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# A Few Characterizations of Topological Spaces with No Infinite Discrete Subspace

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## Abstract

We give several characteristic properties of FAC spaces, namely topological spaces with no infinite discrete subspace. The first one was obtained in 2019 by the first author, and states that every closed set is a finite union of irreducible closed subsets. The full result extends well-known characterizations of posets with no infinite antichain. One of them is that FAC spaces are, equivalently, topological spaces in which every closed set contains a dense Noetherian subspace, or spaces in which every Hausdorff subspace is finite, or in which no subspace has any infinite relatively Hausdorff subset. The latter comes with a nice min-max property, extending an observation of Erdős and Tarski in the case of posets: on spaces with no infinite relatively Hausdorff subset, the cardinalities of relatively Hausdorff subsets are bounded, and the least upper bound is also the least cardinality of a family of closed irreducible subsets that cover the space.

*Keywords:* Ordered set, closure system, Noetherian topological space, well-quasi-order

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## 1. Introduction

A topological space  $T := (E, \mathcal{F})$ , where  $\mathcal{F}$  is the set of closed subsets, is *Noetherian* if every descending sequence of closed subspaces is stationary. A subset  $X$  of  $E$  is *discrete* if and only if the induced topology on  $X$  is the discrete topology, namely if every subset of  $X$  is closed with respect to the induced topology. A closed subset is *irreducible* if it is non-empty and not the union of two proper closed subsets. Our main objective is to show the following.

**Theorem 1.** *The following properties are equivalent for a topological space  $T := (E, \mathcal{F})$ .*

- (i) *No infinite subset of  $E$  is discrete;*
- (ii) *Every closed set is a finite union of irreducible closed subsets;*
- (iii) *Every closed set contains a dense subset on which the induced topology is Noetherian.*

The equivalence between (i) and (ii) was proved by the first author in 2019 [10]. The resulting spaces were called FAC spaces there, since they are a topological generalization of posets with the *finite antichain condition*, namely those that have no infinite antichain. Hence, compared to [10], (iii) is a new, equivalent definition of FAC spaces, itself inspired from a well-known equivalent characterization of posets with the finite antichain condition (see Section 3.5). We will also mention a few other equivalent conditions in Theorem 14, which, as we will see, are connected to a nice min-max property, which we will state in Proposition 13.

We give a proof of the equivalence of (iii) with (i) and (ii) in Section 2. That also gives an alternative proof of the equivalence between (i) and (ii), and will stress the importance of the notion of infinite separating chain of closed sets, inspired from work by Chakir and Pouzet [3, 2]. We make additional remarks in Section 3, relating the result to other characterizations of FAC spaces, to Noetherian spaces, to a related result of Galvin, Milner and Pouzet in the larger context of closure operators, to lattice properties, and finally to a min-max result due to Erdős and Tarski [5], which will lead us to the topological min-max theorem announced above.

## 2. The proof

We mimic the proof of a similar, fairly well-known result for posets, which we will give in Section 3.5. The significant part is the implication (i)  $\Rightarrow$  (iii).

We recall that a *closure system* is a pair  $(E, \varphi)$  where  $\varphi$  (the *closure*) is a map from the power set of  $E$  into itself which is extensive, order-preserving and idempotent [4, Section II.1]. A closure is *topological* if and only if it commutes with finite unions. The fixed points of a topological closure form the lattice of closed sets of a topology, and conversely, the usual closure operator of a topological space is a topological closure.

Given a closure system  $(E, \varphi)$ , a subset  $C \subseteq E$  is *closed* if  $\varphi(C) = C$ ; it is *independent* if  $x \notin \varphi(C \setminus \{x\})$  for every  $x \in C$ ; it is *generating* if  $\varphi(C) = E$ . When  $\varphi$  is a topological closure, a set is independent if and only if it is discrete, and the generating sets are called *dense*.

The closure  $\varphi_{\uparrow E'}$  induced by  $\varphi$  on a subset  $E'$  of  $E$  is defined by  $\varphi_{\uparrow E'}(X) := \varphi(X) \cap E'$  for every  $X \subseteq E'$ .

Although our results will only apply to topological closures, we use the language of (general) closures so as to be able to relate our results to a theorem of Galvin, Milner and Pouzet in Section 3.3.

We will use the following notion and result, adapted from [3, Section 3, p.7], see also [2, p.25]. A non-empty chain  $\mathcal{I}$  of closed sets of  $(E, \varphi)$  is *separating* if for every  $I \in \mathcal{I} \setminus \{\bigcup \mathcal{I}\}$  and every finite set  $F \subseteq \bigcup \mathcal{I} \setminus I$ , there is a set  $J \in \mathcal{I}$  such that  $I \not\subseteq \varphi(F \cup J)$ . Note that  $J$  is necessarily included in  $I$ , and strictly so: otherwise, since  $\mathcal{I}$  is a chain,  $I$  would be included in  $J$ , hence in  $\varphi(F \cup J)$ . In particular, a separating chain  $\mathcal{I}$  cannot have a smallest element  $I \neq \bigcup \mathcal{I}$ ; an *infinite* separating chain cannot have a smallest element at all.

We illustrate the notion in Figure 1. Here we assume that  $\varphi$  is a topological closure operator, so that it commutes with unions;  $\mathcal{I}$  is a countable chain  $I_0 \supseteq I_1 \supseteq \dots \supseteq I_n \supseteq \dots$ , we consider  $I := I_n$ , and  $F$  is the finite set of bullets in the top tier of the figure. Below left, we show  $\varphi(F \cup I_{n+1})$ , which is the union of  $I_{n+1}$  with the closures of each of the points in  $F$  (each closure of a point being shown as a triangular region that extends below the point), and  $\varphi(F \cup I_{n+1})$  contains  $I_n$ ; but  $\varphi(F \cup I_{n+2})$ , shown below right in the figure, does not, and therefore we can take  $J := I_{n+2}$ .

The following holds for every closure system, not necessarily topological.

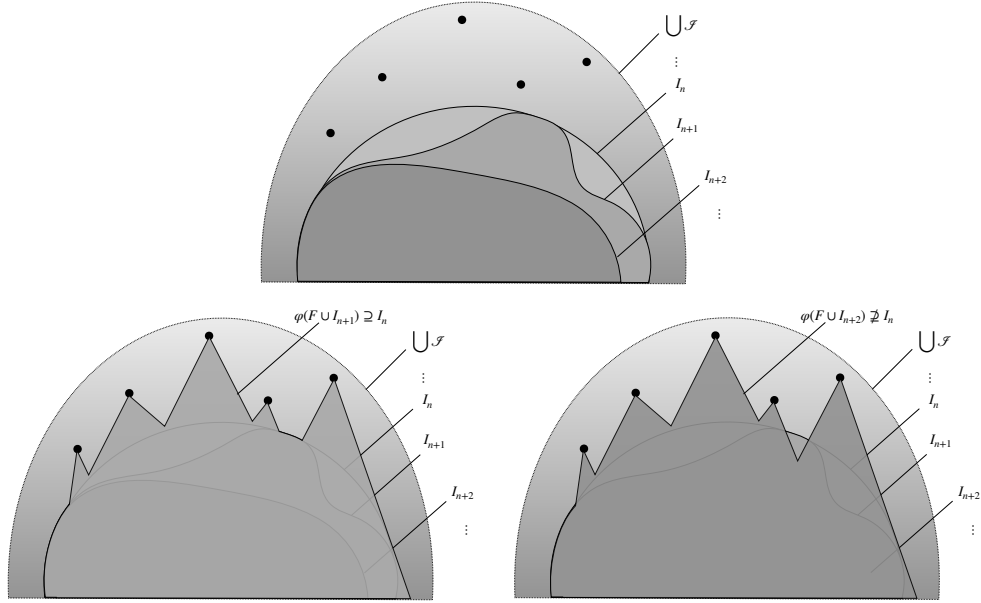


Figure 1: A separating chain

**Lemma 1.** *A closure system  $(E, \varphi)$  contains an infinite independent set if and only if  $E$  contains a subset  $E'$  that contains an infinite separating chain of closed sets with respect to the induced closure.*

PROOF. Any infinite independent set contains a countably infinite independent subset, so we may as well assume a given infinite independent set  $X$  of the form  $\{x_n : n < \omega\}$ , where  $x_m \neq x_n$  for all  $m \neq n$ . Set  $E' := X$ . Then the chain  $\mathcal{I} = \{I_n : n < \omega\}$ , where  $I_n := \varphi_{\uparrow E'}(X \setminus \{x_i : i < n\})$ , is separating in  $E'$ . Indeed, first  $\bigcup \mathcal{I} = I_0 = E' = X$ , next every  $I \in \mathcal{I} \setminus \{\bigcup \mathcal{I}\}$  is an  $I_n$  for some  $n \geq 1$ . For every finite set  $F$  of points of  $I_0 \setminus I_n$ , define  $J$  as  $I_{n+1}$ . Since  $x_n$  is different from every  $x_i$ ,  $i < n$ ,  $x_n$  is in  $X \setminus \{x_i : i < n\}$  hence in  $I_n$ . It follows that  $x_n$  is not in  $F$ . It is not in  $J$  either, because  $J \subseteq \varphi(X \setminus \{x_i : i < n+1\}) \subseteq \varphi(X \setminus \{x_n\})$ , and  $X$  is independent. Therefore  $x_n$  is not in  $F \cup J$ . We rewrite that as  $F \cup J \subseteq X \setminus \{x_n\}$ , and conclude that  $\varphi(F \cup J) \subseteq \varphi(X \setminus \{x_n\})$ . Since  $X$  is independent again,  $x_n$  cannot be in  $\varphi(F \cup J)$ . However,  $x_n$  is in  $I_n$ , so  $I_n \not\subseteq \varphi(F \cup J)$ .

Conversely, let  $E'$  be a subset of  $E$  such that the induced closure  $\varphi' := \varphi_{\uparrow E'}$  on  $E'$  contains an infinite separating chain  $\mathcal{I}$  of closed sets. We construct an infinite independent subset for the induced closure  $\varphi'$ , and therefore also

for the closure  $\varphi$ . To this end, we will define inductively an infinite sequence  $x_0, I_0, \dots, x_n, I_n, \dots$  such that  $I_0 \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$ ,  $x_0 \in \cup \mathcal{I} \setminus I_0$  and such that, for every  $n \geq 1$ :

$$(a_n) \quad I_n \in \mathcal{I};$$

$$(b_n) \quad I_n \subset I_{n-1};$$

$$(c_n) \quad x_n \in I_{n-1} \setminus \varphi'(\{x_0, \dots, x_{n-1}\} \cup I_n).$$

Since  $\mathcal{I}$  is infinite,  $\mathcal{I} \setminus \{\cup \mathcal{I}\} \neq \emptyset$ . We choose  $I_0 \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$  and  $x_0 \in \cup \mathcal{I} \setminus I_0$  arbitrarily. Now let  $n \geq 1$ . Let us assume that  $x_k, I_k$  are defined and satisfy  $(a_k)$ ,  $(b_k)$ ,  $(c_k)$  for every  $k \leq n-1$ , and let us define  $I := I_{n-1}$  and  $F := \{x_0, \dots, x_{n-1}\}$ . Since  $I \in \mathcal{I}$  and  $F$  is a finite subset of  $\cup \mathcal{I} \setminus I$ , there is some  $J \in \mathcal{I}$  such that  $I \not\subseteq \varphi'(F \cup J)$ . The set  $J$  is a proper subset of  $I$ : otherwise, since  $\mathcal{I}$  is a chain, we would have  $I \subseteq J \subseteq F \cup J \subseteq \varphi'(F \cup J)$ . We pick  $z \in I \setminus \varphi'(F \cup J)$ , and we set  $x_n := z$ ,  $I_n := J$ .

It remains to check that the set  $X := \{x_n : n < \omega\}$  is independent. For every  $x \in X$ , say  $x = x_n$ , we know that  $x_n$  is in  $I_{n-1}$  and not in the closed set  $C := \varphi'(\{x_0, \dots, x_{n-1}\} \cup I_n)$ , by  $(c_n)$ . The set  $C$  contains  $x_0, \dots, x_{n-1}$ . For every  $k > n+1$ ,  $I_{k-1} \subset I_n$  by  $(b_{k-1}), \dots, (b_{n+1})$ , and  $x_k \in I_{k-1}$  by  $(c_k)$ , so  $I_n$  contains every  $x_k$  with  $k > n+1$ . It also contains  $x_{n+1}$ , and since  $I_n \subseteq C$ ,  $C$  contains every  $x_k$  with  $k \geq n+1$ . It follows that  $C$  contains every element of  $X$  except  $x = x_n$ . Therefore  $C \supseteq \varphi'(X \setminus \{x\})$ , from which we conclude that  $x$ , not being in  $C$ , is not in  $\varphi'(X \setminus \{x\})$  either.  $\square$

**Remark 2.** Assuming  $\varphi$  topological, we can take  $E' := E$  in Lemma 1. Indeed, let  $X$  be an infinite independent subset of the form  $\{x_n : n < \omega\}$ , where the points  $x_n$  are pairwise distinct. Let  $I_n := \varphi(X \setminus \{x_i : i < n\})$ . The chain  $\mathcal{I} = \{I_n : n < \omega\}$  is separating, as we now check. First  $\cup \mathcal{I} = I_0 = \varphi(X)$ . Every  $I \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$  is an  $I_n$  for some  $n \geq 1$ . For every finite set  $F$  of points of  $\varphi(X) \setminus I_n$ , we define  $J$  as  $I_{n+1}$  and we check that  $I_n \not\subseteq \varphi(F \cup J)$  by showing that  $x_n$ , which is in  $I_n$ , is not in  $\varphi(F \cup I_{n+1})$ . Since  $\varphi$  is topological, it suffices to show that  $x_n$  is neither in  $\varphi(F)$  nor in  $\varphi(I_{n+1}) = I_{n+1}$ . The latter—that  $x_n$  is not in  $I_{n+1}$ —is clear. As for the former, we note that  $I_n \cup \varphi(X \setminus \{x_n\}) = \varphi((X \setminus \{x_i : i < n\}) \cup (X \setminus \{x_n\})) = \varphi(X)$ , using the fact that  $\varphi$  is topological. That implies  $\varphi(X) \setminus I_n \subseteq \varphi(X \setminus \{x_n\})$ , so  $F \subseteq \varphi(X \setminus \{x_n\})$ , whence  $\varphi(F) \subseteq \varphi(X \setminus \{x_n\})$ . If  $x_n$  were in  $\varphi(F)$ , it would then be in  $\varphi(X \setminus \{x_n\})$ , and that is impossible since  $X$  is independent.

**Remark 3.** In general, namely if  $\varphi$  is not topological, we cannot take  $E := E'$ .

Let us consider  $E := \mathbb{P}(\mathbb{N})$ , and let us define  $\varphi$  by  $\varphi(\mathcal{A}) := \mathbb{P}(\bigcup \mathcal{A})$  for every  $\mathcal{A} \in \mathbb{P}(E)$ . The closed subsets of  $E$  are exactly the sets of the form  $\mathbb{P}(A)$  with  $A \subseteq \mathbb{N}$ . There is an infinite independent set, say  $X := \{\{n\} \mid n \in \mathbb{N}\}$ . However, no subset of  $E$  has any separating chain of closed sets  $\mathcal{I}$ . Indeed, let us assume that one existed, and let us pick any  $I \in \mathcal{I} \setminus \{\bigcup \mathcal{I}\}$ . Since  $I$  is closed,  $I = \mathbb{P}(A)$  for some  $A \subseteq \mathbb{N}$ . We take  $F := \{A\}$ : for every  $J \in \mathcal{I}$ ,  $\varphi(F \cup J) \supseteq \varphi(F) = \mathbb{P}(A) = I$ , so  $\mathcal{I}$  cannot be separating.

In a quasi-ordered set  $E$ , for every  $A \subseteq E$ , we write  $\downarrow A$  for  $\{x \in E : \exists y \in A, x \leq y\}$ . A subset  $A$  is *cofinal* in  $E$  if and only if  $E = \downarrow A$  [6, Chapter 2, 5.1, p.44]. An *initial segment* is a subset  $I$  of  $E$  such that  $I = \downarrow I$ . The *finitely generated* initial segments are the sets of the form  $\downarrow A$ ,  $A$  finite. We write  $\downarrow x$  for  $\downarrow \{x\}$ .

Every closure system  $E$  is quasi-ordered by  $x \leq y$  if and only if  $x \in \varphi(\{y\})$ . For every  $x \in X$ ,  $\downarrow x = \varphi(\{x\})$ . Every closed set is an initial segment. If  $\varphi$  is topological, then every finitely generated initial segment  $\downarrow A$  is equal to  $\varphi(A)$ , hence is closed.

We say that a quasi-ordered set is *well-founded* if and only if it has no infinite strictly descending sequence  $x_0 > x_1 > \dots > x_n > \dots$ , where  $x < y$  if and only if  $x \leq y$  and  $y \not\leq x$ . That extends the same notion on posets.

We will use the following lemma in order to prove the implication (i)  $\Rightarrow$  (iii).

**Lemma 2.** *If  $(E, \varphi)$  is a closure system then  $E$  contains a generating subset  $D$  such that the collection of finitely generated initial segments  $\mathbf{I}_{<\omega}(D)$  of the quasi-ordered set  $(D, \leq_{\downarrow D})$  is well-founded under inclusion.*

PROOF. According to a result of Birkhoff, the poset  $\mathbf{I}_{<\omega}(P)$  of finitely generated initial segments of a poset  $P$  is well-founded if  $P$  is well-founded [1, Theorem 2, p.182]. This property holds for initial segments of a quasi-ordered set too, since initial segments of a quasi-ordered set are inverse images of initial segments of the order quotient. For the reader's convenience, we give a proof of Birkhoff's result. Let  $\downarrow A_1 \supset \downarrow A_2 \supset \dots \supset \downarrow A_n \supset$  be an infinite descending sequence where each  $A_i$  is finite. We may assume that each  $A_i$  is an antichain. We construct a tree  $T$  whose vertices are finite chains  $\{x_1, x_2, \dots, x_n\}$  where each  $x_i \in A_i$  and  $x_1 \geq x_2 \geq \dots \geq x_n$ . The cardinality of such a set is  $n$ , or some

lower number (if some element appears several times), and is the *depth* of the vertex in  $T$ . The unique parent of a depth  $n$  set,  $n \geq 1$ , is obtained by removing its least element ( $x_n$ , but also  $x_{n-1}$  if that happens to be equal to  $x_n$ , and so on). The empty chain is the root. Every element of each  $A_n$  is the least element of at least one such chain, of depth at most  $n$ . Hence every set  $\downarrow A_n$  appears as the initial segment generated by a finite set of vertices of  $T$ . Since there are infinitely many sets  $\downarrow A_n$ , the tree  $T$  is infinite. Since  $T$  is finitely branching, by König's Lemma [12] it has an infinite branch, and that is an infinite descending sequence of elements: contradiction.

Thus, in order to prove the lemma, it suffices to prove that  $E$  contains a generating subset on which the quasi-order  $\leq$  is well-founded.

A result due to Hausdorff (see [6, Chapter 2, p.57]) states that every poset contains a well-founded cofinal subset. That is also valid for quasi-ordered sets such as  $E$ . Indeed, let us consider the family  $\mathcal{W}$  of all well-founded subsets of the set  $E$ , and let us order it by prefix:  $A \sqsubseteq B$  if and only if  $B \cap \downarrow A = A$ . By Zorn's Lemma, it has a maximal element  $A$ . If  $A$  were not cofinal, there would be a point  $x$  that is not in  $\downarrow A$ . Then  $B := A \cup \{x\}$  would be a strictly larger well-founded subset of  $E$ , contradicting maximality.

Hence let  $D$  be a well-founded cofinal subset of  $E$ . For every  $x \in E$ , there is a point  $y \in D$  such that  $x \leq y$ . In other words,  $x$  is in  $\varphi(\{y\})$ , hence in the larger set  $\varphi(D)$ . Therefore  $\varphi(D) = E$  and  $D$  is generating.  $\square$

*Proof of the implication (i)  $\Rightarrow$  (iii).*

Let  $\varphi$  be the closure associated with the collection of closed sets  $\mathcal{F}$ , and let us remember that it is topological. Let  $C$  be a closed set. We define a dense subset of  $C$  on which the induced topology is Noetherian. This will be  $D$ , as given by Lemma 2. We note that  $D$  is dense in  $C$ , begin a generating set.

Let  $\varphi_{\uparrow D}$  be the closure induced on  $D$ , namely  $\varphi_{\uparrow D}(X) := \varphi(X) \cap D$  for every  $X \subseteq D$ . This is also a topological closure, and we claim that it is Noetherian.

By (i),  $E$  contains no infinite discrete set, so  $D$  does not contain any infinite discrete set either. Let us imagine that  $D$  contained an infinite strictly descending sequence  $I_0 \supset I_1 \supset \dots \supset I_n \supset \dots$  of closed subsets. By Lemma 1, that chain must fail to be separating: there must be an index  $m_1 \geq 1$  and a finite set  $F_1$  of points of  $I_0 \setminus I_{m_1}$  such that, for every  $n < \omega$  (in particular, for every  $n > m_1$ ),  $I_{m_1} \subseteq \varphi_{\uparrow D}(F_1 \cup I_n) = \varphi_{\uparrow D}(F_1) \cup I_n$ . The last equality is because  $\varphi_{\uparrow D}$  is topological, and  $I_n$  is closed. Hence  $m_1 \geq 1$ ,  $F_1 \subseteq I_0 \setminus I_{m_1}$ ,



and for every  $n > m_1$ ,  $I_{m_1} \setminus I_n \subseteq \varphi_{\uparrow D}(F_1)$ . We do the same with the infinite subsequence starting at  $I_{m_1}$ : there is an index  $m_2 \geq m_1 + 1$  and a finite set  $F_2 \subseteq I_{m_1} \setminus I_{m_2}$  such that for every  $n > m_2$ ,  $I_{m_2} \setminus I_n \subseteq \varphi_{\uparrow D}(F_2)$ . Proceeding this way, we obtain indices  $m_{k+1} \geq m_k + 1$  and finite sets  $F_{k+1} \subseteq I_{m_k} \setminus I_{m_{k+1}}$  such that for every  $n > m_{k+1}$ ,  $I_{m_{k+1}} \setminus I_n \subseteq \varphi_{\uparrow D}(F_{k+1})$ , for every  $k \geq 1$ .

Since  $F_{k+1} \subseteq I_{m_k} \setminus I_{m_{k+1}} \subseteq \varphi_{\uparrow D}(F_k)$ , we have  $\varphi_{\uparrow D}(F_{k+1}) \subseteq \varphi_{\uparrow D}(F_k)$ , for every  $k \geq 1$ . It follows that the sequence  $(\varphi_{\uparrow D}(F_k))_{k \geq 1}$  is descending. Since  $D$  was obtained from Lemma 2, that sequence must be finite. Let us pick  $k \geq 2$  such that  $\varphi_{\uparrow D}(F_k) = \varphi_{\uparrow D}(F_{k+1})$ . The set  $F_k$  cannot be empty, since  $\varphi_{\uparrow D}(F_k)$  contains  $I_{m_k} \setminus I_{m_{k+1}}$ , which is non-empty. We pick  $x \in F_k$ . In particular,  $x$  is in  $I_{m_{k-1}} \setminus I_{m_k}$ , hence is not in  $I_{m_k}$ . However,  $x$  is also in  $\varphi_{\uparrow D}(F_k) = \varphi_{\uparrow D}(F_{k+1})$ , and since  $F_{k+1} \subseteq I_{m_k} \setminus I_{m_{k+1}} \subseteq I_{m_k}$ ,  $x$  is also in  $\varphi_{\uparrow D}(I_{m_k}) = I_{m_k}$ : contradiction.  $\square$

*Proof of the implication (iii)  $\Rightarrow$  (ii).*

Let  $\varphi$  be the closure on  $E$ . Let  $C$  be a closed set and  $D$  be a dense subset of  $C$  on which the closure  $\varphi_{\uparrow D}$  is well-founded.

On  $D$  every closed set  $D'$  is a finite union of irreducible closed sets. This fact goes back to Noether, see [1, Chapter VIII, Corollary, p.181]. Indeed, if  $D$  is not such, then, since the collection of closed sets on  $D$  is well-founded, there is a minimal member  $C'$  which is not a finite union of irreducible members. In particular,  $C'$  is non-empty. If  $C'$  is the union of two proper closed subsets, by minimality those closed subsets must be finite unions of irreducible subsets of  $D$ , hence so must be  $C'$ . It follows that  $C'$  is irreducible: contradiction.

Now  $D$  is itself closed in  $D$ , so we can write  $D$  as a finite union of irreducible closed subsets  $C_i$  of  $D$ ,  $1 \leq i \leq n$ . For each  $C_i$ ,  $\varphi(C_i)$  is irreducible in  $E$ , as one easily checks [9, Lemma 8.4.10]. By density and the fact that  $\varphi$  is topological,  $C = \varphi(D) = \bigcup_{i=1}^n \varphi(C_i)$ .  $\square$

*Proof of the implication (ii)  $\Rightarrow$  (i).*

Let  $\varphi$  be the closure on  $E$  again, and let  $X$  be a discrete subspace. We write  $\varphi(X)$  as a finite union of irreducible closed sets  $I_1, \dots, I_n$ .

For each  $x \in X$ ,  $x$  is in some  $I_k$ . We claim that  $I_k = \varphi(\{x\})$ . To this end, we note that  $I_k \subseteq \varphi(X) = \varphi(\{x\}) \cup \varphi(X \setminus \{x\})$ , since  $\varphi$  is topological. Therefore  $I_k$  is equal to the union of the two closed sets  $\varphi(\{x\}) \cap I_k$  and  $\varphi(X \setminus \{x\}) \cap I_k$ . Since  $X$  is discrete, hence independent,  $x$  is not in  $\varphi(X \setminus \{x\})$ , and since  $x \in I_k$ ,  $\varphi(X \setminus \{x\}) \cap I_k$  is a proper closed subset of  $I_k$ . Because

$I_k$  is irreducible,  $\varphi(\{x\}) \cap I_k$  cannot be a proper closed subset of  $I_k$ , so  $\varphi(\{x\}) \cap I_k = I_k$ . This means that  $I_k \subseteq \varphi(\{x\})$ , and the converse inclusion follows from  $x \in I_k$ .

It follows that for any two distinct points  $x, y \in X$ ,  $x$  and  $y$  cannot be in the same  $I_k$ . Otherwise  $\varphi(\{x\}) = \varphi(\{y\})$ , but since  $X$  is independent,  $x$  is not in  $\varphi(X \setminus \{x\})$ , hence not in the smaller set  $\varphi(\{y\})$ . That is impossible since  $\varphi(\{x\}) = \varphi(\{y\})$  contains  $x$ .

Since each  $I_k$  can contain at most one point from  $X$ ,  $X$  is finite.  $\square$

### 3. Remarks and comments

#### 3.1. Other characterizations

A.H. Stone [14, Theorem 2] shows that (i) is equivalent to two further properties: (iv) every open cover of every subspace  $X$  of  $T$  has a finite subfamily whose union is dense in  $X$ , and (v) every continuous real-valued function on every subspace of  $T$  is bounded.

He also shows [14, Theorem 3] that, for every topological space  $X$ , the fact that  $X$  is a finite union of irreducible closed subsets—the special case of (ii) where the closed set is the whole of  $X$ —is equivalent to six other conditions, among which: (a)  $X$  is *semi-irreducible*, namely every collection of pairwise disjoint non-empty open sets is finite, and: (b)  $X$  has only finitely many regular open sets. He also observes that the cardinality of such collections must be bounded [14, Theorem 3, (vi)].

In particular, our condition (ii) is equivalent to: (vi) every closed subspace of  $T$  is semi-irreducible, and to: (vii) every closed subspace of  $T$  has only finitely many regular open sets.

We will come back to this in Section 3.5.

#### 3.2. Noetherian topological spaces

Noetherian topological spaces have been studied for their own sake by A.H. Stone [14]. They are an important basic notion in algebraic geometry, since the spectrum of any Noetherian ring in a Noetherian topological space, with the Zariski topology. They have also found applications in verification, the domain of computer science concerned with finding algorithms that prove properties of other computer systems, automatically [8]. One can consult Section 9.7 of [9], which is devoted to Noetherian topological spaces.

### 3.3. A related result on general closure systems

The implication (i)  $\Rightarrow$  (ii) follows from the following result about closure systems, as we will see. We recall that an *up-directed* subset of a poset  $P$  is a non-empty subset  $A$  of  $P$  such that any two elements of  $A$  have an upper bound in  $A$ , and that an *ideal* is an up-directed initial segment. We always order powersets by inclusion.

**Theorem 4.** [7, Theorem 1.2] *If a closure system  $(E, \varphi)$  contains no infinite independent set then: (\*) there are finitely many pairwise disjoint subsets  $A_i$  ( $i \in I$ ) of  $E$  and, for each  $A_i$ , a proper ideal  $N_i$  of  $\mathbb{P}(A_i)$  such that for every  $X \subseteq \bigcup_{i \in I} A_i$ , the set  $X$  generates  $E$  if and only if  $A_i \cap X \notin N_i$ , for each  $i \in I$ .*

As it will become apparent in Proposition 7, this result specialized to topological closures is just implication (i)  $\Rightarrow$  (ii). Decompositions of topological closures were considered in [13], but this consequence was totally missed.

**Remark 5.** Condition (\*) entails that  $\bigcup_{i \in I} A_i$  generates  $E$ .

**Remark 6.** In Theorem 4, and if  $\varphi$  is topological, we may suppose that:

$$A_i \cap \varphi\left(\bigcup_{j \in I \setminus \{i\}} A_j\right) = \emptyset \text{ for each } i \in I. \quad (1)$$

This is Remark 1 of [13]. We repeat the argument. Let  $A_i$  and  $N_i$  satisfy condition (\*) of Theorem 4. If (1) does not hold, we set  $A'_i := A_i \setminus \varphi(\bigcup_{j \in I \setminus \{i\}} A_j)$  and  $N'_i := N_i \cap \mathbb{P}(A'_i)$ . First, we claim that each  $N'_i$  is a proper ideal of  $A'_i$ . To this end, we set  $X := A'_i \cup \bigcup_{j \neq i} A_j$ . Because  $\varphi$  is topological,  $\varphi(X) = \varphi(A'_i) \cup \varphi(\bigcup_{j \neq i} A_j) \supseteq A'_i \cup \varphi(\bigcup_{j \neq i} A_j) \supseteq A_i$ ; also,  $\varphi(X) \supseteq X \supseteq A_j$  for every  $j \neq i$ , so  $\varphi(X) \supseteq \bigcup_{j \in I} A_j$ . Remark 5 then entails that  $X$  generates  $E$ . By (\*),  $A'_i \cap X = A'_i$  cannot be in  $N_i$ ; therefore  $A'_i$  is not in  $N'_i$ , showing that  $N'_i$  is a proper ideal of  $\mathbb{P}(A'_i)$ .

Next, let  $X \subseteq \bigcup_{i \in I} A'_i$ . Let us assume that  $X$  generates  $E$ . Since  $X \subseteq \bigcup_{i \in I} A_i$ , we may use (\*):  $A_i \cap X \notin N_i$ , for each  $i \in I$ . Hence  $A'_i \cap X = A_i \cap X \notin N'_i$ . Conversely, if  $A'_i \cap X \notin N'_i$  for each  $i \in I$ , then  $A'_i \cap X \notin N_i$ , so  $A_i \cap X = A'_i \cap X \notin N_i$ , hence  $X$  generates  $E$  by (\*).

**Proposition 7.** *Let  $\varphi$  be a topological closure operator on a set  $E$ . Then  $E$  is a finite union of irreducible closed sets iff  $E$  has a decomposition satisfying condition (\*) on generating sets of Theorem 4.*

PROOF. The result is a consequence of the following two claims. Each one establishes one direction of the implication.

**Claim 8.** *Let  $A_i, N_i$  ( $i \in I$ ) be a finite decomposition satisfying condition  $(*)$  on generating sets of Theorem 4. According to Remark 6, we may assume that it satisfies Condition (1). Then, for every subset  $Y$  of  $E$ ,  $Y \in N_i$  iff  $Y \subseteq A_i$  and  $\varphi(Y) \not\subseteq A_i$ . In particular,  $X_i := \varphi(A_i)$  is irreducible.*

PROOF (PROOF OF CLAIM 8.). By Remark 5,  $E = \varphi(\bigcup_{j \in I} A_j)$ . Let us assume that  $Y \in N_i$ . Since  $N_i$  is an initial segment,  $A_i \cap Y$  is in  $N_i$ . For  $X := Y \cup \bigcup_{j \in I \setminus \{i\}} A_j$ ,  $A_i \cap X = A_i \cap Y$  since the sets  $A_j$  are pairwise disjoint, so  $A_i \cap X$  is in  $N_i$ . By  $(*)$ ,  $X$  does not generate  $E$ . If  $A_i \subseteq \varphi(Y)$  then  $E = \varphi(\bigcup_{j \in I} A_j) \subseteq \varphi(\varphi(Y) \cup \bigcup_{j \in I \setminus \{i\}} A_j) = \varphi(Y \cup \bigcup_{j \in I \setminus \{i\}} A_j)$  (since  $\varphi$  is a topological closure operator)  $= \varphi(X)$ , which is impossible. Hence  $A_i \not\subseteq \varphi(Y)$ .

Conversely, if  $\varphi(Y) \not\subseteq A_i$ , then there is a point  $x$  in  $A_i$ —hence not in  $\varphi(\bigcup_{j \in I \setminus \{i\}} A_j)$  by Condition (1)—which is not in  $\varphi(Y)$ , hence not in  $\varphi(Y) \cup \varphi(\bigcup_{j \in I \setminus \{i\}} A_j)$ . The latter is equal to  $\varphi(X)$ , where  $X := Y \cup \bigcup_{j \in I \setminus \{i\}} A_j$ , since  $\varphi$  is topological, so  $X$  does not generate  $E$ . Using  $(*)$ ,  $A_j \cap X$  is in  $N_j$  for some  $j \in I$ . If  $j \neq i$ , then  $A_j \cap X \supseteq A_j$  by definition of  $X$ , and  $A_j \cap X \in N_j$  implies  $A_j \in N_j$ , contradicting the fact that  $N_j$  is proper. Therefore  $j = i$ . This means that  $A_i \cap X$ , which is equal to  $A_i \cap Y$  since the sets  $A_j$  are pairwise disjoint, hence to  $Y$  since  $Y \subseteq A_i$ , is in  $N_i$ .

We finally show that  $X_i = \varphi(A_i)$  is irreducible. Since  $N_i$  is proper,  $A_i$  is non-empty, hence  $X_i$  is non-empty. Let us assume that  $X_i$  is the union of two proper closed subsets  $C_1$  and  $C_2$ . We consider  $Y := C_1 \cap A_i$  (resp.,  $Y := C_2 \cap A_i$ ). Then  $Y \subseteq A_i$  and  $\varphi(Y) \subseteq C_1$  (resp.,  $C_2$ ); in particular,  $\varphi(Y)$  cannot contain  $X_i = \varphi(A_i)$ , hence cannot contain  $A_i$ . By the first part of the claim,  $Y$  is in  $N_i$ . In other words, both  $C_1 \cap A_i$  and  $C_2 \cap A_i$  are in  $N_i$ . Since  $N_i$  is an ideal,  $(C_1 \cap A_i) \cup (C_2 \cap A_i) = (C_1 \cup C_2) \cap A_i = X_i \cap A_i = A_i$  is in  $N_i$ , which is impossible since  $N_i$  is proper.  $\square$

**Claim 9.** *If  $E$  is a finite union of irreducible closed sets, let  $(X_i)_{i \in I}$  be a family of such sets with  $|I|$  minimum. Set  $A_i := X_i \setminus \bigcup_{j \neq i} X_j$  and  $N_i := \{A' \in A_i : \varphi(A') \not\subseteq X_i\}$ . This decomposition satisfies Condition  $(*)$  of Theorem 4.*

PROOF (PROOF OF CLAIM 9.). We check that  $N_i$  is an ideal. Given  $A', B' \in N_i$ ,  $\varphi(A' \cup B') = \varphi(A') \cup \varphi(B')$ , since  $\varphi$  is topological. If that were equal to the whole of  $X_i$ , and since  $\varphi(A')$  and  $\varphi(B')$  are both proper closed subsets

of  $X_i$ ,  $X_i$  would fail to be irreducible. Hence  $\varphi(A' \cup B') \neq X_i$ , so that  $A' \cup B'$  is in  $N_i$ .

Then we check that  $N_i$  is proper, namely that  $A_i$  is not in  $N_i$ . By definition of  $A_i$ ,  $X_i \subseteq A_i \cup (\bigcup_{j \neq i} X_j) \subseteq \varphi(A_i) \cup (\bigcup_{j \neq i} X_j)$ , so  $X_i = (X_i \cap \varphi(A_i)) \cup (X_i \cap (\bigcup_{j \neq i} X_j)) = \varphi(A_i) \cup (X_i \cap (\bigcup_{j \neq i} X_j))$ , a union of two closed subsets. The second one,  $X_i \cap (\bigcup_{j \neq i} X_j)$ , is a proper subset of  $X_i$  since we have chosen a family of least cardinality. Since  $X_i$  is irreducible, the other one cannot be a proper subset. Therefore  $\varphi(A_i) = X_i$ . It follows that  $A_i$  is not in  $N_i$ .

Finally, let  $X \subseteq \bigcup_{i \in I} A_i$ .

If  $A_i \cap X$  belongs to  $N_i$  for no  $i \in I$ , then by definition of  $N_i$ ,  $\varphi(A_i \cap X) = X_i$ , hence  $\varphi(X) = \varphi(\bigcup_{i \in I} A_i \cap X) = \bigcup_{i \in I} \varphi(A_i \cap X) = \bigcup_{i \in I} X_i = E$ .

Conversely, let us assume that  $\varphi(X) = E$ . Since  $X \subseteq \bigcup_{j \in I} A_j$ ,  $X = \bigcup_{j \in I} A_j \cap X$ , and since  $\varphi$  is topological,  $E = \varphi(X) = \bigcup_{j \in I} \varphi(A_j \cap X)$ . We recall that  $E = \bigcup_{i \in I} X_i$ , so for every  $i \in I$ ,  $X_i \subseteq \bigcup_{j \in I} \varphi(A_j \cap X)$ , and since  $X_i$  is irreducible, there is a  $j \in I$  such that  $X_i \subseteq \varphi(A_j \cap X)$ . If  $j \neq i$ , then  $X_i \subseteq \varphi(A_j) \subseteq \varphi(X_j) = X_j$ , which is impossible since we have chosen  $(X_i)_{i \in I}$  of least cardinality. Hence  $j = i$ , meaning that for every  $i \in I$ ,  $X_i \subseteq \varphi(A_i \cap X)$ . Since  $\varphi(A_i \cap X) \subseteq \varphi(A_i) \subseteq \varphi(X_i) = X_i$ ,  $X_i = \varphi(A_i \cap X)$ , and that shows that  $A_i \cap X$  is not in  $N_i$ .  $\square$

### 3.4. Topological properties versus lattice properties

Item (ii) in Theorem 1 is a property about the lattice of closed sets of a topological space: if two topological spaces have isomorphic lattices of closed sets, then they both satisfy (ii) or neither one satisfies it. A property of a space that only depends on the isomorphism class of its lattice of closed sets is called a *lattice property*. Hence (ii) is a lattice property.

Item (i), too, is a lattice property. Indeed, as it is well-known, the existence of an infinite discrete subspace (or more generally of an infinite independent subset for a closure system) amounts to the existence of an embedding of  $\mathbb{P}(\mathbb{N})$ , the collection of subsets of  $\mathbb{N}$  ordered by inclusion, into the lattice of closed sets.

It not clear that item (iii) is a lattice property without having a proof of Theorem 1. In order to see why, let us consider the *sobrification*  $X^s$  of a topological space  $X$ , see Section 8.2.3 of [9] for example. This is a construction that naturally occurs through the contravariant duality between topological spaces and frames. To say it briefly,  $X^s$  is the free sober space over  $X$  [9, Theorem 8.2.44]. A *sober space* is a  $T_0$  space in which the irreducible closed subsets are the closures of single points. The sobrification  $X^s$  can be

obtained as the collection of irreducible closed subsets of  $X$ , with the topology whose open sets are (exactly) the sets  $\diamond U$  defined as  $\{C \text{ irreducible closed} \mid C \cap U \neq \emptyset\}$ , where  $U$  ranges over the open subsets of  $X$ . The map  $U \mapsto \diamond U$  is then an order-isomorphism [9, Lemma 8.2.26]. It follows that a lattice property cannot distinguish  $X$  from its sobrification  $X^s$ . And density, as used in the statement of item (iii), is not a lattice property: any dense subset of  $\mathbb{N}$  with the cofinite topology (whose closed sets are the finite subsets of  $\mathbb{N}$  plus  $\mathbb{N}$  itself) must be infinite, but  $\mathbb{N}^s$ , which is homeomorphic to the space obtained by adding a new point  $\infty$  to  $\mathbb{N}$ , and whose closed sets are the finite subsets of  $\mathbb{N}$  plus  $\mathbb{N}^s$ , has a one-point dense subset,  $\{\infty\}$ , but an isomorphic lattice of closed sets.

### 3.5. The case of posets

Theorem 1 has a well-known predecessor in the theory of posets. It is worth to recall it.

Let  $P$  be a poset, and  $A$  be a subset of  $P$ . An *upper bound* of  $A$  is any  $z \in P$  such that  $x \leq z$  for every  $x \in A$ . Two elements are *compatible* if they have a common upper bound, and *incompatible* otherwise. The set  $A$  is *up-independent* if all its elements are pairwise incompatible; it is *consistent* if all its elements are pairwise compatible.

The *final segments* of  $P$  are the initial segments of  $P^d$ , the opposite order; we denote by  $\uparrow A$ , resp.  $\uparrow a$ , the final segment of  $P$  generated by  $A \subseteq P$ , resp.  $a \in P$ .

The set  $\mathbf{I}(P)$  of initial segments of  $P$  is the set of closed sets of a topology, the Alexandroff topology. In this setting, a subset  $A$  is discrete if and only if it is an antichain,  $A$  is dense if and only if it is cofinal, and  $A$  is irreducible if and only if it is an ideal.

A poset  $P$  is *well-quasi-ordered* (w.q.o. for short) if it is well-founded and contains no infinite antichain. According to Higman [11],  $P$  is w.q.o. iff  $\mathbf{I}(P)$  is well-founded.

We recall the following result (see [6, Chapter 4]):

**Theorem 10.** *The following properties are equivalent for a poset  $P$ :*

- (a)  $P$  contains no infinite antichain;
- (b) every initial segment of  $P$  is a finite union of ideals;
- (c) every initial segment of  $P$  contains a cofinal subset which is well-quasi-ordered.

PROOF. This is just Theorem 1 applied to  $P$  with the Alexandroff topology, provided one notes that a poset is well-quasi-ordered if and only if it is Noetherian in its Alexandroff topology. But the proof simplifies.

(a)  $\Rightarrow$  (c). Let  $P'$  be an initial segment. By an already cited result of Hausdorff,  $P'$  contains a well-founded cofinal subset  $A$ . Since  $P$  has no infinite antichain,  $P'$  has no infinite antichain; being well-founded it is w.q.o.

(c)  $\Rightarrow$  (b). Let  $P'$  be an initial segment and  $A$  be a cofinal subset of  $P'$  which is w.q.o. Being w.q.o.,  $A$  is a finite union of ideals  $I_1, \dots, I_q$ . This is a basic result of the theory of w.q.o. [1, Chapter VIII, Corollary, p.181]. Indeed, as in the proof of implication (iii)  $\Rightarrow$  (ii) of Theorem 1, replacing “closed” by “initial segment” and “irreducible closed” by “ideal”, if  $A$  is not such, then, since  $\mathbf{I}(A)$  is well-founded, there is a minimal member  $A' \in \mathbf{I}(A)$  which is not a finite union of ideals. This  $A'$  is irreducible, hence is an ideal: contradiction.

Now  $P' = \downarrow A = \downarrow I_1 \cup \dots \cup \downarrow I_q$  and the set  $\downarrow I_i$  are ideals of  $P'$ .

(b)  $\Rightarrow$  (a). Let  $A$  be an antichain of  $P$ . An ideal  $I$  of  $\downarrow A$  cannot contain more than one element of  $A$ . Since  $\downarrow A$  is a finite union of ideals,  $A$  is finite. □

A direct proof of (a)  $\Rightarrow$  (b) can be obtained from a special case of a result of Erdős and Tarski [5]. This special case is a prototypical min-max result which has been overlooked, and which states the following:

**Remark 11 (Erdős-Tarski).** If a poset  $P$  contains no infinite up-independent set then there is a finite upper bound on the cardinality of up-independent sets. In this case, the maximum cardinality of up-independent sets, the least number of ideals whose union is  $P$  and the least number of consistent sets whose union is  $P$  are equal.

We will dispense with the proof, as it will be a consequence of Proposition 13 below. Now, assuming (a), we can prove (b) as follows. Let  $I$  be an initial segment of  $P$ . Since by (a) there is no infinite antichain in  $P$ ,  $I$  does not contain any infinite antichain either. In the subposet  $I$ , the up-independent subsets are all finite, since every up-independent subset is an antichain. Their maximum cardinality is then the least number of ideals whose union is  $I$ , by Remark 11.

Just as Theorem 1 generalizes Theorem 10, we will generalize Remark 11 to the topological setting in Proposition 13.

In the topological setting, we replace up-independence with *relative Hausdorffness*, defined as follows. A subset  $L$  of a topological space  $T := (X, \mathcal{F})$

is *relatively Hausdorff* if and only if for every two distinct points  $x$  and  $y$  of  $L$ , we can find two disjoint open neighborhoods  $U$  of  $x$  and  $V$  of  $y$  in  $X$  (not in the subspace  $L$  itself, whence “relative”). In particular,  $X$  is relatively Hausdorff in itself if and only if it is Hausdorff. Given a poset  $P$  in its Alexandroff topology, every point  $x$  has a smallest open neighborhood, which is  $\uparrow x$ . Then  $L \subseteq P$  is relatively Hausdorff if and only if for every two distinct points  $x$  and  $y$  of  $L$ ,  $\uparrow x$  and  $\uparrow y$  are disjoint in  $P$ , if and only if  $L$  is up-independent.

**Remark 12.** Every relatively Hausdorff subset  $L$  of a topological space  $X$  is Hausdorff in the subspace topology. However, the reverse implication fails in general. In order to see this, let us consider any space  $X^\top$  obtained by adding an element  $\top$  to a Hausdorff space  $X$  and requiring that the non-empty open subsets of  $X^\top$  are those of the form  $U \cup \{\top\}$ , where  $U$  ranges over the open subsets of  $X$ . Then  $X$  is Hausdorff in the subspace topology from  $X^\top$ , but is not relatively Hausdorff in  $X^\top$  unless  $X$  has at most one element.

We also replace consistency by hyperconnectedness. A *hyperconnected* space is a non-empty space in which any two non-empty open sets intersect. In a poset  $P$  with the Alexandroff topology, a subspace  $Q$  is hyperconnected in the induced topology if and only if the smallest open neighborhoods  $\uparrow x$  and  $\uparrow y$  of any two points  $x$  and  $y$  intersect, if and only if  $Q$  is consistent. Hence the following has Remark 11 as a special case.

**Proposition 13 (Min-max).** *In a topological space  $T := (X, \mathcal{F})$  with no infinite relatively Hausdorff subset, there is a finite upper bound on the cardinalities of relatively Hausdorff subsets. In this case, the following numbers exist and are equal:*

- (a) *the maximum cardinality of relatively Hausdorff subsets of  $X$ ;*
- (b) *the maximum number of pairwise disjoint non-empty open subsets of  $X$ .*
- (c) *the least number of irreducible closed subsets whose union is  $X$ ;*
- (d) *the least number of hyperconnected subspaces of  $X$  whose union is  $X$ .*

PROOF. We write  $\varphi$  for closure in  $T$ . We proceed by making a series of observations.



*Observation 1.* Given any relatively Hausdorff subset  $Y$  of  $X$ , say of cardinality  $n$ , we can find  $n$  pairwise disjoint non-empty open subsets of  $X$ . In order to see this, let us write  $Y$  as  $\{y_1, \dots, y_n\}$ . Since  $Y$  is relatively Hausdorff, for every pair of indices  $i, j$  with  $1 \leq i < j \leq n$ , there are disjoint open sets  $U_{ij}$  and  $V_{ij}$  such that  $y_i \in U_{ij}$  and  $y_j \in V_{ij}$ . For each  $y := y_i$  in  $Y$ , let  $U_y := \bigcap_{j>i} U_{ij} \cap \bigcap_{j<i} V_{ji}$ . Then  $y \in U_y$ , and the sets  $U_y$  are pairwise disjoint.

*Observation 2.* Given  $n$  pairwise disjoint non-empty open subsets  $U_1, \dots, U_n$  of  $X$ , there is a relatively Hausdorff subset  $Y$  of  $X$  of cardinality  $n$ : we simply pick one point from each  $U_i$ .

*Observation 3.* Every irreducible closed subset of  $X$  is hyperconnected. Indeed, let  $C$  be irreducible closed in  $X$ . We consider two non-empty open subsets of  $C$ , necessarily of the form  $U \cap C$  and  $V \cap C$ , where  $U$  and  $V$  are open in  $X$ . Then  $C \setminus U$  and  $C \setminus V$  are proper closed subsets of  $C$ , and since  $C$  is irreducible,  $C \neq (C \setminus U) \cup (C \setminus V)$ ; in other words,  $U \cap V \neq \emptyset$ .

*Observation 4.* The closure (in  $T$ ) of every hyperconnected subspace is irreducible closed. Indeed, let  $U$  be a hyperconnected subspace of  $T$ , and let us assume that we can write  $\varphi(U)$  as the union of two proper closed subsets  $C_0$  and  $C_1$ . Since  $C_0$  and  $C_1$  are proper,  $V_0 := X \setminus C_0$  and  $V_1 := X \setminus C_1$  both intersect  $\varphi(U)$ . Hence they both intersect  $U$ . It follows that  $U \cap V_0$  and  $U \cap V_1$  are non-empty open subsets of  $U$ , and since  $U$  is hyperconnected, they must intersect. But  $(U \cap V_0) \cap (U \cap V_1) = U \setminus (C_0 \cup C_1) \subseteq \varphi(U) \setminus (C_0 \cup C_1) = \emptyset$ , which is a contradiction; so  $\varphi(U)$  is irreducible.

*Observation 5.* Every non-empty subset  $U$  of  $X$  contains a hyperconnected open subset. Otherwise,  $U$  is bad, where we call *bad* any non-empty open subset of  $X$  with no hyperconnected open subset. We form an infinite binary tree  $\mathcal{T}$  as follows, whose vertices  $s$  are all labeled by bad sets  $V_s$ . (Here  $s$  ranges over the finite 0-1 strings, and we write  $\epsilon$  for the empty string,  $s0$  and  $s1$  for the string  $s$  with 0, resp. 1, appended to its end.) Its root is  $V_\epsilon := U$ . Given any vertex  $s$  labeled with a bad set  $V_s$ , since  $V_s$  itself is not hyperconnected, there are two non-empty open subsets  $V_{s0}$  and  $V_{s1}$  of  $V$  whose intersection is empty. The sets  $V_{s0}$  and  $V_{s1}$  are themselves bad, since they are included in the bad set  $V_s$ . We label the two successors  $s0$  and  $s1$  of  $s$  by  $V_{s0}$  and by  $V_{s1}$  respectively. Now that  $\mathcal{T}$  has been built, the collection  $\{V_{0^n 1} : n \in \mathbb{N}\}$  consists of infinitely many pairwise disjoint non-empty open sets. We pick  $x_n \in V_{0^n 1}$  for each  $n \in \mathbb{N}$ , and then  $\{x_n : n \in \mathbb{N}\}$  is an infinite relatively Hausdorff subset of  $X$ , since the open sets  $V_{0^n 1}$  are pairwise disjoint; and this is impossible. This shows that  $U$  cannot be bad, proving the claim.

*Observation 6.* Let us call *sieve* any collection  $S$  of pairwise disjoint hy-

perconnected open sets. We note that every sieve is finite, by Observation 2. Every sieve  $S$  is included in a maximal sieve  $S_{\max}$ , with respect to inclusion, by Zorn's Lemma, and if  $S_{\max}$  is any maximal sieve, say of cardinality  $n$ , then we can find a minimal cover of  $X$  by irreducible closed sets, of cardinality  $n$ .

This is proved as follows. We claim that  $\bigcup_{U \in S_{\max}} U$  is dense in  $X$ . Otherwise  $X \setminus \varphi(\bigcup_{U \in S_{\max}} U)$  would be a non-empty open set, hence it would contain a hyperconnected open set  $V$  (by Observation 5), and then  $S_{\max} \cup \{V\}$  would be a strictly larger sieve, contradicting maximality.

For every  $U \in S_{\max}$ ,  $U$  is hyperconnected, so  $\varphi(U)$  is irreducible closed, by Observation 4. Since  $\bigcup_{U \in S_{\max}} U$  is dense in  $X$  and since  $\varphi$  is topological,  $X$  is covered by the  $n$  irreducible closed sets  $\varphi(U)$ ,  $U \in S_{\max}$ .

The cover  $\{\varphi(U) \mid U \in S_{\max}\}$  is minimal, namely removing any element would fail to produce a cover. Indeed, let  $U$  be any element of  $S_{\max}$ , and let us pick a point  $x$  from  $U$ . Then  $x \notin \bigcup_{V \in S_{\max} \setminus \{U\}} \varphi(V)$ , since otherwise  $U$  would intersect  $\varphi(V)$  for some  $V \in S_{\max} \setminus \{U\}$ , hence also  $V$  itself.

*Observation 7.* Any two minimal finite covers  $C_1, \dots, C_m$  and  $C'_1, \dots, C'_n$  of  $X$  by irreducible closed subsets have the same cardinality.

For each  $i \in \{1, \dots, m\}$ ,  $C_i \subseteq X = \bigcup_{j=1}^n C'_j$ . Since  $C_i$  is irreducible, it is easy to see that  $C_i \subseteq C'_j$  for some  $j \in \{1, \dots, n\}$ . We pick one such  $j$  and call it  $f(i)$ ; therefore  $f$  is a map from  $\{1, \dots, m\}$  to  $\{1, \dots, n\}$  such that  $C_i \subseteq C'_{f(i)}$  for every  $i \in \{1, \dots, m\}$ . Similarly, there is a map  $g: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$  such that  $C'_j \subseteq C_{g(j)}$  for every  $j \in \{1, \dots, n\}$ . For every  $i$ , we have  $C_i \subseteq C'_{f(i)} \subseteq C_{g(f(i))}$ . If  $i \neq g(f(i))$ , then removing  $C_i$  from the list  $C_1, \dots, C_m$  would still produce a cover of  $X$  by irreducible closed subsets, contradicting minimality; so  $i = g(f(i))$ , for every  $i \in \{1, \dots, m\}$ . Similarly,  $j = f(g(j))$  for every  $j \in \{1, \dots, n\}$ . Therefore  $f$  and  $g$  are mutually inverse, and  $m = n$ .

*Observation 8.* Given any minimal finite cover of  $X$  by irreducible closed subsets  $C_1, \dots, C_n$ , there is a relatively Hausdorff subset of  $X$  of cardinality  $n$ . Indeed, for every  $i \in \{1, \dots, n\}$ , by minimality  $C_i \not\subseteq \bigcup_{k \neq i} C_k$ , so we can pick a point  $x_i \in C_i \setminus \bigcup_{k \neq i} C_k$ . The subset  $Y := \{x_1, \dots, x_n\}$  is relatively Hausdorff, since for all  $i < j$ ,  $x_i$  and  $x_j$  are separated by the disjoint open sets  $X \setminus \bigcup_{k \neq i} C_k$  and  $X \setminus \bigcup_{k \neq j} C_k$  in  $X$ .

Those observations allow us to prove the proposition as follows. By Observations 1 and 2, the numbers that we can obtain as cardinalities of relatively Hausdorff subsets of  $X$  or as cardinalities of collections of pairwise disjoint non-empty subsets of  $X$  are the same. By Observation 5, those are also the possible cardinalities of sieves, and by Observation 6 they are

bounded from above; let  $n_{\max}$  be their maximum. Observation 6 also tells us that  $n_{\max} \leq n_{\min}$  (and that  $n_{\min} < \infty$ ), where  $n_{\min}$  is the cardinality of some minimal cover of  $X$  by irreducible closed sets. By Observation 7,  $n_{\min}$  is, equivalently, the least number of irreducible closed sets needed to cover  $X$ . Observation 8 tells us that  $n_{\min} \leq n_{\max}$ , so  $n_{\min} = n_{\max}$ .

Finally, if  $n$  hyperconnected sets suffice to cover  $X$ , then their closures are irreducible closed and cover  $X$  by Observation 4, and conversely, if  $n$  irreducible closed sets cover  $X$ , then they are hyperconnected by Observation 3, so  $n_{\min}$  is also the least number of hyperconnected sets covering  $X$ .  $\square$

We have already mentioned that the Erdős-Tarski result (Remark 11) is the special case of the equality of the numbers mentioned in items (a) and (d) of Proposition 13, when  $X$  is a poset  $P$  with its Alexandroff topology. While the Erdős-Tarski result implies the (a)  $\Rightarrow$  (b) direction of Theorem 10, it is not clear that Proposition 13 would entail the (i)  $\Rightarrow$  (ii) direction of Theorem 1. In analogy with the setting of posets, where we had used the fact that every up-independent subset is an antichain, we would need to say that every relatively Hausdorff subset is discrete, but this is clearly wrong. (Consider any non-discrete Hausdorff space.) However, this does hold in FAC spaces, as we now see. Since the proof uses Proposition 2.7 of [10], which relies on Theorem 2.1 there, which itself is the equivalence of (i) and (ii) of Theorem 1, it would be a fallacy to use it to derive (i)  $\Rightarrow$  (ii), though. But we obtain the following, which we offer as our conclusion. (A KC-space is a space in which every compact subset is closed.)

**Theorem 14.** *A topological space  $T := (E, \mathcal{F})$  is a FAC space, namely satisfies the equivalent conditions (i), (ii) or (iii) of Theorem 1, or A.H. Stone's equivalent conditions (iv)–(vii) (see Section 3.1), if and only if it satisfies any of the following equivalent conditions:*

- (viii) *No infinite subspace of  $T$  is both sober and  $T_1$ ;*
- (ix) *No infinite subspace of  $T$  is a KC-space;*
- (x) *No infinite subspace of  $T$  is Hausdorff;*
- (xi) *For no subspace  $X$  of  $T$  is there any infinite subset of  $X$  that is relatively Hausdorff in  $X$ .*

*Then the min-max conditions of Proposition 13 hold on every subspace  $X$  of  $T$ .*

PROOF. Let  $\varphi$  be closure in  $T$ . We claim that if  $T = (E, \mathcal{F})$  is a FAC space, then every subspace  $L$  of  $E$  is a FAC space. Indeed, given any infinite subset  $A$  of  $L$ ,  $A$  is not discrete in  $E$  since  $E$  is FAC, so there is a point  $x$  in  $A$  such that  $x \in \varphi(A \setminus \{x\})$ . Then  $x \in \varphi(A \setminus \{x\}) \cap L = \varphi_{\uparrow L}(A \setminus \{x\})$ , showing that  $A$  is not discrete in  $L$ .

Proposition 2.7 of [10] states that, for a FAC space, it is equivalent to be sober and  $T_1$ , or a KC-space, or Hausdorff, or finite and discrete. Hence, if  $T$  is a FAC space, then (viii)–(x) hold. Since every relatively Hausdorff subset is Hausdorff as a subspace (see Remark 12), the implication (x)  $\Rightarrow$  (xi) follows. The implications (viii)  $\Rightarrow$  (x) and (ix)  $\Rightarrow$  (x) follow from the fact that every Hausdorff space is sober,  $T_1$ , and a KC-space. Finally, if (xi) holds, then Proposition 13 applies, and item (c) of that proposition (applied to the case of a closed subspace  $X$ ) implies item (ii) in Theorem 1, showing that  $T$  is a FAC space.  $\square$

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