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The effect of close elections on the likelihood of voting paradoxes: Further results in three-candidate elections

September 25, 2023

Abstract

The impact of election closeness on the likelihood of monotonicity paradox has recently been studied by some authors (e.g., [Lepelley et al., 2018](#); [Miller, 2017](#)). It was shown that the frequency of such a paradox significantly increases as elections become more closely contested. This paper aims to analyze the impact of election closeness on other best-known paradoxes of voting. Based on the Impartial Anonymous Culture (IAC) assumption, our preliminary results show that closeness has also a significant effect on the likelihood of observing the studied voting paradoxes in the class of scoring rules and scoring elimination rules.

Keywords: Voting paradoxes · Election closeness · Scoring Rules · Scoring Elimination Rules · Probability.

JEL classification: D71, D72.

L'impact des élections serrées sur la probabilité de paradoxes de vote : Autres résultats dans le cas des élections à trois candidats

Résumé

L'impact des élections serrées sur la probabilité de paradoxe de monotonie a été récemment étudié par certains auteurs (e.g., [Lepelley et al., 2018](#); [Miller, 2017](#)). Il a été démontré que la fréquence d'un tel paradoxe augmente considérablement à mesure que les élections deviennent plus disputées. Le présent papier vise à analyser l'impact des élections serrées sur d'autres paradoxes de vote bien connus. En se basant sur l'hypothèse de la culture neutre et anonyme (IAC), nos résultats montrent que des élections serrées ont un effet significatif sur la probabilité d'observer les paradoxes de vote étudiés dans la classe des règles de scores et des règles de scores avec éliminations.

Mots-clés : Paradoxes de vote · Elections serrées · Règles de scores · Règles de scores avec éliminations · Probabilité.

1 Introduction

An election is a pivotal moment in the democratic life of any institution, country, or group of people. During this crucial event, voters express their views on various alternatives such as manifestos, platforms, or propositions. They do so through an electoral process that aims to select or designate the individuals or laws that will govern collective affairs. In the best-case scenarios, the election result is a clear victory for a candidate or party. However, this is not always the case. Complications can arise when election results are closely contested. A narrow or marginal vote gap between the contenders can lead to intense disputes that may undermine the electoral process.

American politics, for instance, frequently experiences such situations. Among the numerous well-known examples of such situations, let us recall a few of them. The recount of votes in Virginia’s 94th district during the 2017 elections for the Virginia House of Delegates was marked by several astonishing twists and turns, first in favour of David Yancey, the Republican candidate, then shifting to his rival, Shelly Simonds, the Democrat, before finally ending in a tie following a court decision. In this district, the Democrat candidate, Shelly Simonds, was declared the winner by a single vote after a recount on December 19, 2017, with a tally of 11,608 votes for her and 11,607 for her opponent. However, the following day brought a new development when a tribunal composed of three judges refused to certify this recount, as the Republicans had raised concerns about the validity of a ballot with more than one inscription. After extensive deliberation, the judges awarded the vote to David Yancey, resulting in a tie of 11,608 votes for each candidate. Simonds was unsuccessful in her legal challenge against the disputed ballot. According to historical records, the 1960 presidential election between J.F. Kennedy and R. Nixon was the closest in terms of the popular vote since 1916. Kennedy won 49.72% of the vote, while Nixon received 49.55%. It is worth noting that there have been other elections in which a candidate won the electoral college while losing the popular vote, often by a very narrow margin. In 2000, for example, G.W. Bush won 271 electoral votes compared to 266 for Al Gore. The 2020 presidential election can also be taken as an example, with several disputes raised by candidate D. Trump and his supporters. The actions that followed, including the attack on the Capitol, dealt a significant blow to the electoral process and democracy.

These historical facts illustrate that when an election does not produce a clear-cut victory for a candidate, it can sometimes lead to undesirable situations. More recently, [Miller \(2017\)](#) highlighted that close elections can create opportunities for strategic behavior under certain voting rules. In his study on the conditions under which Instant Runoff Voting,¹ could lead to the monotonicity paradox,² [Miller \(2017\)](#) conducted numerical simulations and concluded that the frequency of the monotonicity paradox increases as elections become even more closely contested. This finding is further supported by [Lepelley et al. \(2018\)](#) who examined other voting rules in addition to the Plurality Elimination Rule such as the Negative Plurality

¹Also known as Alternative Vote or Plurality Elimination Rule.

²The monotonicity paradox occurs whenever some voters improve (resp. deteriorate) the ranking of a candidate in their individual preferences, all else unchanged, the position of this candidate is deteriorating (resp. improving) in the social ordering.

Elimination Rule and the Borda Elimination Rule. All of these voting rules will be defined later. In both [Miller \(2017\)](#) and [Lepelley et al. \(2018\)](#), the closeness of an election is quantified by the average number of points received by the last-ranked candidate, i.e., the candidate who receives the lowest total score. A higher value of this ratio indicates a more closely contested election. [Diss et al. \(2021\)](#) concluded that increased election closeness has a negative impact on the Condorcet efficiency³ of the five voting rules they analyzed, which are the Plurality Rule, the Borda Rule, the Negative Plurality Rule, the Plurality Elimination Rule, and the Negative Plurality Elimination Rule. Their results suggest that Condorcet efficiency tends to decrease as an election becomes more closely contested. It follows that a close election may accentuate the occurrence of certain electoral phenomena or paradoxes.

In this paper, we aim to provide further results to show that the effect of close election is not limited to the electoral events mentioned in [Diss et al. \(2021\)](#), [Lepelley et al. \(2018\)](#), and [Miller \(2017\)](#). We focus on three-candidate elections and we consider a wide range of voting events (defined in Section 2.3) and six well-known voting rules (Plurality Rule, Borda Rule, Negative Plurality Rule, Plurality Elimination Rule, Negative Plurality Elimination Rule and the Borda Elimination Rule). For each of the considered voting rules, we derive limiting probability representations for all the voting paradoxes considered, taking into account the factor of closeness. For our analysis, we consider large electorates, as in [Diss et al. \(2021\)](#) and [Lepelley et al. \(2018\)](#), and we assume that voters' preferences are distributed according to a version of the well-studied Impartial and Anonymous Culture (IAC) hypothesis. This is one of the well-known assumptions under which probability calculations are often driven in the literature of social choice theory. We will say more about this assumption later.

The rest of the paper is organized as follows: In Section 2, we introduce the basic notation and definitions; we also define all the voting rules and voting paradoxes we focus on. In Section 3, we derive from our computations, the impact of elections closeness on the likelihood of the voting paradoxes for large electorates. Section 4 concludes. The probability representations that we obtain from our calculations are given in the appendix.

2 Definitions and notation

2.1 Preferences

Consider a group of $n \geq 2$ individuals (voters, decision-makers, judges, etc.) and a set of three alternatives or candidates A , B , and C . The individuals rank the candidates from the most desirable candidate to the least desirable one. Each ranking is then assumed to be a linear order, i.e., a transitive, antisymmetric, and total relation. It is also assumed that each voter votes sincerely and acts according to her true preferences. In three-candidate elections, there are six possible strict rankings, which are displayed in Table 1. In this table, it is indicated that n_1 voters have the ranking ABC , which means that they rank candidate A

³The Condorcet efficiency of a voting rule is its propensity to elect the Condorcet winner when she exists. A Condorcet winner, when she exists, is a candidate who defeats each of the other candidates in pairwise comparisons.

at the top followed by candidate B and candidate C is the least preferred. In this framework, a *voting situation* is defined by the vector $\tilde{n} = (n_1, \dots, n_6)$ which indicates the number of voters endowed with each linear order such that $\sum_{i=1}^6 n_i = n$.

Table 1: The six possible strict rankings in three-candidate elections

$n_1 : ABC$	$n_2 : ACB$	$n_3 : BAC$	$n_4 : BCA$	$n_5 : CAB$	$n_6 : CBA$
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Given two candidates A and B , we will write AMB to say that candidate A is majority preferred over candidate B , which means that there are more than half of the voters who prefer A to B ; we can also say that candidate B is majority dominated by candidate A . If a candidate is majority preferred over each of the candidates, she is said to be the *Condorcet winner*. If a candidate is majority dominated by each of the other candidates, she is said to be the *Condorcet loser*. Note that the Condorcet winner (or the Condorcet loser) does not always exist since the majority relation \mathbf{M} can be *cyclic*; in such a case we can get a first possible cycle where AMB , BMC , and CMA or a second possible cycle where BMA , AMC , and CMB .

2.2 Voting rules

A *scoring rule* is a voting system that assigns points to candidates according to their rank in voters' preferences and the winner is the candidate with the highest score (total number of points). In this paper, we focus on three well-known scoring rules (Plurality Rule, Negative Plurality Rule and Borda Rule) and three well-known scoring elimination rules (Plurality Elimination Rule, Negative Plurality Elimination Rule and Borda Elimination Rule).

The Plurality Rule (henceforth PR) only takes into account the voters' first choice; that is, the winner is the candidate with the highest total number of first places in the voters' preferences. Under the Negative Plurality Rule (henceforth NPR), the winner is the candidate with the lowest number of last places in the voters' rankings. In three-candidate elections, the Borda Rule (henceforth BR) considers that a candidate receives two points each time she is ranked first, one point for each second place, and zero point for each last place; the winner is the candidate with the highest total score. For a given voting rule, we simply denote by $S(A)$ the score of candidate A . Using the labels from Table 1, the scores of the candidates under each of the three voting rules are given in Table 2.

Table 2: The scores of the candidates according to Table 1

		Voting rules		
Scores		PR	BR	NPR
	$S(A)$	$n_1 + n_2$	$2n_1 + 2n_2 + n_3 + n_5$	$n_1 + n_2 + n_3 + n_5$
	$S(B)$	$n_3 + n_4$	$n_1 + 2n_3 + 2n_4 + n_6$	$n_1 + n_3 + n_4 + n_6$
	$S(C)$	$n_5 + n_6$	$n_2 + n_4 + 2n_5 + 2n_6$	$n_2 + n_4 + n_5 + n_6$

In a three-candidate election, a scoring elimination rule operates in two steps or rounds. In the first round, the candidate with the lowest score is eliminated and the two remaining candidates go into a second round where the winner is the one who obtains the majority of votes.⁴ Thus, under the Plurality Elimination Rule (henceforth PER), the eliminated candidate is the one with the lowest Plurality score; under the Negative Plurality Elimination Rule (henceforth NPER), the eliminated candidate is the one with the lowest Negative Plurality score; under the Borda Elimination Rule (henceforth BER), the eliminated candidate is the one with the lowest Borda score.

2.3 Voting paradoxes under consideration

A voting paradox is an undesirable outcome that can be observed under a given voting rule and that may be considered as surprising or counterintuitive. Since the seminal works of [Arrow \(1951\)](#), [Gibbard \(1973\)](#), and [Satterthwaite \(1975\)](#), it is known that there is no voting rule that is free from any voting paradox. The social choice literature defines and describes a large number of voting paradoxes; for a non exhaustive review of these paradoxes, the reader may refer to the books by [Diss and Merlin \(2021\)](#), [Felsenthal \(2012\)](#), [Felsenthal and Nurmi \(2018\)](#), [Gehrlein and Lepelley \(2011, 2017\)](#), [Nurmi \(1987, 1999\)](#), and [Saari \(1994, 2001, 2008\)](#). In this paper, we deal with the following voting events, which are widely studied in the literature of social choice theory:

Strong Borda Paradox: This paradox describes a situation where the Condorcet loser exists and is elected. It is known from [Fishburn and Gehrlein \(1976\)](#) that BR is the only scoring rule that cannot elect the Condorcet loser when she exists. In other words, BR never exhibits the Strong Borda Paradox.

Strict Borda Paradox: This paradox is related to situations where the Condorcet loser is elected and, at the same time, the Condorcet winner is ranked last. [Daunou \(1803\)](#) showed that BR never ranks the Condorcet winner in the last position in an election (see also [Gärdenfors, 1973](#); [Smith, 1973](#)), which means that this paradox never occurs with BR.

⁴Notice that ties are not possible in our framework as we consider large electorates.

Absolute Majority Winner Paradox: This paradox occurs when a candidate is not elected despite being ranked first by more than half of the voters. It is obvious that a candidate ranked first by more than half of the voters is the Plurality winner; so, the Absolute Majority Winner Paradox never appears under PR.

Absolute Majority Loser Paradox: This paradox is a special case of the Strong Borda Paradox. It occurs when a candidate may be elected despite she is ranked last by more than half of the voters. Since a candidate ranked last by more than half of the voters is also a Condorcet loser, it follows that BR never exhibits this paradox.

Reversal Bias Paradox: This paradox, also known as the *Preference Inversion Paradox*, occurs when the election winner remains unchanged after reverting all the preferences of the voters. According to [Saari and Barney \(2003\)](#), BR is the only scoring rule that is not vulnerable to this paradox.

Sincere Truncation: A voting rule is said to be vulnerable to the sincere truncation if there are some configurations of ballots such that there is at least one voter who prefers the outcome obtained by submitting a sincere but incomplete ranking (truncated ranking) to the outcome obtained by submitting a complete sincere ranking ([Brams, 1982](#); [Fishburn and Brams, 1983, 1984](#)). Among the family of scoring rules, only PR is immune to this paradox.

Positive Abstention Paradox: This paradox occurs when some of the voters who place a losing candidate at the top of their ranking are deleted, and this candidate becomes the winner.

Negative Abstention Paradox: This paradox occurs when some of the voters who ranked the winning candidate last are deleted, making that candidate a loser.

Positive Participation Paradox: This paradox describes an electoral situation in which the winning candidate is made a loser by the addition of some voters who place the winning candidate at the top of their rankings.

Negative Participation Paradox: This paradox occurs when some voters who ranked a losing candidate at the bottom of their rankings are added to make that candidate the winner.

Table 3 gives us a glimpse of whether or not the above voting rules are vulnerable to the paradoxes just mentioned. In this table, a “yes” indicates that the voting rule is vulnerable to the corresponding paradox and a “no” means the opposite.

Table 3: Voting rules and paradoxes

<i>Voting Paradoxes</i>	<i>Voting rules</i>					
	PR	BR	NPR	PER	BER	NPER
Strong Borda	yes	no	yes	no	no	no
Strict Borda	yes	no	yes	no	no	no
Absolute Majority Winner	no	yes	yes	no	no	yes
Absolute Majority Loser	yes	no	no	no	no	no
Reversal Bias	yes	no	yes	yes	yes	yes
Sincere Truncation	no	yes	yes	no	yes	yes
Negative Abstention	no	no	no	yes	yes	no
Positive Abstention	no	no	no	no	yes	yes
Negative Participation	no	no	no	yes	yes	no
Positive Participation	no	no	no	no	yes	yes

The probabilistic approach applied to voting rules is often seen as complementary to the normative approach, which evaluates voting rules based on a set of criteria they either meet or fail to satisfy. The probabilistic approach quantifies the frequency with which a voting rule violates each of the considered criterion that it fails to satisfy. The probabilities of the occurring voting paradoxes can be used as criteria in order to evaluate voting rules. For this reason, the social choice literature, rooted in the probabilistic approach to voting rules, is replete with results on the likelihood of electoral events. The sheer volume of these results makes it impossible to provide a comprehensive summary in this paper. It is important to note that the majority of these findings are based on two primary assumptions: the Impartial and Anonymous Culture (IAC) and the Impartial Culture (IC). In this paper, we will solely focus on the IAC assumption. For further details on the Impartial Culture and other assumptions, interested readers can refer to the books by [Gehrlein and Lepelley \(2011, 2017\)](#); [Diss and Merlin \(2021\)](#) and [Regenwetter et al. \(2006\)](#). Under IAC, it is assumed that each voting situation $\tilde{n} = (n_1, \dots, n_6)$ is equally likely to occur. This hypothesis was introduced by [Kuga and Hiroaki \(1974\)](#) and later developed by [Gehrlein and Fishburn \(1976\)](#). According to this assumption, the likelihood of a given event X is calculated in respect with the following ratio:

$$\frac{\text{Number of voting situations in which event } X \text{ is likely}}{\text{Total number of possible voting situations}} \quad (1)$$

Based on the results found in the literature, Table 4 recalls for three-candidate elections, the limiting probability of each voting paradox considered here, specifically under the voting

rules we have focused on.

Table 4: Limiting probabilities of paradoxes under IAC in three-candidate elections

<i>Voting Paradoxes</i>	<i>Voting rules</i>					
	PR	BR	NPR	PER	BER	NPER
Strong Borda	0.0296 ^a	0	0.0315 ^a	0	0	0
Strict Borda	0.0111 ^b	0	0.0111 ^b	0	0	0
Absolute Majority Winner	0	0.0370 ^c	0.3919 ^c	0	0	0.0247 ^d
Absolute Majority Loser	0.0247 ^c	0	0	0	0	0
Reversal Bias	0.0926 ^e	0	0.2037 ^e	0.0284 ^f	0.0417 ^f	0.0127 ^f
Sincere Truncation	0	0.3664 ^g	0.7222 ^g	0	0.0682 ^g	0.1597 ^g
Negative Abstention	0	0	0	0.0408 ^h	0.0139 ⁱ	0
Positive Abstention	0	0	0	0	0.0104 ⁱ	0.0425 ^h
Negative Participation	0	0	0	0.0729 ^h	0.0208 ⁱ	0
Positive Participation	0	0	0	0	0.0139 ⁱ	0.0382 ^h

Sources: a. [Gehrlein \(2002\)](#); b. [Gehrlein and Lepelley \(2010\)](#); c. [Diss et al. \(2018\)](#); d. author’s calculations; e. [Bubboloni et al. \(2020\)](#); f. [Belayadi and Mbih \(2021\)](#); g. [Kamwa and Moyouwou \(2021\)](#), [Kamwa \(2022\)](#); h. [Lepelley and Merlin \(2001\)](#); i. [Wilson and Pritchard \(2007\)](#), [Kamwa et al. \(2021\)](#).

Table 4 highlights the varying susceptibility of different paradoxes depending on the choice of voting rules. It should be noted, however, that these probabilities are far from being negligible. It is also important to remember that these probabilities do not take into account whether the election results are close or not. It is therefore interesting to consider what would happen to these probabilities if the closeness of the election results were taken into account. An increase in these probabilities would confirm the observations made in the literature that closeness tends to favour the occurrence of voting paradoxes.

2.4 An index to reflect closeness in elections

As a reminder, the main focus of our paper is to assess how the closeness of election outcomes affects the likelihood of the voting paradoxes we have defined earlier. To do so, we adopt a closeness index, following the methodology of [Diss et al. \(2021\)](#), [Lepelley et al. \(2018\)](#), and [Miller \(2017\)](#). Within a given voting rule, this index is calculated as the ratio between the score received by the candidate ranked last and the sum of the scores received by all the competing candidates.

For scenarios involving three candidates A , B , and C , suppose without loss of generality that candidate C receives the lowest score according to a given voting rule. In this case, the

closeness index is computed as follows:

$$\mathcal{I} = \frac{S(C)}{S(A) + S(B) + S(C)} \quad (2)$$

Let α , γ , and β denote the closeness indices under PR, NPR, and BR, respectively. Given the rankings in Table 1 and the scores in Table 2, if candidate C has the worst score, we derive the closeness index under each of the three scoring rules as follows:

- under PR: $\alpha = \frac{n_5+n_6}{n}$ with $0 \leq \alpha \leq \frac{1}{3}$;
- under BR: $\beta = \frac{n_2+n_4+2(n_5+n_6)}{3n}$ with $0 \leq \beta \leq \frac{1}{3}$;
- under NPR: $\gamma = \frac{n_2+n_4+n_5+n_6}{2n}$ with $0 \leq \gamma \leq \frac{1}{3}$.

3 Impact of closeness on the limiting probability of voting paradoxes

In order to assess the influence of closeness on the probabilities of paradox occurrence, it is essential to represent these probabilities as functions of the closeness index. This task requires a revision of the IAC assumption for the specific purpose under consideration. Recall that we assume without loss of generality that candidate C receives the lowest score under a given voting rule. Given the closeness indices α , β , and γ that have been defined in Section 2.4, the \mathcal{I} -IAC assumption (for $\mathcal{I} = \alpha, \beta, \gamma$) is defined as follows: all voting scenarios in which candidate C is ranked last are considered equiprobable. To derive our probabilities, we use the *parameterized Barvinok's algorithm* developed by Verdoolaeye et al. (2004). This algorithm is encoded to compute the number of lattice points in a rational convex polytope and the output is given in the form of *Ehrhart polynomials* (Ehrhart, 1962, 1967). For more details on this algorithm and the related subjects, the reader may refer to the works of Barvinok and Pommersheim (1999), Bruynooghe et al. (2005), Clauss and Loechner (1998), Lepelley et al. (2008), and Wilson and Pritchard (2007). We report in the Appendix all of the limiting probabilities that we obtain from our calculations. We have chosen to skip the computation details; nonetheless, they are available upon simple request. We concentrate here on commenting on our results regarding the impact of closeness on the limiting probability of the voting paradoxes under consideration. In order to present our results, we separately address the case of each of the voting rules covered by our analysis.

3.1 Closeness and the Plurality Rule

Among the paradoxes presented in Table 3, it was noted that PR was found to be susceptible to the Strong Borda Paradox, the Strict Borda Paradox, the Absolute Majority Loser Paradox, and the Reversal Bias Paradox. Figure 1 provides a comprehensive view of how closeness, measured by the index α , affects the limiting probabilities of these paradoxes.

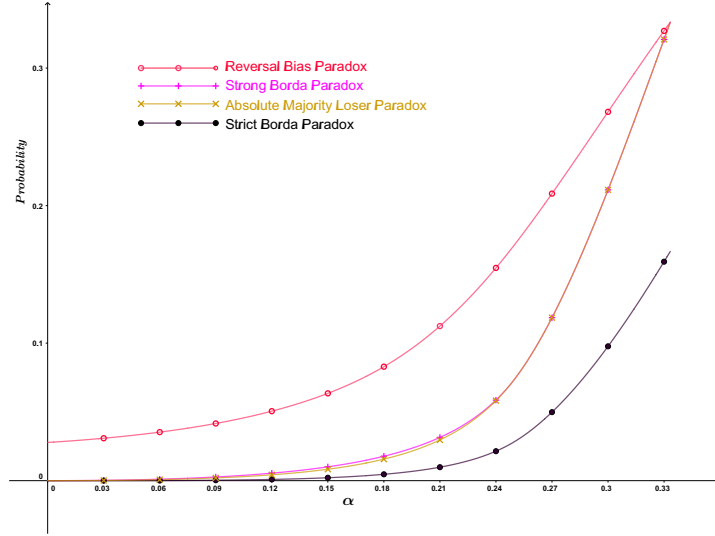


Figure 1: Closeness and voting paradoxes afflicting PR

When $\alpha = 0$ (i.e., candidate C gets no points), only the Reversal Bias Paradox remains viable, while the other paradoxes disappear. In particular, as elections become closer (where α increases from 0 to $\frac{1}{3}$), PR becomes more likely to lead to each of the four paradoxes. It is interesting to note that the limiting probabilities of the Strong Borda Paradox and the Absolute Majority Loser Paradox become quite similar when closeness is taken into account. For reference, Table 4 shows that the limiting probability of the Strong Borda Paradox is 2.96%, the Absolute Majority Loser Paradox is 2.47%, the Strict Borda Paradox is 1.11%, and the Reversal Bias Paradox is 9.26% without considering closeness. However, these probabilities can be significantly exceeded and amplified depending on the precise values of the closeness index α . Closeness appears to mitigate the occurrence of paradoxes for values of α below certain thresholds, which are approximately 0.1552 for the Strong Borda Paradox, 0.2149 for the Strict Borda Paradox, 0.2016 for the Absolute Majority Loser Paradox, and 0.1913 for the Reversal Bias Paradox. Beyond these values, closeness contributes significantly to the likelihood of each paradox occurring. The impact of closeness on the occurrence of voting paradoxes is summarized in Table 5.

Table 5: Impact of closeness on the likelihood of voting paradoxes under PR

	α	
<i>Voting Paradoxes</i>	a decrease	an increase
Strong Borda	$[0, 0.1552[$	$]0.1552, \frac{1}{3}]$
Strict Borda	$[0, 0.2149[$	$]0.2149, \frac{1}{3}]$
Absolute Majority Loser	$[0, 0.2016[$	$]0.2016, \frac{1}{3}]$
Reversal Bias	$[0, 0.1913[$	$]0.1913, \frac{1}{3}]$

3.2 Closeness and the Borda Rule

For the two paradoxes involving BR, Figure 2 shows the evolution of the probabilities as a function of the closeness index β .

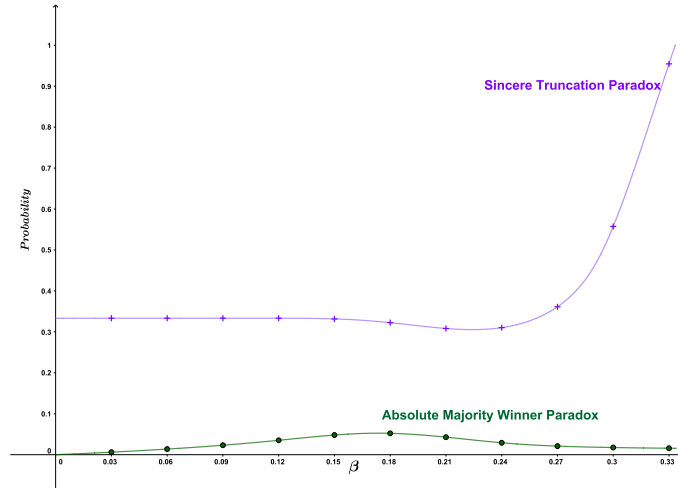


Figure 2: Closeness and voting paradoxes afflicting BR

As the value of β increases, the probabilities of the Absolute Majority Winner Paradox exhibit an interesting pattern. Initially, there is an increasing phase for $\beta < 0.1738$, followed by a decreasing phase for $\beta > 0.1738$. In particular, for $0.1239 \leq \beta < 0.2223$, the probabilities of the Absolute Majority Winner Paradox significantly exceed 3.70%, which is the limiting probability of this paradox without considering closeness. Outside this range, the impact of closeness tends to attenuate the occurrence of the Absolute Majority Winner Paradox, as the probabilities do not exceed 3.70%. A similar trend is observed with the Sincere Truncation Paradox. When β is lower than approximately 0.2716, the probabilities remain below 36.64%, which is the known limit value of this paradox without taking closeness

into account. However, when β exceeds about 0.2716, the probabilities increase rapidly, reaching a peak of 100% as β approaches $\frac{1}{3}$. This phenomenon implies that BR becomes more susceptible to manipulation by truncation as the closeness of the election increases. Consequently, closeness amplifies the occurrence of each of the two paradoxes affecting BR that we consider. In Table 6, we summarize the impact of closeness on the likelihood of paradoxes according to the values of β compared to the case without closeness.

Table 6: Impact of closeness on the likelihood of voting paradoxes under BR

	β	
<i>Voting Paradoxes</i>	a decrease	an increase
Absolute Majority Winner	$[0, 0.1239[\cup]0.2223, \frac{1}{3}]$	$[0.1239, 0.2223[$
Sincere Truncation	$]0, 0.2716[$	$]0.2716, \frac{1}{3}]$

3.3 Closeness and the Negative Plurality Rule

Figure 3 provides an overview of how closeness, represented by γ , affects the probability of voting events under NPR.

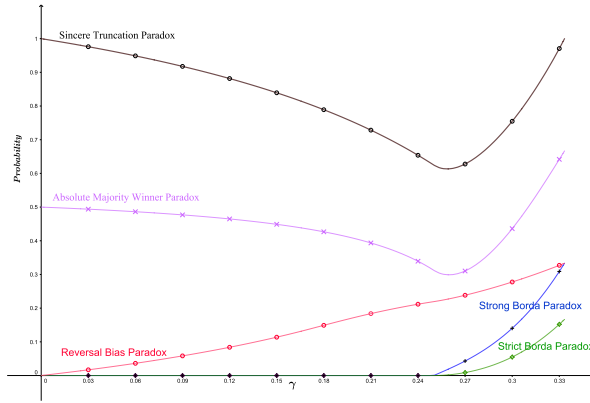


Figure 3: Closeness and voting paradoxes afflicting NPR

We observed that when the candidate ranked last receives less than a quarter of the votes cast ($0 \leq \gamma < \frac{1}{4}$), the Strong Borda Paradox and the Strict Borda Paradox are no longer observed under NPR; this is also true for the Reversal Bias Paradox when $\gamma = 0$. However, beyond these values of γ , each of these paradoxes reappears and seems to increase in frequency as γ increases. In the case of the Sincere Truncation Paradox, its behavior evolves in two distinct phases while maintaining probabilities above 50%: a phase of decrease for values of $\gamma < 0.2586$, followed by a phase of increase for $\gamma > 0.2586$. As γ approaches

$\frac{1}{3}$, the vulnerability of the Sincere Truncation Paradox reaches 100%. In other words, when the candidate ranked last under NPR obtains close to a third of the votes cast, it becomes certain that a group of voters can manipulate the outcome of the vote through truncation of their preferences.

With respect to the Absolute Majority Winner Paradox, the probabilities exhibit a tendency to decrease for values of $\gamma < 0.26$ and then to increase as γ varies. Table 7 summarizes, the impact of closeness on the likelihood of paradoxes following the values of γ in comparison with the case without closeness.

Table 7: Impact of closeness on the likelihood of voting paradoxes under NPR

	γ	
<i>Voting Paradoxes</i>	a decrease	an increase
Strong Borda	$[0, 0.2651[$	$[0.2651, \frac{1}{3}]$
Strict Borda	$[0, 0.2726[$	$[0.2726, \frac{1}{3}]$
Absolute Majority Winner	$]0.2112, 0.2919[$	$[0, 0.2112[\cup]0.2919, \frac{1}{3}]$
Reversal Bias	$[0, 0.2304[$	$[0.2304, \frac{1}{3}]$
Sincere Truncation	$]0.2368, 0.2940[$	$[0, 0.2368[\cup]0.2940, \frac{1}{3}]$

Before we look at the scoring elimination rules, it is worth highlighting an observation about our three scoring rules. For each of these scoring rules, the introduction of closeness as a parameter never changes the order of paradox occurrence among the voting paradoxes that we consider in this paper. To illustrate this point, let us consider the example of BR. Under this rule, the Truncation Paradox is more likely to occur than the Absolute Majority Winner Paradox in the absence of the closeness parameter as shown in Table 4. Importantly, this order of events remains consistent even when closeness is taken into account. However, this is not always the case with our three scoring elimination rules, as we will see later.

3.4 Closeness and the Plurality Elimination Rule

From Table 4, we can see that the limiting probabilities of the three voting paradoxes associated with PER in the absence of any closeness index are as follows: 4.08% for the Negative Abstention Paradox, 7.29% for the Negative Participation Paradox, and only 0.0039% for the Reversal Bias Paradox. Figure 4 gives us a comparative view, taking into account different levels of closeness, and illustrates its impact on the occurrence of each of these three paradoxes.

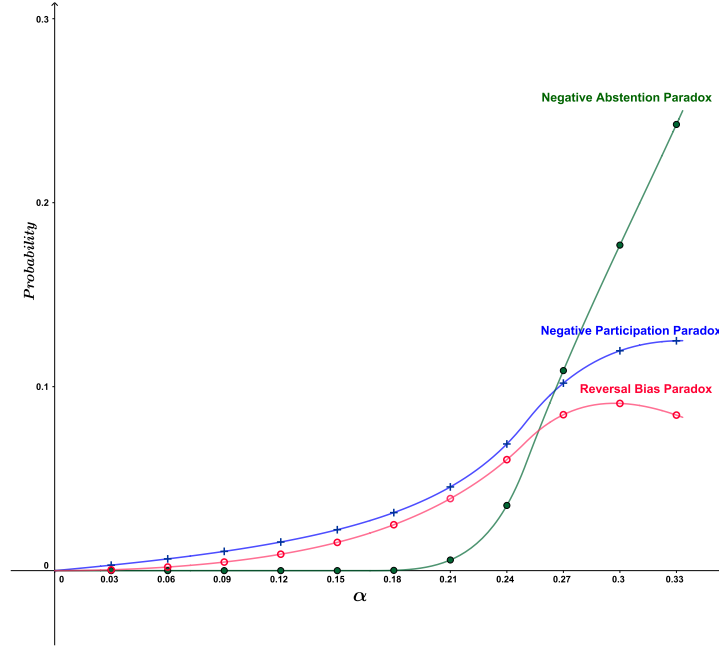


Figure 4: Closeness and voting paradoxes afflicting PER

We can see that the limiting probability of the Reversal Bias Paradox remains low for closeness levels up to $\alpha \leq 0.1882$ and it does exceed the probability value obtained when closeness is not considered. However, for $0.1882 \leq \alpha < \frac{1}{3}$, the introduction of proximity tends to significantly amplify the occurrence of the paradox, almost doubling its probability compared to the case without closeness. With respect to the Negative Abstention Paradox and the Negative Participation Paradox, we observe an increase in the probabilities of occurrence for values of α above about 0.2428 and 0.2437, respectively. Below these thresholds, the probabilities of these voting paradoxes remain below their values obtained when closeness is not taken into account. The impact of closeness on the occurrence of voting paradoxes under PER is summarized in Table 8.

Table 8: Impact of closeness on the likelihood of voting paradoxes under PER

	α	
<i>Voting Paradoxes</i>	a decrease	an increase
Negative Abstention	$[0, 0.2428[$	$[0.2428, \frac{1}{3}]$
Negative Participation	$[0, 0.2437[$	$[0.2437, \frac{1}{3}]$
Reversal Bias	$[0, 0.1882[$	$[0.1882, \frac{1}{3}]$

As shown in Table 4, of the three paradoxes considered, the one most likely to occur under PER is the Negative Participation Paradox, while the Reversal Bias Paradox is least likely. However, our results show that this hierarchy does not hold across all values of α . While this hierarchy remains intact for $\alpha > \frac{1}{4}$, it undergoes a shift for $\alpha < \frac{1}{4}$, where the Negative Participation Paradox becomes the least vulnerable of the three paradoxes.

3.5 Closeness and the Borda Elimination Rule

Let us recall that, without taking into account the closeness criterion, the limiting probabilities of the four voting paradoxes associated with BER are as follows: 6.82% for the Truncation Paradox, 1.39% for the Negative Abstention Paradox, 2.08% for the Negative Participation Paradox, and 1.04% for the Positive Abstention Paradox. However, when we introduce the closeness index β into the equation, some interesting observations emerge. Firstly, it is worth noting that the Reversal Bias Paradox is possible to occur under BER only when $\frac{1}{6} \leq \beta < \frac{1}{3}$. In addition, we found that both the Negative Abstention Paradox and the Positive Participation Paradox lead to approximately the same probabilities as when closeness is not taken into account. A visual representation of these results is displayed in Figure 5.

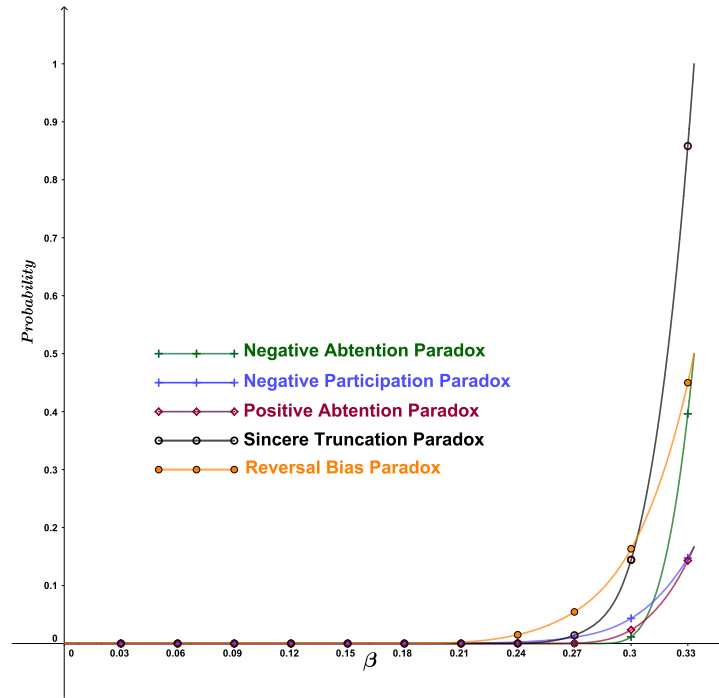


Figure 5: Closeness and voting paradoxes afflicting BER

An interesting observation is that when the score of the candidate ranked last represents less than 25% of the votes cast ($0 \leq \beta < \frac{1}{4}$), the Positive Abstention Paradox no longer

occurs under BER. Similar trends are observed for other paradoxes, where the presence of closeness significantly reduces the probability of their occurrence: the Negative Abstention Paradox, the Negative Participation Paradox, and the Truncation Paradox disappear when $\beta < \frac{5}{18}$, $\beta < \frac{1}{6}$, and $\beta < \frac{2}{9}$, respectively.

Table 9 provides a comparison between the probabilities of paradoxes with and without considering closeness. It is clear that over a substantial part of the β range, the introduction of closeness greatly reduces the occurrence of these paradoxes, and in some cases eliminates them altogether.

Table 9: Impact of closeness on the likelihood of voting paradoxes under BER

	β	
<i>Voting Paradoxes</i>	a decrease	an increase
Sincere Truncation	$[0, 0.2904[$	$[0.2904, \frac{1}{3}]$
Reversal Bias	$[0, 0.2815[$	$[0.2815, \frac{1}{3}]$
Negative Abstention	$[0, 0.3010[$	$[0.3010, \frac{1}{3}]$
Negative Participation	$[0, 0.2843[$	$[0.2843, \frac{1}{3}]$
Positive Abstention	$[0, 0.2911[$	$[0.2911, \frac{1}{3}]$

Comparing our results with those presented in Table 4, it is clear that the Sincere Truncation Paradox consistently retains its position as the most likely to occur, regardless of whether or not closeness is taken into account. However, the hierarchy in terms of occurrence probabilities is not always maintained for the other paradoxes. In particular, for certain values of the index β , the order of paradox occurrence probabilities varies when closeness is taken into account.

3.6 Closeness and the Negative Plurality Elimination Rule

Our calculations have revealed a remarkable phenomenon under NPER: when the candidate eliminated in the first round receives less than a quarter of the votes cast, none of the five considered paradoxes can be observed. However, as γ exceeds the threshold of 0.25, the probabilities of occurrence increase dramatically, reaching 62.50% for the Sincere Truncation Paradox, 33.33% for the Absolute Majority Winner Paradox, 25% for the Positive Participation Paradox, 12.50% for the Positive Abstention Paradox, and 8.33% for the Reversal Bias Paradox. This striking trend is illustrated in Figure 6 as a function of the closeness index γ .

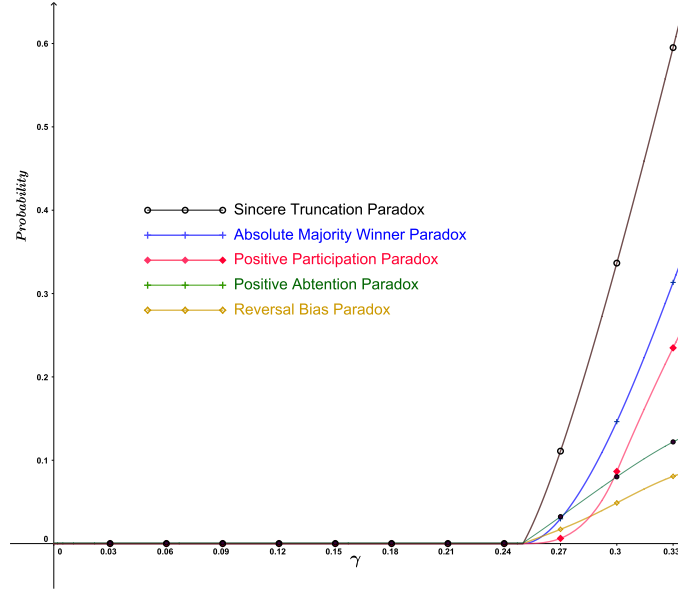


Figure 6: Closeness and voting paradoxes afflicting NPER

The impact of closeness on the occurrence of voting paradoxes is summarized in Table 10. This table provides an insight into the values of γ for which considering the closeness of the election results in either increased or decreased probabilities compared to scenarios without closeness (as shown in Table 4).

Table 10: Impact of closeness on the likelihood of voting paradoxes under NPER

	γ	
<i>Voting Paradoxes</i>	a decrease	an increase
Absolute Majority Winner	$[0, 0.2680[$	$[0.2680, \frac{1}{3}]$
Reversal Bias	$[0, 0.2649[$	$[0.2649, \frac{1}{3}]$
Positive Abstention	$[0, 0.2768[$	$[0.2768, \frac{1}{3}]$
Positive Participation	$[0, 0.2879[$	$[0.2879, \frac{1}{3}]$
Sincere Truncation	$[0, 0.2771[$	$[0.2771, \frac{1}{3}]$

Comparing our results with those in Table 4, we see that the Sincere Truncation Paradox consistently remains the most likely to occur, whether or not election closeness is taken into account. However, for the other paradoxes, especially when γ is less than about 0.29, the hierarchy in terms of probability of occurrence does not always follow the same pattern.

4 Conclusion

The aim of this paper was to provide further insights into the impact of closely contested elections on the likelihood of different voting paradoxes in three-candidate elections in the case of large electorates. Our analysis focused on assessing the likelihood of ten well-known voting paradoxes and covered both scoring rules and scoring elimination rules. Our methodology in this study was inspired by previous research, such as [Miller \(2017\)](#), [Lepelley et al. \(2018\)](#), and [Diss et al. \(2021\)](#), where we use the same index to measure the degree of closeness of elections. This index is calculated as the proportion of scores received by the candidate with the fewest scores.

In order to achieve our goal, we calculated the limiting probability representations for each paradox that could occur under different voting rules, taking into account different degrees of election closeness. These representations served as a basis for examining how the closeness of the election affects the likelihood of these paradoxes occurring. We identified scenarios in which the introduction of the closeness index either increased or decreased the likelihood of paradoxes, depending on particular values of election closeness.

Our analysis also revealed an interesting insight. When we examined one of the three scoring rules in conjunction with the corresponding voting paradoxes, we observed that regardless of the degree of election closeness, accounting for closeness preserved the hierarchy among these paradoxes in terms of their likelihood of occurrence compared to situations without election closeness. It is worth noting, however, that this observation did not hold universally for all scoring elimination rules, especially at certain levels of election closeness.

Appendix: Probability representations

A- Plurality Rule

Given a three-candidate election with large electorates and α the average number of points obtained by the last ranked candidate under PR,

- the probability of the Strong Borda Paradox (SgBP) under PR is given as follows:

$$P_{SgBP}^{\infty}(\alpha) = \begin{cases} \frac{\alpha^2(4-3\alpha)}{16(11\alpha^3-4\alpha^2-3\alpha+1)} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{255\alpha^3-255\alpha^2+75\alpha-7}{8(18\alpha^3-18\alpha^2+6\alpha-1)} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$$

- the probability of the Strict Borda Paradox (StBP) under PR is given as follows:

$$P_{StBP}^{\infty}(\alpha) = \begin{cases} \frac{5\alpha^3}{176\alpha^3-64\alpha^2-48\alpha+16} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{87\alpha^3-99\alpha^2+31\alpha-3}{8(18\alpha^3-18\alpha^2+6\alpha-1)} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$$

- the probability of the Absolute Majority Loser Paradox (AMLPL) under PR is given as follows:

$$P_{AMLPL}^{\infty}(\alpha) = \begin{cases} \frac{\alpha^2 (4 - 3\alpha)}{8(54\alpha^3 - 12\alpha + 3 - 8\alpha^2)} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{255\alpha^3 - 255\alpha^2 + 75\alpha - 7}{8(18\alpha^3 - 18\alpha^2 + 6\alpha - 1)} & \text{for } \frac{1}{4} \leq \alpha \leq \frac{1}{3} \end{cases}$$

- the probability of the Reversal Bias Paradox (RBP) under PR is given as follows:

$$P_{RBP}^{\infty}(\alpha) = \begin{cases} \frac{-9\alpha^3 + 9\alpha^2 + 1}{36(3\alpha - 1)(3\alpha^2 - 1)} & \text{for } 0 \leq \alpha < \frac{1}{6} \\ \frac{-2619\alpha^4 + 1755\alpha^3 - 432\alpha^2 + 51\alpha - 2}{108\alpha(3\alpha - 1)(3\alpha^2 - 1)} & \text{for } \frac{1}{6} \leq \alpha < \frac{1}{4} \\ \frac{1431\alpha^3 - 1242\alpha^2 + 306\alpha - 25}{108\alpha(3\alpha^2 - 1)} & \text{for } \frac{1}{4} \leq \alpha < \frac{1}{3} \end{cases}$$

B- Borda Rule

Consider a three-candidate election with large electorates and β the average number of points obtained by the last ranked candidate under BR,

- the probability of the Absolute Majority Winner Paradox (AMWP) under BR is given as follows:

$$P_{AMWP}^{\infty}(\beta) = \begin{cases} \frac{-3\beta}{16(3\beta - 1)} & \text{for } 0 \leq \beta < \frac{1}{9} \\ \frac{6075\beta^4 - 2916\beta^3 + 486\beta^2 - 36\beta + 1}{2592(3\beta - 1)\beta^3} & \text{for } \frac{1}{9} \leq \beta < \frac{1}{6} \\ \frac{-9477\beta^4 + 7452\beta^3 - 2106\beta^2 + 252\beta - 11}{3(19440\beta^4 - 12096\beta^3 + 2808\beta^2 - 312\beta + 13)} & \text{for } \frac{1}{6} \leq \beta < \frac{2}{9} \\ \frac{(-45\beta + 7)(3\beta - 1)^3}{33048\beta^4 - 34560\beta^3 + 12744\beta^2 - 1992\beta + 115} & \text{for } \frac{2}{9} \leq \beta < \frac{1}{4} \\ \frac{(45\beta - 7)(3\beta - 1)^3}{8424\beta^4 - 20736\beta^3 + 13176\beta^2 - 3192\beta + 263} & \text{for } \frac{1}{4} \leq \beta < \frac{5}{18} \\ \frac{45\beta - 7}{24(175\beta - 37)} & \text{for } \frac{5}{18} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Sincere Truncation Paradox (STP) under BR is given as follows:

$$P_{STP}^{\infty}(\beta) = \begin{cases} \frac{1}{3} & \text{for } 0 \leq \beta < \frac{1}{9} \\ \frac{9477\beta^4 - 4212\beta^3 + 486\beta^2 - 36\beta + 1}{972\beta^3(9\beta - 4)} & \text{for } \frac{1}{9} \leq \beta < \frac{1}{6} \\ \frac{-165969\beta^4 + 184356\beta^3 - 65934\beta^2 + 7956\beta - 349}{3572100\beta^4 - 2313360\beta^3 + 544320\beta^2 - 60480\beta + 2520} & \text{for } \frac{1}{6} \leq \beta < \frac{2}{7} \\ \frac{165969\beta^4 - 184356\beta^3 + 65934\beta^2 - 7956\beta + 349}{(7560\beta - 2520)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{2}{7} \leq \beta < \frac{8}{27} \\ \frac{-19570491\beta^4 + 22367340\beta^3 - 9588618\beta^2 + 1827324\beta - 130567}{(1512\beta - 504)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{8}{27} \leq \beta < \frac{1}{3} \end{cases}$$

C- Negative Plurality Rule

Consider a three-candidate election with large electorates and γ the average number of points obtained by the last ranked candidate under NPR,

- the probability of the Strong Borda Paradox under NPR is given as follows:

$$P_{SgBP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{(4\gamma - 1)(24\gamma^2 - 9\gamma + 1)}{2(144\gamma^3 - 144\gamma^2 + 48\gamma - 5)} & \text{for } \frac{1}{4} \leq \gamma \leq \frac{1}{3} \end{cases}$$

- the probability of the Strict Borda Paradox under NPR is given as follows:

$$P_{StBP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{3\gamma(4\gamma - 1)^2}{2(144\gamma^3 - 144\gamma^2 + 48\gamma - 5)} & \text{for } \frac{1}{4} \leq \gamma \leq \frac{1}{3} \end{cases}$$

- the probability of the Absolute Majority Winner Paradox under NPR is given as follows:

$$P_{AMWP}^{\infty}(\gamma) = \begin{cases} \frac{27\gamma - 8}{16(3\gamma - 1)} & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-222\gamma^3 + 222\gamma^2 - 69\gamma + 7}{2(144\gamma^3 - 144\gamma^2 + 48\gamma - 5)} & \text{for } \frac{1}{4} \leq \gamma \leq \frac{1}{3} \end{cases}$$

- the probability of the Reversal Bias Paradox (RBP) under NPR is given as follows:

$$P_{RBP}^{\infty}(\gamma) = \begin{cases} \frac{-17\gamma}{32(2\gamma-1)} & \text{for } 0 \leq \gamma < \frac{1}{6} \\ \frac{837\gamma^4 - 432\gamma^3 + 12\gamma - 1}{864\gamma^3(2\gamma-1)} & \text{for } \frac{1}{6} \leq \gamma < \frac{1}{4} \\ \frac{-2025\gamma^3 + 1485\gamma^2 - 225\gamma + 1}{432(2\gamma-1)(6\gamma^2-6\gamma+1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Sincere Truncation Paradox (STP) under NPR is given as follows:

$$P_{STP}^{\infty}(\gamma) = \begin{cases} \frac{11\gamma - 4}{4(2\gamma-1)} & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-39\gamma^3 + 39\gamma^2 - 11\gamma + 1}{2(2\gamma-1)(6\gamma^2-6\gamma+1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

D- Plurality Elimination Rule

Consider a three-candidate election with large electorates and α the average number of points obtained by the last ranked candidate under PR,

- the probability of the Reversal Bias Paradox (RBP) under PER is given as follows:

$$P_{RBP}^{\infty}(\alpha) = \begin{cases} \frac{-\alpha^2(2\alpha-1)}{2(3\alpha-1)(3\alpha^2-1)} & \text{for } 0 \leq \alpha < \frac{1}{6} \\ \frac{-2160\alpha^4 + 864\alpha^3 - 12\alpha + 1}{864\alpha(3\alpha-1)(3\alpha^2-1)} & \text{for } \frac{1}{6} \leq \alpha < \frac{1}{4} \\ \frac{-810\alpha^3 + 918\alpha^2 - 333\alpha + 37}{108\alpha(3\alpha^2-1)} & \text{for } \frac{1}{4} \leq \alpha < \frac{1}{3} \end{cases}$$

- the probability of the Negative Abstention Paradox (NAP) under PER is given as follows:

$$P_{NAP}^{\infty}(\alpha) = \begin{cases} 0 & \text{for } 0 \leq \alpha < \frac{1}{6} \\ \frac{(2\alpha+1)(6\alpha-1)^3}{64\alpha(3\alpha-1)(3\alpha^2-1)} & \text{for } \frac{1}{6} \leq \alpha < \frac{1}{4} \\ \frac{-15\alpha^3 + 15\alpha^2 - 7\alpha + 1}{4\alpha(3\alpha^2-1)} & \text{for } \frac{1}{4} \leq \alpha < \frac{1}{3} \end{cases}$$

- the probability of the Negative Participation Paradox (NPP) under PER is given as follows:

$$P_{NPP}^{\infty}(\alpha) = \begin{cases} \frac{\alpha (3\alpha^2 - 8\alpha + 6)}{(192\alpha - 64)(3\alpha^2 - 1)} & \text{for } 0 \leq \alpha < \frac{1}{4} \\ \frac{-255\alpha^3 + 339\alpha^2 - 141\alpha + 17}{64\alpha(3\alpha^2 - 1)} & \text{for } \frac{1}{4} \leq \alpha < \frac{1}{3} \end{cases}$$

E- Borda Elimination Rule

Consider a three-candidate election with large electorates and β the average number of points obtained by the last ranked candidate under BR,

- the probability of the Sincere Truncation Paradox (STP) of BER is given as follows:

$$P_{STP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{2}{9} \\ \frac{(9\beta - 2)^4}{8(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{2}{9} \leq \beta < \frac{1}{4} \\ \frac{-7614\beta^4 + 9072\beta^3 - 3888\beta^2 + 720\beta - 49}{16(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{1}{4} \leq \beta < \frac{5}{18} \\ \frac{-1328238\beta^4 + 1574640\beta^3 - 695952\beta^2 + 136080\beta - 9941}{144(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{5}{18} \leq \beta < \frac{7}{24} \\ \frac{-5143095\beta^4 + 6012792\beta^3 - 2634120\beta^2 + 512568\beta - 37384}{72(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{7}{24} \leq \beta < \frac{8}{27} \\ \frac{-6588\beta^3 + 5229\beta^2 - 1380\beta + 121}{621\beta^3 - 459\beta^2 + 99\beta - 7} & \text{for } \frac{8}{27} \leq \beta < \frac{11}{36} \\ \frac{-6588\beta^3 + 5229\beta^2 - 1380\beta + 121}{621\beta^3 - 459\beta^2 + 99\beta - 7} & \text{for } \frac{11}{36} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Reversal Bias Paradox (RBP) of BER is given as follows:

$$P_{RBP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{1}{6} \\ -\frac{(6\beta - 1)^4}{22680\beta^4 - 14688\beta^3 + 3456\beta^2 - 384\beta + 16} & \text{for } \frac{1}{6} \leq \beta < \frac{2}{9} \\ -\frac{9234\beta^4 - 9072\beta^3 + 3240\beta^2 - 504\beta + 29}{(144\beta - 48)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{2}{9} \leq \beta < \frac{5}{18} \\ -\frac{6345\beta^3 - 4869\beta^2 + 1257\beta - 109}{4968\beta^3 - 3672\beta^2 + 792\beta - 56} & \text{for } \frac{5}{18} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Negative Abstention Paradox (NAP) and Positive Participation Paradox (PPP) of BER is given as follows:

$$P_{NAP}^{\infty}(\beta) = P_{PPP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{5}{18} \\ \frac{(18\beta - 5)^4}{12(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{5}{18} \leq \beta < \frac{8}{27} \\ \frac{(-45\beta + 13)(1053\beta^2 - 612\beta + 89)}{4(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{8}{27} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Negative Participation Paradox (NPP) of BER is given as follows:

$$P_{NPP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{1}{6} \\ \frac{-(6\beta - 1)^4}{24(2835\beta^4 - 1836\beta^3 + 432\beta^2 - 48\beta + 2)} & \text{for } \frac{1}{6} \leq \beta < \frac{2}{9} \\ \frac{(6\beta - 1)^4}{48(3\beta - 1)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{2}{9} \leq \beta < \frac{5}{18} \\ \frac{-(4\beta - 1)(180\beta^2 - 96\beta + 13)}{621\beta^3 - 459\beta^2 + 99\beta - 7} & \text{for } \frac{5}{18} \leq \beta < \frac{1}{3} \end{cases}$$

- the probability of the Positive Abstention Paradox (PAP) of BER is given as follows:

$$P_{PAP}^{\infty}(\beta) = \begin{cases} 0 & \text{for } 0 \leq \beta < \frac{1}{4} \\ \frac{27 (4\beta - 1)^4}{(24\beta - 8)(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{1}{4} \leq \beta < \frac{5}{18} \\ \frac{-(15\beta - 4)(234\beta^2 - 126\beta + 17)}{3(621\beta^3 - 459\beta^2 + 99\beta - 7)} & \text{for } \frac{5}{18} \leq \beta < \frac{1}{3} \end{cases}$$

F- Negative Plurality Elimination Rule

Consider a three-candidate election with large electorates and γ the average number of points obtained by the last ranked candidate under NPR,

- the probability of the Absolute Majority Winner Paradox (AMWP) under NPER is given as follows:

$$P_{AMWP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-3(2\gamma - 1)(-1 + 4\gamma)^2}{144\gamma^3 - 144\gamma^2 + 48\gamma - 5} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Reversal Bias Paradox (RBP) under NPER is given as follows:

$$P_{RBP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{(4\gamma - 1)(8640\gamma^3 - 6480\gamma^2 + 1620\gamma - 139)}{1152(3\gamma - 1)(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{7}{24} \\ \frac{-2376\gamma^3 + 2160\gamma^2 - 639\gamma + 62}{108(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{7}{24} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Sincere Truncation Paradox (STP) under NPER is given as follows:

$$P_{STP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-(4\gamma - 1)(120\gamma^2 + 84\gamma - 13)}{8(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Positive Abstention Paradox (PAP) under NPER is given as follows:

$$P_{PAP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{-3(4\gamma - 1)(11\gamma^2 - 7\gamma + 1)}{8(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{1}{3} \end{cases}$$

- the probability of the Positive Participation Paradox (PPP) under NPER is given as follows:

$$P_{PPP}^{\infty}(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{4} \\ \frac{(44\gamma - 15)(4\gamma - 1)^3}{16(3\gamma - 1)(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{1}{4} \leq \gamma < \frac{3}{10} \\ \frac{-132\gamma^3 + 120\gamma^2 - 34\gamma + 3}{4(2\gamma - 1)(6\gamma^2 - 6\gamma + 1)} & \text{for } \frac{3}{10} \leq \gamma < \frac{1}{3} \end{cases}$$

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