim_{n \rightarrow \infty} T_k^\pm(\varphi + \psi_n) \leq 2C.$$

The inequality follows from applying (4.14) (possible as $\varphi + \psi_n$ is nonnegative for a sufficiently large n) followed by using $|\varphi + \psi_n| \leq 2$ and the constant C from Remark 4.4. The linearity of T_k^\pm then implies $|T_k^\pm(\varphi)| \leq 2C\|\varphi\|_{L^\infty}$ for any $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and so by the Riesz representation theorem

(see e.g. [100]) there exists a signed Borel measure ν_k^\pm , of total variation norm $\|\nu_k^\pm\| \leq 2C$, such that $T_k^\pm(\varphi) = \int_{[0,T] \times \mathbb{R}^d} \varphi d\nu_k^\pm$ for all $\varphi \in C_c^\infty([0,T] \times \mathbb{R}^d)$. That this relation can be extended to $\varphi \in C_b^\infty([0,T] \times \mathbb{R}^d)$ follows by inserting $\varphi\psi_n$, with ψ_n as before, and letting $n \rightarrow \infty$; each side converges to the canonical limit.

As it is clear that ν_k^\pm integrates to zero, it remains to prove that it is non-positive on $[0,T] \times (\mathbb{R}^d \setminus \Omega)$. This can be seen as follows: Let ζ_δ denote a boundary layer sequence and observe that we for any $0 \leq \varphi \in C_c^\infty([0,T] \times \mathbb{R}^d)$ have

$$\int_{[0,T] \times (\mathbb{R}^d \setminus \Omega)} \varphi d\nu_k^\pm = \lim_{\delta \rightarrow 0} \int_{[0,T] \times \mathbb{R}^d} (1 - \zeta_\delta)\varphi d\nu_k^\pm = \lim_{\delta \rightarrow 0} T_k^\pm((1 - \zeta_\delta)\varphi) \leq 0,$$

where we used (4.14). \square

We now demonstrate three forms of boundary regularity which are consequences of the general identity (4.25). They will be referred to as *boundary integrability*, *weak trace* and *boundary condition*. The two latter regularity properties have local analogs; in [89] they are referred to as ‘weak normal trace’ and ‘boundary condition’ respectively. The former however, boundary integrability, has no local analog in neither [89] nor [87], and is in this sense a truly nonlocal feature of the equation.

4.2.1. Boundary integrability. Consider (4.25) when $k = -\|\bar{u}^c\|_{L^\infty(M)}$ in the (+) case and $k = \|\bar{u}^c\|_{L^\infty(M)}$ in the (−) case; if we add the two resulting identities and let ν denote the sum of the corresponding measures, we get

$$(4.26) \quad \int_{[0,T] \times \mathbb{R}^d} \varphi d\nu = - \int_Q [|u - \bar{u}^c| \varphi_t + F(u, \bar{u}^c) \cdot \nabla \varphi] dx dt + \int_M \mathcal{B}[|b(u) - b(\bar{u}^c)|, \varphi] dx dt,$$

for all $\varphi \in C_b^\infty([0,T] \times \mathbb{R}^d)$. Here $F := F^+ + F^-$ so that

$$F(u, \bar{u}^c) = \text{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c))$$

and ν is a finite signed Borel measure which integrates to zero and whose total variation norm satisfies $\|\nu\| \leq 4C$ with C as in (4.28). This identity puts a restriction on how much $b(u)$ can differ from $b(\bar{u}^c)$ close to Γ as shown in the following proposition.

Proposition 4.6 (Boundary integrability). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then*

$$(4.27) \quad \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} |b(u(t,x)) - b(\bar{u}^c(t,x))| d\mu(z) dx dt \leq C,$$

where

$$(4.28) \quad C := |\Omega|T(1 + L_f)L_{\bar{u}^c} + |\Omega|(\|u_0\|_{L^\infty(\Omega)} + \|\bar{u}^c\|_{L^\infty(M)}) + \left(\sup_c \|\mathcal{L}[b(\bar{u}^c \wedge c)]\|_{L^1(Q)} \right) \vee \left(\sup_c \|\mathcal{L}[b(\bar{u}^c \vee c)]\|_{L^1(Q)} \right).$$

Proof. Let $(\bar{\zeta}_\delta)_{\delta>0}$ be an outer boundary layer sequence (cf. (2.8)). Replacing φ by $\bar{\zeta}_\delta$ in (4.26), we get

$$(4.29) \quad \int_M \mathcal{B}[|b(u) - b(\bar{u}^c)|, \bar{\zeta}_\delta] dx dt = \int_{[0,T] \times \mathbb{R}^d} \bar{\zeta}_\delta d\nu \leq 2C,$$

where the inequality follows as the positive part of ν is bounded by $2C$. For brevity, we now set $g(t, x) := |b(u(t, x)) - b(\bar{u}^c(t, x))|$ for $(t, x) \in M$. Writing out the \mathcal{B} term, we find

$$(4.30) \quad \int_M \mathcal{B}[g, \bar{\zeta}_\delta] \, dx \, dt = \int_M \int_{\mathbb{R}^d} \left(g(t, x+z) - g(t, x) \right) \left(\bar{\zeta}_\delta(x+z) - \bar{\zeta}_\delta(x) \right) \, d\mu(z) \, dx \, dt.$$

As g has finite energy (it is dominated by that of $b(u) - b(\bar{u}^c)$) and $\bar{\zeta}_\delta \in C_c^\infty(\mathbb{R}^d)$, the integrand on the right-hand side is absolutely integrable, and so we need not worry about the order of integration. Exploiting that $\bar{\zeta}_\delta = 1$ in $\bar{\Omega}$ and that $g(t, x) = 0$ for almost every $x \in \Omega^c$, the integral in (4.30) reduces to

$$\begin{aligned} & \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} g(t, x)(1 - \bar{\zeta}_\delta(x+z)) \, d\mu(z) \, dx \, dt + \int_0^T \int_{\substack{x+z \in \Omega \\ x \in \Omega^c}} g(t, x+z)(1 - \bar{\zeta}_\delta(x)) \, d\mu(z) \, dx \, dt \\ &= 2 \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} g(t, x)(1 - \bar{\zeta}_\delta(x+z)) \, d\mu(z) \, dx \, dt, \end{aligned}$$

where the equality holds if we in the second integral perform the change of variables $x \mapsto x-z$ followed by $z \mapsto -z$. Inserting this on the left-hand side of (4.29) and letting $\delta \rightarrow 0$, (4.27) follows by Fatou's lemma. \square

A corollary of the previous proposition is the following \mathcal{B} -pairing property of $b(u) - b(\bar{u}^c)$.

Corollary 4.7. *Under the assumptions of Proposition 4.6, the integrand of $\mathcal{B}[b(u) - b(\bar{u}^c), \varphi \chi_\Omega]$ is absolutely integrable with respect to $d\mu(z) \, dx \, dt$ for all $\varphi \in C^\infty([0, T] \times \mathbb{R}^d)$.*

Proof. Similarly to the previous proof, we set $g(t, x) := b(u(t, x)) - b(\bar{u}^c(t, x))$ for brevity, and compute

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} |g(t, x+z) - g(t, x)| |(\varphi \chi_\Omega)(t, x+z) - (\varphi \chi_\Omega)(t, x)| \, d\mu(z) \, dx \, dt \\ &= \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega}} |g(t, x+z) - g(t, x)| |\varphi(t, x+z) - \varphi(t, x)| \, d\mu(z) \, dx \, dt \\ & \quad + \int_0^T \int_{\substack{x \in \Omega \\ x+z \in \Omega^c}} |g(t, x)| |\varphi(t, x)| \, d\mu(z) \, dx \, dt \\ & \quad + \int_0^T \int_{\substack{x \in \Omega^c \\ x+z \in \Omega}} |g(t, x+z)| |\varphi(t, x+z)| \, d\mu(z) \, dx \, dt, \end{aligned}$$

where we used that both g and $\varphi \chi_\Omega$ are supported in Q . The first integral on the right-hand side is finite as g is of finite energy (Proposition 4.1) and φ is smooth, while the two latter integrals (which by a change of variables can be seen to coincide) are finite due to Proposition 4.6. \square

4.2.2. *Weak trace.* We next establish a weak trace identity resulting from (4.26).

Proposition 4.8 (Weak trace). *Assume (A_Ω) – (A_μ) and u is an entropy solution of (1.1). Then there is a finite and non-positive Borel measure ν on $\bar{\Gamma} = [0, T] \times \partial\Omega$ such that: For every $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$, $r > 0$, and boundary layer sequence $(\zeta_\delta)_{\delta>0}$ we have*

$$(4.31) \quad \begin{aligned} & \lim_{\delta \rightarrow 0} \left(\int_Q F(u, \bar{u}^c) \cdot (\nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta] \varphi \, dx \, dt \right) \\ &= \int_{\bar{\Gamma}} (1 - \zeta) \varphi \, d\nu - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] \varphi \, dx \, dt, \end{aligned}$$

where ζ is the limit of ζ_δ on $\partial\Omega$ and where $F(u, \bar{u}^c) = \text{sgn}(u - \bar{u}^c)(f(u) - f(\bar{u}^c))$.

Proof. If we in (4.26) choose $(1 - \zeta_\delta)\varphi$ as the test function, we get after a little rewriting

$$(4.32) \quad \begin{aligned} & \int_Q F(u, \bar{u}^c) \cdot (\nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta] \varphi \, dx \, dt \\ &= \int_{[0, T] \times \mathbb{R}^d} (1 - \zeta_\delta) \varphi \, d\nu + \int_Q \left[|u - \bar{u}^c| \varphi_t + F(u, \bar{u}^c) \cdot \nabla \varphi \right] (1 - \zeta_\delta) \, dx \, dt \\ & \quad - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \varphi] (1 - \zeta_\delta) \, dx \, dt - \int_M \mathcal{B}^{\geq r} [|b(u) - b(\bar{u}^c)|, (1 - \zeta_\delta) \varphi] \, dx \, dt, \end{aligned}$$

where we decomposed $\mathcal{B} = \mathcal{B}^{\geq r} + \mathcal{B}^{<r}$ for some fixed $r > 0$ and used the product rule for $\mathcal{B}^{<r}$ (Proposition 3.3). Letting $\delta \rightarrow 0$ in (4.32) we obtain

$$(4.33) \quad \begin{aligned} & \lim_{\delta \rightarrow 0} \left(\int_Q F(u, \bar{u}^c) \cdot (\nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta] \varphi \, dx \, dt \right) \\ &= \int_{\bar{\Gamma}} (1 - \zeta) \varphi \, d\nu + \int_{[0, T] \times (\mathbb{R}^d \setminus \bar{\Omega})} \varphi \, d\nu - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \varphi] (1 - \chi_\Omega) \, dx \, dt \\ & \quad - \int_M \mathcal{B}^{\geq r} [|b(u) - b(\bar{u}^c)|, (1 - \chi_\Omega) \varphi] \, dx \, dt. \end{aligned}$$

It remains to show that the right-hand side of (4.33) coincides with that of (4.31). Consider the special case of an outer boundary layer sequence $(\bar{\zeta}_\delta)_{\delta > 0}$ (cf. (2.8)). Using that $\bar{\zeta}_\delta = 1$ on $\bar{\Omega}$, (4.33) reads

$$(4.34) \quad \begin{aligned} & - \lim_{\delta \rightarrow 0} \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \bar{\zeta}_\delta] \varphi \, dx \, dt \\ &= \int_{[0, T] \times (\mathbb{R}^d \setminus \bar{\Omega})} \varphi \, d\nu - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \varphi] (1 - \chi_\Omega) \, dx \, dt \\ & \quad - \int_M \mathcal{B}^{\geq r} [|b(u) - b(\bar{u}^c)|, (1 - \chi_\Omega) \varphi] \, dx \, dt. \end{aligned}$$

But in the case of an outer boundary layer sequence, the limit on the left-hand side of (4.34) may be computed explicitly giving

$$(4.35) \quad - \lim_{\delta \rightarrow 0} \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \bar{\zeta}_\delta] \varphi \, dx \, dt = - \int_M \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] \varphi \, dx \, dt.$$

For brevity we skip this computation as it is almost identical to the proof of Proposition 4.6; one may replace the use of Fatou's lemma by dominated convergence (in the case when φ changes sign) due to the integrability provided by Proposition 4.6. In conclusion, the three last terms on the right-hand side of (4.33) may be replaced by the right-hand side of (4.35) and so we attain (4.31). \square

Remark 4.9. In the uniqueness proof presented in the next section, the use of Proposition 4.8 will occur inside a second integral over a second set of variables on which φ (also) depends. Thus, in order to justify this use of the weak trace (through dominated convergence) we make the following observation: If we on the right-hand side of (4.32) took the supremum over $\delta > 0$ and $\varphi \in C_c^2(M)$ with $\|\varphi\|_{C^2(M)} \leq 1$, we would get a finite value. This fact is easily seen when exploiting the uniform bound $|\zeta_\delta| \leq 1$, the total variation bound $\|\nu\| \leq 4C$ with C as in (4.28), the finite energy of $|b(u) - b(\bar{u}^c)|$ and the fact that $\mathcal{B}^{\geq r}$ is a zero order operator.

4.2.3. *Boundary condition.* Recall that $F = F^+ + F^-$, and define the quantities

$$(4.36) \quad \begin{aligned} \mathcal{F} &= \mathcal{F}(u, \bar{u}^c, k) := F(u, \bar{u}^c) + F(u, k) - F(\bar{u}^c, k), \\ \Sigma &= \Sigma(u, \bar{u}^c, k) := |b(u) - b(\bar{u}^c)| + |b(u) - b(k)| - |b(\bar{u}^c) - b(k)|. \end{aligned}$$

As was observed in [89], these quantities may alternatively be written as

$$(4.37) \quad \begin{aligned} \mathcal{F}/2 &= F^+(u, \bar{u}^c \vee^+ k) + F^-(u, \bar{u}^c \vee^- k), \\ \Sigma/2 &= (b(u) - b(\bar{u}^c \vee^+ k))^+ + (b(u) - b(\bar{u}^c \vee^- k))^- , \end{aligned}$$

which is verified by considering the three cases $u < \bar{u}^c \vee^- k$, $u > \bar{u}^c \vee^+ k$, and $u \in [\bar{u}^c \vee^- k, \bar{u}^c \vee^+ k]$.

Proposition 4.10 (Boundary condition). *Assume (A_Ω) – (A_μ) and u is an entropy of (1.1). For all inner boundary layer sequences $(\zeta_\delta)_{\delta>0}$, $k \in \mathbb{R}$, $r > 0$, and $0 \leq \varphi \in C^\infty([0, T] \times \mathbb{R}^d)$,*

$$(4.38) \quad \limsup_{\delta \rightarrow 0} \left(\int_Q (\mathcal{F} \cdot \nabla \zeta_\delta) \varphi \, dx \, dt - \int_M \mathcal{B}^{<r}[\Sigma, \zeta_\delta] \varphi \, dx \, dt \right) \leq 0.$$

Proof. For a fixed k , the (+) and (–) case of (4.25) results in two identities, each featuring their own measure ν_k^+ and ν_k^- . Setting $\nu_k := \nu_k^+ + \nu_k^-$, and adding these two identities together, we get for any $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$

$$(4.39) \quad \begin{aligned} \int_{[0, T] \times \mathbb{R}^d} \varphi \, d\nu_k &= - \int_Q \left[(u - (\bar{u}^c \vee^+ k))^+ + (u - (\bar{u}^c \vee^- k))^- \right] \varphi_t \, dx \, dt \\ &\quad - \frac{1}{2} \int_Q \mathcal{F} \cdot \nabla \varphi \, dx \, dt + \frac{1}{2} \int_M \mathcal{B}[\Sigma, \varphi] \, dx \, dt, \end{aligned}$$

where we used the formula (4.37). Next, we substitute $\varphi \mapsto (1 - \zeta_\delta) \bar{\zeta}_\delta \varphi$ in (4.39), where the the new φ is as before, while ζ_δ and $\bar{\zeta}_\delta$ represents respectively an inner boundary layer sequence and an outer boundary layer sequence (cf. (2.9) and (2.8)). By the product rules for ∇ and \mathcal{B} we find

$$\begin{aligned} - \int_Q \mathcal{F} \cdot \nabla ((1 - \zeta_\delta) \bar{\zeta}_\delta \varphi) \, dx \, dt &= \int_Q (\mathcal{F} \cdot \nabla \zeta_\delta) \varphi \, dx \, dt - \int_Q (\mathcal{F} \cdot \nabla \varphi) (1 - \zeta_\delta) \, dx \, dt, \\ \int_M \mathcal{B}[\Sigma, (1 - \zeta_\delta) \varphi] \, dx \, dt &= \int_M \mathcal{B}[\Sigma, (1 - \zeta_\delta) \bar{\zeta}_\delta] \varphi \, dx \, dt + \int_M \mathcal{B}[\Sigma, \varphi] (1 - \zeta_\delta) \bar{\zeta}_\delta \, dx \, dt, \end{aligned}$$

where we for the first identity used that $\bar{\zeta}_\delta = 1$ in $\bar{\Omega}$. Using these in (4.39) (after the substitution) together with some rewriting yields

$$(4.40) \quad \begin{aligned} &\int_Q (\mathcal{F} \cdot \nabla \zeta_\delta) \varphi \, dx \, dt + \int_M \mathcal{B}^{<r}[\Sigma, (1 - \zeta_\delta) \bar{\zeta}_\delta] \varphi \, dx \, dt \\ &= 2 \int_{[0, T] \times \mathbb{R}^d} (1 - \zeta_\delta) \bar{\zeta}_\delta \varphi \, d\nu_k - \int_M \mathcal{B}^{\geq r}[\Sigma, (1 - \zeta_\delta) \bar{\zeta}_\delta] \varphi \, dx \, dt \\ &\quad + 2 \int_Q \left[(u - (\bar{u}^c \vee^+ k))^+ + (u - (\bar{u}^c \vee^- k))^- \right] (1 - \zeta_\delta) \bar{\zeta}_\delta \varphi_t \, dx \, dt \\ &\quad + \int_Q (\mathcal{F} \cdot \nabla \varphi) (1 - \zeta_\delta) \, dx \, dt - \int_M \mathcal{B}[\Sigma, \varphi] (1 - \zeta_\delta) \bar{\zeta}_\delta \, dx \, dt, \end{aligned}$$

where we decomposed $\mathcal{B} = \mathcal{B}^{\geq r} + \mathcal{B}^{<r}$ for some fixed $r > 0$. As $\lim_{\delta \rightarrow 0} (1 - \zeta_\delta(x)) \bar{\zeta}_\delta(x) = \chi_{\partial\Omega}(x)$, we see that all the terms on the right-hand side of (4.40) vanish when $\delta \rightarrow 0$ except for the very first one.

However, ν_k is by Corollary 4.5 non-positive on $[0, T] \times \partial\Omega$ and so we infer from (4.40) that

$$(4.41) \quad \lim_{\delta \rightarrow 0} \left(\int_Q (\mathcal{F} \cdot \nabla \zeta_\delta) \varphi \, dx \, dt + \int_M \mathcal{B}^{<r}[\Sigma, (1 - \zeta_\delta) \bar{\zeta}_\delta] \varphi \, dx \, dt \right) \leq 0.$$

Finally, using the identity $\zeta_\delta \bar{\zeta}_\delta = \zeta_\delta$ we have

$$(4.42) \quad \mathcal{B}^{<r}[\Sigma, (1 - \zeta_\delta) \bar{\zeta}_\delta] = \mathcal{B}^{<r}[\Sigma, \bar{\zeta}_\delta] - \mathcal{B}^{<r}[\Sigma, \zeta_\delta] \geq -\mathcal{B}^{<r}[\Sigma, \zeta_\delta],$$

where the positivity of $\mathcal{B}^{<r}[\Sigma, \bar{\zeta}_\delta]$ follows from a similar calculation as done for $\mathcal{B}^{<r}[|b(u) - b(\bar{u}^c)|, \bar{\zeta}_\delta]$ in the proof of Proposition 4.6 (note that Σ is also nonnegative and supported in Q). Combining (4.41) with (4.42) we attain (4.38). \square

Remark 4.11. We here make an analogous observation for the boundary condition as was done in Remark 4.9 for the weak trace. If we on the right-hand side of (4.40) took the supremum over $k \in \mathbb{R}$, $\delta \in (0, 1)$ and $\varphi \in C_c^1(M)$ with $\|\varphi\|_{C^1(M)} \leq 1$, we would get a finite value. This follows by exploiting the uniform bounds $|\zeta_\delta|, |\bar{\zeta}_\delta| \leq 1$, the total variation bound $\|\nu_k\| \leq 4C$ with C as in (4.28) and the bounds $|\mathcal{F}| \leq 2L_f|u - \bar{u}^c|$ and $|\Sigma| \leq 2|b(u) - b(\bar{u}^c)|$ which can be read from (4.37); this last bound also implies that the energy of Σ is less than twice that of $b(u) - b(\bar{u}^c)$.

5. UNIQUENESS OF ENTROPY SOLUTIONS

This section is devoted to proving the uniqueness of entropy solutions of (1.1). The proof deploys the classical Kruskov's 'doubling of variables' device, and follows a similar path as the one developed in the work of Otto [91], Michel and Vovelle [89] and Mascia et al. [87]. Its length is due to the many non-trivial limits that need to be computed. To get a rough understanding of the coming proof, we give a formal overview of its steps:

For two entropy solutions u, v with the same exterior data $\bar{u}^c = \bar{v}^c$ the proof starts off classically by doubling the variables, $u = u(t, x)$ and $v = v(s, y)$, followed by combining the two entropy inequalities for a nonnegative test function in the variables (t, x, s, y) which is compactly supported in $Q \times Q$. The test function is chosen as a product of boundary layer sequences (for compact support) and a standard mollifier in the variables $t - s$ and $x - y$.

Just as in the classical hyperbolic case [78] the mollifier commutes with the symmetric operators $(\partial_t + \partial_s)$ and $(\nabla_x + \nabla_y)$, which is necessary for avoiding blow up when we later let the mollifier approximate $\delta(t - s, x - y)$. In the degenerate parabolic-hyperbolic case, the Δ -term poses some challenge at this step since $\Delta_x + \Delta_y$ does *not* commute with the mollifier. This difficulty was overcome by Carrillo who in [35] showed that the necessary 'cross-terms' could be safely introduced into the entropy inequality, replacing this operator by the more appropriate $(\nabla_x + \nabla_y) \cdot (\nabla_x + \nabla_y)$. In our case, a similar difficulty arises for \mathcal{L} , but only for its singular part (which has been subjected to a convex inequality). Of course, this singular part vanishes as $r \rightarrow 0$, which is how the proof proceeds. The remaining 'good' part of \mathcal{L} is then written in the appropriate (x, y) -symmetric form using Proposition D.1 and partially shifted on to the test function to avoid blow up as $r \rightarrow 0$.

We then let the support of the test function expand up to the boundary. Here we introduce a new $r > 0$ to divide our symmetric \mathcal{L} up into a singular and non-singular part. As the boundary layers converge to the characteristic function on Ω , we avoid potential blow up of the convection term and the singular part of \mathcal{L} by resorting to the boundary condition of Proposition 4.10. Essentially, this condition lets us exchange cumbersome differences of u and v with more manageable ones, such as a difference of u and \bar{u}^c which admits a weak trace by Proposition 4.8.

Finally, we 'go to the diagonal' by letting the mollifier tend to $\delta(t - s, x - y)$ in an appropriate sense. Here we again deploy the weak trace result of Proposition 4.8 to compute some of the limits and,

conveniently, these new trace-terms cancel out the ones from before, which could not else be dropped due to having a ‘bad’ sign. After the completion of this process, the desired conclusion is reached modulo some debris from the singular part of \mathcal{L} . But these bad terms are small by Proposition 4.6 (boundary integrability) and vanish when we, once more, let $r \rightarrow 0$ which completes the proof.

Theorem 5.1 (Uniqueness). *Assume (A_Ω) – (A_μ) . Let u, v be entropy solutions of (1.1) in the sense of Definition 2.2 with initial data u_0, v_0 and exterior data $u^c = v^c$. Then for a.e. $t \in (0, T)$ we have*

$$\int_{\Omega} |u(t, x) - v(t, x)| \, dx \leq \int_{\Omega} |u_0(x) - v_0(x)| \, dx.$$

In particular, $u = v$ a.e. in $(0, T) \times \mathbb{R}^d$ when $u_0 = v_0$.

Remark 5.2. We can pick the same C^2 -extension for both u^c and v^c , i.e., $\bar{u}^c = \bar{v}^c$. This is unproblematic, as all the a priori calculations of Section 4 can be carried out for any smooth extension.

Proof of Theorem 5.1. 1) Doubling the variables. We begin with the standard approach of doubling the variables: Consider u and v functions in (t, x) and (s, y) respectively, and let $0 \leq \varphi \in C_c^\infty(Q_x \times Q_y)$, where Q_x and Q_y are the set Q parameterized over (t, x) and (s, y) respectively (similar notation will also be used for the sets M, Γ, Ω and $\partial\Omega$ throughout the proof). Considering $(s, y) \in M_y$ fixed, we may in the entropy inequalities (2.6) for u let the constant be replaced with $v(s, y)$ and the test function replaced with $(t, x) \mapsto \varphi(t, x, s, y)$. Combining the (+) and (–) case, we obtain

$$(5.1) \quad \begin{aligned} & - \int_{Q_x} |u - v| \partial_t \varphi + F(u, v) \cdot \nabla_x \varphi \\ & - \int_{M_x} \mathcal{L}_x^{\geq r}[b(u)] \operatorname{sgn}(u - v) \varphi + |b(u) - b(v)| \mathcal{L}_x^{< r}[\varphi] \leq 0, \end{aligned}$$

where we for brevity have suppressed the measures $dt dx$ and have added sub-indexes to clarify in which variables the operators act. As (5.1) holds for all $(s, y) \in M_y$, it can be integrated over this set, so that we obtain

$$(5.2) \quad \begin{aligned} & - \int_{Q_x \times Q_y} |u - v| \partial_t \varphi + F(u, v) \cdot \nabla_x \varphi \\ & - \int_{M_x \times M_y} \mathcal{L}_x^{\geq r}[b(u)] \operatorname{sgn}(u - v) \varphi + |b(u) - b(v)| \mathcal{L}_x^{< r}[\varphi] \leq 0, \end{aligned}$$

where we for the first integral used that φ is supported in $Q_x \times Q_y$. Swapping the role of u and v (and the role of (t, x) and (s, y)) we get the analogous inequality

$$(5.3) \quad \begin{aligned} & - \int_{Q_x \times Q_y} |v - u| \partial_s \varphi + F(v, u) \cdot \nabla_y \varphi \\ & - \int_{M_x \times M_y} \mathcal{L}_y^{\geq r}[b(v)] \operatorname{sgn}(v - u) \varphi + |b(v) - b(u)| \mathcal{L}_y^{< r}[\varphi] \leq 0. \end{aligned}$$

Combining (5.2) with (5.3) and using that $\mathcal{L}_{x+y}^{\geq r}[b(u) - b(v)] = \mathcal{L}_x^{\geq r}[b(u)] - \mathcal{L}_y^{\geq r}[b(v)]$ we obtain

$$(5.4) \quad \begin{aligned} & - \int_{Q_x \times Q_y} |u - v| (\partial_t + \partial_s) \varphi + F(u, v) \cdot (\nabla_x + \nabla_y) \varphi \\ & - \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r}[b(u) - b(v)] \operatorname{sgn}(u - v) \varphi + |b(u) - b(v)| (\mathcal{L}_x^{< r} + \mathcal{L}_y^{< r})[\varphi] \leq 0, \end{aligned}$$

where we used that $F(u, v) = F(v, u)$ and that $\text{sgn}(u - v) = -\text{sgn}(v - u)$. Exploiting the second inequality of Corollary C.2 with the self-adjointness of $\mathcal{L}_{x+y}^{\geq r}$ (Proposition 3.1) we see that

$$\begin{aligned} \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [b(u) - b(v)] \text{sgn}(u - v) \varphi &\leq \int_{M_x \times M_y} \mathcal{L}_{x+y}^{\geq r} [|b(u) - b(v)|] \varphi \\ &= \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} [\varphi]. \end{aligned}$$

Using this in (5.4) and further letting $r \rightarrow 0$, we conclude

$$(5.5) \quad - \int_{Q_x \times Q_y} \left(|u - v| (\partial_t + \partial_s) \varphi + F(u, v) \cdot (\nabla_x + \nabla_y) \varphi \right) - \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{L}_{x+y} [\varphi] \right) \leq 0.$$

2) Going to the boundary. With a slight abuse of notation, we let $r > 0$ represent a new arbitrary constant so that we may (again) split up the operator $\mathcal{L}_{x+y} = \mathcal{L}_{x+y}^{\geq r} + \mathcal{L}_{x+y}^{< r}$. By Proposition 3.1 we may shift $\mathcal{L}_{x+y}^{< r}$ to $\mathcal{B} < r_{x+y}$ so to get

$$- \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{< r} [\varphi] = \int_{M_x \times M_y} \mathcal{B}_{x+y}^{< r} [|b(u) - b(v)|, \varphi].$$

The integral on the right hand side is indeed well-defined; In fact the integrand of $\mathcal{B}^{< r} [|b(u) - b(v)|, \varphi]$ is absolutely integrable (c.f. (3.10)) as follows by a triangle inequality argument and the (locally) bounded energy of $b(u)$ and $b(v)$ (see Remark 2.6). Using this relation in (5.5) and rearranging, we get

$$(5.6) \quad \begin{aligned} &- \int_{Q_x \times Q_y} |u - v| (\partial_t + \partial_s) \varphi - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} [\varphi] \\ &\leq \int_{Q_x \times Q_y} F(u, v) \cdot (\nabla_x + \nabla_y) \varphi - \int_{M_x \times M_y} \mathcal{B}_{x+y}^{< r} [|b(u) - b(v)|, \varphi]. \end{aligned}$$

Next, we set

$$\varphi(t, x, s, y) = \zeta_\delta(x) \zeta_\delta(y) \rho(t - s, x - y) \alpha(t),$$

where $(\zeta_\delta)_{\delta > 0}$ is an inner boundary layer sequence satisfying $\sup_{\delta > 0} \|\nabla \zeta_\delta\|_{L^1(\Omega)} < \infty$ (cf. (2.7)), and where both $\rho \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ and $\alpha \in C_c^\infty((0, T))$ are nonnegative. We assume that the support of ρ is contained in a sufficiently small ball around zero so that φ is indeed compactly supported in $Q_x \times Q_y$. For brevity, we shall write $\zeta_\delta^x, \zeta_\delta^y, \rho, \alpha$ to mean the expressions $\zeta_\delta(x), \zeta_\delta(y), \rho(t - s, x - y), \alpha(t)$ respectively. Inserting for φ in (5.6) we get

$$(5.7) \quad \begin{aligned} &- \int_{Q_x \times Q_y} |u - v| \zeta_\delta^x \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} [\zeta_\delta^x \zeta_\delta^y] \rho \alpha \\ &\leq \int_{Q_x \times Q_y} F(u, v) \cdot (\nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_{x, x+y}^{< r} [|b(u) - b(v)|, \zeta_\delta^x \zeta_\delta^y] \rho \alpha \\ &\quad + \int_{Q_x \times Q_y} F(u, v) \cdot (\nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_{y, x+y}^{< r} [|b(u) - b(v)|, \zeta_\delta^x \zeta_\delta^y] \rho \alpha, \end{aligned}$$

where we used the cross-terms formula for $\mathcal{B}_{x+y}^{< r}$ from Proposition D.1 and the symmetry of ρ to move it outside of the respective operators.

We now wish to let $\delta \rightarrow 0$; observe that the left-hand side of (5.7) has a well-defined limit as $\mathcal{L}_{x+y}^{\geq r}$ is a zero order operator. Thus, letting $\delta \rightarrow 0$ we obtain

$$(5.8) \quad - \int_{Q_x \times Q_y} |u - v| \rho \alpha' - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} \left[\chi_\Omega^x \chi_\Omega^y \right] \rho \alpha \leq \lim_{\delta \rightarrow 0} (I_u + I_v),$$

where χ_Ω^x and χ_Ω^y denote the characteristic function χ_Ω as a function in x and y respectively and where I_u and I_v denote the first and second line on the right-hand side of (5.7). That $I_u + I_v$ admits a limit as $\delta \rightarrow 0$ will be proved by breaking it up into eighth appropriate terms.

We begin by exploiting the product rule given by Lemma D.2 to see that I_u may be rewritten

$$\begin{aligned} I_u &= \int_{Q_x \times Q_y} F(u, v) \cdot (\nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_x^{< r} \left[|b(u) - b(v)|, \zeta_\delta^x \right] \zeta_\delta^y \rho \alpha \\ &\quad - \int_{M_x \times M_y} |b(u) - b(v)| \left(\mathcal{B}_y^{< r} \left[\zeta_\delta^y, \rho \right] \zeta_\delta^x - \mathcal{B}_{x,y}^{< r} \left[\zeta_\delta^x, \zeta_\delta^y \right] \rho \right) \alpha, \end{aligned}$$

and by adding and subtracting terms we further obtain

$$\begin{aligned} (5.9) \quad I_u &= \int_{Q_x \times Q_y} (\mathcal{F}_u \cdot \nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_x^{< r} \left[\Sigma_u, \zeta_\delta^x \right] \zeta_\delta^y \rho \alpha \\ &\quad - \int_{M_x \times M_y} |b(u) - b(v)| \left(\mathcal{B}_y^{< r} \left[\zeta_\delta^y, \rho \right] \zeta_\delta^x - \mathcal{B}_{x,y}^{< r} \left[\zeta_\delta^x, \zeta_\delta^y \right] \rho \right) \alpha \\ &\quad - \int_{Q_x \times Q_y} (F(u, \bar{u}^c) - F(v, \bar{u}^c)) \cdot (\nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha \\ &\quad + \int_{M_x \times M_y} \mathcal{B}_x^{< r} \left[|b(u) - b(\bar{u}^c)| - |b(v) - b(\bar{u}^c)|, \zeta_\delta^x \right] \zeta_\delta^y \rho \alpha, \end{aligned}$$

where $\mathcal{F}_u = \mathcal{F}(u(t, x), v(s, y), \bar{u}^c(t, x))$, and $\Sigma_u = \Sigma(u(t, x), v(s, y), \bar{u}^c(t, x))$, and with Σ, \mathcal{F} as in (4.36). Analogously, we may write I_v as

$$\begin{aligned} (5.10) \quad I_v &= \int_{Q_x \times Q_y} (\mathcal{F}_v \cdot \nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_y^{< r} \left[\Sigma_v, \zeta_\delta^y \right] \zeta_\delta^x \rho \alpha \\ &\quad - \int_{M_x \times M_y} |b(u) - b(v)| \left(\mathcal{B}_x^{< r} \left[\zeta_\delta^x, \rho \right] \zeta_\delta^y - \mathcal{B}_{x,y}^{< r} \left[\zeta_\delta^x, \zeta_\delta^y \right] \rho \right) \alpha \\ &\quad - \int_{Q_x \times Q_y} (F(v, \bar{v}^c) - F(u, \bar{v}^c)) \cdot (\nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha \\ &\quad + \int_{M_x \times M_y} \mathcal{B}_y^{< r} \left[|b(v) - b(\bar{v}^c)| - |b(u) - b(\bar{v}^c)|, \zeta_\delta^y \right] \zeta_\delta^x \rho \alpha, \end{aligned}$$

where $\mathcal{F}_v = \mathcal{F}(v(s, y), u(t, x), \bar{v}^c(s, y))$, and $\Sigma_v = \Sigma(v(s, y), u(t, x), \bar{v}^c(s, y))$. Note that, although \bar{u}^c and \bar{v}^c coincides as functions, they depend on the variables (t, x) and (s, y) respectively. Finally, by grouping terms from (5.9) and (5.10) we can further write

$$(5.11) \quad I_u + I_v = J_u^0 + J_v^0 + J_u^1 + J_v^1 + J_u^2 + J_v^2 + J_u^3 + J_v^3,$$

where we have introduced

$$\begin{aligned}
J_u^0 &:= \int_{Q_x \times Q_y} (\mathcal{F}_u \cdot \nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_x^{<r} [\Sigma_u, \zeta_\delta^x] \zeta_\delta^y \rho \alpha, \\
J_v^0 &:= \int_{Q_x \times Q_y} (\mathcal{F}_v \cdot \nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha - \int_{M_x \times M_y} \mathcal{B}_y^{<r} [\Sigma_v, \zeta_\delta^y] \zeta_\delta^x \rho \alpha, \\
J_u^1 &:= - \int_{Q_x \times Q_y} (F(u, \bar{u}^c) \cdot \nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha + \int_{M_x \times M_y} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta^x] \zeta_\delta^y \rho \alpha, \\
J_v^1 &:= - \int_{Q_x \times Q_y} (F(v, \bar{v}^c) \cdot \nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha + \int_{M_x \times M_y} \mathcal{B}_y^{<r} [|b(v) - b(\bar{v}^c)|, \zeta_\delta^y] \zeta_\delta^x \rho \alpha, \\
J_u^2 &:= \int_{Q_x \times Q_y} (F(u, \bar{v}^c) \cdot \nabla_y \zeta_\delta^y) \zeta_\delta^x \rho \alpha - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{B}_y^{<r} [\zeta_\delta^y, \rho] \zeta_\delta^x \alpha, \\
J_v^2 &:= \int_{Q_x \times Q_y} (F(v, \bar{u}^c) \cdot \nabla_x \zeta_\delta^x) \zeta_\delta^y \rho \alpha - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{B}_x^{<r} [\zeta_\delta^x, \rho] \zeta_\delta^y \alpha, \\
J_u^3 &:= \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\zeta_\delta^x, \zeta_\delta^y] - \mathcal{B}_y^{<r} [|b(u) - b(\bar{v}^c)|, \zeta_\delta^y] \zeta_\delta^x \right) \rho \alpha, \\
J_v^3 &:= \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\zeta_\delta^x, \zeta_\delta^y] - \mathcal{B}_x^{<r} [|b(v) - b(\bar{u}^c)|, \zeta_\delta^x] \zeta_\delta^y \right) \rho \alpha.
\end{aligned}$$

The limit as $\delta \rightarrow 0$ of each of these terms will now be examined.

3) The limit of J_u^0 and J_v^0 . We will not establish the exact limits of these terms. We only demonstrate that the limits are non-positive which suffices for our purpose. Starting with J_u^0 , we observe that it can be written

$$J_u^0 = \int_{Q_y} \left[\int_{Q_x} (\mathcal{F}_u \cdot \nabla_x \zeta_\delta^x) \rho \alpha - \int_{M_x} \mathcal{B}_x^{<r} [\Sigma_u, \zeta_\delta^x] \rho \alpha \right] \zeta_\delta^y.$$

By the boundary condition for u (Proposition 4.10) the square bracket has a nonnegative limit as $\delta \rightarrow 0$. Moreover, as Remark 4.11 points out, the bracket is uniformly bounded on Q_y as $\delta \rightarrow 0$. By dominated convergence we conclude that

$$\lim_{\delta \rightarrow 0} J_u^0 \leq 0.$$

An analogous calculation gives the corresponding result $\lim_{\delta \rightarrow 0} J_v^0 \leq 0$.

4) The limit of J_u^1 and J_v^1 . Starting with J_u^1 , we again do a little rewriting to see that

$$J_u^1 = \int_{Q_y} \left[\int_{M_x} \left(-F(u, \bar{u}^c) \cdot \nabla_x \zeta_\delta^x + \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \zeta_\delta^x] \right) \rho \alpha \right] \zeta_\delta^y,$$

As ζ_δ^x is a boundary layer sequence satisfying $\zeta_\delta^x = 0$ on $\partial\Omega_x$, the limit of the inner integral is given by the weak trace result of Proposition 4.8. By dominated convergence and Remark 4.9 this limit may be taken inside the integral and we get

$$\lim_{\delta \rightarrow 0} J_u^1 = L_u^1 := - \int_{\Gamma_x \times Q_y} \rho \alpha \, d\nu_u + \int_{M_x \times Q_y} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega^x] \rho \alpha,$$

where ν_u is a finite Borel measure on $\bar{\Gamma}_x$, but we integrate over Γ_x as α is compactly supported in $(0, T)$. As before, we write χ_Ω^x to stress the x -dependence of the characteristic function.

Analogously, we also obtain

$$\lim_{\delta \rightarrow 0} J_v^1 = L_v^1 := - \int_{Q_x \times \Gamma_y} \rho \alpha \, d\nu_v + \int_{Q_x \times M_y} \mathcal{B}_y^{<r} \left[|b(v) - b(\bar{v}^c)|, \chi_\Omega^y \right] \rho \alpha.$$

5) *The limit of J_u^2 and J_v^2 .* As before, we start with J_u^2 and rewrite it as

$$\begin{aligned} J_u^2 &= \int_{Q_x} \left(\int_{Q_y} (F(u, \bar{v}^c) \cdot \nabla_y \zeta_\delta^y) \rho \alpha \right) \zeta_\delta^x \\ &\quad - \int_{|z| < r} \left(\int_{M_x \times M_y} |b(u) - b(v)| (\zeta_\delta^y(y+z) - \zeta_\delta^y(y)) (\rho(x-y-z) - \rho(x-y)) \alpha \zeta_\delta^x \right) d\mu(z), \end{aligned}$$

where we for the second integral used Fubini's theorem and wrote out only the arguments featuring translations in z . For the first integral, we exploit the regularity of \bar{v}^c , ρ , α and f to get

$$\lim_{\delta \rightarrow 0} \int_{Q_x} \left(\int_{Q_y} (F(u, \bar{v}^c) \cdot \nabla_y \zeta_\delta^y) \rho \alpha \right) \zeta_\delta^x = - \int_{Q_x \times \Gamma_y} (F(u, \bar{v}^c) \cdot \hat{n}) \rho \alpha,$$

where \hat{n} is an outward-pointing normal vector on $\partial\Omega$. Here we used dominated convergence; this is justified as the inner integral is uniformly bounded over Q_x and $\delta > 0$ (we recall that $\sup_{\delta > 0} \|\nabla \zeta_\delta\|_{L^1(\Omega)} < \infty$). For the second part of J_u^2 we again wish to deploy dominated convergence; the inner integral has a canonical limit for fixed z , and for all $\delta > 0$ it is dominated by

$$\|b(u) - b(v)\|_{L^\infty(M_x \times M_y)} \sup_{\delta > 0} \left(\|\nabla \zeta_\delta^y\|_{L^1(M_y)} |z| \wedge 2 \|\zeta_\delta^y\|_{L^1(M_y)} \right) \left(L_\rho |z| \wedge 2 \|\rho\|_{L^\infty(M_x \times M_y)} \right) \|\alpha \zeta_\delta^x\|_{L^1(M_x)},$$

where L_ρ is a Lipschitz constant for ρ on $M_x \times M_y$. As this bound can further be bounded by $C(|z|^2 \wedge 1)$, for an appropriate C , we infer from dominated convergence that the second part of J_u^2 admits the limit

$$- \int_{|z| < r} \left(\int_{M_x \times M_y} |b(u) - b(v)| (\chi_\Omega^y(y+z) - \chi_\Omega^y(y)) (\rho(x-y-z) - \rho(x-y)) \alpha \chi_\Omega^x \right) d\mu(z),$$

as $\delta \rightarrow 0$. Fubini's theorem may again be applied to move the integral in z inside, and so we conclude

$$\lim_{\delta \rightarrow 0} J_u^2 = L_u^2 := - \int_{Q_x \times \Gamma_y} (F(u, \bar{v}^c) \cdot \hat{n}) \rho \alpha - \int_{Q_x \times M_y} |b(u) - b(v)| \mathcal{B}_y^{<r} \left[\chi_\Omega^y, \rho \right] \alpha,$$

and by an analogous argument we also obtain

$$\lim_{\delta \rightarrow 0} J_v^2 = L_v^2 := - \int_{\Gamma_x \times Q_y} (F(v, \bar{u}^c) \cdot \hat{n}) \rho \alpha - \int_{M_x \times Q_y} |b(u) - b(v)| \mathcal{B}_x^{<r} \left[\chi_\Omega^x, \rho \right] \alpha.$$

6) *The limit of J_u^3 and J_v^3 .* We begin by studying J_u^3 . By Fubini's theorem, we rewrite J_u^3 as

$$\begin{aligned} J_u^3 &= \int_{|z| < r} \left(\int_{M_x \times M_y} \left[|b(u) - b(v)| \left(\zeta_\delta^x(x+z) - \zeta_\delta^x(x) \right) \left(\zeta_\delta^y(y+z) - \zeta_\delta^y(y) \right) \right. \right. \\ &\quad \left. \left. - \left(|b(u) - b(\bar{v}^c(y+z))| - |b(u) - b(\bar{v}^c(y))| \right) \left(\zeta_\delta^y(y+z) - \zeta_\delta^y(y) \right) \zeta_\delta^x \right] \rho \alpha \right) d\mu(z), \end{aligned}$$

where only the arguments featuring translations in z have been written out. For fixed z it is obvious that the inner integral converges to its canonical limit as $\delta \rightarrow 0$. Moreover, the inner integral admits

the following bound

$$(5.12) \quad |z|^2 \left(\|b(u) - b(v)\|_{L^\infty(M_x \times M_y)} \|\nabla \zeta_\delta^x\|_{L^1(\mathbb{R}^d)} \|\nabla \zeta_\delta^y\|_{L^1(\mathbb{R}^d)} \right. \\ \left. + L_{b(\bar{v}^c)} \|\nabla \zeta_\delta^y\|_{L^1(\mathbb{R}^d)} \|\zeta_\delta^x\|_{L^1(\mathbb{R}^d)} \right) \|\rho\alpha\|_{L^\infty(M_x \times M_y)} \leq C|z|^2,$$

where $L_{b(\bar{v}^c)}$ is a local Lipschitz constant of $b(\bar{v}^c)$ and where C is some large constant independent of $\delta > 0$ as $\sup_\delta \|\nabla \zeta_\delta\|_{L^1(\mathbb{R}^d)} < \infty$ and $\sup_\delta \|\zeta_\delta\|_{L^1(\mathbb{R}^d)} \leq |\Omega|$. As this last bound is integrable on $|z| < r$ with respect to $d\mu$, it follows by dominated convergence that

$$\lim_{\delta \rightarrow 0} J_u^3 = L_u^3 := \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\chi_\Omega^x, \chi_\Omega^y] - \mathcal{B}_y^{<r} [b(u) - b(\bar{v}^c), \chi_\Omega^y] \chi_\Omega^x \right) \rho\alpha.$$

And by an analogous argument, we also obtain

$$\lim_{\delta \rightarrow 0} J_v^3 = L_v^3 := \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{B}_{x,y}^{<r} [\chi_\Omega^x, \chi_\Omega^y] - \mathcal{B}_x^{<r} [b(v) - b(\bar{v}^c), \chi_\Omega^x] \chi_\Omega^y \right) \rho\alpha.$$

7) *Going to the diagonal.* In summary, we have so far established the following inequality

$$(5.13) \quad - \int_{Q_x \times Q_y} |u - v| \rho\alpha' - \int_{M_x \times M_y} |b(u) - b(v)| \mathcal{L}_{x+y}^{\geq r} [\chi_\Omega^x, \chi_\Omega^y] \rho\alpha \\ \leq L_u^1 + L_v^1 + L_u^2 + L_v^2 + L_u^3 + L_v^3,$$

which follows from (5.8) and (5.11) together with the above established limits. Observe that we have chosen to omit L_u^0 and L_v^0 from the right-hand side as these terms are non-positive. We now wish to insert a more explicit expression for ρ . For two small fixed parameters $\epsilon, \varepsilon > 0$ we set

$$\rho(t - s, x - y) = \theta_\epsilon(t - s) \tilde{\rho}_\varepsilon(x - y),$$

where θ_ϵ and ρ_ε are standard mollifiers in \mathbb{R} and \mathbb{R}^d respectively. More precisely, we have $\theta_\epsilon = \epsilon^{-1} \theta(\cdot/\epsilon)$ and $\tilde{\rho}_\varepsilon = \varepsilon^{-d} \tilde{\rho}(\cdot/\varepsilon)$, where θ and $\tilde{\rho}$ are positive, smooth and symmetric functions supported in the unit ball of \mathbb{R} and \mathbb{R}^d that additionally satisfy $\int_{\mathbb{R}} \theta dt = \int_{\mathbb{R}^d} \tilde{\rho} dx = 1$. For notational simplicity, we shall write ρ instead of $\tilde{\rho}$, and the abbreviations θ_ϵ and ρ_ε to mean the expressions $\theta_\epsilon(t - s)$ and $\rho_\varepsilon(x - y)$. If we on the left-hand side of (5.13) replace the (old) ρ with the new expression $\theta_\epsilon \rho_\varepsilon$ we get

$$(5.14) \quad - \int_{Q_x \times Q_y} |u(t, x) - v(s, y)| \theta_\epsilon \rho_\varepsilon \alpha' - \int_{M_x \times M_y} |b(u(t, x)) - b(v(s, y))| \mathcal{L}_{x+y}^{\geq r} [\chi_\Omega^x, \chi_\Omega^y] \theta_\epsilon \rho_\varepsilon \alpha,$$

where the arguments of u and v have been included to clarify the next computation. We wish to *go to the diagonal*, i.e. to let $\varepsilon, \epsilon \rightarrow 0$. This task is fairly straightforward for the above expression. Indeed, subtracting

$$(5.15) \quad - \int_{Q_x \times Q_y} |u(t, x) - v(t, x)| \theta_\epsilon \rho_\varepsilon \alpha' - \int_{M_x \times M_y} |b(u(t, x)) - b(v(t, x))| \mathcal{L}_x^{\geq r} [\chi_\Omega^x] \theta_\epsilon \rho_\varepsilon \alpha,$$

from (5.14) and making the substitution of variables $s = t + \tau$ and $y = x + \xi$, a (mostly) standard calculation shows that the result is bounded by

$$\begin{aligned} & \sup_{\substack{|\tau| < \epsilon \\ |\xi| < \epsilon}} \|v(\cdot + \tau, \cdot + \xi) - v\|_{L^1(Q_y)} \|\alpha'\|_{L^\infty((0,T))} \\ & + \sup_{\substack{|\tau| < \epsilon \\ |\xi| < \epsilon}} \|b(v)(\cdot + \tau, \cdot + \xi) - b(v)\|_{L^1(Q_y)} C_r 2 \|\chi_\Omega^x\|_{L^\infty(M_x)} \|\alpha\|_{L^\infty(M_x)} \\ & + \|b(u) - b(v)\|_{L^\infty(M_x \times M_y)} C_r \sup_{|\xi| < \epsilon} \|\chi_\Omega^x(\cdot)(\chi_\Omega^x(\cdot + \xi) - 1)\|_{L^1(M_x)} \|\alpha\|_{L^\infty((0,T))}, \end{aligned}$$

where $C_r = \mu(\{|z| \geq r\})$. By translation regularity of L^1_{loc} -functions, this last bound tends to zero as $\epsilon, \epsilon \rightarrow 0$. In particular, the limit of (5.14) as $\epsilon, \epsilon \rightarrow 0$ coincides with that of (5.15) which is given by

$$(5.16) \quad - \int_Q |u(t, x) - v(t, x)| \alpha'(t) \, dx \, dt - \int_M |b(u(t, x)) - b(v(t, x))| \mathcal{L}^{\geq r}[\chi_\Omega](x) \alpha(t) \, dx \, dt.$$

Observe next that the second integral in (5.16) is non-positive. This follows as $|b(u(t, x)) - b(v(t, x))|$ is nonnegative and supported in Q (since $u^c = v^c$ a.e. in Q^c), while $\mathcal{L}^{\geq r}[\chi_\Omega]$ is non-positive in Q due to χ_Ω taking its maximum in Q . We conclude then from (5.13) and the above analysis that

$$(5.17) \quad - \int_Q |u(t, x) - v(t, x)| \alpha'(t) \, dx \, dt \leq \lim_{r \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} \left[L_u^1 + L_v^1 + L_u^2 + L_v^2 + L_u^3 + L_v^3 \right].$$

The remainder of the proof is to show that the right-hand side is non-positive. For clarity, we restate the definitions of the right-hand-side terms, with the old ρ replaced by $\theta_\epsilon \rho_\epsilon$. They are

$$\begin{aligned} L_u^1 &= - \int_{\Gamma_x \times Q_y} \theta_\epsilon \rho_\epsilon \alpha \, d\nu_u + \int_{M_x \times Q_y} \mathcal{B}_x^{< r} \left[|b(u) - b(\bar{u}^c)|, \chi_\Omega^x \right] \theta_\epsilon \rho_\epsilon \alpha, \\ L_v^1 &= - \int_{Q_x \times \Gamma_y} \theta_\epsilon \rho_\epsilon \alpha \, d\nu_v + \int_{Q_x \times M_y} \mathcal{B}_y^{< r} \left[|b(v) - b(\bar{v}^c)|, \chi_\Omega^y \right] \theta_\epsilon \rho_\epsilon \alpha, \\ L_u^2 &= - \int_{Q_x \times \Gamma_y} (F(u, \bar{v}^c) \cdot \hat{n}) \theta_\epsilon \rho_\epsilon \alpha - \int_{Q_x \times M_y} |b(u) - b(v)| \mathcal{B}_y^{< r} \left[\chi_\Omega^y, \rho_\epsilon \right] \theta_\epsilon \alpha, \\ L_v^2 &= - \int_{\Gamma_x \times Q_y} (F(v, \bar{u}^c) \cdot \hat{n}) \theta_\epsilon \rho_\epsilon \alpha - \int_{M_x \times Q_y} |b(u) - b(v)| \mathcal{B}_x^{< r} \left[\chi_\Omega^x, \rho_\epsilon \right] \theta_\epsilon \alpha, \\ L_u^3 &= \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{B}_{x,y}^{< r} \left[\chi_\Omega^x, \chi_\Omega^y \right] - \mathcal{B}_y^{< r} \left[|b(u) - b(\bar{v}^c)|, \chi_\Omega^y \right] \chi_\Omega^x \right) \theta_\epsilon \rho_\epsilon \alpha, \\ L_v^3 &= \int_{M_x \times M_y} \left(|b(u) - b(v)| \mathcal{B}_{x,y}^{< r} \left[\chi_\Omega^x, \chi_\Omega^y \right] - \mathcal{B}_x^{< r} \left[|b(v) - b(\bar{u}^c)|, \chi_\Omega^x \right] \chi_\Omega^y \right) \theta_\epsilon \rho_\epsilon \alpha. \end{aligned}$$

We will study each term individually as was done with the J -terms.

8) *The limit of L_u^1 and L_v^1 .* Starting with L_u^1 , we introduce

$$(5.18) \quad \varrho_\epsilon(x) := \int_\Omega \rho_\epsilon(x - y) \, dy.$$

Note that ϱ_ε is smooth, has compact support, and is bounded between 0 and 1 on \mathbb{R}^d . Moreover, it satisfies

$$(5.19) \quad \lim_{\varepsilon \rightarrow 0} \varrho_\varepsilon(x) = \begin{cases} 1, & x \in \Omega, \\ \frac{1}{2}, & x \in \partial\Omega, \\ 0, & x \in \Omega^c \setminus \partial\Omega, \end{cases}$$

by the regularity of the boundary. With this notation, we let $\varepsilon \rightarrow 0$ and find that L_u^1 can be written

$$(5.20) \quad \lim_{\varepsilon \rightarrow 0} L_u^1 = - \int_{\Gamma_x} \varrho_\varepsilon \alpha \, d\nu_u + \int_{M_x} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega^x] \varrho_\varepsilon \alpha.$$

By Corollary 4.7, the expression $\mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega^x]$ is well defined in $L^1(M_x)$ and must consequently vanish (in L^1 sense) as $r \rightarrow 0$. We conclude from (5.19), (5.20) and dominated convergence that

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{\varepsilon \rightarrow 0} L_u^1 = -\frac{1}{2} \int_{\Gamma} \alpha \, d\nu_u,$$

where we used that α has compact support in $(0, T)$ so that and that $\lim_{\varepsilon \rightarrow 0} \int_0^T \theta_\varepsilon(t-s) \, ds = 1$ for all $t \in (0, T)$.

For L_v^1 the analysis is analogous, except for the small difference that α goes from being a function in t to one in s as $\varepsilon \rightarrow 0$. This leads to the analogous conclusion

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{\varepsilon \rightarrow 0} L_v^1 = -\frac{1}{2} \int_{\Gamma} \alpha \, d\nu_v.$$

9) The limit of L_u^2 and L_v^2 . These two limits are more laborious than the previous two. Starting with L_u^2 , we first note that the limit $\varepsilon \rightarrow 0$ is standard, giving

$$(5.21) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} L_u^2 &= - \int_0^T \int_{\Omega_x \times \partial\Omega_y} (F(u, \bar{v}^c) \cdot \hat{n}) \alpha \rho_\varepsilon \, dx \, d\sigma(y) \, dt \\ &\quad - \int_0^T \int_{\Omega_x \times \mathbb{R}_y^d} |b(u) - b(v)| \mathcal{B}_y^{<r} [\chi_\Omega^y, \rho_\varepsilon] \alpha \, dx \, dy \, dt. \end{aligned}$$

where v and \bar{v}^c on the right-hand side of (5.21) are functions in (t, y) . We wish to replace $F(u(t, x), \bar{v}^c(t, y))$ with $F(u(t, x), \bar{u}^c(t, x))$ in the first integral on the right-hand side of (5.21). To do so, we recall that \bar{v}^c is the same function as \bar{u}^c (but in different variables) so we may compute

$$(5.22) \quad \begin{aligned} &\left| \int_0^T \int_{\Omega_x \times \partial\Omega_y} (F(u, \bar{v}^c) - F(u, \bar{u}^c)) \cdot \hat{n} \alpha \rho_\varepsilon \, dx \, d\sigma(y) \, dt \right| \\ &\leq L_f \int_0^T \int_{\Omega_x \times \partial\Omega_y} |\bar{v}^c(t, y) - \bar{u}^c(t, x)| \alpha(t) \rho_\varepsilon(x-y) \, dx \, d\sigma(y) \, dt \\ &\leq L_f L_{\bar{u}^c} T \|\alpha\|_{L^\infty((0, T))} \int_{\Omega_x \times \partial\Omega_y} |x-y| \rho_\varepsilon(x-y) \, dx \, d\sigma(y), \end{aligned}$$

where L_f and $L_{\bar{u}^c}$ are local Lipschitz constants of f and \bar{u}^c . Using that $\int_{\mathbb{R}^d} |x-y| \rho_\varepsilon(x-y) \, dx \leq \varepsilon$ we note that the right-most side of (5.22) is of size $\lesssim \varepsilon$. Moreover, by the divergence theorem we have the identity

$$- \int_{\partial\Omega_y} \hat{n} \rho_\varepsilon \, d\sigma(y) = \int_{\Omega_y} -\nabla_y \rho_\varepsilon \, dy = \int_{\Omega_y} \nabla_x \rho_\varepsilon \, dy = \nabla_x \varrho_\varepsilon,$$

where ϱ_ε is as in (5.18). Summarizing, the first part of (5.21) can be written

$$(5.23) \quad - \int_0^T \int_{\Omega_x \times \partial\Omega_y} (F(u, \bar{v}^c) \cdot \hat{n}) \alpha \rho_\varepsilon \, dx \, d\sigma(y) \, dt = \int_{Q_x} F(u, \bar{u}^c) \cdot \nabla_x \varrho_\varepsilon \alpha \, dx \, dt + O(\varepsilon).$$

Next, we shall find an analogous representation for the second integral in (5.21). For brevity, we will in the coming analysis ignore α and the integral in t and suppress the t -dependence of the remaining expressions. We begin by observing that

$$(5.24) \quad \begin{aligned} & \left| \int_{\Omega_x \times \mathbb{R}_y^d} \left(|b(u) - b(v)| - |b(u) - b(\bar{u}^c)| \right) \mathcal{B}_y^{<r} \left[\chi_\Omega^y, \rho_\varepsilon \right] \, dx \, dy \right| \\ & \leq \int_{\Omega_x \times \mathbb{R}_y^d} \left| (b(v) - b(\bar{v}^c)) \mathcal{B}_y^{<r} \left[\chi_\Omega^y, \rho_\varepsilon \right] \right| \, dx \, dy \\ & \quad + \int_{\Omega_x \times \mathbb{R}_y^d} \left| (b(\bar{u}^c) - b(\bar{v}^c)) \mathcal{B}_y^{<r} \left[\chi_\Omega^y, \rho_\varepsilon \right] \right| \, dx \, dy. \end{aligned}$$

We show that both of the expressions on the right hand side of (5.24) vanish as $\varepsilon \rightarrow 0$. For the first one, we exploit that $|b(v) - b(\bar{v}^c)|$ is zero for $y \in \Omega^c$ and obtain

$$\begin{aligned} & \int_{\Omega_x \times \mathbb{R}_y^d} \left| (b(v) - b(\bar{v}^c)) \mathcal{B}_y^{<r} \left[\chi_\Omega^y, \rho_\varepsilon \right] \right| \, dx \, dy \\ & \leq \int_{\substack{x, y \in \Omega \\ y+z \in \Omega^c \\ |z| < r}} \frac{1}{2} |b(v(y)) - b(\bar{v}^c(y))| |\rho_\varepsilon(x - y - z) - \rho_\varepsilon(x - y)| \, dx \, dy \, d\mu(z) \\ & \leq \int_{\substack{y \in \Omega \\ y+z \in \Omega^c \\ |z| < r}} |b(v(y)) - b(\bar{v}^c(y))| \, dy \, d\mu(z) \rightarrow 0, \quad r, \varepsilon \rightarrow 0, \end{aligned}$$

where the limit holds by the boundary integrability of Proposition 4.6. The second expression on the right-hand side of (5.24) is more tricky. First, recalling that \bar{u}^c and \bar{v}^c is the same function and differ only in their arguments (they depend on (t, x) and (t, y) respectively) we compute

$$(5.25) \quad \begin{aligned} & \int_{\Omega_x \times \mathbb{R}_y^d} \left| (b(\bar{u}^c) - b(\bar{v}^c)) \mathcal{B}_y^{<r} \left[\chi_\Omega^y, \rho_\varepsilon \right] \right| \, dx \, dy \\ & \leq \frac{L_{b(\bar{u}^c)}}{2} \int_{\substack{x \in \Omega, \\ y \in \mathbb{R}^d, \\ |z| < r}} |x - y| |\chi_\Omega(y + z) - \chi_\Omega(y)| |\rho_\varepsilon(x - y - z) - \rho_\varepsilon(x - y)| \, dx \, dy \, d\mu(z), \end{aligned}$$

where $L_{b(\bar{u}^c)}$ is a local Lipschitz constant of $b(\bar{u}^c)$. As the integrand of the previous integral is globally nonnegative, we expand the domain of integration to attain an upper bound. Ignoring the factor $L_{b(\bar{u}^c)}/2$, the previous integral is then bounded by

$$\begin{aligned} & \int_{\substack{x, y \in \mathbb{R}^d, \\ |z| < r}} |x - y| |\chi_\Omega(y + z) - \chi_\Omega(y)| |\rho_\varepsilon(x - y - z) - \rho_\varepsilon(x - y)| \, dx \, dy \, d\mu(z) \\ & = \int_{\substack{x, y \in \mathbb{R}^d, \\ |z| < r}} |x| |\chi_\Omega(y + z) - \chi_\Omega(y)| |\rho_\varepsilon(x - z) - \rho_\varepsilon(x)| \, dx \, dy \, d\mu(z) \\ & \leq |\chi_\Omega|_{TV(\mathbb{R}^d)} \int_{\substack{x \in \mathbb{R}^d, \\ |z| < r}} |x| |\rho_\varepsilon(x - z) - \rho_\varepsilon(x)| \, |z| \, dx \, d\mu(z), \end{aligned}$$

where we recall that $|\chi_\Omega|_{TV(\mathbb{R}^d)} = |\partial\Omega|_{\mathcal{H}^{d-1}} < \infty$ (see comment after Proposition 3.8). For the quantity $|\rho_\varepsilon(x-z) - \rho_\varepsilon(x)|$ we have the estimate

$$|\rho_\varepsilon(x-z) - \rho_\varepsilon(x)| \leq \frac{1}{\varepsilon^d} \left(2\|\rho\|_{L^\infty(\mathbb{R}^d)} \wedge \frac{|z|}{\varepsilon} \|\nabla\rho\|_{L^\infty(\mathbb{R}^d)} \right) \lesssim \frac{|z|}{\varepsilon^d(|z| + \varepsilon)},$$

where the last bound exploits that $(1 \wedge p) \leq 2p/(1+p)$ when $p \geq 0$, giving the desired conclusion for $p = |z|/\varepsilon$. Moreover, the support of said quantity is contained in $\{(x, z): |x| \wedge |x-z| < \varepsilon\}$; for fixed z , this set is of measure $\simeq \varepsilon^d$, and any x in it satisfies $|x| < |z| + \varepsilon$. Putting this together yields

$$\int_{x \in \mathbb{R}^d} |x| |\rho_\varepsilon(x-z) - \rho_\varepsilon(x)| dx \lesssim \varepsilon^d (|z| + \varepsilon) \frac{|z|}{\varepsilon^d (|z| + \varepsilon)} = |z|,$$

so that we may conclude

$$\int_{\substack{x \in \mathbb{R}^d \\ |z| < r}} |x| |\rho_\varepsilon(x-z) - \rho_\varepsilon(x)| |z| dx d\mu(z) \lesssim \int_{|z| < r} |z|^2 d\mu(z) \rightarrow 0, \quad r, \varepsilon \rightarrow 0.$$

To summarize, we have just demonstrated that both terms on the right-hand side of (5.24) vanish when sending $\varepsilon \rightarrow 0$ followed by $r \rightarrow 0$. For the term we subtracted on the left-hand side of (5.24) we use the relation between $\mathcal{B}^{<r}$ and $\mathcal{L}^{<r}$ from Proposition 3.1 to compute

$$\begin{aligned} \int_{\Omega_x \times \mathbb{R}_y^d} |b(u) - b(\bar{u}^c)| \mathcal{B}_y^{<r} [\chi_\Omega^y, \rho_\varepsilon] dx dy &= - \int_{\Omega_x \times \Omega_y} |b(u) - b(\bar{u}^c)| \mathcal{L}_y^{<r} [\rho_\varepsilon] dx dy \\ &= - \int_{\Omega_x} |b(u) - b(\bar{u}^c)| \mathcal{L}_x^{<r} [\varrho_\varepsilon] dx \\ &= \int_{\mathbb{R}_x^d} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \varrho_\varepsilon] dx, \end{aligned}$$

where ϱ_ε is as defined in (5.18) and where the compact support of $|b(u) - b(\bar{u}^c)|$ justified expanding the domain of integration from Ω_x to \mathbb{R}_x^d (so to shift $\mathcal{L}_x^{<r}$ over to $\mathcal{B}_x^{<r}$). All in all, analogous to (5.23), we see that the second integral in (5.21) can be written

$$(5.26) \quad - \int_0^T \int_{\Omega_x \times \mathbb{R}_y^d} |b(u) - b(v)| \mathcal{B}_y^{<r} [\chi_\Omega^y, \rho_\varepsilon] \alpha dx dy dt = \int_{M_x} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \varrho_\varepsilon] \alpha dx dt + o(1),$$

where the o -term is such that $\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} o(1) = 0$. Finally, combining (5.21) with (5.23) and (5.26) we conclude that

$$(5.27) \quad \begin{aligned} &\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_u^2 \\ &= \lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left(\int_{Q_x} F(u, \bar{u}^c) \cdot \nabla_x \varrho_\varepsilon \alpha - \int_{M_x} \mathcal{B}_x^{<r} [|b(u) - b(\bar{u}^c)|, \varrho_\varepsilon] \alpha \right). \end{aligned}$$

Observe that $(\varrho_\varepsilon)_{\varepsilon > 0}$ is a boundary layer sequence satisfying $\lim_{\varepsilon \rightarrow 0} \varrho_\varepsilon = \frac{1}{2}$ pointwise on $\partial\Omega$. By the weak trace result of Proposition 4.8 we conclude

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_u^2 = \frac{1}{2} \int_\Gamma \alpha d\nu_u,$$

where we used that $\lim_{r \rightarrow 0} \mathcal{B}^{<r} [|b(u) - b(\bar{u}^c)|, \chi_\Omega] = 0$ in $L^1(M)$ as follows from Corollary 4.7. By analogous arguments, we also find that

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_v^2 = \frac{1}{2} \int_\Gamma \alpha d\nu_v.$$

Before moving on, note that we have proved

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left(L_u^1 + L_v^1 + L_u^2 + L_v^2 \right) = 0.$$

10) The limit of L_u^3 and L_v^3 . We demonstrate that both of these limits vanish. Starting with L_u^3 , we begin as before by going to the diagonal in the time variable

$$(5.28) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} L_u^3 &= \int_0^T \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} |b(u) - b(v)| \mathcal{B}_{x,y}^{<r} \left[\chi_\Omega^x, \chi_\Omega^y \right] \rho_\varepsilon \alpha \, dx \, dy \, dt \\ &\quad - \int_0^T \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} \mathcal{B}_y^{<r} \left[|b(u) - b(\bar{v}^c)|, \chi_\Omega^y \right] \chi_\Omega^x \rho_\varepsilon \alpha \, dx \, dy \, dt, \end{aligned}$$

where the time dependence of v and \bar{v}^c is now in t . For brevity, we again ignore α , the integral in t and arguments in t .

The second integral in (5.28) is easily seen to vanish as $r, \varepsilon \rightarrow 0$. Indeed, as $b(u)$ depends on x and not y it follows by a triangle argument that

$$(5.29) \quad \begin{aligned} &\left| \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} \mathcal{B}_y^{<r} \left[|b(u) - b(\bar{v}^c)|, \chi_\Omega^y \right] \chi_\Omega^x \rho_\varepsilon \, dx \, dy \right| \\ &\leq \frac{1}{2} \int_{\substack{x, y \in \mathbb{R}^d \\ |z| < r}} |b(\bar{v}^c(y+z)) - b(\bar{v}^c(y))| |\chi_\Omega(y+z) - \chi_\Omega(y)| \rho_\varepsilon(x-y) \, dx \, dy \, d\mu(z) \\ &\leq \frac{L_{b(\bar{v}^c)}}{2} \int_{\substack{y \in \mathbb{R}^d \\ |z| < r}} |z| |\chi_\Omega(y+z) - \chi_\Omega(y)| \, dy \, d\mu(z) \\ &\leq \frac{L_{b(\bar{v}^c)}}{2} |\chi_\Omega|_{TV} \int_{|z| < r} |z|^2 \, d\mu(z) \rightarrow 0 \quad r, \varepsilon \rightarrow 0, \end{aligned}$$

where $L_{b(\bar{v}^c)}$ denotes a local Lipschitz constant of \bar{v}^c . Thus, we turn our attention to the first integral in (5.28). We begin by a similar argument as was used for the second integral in (5.21): observe that

$$(5.30) \quad \begin{aligned} &\left| \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} \left(|b(u) - b(v)| - |b(\bar{u}^c) - b(\bar{v}^c)| \right) \mathcal{B}_{x,y}^{<r} \left[\chi_\Omega^x, \chi_\Omega^y \right] \rho_\varepsilon \, dx \, dy \right| \\ &\leq \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} \left(|b(u) - b(\bar{u}^c)| + |b(v) - b(\bar{v}^c)| \right) \left| \mathcal{B}_{x,y}^{<r} \left[\chi_\Omega^x, \chi_\Omega^y \right] \right| \rho_\varepsilon \, dx \, dy. \end{aligned}$$

We now show that the right-hand side vanishes as $r, \varepsilon \rightarrow 0$. Focusing on the $|b(u) - b(\bar{u}^c)|$ term (which is supported in Ω_x) we have

$$\begin{aligned} &\int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} |b(u) - b(\bar{u}^c)| \left| \mathcal{B}_{x,y}^{<r} \left[\chi_\Omega^x, \chi_\Omega^y \right] \right| \rho_\varepsilon \, dx \, dy \\ &= \frac{1}{2} \int_{\substack{x \in \Omega \\ x+z \in \Omega^c \\ |z| < r}} |b(u(x)) - b(\bar{u}^c(x))| \left(\int_{\mathbb{R}_y^d} |\chi_\Omega(y+z) - \chi_\Omega(y)| \rho_\varepsilon \, dy \right) \, dx \, d\mu(z) \\ &\leq \int_{\substack{x \in \Omega \\ x+z \in \Omega^c \\ |z| < r}} |b(u(x)) - b(\bar{u}^c(x))| \, dx \, d\mu(z) \rightarrow 0, \quad r, \varepsilon \rightarrow 0, \end{aligned}$$

where the limit holds by the boundary integrability of Proposition 4.6. As a corresponding computation can be carried out for the $|b(v) - b(\bar{v}^c)|$ term, we conclude that the right hand side of (5.30) vanishes as $r, \varepsilon \rightarrow 0$. Thus, we may replace $|b(u) - b(v)|$ with the more regular $|b(\bar{u}^c) - b(\bar{v}^c)|$ when seeking the

limit of the first integral in (5.28). Moreover, by a similar argument used to prove Proposition 3.1 we may shift $\mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, \cdot]$ from χ_Ω^y over to $|b(\bar{u}^c) - b(\bar{v}^c)|\rho_\varepsilon$ giving

$$\int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} |b(\bar{u}^c) - b(\bar{v}^c)| \mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, \chi_\Omega^y] \rho_\varepsilon \, dx \, dy = \int_{\mathbb{R}_x^d \times \mathbb{R}_y^d} \mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, |b(\bar{u}^c) - b(\bar{v}^c)|\rho_\varepsilon] \chi_\Omega^y \, dx \, dy.$$

And by a triangle inequality argument, we find

$$\begin{aligned} & \int_{\mathbb{R}_x^d \times \Omega_y} \left| \mathcal{B}_{x,y}^{<r}[\chi_\Omega^x, |b(\bar{u}^c) - b(\bar{v}^c)|\rho_\varepsilon] \right| \, dx \, dy \\ & \leq \int_{\mathbb{R}_x^d \times \Omega_y} \int_{|z| < r} |\chi_\Omega(x+z) - \chi_\Omega(x)| |b(\bar{v}^c(y+z)) - b(\bar{v}^c(y))| \rho_\varepsilon(x-y-z) \, d\mu(z) \, dx \, dy \\ & \quad + \int_{\mathbb{R}_x^d \times \Omega_y} \int_{|z| < r} |\chi_\Omega(x+z) - \chi_\Omega(x)| |b(\bar{u}^c) - b(\bar{v}^c)| \rho_\varepsilon(x-y-z) - \rho_\varepsilon(x-y) \, d\mu(z) \, dx \, dy. \end{aligned}$$

The terms on the right hand side are of similar form as the ones from (5.29) and (5.25) respectively; we infer that both vanish as $r, \varepsilon \rightarrow 0$.

In conclusion, both integrals on the right-hand side of (5.28) vanish when $r, \varepsilon \rightarrow 0$, and so

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_u^3 = 0.$$

By analogous arguments, we also obtain

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{\epsilon \rightarrow 0} L_v^3 = 0.$$

11) Concluding. Inserting these limits in (5.17), we see

$$(5.31) \quad - \int_Q |u(t, x) - v(t, x)| \alpha'(t) \, dx \, dt \leq 0.$$

This in turn gives for a.e. $\tau \in (0, T)$ the L^1 -contraction

$$\int_\Omega |u(\tau, x) - v(\tau, x)| \, dx \leq \int_\Omega |u_0(x) - v_0(x)| \, dx.$$

Indeed, this standard implication follows by letting $\alpha \rightarrow \chi_{(0, \tau)}$ pointwise while $\|\alpha'\|_{L^1((0, T))} \leq 2$; the initial L^1 -continuity of u and v (Lemma 2.5) and the Lebesgue differentiation theorem then gives the above inequality for a.e. $\tau \in (0, T)$. With this, uniqueness is established. \square

6. EXISTENCE OF ENTROPY SOLUTIONS

In this section we prove the existence of entropy solutions of problem (1.1). The existence proof starts from known results for non-viscous conservation laws [94], to results for viscous equations with zero-order \mathcal{L} obtained from a fixed point argument. Then it proceeds via two successive approximations (approximation by zero-order operators followed by vanishing viscosity) to the most general case where \mathcal{L} is any symmetric Lévy operator whose bilinear form induce compactness in L^2 . This very general class of operators include all fractional Laplacians, logarithmic Laplacians, and practically any (symmetric) positive nonlocal operator seen in applications.

A key requirement for the proof to work is the ability to pass to the limit in approximate problems: Let $(u_n)_{n \in \mathbb{N}}$ be the sequence of entropy solutions of

$$(6.1) \quad \begin{cases} \partial_t u + \operatorname{div}(f(u)) = \mathcal{L}_n[b(u)] & \text{in } Q, \\ u = u^c & \text{on } Q^c, \\ u(0, \cdot) = u_0 & \text{on } \Omega, \end{cases}$$

where $\mathcal{L}_n \rightarrow \mathcal{L}$ in some appropriate sense. Then we expect that $u_n \rightarrow u$, the entropy solution of the limiting problem (1.1). But lacking sufficiently good L^1 -contraction estimates, we are forced to consider nonlinear weak- \star limits (defined below) giving rise to the auxiliary solution concept of an entropy-*process* solution [60, Section 6.9]. We will argue that our uniqueness result extends to entropy-process solutions which, conveniently, will imply that they coincide with standard entropy solutions.

One novel difficulty, is dealing with terms whose dependence on a weakly convergent sequence is both nonlinear *and* nonlocal. More precisely, suppose a sequence $(u_n)_{n \in \mathbb{N}}$ converges to v in nonlinear weak- \star sense. Then each integral in the entropy formulation (2.6) admits a well-defined limit, when replacing u by u_n and letting $n \rightarrow \infty$, except for the one featuring $\text{sgn}^\pm(u-k)\mathcal{L}^{\geq r}[b(u)]$. Even though convergence of this integral can be obtained (from a subsequence), such a limit lacks any obvious connection to v , and this is the real problem. To resolve the issue, we will require *strong* convergence of the sequence $(b(u_n))_{n \in \mathbb{N}}$; such strong compactness morally comes from an energy estimate (cf. Theorem 4.1) if the operator \mathcal{L} is strong/non-degenerate enough. We will show two results in this direction: (i) for \mathcal{L} obtained as limits of bounded truncated operators, and (ii) for general \mathcal{L} inducing L^2 compactness.

The main results of this section are:

- (1) Existence of entropy solutions of (1.1) for zero-order \mathcal{L} in Theorem 6.17.
- (2) Existence for symmetric \mathcal{L} inducing compactness in L^2 in Theorem 6.20.
- (3) L^p -convergence of the vanishing viscosity method in Proposition 6.21.

The plan of the rest of this section is then the following:

- 6.1 Nonlinear weak- \star convergence/compactness and entropy-process solutions.
- 6.2 Stability of solutions $(u_n)_n$ of (6.1) under variations of \mathcal{L}_n .
- 6.3 Compactness of approximate solutions, weak for $(u_n)_n$ and strong in time for $(b(u_n))_n$.
- 6.4 Strong compactness in space for $(b(u_n))_n$.
- 6.5 Existence of solutions: (i) for bounded \mathcal{L} , (ii) via approximation by bounded \mathcal{L} , and (iii) via vanishing viscosity.

6.1. Nonlinear weak- \star convergence and entropy-process solutions. To efficiently handle the convergence of approximate entropy solutions, we introduce some tools following [60, Section 6.9].

Definition 6.1 (Nonlinear weak- \star convergence). Let U be an open subset of \mathbb{R}^{d+1} , $(u_n)_{n \in \mathbb{N}} \subset L^\infty(U)$, and $u \in L^\infty(U \times (0, 1))$. Then $(u_n)_{n \in \mathbb{N}}$ converges towards u in the nonlinear weak- \star $L^\infty(U)$ sense if

$$(6.2) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \int_U \psi(u_n(t, x)) \varphi(t, x) \, dx \, dt \\ &= \int_0^1 \int_U \psi(u(t, x, a)) \varphi(t, x) \, dx \, dt \, da, \quad \text{for all } \varphi \in L^1(U) \text{ and all } \psi \in C(\mathbb{R}). \end{aligned}$$

Remark 6.2.

- (a) When $u_n \rightarrow u$ in the nonlinear weak- \star $L^\infty(U)$ sense, then

$$u_n \rightarrow \int_0^1 u(\cdot, \cdot, a) \, da$$

in the weak- \star $L^\infty(U)$ sense, and the sequence $(u_n)_n$ is bounded in $L^\infty(U)$.

- (b) Any bounded sequence in $L^\infty(U)$ has a subsequence converging in the nonlinear weak- \star $L^\infty(U)$ sense. See Proposition 6.4 in [60] for a proof.
- (c) If a nonlinear weak- \star limit u of $(u_n)_n$ does not depend on a , i.e. $u \in L^\infty(U)$, then $u_n \rightarrow u$ in $L^p_{\text{loc}}(U)$ for every $p \in [1, \infty)$. See Remark 6.16 in [60] for a proof.
- (d) Nonlinear weak- \star convergence is equivalent to Young measure convergence of [53].

A sequence of entropy solutions $(u_n)_{n \in \mathbb{N}}$ that is bounded in L^∞ will therefore converge nonlinear weak- \star to a limit

$$\int_0^1 u(\cdot, \cdot, a) da,$$

and this convergence is sometimes strong enough to characterise in which sense the limit satisfy an entropy formulation. This naturally leads to the concept of entropy-*process* solution, introduced in different settings e.g. in [60, Eq. (5.36), p. 139], [111, Def. 3], and [89, Def. 3.1]. This concept allows for a family of functions $u_a \in L^\infty(M)$, for $a \in (0, 1)$, to be considered a solution of (1.1) if it satisfies an a -averaged version of the entropy inequalities (2.6).

Definition 6.3 (Entropy-process solution). We say that $u = u(t, x, a) \in L^\infty(M \times (0, 1))$ is an *entropy-process solution* of (1.1) if:

- (a) (Averaged entropy inequalities in \overline{Q}) For all $r > 0$, and all $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ satisfying

$$(b(u^c) - b(k))^\pm \varphi = 0 \quad \text{a.e. in } Q^c,$$

we have the inequality

$$\begin{aligned} & - \int_0^1 \int_Q \left((u - k)^\pm \varphi_t + F^\pm(u, k) \cdot \nabla \varphi \right) dx dt da \\ (6.3) \quad & - \int_0^1 \int_M \left(\mathcal{L}^{\geq r} [b(u) - b(k)] \text{sgn}^\pm(u - k) \varphi \chi_\Omega + (b(u) - b(k))^\pm \mathcal{L}^{< r} [\varphi] \right) dx dt da \\ & \leq \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) dx + L_f \int_\Gamma (\overline{u}^c - k)^\pm \varphi d\sigma(x) dt. \end{aligned}$$

- (b) (Data in Q^c) $u(t, x, a) = u^c(t, x)$ for a.e. $(t, x, a) \in (0, T) \times \Omega^c \times (0, 1)$.

- (c) (Non-degenerate points) The expression $b(u(t, x, a))$ does not depend on a .

Remark 6.4. By (c) we see that u must be constant in a whenever $b'(u) > 0$. In particular, if u is (essentially) independent of a for a.e. $(t, x) \in Q$, then $\tilde{u}(t, x) := \int_0^1 u(t, x, a) da$ is the unique entropy solution of (1.1), as can be seen by comparing Definition 2.2 with the above.

Our uniqueness proof for entropy solution can be adapted to entropy-process solutions of (1.1).

Theorem 6.5 (Uniqueness of entropy-process solutions). *Assume (A_Ω) – (A_μ) . Let u, v be entropy-process solutions of (1.1) with $u_0 = v_0$ and $\overline{u}^c = \overline{v}^c$. Then there exists $w \in L^\infty(Q)$ such that*

$$u(t, x, a) = w(t, x) = v(t, x, \tilde{a}) \quad \text{for a.e. } (t, x, a, \tilde{a}) \in Q \times (0, 1)^2.$$

Moreover, w is the unique entropy solution of (1.1).

Proof. Under our assumptions entropy-process solutions satisfy the same a priori estimates as those given for entropy solutions in Section 4. The proofs follow the *exact* same steps, after integrating all expressions over $a \in (0, 1)$ (expressions independent of a are unaffected). Consequently, if we double variables setting $u = u(t, x, a)$ and $v = v(s, y, \tilde{a})$, we can follow the steps of the uniqueness proof of Section 5 (integrating over $a, \tilde{a} \in (0, 1)$) to conclude that

$$(6.4) \quad \int_0^1 \int_0^1 \int_\Omega |u(t, x, a) - v(t, x, \tilde{a})| dx da d\tilde{a} \leq 0,$$

for a.e. $t \in (0, T)$. Thus, u and v must coincide with $w(t, x) := \int_0^1 u(t, x, a) da$ a.e. in $Q \times (0, 1)$. As explained in Remark 6.4, w is then necessarily an entropy solution of (1.1). \square

6.2. Stability of entropy solutions under variations of \mathcal{L} . Assume that $(u_n)_{n \in \mathbb{N}}$ is a sequence of entropy solutions of (6.1). If $\mathcal{L}_n \rightarrow \mathcal{L}$ and $u_n \rightarrow u$, then we expect under certain assumptions that u is the solution of the limiting problem (1.1).

To be precise, let $(\mathcal{L}_n)_n$ and \mathcal{L} be a sequence of operators defined by (1.2) with Lévy measures $(\mu_n)_n$ and μ satisfying (A_μ) and

$$(A_\mu\text{-lim}) \quad \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d|\mu_n - \mu|(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where $|\nu| = \nu^+ + \nu^-$ is the total variation measure of a signed measure ν . By $(A_\mu\text{-lim})$, $(\mu_n)_n$ satisfies (A_μ) uniformly in n and $\mathcal{L}_n \phi \rightarrow \mathcal{L} \phi$ in $L^p(\mathbb{R}^d)$ for $\phi \in C_c^\infty(\mathbb{R}^d)$, $p \in [1, \infty]$:

$$(6.5) \quad M := \sup_n \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d\mu_n(z) < \infty,$$

$$(6.6) \quad \|\mathcal{L} \phi_n\|_p \leq 2M \|\phi\|_{W^{2,p}}, \quad \text{and} \quad \|\mathcal{L}_n \phi - \mathcal{L} \phi\|_p \leq 2\|\phi\|_{W^{2,p}} \int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d|\mu_n - \mu|(z).$$

Boundedness of μ_n is immediate, while the L^p bound follows as in the proof of Lemma 3.4. We will then show that a nonlinear weak- \star limit u of $(u_n)_n$ is an entropy solution of (1.1) provided we have the following strong L^2 convergence of $(b(u_n))_n$:

$$(\text{Str-}L^2) \quad \begin{cases} b(u(t, x, a)) \text{ does not depend on } a, \text{ and for a.e. } a, \\ \|b(u_n) - b(u(\cdot, \cdot, a))\|_{L^2(\mathbb{R}^{d+1})} = \left\| b(u_n) - \int_0^1 b(u(\cdot, \cdot, a)) \, da \right\|_{L^2(\mathbb{R}^{d+1})} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{cases}$$

Proposition 6.6 (Stability of entropy solutions w.r.t. \mathcal{L}). *Assume (A_Ω) - (A_μ) , $(A_\mu\text{-lim})$, $(u_n)_{n \in \mathbb{N}} \subset L^\infty(Q)$ is a sequence of entropy solutions of (6.1), $u \in L^\infty(Q \times (0, 1))$ is its nonlinear weak- \star limit, and $(\text{Str-}L^2)$ holds.*

Then $u(t, x, a)$ is independent of a and $u \in L^\infty(Q)$ is the unique entropy solution of (1.1).

By Remark 6.2 (c) it follows a posteriori that $u_n \rightarrow u$ in $L^p(Q)$ for all $p \in [1, \infty)$.

Proof. If we can show that u is an entropy-process solution of (1.1), we are done since by uniqueness (Theorem 6.5) u will not depend on a and then be an entropy solution of (1.1) (Remark 6.4).

Let us check that u is an entropy-process solution of (1.1). Since u_n is an entropy solution of (6.1), for all $k \in \mathbb{R}$, $r > 0$, and $0 \leq \varphi \in C_c^\infty(M)$ satisfying

$$(6.7) \quad (b(u^c) - b(k))^\pm \varphi = 0, \quad \text{a.e. in } Q^c,$$

the following entropy inequality then holds:

$$(6.8) \quad \begin{aligned} & - \int_Q \left((u_n - k)^\pm \partial_t \varphi + F^\pm(u_n, k) \cdot \nabla \varphi \right) dx \, dt \\ & - \int_Q \text{sgn}^\pm(u_n - k) \mathcal{L}_n^{\geq r} [b(u_n)] \varphi \, dx \, dt - \int_M (b(u_n) - b(k))^\pm \mathcal{L}_n^{\leq r} [\varphi] \, dx \, dt \\ & \leq \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) \, dx + L_f \int_\Gamma (\bar{u}^c - k)^\pm \varphi \, d\sigma(x) \, dt. \end{aligned}$$

We send $n \rightarrow \infty$ in (6.8) to show that u satisfies (6.3). Note that (6.7) and the two last terms in (6.8) are independent of n , and the first term in (6.8) converges to the corresponding term in (6.3) by

nonlinear weak- \star convergence of u_n (6.27) since $(r - k)^\pm$ and $F^\pm(r, k)$ are continuous. The second nonlocal term in (6.8) converges to

$$- \int_0^1 \int_M (b(u) - b(k))^\pm \mathcal{L}^{<r}[\varphi] \, dx \, dt \, da$$

because of strong L^2 convergence of $b(u_n)$ to $\int_0^1 b(u) \, da$ and of $\mathcal{L}_n^{<r}[\varphi]$ to $\mathcal{L}^{<r}[\varphi]$ when $n \rightarrow \infty$. This follows by (Str- L^2), (A_μ -lim), and the bound (6.6) with $p = 2$ (which holds for $\mathcal{L}_n^{<r}$ with $1_{|z| < r} \mu_n$).

We now estimate the first nonlocal term in (6.8). For a moment we think of sgn^\pm as a continuous function. By adding and subtracting terms and applying the triangle inequality,

$$(6.9) \quad \begin{aligned} & \left| \int_M \text{sgn}^\pm(u_n - k) \mathcal{L}_n^{\geq r}[b(u_n)] \varphi \chi_\Omega \, dx \, dt - \int_0^1 \int_M \text{sgn}^\pm(u - k) \mathcal{L}^{\geq r}[b(u)] \varphi \chi_\Omega \, dx \, dt \, da \right| \\ & \leq \int_M |\text{sgn}^\pm(u_n - k)| \times \\ & \quad \times \left(\left| \mathcal{L}_n^{\geq r}[b(u_n)] - \mathcal{L}^{\geq r}[b(u_n)] \right| + \left| \mathcal{L}^{\geq r}[b(u_n)] - \int_0^1 \mathcal{L}^{\geq r}[b(u)] \, da \right| \right) \varphi \chi_\Omega \, dx \, dt \\ & \quad + \left| \int_0^1 \int_M (\text{sgn}^\pm(u_n - k) - \text{sgn}^\pm(u - k)) \mathcal{L}^{\geq r}[b(u)] \varphi \chi_\Omega \, dx \, dt \right| \end{aligned}$$

The last term goes to zero as $n \rightarrow \infty$ by nonlinear weak- \star convergence (6.27) since $\mathcal{L}^{\geq r}[b(u)] \varphi \chi_\Omega \in L^1(M)$. By the Cauchy-Schwarz inequality, and the inequality⁷

$$\|(\mathcal{L}_n^{\geq r} - \mathcal{L}^{\geq r})[\psi]\|_{L^2} \leq 2\|\psi\|_{L^2} \int_{|z| \geq r} d|\mu_n - \mu|(z),$$

the other term is bounded by

$$\begin{aligned} & \left(\left\| (\mathcal{L}_n^{\geq r}[b(u_n)] - \mathcal{L}^{\geq r}[b(u_n)]) \right\|_{L^2(M)} + \left\| \mathcal{L}^{\geq r} \left[b(u_n) - \int_0^1 b(u) \, da \right] \right\|_{L^2(M)} \right) \|\varphi\|_{L^\infty} \|\chi_\Omega\|_{L^2(M)} \\ & \leq 2\|\varphi\|_{L^\infty} \|\chi_\Omega\|_{L^2} \left(\|b(u_n)\|_{L^2} \int_{|z| \geq r} d|\mu_n - \mu|(z) + \left\| b(u_n) - \int_0^1 b(u) \, da \right\|_{L^2} \int_{|z| \geq r} d\mu_\alpha(z) \right), \end{aligned}$$

where $\|\chi_\Omega\|_{L^2(M)} = (T|\Omega|)^{\frac{1}{2}}$. These terms converge to zero by (A_μ -lim) and since $b(u_n)$ converges (and is bounded) in L^2 by (Str- L^2).

Since $\text{sgn}^\pm(\cdot - k)$ is not a continuous function, we need an approximation argument. Note first that if the \pm entropy inequalities (6.8) is satisfied for (φ, k) satisfying (6.7), then it is also satisfied for (φ, ξ) for any $\pm \xi \geq \pm k$. Hence integrating from $\xi = k$ to $\xi = k + \varepsilon$ in the (+) case and $\xi = k - \varepsilon$ to $\xi = k$ in the (-) case and dividing by ε , we find that

$$(6.10) \quad \begin{aligned} & - \int_Q (u_n - k)_\varepsilon^\pm \partial_t \varphi + F_\varepsilon^\pm(u_n, k) \nabla \varphi \, dx \, dt \\ & - \int_Q \text{sgn}_\varepsilon^\pm(u_n - k) \mathcal{L}_n^{\geq r}[b(u_n)] \varphi \, dx \, dt - \int_M (b(u_n) - b(k))_\varepsilon^\pm \mathcal{L}_n^{<r}[\varphi] \, dx \, dt \\ & \leq \int_\Omega (u_0 - k)_\varepsilon^\pm \varphi(0, \cdot) \, dx + L_f \int_\Gamma (\bar{u}^c - k)_\varepsilon^\pm \varphi \, d\sigma(x) \, dt, \end{aligned}$$

⁷This inequality follows from the triangle and Minkowski integral inequalities (cf. the proof of Lemma 2.1 in [58]).

where $g_\varepsilon^+(k) = \frac{1}{\varepsilon} \int_k^{k+\varepsilon} g^+(\xi) d\xi$ and $g_\varepsilon^-(k) = \frac{1}{\varepsilon} \int_{k-\varepsilon}^k g^-(\xi) d\xi$ for $g^\pm(k) = (a-k)^\pm$, $\text{sgn}^\pm(a-k)$, and $F^\pm(a, k)$. Note that these functions are all continuous. Fixing $\varepsilon > 0$, we can now argue as above and send $n \rightarrow \infty$ in (6.10) to get

$$\begin{aligned} & - \int_0^1 \int_Q (u-k)_\varepsilon^\pm \partial_t \varphi + F_\varepsilon^\pm(u, k) \nabla \varphi \, dx \, dt \, da \\ & - \int_0^1 \int_Q \text{sgn}_\varepsilon^\pm(u-k) \mathcal{L}^{\geq r}[b(u)] \varphi \, dx \, dt \, da - \int_0^1 \int_M (b(u) - b(k))_\varepsilon^\pm \mathcal{L}^{< r}[\varphi] \, dx \, dt \, da \\ & \leq \int_\Omega (u_0 - k)_\varepsilon^\pm \varphi(0, \cdot) \, dx + L_f \int_\Gamma (\bar{u}^c - k)_\varepsilon^\pm \varphi \, d\sigma(x) \, dt. \end{aligned}$$

Finally, we send $\varepsilon \rightarrow 0$ to get inequality (6.3), using the dominated convergence theorem and the pointwise convergence of $g_\varepsilon^\pm \rightarrow g^\pm$ for $g^\pm = (a - \cdot)^\pm$, $\text{sgn}^\pm(a - \cdot)$, and $F^\pm(a, \cdot)$. \square

6.3. Compactness for entropy solutions under variations of \mathcal{L} . As in the previous section, we let \mathcal{L}_n be an approximation of \mathcal{L} satisfying $(A_\mu\text{-lim})$ and u_n the corresponding entropy solution of the Dirichlet problem (6.1). First note that since each u_n admits the data (u_0, u^c) Lemma 2.4 implies the uniform bound:

Corollary 6.7. *Assume (A_Ω) – (A_μ) , $(A_\mu\text{-lim})$ and u_n is the entropy solution of (6.1). Then*

$$\text{ess inf}_\Omega u_0 \wedge \inf_{Q^c} u^c \leq u_n(t, x) \leq \text{ess sup}_\Omega u_0 \vee \sup_{Q^c} u^c.$$

From this result and the discussion in Section 6.1 (cf. Remark 6.2), $(u_n)_n$ is compact in the nonlinear weak- \star sense. Now we proceed to prove strong compactness of $(b(u_n))_n$. Let $(\gamma_n)_{n \in \mathbb{N}} \in L^2(\mathbb{R}^{1+d})$ be defined by

$$(6.11) \quad \gamma_n(t, x) := \begin{cases} b(u_n(t, x)) - b(\bar{u}^c(t, x)) & \text{a.e. in } (0, T) \times \mathbb{R}^d, \\ 0, & \text{a.e. in } (0, T)^c \times \mathbb{R}^d. \end{cases}$$

Clearly $(\gamma_n)_n$ converges in $L^2(\mathbb{R}^{1+d})$ if, and only if, $(b(u_n))_n$ converges in $L^2(Q)$. But we will work with the former as its global definition allows for the use of Kolmogorov's compactness theorem: As $(\gamma_n)_n$ is supported on the bounded set Q , the theorem tells us that $(\gamma_n)_n$ is precompact in $L^2(\mathbb{R}^{1+d})$ if, and only if, it is equicontinuous with respect to translations in L^2 .

We now give uniform time-translation estimates for γ_n , conditioned on translation estimates in space.

Proposition 6.8. *Assume (A_Ω) – (A_μ) , $(A_\mu\text{-lim})$, u_n is the entropy solution of (6.1), and γ_n is defined by (6.11). If $(\gamma_n)_n$ is equicontinuous with respect to space-translations*

$$(6.12) \quad \sup_n \|\gamma_n(\cdot, \cdot + h) - \gamma_n(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})} \rightarrow 0 \quad \text{as } h \rightarrow 0,$$

then it is equicontinuous with respect to time-translations

$$(6.13) \quad \sup_n \|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})} \rightarrow 0 \quad \text{as } \tau \rightarrow 0.$$

This estimate follows from the a weak formulation of the equation and corresponding weak estimate for $\partial_t u_n$, combined with a spatial regularisation argument for u_n exploiting the (assumed) regularity of $b(u_n)$. Our regularizations will be defined by mollification. Let $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d} \rho(\frac{x}{\varepsilon})$ where $0 \leq \rho \in C_c^\infty(\mathbb{R}^d)$ is supported in $B_1(0)$ and integrates to one, and define the mollified solutions

$$u_{n,\varepsilon}(t, x) := (u_n(t, \cdot) *_{x} \rho_\varepsilon)(x) = \int_{\mathbb{R}^d} u_n(t, y) \rho_\varepsilon(x - y) \, dy.$$

To upgrade the resulting L^1 estimate of $\partial_t u_{n,\varepsilon}$ to the L^2 -setting of $b(u_n)$ considered in the proposition, we need to control L^2 norms of $b(u_{n,\varepsilon}) - b(u_n)$ in terms of the L^2 translations of $b(u_n)$. We first state and prove this result and then give the proof of Proposition 6.8.

Lemma 6.9. *Assume (A_Ω) – (A_μ) , $(A_\mu\text{-lim})$, $\varepsilon > 0$, and u_n is the entropy solution of (6.1). For a.e. $t \in (0, T)$ and a.e. $x \in \Omega_{-\varepsilon}$,*

$$(6.14) \quad |b(u_{n,\varepsilon}(t, x)) - b(u_n(t, x))|^2 \leq C |b(u_n(t, \cdot)) - b(u_n(t, x))| * \rho_\varepsilon(x),$$

where $C = 2L_b(\|u_0\|_{L^\infty(\Omega)} \vee \|u^c\|_{L^\infty(Q^c)})$.

Proof. We will use Lemma F.1. Fix $(t, x) \in (0, T) \times \Omega_{-\varepsilon}$ and define $w(y) := u_n(t, y) - u_n(t, x)$. Note that $|w| \leq R$ where $R := 2(\|u_0\|_{L^\infty(\Omega)} \vee \|u^c\|_{L^\infty(Q^c)})$ by the L^∞ -bound Corollary 6.7. Define further the probability measure κ on $[-R, R]$ as the w -pushforward measure

$$\kappa(I) := \int_{w^{-1}(I)} \rho_\varepsilon(x - y) \, dy \quad \text{for Borel } I \subseteq [-R, R].$$

For any continuous function $g: [-R, R] \rightarrow \mathbb{R}$, we then get

$$(6.15) \quad \int_{[-R, R]} g(s) \, d\kappa(s) = \int_{\mathbb{R}^d} g(w(y)) \rho_\varepsilon(x - y) \, dy = \int_{\mathbb{R}^d} g(u(t, y) - u(t, x)) \rho_\varepsilon(x - y) \, dy.$$

Letting $h: [-R, R] \rightarrow [0, \infty)$ be defined by

$$h(s) := |b(s + u_n(t, x)) - b(u_n(t, x))|,$$

we infer from (6.15) that

$$\begin{aligned} \bar{s} &:= \int_{[-R, R]} s \, d\kappa(s) = u_\varepsilon(t, x) - u(t, x), \\ \bar{h} &:= \int_{[-R, R]} h(s) \, d\kappa(s) = |b(u_n(t, \cdot)) - b(u_n(t, x))| * \rho_\varepsilon(x). \end{aligned}$$

By Lemma F.1 it follows that $h(\bar{s})^2 \leq L_b R \bar{h}$ which is precisely (6.14). \square

Proof of Proposition 6.8.

1) Lemma A.1 (b) holds for entropy solutions of (6.1), i.e., entropy solutions are very weak solutions in the sense of (A.2).

2) Now we estimate time translations of $u_{n,\varepsilon}$ via the estimate for $\partial_t u_n$ given by the weak formulation. From (A.2) with y instead of x and $\varphi(t, y) := \chi_{(t_1, t_2)}(t) \rho_\varepsilon(x - y)$ for $t_1, t_2 \in (0, T)$, we find that

$$u_{n,\varepsilon}(t_2, x) - u_{n,\varepsilon}(t_1, x) = \int_{t_1}^{t_2} \left(f(u_n) * \nabla \rho_\varepsilon(x) + b(u_n) * \mathcal{L}_n[\rho_\varepsilon(x)] \right) dt \quad \text{for } x \in \Omega_{-\varepsilon}.$$

Since $\Omega_{-\varepsilon} = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$ and $\text{supp } \rho_\varepsilon \subset B_\varepsilon(0)$, the convolutions are well-defined. To make this argument rigorous, we must approximate $\chi_{(t_1, t_2)}$ and then pass to the limit. The result will then hold in Lebesgue points t_1, t_2 of the time slices of u_ε and hence for a.e. $t_1, t_2 \in (0, T)$.⁸ Integrating over $\Omega_{-\varepsilon}$ we get

$$\int_{\Omega_{-\varepsilon}} |u_{n,\varepsilon}(t_2, x) - u_{n,\varepsilon}(t_1, x)| \, dx \leq |t_2 - t_1| |\Omega| \left(\|f(u_n)\|_{L^\infty(Q)} \|\nabla \rho_\varepsilon\|_{L^1(\mathbb{R}^d)} + \|b(u_n)\|_{L^\infty(M)} \|\mathcal{L}[\rho_\varepsilon]\|_{L^1(\mathbb{R}^d)} \right).$$

⁸Cf. e.g. [1, Lemma B.1] for more details.

Since by standard estimates, $\|\nabla\rho_\varepsilon\|_{L^1(\mathbb{R}^d)} = \varepsilon^{-1}\|\nabla\rho\|_{L^1(\mathbb{R}^d)}$ and

$$\|\mathcal{L}[\rho_\varepsilon]\|_{L^1(\mathbb{R}^d)} \leq \frac{1}{2}\|D^2\rho\|_{L^1(\mathbb{R}^d)}\frac{1}{\varepsilon^2}\int_{|z|<1}|z|^2 d\mu_n(z) + 2\|\rho\|_{L^1(\mathbb{R}^d)}\int_{|z|\geq 1}d\mu(z),$$

this means that for a.e. $t, t + \tau \in (0, T)$,

$$(6.16) \quad \sup_{n \in \mathbb{N}} \int_{\Omega_{-\varepsilon}} |u_{n,\varepsilon}(t + \tau, x) - u_{n,\varepsilon}(t, x)| dx \leq C|\tau|(1 + \varepsilon^{-1} + \varepsilon^{-2}),$$

where $C := \sup_{n \in \mathbb{N}} 2|\Omega| \int_{\mathbb{R}^d} (|z|^2 \wedge 1) d\mu(z) \|\rho\|_{W^{2,1}} (\|f(u_n)\|_{L^\infty} + \|b(u_n)\|_{L^\infty} + \|b(u^c)\|_{L^\infty}) < \infty$ by assumptions and Corollary 6.7.

3) We prove the L^2 time translation result for γ_n . Assume $0 < \tau < T$. We split the domain,

$$\begin{aligned} \|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})} &= \|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2((0, T-\tau) \times (\Omega \setminus \Omega_{-\varepsilon}))} \\ &\quad + \|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2((0, T-\tau) \times \Omega_{-\varepsilon})} + \|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2((T-\tau, T) \times \Omega)} + 0, \end{aligned}$$

and estimate the different terms. For the two ‘boundary’ terms, we find that

$$\begin{aligned} \|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2((0, T-\tau) \times (\Omega \setminus \Omega_{-\varepsilon}))} &\leq (2T|\Omega \setminus \Omega_{-\varepsilon}|)^{\frac{1}{2}} (\|b(u_n)\|_{L^\infty(Q)} + \|b(\bar{u}^c)\|_{L^\infty(Q)}), \\ \|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2((T-\tau, T) \times \Omega)} &\leq (2|\tau||\Omega|)^{\frac{1}{2}} (\|b(u_n)\|_{L^\infty(Q)} + \|b(\bar{u}^c)\|_{L^\infty(Q)}), \end{aligned}$$

where $|\Omega \setminus \Omega_{-\varepsilon}| \leq C_{\partial\Omega}\varepsilon$ by (A $_{\Omega}$). To estimate the ‘interior’ term, we add and subtract $b(u_{n,\varepsilon})$ -terms, use Lipschitz regularity of b and \bar{u}^c , the uniform L^∞ -bound on u_n , and time regularity of $u_{n,\varepsilon}$ (6.16):

$$\begin{aligned} &\|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2((0, T-\tau) \times \Omega_{-\varepsilon})} \\ &\leq \|b(u_n(\cdot + \tau, \cdot)) - b(u_{n,\varepsilon}(\cdot + \tau, \cdot))\|_{L^2((0, T-\tau) \times \Omega_{-\varepsilon})} + \|b(u_{n,\varepsilon}(\cdot + \tau, \cdot)) - b(u_{n,\varepsilon}(\cdot, \cdot))\|_{L^2((0, T-\tau) \times \Omega_{-\varepsilon})} \\ &\quad + \|b(u_{n,\varepsilon}(\cdot, \cdot)) - b(u_n(\cdot, \cdot))\|_{L^2((0, T-\tau) \times \Omega_{-\varepsilon})} + \|b(\bar{u}^c(\cdot + \tau, \cdot)) - b(\bar{u}^c(\cdot, \cdot))\|_{L^2((0, T-\tau) \times \Omega_{-\varepsilon})} \\ &\leq 2\|b(u_{n,\varepsilon}(\cdot, \cdot)) - b(u_n(\cdot, \cdot))\|_{L^2((0, T) \times \Omega_{-\varepsilon})} + L_b \|u_n\|_{L^\infty(Q)}^{\frac{1}{2}} \|u_{n,\varepsilon}(\cdot + \tau, \cdot) - u_{n,\varepsilon}(\cdot, \cdot)\|_{L^1((0, T-\tau) \times \Omega_{-\varepsilon})} \\ &\quad + L_b \|\partial_t \bar{u}^c\|_{L^\infty(Q)} \tau |Q|^{\frac{1}{2}} \\ &\leq 2I + C(\tau^{\frac{1}{2}}(1 + \varepsilon^{-1}) + \tau). \end{aligned}$$

Here the constant C is independent of n, ε, τ . From Lemma 6.9, a change of variables and Fubini, Cauchy-Schwarz, the definition of γ_n , the space translation estimate (6.12) on γ_n , and the regularity of \bar{u}^c , we find that

$$\begin{aligned} I^2 &\leq C \int_0^T \int_{\Omega_{-\varepsilon}} \int_{\Omega} |b(u_n(t, y)) - b(u_n(t, x))| \rho_\varepsilon(x - y) dy dx dt \\ &\leq C \sup_{|y| \leq \varepsilon} \|b(u_n(\cdot, \cdot + y)) - b(u_n(\cdot, \cdot))\|_{L^1((0, T) \times \Omega_{-\varepsilon})} \\ &\leq C(T|\Omega|)^{\frac{1}{2}} \left(\sup_{|y| \leq \varepsilon} \|\gamma_n(\cdot, \cdot + y) - \gamma_n(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})} + L_b \sup_{|y| \leq \varepsilon} \|\bar{u}^c(\cdot, \cdot + y) - \bar{u}^c(\cdot, \cdot)\|_{L^2((0, T) \times \Omega_{-\varepsilon})} \right) \\ &\leq o_\tau(1) + o_\varepsilon(1), \end{aligned}$$

where $o_\tau(1)$ and $o_\varepsilon(1)$ are independent of n, ε and n, τ respectively. Combining all estimates, we get

$$\sup_n \|\gamma_n(\cdot + \tau, \cdot) - \gamma_n(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})} \leq o_\tau(1) + o_\varepsilon(1) + C(|\tau|^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} + |\tau| + |\tau|^{\frac{1}{2}}(1 + \varepsilon^{-1})).$$

The same bound holds for $-T < \tau < 0$, so (6.13) follows after taking (e.g.) $\varepsilon = |\tau|^{1/3}$. \square

6.4. Compactness in space and condition (Str- L^2). Next we will prove space translation estimates (6.12) for γ_n . Since we do not have an L^1 -contraction result here, we will use the energy estimate Proposition 4.1 for this purpose.

Lemma 6.10. *Assume (A_Ω) – (A_μ) , $(A_\mu\text{-lim})$, u_n is the entropy solution of (6.1), and γ_n is defined by (6.11). Then the sequence $(\gamma_n)_{n \in \mathbb{N}}$ has uniformly bounded energy, that is*

$$(6.17) \quad \sup_n \int_0^T \int_{\mathbb{R}^d} \mathcal{B}_n[\gamma_n, \gamma_n] dx dt < \infty,$$

where \mathcal{B}_n is the bilinear operator corresponding to \mathcal{L}_n , cf. (2.1).

Proof. Recall that $\gamma_n(t, x) = b(u_n(t, x)) - b(\bar{u}^c(t, x))$ in $M = (0, T) \times \mathbb{R}^d$. By the above assumptions and Proposition 4.1, the bound (4.1) holds for the solution u_n of (6.1), with a right-hand side that is bounded uniformly in n . By the assumptions this is immediate for the two first terms, and now we check the third term:

$$\left| \int_Q \mathcal{L}_n[b(\bar{u}^c)](b(u_n) - b(\bar{u}_n^c)) dx dt \right| \leq \|\mathcal{L}_n[b(\bar{u}^c)]\|_{L^1(Q)} (\|b(u_n)\|_{L^\infty} + \|b(\bar{u}^c)\|_{L^\infty}).$$

The first factor is bounded in n by Corollary 3.7 and the second by Corollary 6.7. \square

It follows that γ_n is uniformly bounded in the Sobolev space $L^2(0, T; H_0^{\mathcal{L}}(\Omega)) \subset L^2(Q)$ where the norm of $H_0^{\mathcal{L}}$ is defined in (2.4). We now give a sufficient and necessary condition for $H_0^{\mathcal{L}}$ to be a compact subset of L^2 . We need the symbol/Fourier multiplier $-m(\xi)$ of \mathcal{L} , where the *positive* function $m: \mathbb{R}^d \rightarrow \mathbb{R}$ is defined as

$$(6.18) \quad m(\xi) = \int_{\mathbb{R}^d} [1 - \cos(\xi \cdot z)] d\mu(z).$$

Proposition 6.11. *Assume (A_μ) , let $H_0^{\mathcal{L}}(\Omega)$ be defined by (2.4), and let m denote the symbol of $-\mathcal{L}$. Then the embedding $H_0^{\mathcal{L}}(\Omega) \hookrightarrow L^2(\Omega)$ is compact if, and only if,*

$$(6.19) \quad \liminf_{|\xi| \rightarrow \infty} m(\xi) = \infty.$$

This could very well be a classical result, but we have not found it in the literature. A proof and discussion is given in Appendix E. The L^2 space translation estimate (6.12) is essentially a consequence of this result, at least it follows by modifying the proof to our parabolic setting.

Lemma 6.12. *Assume (A_Ω) – (A_μ) , $(A_\mu\text{-lim})$, u_n is the entropy solution of (6.1), γ_n is defined by (6.11), and m and m_n are the symbols of $-\mathcal{L}$ defined in (6.18) and $-\mathcal{L}_n$ defined in (6.1). Then the sequence $(\gamma_n)_{n \in \mathbb{N}}$ satisfies the L^2 space translation estimate (6.12) if*

$$(6.20) \quad \liminf_{|\xi| \rightarrow \infty} \inf_n m_n(\xi) = \infty.$$

Proof. Suppose first that m satisfies (6.19), and define $\rho: [0, \infty) \rightarrow [0, \infty)$ by

$$\rho(R) := \inf_{|\xi| \geq R} \inf_n m_n(\xi),$$

which must tend to infinity as $R \rightarrow \infty$. For any $n \in \mathbb{N}$, $y \in \mathbb{R}^d$, and (large) $R > 0$, we have by Plancherel's theorem in the space variables,

$$\begin{aligned} 2\pi \|\gamma_n - \gamma_n(\cdot, \cdot + y)\|_{L^2(\mathbb{R}^{d+1})}^2 &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} |1 - e^{i\xi \cdot y}|^2 |\hat{\gamma}_n(t, \xi)|^2 d\xi dt \\ &\leq (R|y|)^2 \int_{\mathbb{R}} \int_{|\xi| < R} |\hat{\gamma}_n(t, \xi)|^2 d\xi dt + \frac{4}{\rho(R)} \int_{\mathbb{R}} \int_{|\xi| \geq R} m_n(\xi) |\hat{\gamma}_n(t, \xi)|^2 d\xi dt \\ &\leq \left[(R|y|)^2 + \frac{4}{\rho(R)} \right] \int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 + m_n(\xi)) |\hat{\gamma}_n(t, \xi)|^2 d\xi dt. \end{aligned}$$

Here the last integral equals $\|\gamma_n\|_{L^2(\mathbb{R}; H_0^{\mathcal{L}^n}(\Omega))}$. Defining $C = \sup_n \|\gamma_n\|_{L^2(\mathbb{R}; H_0^{\mathcal{L}^n}(\Omega))}$, which by Lemma 6.10 is bounded, the previous calculation shows that we for any $y \in \mathbb{R}^d$ have

$$\sup_n \|\gamma_n - \gamma_n(\cdot + y)\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C}{2\pi} \inf_{R>0} \left[(R|y|)^2 + \frac{4}{\rho(R)} \right].$$

As the right-hand side can be bounded by for example setting $R = |y|^{-\frac{1}{2}}$ (making the right-hand side vanish as $|y| \rightarrow 0$), it follows that the family $(\gamma_n)_{j \in \mathbb{N}}$ is equicontinuous with respect to translation. \square

We also need space translation estimates from uniform energy estimates for a family of truncated operators. Since these operators do not satisfy (6.20), we need stronger assumptions on μ and \mathcal{L} . (See Example E.1 for a Lévy operator that satisfies (6.19), but whose truncations fail the conclusion of Lemma 6.13.)

Lemma 6.13. *Assume (A_μ) , μ is absolutely continuous with $\frac{d\mu}{dz} \geq \frac{1}{|z|^{d\nu}(|z|)}$ for a non-decreasing $\nu \geq 0$ with $\nu(0^+) = 0$, and let \mathcal{L} and $\mathcal{B}^{>(1/n)}$ be given by (1.2) and (2.2). If $(\phi_n)_{n \in \mathbb{N}} \subset L^2(\mathbb{R}^d)$ satisfy*

$$\sup_n \int_{\mathbb{R}^d} \mathcal{B}^{>(1/n)}[\phi_n, \phi_n] dx =: C < \infty,$$

then $(\phi_n)_{n \in \mathbb{N}}$ is equicontinuous with respect to L^2 -translations:

$$\lim_{|y| \rightarrow 0} \sup_n \|\phi_n(\cdot + y) - \phi_n\|_{L^2(\mathbb{R}^d)} = 0.$$

Remark 6.14.

- (a) By setting $\nu(r) = \infty$ for $r > 1$, we see that the condition on $\frac{d\mu}{dz}$ is a local condition near the singularity of μ at $z = 0$.
- (b) The $\mathcal{B}^{>(1/n)}$ -condition is a uniform in n bound on the bilinear form of truncated operator $\mathcal{L}^{>\frac{1}{n}}$. When $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$, it is an approximate $H^{\frac{\alpha}{2}}$ -estimate in space.

Lemma 6.15. *Under the assumptions of Lemma 6.13, and in addition, (A_Ω) - (A_{u_0}) , $(A_\mu\text{-lim})$, u_n is the entropy solution of (6.1), and γ_n is defined by (6.11). If $(\gamma_n)_{n \in \mathbb{N}} \subset L^2(Q)$ satisfy*

$$\sup_n \int_0^T \int_{\mathbb{R}^d} \mathcal{B}^{>(1/n)}[\gamma_n(t, \cdot), \gamma_n(t, \cdot)] dx dt < \infty,$$

then $(\gamma_n)_{n \in \mathbb{N}}$ is equicontinuous with respect to L^2 -translations:

$$\lim_{|y| \rightarrow 0} \sup_n \int_0^T \|\gamma_n(t, \cdot + y) - \gamma_n(t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 dt = 0.$$

Proof. The proof is the same as for Lemma 6.13 after redefining $\tau_n(y)$ in the obvious way. \square

Proof of Lemma 6.13. For brevity define for each $n \in \mathbb{N}$ the function $\tau_n: \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\tau_n(y) := \|\phi_n(\cdot + y) - \phi_n\|_{L^2(\mathbb{R}^d)},$$

and observe that τ_n is sub-additive by the triangle inequality and a change of variables. Therefore, with $|B^d|$ denoting the volume of the unit ball in \mathbb{R}^d , for any $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$ we have

$$|B^d| \frac{|y|^d}{4^d} \tau_n(y)^2 = \int_{|z| < \frac{|y|}{4}} \tau_n(y)^2 dz \leq 2 \int_{|z| < \frac{|y|}{4}} (\tau_n(\frac{y}{2} + z)^2 + \tau_n(\frac{y}{2} - z)^2) dz = 4 \int_{|z| < \frac{|y|}{4}} \tau_n(\frac{y}{2} - z)^2 dz.$$

By a change of variables,

$$\int_{|z| < \frac{|y|}{4}} \tau_n(\frac{y}{2} - z)^2 dz \leq \int_{\frac{|y|}{4} < |z| < \frac{3|y|}{4}} \tau_n(z)^2 dz \leq \left(\frac{3|y|}{4}\right)^d \nu\left(\frac{3|y|}{4}\right) \int_{\frac{|y|}{4} < |z| < \frac{3|y|}{4}} \frac{\tau_n(z)^2}{|z|^d \nu(|z|)} dz.$$

If $|y| > \frac{4}{n}$, then by Fubini-Tonelli, the lower bound on $\frac{d\mu}{dz}$, and the definition and bound on $\mathcal{B}^{>(1/n)}$,

$$\int_{\frac{|y|}{4} < |z| < \frac{3|y|}{4}} \frac{\tau_n(z)^2}{|z|^d \nu(|z|)} dz \leq \int_{\mathbb{R}^d} \mathcal{B}^{>(1/n)}[\phi_n, \phi_n] dx \leq C,$$

and we conclude from the above estimates that

$$\tau_n(y) \leq c(d, C) \nu\left(\frac{3|y|}{4}\right)^{\frac{1}{2}}.$$

If instead $|y| \leq \frac{4}{n}$, we pick an $x \in \mathbb{R}^d$ such that $|x| = \frac{8}{n}$, and use the sub-additivity of τ_n and twice the estimate for $|y| \geq \frac{4}{n}$ to see that

$$\tau_n(y) \leq \tau_n(y - x) + \tau(x) \leq c(d, C) \left(\nu\left(\frac{3|y - x|}{4}\right)^{\frac{1}{2}} + \nu\left(\frac{3|x|}{4}\right)^{\frac{1}{2}} \right) \leq 2c(d, C) \nu\left(\frac{9}{n}\right)^{\frac{1}{2}}.$$

Hence we conclude that for all $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$,

$$(6.21) \quad \tau_n(y) \leq 2c(d, C) \nu\left(\frac{9}{n} \vee \frac{3|y|}{4}\right)^{\frac{1}{2}}.$$

Finally, as $\lim_{|y| \rightarrow 0} \tau_n(y) = 0$ for all $n \in \mathbb{N}$ (by continuity of translations in L^2), we have for any fixed $m \in \mathbb{N}$,

$$\limsup_{|y| \rightarrow 0} \sup_{n \leq m} \tau_n(y) = 0.$$

By (6.21) we then find that

$$\limsup_{|y| \rightarrow 0} \sup_n \tau_n(y) = \left(\limsup_{|y| \rightarrow 0} \sup_{n \geq m} \tau_n(y) \right) \vee \left(\limsup_{|y| \rightarrow 0} \sup_{n \leq m} \tau_n(y) \right) \leq 2c(d, C) \nu\left(\frac{9}{m}\right)^{\frac{1}{2}},$$

and the lemma follows since m is arbitrary and $\lim_{r \rightarrow 0^+} \nu(r) = 0$. \square

6.5. Existence of entropy solutions. Note that by Lemma 2.4, the solutions we seek will take values in the interval

$$\left[\operatorname{ess\,inf}_{\Omega} u_0 \wedge \operatorname{ess\,inf}_{Q^c} u^c, \operatorname{ess\,sup}_{\Omega} u_0 \vee \operatorname{ess\,sup}_{Q^c} u^c \right].$$

Modifications of f, b outside this interval will have no effect on the solutions of the problem. Without loss of generality, we may therefore assume f and b to be globally Lipschitz continuous and bounded. With this in mind we start the existence proof from the following local problem:

$$(6.22) \quad \begin{cases} u_t + \operatorname{div}(f(u)) = g, & \text{in } Q, \\ u = \bar{u}^c & \text{on } \Gamma, \\ u(0, \cdot) = u_0 & \text{on } \Omega. \end{cases}$$

Lemma 6.16 ([94]). *Assume (A_Ω) , (A_f) , (A_{u^c}) , (A_{u_0}) , $g \in L^1(Q)$, and f globally Lipschitz and bounded. Then*

- (a) *There exists a unique renormalised entropy solution $u \in C(0, T; L^1(\Omega))$ of (6.22).*
 (b) *Two renormalised entropy solutions $u, \tilde{u} \in L^1(Q)$ of (6.22) with source terms $g, \tilde{g} \in L^1(Q)$, satisfy*

$$(6.23) \quad \int_{\Omega} |u - \tilde{u}|(t) \, dx \leq \int_0^t \int_{\Omega} |g - \tilde{g}| \, dx \, dt \quad \text{for } t \in (0, T).$$

- (c) *If in addition $g \in L^\infty(Q)$, then $u \in L^\infty(Q)$, and u is an entropy solution of (6.22): For all $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$,*

$$(6.24) \quad \begin{aligned} & - \int_Q (u - k)^\pm \varphi_t + F^\pm(u, k) \cdot \nabla \varphi + \operatorname{sgn}^\pm(u - k) g \varphi \, dx \, dt \\ & \leq \int_{\Omega} (u_0 - k)^\pm \varphi(0, \cdot) \, dx + L_f \int_{\Gamma} (\bar{u}^c - k)^\pm \varphi \, d\sigma(x) \, dt. \end{aligned}$$

Proof. Since Ω, u, u_0, f and g satisfy the assumptions of [94], existence and uniqueness of renormalised entropy solutions of (6.22) follow from [94, Theorem 2.3] and the L^1 -contraction from [94, Theorem 3.1]. Under our assumptions (data in L^∞) the renormalized entropy solutions of [94] are entropy solutions and hence satisfy (6.24), see [94, Remark 2.2]. The solution u is bounded by (adapting the proof of) Lemma 2.4. \square

From this result and a fixed point argument, we now prove existence for (1.1) when μ and \mathcal{L} are bounded.

Theorem 6.17 (Existence for zero-order \mathcal{L}). *Assume (A_Ω) – (A_{u_0}) , f, b globally Lipschitz and bounded, and $\mu \geq 0$ is a symmetric Radon measure such that $\mu(\mathbb{R}^d) < \infty$. Then there exists an entropy solution u of (1.1).*

Proof. For $v \in L^1(Q)$ we define a map I by $I[v] = u$, the renormalised entropy solution of (6.22) with $g = \mathcal{L}[b(\bar{v})]$ where the extension \bar{v} is given by $\bar{v}|_Q = v$ and $\bar{v}|_{Q^c} = u^c$. Since $\bar{v} \in L^1_{\text{loc}}$ and b and μ are bounded, the Q -restriction of $g \in L^\infty(Q) \subset L^1(Q)$. Since $C(0, T; L^1(\Omega)) \subset L^1(Q)$, it follows by Lemma 6.16 that $I: C(0, T; L^1(\Omega)) \rightarrow C(0, T; L^1(\Omega))$ is well-defined, injective, and

$$(6.25) \quad \|(I[v] - I[w])(t)\|_{L^1(\Omega)} \leq \int_0^t \|\mathcal{L}[b(\bar{v}) - b(\bar{w})](s)\|_{L^1(\Omega)} \, ds \leq 2L_b \|\mu\| \int_0^t \|(v - w)(s)\|_{L^1(\Omega)} \, ds,$$

for $t \in (0, T)$ where L_b is the Lipschitz constant of b and $\|\mu\|$ the total mass of μ .

Now we define a fixed point iteration by $u_{k+1} = I[u_k]$ for $k \in \mathbb{N}_0$ and $u_0 = 0$. From (6.25) we get

$$\begin{aligned} \|(u_{k+1} - u_k)(t)\|_{L^1(\Omega)} & \leq 2L_b \|\mu\| \int_0^t \|(u_k - u_{k-1})(s)\|_{L^1(\Omega)} \, ds \\ & \leq (2L_b \|\mu\|)^k \int_0^T \int_0^{s_1} \cdots \int_0^{s_k} \|u_1(s_{k+1})\|_{L^1(\Omega)} \, ds_{k+1} \cdots ds_1 \\ & \leq \frac{(2L_b \|\mu\| T)^k}{k!} \max_{t \in [0, T]} \|u_1(t)\|_{L^1(\Omega)}, \end{aligned}$$

where the last term is finite since $u_1 = I[0] \in C(0, T; L^1(\Omega))$ by Lemma 6.16. Thus $(u_k)_{k \in \mathbb{N}}$ is Cauchy and converges to some u in $C(0, T; L^1(\Omega))$. Taking the limit as $k \rightarrow \infty$ in $u_{k+1} = I(u_k)$, we conclude that $u = I(u)$, i.e. u is a renormalized entropy solution of (6.22) with source term $g = -\mathcal{L}[b(\bar{u})]$ where \bar{u} is the extension of u to M by $u|_{Q^c} = u^c$. Since u_0 and g are bounded (b is bounded and

$\mathcal{L} : L^\infty \rightarrow L^\infty$, u is bounded and an entropy solution of (6.22) as defined in (6.24): For all $k \in \mathbb{R}$ and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$

$$(6.26) \quad \begin{aligned} & \int_Q -(u-k)^\pm \varphi_t - F^\pm(u, k) \cdot \nabla \varphi - \operatorname{sgn}^\pm(u-k) \mathcal{L}[b(\bar{u})] \varphi \, dx \, dt \\ & \leq \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) \, dx + L_f \int_\Gamma (\bar{u}^c - k)^\pm \varphi \, d\sigma(x) \, dt. \end{aligned}$$

It remains to show that u is in fact an entropy solution of (1.1) according to Definition 2.2; we need only show that inequality (6.26) implies (2.6) for all $r > 0$, since u is bounded and $u|_{Q^c} = u^c$ (this is implicit from the use of \bar{u} in the g -term). Take $r > 0$, $k \in \mathbb{R}$, and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ satisfying

$$(b(u^c) - b(k))^\pm \varphi = 0 \quad \text{in} \quad Q^c.$$

Since $\mu(\mathbb{R}^d) < \infty$, it immediately follows that for $(t, x) \in Q^c$,

$$\mathcal{L}^{<r}[(b(u) - b(k))^\pm] \varphi = \int_{|z| < r} \left((b(u(\cdot, \cdot + z)) - b(k))^\pm - (b(u) - b(k))^\pm \right) \varphi \, d\mu(z) \geq 0,$$

and then a convex inequality (cf. Corollary C.2) and symmetry of $\mathcal{L}^{<r}$ yields

$$\begin{aligned} & \int_Q \operatorname{sgn}^\pm(u-k) \mathcal{L}^{<r}[b(u)] \varphi \, dx \, dt \leq \int_Q \mathcal{L}^{<r}[(b(u) - b(k))^\pm] \varphi \, dx \, dt \\ & \leq \int_M \mathcal{L}^{<r}[(b(u) - b(k))^\pm] \varphi \, dx \, dt = \int_M ((b(u) - b(k))^\pm) \mathcal{L}^{<r}[\varphi] \, dx \, dt. \end{aligned}$$

Hence, since $\mathcal{L} = \mathcal{L}^{<r} + \mathcal{L}^{\geq r}$, we conclude that

$$\int_Q \operatorname{sgn}^\pm(u-k) \mathcal{L}[b(u)] \varphi \, dx \, dt \leq \int_M (b(u) - b(k))^\pm \mathcal{L}^{<r}[\varphi] \, dx \, dt + \int_Q \operatorname{sgn}^\pm(u-k) \mathcal{L}^{\geq r}[b(u)] \varphi \, dx \, dt.$$

Inserting this in (6.26), we get (2.6). \square

We now show existence of entropy solutions of (1.1) when $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$. In this case \mathcal{L} is an unbounded operator with an unbounded absolutely continuous Lévy measure with density

$$\frac{d\mu_\alpha}{dz}(z) = \frac{C_{d,\alpha}}{|z|^{d+\alpha}}.$$

We approximate \mathcal{L} by the sequence of bounded Lévy operators $\mathcal{L}^{\geq \frac{1}{n}}$, $n \in \mathbb{N}$ (and hence μ_α by $\mu_n := \mu_\alpha \chi_{\{|z| \geq \frac{1}{n}\}}$), and the solution u of (1.1) by solutions u_n of (6.1). Note that μ_n satisfies $(A_\mu\text{-lim})$ by the dominated convergence theorem:

$$\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \, d|\mu_n(z) - \mu_\alpha(z)| = \int_{|z| < \frac{1}{n}} (|z|^2 \wedge 1) \, d\mu_\alpha(z) \rightarrow 0.$$

We now prove existence as a consequence of existence for bounded \mathcal{L} and the compactness and stability results of Sections 6.2–6.4.

Proposition 6.18 (Existence for $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$). *Assume (A_Ω) – (A_{u_0}) , and $\mathcal{L} = -(-\Delta)^{\frac{\alpha}{2}}$. Then there exists an entropy solution u of (1.1).*

Remark 6.19. From the proof below, it follows that this result holds for a large class of Lévy operators \mathcal{L} . The only requirement is (A_μ) and the lower bound on $\frac{d\mu}{dz}$ in Lemma 6.15, see also Remark 6.14 (a). These assumptions are satisfied for most symmetric Lévy operators appearing in applications.

Proof of Proposition 6.18.

1) *Existence for (6.1).* Since \mathcal{L}_n and μ_n are bounded, there is a unique entropy solution u_n of (6.1) by Theorem 6.17.

2) *Nonlinear weak- \star compactness and limit of $(u_n)_n$.* Since $(A_\mu\text{-lim})$ holds, u_n is uniformly bounded in $L^\infty(Q)$ by Corollary 6.7. Then since u_n coincides with $u^c \in L^\infty$ in Q^c , by Remark 6.2 (b) there is a function $u \in L^\infty(M \times (0, 1))$ and a subsequence (also called) $(u_n)_n$ such that

$$(6.27) \quad u_n \rightarrow u \quad \text{in the nonlinear weak-}\star L^\infty(\mathbb{R}^{d+1}) \text{ sense as } n \rightarrow \infty,$$

cf. Definition 6.1.

3) *Translation estimates for γ_n in $L^2(\mathbb{R}^{d+1})$.* Let γ_n be defined in (6.11). Since $(A_\mu\text{-lim})$ holds, the assumptions of Lemma 6.10 are satisfied, and hence the energy estimates (6.17) holds. Moreover, the the assumptions of Lemma 6.15 on μ are satisfied with $\nu(z) = |z|^\alpha$. Hence by Lemma 6.15, the L^2 space translation estimate (6.12) holds. We can then conclude from Proposition 6.8 that the L^2 -translation estimates in time (6.13) also holds. Hence $\sup_n \|\gamma_n(\cdot + \tau, \cdot + h) - \gamma_n(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})} = o_h(1) + o_\tau(1)$.

4) *Precompactness of γ_n in $L^2(\mathbb{R}^{d+1})$.* L^2 -translations of $(\gamma_n)_n$ are equicontinuous by step 3), and since γ_n is uniformly bounded by step 2) and 0 outside the compact set Q , the sequence $(\gamma_n)_n$ is tight in L^p for $p \in [1, \infty)$. Hence we can use the Fréchet-Kolmogorov-Riesz compactness theorem in $L^2(\mathbb{R}^{d+1})$ to conclude that there exists a subsequence (still denoted by) $(\gamma_n)_{n \in \mathbb{N}}$ and a limit $\gamma_* \in L^2(\mathbb{R}^{d+1})$ such that

$$\gamma_n \rightarrow \gamma_* \quad \text{in } L^2(\mathbb{R}^{d+1}) \text{ as } n \rightarrow \infty.$$

5) *Strong L^2 convergence of $b(u_n)$ and assumption $(\text{Str-}L^2)$.* Since $u_n \rightarrow u$ in the nonlinear weak- $\star L^\infty(\mathbb{R}^{d+1})$ sense, $b(u_n) \rightarrow b(u)$ in the same sense (replace ψ by $\psi \circ b$ in the definition), and hence

$$\gamma_n \rightarrow [b(u) - b(\bar{u}^c)] \mathbf{1}_{t \in (0, T)} \quad \text{in the nonlinear weak-}\star L^\infty(\mathbb{R}^{d+1}) \text{ sense as } n \rightarrow \infty.$$

By step 4) and uniqueness of limits (Lemma 4.1 in [61]), it then follows that

$$\gamma_*(t, x) = [b(u(t, x, a)) - b(\bar{u}^c(t, x))] \mathbf{1}_{t \in (0, T)} \quad \text{for a.e. } (t, x, a) \in \mathbb{R}^{d+1} \times (0, 1).$$

Then $b(u(t, x, a))$ cannot depend on a , and since $b(u_n) = b(\bar{u}^c)$ in Ω^c , $b(u_n) \rightarrow b(u(\cdot, \cdot, a)) = \int_0^1 b(u(\cdot, \cdot, a)) da$ in $L^2(\mathbb{R}^{d+1})$ as $n \rightarrow \infty$ for a.e. $a \in (0, 1)$. We have proved that $(\text{Str-}L^2)$ holds.

6) *Stability and identification of limit as entropy solution.* Since both $(A_\mu\text{-lim})$ and $(\text{Str-}L^2)$ hold, we can conclude from the stability result Proposition 6.6 that the nonlinear weak- \star limit u of $(u_n)_n$ (i) do not depend on a and belong to $L^\infty(Q)$, and (ii) is the unique entropy solution of (1.1). \square

Finally we show existence of entropy solutions of (1.1) for operators inducing L^2 compactness, i.e. that satisfy the symbol condition (6.19), see Section 6.4. To do that we will approximate \mathcal{L} by

$$\mathcal{L}_n = \mathcal{L} - \frac{1}{n}(-\Delta)^{\frac{\alpha}{2}},$$

and problem (1.1) by problem (6.1). In this case $\mu_n := \mu + \frac{1}{n}\mu_\alpha$, and $(A_\mu\text{-lim})$ trivially holds. Moreover, denoting $-m_n$ for the symbol of \mathcal{L}_n (cf. (6.18)) we get

$$(6.28) \quad m_n(\xi) = m(\xi) + \frac{1}{n}m_\alpha(\xi) \quad \text{where} \quad m_\alpha(\xi) = c|\xi|^\alpha,$$

and where m is the symbol of $-\mathcal{L}$. Thus, $\inf_n m_n(\xi) = m(\xi)$ and so the family $(m_n)_{n \in \mathbb{N}}$ satisfies the L^2 -compactness condition (6.20) as $\liminf_{\xi \rightarrow \infty} m(\xi) = \infty$. From Proposition 6.18 and Remark 6.19, there is a unique entropy solution u_n of (6.1) with the current definition of \mathcal{L}_n . At this point we can argue exactly as in the proof of Proposition 6.18, only replacing the L^2 -translation result Lemma 6.15

by Lemma 6.12. The conclusion is the existence of $u = \lim_n u_n$ in the nonlinear weak- \star sense, such that $u \in L^\infty(Q)$ is the entropy solution of (1.1).

Theorem 6.20 (Existence). *Assume (A_Ω) – (A'_μ) . Then there exists an entropy solution u of (1.1).*

We end this paper by providing a convergence result for the fractional-nonlinear vanishing viscosity method (2.10). We show that the solutions u_n of (2.10) converge to the solution u of the boundary value problem for the scalar conservation law (1.3). In other words that solutions of (1.1) with $\mathcal{L} = -\frac{1}{n}(-\Delta)^{\frac{\alpha}{2}}$ converge to solutions of (1.1) with $\mathcal{L} = 0$ as $n \rightarrow 0$. This result is a simple byproduct of the proof of Proposition 6.18, and the fact that nonlinear weak- \star convergence a posteriori imply L^p -convergence when the limit does not depend on a and belongs to L^∞ (see Remark 6.2 (c)).

Proposition 6.21 (Vanishing viscosity). *Assume (A_Ω) – (A_{u_0}) and u and u_n are entropy solutions of (1.3) and (2.10). Then*

$$u_n \rightarrow u \quad \text{as } n \rightarrow \infty \quad \text{in } L^p(Q) \quad \text{for all } p \in [1, \infty).$$

Remark 6.22. The result holds for all the operators \mathcal{L} discussed in Remark 6.19.

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APPENDIX A. ENTROPY SOLUTIONS ARE A GENERALIZED SOLUTION CONCEPT

Entropy solutions in the sense of Definition Definition 2.2 naturally contain classical solutions, and they are more restrictive than very weak solutions.

Lemma A.1. *Assume (A_Ω) – (A_μ) . Then:*

- (a) *Classical solutions of (1.1) are entropy solutions in the sense of Definition 2.2.*
- (b) *Entropy solutions in the sense of Definition 2.2 satisfy*

$$\int_M u \partial_t \varphi + f(u) \cdot \nabla \varphi + b(u) \mathcal{L}[\varphi] \, dx \, dt = 0,$$

for all $\varphi \in C_c^\infty(Q)$, i.e., they are (very) weak solutions.

Proof. (a) Let us assume that $u \in C_b^2([0, T] \times \mathbb{R}^d)$ solves (1.1). It remains to check part (a) of Definition 2.2. To that end, multiply the PDE by $\text{sgn}_\epsilon^\pm(u - k)\varphi$, where

$$\text{sgn}_\epsilon^\pm(s) = \begin{cases} 0, & \pm s \leq 0, \\ \frac{s}{\epsilon}, & 0 \leq \pm s \leq \epsilon, \\ \pm 1, & \epsilon \leq \pm s, \end{cases}$$

and $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ satisfies (2.5). With the short-hand notation

$$(\psi)_\epsilon^\pm = \int_0^\psi \text{sgn}_\epsilon^\pm(\xi) \, d\xi,$$

for (real valued) expressions ψ , we get after integrating the PDE over Q (and performing several chain rules and integration by parts)

$$\begin{aligned}
& - \int_Q \left((u-k)_\epsilon^\pm \partial_t \varphi + \operatorname{sgn}_\epsilon^\pm(u-k)(f(u)-f(k)) \cdot \nabla \varphi \right) dx dt \\
& - \int_Q \mathcal{L}^{\geq r}[b(u)] \operatorname{sgn}_\epsilon^\pm(u-k) \varphi dx dt - \int_M (b(u)-b(k))_\epsilon^\pm \mathcal{L}^{< r}[\varphi] dx dt \\
& = \int_\Omega (u_0 - k)_\epsilon^\pm \varphi(0, \cdot) dx + L_f \int_\Gamma (\bar{u}^c - k)_\epsilon^\pm \varphi d\sigma(x) dt \\
& \quad - \int_\Gamma \left(\operatorname{sgn}_\epsilon^\pm(u-k)(f(u)-f(k)) \cdot \hat{n} + L_f(u-k)_\epsilon^\pm \right) \varphi d\sigma(x) dt \\
& \quad + \int_M \left(\mathcal{L}^{< r}[b(u)] \operatorname{sgn}_\epsilon^\pm(u-k) \chi_\Omega - \mathcal{L}^{< r}[(b(u)-b(k))_\epsilon^\pm] \right) \varphi dx dt \\
& \quad + \int_Q (f(u)-f(k)) \cdot (\nabla u)(\operatorname{sgn}_\epsilon^\pm)'(u-k) \varphi dx dt.
\end{aligned}$$

We then pick a weak- \star converging sequence as $\epsilon \rightarrow 0$ to obtain

$$\begin{aligned}
& - \int_Q \left((u-k)^\pm \partial_t \varphi + F^\pm(u, k) \cdot \nabla \varphi \right) dx dt \\
& - \int_M \mathcal{L}^{\geq r}[b(u)-b(k)] \operatorname{sgn}^\pm(u-k) \varphi \chi_\Omega dx dt - \int_M (b(u)-b(k))^\pm \mathcal{L}^{< r}[\varphi] dx dt \\
& = \int_\Omega (u_0 - k)^\pm \varphi(0, \cdot) dx + L_f \int_\Gamma (\bar{u}^c - k)^\pm \varphi d\sigma(x) dt - \int_M \varphi d\mu^{\pm, r},
\end{aligned}$$

where

$$\begin{aligned}
\mu^{\pm, r} & := \left(\hat{n} \cdot F^\pm(u, k) + L_f(u-k)^\pm \right) \delta_{(t,x) \in \Gamma} \\
& \quad - \left(\mathcal{L}^{< r}[b(u)-b(k)] \operatorname{sgn}^\pm(u-k) \chi_\Omega - \mathcal{L}^{< r}[(b(u)-b(k))^\pm] \right).
\end{aligned}$$

We claim that this measure is nonnegative to finish the proof.

To prove the claim, we note that the first term in $\mu^{\pm, r}$ is immediately nonnegative since f is Lipschitz. The second term requires a bit more work. As $b' \geq 0$, $\operatorname{sgn}^\pm(u-k)(b(u)-b(k)) = \operatorname{sgn}^\pm(b(u)-b(k))(b(u)-b(k))$, and we need to consider the following expression:

$$(A.1) \quad - \int_M \left(\mathcal{L}^{< r}[b(u)-b(k)] \operatorname{sgn}^\pm(b(u)-b(k)) \chi_\Omega - \mathcal{L}^{< r}[(b(u)-b(k))^\pm] \right) \varphi dx dt.$$

Since φ satisfies (2.5), which we reformulate as

$$\operatorname{sgn}^\pm(b(u^c) - b(k)) \varphi = 0, \quad \text{a.e. in } Q^c,$$

we may further omit the χ_Ω from (A.1). Finally, by the pointwise convex inequality (cf. Corollary C.2)

$$\mathcal{L}^{< r}[b(u)-b(k)] \operatorname{sgn}^\pm(b(u)-b(k)) \leq \mathcal{L}^{< r}[(b(u)-b(k))^\pm],$$

the above measure is nonnegative when multiplied by φ , i.e.,

$$- \int_M \varphi d\mu^{\pm, r} \leq 0,$$

which proves the claim.

(b) The proof follows by Definition 2.2, the convex inequality $\mathcal{L}^{\geq r}[\psi] \operatorname{sgn}^{\pm} \psi \leq \mathcal{L}^{\geq r}[\psi^{\pm}]$ (cf. Corollary C.2), and taking k to be

$$k^{\pm} = \pm(\|u_0\|_{L^\infty(\Omega)} + \|u^c\|_{L^\infty(Q^c)}).$$

Indeed, since $k^- \leq u \leq k^+$ by Lemma 2.4, for $0 \leq \varphi \in C_c^\infty(Q)$ we then conclude from (2.6) that

$$\mp \int_Q \left(u \partial_t \varphi + f(u) \cdot \nabla \varphi \right) + \int_M b(u) \mathcal{L}[\varphi] \leq \mp \int_Q \left(k^{\pm} \partial_t \varphi + f(k^{\pm}) \cdot \nabla \varphi \right),$$

where the non-singular operator $\mathcal{L}^{\geq r}$ has been shifted over to φ using that $\mathcal{L}^{\geq r}$ is self adjoint (Proposition 3.1) followed by letting $r \rightarrow 0$. As φ has compact support in Q , the right-hand side is zero and thus

$$(A.2) \quad - \int_Q \left(u \partial_t \varphi + f(u) \cdot \nabla \varphi \right) dx dt + \int_M b(u) \mathcal{L}[\varphi] dx dt = 0.$$

The general case when $\varphi \in C_c^\infty(Q)$ holds by picking $0 \leq \psi \in C_c^\infty(Q)$ such that $0 \leq \phi := \psi - \varphi$ and using that (A.2) holds for ϕ and ψ . \square

APPENDIX B. PROOFS OF BASIC PROPERTIES OF ENTROPY SOLUTIONS

We will prove the maximum principle and time continuity at $t = 0$ for entropy solutions.

Proof of Lemma 2.4. In the entropy inequality (2.6) for u we set k in the (+) case to $k^+ := \sup_\Omega u_0 \vee \sup_{Q^c} u^c$ and in the (-) case to $k^- := \inf_\Omega u_0 \wedge \inf_{Q^c} u^c$. Then the condition (2.5) is automatically satisfied for every $0 \leq \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$. We then set $\varphi(t, x) = \theta(t)\phi(x)$ with $0 \leq \theta \in C_c^\infty([0, T])$ and with $0 \leq \phi \in C_c^\infty(\mathbb{R}^d)$, and where $\phi = 1$ in Ω and $\phi \leq 1$ in Ω^c . By sending $r \rightarrow \infty$ in (2.6) we find that

$$- \int_Q (u - k^{\pm})^{\pm} \theta' dx dt - \int_M (b(u) - b(k^{\pm}))^{\pm} \mathcal{L}[\phi] \theta dx dt \leq 0.$$

As $(b(u) - b(k^{\pm}))^{\pm} = 0$ a.e. in Q^c and $\mathcal{L}[\phi] \leq 0$ in Q (ϕ attains its global maximum on every point in Q), the integrand of the second integral is clearly non-positive and we conclude that

$$- \int_Q (u - k^{\pm})^{\pm} \theta' dx dt \leq 0, \quad \implies \quad \int_Q (u - k^{\pm})^{\pm} dx dt \leq 0,$$

where the last implication follows from setting $\theta(t) = T - t$, which can be approximated by nonnegative elements of $C_c^\infty([0, T])$. This concludes the proof. \square

Proof of Lemma 2.5. We follow Lemma 7.41 in [86]. Let $(\theta_\epsilon)_\epsilon \subset C_c^\infty([0, T])$ be sequence of functions satisfying $0 \leq \theta_\epsilon(t) \leq \theta_\epsilon(0) = 1$, $\operatorname{supp}(\theta_\epsilon) \subset [0, \epsilon]$, and $\theta_\epsilon(t) - \theta_{\epsilon'}(t) \geq 0$ when $\epsilon \geq \epsilon' > 0$. For any $0 \leq \beta \in C_c^\infty(\Omega)$, we set $\varphi(t, x) = (\theta_\epsilon - \theta_{\epsilon'})(t)\beta(x)$ in (2.6), add the inequalities for $(\cdot)^+$ and $(\cdot)^-$, and since all terms are bounded and $\theta_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, we observe that

$$(B.1) \quad \limsup_{\epsilon \rightarrow 0} \sup_{\epsilon' \in (0, \epsilon)} - \iint_Q |u(t, x) - k| \beta(x) (\theta'_\epsilon - \theta'_{\epsilon'})(t) dx dt \leq 0,$$

or equivalently, $\limsup_{\epsilon \rightarrow 0} \iint_Q |u(t, x) - k| \beta(x) (-\theta'_\epsilon)(t) dx dt \leq \liminf_{\epsilon' \rightarrow 0} \iint_Q |u(t, x) - k| \beta(x) (-\theta'_{\epsilon'})(t) dx dt$.

In the same way, but with $\varphi(t, x) = \theta_\epsilon(t)\beta(x)$, we find that

$$(B.2) \quad \lim_{\epsilon \rightarrow 0} \int_0^\epsilon \left(\int_\Omega |u(t, x) - k| \beta(x) dx \right) (-\theta'_\epsilon(t)) dt \leq \int_\Omega |u_0(x) - k| \beta(x) dx,$$

where the limit on the left-hand side exists by (B.1). It then follows that

$$(B.3) \quad \operatorname{ess\,lim}_{t \rightarrow 0^+} \int_{\Omega} |u(t, x) - k|\beta(x) \, dx \leq \int_{\Omega} |u_0(x) - k|\beta(x) \, dx.$$

Indeed, assume (B.3) is false: There is an $s > 0$ and a set $E \subset [0, T]$ with $\operatorname{meas}\{E \cap (0, \delta)\} > 0$ for all $\delta > 0$, such that $\int_{\Omega} |u(t, x) - k|\beta(x) \, dx > \int_{\Omega} |u_0(x) - k|\beta(x) \, dx + s$ for $t \in E$. Let $\theta_{\epsilon}(t) = 1 - \int_0^t \psi(s) \, ds$ and $\psi(s) = \frac{1}{\operatorname{meas} E_{\epsilon}} 1_{E_{\epsilon}} * \rho_{\delta}(t)$ where $\delta \in (0, \epsilon)$, $E_{\epsilon} := E \cap (0, \epsilon)$, $\rho_{\delta} = \frac{1}{\delta} \rho(\frac{\cdot}{\delta})$, and $0 \leq \rho \in C_c^{\infty}((0, T))$ with $\int_0^T \rho \, dt = 1$. Integration in time against $\theta'_{\epsilon} = -\psi$ and sending $\delta \rightarrow 0$ then gives for all $\epsilon > 0$,

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_0^{\epsilon} \left(\int_{\Omega} |u(t, x) - k|\beta(x) \, dx \right) (-\theta'_{\epsilon}(t)) \, dt &= \frac{1}{\operatorname{meas} E_{\epsilon}} \int_{E_{\epsilon}} \int_{\Omega} |u(t, x) - k|\beta(x) \, dx \, dt \\ &> \int_{\Omega} |u_0(x) - k|\beta(x) \, dx + s, \end{aligned}$$

which contradicts (B.2).

As (B.3) holds for all $k \in (-\infty, \infty)$ and $0 \leq \beta \in C_c^{\infty}(\Omega)$, we can ‘double the variables’; we set $k = u_0(y)$ and $\beta(x) \mapsto \beta(x)\rho_{\epsilon}(x - y)$ where the new β is as the old while ρ_{ϵ} is a sequence of positive symmetric mollifiers of support in a ball of radius $\epsilon > 0$. Integrating over $y \in \Omega$ and letting $\epsilon \rightarrow 0$ we find that

$$(B.4) \quad \lim_{\epsilon \rightarrow 0} \int_{\Omega} \operatorname{ess\,lim}_{t \rightarrow 0^+} \left(\int_{\Omega} |u(t, x) - u_0(y)|\beta(x)\rho_{\epsilon}(x - y) \, dx \right) dy = 0.$$

By dominated convergence, we exchange the inner limit and the outer integral in (B.4), and since

$$\begin{aligned} \sup_{t \in (0, T)} \left| \iint_{\Omega \times \Omega} |u(t, x) - u_0(y)|\beta(x)\rho_{\epsilon}(x - y) \, dx \, dy - \int_{\Omega} |u(t, x) - u_0(x)|\beta(x) \, dx \right| \\ \leq \iint_{\Omega \times \Omega} |u_0(x) - u_0(y)|\beta(x)\rho_{\epsilon}(x - y) \, dx \, dy \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0, \end{aligned}$$

we then conclude that

$$\operatorname{ess\,lim}_{t \rightarrow 0^+} \int_{\Omega} |u(t, x) - u_0(x)|\beta(x) \, dx = 0.$$

Since $0 \leq \beta \in C_c^{\infty}(\Omega)$ was arbitrary, it follows that $\operatorname{ess\,lim}_{t \rightarrow 0^+} \|u(t, \cdot) - u_0\|_{L^1(K)} = 0$ for any compact set $K \subset \Omega$. Moreover, for $K \subset \Omega$ and a.e. $t \in (0, T)$, we also have

$$\|u(t, \cdot) - u_0\|_{L^1(\Omega)} \leq (\|u\|_{L^{\infty}(Q)} + \|u_0\|_{L^{\infty}(\Omega)})|\Omega \setminus K| + \|u(t, \cdot) - u_0\|_{L^1(K)},$$

and so by inner regularity of Ω the result follows. \square

APPENDIX C. CONVEX INEQUALITY FOR NONLOCAL OPERATORS

Lemma C.1. *Assume $\eta : \mathbb{R} \rightarrow \mathbb{R}$ is convex. Then, for any real numbers v, w , we have*

$$\eta(v) - \eta(w) \geq \eta'(w)(v - w),$$

where η' is a subderivative of η , that is, $\eta'(w) \in [\eta'(w-), \eta'(w+)]$.

In the following corollary we work with a zero order Lévy operator (i.e. the Lévy measure satisfies $\mu(\mathbb{R}^d) < \infty$) both for simplicity and because we only apply the result in such situations. However, by a truncation argument the corollary can be extended to the general situation provided sufficient regularity of the components.

Corollary C.2. *Let \mathcal{L} be a zero order Lévy operator and $\eta: \mathbb{R} \rightarrow \mathbb{R}$ convex. For any $\phi \in L^\infty(\mathbb{R}^d)$ the inequality*

$$\mathcal{L}[\phi]\eta'(\phi) \leq \mathcal{L}[\eta(\phi)] \quad \text{a.e.} \quad \text{in } \mathbb{R}^d.$$

In the special case $\eta = (\cdot)^\pm$ with $h: \mathbb{R} \rightarrow \mathbb{R}$ a non-decreasing function we have: For any $\phi, \psi \in \mathbb{R}^d$ the inequality

$$\mathcal{L}[h(\phi) - h(\psi)]\text{sgn}^\pm(\phi - \psi) \leq \mathcal{L}[(h(\phi) - h(\psi))^\pm] \quad \text{a.e.} \quad \text{in } \mathbb{R}^d.$$

Proof. The first inequality is a direct application of Lemma C.1 and so we focus on the second one. For a.e. $x \in \mathbb{R}^d$ we have

$$\begin{aligned} \mathcal{L}[h(\phi) - h(\psi)]\text{sgn}^\pm(\phi - \psi) &= \int_{\mathbb{R}^d} \left((h(\phi) - h(\psi))(x+z) - (h(\phi) - h(\psi))(x) \right) \text{sgn}^\pm((\phi - \psi)(x)) \, d\mu(z) \\ &\leq \int_{\mathbb{R}^d} \left((h(\phi) - h(\psi))^\pm(x+z) - (h(\phi) - h(\psi))^\pm(x) \right) \, d\mu(z), \end{aligned}$$

where the inequality, which holds for the integrands, can be seen as follows: If $h(\phi(x)) = h(\psi(x))$ then both integrands are zero, whereas if $h(\phi(x)) \neq h(\psi(x))$ then we may replace $\text{sgn}^\pm(\phi - \psi)(x)$ by $\text{sgn}^\pm(h(\phi) - h(\psi))(x)$ (the two coincide as h is non-decreasing) resulting in the inequality when using that $v \text{sgn}^\pm(v) = (v)^\pm$ and $w \text{sgn}^\pm(v) \leq (w)^\pm$ for any $v, w \in \mathbb{R}$. \square

APPENDIX D. ADDITIONAL IDENTITIES FOR THE BILINEAR OPERATOR

We here show that the expression $\mathcal{B}_{x+y}[\phi, \psi]$ is somewhat analogous to the expression

$$(\nabla_x \phi + \nabla_y \phi) \cdot (\nabla_x \psi + \nabla_y \psi),$$

and that we, in particular, need not introduce ‘‘cross-terms’’ for the diffusion in the doubling of variables argument as is needed in the local case (c.f. [35, 72, 87, 89]).

For convenient notation, we introduce two operators: Given a Lévy measure μ on \mathbb{R}^d and two functions ϕ, ψ in the variables $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ we formally define

$$\begin{aligned} \mathcal{B}_{x,x+y}[\phi, \psi](x, y) &= \lim_{r \searrow 0} \frac{1}{2} \int_{|z| \geq r} \left(\phi(x+z, y) - \phi(x, y) \right) \left(\psi(x+z, y+z) - \psi(x, y) \right) \, d\mu(z), \\ \mathcal{B}_{y,x+y}[\phi, \psi](x, y) &= \lim_{r \searrow 0} \frac{1}{2} \int_{|z| \geq r} \left(\phi(x, y+z) - \phi(x, y) \right) \left(\psi(x+z, y+z) - \psi(x, y) \right) \, d\mu(z), \end{aligned} \tag{D.1}$$

which are non-local analogues of $\nabla_x \phi \cdot (\nabla_x \psi + \nabla_y \psi)$ and $\nabla_y \phi \cdot (\nabla_x \psi + \nabla_y \psi)$.

Proposition D.1 (Cross-terms formula). *Assume (A_μ) . Let $\phi, \psi \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be functions in (x, y) and let ψ have compact support. Then*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x+y}[\phi, \psi] \, dx \, dy = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x,x+y}[\phi, \psi] + \mathcal{B}_{y,x+y}[\phi, \psi] \, dx \, dy,$$

provided the integrands are well defined in $L^1(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. Fix $r > 0$. Observe that the integrands in the following argument are absolutely integrable with respect to $dx \, dy \, d\mu(z)$ due to the compact support of ψ and the truncation of our operators; in

particular we are free to deploy Fubini's theorem. We compute

$$\begin{aligned}
& 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x,x+y}^{\geq r}[\phi, \psi] \, dx \, dy \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y)) (\psi(x+z, y+z) - \psi(x, y)) \, d\mu(z) \, dx \, dy \\
&= \int_{|z| \geq r} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\phi(x, y-z) - \phi(x-z, y-z)) \psi(x, y) \, dx \, dy \, d\mu(z) \\
&\quad - \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y)) \psi(x, y) \, d\mu(z) \, dx \, dy \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x, y) - \phi(x-z, y-z)) \psi(x, y) \, d\mu(z) \, dx \, dy \\
&\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x, y-z) - \phi(x+z, y)) \psi(x, y) \, d\mu(z) \, dx \, dy.
\end{aligned}$$

By the change of variables $z \mapsto -z$ and the symmetry of $d\mu(z)$ we see that the first integral on the right-hand side coincides with

$$- \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{L}_{x+y}^{\geq r}[\phi] \psi \, dx \, dy = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x+y}^{\geq r}[\phi, \psi] \, dx \, dy,$$

where the identity follows from Proposition 3.1. By a similar computation for the $\mathcal{B}_{y,x+y}^{\geq r}$ operator, we conclude

$$\begin{aligned}
& \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\mathcal{B}_{x,x+y}^{\geq r} + \mathcal{B}_{y,x+y}^{\geq r} \right) [\phi, \psi] \, dx \, dy \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x+y}^{\geq r}[\phi, \psi] \, dx \, dy \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x, y-z) - \phi(x, y+z)) \psi(x, y) \, d\mu(z) \, dx \, dy \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x-z, y) - \phi(x+z, y)) \psi(x, y) \, d\mu(z) \, dx \, dy.
\end{aligned}$$

The two last integrals cancel out, as can be seen by substituting $z \mapsto -z$ in either, and so the result follows by letting $r \rightarrow 0$. \square

For the same setting as above, we introduce one final operator formally by

$$(D.2) \quad \mathcal{B}_{x,y}[\phi, \psi](x, y) = \lim_{r \searrow 0} \frac{1}{2} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y)) (\psi(x, y+z) - \psi(x, y)) \, d\mu(z),$$

which is a non-local analogue of $\nabla_x \phi \cdot \nabla_y \psi$.

Lemma D.2 (An auxiliary product rule). *Assume (A_μ) . Let $\phi \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a function in (x, y) , and let $\zeta^x, \zeta^y, \rho \in L^\infty(\mathbb{R}^d)$ be functions in x, y , and $x-y$ respectively. Let further ζ^x and ζ^y have compact support. Then we have the product rule*

$$\begin{aligned}
& \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x,x+y}[\phi, \zeta^x \zeta^y] \rho \, dx \, dy \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_x[\phi, \zeta^x] \zeta^y \rho \, dx \, dy + \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi \left(\mathcal{B}_y[\zeta^y, \rho] \zeta^x - \mathcal{B}_{x,y}[\zeta^x, \zeta^y] \rho \right) \, dx \, dy,
\end{aligned}$$

provided the integrands are well-defined in $L^1(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. Fix $r > 0$. The compact support of ζ^x, ζ^y guarantees the following integrals to be well-defined. Adding and subtracting $\zeta^x(x+z)\zeta^y(y)$ to the integrand, and performing a few changes of variables, we compute

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_{x,x+y}^{\geq r} [\phi, \zeta^x \zeta^y] \rho \, dx \, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y)) (\zeta^x(x+z)\zeta^y(y+z) - \zeta^x(x)\zeta^y(y)) \rho(x-y) \, d\mu(z) \, dx \, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y)) (\zeta^x(x+z) - \zeta^x(x)) \zeta^y(y) \rho(x-y) \, d\mu(z) \, dx \, dy \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y)) (\zeta^y(y+z) - \zeta^y(y)) \zeta^x(x+z) \rho(x-y) \, d\mu(z) \, dx \, dy. \end{aligned}$$

Note that the first integral on the right-hand side coincides with $\int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{B}_x^{\geq r} [\phi, \zeta^x] \zeta^y \rho \, dx \, dy$. For the second integral, we split it up and perform a change of variables giving

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} (\phi(x+z, y) - \phi(x, y)) (\zeta^y(y+z) - \zeta^y(y)) \zeta^x(x+z) \rho(x-y) \, d\mu(z) \, dx \, dy \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} \phi(x, y) (\zeta^y(y+z) - \zeta^y(y)) \zeta^x(x) \rho(x-z-y) \, d\mu(z) \, dx \, dy \\ & \quad - \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{|z| \geq r} \phi(x, y) (\zeta^y(y+z) - \zeta^y(y)) \zeta^x(x+z) \rho(x-y) \, d\mu(z) \, dx \, dy, \end{aligned}$$

If we further add and subtract the integral of $\phi(x, y) (\zeta^y(y+z) - \zeta^y(y)) \zeta^x(x) \rho(x-y)$ to this right-hand side we get

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi \mathcal{B}_y^{\geq r} [\zeta^y, \rho] \zeta^x \, dx \, dy - \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi \mathcal{B}_{x,y}^{\geq r} [\zeta^x, \zeta^y] \rho \, dx \, dy,$$

where we used that $\rho(x-z-y) = \rho(x-(y+z))$. Letting $r \rightarrow 0$ gives the result. \square

APPENDIX E. THE LÉVY OPERATORS INDUCING TRANSLATION REGULARITY

Lévy operators are special instances of Fourier multipliers with negative symbols. When using the following normalization of the Fourier transform

$$\hat{\phi}(\xi) := \int_{\mathbb{R}^d} \phi(x) e^{-i\xi \cdot x} \, dx,$$

a Lévy operator \mathcal{L} admits the symbol $-m(\xi)$ where the *positive* function $m: \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by

$$(6.18) \quad m(\xi) = \int_{\mathbb{R}^d} [1 - \cos(\xi \cdot z)] \, d\mu(z),$$

with μ being the corresponding Lévy measure of \mathcal{L} . Note that (6.18) is well-defined as the integrand is bounded by $2 \wedge (\frac{1}{2}|\xi|^2|z|^2)$. That m is the symbol of $-\mathcal{L}$ or, equivalently, that $-m$ is the symbol of

\mathcal{L} , can be seen as follows: For any Schwartz-function $\phi \in \mathcal{S}(\mathbb{R}^d)$ we compute

$$\begin{aligned}
\widehat{\mathcal{L}[\phi]}(\xi) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{1}{2}\phi(x+z) + \frac{1}{2}\psi(x-z) - \psi(x) \right] e^{-i\xi \cdot x} d\mu(z) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{1}{2}\psi(x+z) + \frac{1}{2}\psi(x-z) - \psi(x) \right] e^{-i\xi \cdot x} dx d\mu(z) \\
&= \int_{\mathbb{R}^d} \left[\frac{1}{2}(e^{i\xi \cdot z} + e^{-i\xi \cdot z}) - 1 \right] d\mu(x) \hat{\psi}(\xi) \\
&= \int_{\mathbb{R}^d} [\cos(\xi \cdot z) - 1] d\mu(z) \hat{\psi}(\xi) \\
&= -m(\xi) \hat{\psi}(\xi)
\end{aligned}$$

where we in the second line used Fubini's theorem. Observe also that we can exploit the identity $|1 - e^{i\xi \cdot z}|^2 = 2[1 - \cos(\xi \cdot z)]$ and Plancherel's Theorem to attain for any $\phi \in H_0^{\mathcal{L}}(\Omega)$ (cf. (2.4)) the identity

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathcal{B}[\phi, \phi] dx &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(x) - \phi(x+z))^2 d\mu(z) dx \\
&= \frac{1}{4\pi} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |1 - e^{i\xi \cdot z}|^2 |\hat{\phi}(\xi)|^2 d\xi d\mu(z) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [1 - \cos(\xi \cdot z)] d\mu(z) |\hat{\phi}(\xi)|^2 d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^d} m(\xi) |\hat{\phi}(\xi)|^2 d\xi.
\end{aligned}
\tag{E.1}$$

In the next proposition, we give a necessary and sufficient condition on m for when $H_0^{\mathcal{L}}(\Omega)$ embeds compactly in $L^2(\Omega)$.

Proposition 6.11. *Assume (A_μ) , let $H_0^{\mathcal{L}}(\Omega)$ be defined by (2.4), and let m denote the symbol of $-\mathcal{L}$ (see (6.18)). Then the embedding $H_0^{\mathcal{L}}(\Omega) \hookrightarrow L^2(\Omega)$ is compact if, and only if,*

$$\liminf_{|\xi| \rightarrow \infty} m(\xi) = \infty.
\tag{6.19}$$

Proof. Suppose first that m satisfies (6.19), and define $\rho: [0, \infty) \rightarrow [0, \infty)$ by

$$\rho(R) := \inf_{|\xi| \geq R} m(\xi),$$

which must tend to infinity as $R \rightarrow \infty$. As $L^2(\Omega)$ is a metric space, we need only prove that a bounded sequence $(\phi_j)_{j \in \mathbb{N}} \subset H_0^{\mathcal{L}}(\Omega)$ admits a subsequence converging in $L^2(\mathbb{R}^d)$. For such a sequence, observe that we for any $j \in \mathbb{N}$, $y \in \mathbb{R}^d$ and any $R > 0$ have by Plancherel's theorem

$$\begin{aligned}
2\pi \|\phi_j - \phi_j(\cdot + y)\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} |1 - e^{i\xi \cdot y}|^2 |\hat{\phi}_j(\xi)|^2 d\xi \\
&\leq (R|y|)^2 \int_{|\xi| < R} |\hat{\phi}_j(\xi)|^2 d\xi + \frac{4}{\rho(R)} \int_{|\xi| \geq R} m(\xi) |\hat{\phi}_j(\xi)|^2 d\xi \\
&\leq \left[(R|y|)^2 + \frac{4}{\rho(R)} \right] \|\phi_j\|_{H_0^{\mathcal{L}}(\Omega)}^2.
\end{aligned}$$

Defining $C = \sup_j \|\phi_j\|_{H_0^{\mathcal{L}}(\Omega)}$, which by assumption is bounded, the previous calculation shows that we for any $y \in \mathbb{R}^d$ have

$$\sup_j \|\phi_j - \phi_j(\cdot + y)\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{C}{2\pi} \inf_{R>0} \left[(R|y|)^2 + \frac{4}{\rho(R)} \right].$$

Setting $R = |y|^{-\frac{1}{2}}$, the right-hand side vanish as $|y| \rightarrow 0$ and so the family $(\phi_j)_{j \in \mathbb{N}}$ is equicontinuous with respect to translation. The family is also ‘equitight’ as all functions are supported in Ω . By the Fréchet–Kolmogorov theorem, $(\phi_j)_{j \in \mathbb{N}}$ then admits a sub-sequence converging in $L^2(\Omega)$.

Next, assume m fails to satisfy (6.19); i.e. there is a sequence $(\xi_j)_{j \in \mathbb{N}} \subset \mathbb{R}^d$ so that $|\xi_j| \rightarrow \infty$ as $j \rightarrow \infty$ but $\sup_j m(\xi_j) \leq c_0$ for some finite c_0 . We shall construct a bounded sequence $(\phi_j)_{j \in \mathbb{N}} \subset H_0^{\mathcal{L}}(\Omega)$ for which there is no sub-sequence converging in $L^2(\Omega)$. Without loss of generality, we assume the corresponding measure μ of \mathcal{L} to be supported in the unit ball of \mathbb{R}^d ; after all, the space induced by \mathcal{L} is equivalent to that of $\mathcal{L}^{<1} = \mathcal{L} - \mathcal{L}^{\geq 1}$ as $\mathcal{L}^{\geq 1}$ is a zero order operator. We then see from its definition that m is an analytic function whose double derivative in any direction $\nu \in \mathbb{S}^{d-1}$ satisfies

$$\left| \frac{\partial^2}{\partial s^2} m(\xi + s\nu) \right|_{s=0} \leq \int_{|z| \leq 1} |z|^2 d\mu(z) =: c_1.$$

Without loss of generality, assume also that Ω includes zero in its interior and select a non-zero, radially symmetric function $\varphi \in C_c^\infty(\Omega)$. Consider the sequence $(\phi_j)_{j \in \mathbb{N}} \subset L^2(\Omega)$ defined by

$$\phi_j(x) := \cos(\xi_j \cdot x) \varphi(x), \quad \implies \quad \widehat{\phi_j}(\xi) = \frac{1}{2}(\widehat{\varphi}(\xi + \xi_j) + \widehat{\varphi}(\xi - \xi_j)),$$

where $(\xi_j)_{j \in \mathbb{N}}$ is the sequence from before. Using (E.1), the symmetry of m and $\widehat{\varphi}$, and performing two change of variables we compute

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{B}[\phi_j, \phi_j] dx &= \frac{1}{8\pi} \int_{\mathbb{R}^d} m(\xi) |\widehat{\varphi}(\xi + \xi_j) + \widehat{\varphi}(\xi - \xi_j)|^2 d\xi \\ &\leq \frac{1}{2\pi} \int_{\mathbb{R}^d} m(\xi + \xi_j) |\widehat{\varphi}(\xi)|^2 d\xi \\ &\leq \frac{1}{2\pi} \int_{\mathbb{R}^d} \left(c_0 + \nabla m(\xi_j) \cdot \xi + \frac{c_1}{2} |\xi|^2 \right) |\widehat{\varphi}(\xi)|^2 d\xi, \end{aligned}$$

where the last inequality follows by Taylor expansion. As φ is smooth and rapidly decaying, so is $\widehat{\varphi}$ and thus the last integral is well-defined. Moreover, as φ is symmetric (about zero) so is $\widehat{\varphi}$ and consequently $\int_{\mathbb{R}^d} \nabla m(\xi_j) \cdot \xi |\widehat{\varphi}(\xi)|^2 d\xi = 0$. We conclude

$$\sup_j \|\phi_j\|_{H_0^{\mathcal{L}}(\Omega)}^2 \leq \frac{1}{2\pi} \int_{\mathbb{R}^d} \left(1 + c_0 + \frac{c_1}{2} |\xi|^2 \right) |\widehat{\varphi}(\xi)|^2 d\xi < \infty,$$

that is, the sequence $(\phi_j)_{j \in \mathbb{N}}$ is bounded in $H_0^{\mathcal{L}}(\Omega)$. However, no subsequence of $(\phi_j)_{j \in \mathbb{N}}$ can converge in $L^2(\Omega)$ as no subsequence of $(\widehat{\varphi}(\cdot - \xi_j))_{j \in \mathbb{N}}$ can converge in $L^2(\mathbb{R}^d)$. This concludes the proof. \square

Example E.1. We conclude the section with an example of a symmetric Lévy operator \mathcal{L} for which $H_0^{\mathcal{L}}(\Omega)$ compactly embeds in $L^2(\mathbb{R}^d)$, but where the conclusion of Lem. 6.13 fails. For simplicity we restrict our attention to the one-dimensional case ($d = 1$), but we note that the example generalizes to higher dimensions.

Let \mathcal{L} be the symmetric Lévy operator whose corresponding Lévy measure μ on \mathbb{R} is given by

$$\mu = \sum_{k=0}^{\infty} 2^{k-1} \left(\delta(x + \frac{1}{2^k}) + \delta(x - \frac{1}{2^k}) \right).$$

Clearly μ is nonnegative and symmetric, and it satisfies $\int_{\mathbb{R}} (|z|^2 \wedge 1) d\mu(z) < \infty$ since

$$\int_{\mathbb{R}} |z|^2 d\mu(z) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2.$$

Moreover, the corresponding symbol m of $-\mathcal{L}$ is given by

$$m(\xi) = \int_{\mathbb{R}} \left[1 - \cos(\xi z)\right] d\mu(z) = \sum_{k=0}^{\infty} 2^k \left[1 - \cos\left(\frac{\xi}{2^k}\right)\right].$$

For any $n \in \mathbb{N}$ the values $|\xi| \in [2^n, 2^{n+1}]$ satisfy $|\xi/2^n| \in [1, 2]$ and consequently

$$\left[1 - \cos\left(\frac{\xi}{2^n}\right)\right] \geq [1 - \cos(1)], \quad \implies \quad m(\xi) \geq 2^n [1 - \cos(1)].$$

Thus, $\liminf_{|\xi| \rightarrow \infty} m(\xi) = \infty$ and so Prop. 6.11 implies that $H_0^{\mathcal{L}}(\Omega)$ embeds compactly in $L^2(\Omega)$. Consider now for a fixed $j \in \mathbb{N}$ the truncated operator $\mathcal{L}^{>2^{-j}}$ whose corresponding measure and symbol are given by

$$\mu_j = \sum_{k=0}^{j-1} 2^{k-1} \left(\delta\left(x + \frac{1}{2^k}\right) + \delta\left(x - \frac{1}{2^k}\right)\right), \quad m_j(\xi) = \sum_{k=0}^{j-1} 2^k \left[1 - \cos\left(\frac{\xi}{2^k}\right)\right].$$

Setting $\xi_j := \pi 2^j$ we observe that $m_j(\xi_j) = 0$. Assuming without loss of generality that Ω includes zero as an interior point, we may pick a nonzero, radially symmetric function $\varphi \in C_c^\infty(\Omega)$ and define the sequence $(\phi_j)_{j \in \mathbb{N}} \subset L^2(\Omega)$ by

$$\phi_j(x) = \cos(\xi_j x) \varphi(x), \quad \implies \quad \widehat{\phi_j}(\xi) = \frac{1}{2} (\widehat{\varphi}(\xi + \xi_j) + \widehat{\varphi}(\xi - \xi_j)).$$

Arguing similarly as in the second part of the proof of Prop. 6.11, we conclude that

$$\sup_j \int_{\mathbb{R}} \mathcal{B}^{>2^{-j}}[\phi_j, \phi_j] dx < \infty.$$

The sequence $(\phi_j)_{j \in \mathbb{N}}$ is, however, not equicontinuous with respect to translation as this would (together with their equitightness) imply there is a convergent subsequence, and no subsequence of $(\widehat{\varphi}(\xi - \xi_j))_{j \in \mathbb{N}}$ can converge. We conclude that the corresponding result of Lemma 6.13 does not hold for \mathcal{L} .

APPENDIX F. A TECHNICAL LEMMA

Lemma F.1. *Assume $R > 0$, κ is a Borel probability measure on $[-R, R]$, $h: [-R, R] \rightarrow [0, \infty)$ is Lipschitz with Lipschitz-constant L , non-increasing on $[-R, 0]$, non-decreasing on $[0, R]$, and $h(0) = 0$. The two κ -means $\bar{s} := \int_{[-R, R]} s d\kappa(s)$ and $\bar{h} := \int_{[-R, R]} h(s) d\kappa(s)$ then satisfy*

$$h(\bar{s})^2 \leq LR\bar{h}.$$

Remark F.2. This lemma is comparable to (and can be viewed as a generalization of) Lemma 4.5 in [60] which states that for every $c < d$ and monotone Lipschitz b ,

$$|b(d) - b(c)|^2 \leq 2L_b \int_c^d (b(x) - b(c)) dx.$$

Proof of Lemma F.1. Without loss of generality, we assume both $\bar{s} > 0$ and $h(\bar{s}) > 0$. Observe first that since $h(\bar{s}) \leq h(s)$ for $s \in [\bar{s}, R]$, we get

$$(F.1) \quad h(\bar{s}) - \bar{h} = \left(\int_{[-R, \bar{s}]} + \int_{[\bar{s}, R]} \right) (h(\bar{s}) - h(s)) d\kappa(s) \leq L \int_{[-R, \bar{s}]} (\bar{s} - s) d\kappa(s) + 0.$$

Using that $s - \bar{s}$ has κ -mean zero and $h(\bar{s}) \leq L\bar{s}$ (since $h(0) = 0$), we further compute

$$(F.2) \quad \begin{aligned} L \int_{[-R, \bar{s}]} (\bar{s} - s) \, d\kappa(s) &= L \int_{[\bar{s}, R]} (s - \bar{s}) \, d\kappa(s) \leq (LR - h(\bar{s})) \int_{[\bar{s}, R]} d\kappa(s) \\ &\leq \left(\frac{LR}{h(\bar{s})} - 1 \right) \int_{[\bar{s}, R]} h(s) \, d\kappa(s) \leq \left(\frac{LR}{h(\bar{s})} - 1 \right) \bar{h}. \end{aligned}$$

Combining (F.1) with (F.2) gives the result. \square

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