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# The effect of close elections on the likelihood of voting paradoxes: Further results in three-candidate elections

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## Abstract

The impact of election closeness on the likelihood of monotonicity paradox has recently been studied by some authors (e.g., [Lepelley et al., 2018](#); [Miller, 2017](#)). It was shown that the frequency of such a paradox significantly increases as elections become more closely contested. This paper aims to analyze how the closeness of an election affects other well-known paradoxes of voting. Based on the Impartial Anonymous Culture (IAC) assumption, our preliminary results show that closeness has also a significant effect on the likelihood of observing the studied voting paradoxes in the class of scoring rules and scoring elimination rules.

**Keywords:** Voting paradoxes · Election closeness · Scoring Rules · Scoring Elimination Rules · Probability.

**JEL classification:** D71, D72.

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# L'impact des élections serrées sur la probabilité de paradoxes de vote : Autres résultats dans le cas des élections à trois candidats

## Résumé

L'impact des élections serrées sur la probabilité de paradoxe de monotonie a été récemment étudié par certains auteurs (ex., [Lepelley et al., 2018](#); [Miller, 2017](#)). Il a été démontré que la fréquence d'un tel paradoxe augmente considérablement à mesure que les élections deviennent plus disputées. Le présent papier vise à analyser l'impact des élections serrées sur d'autres paradoxes de vote bien connus. En se basant sur l'hypothèse de la culture neutre et anonyme (IAC), nos résultats montrent que des élections serrées ont un effet significatif sur la probabilité d'observer les paradoxes de vote étudiés dans la classe des règles de scores et des règles de scores avec éliminations.

**Mots-clés** : Paradoxes de vote · Elections serrées · Règles de scores · Règles de scores avec éliminations · Probabilité.

## 1 Introduction

An election stands as a pivotal moment in the democratic life of any institution, country, or group of people. During this crucial event, voters express their views on various alternatives such as manifestos, platforms, or propositions. They do so through an electoral process that aims to select or designate the individuals or laws that will govern collective affairs. In the best-case scenarios, the election result is a clear victory for a candidate or party. However, this is not always the case. Complications can arise when election results are closely contested. A narrow or marginal vote gap between the contenders can lead to intense disputes that may undermine the electoral process.

According to historical records, the 1960 presidential election between J.F. Kennedy and R. Nixon was the closest in terms of the popular vote since 1916. Kennedy won 49.72% of the vote, while Nixon received 49.55%. It is worth noting that there have been other elections in which a candidate won the electoral college while losing the popular vote, often by a very narrow margin. In 2000, for example, G.W. Bush won 271 electoral votes compared to 266 for Al Gore. The 2020 presidential election can also be taken as an example, with several disputes raised by candidate D. Trump and his supporters. The actions that followed, including the attack on the Capitol, dealt a significant blow to the electoral process and democracy. These historical facts illustrate that when an election does not produce a clear-cut victory for a candidate, it can sometimes lead to undesirable situations.

Recently, [Miller \(2017\)](#) highlighted that close elections can create opportunities for strategic behavior under certain voting rules. In his study on the conditions under which

Instant Runoff Voting<sup>1</sup> could lead to the monotonicity paradox,<sup>2</sup> Miller (2017) conducted numerical simulations and concluded that the frequency of the monotonicity paradox increases as elections become even more closely contested.<sup>3</sup> This finding is further supported by Lepelley et al. (2018) who examined other voting rules in addition to the Plurality Elimination Rule such as the Negative Plurality Elimination Rule and the Borda Elimination Rule. All of these voting rules will be defined later.

In both Miller (2017) and Lepelley et al. (2018), the closeness of an election is quantified by the ratio between the total number of points (score) of the candidate ranked last (i.e., the candidate who receives the lowest total score) and the sum of the scores of all the candidates in contention. A higher value of this ratio indicates a more closely contested election.<sup>4</sup> Diss et al. (2021) concluded that increased election closeness has a negative impact on the Condorcet efficiency<sup>5</sup> of the five voting rules they analyzed, which are the Plurality Rule, the Borda Rule, the Negative Plurality Rule, the Plurality Elimination Rule, and the Negative Plurality Elimination Rule. Their results suggest that Condorcet efficiency tends to decrease as an election becomes more closely contested.

In this paper, we aim to provide further results to show that the effect of close election is not limited to the electoral events mentioned in Diss et al. (2021), Lepelley et al. (2018), and Miller (2017). We focus on three-candidate elections and we consider a wide range of voting events (defined in Section 2.3) and six well-known voting rules (Plurality Rule, Borda Rule, Negative Plurality Rule, Plurality Elimination Rule, Negative Plurality Elimination Rule and the Borda Elimination Rule). For each of the considered voting rules, we derive limiting probability representations for all the voting paradoxes considered, taking into account the factor of closeness. For our analysis, we consider large electorates, as in Diss et al. (2021) and Lepelley et al. (2018), and we assume that voters' preferences are distributed according

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<sup>1</sup>Also known as Alternative Vote or Plurality Elimination Rule.

<sup>2</sup>The monotonicity paradox occurs whenever some voters raise (resp. lower) the ranking of a candidate in their individual preferences, all else unchanged, and the position of this candidate deteriorates (or improves) in the overall social ranking.

<sup>3</sup>In a recent paper, Miller (2023) noted that the incidence of spoiler effects (which may or may not be deemed paradoxical) under Plurality Rule and Plurality Elimination Rule, and perhaps Negative Plurality Elimination (Coombs) Rule, also appears to increase with closeness (as measured by the percent of first preferences for the third-place candidate). Spoiler effects imply that who wins an election depends in an unreasonable way on the slate of candidates appearing on the ballot.

<sup>4</sup>It follows that a close election may accentuate the occurrence of certain electoral phenomena or paradoxes. This suggests that the dynamics and intricacies of the electoral process become more pronounced when the margin of victory is narrow. It should be noted that the elimination rules do not follow the same methodology in Miller (2017) and in Lepelley et al. (2018), particularly in the case of three-candidate elections addressed in both studies by these authors. While in Miller (2017), it is checked in the first round of voting whether a candidate obtains more than half of the votes to consider a second round between the top 2 candidates, Lepelley et al. (2018) dispense with this constraint (of majority of votes) by simply eliminating the candidate with the lowest score. It is in the same approach as Lepelley et al. (2018) that we frame our analysis.

<sup>5</sup>The Condorcet efficiency of a voting rule is its propensity to elect the Condorcet winner when she exists. A Condorcet winner, when she exists, is a candidate who defeats each of the other candidates in pairwise comparisons.

to a version of the well-studied Impartial Anonymous Culture (IAC) hypothesis. This is one of the well-known assumptions under which probability calculations are often driven in the literature of social choice theory. We will say more about this assumption later.

The rest of the paper is organized as follows: In Section 2, we introduce the basic notation and definitions; we also define all the voting rules and voting paradoxes we focus on. In Section 3, we derive from our computations the impact of elections closeness on the likelihood of the voting paradoxes for large electorates. Section 4 concludes. The probability representations obtained from our calculations are presented in Appendix 1, while Appendix 2 provides an example of the proof techniques utilized. The remaining proofs employ the same methodology.

## 2 Definitions and notation

### 2.1 Preferences

Consider a group of  $n \geq 2$  individuals (voters, decision-makers, judges, etc.) and a set of three alternatives or candidates  $A$ ,  $B$ , and  $C$ . The individuals rank the candidates from the most desirable candidate to the least desirable one. Each ranking is then assumed to be a linear order, i.e., a transitive, antisymmetric, and total relation. In three-candidate elections, there are six possible strict rankings, which are displayed in Table 1. In this table, it is indicated that  $n_1$  voters have the ranking  $ABC$ , which means that they rank candidate  $A$  at the top followed by candidate  $B$  and candidate  $C$  is the least preferred. In this framework, a *voting situation* is defined by the vector  $\tilde{n} = (n_1, \dots, n_6)$  which indicates the number of voters endowed with each linear order such that  $\sum_{i=1}^6 n_i = n$ .

Table 1: The six possible strict rankings in three-candidate elections

$n_1 : ABC$	$n_2 : ACB$	$n_3 : BAC$	$n_4 : BCA$	$n_5 : CAB$	$n_6 : CBA$
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Given two candidates  $A$  and  $B$ , we will write  $AMB$  to say that candidate  $A$  is majority preferred over candidate  $B$ , which means that there are more than half of the voters who prefer  $A$  to  $B$ ; we can also say that candidate  $B$  is majority dominated by candidate  $A$ . If a candidate is majority preferred over each of the candidates, she is said to be the *Condorcet winner*. If a candidate is majority dominated by each of the other candidates, she is said to be the *Condorcet loser*. Note that the Condorcet winner (or the Condorcet loser) does not always exist since the majority relation  $\mathbf{M}$  can be *cyclic*; in such a case we can get a first possible cycle where  $AMB$ ,  $BMC$ , and  $CMA$  or a second possible cycle where  $BMA$ ,  $AMC$ , and  $CMB$ .

## 2.2 Voting rules

A *scoring rule* is a voting system that assigns points to candidates based on their position in voters' preferences, thereby generating a social ranking of the candidates according to their scores (the total number of points received), and the candidate with the highest score is declared the winner. In this paper, we focus on three well-known scoring rules (Plurality Rule, Negative Plurality Rule and Borda Rule).

Under the Plurality Rule (henceforth PR), each voter gives one point to the candidate ranked in the first position; the winner is the candidate with the highest total number of first positions in the voters' preferences. Under the Negative Plurality Rule (henceforth NPR), each voter gives one point to every candidate except the candidate ranked last, who receives zero points. The aggregated score of each candidate is the sum of all points received, and the winner is the candidate with the highest aggregated score. In three-candidate elections, the Borda Rule (henceforth BR) considers that a candidate receives two points each time she is ranked first, one point for each second place, and zero point for each last place; the winner is the candidate with the highest total score. Under BR, the winner can also be described as the candidate with the highest average rank on the ballots. For a given voting rule, we simply denote by  $S(A)$  the score of candidate  $A$ . Using the labels from Table 1, the scores of the candidates under each of the three voting rules are given in Table 2.

Table 2: The scores of the candidates according to Table 1

		Voting rules		
Scores		PR	BR	NPR
	$S(A)$	$n_1 + n_2$	$2n_1 + 2n_2 + n_3 + n_5$	$n_1 + n_2 + n_3 + n_5$
	$S(B)$	$n_3 + n_4$	$n_1 + 2n_3 + 2n_4 + n_6$	$n_1 + n_3 + n_4 + n_6$
	$S(C)$	$n_5 + n_6$	$n_2 + n_4 + 2n_5 + 2n_6$	$n_2 + n_4 + n_5 + n_6$

We also focus on three well-known scoring elimination rules (Plurality Elimination Rule, Negative Plurality Elimination Rule and Borda Elimination Rule). In a three-candidate election, a scoring elimination rule operates in two steps or rounds. In the first round, the candidate with the lowest score is eliminated and the two remaining candidates go into a second round where the winner is the one who obtains the majority of votes.<sup>6</sup> Thus, under the Plurality Elimination Rule (henceforth PER), the eliminated candidate is the one with the lowest Plurality score; under the Negative Plurality Elimination Rule (henceforth NPER), the eliminated candidate is the one with the lowest Negative Plurality score; under the Borda Elimination Rule (henceforth BER), the eliminated candidate is the one with the lowest Borda score.

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<sup>6</sup>Notice that we do not assume ties since the probability of a tie vanishes in the limit of large electorates.

## 2.3 Voting paradoxes

A voting paradox is an outcome that can be observed under a given voting rule and that may be considered surprising, counterintuitive, or undesirable. Since the seminal works of [Arrow \(1951\)](#), [Gibbard \(1973\)](#), and [Satterthwaite \(1975\)](#), it is known that there is no reasonable voting rule that is free from any voting paradox. The social choice literature defines and describes a large number of voting paradoxes; for a non exhaustive review of these paradoxes, the reader may refer to the works of [Diss and Merlin \(2021\)](#), [Felsenthal \(2012\)](#), [Felsenthal and Nurmi \(2018\)](#), [Gehrlein and Lepelley \(2011, 2017\)](#), [Nurmi \(1987, 1999\)](#), and [Saari \(1994, 2001, 2008\)](#). In this paper, we deal with the following voting events, which are widely studied in the literature of social choice theory:

**Strong Borda Paradox:** This paradox describes a situation where the Condorcet loser exists and is elected. It is known from [Fishburn and Gehrlein \(1976\)](#) that BR is the only scoring rule that cannot elect the Condorcet loser when one exists.<sup>7</sup>

**Strict Borda Paradox:** This paradox is related to situations where the Condorcet loser is elected and, at the same time, the Condorcet winner is ranked last. [Daunou \(1803\)](#) showed that BR never ranks the Condorcet winner in the last position in an election (see also [Gärdenfors, 1973](#); [Smith, 1973](#)), which means that this paradox never occurs with BR.

**Absolute Majority Winner Paradox:** This paradox occurs when a candidate is not elected despite being ranked first by more than half of the voters. It is obvious that a candidate ranked first by more than half of the voters is the Plurality winner; so, the Absolute Majority Winner Paradox never appears under PR.

**Absolute Majority Loser Paradox:** This paradox is a special case of the Strong Borda Paradox. It occurs when a candidate is elected despite being ranked last by more than half of the voters. Since a candidate ranked last by more than half of the voters is also a Condorcet loser, it follows that BR never exhibits this paradox.

**Reversal Bias Paradox:** This paradox, also known as the *Preference Inversion Paradox*, occurs when the election winner is unchanged after reversing the preferences of all the voters. According to [Saari and Barney \(2003\)](#), BR is the only scoring rule that is not vulnerable to this paradox.

**Sincere Truncation:** This paradox appears if there are some configurations of ballots such that there is at least one voter who prefers the outcome obtained by submitting a

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<sup>7</sup>In contrast, the Weak Borda Paradox occurs when the Condorcet loser exists and is not ranked last by plurality rule.

sincere but incomplete ranking<sup>8</sup> to the outcome obtained by submitting a complete sincere ranking (Brams, 1982; Fishburn and Brams, 1983, 1984). Among the family of scoring rules, only PR is immune to this paradox.

**Positive Abstention Paradox:** This paradox occurs when some of the voters who rank a losing candidate first abstain, and this candidate becomes the winner.

**Negative Abstention Paradox:** This paradox occurs when some of the voters who rank the winning candidate last abstain, and this candidate becomes a loser.

**Positive Participation Paradox:** This paradox occurs when the winning candidate is made a loser by the addition of some voters who rank this candidate first.

**Negative Participation Paradox:** This paradox occurs when some voters who rank a losing candidate last are added, and this candidate becomes the winner.

Table 3 gives us a glimpse of whether or not the above voting rules are vulnerable to the paradoxes just mentioned. In this table, a “yes” indicates that the voting rule is vulnerable to the corresponding paradox and a “no” means the opposite.

Table 3: Voting rules and their (non)vulnerability to the discussed paradoxes

<i>Voting Paradoxes</i>	<i>Voting rules</i>					
	PR	BR	NPR	PER	BER	NPER
Strong Borda	yes	no	yes	no	no	no
Strict Borda	yes	no	yes	no	no	no
Absolute Majority Winner	no	yes	yes	no	no	yes
Absolute Majority Loser	yes	no	no	no	no	no
Reversal Bias	yes	no	yes	yes	yes	yes
Sincere Truncation	no	yes	yes	no	yes	yes
Negative Abstention	no	no	no	yes	yes	no
Positive Abstention	no	no	no	no	yes	yes
Negative Participation	no	no	no	yes	yes	no
Positive Participation	no	no	no	no	yes	yes

<sup>8</sup>In this paper, when dealing with truncated rankings, we assume the pessimistic model as in Kamwa (2022) or Kamwa and Moyouwou (2021); this means that for a given voting (scoring) rule, only the candidates mentioned on a ballot receive points according to the logic of the voting rule, while the other truncated candidates receive no points. For the other approaches for dealing with truncated ballots, the reader may refer to Kamwa (2022) and the works cited therein.



The probabilistic approach applied to voting rules is often seen as complementary to the normative approach, which evaluates voting rules based on a set of criteria they either meet or fail to satisfy. The probabilistic approach quantifies the frequency with which a voting rule violates each of the considered criterion that it fails to satisfy. The probabilities of the occurring voting paradoxes can be used as criteria in order to evaluate voting rules. For this reason, the social choice literature, rooted in the probabilistic approach to voting rules, is replete with results on the likelihood of electoral events. The sheer volume of these results makes it impossible to provide a comprehensive summary in this paper. It is important to note that the majority of these findings are based on two primary assumptions: the Impartial Anonymous Culture (IAC) and the Impartial Culture (IC). In this paper, we will solely focus on the IAC assumption. For further details on the Impartial Culture and other assumptions, interested readers can refer to the books by [Gehrlein and Lepelley \(2011, 2017\)](#); [Diss and Merlin \(2021\)](#) and [Regenwetter et al. \(2006\)](#). Under IAC, it is assumed that each voting situation  $\tilde{n} = (n_1, \dots, n_6)$  is equally likely to occur. This hypothesis was introduced by [Kuga and Hiroaki \(1974\)](#) and later developed by [Gehrlein and Fishburn \(1976\)](#). According to this assumption, the likelihood of a given event  $X$  is calculated according to the following ratio:

$$\frac{\text{Number of voting situations in which event } X \text{ occurs}}{\text{Total number of possible voting situations}} \quad (1)$$

Based on the results found in the literature, Table 4 recalls for three-candidate elections, the limiting probability (in the sense that  $n$  to infinity) of each voting paradox considered here, specifically under the voting rules we have focused on.

Table 4: Limiting probabilities of the discussed paradoxes under IAC in three-candidate elections

<i>Voting Paradoxes</i>	<i>Voting rules</i>					
	PR	BR	NPR	PER	BER	NPER
Strong Borda	0.0296 <sup>a</sup>	0	0.0315 <sup>a</sup>	0	0	0
Strict Borda	0.0111 <sup>b</sup>	0	0.0111 <sup>b</sup>	0	0	0
Absolute Majority Winner	0	0.0370 <sup>c</sup>	0.3919 <sup>c</sup>	0	0	0.0247 <sup>d</sup>
Absolute Majority Loser	0.0247 <sup>c</sup>	0	0	0	0	0
Reversal Bias	0.0926 <sup>e</sup>	0	0.2037 <sup>e</sup>	0.0284 <sup>f</sup>	0.0417 <sup>f</sup>	0.0127 <sup>f</sup>
Sincere Truncation	0	0.3664 <sup>g</sup>	0.7222 <sup>g</sup>	0	0.0682 <sup>g</sup>	0.1597 <sup>g</sup>
Negative Abstention	0	0	0	0.0408 <sup>h</sup>	0.0139 <sup>i</sup>	0
Positive Abstention	0	0	0	0	0.0104 <sup>i</sup>	0.0425 <sup>h</sup>
Negative Participation	0	0	0	0.0729 <sup>h</sup>	0.0208 <sup>i</sup>	0
Positive Participation	0	0	0	0	0.0139 <sup>i</sup>	0.0382 <sup>h</sup>

Sources: a. [Gehrlein \(2002\)](#); b. [Gehrlein and Lepelley \(2010\)](#); c. [Diss et al. \(2018\)](#); d. author’s calculations; e. [Bubboloni et al. \(2020\)](#); f. [Belayadi and Mbih \(2021\)](#); g. [Kamwa and Moyouwou \(2021\)](#), [Kamwa \(2022\)](#); h. [Lepelley and Merlin \(2001\)](#); i. [Wilson and Pritchard \(2007\)](#), [Kamwa et al. \(2023\)](#).

Table 4 highlights the varying susceptibility of different paradoxes depending on the choice of voting rules. It should be noted, however, that these probabilities are far from being negligible. It is also important to remember that these probabilities do not take into account whether the election results are close or not. It is therefore interesting to consider what would happen to these probabilities if the closeness of the election results were taken into account. An increase in these probabilities would confirm the observations made in the literature that closeness tends to favour the occurrence of voting paradoxes.

## 2.4 An index to reflect closeness in elections

As a reminder, the main focus of our paper is to assess how the closeness of election outcomes affects the likelihood of the voting paradoxes we have defined earlier. To do so, we adopt a closeness index, following the methodology of [Diss et al. \(2021\)](#), [Lepelley et al. \(2018\)](#), and [Miller \(2017\)](#). Within a given voting rule, this index is calculated as the ratio between the score received by the candidate ranked last and the sum of the scores received by all the competing candidates.

For scenarios involving three candidates  $A$ ,  $B$ , and  $C$ , suppose without loss of generality that candidate  $C$  receives the lowest score according to a given voting rule. In this case, the

closeness index is computed as follows:

$$\mathcal{I} = \frac{S(C)}{S(A) + S(B) + S(C)} \quad (2)$$

Let  $\alpha$ ,  $\gamma$ , and  $\beta$  denote the closeness indices under PR, NPR, and BR, respectively. Given the rankings in Table 1 and the scores in Table 2, if candidate  $C$  has the worst score, we derive the closeness index under each of the three scoring rules as follows:

- under PR:  $\alpha = \frac{n_5+n_6}{n}$  with  $0 \leq \alpha \leq \frac{1}{3}$ ;
- under BR:  $\beta = \frac{n_2+n_4+2(n_5+n_6)}{3n}$  with  $0 \leq \beta \leq \frac{1}{3}$ ;
- under NPR:  $\gamma = \frac{n_2+n_4+n_5+n_6}{2n}$  with  $0 \leq \gamma \leq \frac{1}{3}$ .

When it comes to the elimination rules, remember that we mentioned earlier that the candidate eliminated in the first round is considered to be the candidate with the lowest score. Therefore, it is from this candidate's score that the closeness indices will be built, just as it would have been the case if the election had been a single round. Thus,  $\alpha$  will apply to PER,  $\gamma$  to NPER, and  $\beta$  to BER; this is also consistent with the approach taken by [Lepelley et al. \(2018\)](#).

### 3 Impact of closeness on the limiting probability of voting paradoxes

In order to assess the influence of closeness on the probabilities of paradox occurrence, it is essential to represent these probabilities as functions of the closeness index. This task requires a revision of the IAC assumption for the specific purpose under consideration. Recall that we assume without loss of generality that candidate  $C$  receives the lowest score under a given voting rule. Given the closeness indices  $\alpha$ ,  $\beta$ , and  $\gamma$  that have been defined in Section 2.4, the  $\mathcal{I}$ -IAC assumption (for  $\mathcal{I} = \alpha, \beta, \gamma$ ) is defined as follows: all voting scenarios in which candidate  $C$  is ranked last are considered equiprobable. To derive our probabilities, we use the *parameterized Barvinok's algorithm* developed by [Verdoolaege et al. \(2004\)](#). This algorithm is encoded to compute the number of lattice points in a rational convex polytope and the output is given in the form of *Ehrhart polynomials* ([Ehrhart, 1962, 1967](#)). For more details on this algorithm and the related subjects, the reader may refer to the works of [Barvinok and Pommersheim \(1999\)](#), [Bruynooghe et al. \(2005\)](#), [Clauss and Loechner \(1998\)](#), [Lepelley et al. \(2008\)](#), and [Wilson and Pritchard \(2007\)](#). We report in the Appendix all of the limiting probabilities that we obtain from our calculations. We have chosen to skip the computation details; nonetheless, they are available upon simple request. We concentrate here on commenting on our results regarding the impact of closeness on the limiting probability of the voting paradoxes under consideration. In order to present our results, we separately address the case of each of the voting rules covered by our analysis.

It should be emphasized that the observations we will make regarding the evolution of each probability in relation to the closeness parameter are derived from the study of the functions (theoretical results) presented in the appendices.

### 3.1 Closeness and the Plurality Rule

Among the paradoxes presented in Table 3, it was noted that PR was found to be susceptible to the Strong Borda Paradox, the Strict Borda Paradox, the Absolute Majority Loser Paradox, and the Reversal Bias Paradox. Figure 1 provides a comprehensive view of how closeness, measured by the index  $\alpha$ , affects the limiting probabilities of these paradoxes.

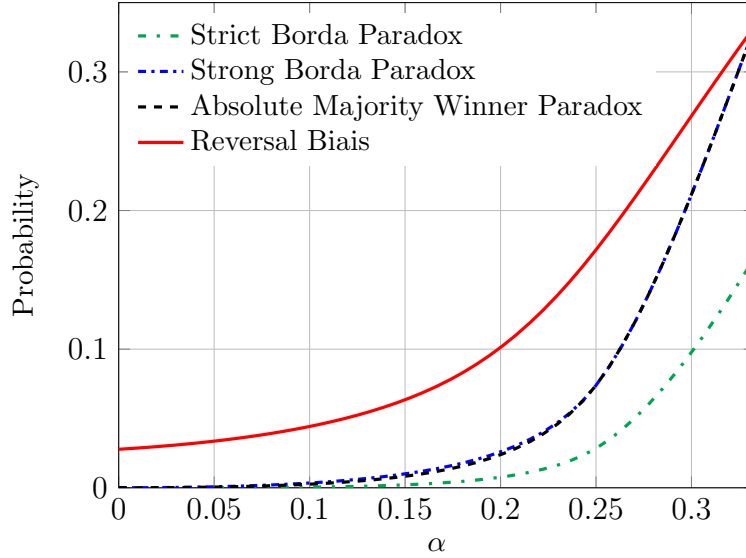


Figure 1: Closeness and voting paradoxes afflicting PR

When  $\alpha = 0$  (i.e., candidate  $C$  gets no points), only the Reversal Bias Paradox remains viable, while the other paradoxes disappear. In particular, as elections become closer (where  $\alpha$  increases from 0 to  $\frac{1}{3}$ ), PR becomes more likely to lead to each of the four paradoxes. It is interesting to note that the limiting probabilities of the Strong Borda Paradox and the Absolute Majority Loser Paradox become quite similar when closeness is taken into account. For reference, Table 4 shows that the limiting probability of the Strong Borda Paradox is 2.96%, the Absolute Majority Loser Paradox is 2.47%, the Strict Borda Paradox is 1.11%, and the Reversal Bias Paradox is 9.26% without considering closeness. However, these probabilities can be significantly exceeded and amplified depending on the precise values of the closeness index  $\alpha$ . It appears that paradoxes do not occur when closeness falls below certain thresholds, which are approximately 0.1552 for the Strong Borda Paradox, 0.2149 for the Strict Borda Paradox, 0.2016 for the Absolute Majority Loser Paradox, and 0.1913 for the Reversal Bias Paradox. Beyond these values, closeness contributes significantly to the likelihood of each paradox occurring.

We can now highlight the comparative effect of including the closeness index in the calculation of probabilities associated with the considered voting paradoxes. The impact of closeness on the probability of voting paradoxes under PR is summarized in Table 5. In this table, and all subsequent ones, we denote  $\mathbf{P}(\cdot) < \mathbf{P}_{\text{IAC}}$  (resp.  $\mathbf{P}(\cdot) > \mathbf{P}_{\text{IAC}}$ ) to indicate that within the specified interval,  $\mathbf{P}(\cdot)$ , the probability adjusted for closeness index, is smaller (resp. larger) than  $\mathbf{P}_{\text{IAC}}$ , the probability without considering the closeness index.

Table 5:  $\mathbf{P}(\alpha)$  vs.  $\mathbf{P}_{\text{IAC}}$  under PR

<i>Voting Paradoxes</i>	$\mathbf{P}(\alpha) < \mathbf{P}_{\text{IAC}}$	$\mathbf{P}(\alpha) > \mathbf{P}_{\text{IAC}}$
Strong Borda	$[0, 0.1552[$	$]0.1552, \frac{1}{3}]$
Strict Borda	$[0, 0.2149[$	$]0.2149, \frac{1}{3}]$
Absolute Majority Loser	$[0, 0.2016[$	$]0.2016, \frac{1}{3}]$
Reversal Bias	$[0, 0.1913[$	$]0.1913, \frac{1}{3}]$

### 3.2 Closeness and the Borda Rule

For the two paradoxes involving BR, Figure 2 shows the evolution of the probabilities as a function of the closeness index  $\beta$ .

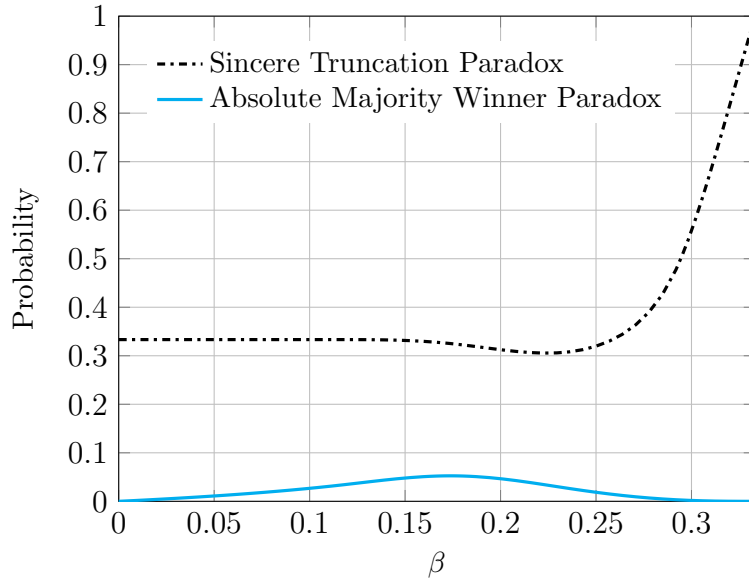


Figure 2: Closeness and voting paradoxes afflicting BR

As the value of  $\beta$  increases, the probabilities of the Absolute Majority Winner Paradox

exhibit an interesting pattern. Initially, there is an increasing phase for  $\beta < 0.1738$ , followed by a decreasing phase for  $\beta > 0.1738$ . When  $\beta = 0$ , it means that  $S(C) = 0$  and the election amounts to a Plurality vote between  $A$  and  $B$ ; hence, AMWP cannot occur. This explains the probability equal to 0. When  $\beta = \frac{1}{3}$ , no candidate can be an absolute majority winner and we get  $S(A) = S(B) = S(C)$ ; hence AMWP cannot occur, so the probability is 0 also. Since we know that AMWP can occur with BR, the two considerations above help to grasp why the curve is not monotonous. In particular, for  $0.1241 \leq \beta < 0.2184$ , the probabilities of the Absolute Majority Winner Paradox significantly exceed 3.70%, which is the limiting probability of this paradox without considering closeness. Outside this range, the impact of closeness tends to attenuate the occurrence of the Absolute Majority Winner Paradox, as the probabilities do not exceed 3.70%.

It is observed with the Sincere Truncation Paradox that when  $\beta$  is lower than approximately 0.2716, the probabilities remain below 36.64%, which is the known limit value of this paradox without taking closeness into account. However, when  $\beta$  exceeds about 0.2716, the probabilities increase rapidly, reaching a peak of 100% as  $\beta$  approaches  $\frac{1}{3}$ . This phenomenon implies that BR becomes more susceptible to manipulation by truncation as the closeness of the election increases. Consequently, closeness increases the likelihood of each of the two paradoxes affecting BR that we examine.

In Table 6, we provide a comparison between the probabilities of paradoxes with and without considering closeness.

Table 6:  $\mathbf{P}(\beta)$  vs.  $\mathbf{P}_{\text{IAC}}$  under BR

<i>Voting Paradoxes</i>	$\mathbf{P}(\beta) < \mathbf{P}_{\text{IAC}}$	$\mathbf{P}(\beta) > \mathbf{P}_{\text{IAC}}$
Absolute Majority Winner	$[0, 0.1241[ \cup ]0.2184, \frac{1}{3}]$	$[0.1241, 0.2184[$
Sincere Truncation	$]0, 0.2716[$	$]0.2716, \frac{1}{3}]$

### 3.3 Closeness and the Negative Plurality Rule

Figure 3 provides an overview of how closeness, represented by  $\gamma$ , affects the probability of voting events under NPR.

We observed that when the candidate ranked last receives less than a quarter of the votes cast ( $0 \leq \gamma < \frac{1}{4}$ ), the Strong Borda Paradox and the Strict Borda Paradox are no longer observed under NPR; this is also true for the Reversal Bias Paradox when  $\gamma = 0$ . However, beyond these values of  $\gamma$ , each of these paradoxes reappears and increase in frequency as  $\gamma$  increases.

In the case of the Sincere Truncation Paradox, its behavior evolves in two distinct phases while maintaining probabilities above 50%: a phase of decrease for values of  $\gamma < 0.2586$ , followed by a phase of increase for  $\gamma > 0.2586$ . As  $\gamma$  approaches  $\frac{1}{3}$ , the vulnerability of the Sincere Truncation Paradox reaches 100%. In other words, when the candidate ranked

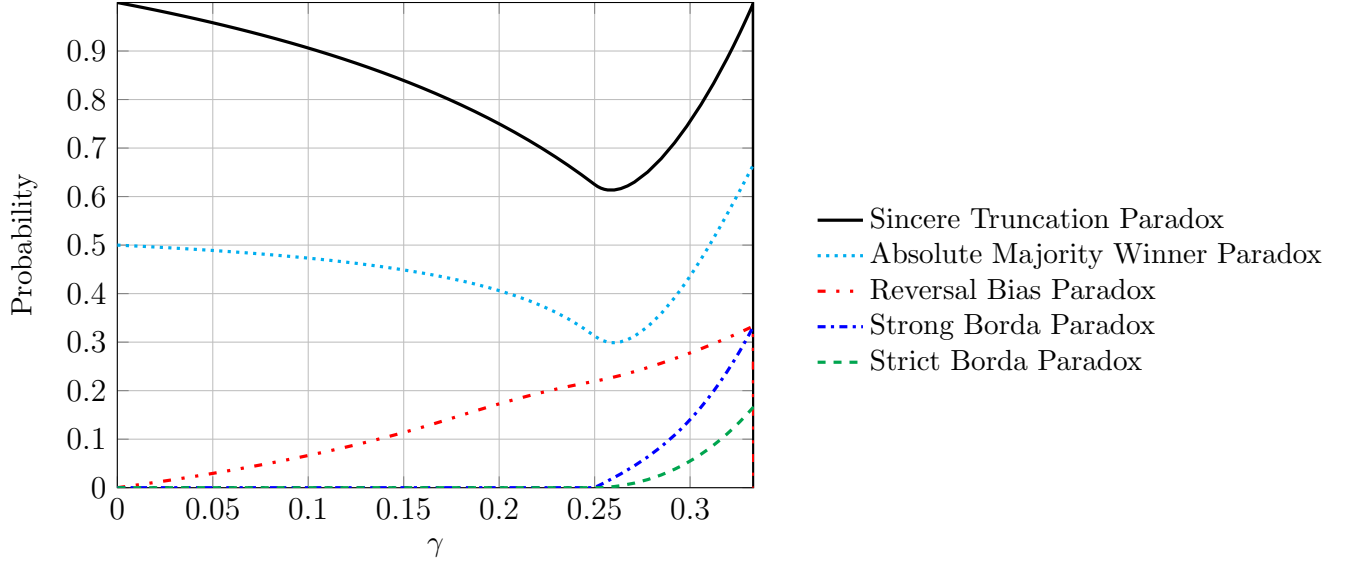


Figure 3: Closeness and voting paradoxes afflicting NPR

last under NPR obtains close to a third of the votes cast, it becomes certain that a group of voters can manipulate the outcome of the vote through truncation of their preferences. When  $\gamma = 0$  (respectively,  $\gamma = \frac{1}{3}$ ), we have  $S(A) = S(B)$  (respectively,  $S(A) = S(B) = S(C)$ ). Assuming that one candidate is the winner, it is sufficient for a single voter who ranks the other candidate first to truncate in order for the paradox to occur. Thus, for  $\gamma = 0$  (respectively,  $\gamma = \frac{1}{3}$ ), the paradox always occurs, leading to a probability of 1. However, since the probability of the paradox is not always 1, there exist values of closeness for which the probability is less than 1. These observations elucidate why the curve depicting the probabilities of the Sincere Truncation Paradox exhibits non-monotonic behavior.

With respect to the Absolute Majority Winner Paradox, the probabilities exhibit a tendency to decrease for values of  $\gamma < 0.26$  and then to increase as  $\gamma$  varies. When there is an Absolute Majority Winner and that  $\gamma = 0$  (resp.  $\gamma = \frac{1}{3}$ ), we get  $S(A) = S(B)$  (resp.  $S(A) = S(B) = S(C)$ ). Given that, there is a 1 chance in 2 (resp. 2 chances in 3) of not choosing the AMW in this case; hence the probability of  $\frac{1}{2}$  (resp.  $\frac{2}{3}$ ). As  $\gamma = 0$  and  $\gamma = \frac{1}{3}$  are the only instances where scores equality occurs among the candidates, it follows that the highest values for the paradox probabilities are attained at these bounds. Consequently, for other closeness values, the probabilities of the Absolute Majority Winner Paradox remain significantly lower compared to those at the boundaries, hence the non-monotonic nature of the curve.

Table 7 provides a comparison between the probabilities of paradoxes with and without considering closeness.

Table 7:  $\mathbf{P}(\gamma)$  vs.  $\mathbf{P}_{\text{IAC}}$  under NPR

<i>Voting Paradoxes</i>	$\mathbf{P}(\gamma) < \mathbf{P}_{\text{IAC}}$	$\mathbf{P}(\gamma) > \mathbf{P}_{\text{IAC}}$
Strong Borda	$[0, 0.2651[$	$[0.2651, \frac{1}{3}]$
Strict Borda	$[0, 0.2726[$	$[0.2726, \frac{1}{3}]$
Absolute Majority Winner	$]0.2112, 0.2919[$	$[0, 0.2112[ \cup ]0.2919, \frac{1}{3}]$
Reversal Bias	$[0, 0.2304[$	$[0.2304, \frac{1}{3}]$
Sincere Truncation	$]0.2368, 0.2940[$	$[0, 0.2368[ \cup ]0.2940, \frac{1}{3}]$

Before we look at the scoring elimination rules, it is worth highlighting an observation about our three scoring rules. For each of these scoring rules, the introduction of closeness as a parameter never changes the relative frequency that the paradoxes occurs – that is, the graphed frequency lines never cross one another. To illustrate this point, let us consider the example of BR. Under this rule, the Truncation Paradox is more likely to occur than the Absolute Majority Winner Paradox in the absence of the closeness parameter as shown in Table 4. Importantly, this order of events remains consistent even when closeness is taken into account. However, this is not always the case with our three scoring elimination rules, as we will see later.

### 3.4 Closeness and the Plurality Elimination Rule

From Table 4, we can see that the limiting probabilities of the three voting paradoxes associated with PER in the absence of any closeness index are as follows: 4.08% for the Negative Abstention Paradox, 7.29% for the Negative Participation Paradox, and only 0.0039% for the Reversal Bias Paradox. Figure 4 gives us a comparative view, taking into account different levels of closeness, and illustrates its impact on the occurrence of each of these three paradoxes.

We can see that the limiting probability of the Reversal Bias Paradox remains low for closeness levels up to  $\alpha \leq 0.1882$  and it does exceed the probability value obtained when closeness is not considered. However, for  $0.1882 \leq \alpha < \frac{1}{3}$ , the introduction of proximity tends to significantly amplify the occurrence of the paradox, almost doubling its probability compared to the case without closeness. With respect to the Negative Abstention Paradox and the Negative Participation Paradox, we observe an increase in the probabilities of occurrence for values of  $\alpha$  above about 0.2428 and 0.2437, respectively. Below these thresholds, the probabilities of these voting paradoxes remain below their values obtained when closeness is not taken into account. Table 8 provides a comparison between the probabilities of paradoxes with and without considering closeness.



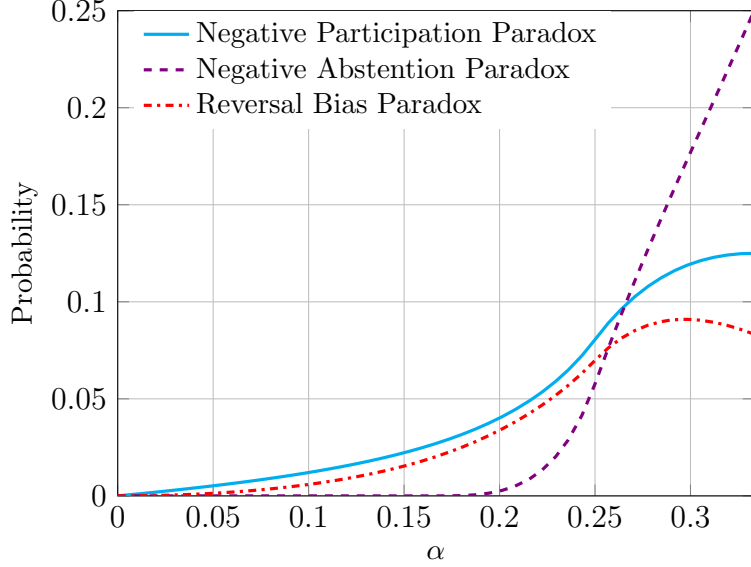


Figure 4: Closeness and voting paradoxes afflicting PER

Table 8:  $\mathbf{P}(\alpha)$  vs.  $\mathbf{P}_{\text{IAC}}$  under PER

<i>Voting Paradoxes</i>	$\mathbf{P}(\alpha) < \mathbf{P}_{\text{IAC}}$	$\mathbf{P}(\alpha) > \mathbf{P}_{\text{IAC}}$
Negative Abstention	$[0, 0.2428[$	$[0.2428, \frac{1}{3}]$
Negative Participation	$[0, 0.2437[$	$[0.2437, \frac{1}{3}]$
Reversal Bias	$[0, 0.1882[$	$[0.1882, \frac{1}{3}]$

As shown in Table 4, of the three paradoxes considered, the one most likely to occur under PER is the Negative Participation Paradox, while the Reversal Bias Paradox is least likely. However, our results show that this hierarchy does not hold across all values of  $\alpha$ . While this hierarchy remains intact for  $\alpha > \frac{1}{4}$ , it undergoes a shift for  $\alpha < \frac{1}{4}$ , where the Negative Participation Paradox becomes the least vulnerable of the three paradoxes.

### 3.5 Closeness and the Borda Elimination Rule

Let us recall that, without taking into account the closeness criterion, the limiting probabilities of the four voting paradoxes associated with BER are as follows: 6.82% for the Truncation Paradox, 1.39% for the Negative Abstention Paradox, 2.08% for the Negative Participation Paradox, and 1.04% for the Positive Abstention Paradox. However, when we introduce the closeness index  $\beta$  into the equation, some interesting observations emerge. Firstly, it is worth noting that the Reversal Bias Paradox is possible to occur under BER

only when  $\frac{1}{6} \leq \beta < \frac{1}{3}$ . In addition, we found that both the Negative Abstention Paradox and the Positive Participation Paradox result in approximately the same probabilities, whether closeness is considered (our results) or not (Table 4). A visual representation of these results is displayed in Figure 5.

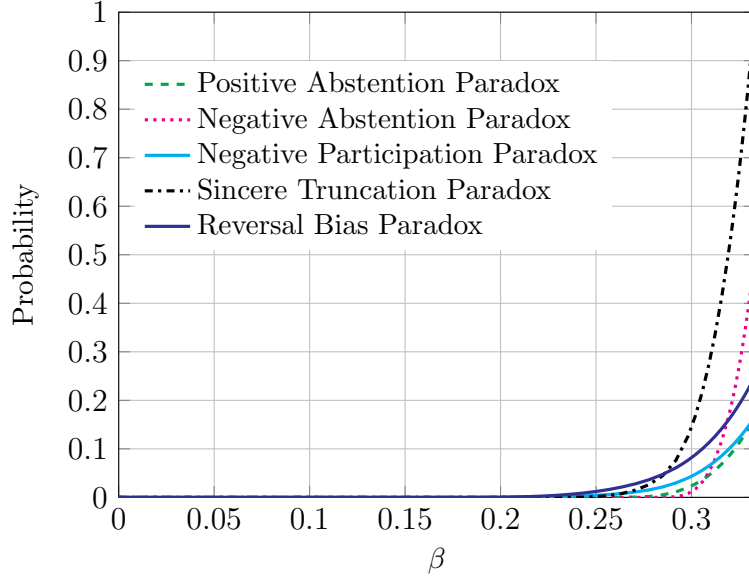


Figure 5: Closeness and voting paradoxes afflicting BER

An interesting observation is that when the score of the candidate ranked last represents less than 25% of the votes cast ( $0 \leq \beta < \frac{1}{4}$ ), the Positive Abstention Paradox no longer occurs under BER. Similar trends are observed for other paradoxes, where the presence of closeness significantly reduces the probability of their occurrence: the Negative Abstention Paradox, the Negative Participation Paradox, and the Truncation Paradox disappear when  $\beta < \frac{5}{18}$ ,  $\beta < \frac{1}{6}$ , and  $\beta < \frac{2}{9}$ , respectively.

Table 9 provides a comparison between the probabilities of paradoxes with and without considering closeness. It is clear that over a substantial part of the  $\beta$  range, the introduction of closeness greatly reduces the occurrence of these paradoxes, and in some cases eliminates them altogether.

Table 9:  $\mathbf{P}(\beta)$  vs.  $\mathbf{P}_{\text{IAC}}$  under BER

<i><b>Voting Paradoxes</b></i>	$\mathbf{P}(\beta) < \mathbf{P}_{\text{IAC}}$	$\mathbf{P}(\beta) > \mathbf{P}_{\text{IAC}}$
Sincere Truncation	$[0, 0.2904[$	$[0.2904, \frac{1}{3}]$
Reversal Bias	$[0, 0.2815[$	$[0.2815, \frac{1}{3}]$
Negative Abstention	$[0, 0.3010[$	$[0.3010, \frac{1}{3}]$
Negative Participation	$[0, 0.2843[$	$[0.2843, \frac{1}{3}]$
Positive Abstention	$[0, 0.2911[$	$[0.2911, \frac{1}{3}]$

Comparing our results with those presented in Table 4, it is clear that the Sincere Truncation Paradox consistently retains its position as the most likely to occur given that  $\beta > \frac{11185}{39332}$ . However, the hierarchy in terms of occurrence probabilities is not always maintained for the other paradoxes. In particular, for certain values of the index  $\beta$ , the order of paradox occurrence probabilities varies when closeness is taken into account.

### 3.6 Closeness and the Negative Plurality Elimination Rule

Our calculations have revealed a remarkable phenomenon under NPER: when the candidate eliminated in the first round receives less than a quarter of the votes cast, none of the five considered paradoxes can be observed. However, as  $\gamma$  exceeds the threshold of 0.25, the probabilities of occurrence increase dramatically, reaching 62.50% for the Sincere Truncation Paradox, 33.33% for the Absolute Majority Winner Paradox, 25% for the Positive Participation Paradox, 12.50% for the Positive Abstention Paradox, and 8.33% for the Reversal Bias Paradox. This striking trend is illustrated in Figure 6 as a function of the closeness index  $\gamma$ .

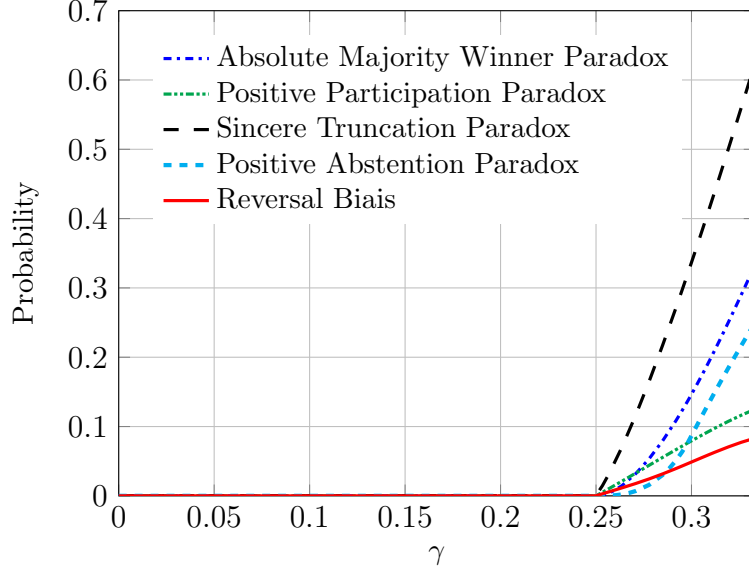


Figure 6: Closeness and voting paradoxes afflicting NPER

The comparison between the probabilities of paradoxes with and without considering closeness is summarized in Table 10. This table provides an insight into the values of  $\gamma$  for which considering the closeness of the election results in either increased or decreased probabilities compared to scenarios without closeness (as shown in Table 4).

Table 10:  $\mathbf{P}(\gamma)$  vs.  $\mathbf{P}_{\text{IAC}}$  under NPER

<i>Voting Paradoxes</i>	$\mathbf{P}(\gamma) < \mathbf{P}_{\text{IAC}}$	$\mathbf{P}(\gamma) > \mathbf{P}_{\text{IAC}}$
Absolute Majority Winner	$[0, 0.2680[$	$[0.2680, \frac{1}{3}]$
Reversal Bias	$[0, 0.2649[$	$[0.2649, \frac{1}{3}]$
Positive Abstention	$[0, 0.2768[$	$[0.2768, \frac{1}{3}]$
Positive Participation	$[0, 0.2879[$	$[0.2879, \frac{1}{3}]$
Sincere Truncation	$[0, 0.2771[$	$[0.2771, \frac{1}{3}]$

Comparing our results with those in Table 4, we see that the Sincere Truncation Paradox consistently remains the most likely to occur, whether or not election closeness is taken into account. However, for the other paradoxes, especially when  $\gamma$  is less than about 0.29, the hierarchy in terms of probability of occurrence does not always follow the same pattern.

## 4 Conclusion

The aim of this paper was to provide further insights into the impact of closely contested elections on the likelihood of different voting paradoxes in three-candidate elections in the case of large electorates. Our analysis focused on assessing the likelihood of ten well-known voting paradoxes and covered both scoring rules and scoring elimination rules. Our methodology in this study was inspired by previous research, such as [Miller \(2017\)](#), [Lepelley et al. \(2018\)](#), and [Diss et al. \(2021\)](#), where we use the same index to measure the degree of closeness of elections. This index is calculated as the proportion of scores received by the candidate with the fewest scores.

In order to achieve our goal, we computed the limiting probability representations for each paradox that could occur under different voting rules, taking into account different degrees of election closeness. These representations served as a basis for examining how the closeness of the election affects the likelihood of these paradoxes occurring. We identified scenarios in which the introduction of the closeness index either increased or decreased the likelihood of paradoxes, depending on particular values of election closeness.

Our analysis also revealed an interesting insight. When we examined one of the three scoring rules in conjunction with the corresponding voting paradoxes, we noticed that regardless of the degree of election closeness, accounting for closeness preserved the hierarchy among these paradoxes in terms of their likelihood of occurrence compared to situations without election closeness. It is worth noting, however, that this observation did not hold universally for all scoring elimination rules, especially at certain levels of election closeness.

## Appendix 1: Probability representations

### A- Plurality Rule

Given a three-candidate election with large electorates and  $\alpha$  the average number of points obtained by the last ranked candidate under PR,

- the probability of the Strong Borda Paradox (SgBP) under PR is:

$$P_{SgBP}^{\infty}(\alpha) = \begin{cases} \frac{\alpha^2(4 - 3\alpha)}{16(11\alpha^3 - 4\alpha^2 - 3\alpha + 1)} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{255\alpha^3 - 255\alpha^2 + 75\alpha - 7}{8(18\alpha^3 - 18\alpha^2 + 6\alpha - 1)} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$$

- the probability of the Strict Borda Paradox (StBP) under PR is:

$$P_{StBP}^{\infty}(\alpha) = \begin{cases} \frac{5\alpha^3}{176\alpha^3 - 64\alpha^2 - 48\alpha + 16} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{87\alpha^3 - 99\alpha^2 + 31\alpha - 3}{8(18\alpha^3 - 18\alpha^2 + 6\alpha - 1)} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$$

- the probability of the Absolute Majority Loser Paradox (AML<sub>P</sub>) under PR is:

$$P_{AML_P}^{\infty}(\alpha) = \begin{cases} \frac{\alpha^2 (4 - 3\alpha)}{8(54\alpha^3 - 12\alpha + 3 - 8\alpha^2)} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{255\alpha^3 - 255\alpha^2 + 75\alpha - 7}{8(18\alpha^3 - 18\alpha^2 + 6\alpha - 1)} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$$

- the probability of the Reversal Bias Paradox (RBP) under PR is:

$$P_{RBP}^{\infty}(\alpha) = \begin{cases} \frac{-9\alpha^3 + 9\alpha^2 + 1}{36(3\alpha - 1)(3\alpha^2 - 1)} & \text{for } 0 \leq \alpha < \frac{1}{6} \\ \frac{-2619\alpha^4 + 1755\alpha^3 - 432\alpha^2 + 51\alpha - 2}{108\alpha(3\alpha - 1)(3\alpha^2 - 1)} & \text{for } \frac{1}{6} \leq \alpha < \frac{1}{4} \\ \frac{1431\alpha^3 - 1242\alpha^2 + 306\alpha - 25}{108\alpha(3\alpha^2 - 1)} & \text{for } \frac{1}{4} \leq \alpha < \frac{1}{3} \end{cases}$$

## B- Borda Rule

Consider a three-candidate election with large electorates and  $\beta$  the average number of points obtained by the last ranked candidate under BR,

- the probability of the Absolute Majority Winner Paradox (AMWP) under BR is:

$$P_{AMWP}^{\infty}(\beta) = \begin{cases} \frac{-3\beta}{16(3\beta - 1)} & \text{for } 0 \leq \beta < \frac{1}{9} \\ \frac{6075\beta^4 - 2916\beta^3 + 486\beta^2 - 36\beta + 1}{2592\beta^3(3\beta - 1)} & \text{for } \frac{1}{9} \leq \beta < \frac{1}{6} \\ \frac{9477\beta^4 - 7452\beta^3 + 2106\beta^2 - 252\beta + 11}{9(432\beta^4 - 576\beta^3 + 216\beta^2 - 24\beta + 1)} & \text{for } \frac{1}{6} \leq \beta < \frac{2}{9} \\ \frac{-1215\beta^4 + 1404\beta^3 - 594\beta^2 + 108\beta - 7}{3(432\beta^4 - 576\beta^3 + 216\beta^2 - 24\beta + 1)} & \text{for } \frac{2}{9} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Sincere Truncation Paradox (STP) under BR is:

$$P_{STP}^{\infty}(\beta) = \begin{cases} \frac{1}{3} & \text{for } 0 \leq \beta < \frac{1}{9} \\ \frac{9477\beta^4 - 4212\beta^3 + 486\beta^2 - 36\beta + 1}{972\beta^3(9\beta - 4)} & \text{for } \frac{1}{9} \leq \beta < \frac{1}{6} \\ \frac{-165969\beta^4 + 184356\beta^3 - 65934\beta^2 + 7956\beta - 349}{3572100\beta^4 - 2313360\beta^3 + 544320\beta^2 - 60480\beta + 2520} & \text{for } \frac{1}{6} \leq \beta < \frac{2}{7} \\ \frac{165969\beta^4 - 184356\beta^3 + 65934\beta^2 - 7956\beta + 349}{(7560\beta - 2520)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{2}{7} \leq \beta < \frac{8}{27} \\ \frac{-19570491\beta^4 + 22367340\beta^3 - 9588618\beta^2 + 1827324\beta - 130567}{(1512\beta - 504)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{8}{27} \leq \beta < \frac{1}{3} \end{cases}$$

### C- Negative Plurality Rule

Consider a three-candidate election with large electorates and  $\gamma$  the average number of points obtained by the last ranked candidate under NPR,

- the probability of the Strong Borda Paradox under NPR is:

$$P_{SgBP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{(4\gamma - 1)(24\gamma^2 - 9\gamma + 1)}{2(144\gamma^3 - 144\gamma^2 + 48\gamma - 5)} & \text{for } \frac{1}{4} \leq \gamma \leq \frac{1}{3} \end{cases}$$

- the probability of the Strict Borda Paradox under NPR is:

$$P_{StBP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{3\gamma(4\gamma - 1)^2}{2(144\gamma^3 - 144\gamma^2 + 48\gamma - 5)} & \text{for } \frac{1}{4} \leq \gamma \leq \frac{1}{3} \end{cases}$$

- the probability of the Absolute Majority Winner Paradox under NPR is:

$$P_{AMWP}^{\infty}(\gamma) = \begin{cases} \frac{27\gamma - 8}{16(3\gamma - 1)} & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-222\gamma^3 + 222\gamma^2 - 69\gamma + 7}{2(144\gamma^3 - 144\gamma^2 + 48\gamma - 5)} & \text{for } \frac{1}{4} \leq \gamma \leq \frac{1}{3} \end{cases}$$

- the probability of the Reversal Bias Paradox (RBP) under NPR is:

$$P_{RBP}^{\infty}(\gamma) = \begin{cases} \frac{-17\gamma}{32(2\gamma-1)} & \text{for } 0 \leq \gamma < \frac{1}{6} \\ \frac{837\gamma^4 - 432\gamma^3 + 12\gamma - 1}{864\gamma^3(2\gamma-1)} & \text{for } \frac{1}{6} \leq \gamma < \frac{1}{4} \\ \frac{-2025\gamma^3 + 1485\gamma^2 - 225\gamma + 1}{432(2\gamma-1)(6\gamma^2-6\gamma+1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Sincere Truncation Paradox (STP) under NPR is:

$$P_{STP}^{\infty}(\gamma) = \begin{cases} \frac{11\gamma-4}{4(2\gamma-1)} & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-39\gamma^3 + 39\gamma^2 - 11\gamma + 1}{2(2\gamma-1)(6\gamma^2-6\gamma+1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

## D- Plurality Elimination Rule

Consider a three-candidate election with large electorates and  $\alpha$  the average number of points obtained by the last ranked candidate under PR,

- the probability of the Reversal Bias Paradox (RBP) under PER is:

$$P_{RBP}^{\infty}(\alpha) = \begin{cases} \frac{-\alpha^2(2\alpha-1)}{2(3\alpha-1)(3\alpha^2-1)} & \text{for } 0 \leq \alpha < \frac{1}{6} \\ \frac{-2160\alpha^4 + 864\alpha^3 - 12\alpha + 1}{864\alpha(3\alpha-1)(3\alpha^2-1)} & \text{for } \frac{1}{6} \leq \alpha < \frac{1}{4} \\ \frac{-810\alpha^3 + 918\alpha^2 - 333\alpha + 37}{108\alpha(3\alpha^2-1)} & \text{for } \frac{1}{4} \leq \alpha < \frac{1}{3} \end{cases}$$

- the probability of the Negative Abstention Paradox (NAP) under PER is:

$$P_{NAP}^{\infty}(\alpha) = \begin{cases} 0 & \text{for } 0 \leq \alpha < \frac{1}{6} \\ \frac{(2\alpha+1)(6\alpha-1)^3}{64\alpha(3\alpha-1)(3\alpha^2-1)} & \text{for } \frac{1}{6} \leq \alpha < \frac{1}{4} \\ \frac{-15\alpha^3 + 15\alpha^2 - 7\alpha + 1}{4\alpha(3\alpha^2-1)} & \text{for } \frac{1}{4} \leq \alpha < \frac{1}{3} \end{cases}$$

- the probability of the Negative Participation Paradox (NPP) under PER is:



$$P_{NPP}^{\infty}(\alpha) = \begin{cases} \frac{\alpha (3\alpha^2 - 8\alpha + 6)}{(192\alpha - 64)(3\alpha^2 - 1)} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{-255\alpha^3 + 339\alpha^2 - 141\alpha + 17}{64\alpha(3\alpha^2 - 1)} & \text{for } \frac{1}{4} \leq \alpha < \frac{1}{3} \end{cases}$$

## E- Borda Elimination Rule

Consider a three-candidate election with large electorates and  $\beta$  the average number of points obtained by the last ranked candidate under BR,

- the probability of the Sincere Truncation Paradox (STP) of BER is:

$$P_{STP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{2}{9} \\ \frac{(9\beta - 2)^4}{8(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{2}{9} \leq \beta < \frac{1}{4} \\ \frac{-7614\beta^4 + 9072\beta^3 - 3888\beta^2 + 720\beta - 49}{16(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{1}{4} \leq \beta < \frac{5}{18} \\ \frac{-1328238\beta^4 + 1574640\beta^3 - 695952\beta^2 + 136080\beta - 9941}{144(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{5}{18} \leq \beta < \frac{7}{24} \\ \frac{-5143095\beta^4 + 6012792\beta^3 - 2634120\beta^2 + 512568\beta - 37384}{72(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{7}{24} \leq \beta < \frac{8}{27} \\ \frac{-6588\beta^3 + 5229\beta^2 - 1380\beta + 121}{621\beta^3 - 459\beta^2 + 99\beta - 7} & \text{for } \frac{8}{27} \leq \beta < \frac{11}{36} \\ \frac{-6588\beta^3 + 5229\beta^2 - 1380\beta + 121}{621\beta^3 - 459\beta^2 + 99\beta - 7} & \text{for } \frac{11}{36} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Reversal Bias Paradox (RBP) of BER is:

$$P_{RBP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{1}{6} \\ -\frac{(6\beta - 1)^4}{22680\beta^4 - 14688\beta^3 + 3456\beta^2 - 384\beta + 16} & \text{for } \frac{1}{6} \leq \beta < \frac{2}{9} \\ -\frac{9234\beta^4 - 9072\beta^3 + 3240\beta^2 - 504\beta + 29}{(144\beta - 48)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{2}{9} \leq \beta < \frac{5}{18} \\ -\frac{6345\beta^3 - 4869\beta^2 + 1257\beta - 109}{4968\beta^3 - 3672\beta^2 + 792\beta - 56} & \text{for } \frac{5}{18} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Negative Abstention Paradox (NAP) and Positive Participation Paradox (PPP) of BER is:

$$P_{NAP}^{\infty}(\beta) = P_{PPP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{5}{18} \\ \frac{(18\beta - 5)^4}{12(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{5}{18} \leq \beta < \frac{8}{27} \\ \frac{(-45\beta + 13)(1053\beta^2 - 612\beta + 89)}{4(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{8}{27} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Negative Participation Paradox (NPP) of BER is:

$$P_{NPP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{1}{6} \\ \frac{-(6\beta - 1)^4}{24(2835\beta^4 - 1836\beta^3 + 432\beta^2 - 48\beta + 2)} & \text{for } \frac{1}{6} \leq \beta < \frac{2}{9} \\ \frac{(6\beta - 1)^4}{48(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{2}{9} \leq \beta < \frac{5}{18} \\ \frac{-(4\beta - 1)(180\beta^2 - 96\beta + 13)}{621\beta^3 - 459\beta^2 + 99\beta - 7} & \text{for } \frac{5}{18} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Positive Abstention Paradox (PAP) of BER is:

$$P_{PAP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{1}{4} \\ \frac{27 (4\beta - 1)^4}{(24\beta - 8)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{1}{4} \leq \beta < \frac{5}{18} \\ \frac{-(15\beta - 4)(234\beta^2 - 126\beta + 17)}{3(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{5}{18} \leq \beta < \frac{1}{3} \end{cases}$$

## F- Negative Plurality Elimination Rule

Consider a three-candidate election with large electorates and  $\gamma$  the average number of points obtained by the last ranked candidate under NPR,

- the probability of the Absolute Majority Winner Paradox (AMWP) under NPER is:

$$P_{AMWP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-3(2\gamma - 1)(-1 + 4\gamma)^2}{144\gamma^3 - 144\gamma^2 + 48\gamma - 5} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Reversal Bias Paradox (RBP) under NPER is:

$$P_{RBP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{(4\gamma - 1)(8640\gamma^3 - 6480\gamma^2 + 1620\gamma - 139)}{1152(3\gamma - 1)(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{7}{24} \\ \frac{-2376\gamma^3 + 2160\gamma^2 - 639\gamma + 62}{108(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{7}{24} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Sincere Truncation Paradox (STP) under NPER is:

$$P_{STP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-(4\gamma - 1)(120\gamma^2 + 84\gamma - 13)}{8(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Positive Abstention Paradox (PAP) under NPER is:

$$P_{PAP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-3(4\gamma - 1)(11\gamma^2 - 7\gamma + 1)}{8(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Positive Participation Paradox (PPP) under NPER is:

$$P_{PPP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{(44\gamma - 15)(4\gamma - 1)^3}{16(3\gamma - 1)(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{3}{10} \\ \frac{-132\gamma^3 + 120\gamma^2 - 34\gamma + 3}{4(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{3}{10} \leq \gamma < \frac{1}{3} \end{cases}$$

## Appendix 2: Detailed proof of the Strong Borda Paradox probability representation under $\alpha$ -IAC

As the proofs of all our propositions almost follow the same pattern, we will only provide the proof of the Strong Borda Paradox probability representation under the Plurality Rule (PR) and  $\alpha$ -IAC. This will enable the reader to grasp the steps taken to derive all the other probability representations. Complete proofs are available upon simple request from the authors.

In our context, the Strong Borda Paradox emerges when the Condorcet Loser (CL), if one exists, is ranked last according to PR. Let us assume, without loss of generality, that candidate  $C$  is the last ranked one under PR. In such a case, the closeness index is given by  $\alpha = \frac{n_5 + n_6}{n} = \frac{k}{n}$ , where  $k$  is the score of candidate  $C$  under PR. In order to compute the Strong Borda Paradox, we first need to count the number of voting situations for which the CL exists under the  $\alpha$ -IAC assumption when candidate  $C$  is the last ranked one under PR. In order to accomplish this goal, we need to consider the three following disjoint events:

$$\begin{aligned} X_1 &= \text{"A is the CL and C is ranked last by PR"} \\ X_2 &= \text{"B is the CL and C is ranked last by PR"} \\ X_3 &= \text{"C is the CL and C is ranked last by PR"} \end{aligned}$$

Recall that a voting situation can be defined by the 6-tuple  $\tilde{n} = (n_1, n_2, \dots, n_6)$ , where  $n_i$  denotes the number of voters endowed with the associated  $i^{th}$  preference ranking, such that  $\sum_{i=1}^6 n_i = n$ . Moreover, under the  $\alpha$ -IAC assumption all possible voting situations  $\tilde{n} = (n_1, n_2, \dots, n_6)$  having a concrete value of  $\alpha$  are equally likely to be observed. Let us denote by  $N_{X_j}(k, n)$  the number of voting situations for which event  $X_j$  is observed under the  $\alpha$ -IAC assumption, i.e., when  $\alpha$  takes a given value. The number  $N_{X_j}(k, n)$  depends on the number of voters  $n$  and the value of  $k$ . Using those notations, the number of voting situations for which the CL exists under the  $\alpha$ -IAC assumption when candidate  $C$  is the last ranked one under PR can be written as follows:

$$N_{X_1}(k, n) + N_{X_2}(k, n) + N_{X_3}(k, n). \quad (3)$$

Due to the symmetry of IAC-like assumptions with respect to candidates, we can easily show that  $N_{X_1}(k, n) = N_{X_2}(k, n)$ . This means that the number of voting situations in (3) can also be written as follows:

$$2 N_{X_1}(k, n) + N_{X_3}(k, n). \quad (4)$$

Thus, all that we have to do is to calculate  $N_{X_1}(k, n)$  and  $N_{X_3}(k, n)$ . Notice first that  $N_{X_1}(k, n)$  corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\left\{ \begin{array}{l} -n_1 - n_2 - n_3 + n_4 + n_5 + n_6 > 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 > 0 \\ n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_5 + n_6 = k \\ n_i \geq 0 \text{ for } i \in \{1, \dots, 6\} \\ k \leq \frac{n}{3} \\ k \geq 0 \end{array} \right. \quad (5)$$

We compute the number of voting situations that fulfill these conditions using the Parametrized Barvinok's algorithm developed by [Verdoolaege et al. \(2004\)](#). This algorithm allows us to quantify the number of integer solutions for systems of (in)equalities with parameters and the output is given in the form of *Ehrhart polynomials* ([Ehrhart, 1962, 1967](#)). For more details on this algorithm, the reader may refer to the works of [Barvinok and Pommersheim \(1999\)](#); [Bruynooghe et al. \(2005\)](#); [Clauss and Loechner \(1998\)](#); [Diss and Mahajne \(2020\)](#); [Lepelley et al. \(2008\)](#); [Verdoolaege et al. \(2004\)](#). This method, along with related methods, has been widely used recently in research focusing on the probability of voting events under IAC-like assumptions. For instance, we can cite the works of [Bubboloni et al. \(2020\)](#); [Diss and Mahajne \(2020\)](#); [Diss and Merlin \(2021\)](#); [Diss et al. \(2023, 2020, 2012\)](#); [Kamwa \(2023, 2022\)](#); [Kamwa and Moyouwou \(2021\)](#); [Kamwa and Valognes \(2017\)](#); [Kamwa et al. \(2023\)](#), among others.

In our study, given the two parameters  $n$  and  $k$ , the number of voting situations for a system is given by bivariate quasi polynomials in  $n$  and  $k$  with 2-periodic coefficients meaning that such coefficients depend on the parity of the parameters  $n$  and  $k$ . Following the notation introduced in [Lepelley et al. \(2008\)](#), we represent these coefficients by a list of 2 rational numbers in square brackets. To illustrate, assume the bracketed list  $[[a, b]_n, [c, d]_n]_k$ . In the case of even  $k$ , the relevant list corresponds to  $[a, b]_n$ . The coefficient will be either  $a$  when  $n$  is even or  $b$  when  $n$  is odd. Similarly, in the case of odd  $k$ , the relevant list is  $[c, d]_n$  and therefore, the coefficient will be either  $c$  when  $n$  is even or  $d$  when  $n$  is odd. The program indicates that the corresponding quasi-polynomial for System (5) is given as follows:

1. If  $0 \leq k \leq \frac{n-4}{4}$ :

$$N_{X_1}(k, n) = \frac{5}{12} k^4 + f_1 k^3 + f_2 k^2 + f_3 k + f_4,$$

where

$$\begin{aligned} f_1 &= -\frac{1}{3} n + \left[ \frac{3}{2}, \frac{7}{6} \right]_n \\ f_2 &= -\frac{5}{4} n + \left[ \frac{19}{12}, \frac{1}{3} \right]_n \\ f_3 &= \frac{1}{48} n^3 + \left[ \frac{1}{8}, \frac{3}{16} \right]_n n^2 + \left[ -\frac{5}{4}, -\frac{15}{16} \right]_n n + \left[ \frac{1}{2}, -\frac{29}{48} \right]_n \\ f_4 &= \frac{1}{48} n^3 + \left[ \frac{1}{8}, \frac{3}{16} \right]_n n^2 + \left[ -\frac{1}{3}, -\frac{1}{48} \right]_n n + \left[ 0, -\frac{3}{16} \right]_n \end{aligned}$$

2. If  $\frac{n}{4} \leq k \leq \frac{n-2}{3}$ :

$$N_{X_1}(k, n) = \frac{15}{4} k^4 + g_1 k^3 + g_2 k^2 + g_3 k + g_4,$$

where

$$\begin{aligned} g_1 &= -5 n + \left[ \frac{1}{2}, \frac{7}{2} \right]_n \\ g_2 &= \frac{19}{8} n^2 + \left[ -1, -\frac{17}{4} \right]_n n + \left[ -\frac{7}{4}, -\frac{3}{8} \right]_n \\ g_3 &= -\frac{1}{2} n^3 + \left[ \frac{3}{8}, \frac{11}{8} \right]_n n^2 + \left[ 1, -\frac{3}{4} \right]_n n + \left[ -\frac{1}{2}, -\frac{9}{8} \right]_n \\ g_4 &= \frac{1}{24} n^4 + \left[ -\frac{1}{24}, -\frac{1}{8} \right]_n n^3 + \left[ -\frac{1}{6}, \frac{5}{24} \right]_n n^2 + \left[ \frac{1}{6}, \frac{1}{8} \right]_n n + \left[ 0, -\frac{1}{4} \right]_n \end{aligned}$$

3. Otherwise,  $N_{X_1}(k, n) = 0$ .

Now,  $N_{X_3}(k, n)$  corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\left\{ \begin{array}{l} n_1 + n_2 + n_3 - n_4 - n_5 - n_6 > 0 \\ n_1 - n_2 + n_3 + n_4 - n_5 - n_6 > 0 \\ n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_5 + n_6 = k \\ n_i \geq 0 \text{ for } i \in \{1, \dots, 6\} \\ k \leq \frac{n}{3} \\ k \geq 0 \end{array} \right. \quad (6)$$

Using again the Parametrized Barvinok's algorithm, the program indicates that the corresponding quasi-polynomial for System (6) is given as follows:

1. If  $0 \leq k \leq \frac{n-4}{4}$ :

$$N_{X_3}(k, n) = k^4 + h_1 k^3 + h_2 k^2 + h_3 k + h_4,$$

where

$$\begin{aligned} h_1 &= [6, 5]_n \\ h_2 &= -\frac{1}{2}n^2 - \frac{7}{2}n + [7, 3]_n \\ h_3 &= \frac{1}{8}n^3 + \left[0, \frac{1}{8}\right]_n n^2 + \left[-\frac{9}{2}, -\frac{29}{8}\right]_n n + \left[2, -\frac{13}{8}\right]_n \\ h_4 &= \frac{1}{8}n^3 + \left[\frac{1}{2}, \frac{5}{8}\right]_n n^2 + \left[-1, -\frac{1}{8}\right]_n n + \left[0, -\frac{5}{8}\right]_n \end{aligned}$$

2. If  $\frac{n-3}{4} \leq k \leq \frac{n-2}{3}$ :

$$N_{X_3}(k, n) = -3k^4 + i_1 k^3 + i_2 k^2 + i_3 k + i_4,$$

where

$$\begin{aligned} i_1 &= 4n + [-4, -1]_n \\ i_2 &= -\frac{7}{4}n^2 + \left[5, \frac{5}{2}\right]_n n + \left[-1, \frac{9}{4}\right]_n \\ i_3 &= \frac{1}{4}n^3 + \left[-2, -\frac{3}{2}\right]_n n^2 + \left[1, -\frac{7}{4}\right]_n n \\ i_4 &= \frac{1}{4}n^3 + \left[-\frac{1}{4}, \frac{1}{4}\right]_n n^2 + \left[0, -\frac{1}{4}\right]_n n + \left[0, -\frac{1}{4}\right]_n \end{aligned}$$

3. Otherwise,  $N_{X_3}(k, n) = 0$ .

In order to calculate the Strong Borda Paradox probability under the  $\alpha$ -IAC assumption, the three following disjoint events have to be taken into consideration:

$X_4$  = “ $A$  is the CL and the winner under PR. Additionally,  $C$  is ranked last by PR”.

$X_5$  = “ $B$  is the CL and the winner under PR. Additionally,  $C$  is ranked last by PR”.

$X_6$  = “ $C$  is the CL and the winner under PR. Additionally,  $C$  is ranked last by PR”.

It follows that the Strong Borda Paradox Probability under the  $\alpha$ -IAC assumption is given in general by the following function in  $n$  and  $k$ :

$$\frac{N_{X_4}(k, n) + N_{X_5}(k, n) + N_{X_6}(k, n)}{2N_{X_1}(k, n) + N_{X_3}(k, n)}. \quad (7)$$

We can show that  $N_{X_6}(k, n) = 0$  since when candidate  $C$  is chosen under PR it cannot be ranked last by this voting rule. Again, due to the symmetry of IAC-like assumptions with respect to candidates, we can also show that  $N_{X_4}(k, n) = N_{X_5}(k, n)$ . It follows that the

Strong Borda Paradox probability under the  $\alpha$ -IAC assumption in (7) can also be calculated as follows:

$$\frac{2 N_{X_4}(k, n)}{2 N_{X_1}(k, n) + N_{X_3}(k, n)}. \quad (8)$$

$N_{X_4}(k, n)$  corresponds to the number of voting situations satisfying the following system of (in)equalities:

$$\left\{ \begin{array}{l} -n_1 - n_2 - n_3 + n_4 + n_5 + n_6 > 0 \\ -n_1 - n_2 + n_3 + n_4 - n_5 + n_6 > 0 \\ n_1 + n_2 - n_3 - n_4 > 0 \\ n_1 + n_2 - n_5 - n_6 > 0 \\ n_3 + n_4 - n_5 - n_6 > 0 \\ n_1 + n_2 + n_3 + n_4 + n_5 + n_6 = n \\ n_5 + n_6 = k \\ n_i \geq 0 \text{ for } i \in \{1, \dots, 6\} \\ k \leq \frac{n}{3} \\ k \geq 0 \end{array} \right. \quad (9)$$

The program indicates that the corresponding quasi-polynomial of System (9) is given as follows:

1. If  $2 \leq k \leq \frac{n-2}{4}$ ,  $N_{X_4}(k, n) = -\frac{1}{64} k^4 + a_1(n, k) k^3 + a_2(n, k) k^2 + a_3(n, k) k + a_4(n, k)$ , where

$$\begin{aligned} a_1(n, k) &= \frac{1}{48} n + \left[ \left[ \frac{5}{48}, 0 \right]_n, \left[ \frac{1}{24}, \frac{1}{16} \right]_n \right]_k \\ a_2(n, k) &= \left[ \left[ -\frac{1}{16}, \frac{1}{16} \right]_n, [0, 0]_n \right]_k n + \left[ \left[ -\frac{3}{16}, \frac{1}{8} \right]_n, \left[ \frac{1}{32}, \frac{1}{32} \right]_n \right]_k \\ a_3(n, k) &= \left[ \left[ \frac{1}{24}, \frac{1}{24} \right]_n, \left[ -\frac{1}{48}, -\frac{1}{48} \right]_n \right]_k n + \left[ \left[ \frac{1}{12}, \frac{1}{8} \right]_n, \left[ -\frac{1}{24}, -\frac{1}{16} \right]_n \right]_k \\ a_4(n, k) &= \left[ [0, 0]_n, \left[ -\frac{1}{64}, -\frac{1}{64} \right]_n \right]_k \end{aligned}$$

2. If  $\frac{n-1}{4} \leq k \leq \frac{n-3}{3}$ ,  $N_{X_4}(k, n) = \frac{255}{64} k^4 + b_1(n, k) k^3 + b_2(n, k) k^2 + b_3(n, k) k + b_4(n, k)$ ,



where

$$\begin{aligned}
b_1(n, k) &= -\frac{85}{16}n + \left[ \left[ \frac{23}{16}, 4 \right]_n, \left[ \frac{11}{8}, \frac{65}{16} \right]_n \right]_k \\
b_2(n, k) &= \frac{5}{2}n^2 + \left[ \left[ -\frac{33}{16}, -\frac{79}{16} \right]_n, [-2, -5]_n \right]_k n + \left[ \left[ -\frac{19}{16}, -\frac{3}{8} \right]_n, \left[ -\frac{31}{32}, -\frac{15}{32} \right]_n \right]_k \\
b_3(n, k) &= -\frac{1}{2}n^3 + \left[ \frac{3}{4}, \frac{7}{4} \right]_n n^2 + \left[ \left[ \frac{3}{8}, -\frac{5}{8} \right]_n, \left[ \frac{5}{16}, -\frac{11}{16} \right]_n \right]_k n + \left[ \left[ -\frac{1}{4}, -\frac{9}{8} \right]_n, \left[ -\frac{3}{8}, -\frac{21}{16} \right]_n \right]_k \\
b_4(n, k) &= \frac{7}{192}n^4 + \left[ -\frac{1}{12}, -\frac{3}{16} \right]_n n^3 + \left[ -\frac{1}{48}, -\frac{19}{96} \right]_n n^2 + \left[ \frac{1}{12}, \frac{3}{16} \right]_n n + \left[ \left[ 0, -\frac{15}{64} \right]_n, \left[ -\frac{1}{64}, -\frac{1}{4} \right]_n \right]_k
\end{aligned}$$

3. Otherwise,  $N_{X_4}(k, n) = 0$ .

Notice that it is possible to represent the above results as functions of the closeness index  $\alpha$ . If we assume large electorates as it is the case in our paper and replace  $k$  by  $\alpha n$  in the above results, we obtain functions in  $\alpha$  by only considering the terms of higher degree in each function. Let us then denote by  $N_{X_j}^\infty(\alpha)$  the number of voting situations for which event  $X_j$  is observed under the  $\alpha$ -IAC assumption with large electorates. It follows that:

$$N_{X_1}^\infty(\alpha) = \begin{cases} \frac{\alpha(20\alpha^3 - 16\alpha^2 + 1)n^4}{48} & \text{for } 0 \leq \alpha \leq \frac{1}{4} \\ \frac{(3\alpha - 1)(30\alpha^3 - 30\alpha^2 + 9\alpha - 1)n^4}{24} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases} \quad (10)$$

and

$$N_{X_3}^\infty(\alpha) = \begin{cases} \frac{\alpha(2\alpha - 1)(4\alpha^2 + 2\alpha - 1)n^4}{8} & \text{for } 0 \leq \alpha \leq \frac{1}{4} \\ \frac{-(3\alpha - 1)(2\alpha - 1)^2 n^4}{4} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases} \quad (11)$$

and

$$N_{X_4}^\infty(\alpha) = \begin{cases} \frac{-\alpha^3(3\alpha - 4)n^4}{192} & \text{for } 0 \leq \alpha \leq \frac{1}{4} \\ \frac{(3\alpha - 1)(255\alpha^3 - 255\alpha^2 + 75\alpha - 7)n^4}{192} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases} \quad (12)$$

We then determine the desired probability, concluding the proof:

$$P_{SgBP}^\infty(\alpha) = \begin{cases} \frac{\alpha^2(4 - 3\alpha)}{16(11\alpha^3 - 4\alpha^2 - 3\alpha + 1)} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{255\alpha^3 - 255\alpha^2 + 75\alpha - 7}{8(18\alpha^3 - 18\alpha^2 + 6\alpha - 1)} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$$

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