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► To cite this version:

Antonin Jacquet. Geodesics cross any pattern in first-passage percolation without any moment assumption and with possibly infinite passage times. 2023. hal-04228899

HAL Id: hal-04228899

<https://hal.science/hal-04228899>

Preprint submitted on 4 Oct 2023

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Geodesics cross any pattern in first-passage percolation without any moment assumption and with possibly infinite passage times

Antonin Jacquet*

Abstract

In first-passage percolation, one places nonnegative i.i.d. random variables $(T(e))$ on the edges of \mathbb{Z}^d . A geodesic is an optimal path for the passage times $T(e)$. Consider a local property of the time environment. We call it a pattern. We investigate the number of times a geodesic crosses a translate of this pattern. When we assume that the common distribution of the passage times satisfies a suitable moment assumption, it is shown in [Antonin Jacquet. Geodesics in first-passage percolation cross any pattern, arXiv:2204.02021, 2023] that, apart from an event with exponentially small probability, this number is linear in the distance between the extremities of the geodesic. This paper completes this study by showing that this result remains true when we consider distributions with an unbounded support without any moment assumption or distributions with possibly infinite passage times. The techniques of proof differ from the preceding article and rely on a notion of penalized geodesic.

1 Introduction

1.1 Settings

Fix an integer $d \geq 2$. In this paper, we consider first passage percolation on the hypercubic lattice \mathbb{Z}^d . We denote by 0 the origin of \mathbb{Z}^d and by \mathcal{E} the set of edges in this lattice. The edges in \mathcal{E} are those connecting two vertices x and y such that $\|x - y\|_1 = 1$. The basic random object consists of a family $T = \{T(e) : e \in \mathcal{E}\}$ of i.i.d. random variables taking values in $[0, \infty]$ and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The random variable $T(e)$ represents the passage time of the edge e . Their common distribution is denoted by \mathcal{L} .

A finite path $\pi = (x_0, \dots, x_k)$ is a sequence of adjacent vertices of \mathbb{Z}^d , i.e. for all $i = 0, \dots, k - 1$, $\|x_{i+1} - x_i\|_1 = 1$. We say that π goes from x_0 to x_k . Sometimes we identify a path with the sequence of edges it visits, writing $\pi = (e_1, \dots, e_k)$ where for $i = 1, \dots, k$, $e_i = \{x_{i-1}, x_i\}$. We say that k is the length of π and we denote $|\pi| = k$. The passage time $T(\pi)$ of a path $\pi = (e_1, \dots, e_k)$ is the sum of the variables $T(e_i)$ for $i = 1, \dots, k$.

We do not exclude the case $\mathcal{L}(\infty) > 0$. In this case, there are vertices between which all paths have an infinite passage time. Thus, we define the following random set:

$$\mathfrak{C} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : \exists \text{ a path } \pi \text{ from } x \text{ to } y \text{ such that } T(\pi) < \infty\}.$$

Throughout the article, we assume that

$$\mathcal{L}([0, \infty)) > p_c, \tag{1.1}$$

where p_c denotes the critical probability for Bernoulli bond percolation model on \mathbb{Z}^d . We refer to [5] for background on percolation. Say that an edge e is open if its passage time $T(e)$ is finite and closed otherwise. Thanks to (1.1), this percolation model is supercritical. Therefore there exists a unique infinite component which we denote by \mathcal{C}_∞ . When $\mathcal{L}(\infty) = 0$, note that every couple of vertices belongs to \mathfrak{C} and that \mathcal{C}_∞ is equal to the whole graph.

Now, for two vertices x and y , we define the geodesic time

$$t(x, y) = \inf\{T(\pi) : \pi \text{ is a path from } x \text{ to } y\}. \tag{1.2}$$

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Note that $\mathfrak{C} = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : t(x, y) \text{ is finite}\}$. A self-avoiding path γ from x to y such that $T(\gamma) = t(x, y)$ is called a geodesic between x and y .

For the following and for the existence of geodesics, we need some assumptions on \mathcal{L} . Let t_{\min} denote the minimum of the support of \mathcal{L} . We extend a definition introduced in [9]. A distribution \mathcal{L} with support in $[0, \infty]$ is called useful if the following holds:

$$\begin{aligned} \mathcal{L}(t_{\min}) &< p_c \text{ when } t_{\min} = 0, \\ \mathcal{L}(t_{\min}) &< \vec{p}_c \text{ when } t_{\min} > 0, \end{aligned} \tag{1.3}$$

where p_c has been introduced above and where \vec{p}_c is the critical probability for oriented Bernoulli bond percolation on \mathbb{Z}^d (see Section 12.8 in [5]). Throughout the article, we also assume that \mathcal{L} is useful.

Geodesics between any pair of vertices belonging to \mathfrak{C} exist with probability one. This is Proposition 4.4 in [2] when $\mathcal{L}(\infty) = 0$ and Proposition A.1 in Appendix A when $\mathcal{L}(\infty) > 0$. Thus, geodesics between any pair of vertices belonging to \mathcal{C}_∞ exist with probability one.

1.2 Patterns

For a set B of vertices, we denote by ∂B its boundary, this is the set of vertices which are in B and which are linked by an edge to a vertex which is not in B . We make an abuse of notation by saying that an edge $e = \{u, v\}$ belongs to a set of vertices if u and v are in this set.

Let L_1, \dots, L_d be non-negative integers. To avoid trivialities we assume that at least one of them is positive. We fix $\Lambda = \prod_{i=1}^d \{0, \dots, L_i\}$ and two distinct vertices u^Λ and v^Λ on the boundary of Λ .

These points u^Λ and v^Λ are called endpoints. Then we fix an event \mathcal{A}^Λ , with positive probability, only depending on the passage time of the edges of Λ . We say that $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ is a pattern. Let $x \in \mathbb{Z}^d$. Define:

- for $y \in \mathbb{Z}^d$, $\theta_x y = y - x$,
- for $e = \{u, v\}$ an edge connecting two vertices u and v , $\theta_x e = \{\theta_x u, \theta_x v\}$.

Similarly, if $\pi = (x_0, \dots, x_k)$ is a path, we define $\theta_x \pi = (\theta_x x_0, \dots, \theta_x x_k)$. Then $\theta_x T$ denotes the environment T translated by $-x$, i.e. the family of random variables indexed by the edges of \mathbb{Z}^d defined for all $e \in \mathcal{E}$ by

$$(\theta_x T)(e) = T(\theta_{-x} e).$$

Let π be a self-avoiding path and $x \in \mathbb{Z}^d$. We say that x satisfies the condition $(\pi; \mathfrak{P})$ if these two conditions are satisfied:

1. $\theta_x \pi$ visits u^Λ and v^Λ , and the subpath of $\theta_x \pi$ between u^Λ and v^Λ is entirely contained in Λ ,
2. $\theta_x T \in \mathcal{A}^\Lambda$.

Note that, if x satisfies the condition $(\pi; \mathfrak{P})$ when π is a geodesic, then the subpath of $\theta_x \pi$ between u^Λ and v^Λ is one of the optimal paths from u^Λ to v^Λ entirely contained in Λ in the environment $\theta_x T$. When the pattern is given, we also say " π takes the pattern in $\theta_{-x} \Lambda$ " for " x satisfies the condition $(\pi; \mathfrak{P})$ ". We denote:

$$N^{\mathfrak{P}}(\pi) = \sum_{x \in \mathbb{Z}^d} \mathbf{1}_{\{x \text{ satisfies the condition } (\pi; \mathfrak{P})\}}. \tag{1.4}$$

Note that the number of terms in this sum is actually bounded from above by the number of vertices in π . If $N^{\mathfrak{P}}(\pi) \geq 1$, we say that π takes the pattern. The aim of the article is to investigate, under reasonable conditions on \mathfrak{P} , the behavior of $N^{\mathfrak{P}}(\gamma)$ for all geodesics γ from 0 to x with $\|x\|_1$ large. The first step is to determine these reasonable conditions, that is why we define the notion of valid patterns.

Definition 1.1. Denote by $\{\varepsilon_1, \dots, \varepsilon_d\}$ the vectors of the canonical basis. An external normal unit vector associated to a vertex z of the boundary of Λ is an element α of the set $\{\pm\varepsilon_1, \dots, \pm\varepsilon_d\}$ such that $z + \alpha$ does not belong to Λ .

Definition 1.2. We say that a pattern is valid if the following three conditions hold:

- \mathcal{A}^Λ has a positive probability,
- when \mathcal{A}^Λ occurs, there exists a path between the two endpoints, entirely contained in Λ , whose passage time is finite,
- one of the following two conditions holds:
 - the support of \mathcal{L} is unbounded,
 - there exist two distinct external normal unit vectors, one associated with u^Λ and one associated with v^Λ .

Remark 1.3.

- The second condition is always satisfied when $\mathcal{L}(\infty) = 0$.
- The existence of the two distinct vectors in the third condition of Definition 1.2 is equivalent to the fact that the endpoints of the pattern belong to two different faces. As explained in Remark 1.3 in [6], a real obstruction can appear when the support of \mathcal{L} is bounded and this third condition is not satisfied.

1.3 Main result and applications

Here is our main result. We assume that one of the following two conditions is satisfied:

$$\begin{aligned} & \mathcal{L}(\infty) > 0 \text{ and } \mathcal{L}([0, \infty)) > p_c, \\ \text{or } & \mathcal{L}(\infty) = 0 \text{ and the support of } \mathcal{L} \text{ is unbounded.} \end{aligned} \tag{1.5}$$

Theorem 1.4. *Let $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ be a valid pattern, assume (1.5) and that \mathcal{L} is useful. Then there exist $\alpha > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ such that for all $x \in \mathbb{Z}^d$,*

$$\mathbb{P}((0, x) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

In [6] we proved the following result.

Theorem A (Theorem 1.4 in [6]). *Let $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ be a valid pattern, assume that \mathcal{L} is useful, $\mathcal{L}(\infty) = 0$ and one of the following two conditions is satisfied:*

- (I) \mathcal{L} has a bounded support,
- (II) \mathcal{L} has an unbounded support and we have

$$\mathbb{E} \min [T_1^d, \dots, T_{2d}^d] < \infty, \tag{1.6}$$

where T_1^d, \dots, T_{2d}^d are independent with distribution \mathcal{L} .

Then there exist $\alpha > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ such that for all $x \in \mathbb{Z}^d$,

$$\mathbb{P}(\exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

Combining Theorems 1.4 and A we immediately get:

Theorem 1.5. *Let $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ be a valid pattern, assume that \mathcal{L} is useful and $\mathcal{L}([0, \infty)) > p_c$. Then there exist $\alpha > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ such that for all $x \in \mathbb{Z}^d$,*

$$\mathbb{P}((0, x) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha \|x\|_1) \leq \beta_1 e^{-\beta_2 \|x\|_1}.$$

Theorem 1.5 is a generalization of Theorem 2.3 in [1] (stated below as Theorem B) since, to get this result, we can take the pattern (reduced to one edge) $\mathfrak{P} = (\{u^\Lambda, v^\Lambda\}, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ where $u^\Lambda = (0, \dots, 0)$, $v^\Lambda = (1, 0, \dots, 0)$ and \mathcal{A}^Λ is the event on which the passage time of the only edge of the pattern is greater than M .

Theorem B (Theorem 2.3 in [1]). *Let \mathcal{L} be a useful distribution on $[0, +\infty)$ with unbounded support. Then, for each M positive there exists $\varepsilon = \varepsilon(M) > 0$ and $\alpha = \alpha(M) > 0$ so that for all x , we have*

$$\mathbb{P} \left(\exists \text{ geodesic } \pi \text{ from } 0 \text{ to } x \text{ such that } \sum_{e \in \pi} \mathbb{1}_{T(e) \geq M} \leq \alpha \|x\|_1 \right) \leq e^{-\varepsilon \|x\|_1}. \quad (1.7)$$

The proof of Theorem 1.4 is given in Section 2. It is partly inspired by the proof of Theorem 2.3 in [1]. The proofs of Theorems 1.4 and A are independent and the only intersection between these two theorems is the case (II) above. Theorem 1.5 is an extension of Theorem 1.4 in [6]. We refer to [6] for an account of the history of such results and for applications. As an example of application we prove the following result, which is a generalization of the main result of [9].

The van den Berg-Kesten comparison principle without any moment assumption. Let \mathcal{L} and $\tilde{\mathcal{L}}$ be two distributions taking values in $[0, \infty]$ such that:

(H1) \mathcal{L} is useful,

(H2) $\mathcal{L}([0, \infty)) > p_c$ and $\tilde{\mathcal{L}}([0, \infty)) > p_c$,

(H3) $\mathcal{L} \neq \tilde{\mathcal{L}}$,

(H4) there exists a couple of random variables $\tau, \tilde{\tau}$ on some probability space, with marginal distributions \mathcal{L} and $\tilde{\mathcal{L}}$, respectively, and satisfying

$$\mathbb{E}[\tilde{\tau}|\tau] \leq \tau. \quad (1.8)$$

We consider a family $T = \{T(e) : e \in \mathcal{E}\}$ of i.i.d. random variables with distribution \mathcal{L} and another family $\tilde{T} = \{\tilde{T}(e) : e \in \mathcal{E}\}$ of i.i.d. random variables with distribution $\tilde{\mathcal{L}}$. The geodesic time defined at (1.2) is denoted by t in the environment T and by \tilde{t} in the environment \tilde{T} . With these assumptions, a time constant for each distribution can be defined thanks to [3]. We refer to [3] for an extensive account. Here we recall what we need for our purpose. By (H2), there exists $M \in \mathbb{R}$ such that

$$\mathcal{L}([0, M]) > p_c \text{ and } \tilde{\mathcal{L}}([0, M]) > p_c. \quad (1.9)$$

Fix such a M . Let \mathcal{C}_M (resp. $\tilde{\mathcal{C}}_M$) be the infinite cluster for the Bernoulli percolation ($\mathbb{1}_{\{T(e) \leq M\}}$, $e \in \mathcal{E}$) (resp. ($\mathbb{1}_{\{\tilde{T}(e) \leq M\}}$, $e \in \mathcal{E}$)) which exists and is unique a.s. To any $x \in \mathbb{R}^d$, we associate a random point $\varphi(x)$ (resp. $\tilde{\varphi}(x)$) in \mathcal{C}_M (resp. in $\tilde{\mathcal{C}}_M$) such that $\|x - \varphi(x)\|_1$ (resp. $\|x - \tilde{\varphi}(x)\|_1$) is minimal, with a deterministic rule to break ties.

Theorem 1 in [3] gives the existence of two deterministic functions $\mu : \mathbb{R}^d \rightarrow [0, \infty)$ and $\tilde{\mu} : \mathbb{R}^d \rightarrow [0, \infty)$ such that

$$\forall x \in \mathbb{Z}^d, \lim_{n \rightarrow \infty} \frac{t(\varphi(0), \varphi(nx))}{n} = \mu(x) \text{ a.s. and in } L^1, \text{ and } \lim_{n \rightarrow \infty} \frac{\tilde{t}(\tilde{\varphi}(0), \tilde{\varphi}(nx))}{n} = \tilde{\mu}(x) \text{ a.s. and in } L^1. \quad (1.10)$$

Theorem 4 in [3] ensures that the functions μ and $\tilde{\mu}$ do not depend on the choice of the constant M satisfying (1.9). Furthermore, when

$$\mathbb{E} \min[\tau_1, \dots, \tau_{2d}] < \infty, \quad (1.11)$$

where τ_1, \dots, τ_{2d} are i.i.d. copies of τ , Theorem 4 in [3] also ensures that for all $x \in \mathbb{R}^d$,

$$\lim_{n \rightarrow \infty} \frac{t(0, \lfloor nx \rfloor)}{n} = \mu(x) \text{ a.s. and in } L^1. \quad (1.12)$$

This is the usual definition of the time constant. We refer to Theorem 2.18 in [7] and Section 2.1 in [2] for more details on the result (1.12). The same holds for the environment \tilde{T} if (1.11) holds for $2d$ i.i.d. copies of $\tilde{\tau}$.

Remark 1.6. *We warn the reader that notations \tilde{T} and $\tilde{\mu}$ are used in [3] with a different meaning. We refer in particular to Remark 2 in [3] for explanations.*

We can now state the van den Berg-Kesten comparison principle for these time constants.

Theorem 1.7 (Extension of the van den Berg-Kesten comparison principle). *Let \mathcal{L} and $\tilde{\mathcal{L}}$ be two distributions taking values in $[0, \infty]$ satisfying $(\mathcal{H}1)$, $(\mathcal{H}2)$, $(\mathcal{H}3)$ and $(\mathcal{H}4)$. For all $x \in \mathbb{Z}^d$ such that $x \neq 0$,*

$$\tilde{\mu}(x) < \mu(x). \quad (1.13)$$

The proof of Theorem 1.7 is given in Section 3. In [9], van den Berg and Kesten prove the following theorem.

Theorem C (Theorem 2.9 in [9]). *Let \mathcal{L} and $\tilde{\mathcal{L}}$ be two distributions taking values in $[0, \infty)$, having a finite first moment, satisfying¹ $(\mathcal{H}1)$, $(\mathcal{H}3)$ and $(\mathcal{H}4)$. Then,*

$$\tilde{\mu}(\varepsilon_1) < \mu(\varepsilon_1).$$

Theorem 1.7 is an extension of Theorem C. With Theorem 1.2 in [8], Marchand extends Theorem C in another direction.

Theorem D (Theorem 1.2 in [8]). *Assume that $d = 2$ and let \mathcal{L} and $\tilde{\mathcal{L}}$ be two distributions taking values in $[0, \infty)$, such that $\mathcal{L}(0) < p_c$, satisfying² $(\mathcal{H}3)$ and $(\mathcal{H}4)$. Then,*

$$\tilde{\mu}(\varepsilon_1) < \mu(\varepsilon_1).$$

In dimension 2, the result of Marchand is stronger than Theorem C on two aspects: on the one hand, there is no moment assumption and on the other hand, the condition

$$\mathcal{L}(t_{\min}) < \overline{p}_c \text{ when } t_{\min} > 0 \quad (1.14)$$

is removed. When $t_{\min} > 0$ and $\mathcal{L}(t_{\min}) > \overline{p}_c$, the problem involves oriented percolation, where the open edges correspond to those with the smallest time values. In this context, the largest part of each geodesic linking the origin to a distant point within the cone of percolation is a directed path made of minimal edges, highlighting a distinct behavior.

For a point x inside this cone, $\mu(x) = t_{\min}\|x\|_1$. Moreover, when $\tilde{t}_{\min} = t_{\min}$ and the $(\mathcal{H}2)$ condition is met, $\tilde{\mu}(x) = t_{\min}\|x\|_1$ as well. Notably, (1.13) does not apply to such x values. Establishing (1.13) for x outside the cone, like ε_1 , requires specific arguments, in particular large deviations for supercritical oriented percolation. We have opted not to explore this case in this article. We refer to [8], and more specifically to Theorem 1.5 in [8], for further explanations.

1.4 Sketch of the proof

In this section, we give an informal sketch of the proof of Theorem 1.4. Fix a pattern \mathfrak{P} and $x \in \mathbb{Z}^d$ with $\|x\|$ large. Consider the event:

$$\mathcal{M} = \{(0, x) \in \mathfrak{C} \text{ and there exists a geodesic from } 0 \text{ to } x \text{ which does not take the pattern}\}.$$

The aim is to prove that \mathcal{M} has a probability small enough in $\|x\|$. More precisely, we want to prove

$$\mathbb{P}(\mathcal{M}) \ll \frac{1}{\|x\|^{d-1}}. \quad (1.15)$$

From this result, by a standard re-normalization argument, we easily get Theorem 1.4 (see Proposition 1.8 in Section 1.5 for a formal statement of (1.15)).

¹It is stated in [9] with the definition of a distribution more variable than another, but Theorem 2.6 in [9] ensures that the fact that $\tilde{\mathcal{L}}$ is more variable than \mathcal{L} is equivalent to $(\mathcal{H}4)$ when \mathcal{L} and $\tilde{\mathcal{L}}$ have a finite first moment.

²It is also stated in [8] with the definition of a distribution more variable than another, but Lemma 6.1 in [8] ensures that this definition is also equivalent to $(\mathcal{H}4)$ when \mathcal{L} and $\tilde{\mathcal{L}}$ takes value in $[0, \infty)$.

General idea. As in [6], to get (1.15), the idea is to define a suitable event \mathcal{G} and a suitable sequence of events $\mathcal{M}(\ell)$ for $0 \leq \ell \leq q$ such that, for some positive constant $c < 1$,

1. $q \geq c\|x\|$,

2. $\mathcal{M} \subset \mathcal{M}(q) \cup \mathcal{G}^c$ where $\mathbb{P}(\mathcal{G}^c) \ll \frac{1}{\|x\|^{d-1}}$,

3. for all $\ell \geq 1$,

$$\mathbb{P}(\mathcal{M}(\ell)) \leq c\mathbb{P}(\mathcal{M}(\ell - 1)). \quad (1.16)$$

If the above holds, we get $\mathbb{P}(\mathcal{M}) \leq c^{c\|x\|_1} + \mathbb{P}(\mathcal{G}^c)$, which allows us to conclude.

Penalized geodesics. We now introduce the notion of penalized path. This is an idea which comes from the article [1] by Andjel and Vares in their proof of Theorem 1.7. A penalized path is a path which does not take the pattern. In other words, this is a path π such that $N^{\mathfrak{P}}(\pi) = 0$. This allows us to define the penalized passage time for every $z \in \mathbb{Z}^d$:

$$t_P(0, z) = \inf\{T(\pi) : \pi \text{ is a penalized path from } 0 \text{ to } z\},$$

with the convention $\inf \emptyset = \infty$. Then, for every $z \in \mathbb{Z}^d$, if it exists, a penalized geodesic from 0 to z is a penalized path γ from 0 to z such that $T(\gamma) = t_P(0, z)$. With these definitions, we have

$$\mathcal{M} \subset \{(0, x) \in \mathfrak{C} \text{ and } t_P(0, x) = t(0, x)\}. \quad (1.17)$$

Shortcuts. A good way to get that the event $\{(0, x) \in \mathfrak{C} \text{ and } t_P(0, x) = t(0, x)\}$ does not occur is to prove that a penalized geodesic has a shortcut. The formal definition of a shortcut is given in Section 2.1.4. Informally, a shortcut for a penalized geodesic γ is a path going from a vertex u of γ to another vertex v of γ which takes the pattern and which has a passage time lower than the passage time of the subpath of γ going from u to v . Hence, if a penalized geodesic γ from 0 to x has a shortcut, it implies that there exists a path from 0 to x which is not penalized and such that its passage time is strictly lower than the passage time of γ . It gives $t(0, x) < t_P(0, x)$.

Events \mathcal{G} and $\mathcal{M}(\ell)$. A successful box for a path π is a box satisfying one of the following two conditions:

- it is a typical box,
- the path π has a shortcut taking the pattern inside the box.

We say that a box is shortcut-equipped for a path π or not shortcut-equipped for π depending on whether the second condition is satisfied or not. We define \mathcal{G} as the event on which $(0, x) \in \mathfrak{C}$ and there exists a penalized geodesic π whose passage time is finite and which crosses at least q successful boxes for π . On this event, we define the selected penalized geodesic denoted by γ : it is the first (for an arbitrary deterministic order) of the penalized geodesics satisfying the condition which appears in the definition of \mathcal{G} . We define the sequence of successful boxes crossed by γ as the sequence of the first q successful boxes crossed by γ indexed in the order in which they are crossed by γ . It allows us to define, for every $\ell \in \{1, \dots, q\}$ the event

$$\mathcal{M}(\ell) = \mathcal{G} \cap \{\text{the } \ell \text{ first successful boxes of the sequence of } \gamma \text{ are not shortcut-equipped for } \gamma\}.$$

This gives us ℓ opportunities to modify the environment in each of these ℓ typical boxes to create a shortcut for γ . The aim is now reduced to proving (1.16).

k -boxes. We describe a small change of the plan above. This change does not create any complications. In particular, the entire plan describe above works with these new objects. The advantage is to avoid a number of complications, such as, for example, those related to what happens at the boundary of a box when we modify the environment in it.

The idea is to only consider a family of boxes (called the k -boxes, see Section 2.1.4) that partitions \mathbb{Z}^d . We also replace "penalized paths" -the paths which does not take the pattern- by " k -penalized paths" -the paths which does not take any pattern contained in a k -box-. We similarly replace "penalized geodesics" by " k -penalized geodesics" : the geodesics which does not take any pattern contained in a k -box. We say " k -geodesic" instead of " k -penalized geodesic" for short.

Modification and stability. For all ℓ , we have $\mathcal{M}(\ell) \subset \mathcal{M}(\ell - 1)$. Thus (1.16) is equivalent to the existence of a constant $\eta > 0$ (by taking $\eta = \frac{1}{c} - 1$) such that

$$\mathbb{P}(\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)) \geq \eta \mathbb{P}(\mathcal{M}(\ell)). \quad (1.18)$$

Fix $\ell \in \{1, \dots, q\}$ and denote by B_s the ℓ -th successful k -box crossed by γ . The aim is to prove (1.18). The idea is to resample the passage times of edges of B_s in an environment in which $\mathcal{M}(\ell)$ occurs to get a new environment in which $\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)$ occurs. When the resampled passage times satisfy good conditions (to be determined), the following properties are satisfied:

1. The event \mathcal{G} still occurs and the selected k -geodesic is still γ in the new environment.
2. The box B_s is shortcut-equipped for γ in the new environment.
3. The sequence of successful boxes crossed by γ is the same in the two environments.
4. The event $\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)$ occurs in the new environment.

By the fourth property, we get roughly

$$\mathbb{P}(\mathcal{M}(\ell))\mathbb{P}(\text{good conditions on the resampled passage times}) \leq \mathbb{P}(\mathcal{M}(\ell - 1) \setminus \mathcal{M}(\ell)).$$

Since η is fixed according to the probability of the good conditions on the resampled passage times, which is positively bounded from below independently of the box, we get (1.18). See the proof of Lemma 2.12 using Lemma 2.13 in Section 2.2.1.

The third property follows from the first two. Indeed, the box B_s is typical in the first environment (since it is a successful box for γ and it is not shortcut-equipped for γ as $\mathcal{M}(\ell)$ holds) and is shortcut-equipped for γ in the new environment. Furthermore, the other boxes have the same status (successful or not for γ) in the two environments since the passage times of the edges of the other boxes have not been modified.

The fourth property follows from the first three by similar ideas. We thus see that, in order to get the fourth property, we do not only need to get the second one. We also need the first and the third ones. We call these two additional properties "stability properties". The proof is thus reduced to getting the first two properties.

Some more details. Recall that we assume (1.5). There are two cases to be considered differently:

(INF) $\mathcal{L}(\infty) > 0$,

(FU) $\mathcal{L}(\infty) = 0$ and the support of \mathcal{L} is unbounded.

(INF) stands for "infinite" and (FU) stands for "finite unbounded".

In what follows, when we say "after the modification" or "in the new environment", we mean "in the new environment where passage times of the edges in the box B_s have been resampled and on the event where the resampled passage times satisfy some good properties that we do not explicit here". In this paragraph, we focus on the first property of the previous paragraph. To get it, it is sufficient to prove the following properties:

- (i) The path γ still has a finite passage time in the new environment. There are no difficulties with this property.

(ii) The path γ remains a k -penalized path in the new environment. Since the passage times of the k -boxes different from B_s have not been modified, it is sufficient to prove that γ does not take the pattern in B_s in the new environment. This is based on the two following ideas.

- We identify *forbidden zones* which are subsets of B_s where γ can not go. In the case (INF) the forbidden zones are simply balls whose edges have infinite passage time. In the case (FU) we refer to Lemma 2.5. By definition, a typical box possesses many forbidden zones (see the third item of the definition of a typical box in Section 2.1.2).
- We make sure that, after a successful modification, there is a *unique* pattern inside B_s . The uniqueness is ensured by replacing the original pattern by a new larger pattern, containing the original one, and by requiring that the behavior of passage times in the boundary of the new pattern is very atypical (see Lemma 2.4 and in particular its last item). In the case (FU) we just require that the passage times on the boundary of the new pattern are very high and contained in a special interval (see Remark 2.3 and (AF-4')). These will be the unique edges with passage times in this interval after the modification. In the case (INF) we require the existence of a large connected component of edges with finite passage time (see Definition 2.1 and (AI-4)). This will be the unique such large component not touching the boundary of B_s after the modification.

We place the pattern in a forbidden zone. By this we mean that, after the modification, the pattern lies in what was a forbidden zone before the modification. Recall that the pattern is unique in B_s and that γ does not enter into forbidden zones. Therefore γ does not take the pattern in B_s after the modification.

(iii) A k -penalized path π with finite passage time in the new environment is also a k -penalized path in the initial environment. Once again, it is sufficient to prove that π does not take a pattern entirely contained in B_s in the initial environment. The proof differs between the case (INF) and the case (FU).

- In the case (FU), this is a consequence of the fact that B_s is a typical box and that there is no pattern in a typical box. Indeed, the passage times on the boundary of the pattern are bigger (in the case (FU)) than they can be in a typical box.
- In the case (INF), it comes from the fact that, in the new environment, a path with a finite passage time taking edges in B_s is very constrained (see Figure 4 where a path with a finite passage time can only take edges of the green, red, blue and orange parts). If π does not take the pattern in the new environment, it can only take edges of γ . Since γ does not take a pattern entirely contained in B_s in the initial environment, neither does π .

(iv) A k -penalized path in the new environment has a passage time greater than or equal to the passage time of γ in the new environment. The proof comes from the fact that a path with a reasonable passage time in the new environment is very constrained in B_s . The edges that do not belong to γ , the shortcut for γ or the unique pattern in B_s have a prohibitive passage time in the case (FU) and an infinite passage time in the case (INF). Based on this observation and on the fact that γ has a lower passage time in the new environment than in the initial environment, we simply prove that a k -penalized path can not save more time than γ during the modification.

(v) With the same ideas as above, we also prove that a k -geodesic in the new environment is also a k -geodesic in the initial environment.

Indeed, by (i), (ii) and (iv) we get that γ is a k -geodesic with a finite passage time in the new environment. Using the same arguments as before, we get that γ crosses at least q successful k -boxes in the new environment. Hence, the event \mathcal{G} occurs in the new environment. Furthermore, we also get that every k -geodesic crossing at least q successful k -boxes in the new environment crosses at least q successful k -boxes in the initial environment. Adding (iii) and (v), we get that the set of potential selected k -geodesics in the new environment is contained in the set of potential selected k geodesic in the initial environment, and then γ remains the first geodesic (and thus the selected one) among the geodesics of this set.

Advantages of a strategy using penalized geodesics. In [6] the proof does not rely on penalized geodesics. Using penalized geodesics has two main advantages:

- In [6], proving the result for all geodesics (and not only for one selected geodesic) requires further technicalities (see the use of concentric annuli in Section 2.1 in [6]). Here, it comes for free from the fact that the existence of one k -geodesic having a shortcut implies that $t(0, x) < t_P(0, x)$ and thus that every geodesic from 0 to x takes the pattern.
- It allows us to remove Assumption (1.6). Indeed, in item 3 in the paragraph on the modification and stability above, we need to have the same sequence of successful boxes crossed by γ in the two environments. Assume that we do not use penalized geodesics and, to make things easier, assume (only in this item) that we are in the case where there is a unique geodesic between any couple of vertices. Then the modification consists in replacing a subpath (denoted by $\bar{\gamma}$) of the geodesic from 0 to x (denoted by γ) by a path (denoted by π) with a shorter passage time which takes the pattern. It implies that in the new environment, $\bar{\gamma}$ does not belong to the new geodesic (which is the concatenation of the part of γ from 0 to π , then π and then the part of γ from π to x). It can create a problem of stability if a box of the sequence of successful boxes crossed by γ was crossed by $\bar{\gamma}$: the sequence of successful boxes crossed by the geodesic from 0 to x would not be the same in the two environments. To avoid this problem when we do not use penalized geodesics, we use the Cox-Durrett shape theorem (Theorem 2.16 in [2]) in order to control the length of geodesics excursions from a box (see for example the proof of Lemma 2.1 in [6]). This is why we need Assumption (1.6) in the strategy developed in [6].

Hence, by making a modification which guarantees that the penalized geodesic is the same in the initial environment as in the modified one, we avoid this problem without using the Cox-Durrett shape theorem and thus without requiring Assumption (1.6).

1.5 Organization of the proof of Theorem 1.4

Recall that, in this article, we assume (1.5). One can check, using a standard re-normalization argument, that Theorem 1.4 is a simple consequence of the following proposition (see for example the proof of Theorem 2.3 in [1]).

Proposition 1.8. *Let $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ be a valid pattern. Assume (1.5) and that \mathcal{L} is useful. Then there exist $C > 0$ and $D > 0$ such that for all $n \geq 0$, for all x such that $\|x\|_1 = n$,*

$$\mathbb{P}((0, x) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } 0 \text{ to } x \text{ such that } N^{\mathfrak{P}}(\gamma) = 0) \leq De^{-Cn}. \quad (1.19)$$

Thus, the aim is now to prove Proposition 1.8. Recall that there are two cases to be considered differently:

(INF) $\mathcal{L}(\infty) > 0$,

(FU) $\mathcal{L}(\infty) = 0$ and the support of \mathcal{L} is unbounded.

The proof of Proposition 1.8 is the aim of Section 2. This section is divided in two parts.

Section 2.1 is devoted to patterns and typical boxes. We replace the original pattern by a larger pattern containing the original one and which satisfies several assumptions. Some of the assumptions are simply convenient: they simplify some parts of the proof. The assumption on the boundary is more crucial as explained in item (ii) in the paragraph "Some more details" in Section 1.4. In the case (FU), the requirement on the passage times on the boundary depends on the size of the boxes we consider in the proof. But the size of the boxes depends on the notion of typical boxes which in turn depends on parts of the definition of the pattern we consider. The definitions are thus intertwined. This is why we first start defining the new pattern in Section 2.1.1 (postponing the boundary conditions in the case (FU)), we then define and study typical boxes in Section 2.1.2 and we finally choose the boundary of the pattern in the case (FU) in Section 2.1.3. We then introduce the notions of k -penalized paths, shortcuts and successful boxes in Section 2.1.4.

The second part of Section 2 is divided in four parts. In Section 2.2.1, the proof of Proposition 1.8 is reduced to the proof of a key lemma : Lemma 2.13. In this lemma, we introduce some sets of edges

which correspond to the edges whose passage times have to be modified. The exact definitions of these sets are postponed to Section 2.2.2. It corresponds to the modification we want to make. The more difficult part in the proof of Lemma 2.13 is item (iii), which is the key to get (1.18). To get this item and also item (i), we state and prove several properties which are consequences of the modification in Section 2.2.3 before using them to conclude in Section 2.2.4.

1.6 Some tools and notations

In this subsection, we recall some results and fix some notations. First, we denote by \mathbb{N} the set of all non-negative integers, by \mathbb{N}^* the set $\mathbb{N} \setminus \{0\}$, and by \mathbb{R}_+ the set of all $x \in \mathbb{R}$ such that $x \geq 0$.

For a self-avoiding³ path $\pi = (x_0, \dots, x_k)$ going from x_0 to x_k , we say that x_i is visited by π before x_j if $i < j$; we say that an edge $\{x_i, x_{i+1}\}$ is visited before an edge $\{x_j, x_{j+1}\}$ if $i < j$. A subpath of π going from x_i to x_j (where $i, j \in \{0, \dots, k\}$ and $i < j$) is the path (x_i, \dots, x_j) and is denoted by π_{x_i, x_j} .

We say that a path, whose endpoints are denoted by u and v , is oriented if this path has exactly $\|u - v\|_1$ edges. In other words, its number of edges is minimal among those of all the other paths linking u and v .

For a set B of vertices, we denote by ∂B its boundary, this is the set of vertices of B which can be linked by an edge to a vertex which is not in B . We denote by B^c the set of all vertices which does not belong to B . When we define a set of vertices of \mathbb{Z}^d , sometimes we also want to say that an edge belongs to this set. So we make an abuse of notation by saying that an edge $e = \{u, v\}$ belongs to a set of vertices if u and v are in this set. Since now a subset B of \mathbb{Z}^d can be seen as a set of vertices or as a set of edges, we denote by $|B|_v$ the number of vertices of B and by $|B|_e$ its number of edges.

Then, for all $c \in \mathbb{Z}^d$ and $r \in \mathbb{R}_+$, we denote

$$\begin{aligned} B_\infty(c, r) &= \{u \in \mathbb{Z}^d : \|u - c\|_\infty \leq r\}, \\ B_1(c, r) &= \{u \in \mathbb{Z}^d : \|u - c\|_1 \leq r\}, \end{aligned}$$

and for $n \in \mathbb{N}^*$, we denote by Γ_n the boundary of $B_1(0, n)$, i.e.

$$\Gamma_n = \{u \in \mathbb{Z}^d : \|u\|_1 = n\}. \quad (1.20)$$

Constants related to the distribution. One can check that Lemma 5.5 in [9] can be adapted for a useful distribution \mathcal{L} such that $\mathcal{L}(\infty) > 0$ and $\mathcal{L}([0, \infty)) > p_c$. Thus there exist $\delta = \delta(\mathcal{L}) > 0$ and $D_0 = D_0(\mathcal{L})$ fixed for the remaining of the article such that for all $u, v \in \mathbb{Z}^d$,

$$\mathbb{P}(\text{there exists a path } \pi \text{ from } u \text{ to } v \text{ such that } T(\pi) \leq (t_{\min} + \delta)\|u - v\|_1) \leq e^{-D_0\|u - v\|_1}. \quad (1.21)$$

Furthermore, when $t_{\min} = 0$,

- even if it means reducing δ , in the cases (INF) and (FU), we assume that $\delta > 0$ is such that $\mathcal{L}([0, \delta]) < p_c$,
- even if it means reducing δ , in the case (INF), we can fix ν_0 such that

$$\nu_0 > \delta \text{ and } \mathcal{L}((\delta, \nu_0)) > 0. \quad (1.22)$$

Note that it is possible since in the case (INF), we have $\mathcal{L}(0) + \mathcal{L}(\infty) < 1$. Indeed, it comes from the fact that \mathcal{L} is useful and that $\mathcal{L}([0, \infty)) > p_c$.

Then, still in the case where $t_{\min} = 0$, we fix

$$\beta > 0, \beta' > 0 \text{ and } \rho > 0 \quad (1.23)$$

such that (1.24) below holds with $\tau = \delta$. The existence of such constants is guaranteed by Lemma 1.9 below whose proof is given in Appendix B.

Lemma 1.9. *Assume that \mathcal{L} is useful and that $t_{\min} = 0$. Let $\tau > 0$ such that $\mathcal{L}([0, \tau]) < p_c$. Then there exists $\beta > 0$, $\beta' > 0$ and $\rho > 0$ such that for all $v, w \in \mathbb{Z}^d$,*

$$\mathbb{P}(\text{there exists a self-avoiding path from } v \text{ to } w \text{ taking at most } \rho\|v - w\|_1 \text{ edges } e \text{ such that } T(e) > \tau) \leq \beta' e^{-\beta\|v - w\|_1}. \quad (1.24)$$

³The definition can be extended to not necessarily self-avoiding paths by saying that a vertex x is visited by π before y if there exists $i_0 \in \{0, \dots, k\}$ such that $x_{i_0} = x$ and for all $j \in \{0, \dots, k\}$, $x_j = y$ implies that $j > i_0$.

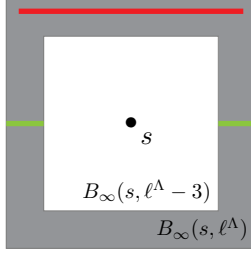


Figure 1: Example of a ball satisfying the boundary condition. The set $\mathcal{S}_{s, \ell^\Lambda}$ is represented in red. The edges of the gray area have an infinite passage time and the edges in green and red have a finite passage time.

2 Proof of Proposition 1.8

Let $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ be a valid pattern. We assume (1.5) and that \mathcal{L} is useful. Thus one of the cases (INF) or (FU) stated in Section 1.5 is realized. The proofs in these two cases are almost the same. However, throughout this section, it will sometimes be necessary to distinguish the cases.

2.1 Settings for the proof

2.1.1 Assumptions on the patterns

We begin by making some assumptions on \mathfrak{P} for the remaining of the proof. At first sight, these assumptions can be seen as a restriction but Lemma 2.4 guarantees that we can make them with no loss of generality.

In the case (INF).

Definition 2.1 (Boundary condition). *For every $s \in \mathbb{Z}^d$ and $\ell^\Lambda \geq 3$, define the set $\mathcal{S}_{s, \ell^\Lambda}$ as the set of edges belonging to the path going from $s - (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$ to $s + (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$ in the shortest way by $2(\ell^\Lambda - 1)$ steps in the direction ε_1 .*

Then, we say that $B_\infty(s, \ell^\Lambda)$ satisfies the boundary condition in the environment T if for all edges e belonging to $B_\infty(s, \ell^\Lambda)$ but not to $B_\infty(s, \ell^\Lambda - 3)$,

- *either e belongs to $\mathcal{S}_{s, \ell^\Lambda} \cup (s + \mathbb{Z}\varepsilon_1)$ and $T(e)$ is finite,*
- *or $T(e)$ is infinite.*

Remark 2.2. *Let $s \in \mathbb{Z}^d$ and $\ell^\Lambda \geq 3$. If $B_\infty(s, \ell^\Lambda)$ satisfies the boundary condition in the environment T , then there is no path from $\partial B_\infty(s, \ell^\Lambda)$ to $\partial B_\infty(s, \ell^\Lambda)$ with finite passage time which takes an edge of $\mathcal{S}_{s, \ell^\Lambda}$ (see Figure 1 for a representation of the boundary condition in two dimensions).*

Let us consider the following assumptions:

(AI-1) there exists an integer $\ell^\Lambda \geq 3$, fixed for the remaining of the proof, such that⁴ $\Lambda = B_\infty(0, \ell^\Lambda)$,

(AI-2) $u^\Lambda = -\ell^\Lambda \varepsilon_1$ and $v^\Lambda = \ell^\Lambda \varepsilon_1$,

(AI-3) there exist a constant $T^\Lambda > 0$ and a path π_∞ from u^Λ to v^Λ entirely contained in Λ such that, when \mathcal{A}^Λ occurs,

$$T(\pi_\infty) < T^\Lambda. \quad (2.1)$$

(AI-4) if \mathcal{A}^Λ occurs, Λ satisfies the boundary condition

⁴We make a very slight abuse of notation: we also consider patterns where 0 is in the center of Λ .

In the case (FU). Let us consider the following assumptions:

- (AF-1) there exists an integer $\ell^\Lambda > 0$, fixed for the remaining of the proof, such that $\Lambda = B_\infty(0, \ell^\Lambda)$,
- (AF-2) $u^\Lambda = -\ell^\Lambda \varepsilon_1$ and $v^\Lambda = \ell^\Lambda \varepsilon_1$,
- (AF-3) when \mathcal{A}^Λ occurs, there exists a constant M^Λ such that for every edge e belonging to Λ but not to $\partial\Lambda$, $T(e) \leq M^\Lambda$,
- (AF-4) for all $M > 0$, the event $\mathcal{A}^\Lambda \cap \{\forall e \in \partial\Lambda, T(e) > M\}$ has a positive probability.

Remark 2.3. *The aim of wanting the pattern to satisfy the condition (AF-4) above is to be able to choose the passage times of the edges on its boundary once we have fixed some constants. Thus, from Section 2.1.3 onwards the condition (AF-4) is replaced by the condition (AF-4') stated at this point.*

In the two cases.

Lemma 2.4. *Let $\mathfrak{P}_0 = (\Lambda_0, u_0^\Lambda, v_0^\Lambda, \mathcal{A}_0^\Lambda)$ be a valid pattern. There exists a pattern $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ such that:*

- $\Lambda_0 \subset \Lambda$,
- $\mathbb{P}(\mathcal{A}^\Lambda)$ is positive,
- on \mathcal{A}^Λ , any path from u^Λ to v^Λ optimal for the passage time among the paths entirely inside Λ contains a subpath from u_0^Λ to v_0^Λ entirely inside Λ_0 ,
- $\mathcal{A}^\Lambda \subset \mathcal{A}_0^\Lambda$,
- in the case (INF), \mathfrak{P} satisfies the conditions (AI-1), (AI-2), (AI-3) and (AI-4), and in the case (FU), \mathfrak{P} satisfies the conditions (AF-1), (AF-2), (AF-3) and (AF-4).

Consider a valid pattern \mathfrak{P}_0 and a pattern \mathfrak{P} satisfying the conditions of Lemma 2.4 above. Then, by this lemma, for every path π , if a vertex x satisfies the condition $(\pi; \mathfrak{P})$, x satisfies the condition $(\pi; \mathfrak{P}_0)$. Thus we get $N^{\mathfrak{P}_0}(\pi) \geq N^{\mathfrak{P}}(\pi)$ and to prove Proposition 1.8 for the pattern \mathfrak{P}_0 , it is sufficient to prove it for the pattern \mathfrak{P} . That is why from now on, we can assume that the pattern \mathfrak{P} introduced at the beginning of Section 2 satisfies the conditions (AI-1), (AI-2), (AI-3), (AI-4) in the case (INF) and the conditions (AF-1), (AF-2), (AF-3) and (AF-4) in the case (FU). The proof of Lemma 2.4 is postponed in Appendix C.

For the remaining of the proof, fix ℓ^Λ given by (AI-1) and (AF-1). In the case (INF), fix T^Λ given by (AI-3) and in the case (FU), fix

$$M^\Lambda \text{ satisfying (AF-3) and } T^\Lambda > |\Lambda|_e M^\Lambda. \quad (2.2)$$

2.1.2 Typical boxes

Recall that δ is fixed at (1.21) and that, when $t_{\min} = 0$, ν_0 is fixed at (1.22).

Technical lemma. We state in this paragraph the lemma used to create a "forbidden zone" in the case (FU) (see item (ii) of the seventh paragraph of Section 1.4).

Fix r_P an integer such that

$$r_P > \max \left(\ell^\Lambda + 2, \frac{2(|B_\infty(0, \ell^\Lambda + 1)|_e(t_{\min} + 1) + T^\Lambda)}{\delta}, \frac{|B_\infty(0, \ell^\Lambda + 1)|_e(t_{\min} + 1) + T^\Lambda + 1}{2t_{\min}} \right). \quad (2.3)$$

Lemma 2.5. *In the case (FU), we can define an event \mathcal{T} , whose probability is positive, only depending on the edges of $B_\infty(0, r_P)$ and such that for all $x, y \in \partial B_\infty(0, r_P)$, for every self-avoiding path π going from x to y and using only edges in $\partial B_\infty(0, r_P)$ and every path $\tilde{\pi}$ going from x to y using edges of $B_\infty(0, r_P)$ and at least one edge which is not in $\partial B_\infty(0, r_P)$, we have*

$$T(\pi) < T(\tilde{\pi}).$$

Proof. Fix

$$\nu > |\partial B_\infty(0, r_P)|_e(t_{\min} + 1).$$

Define the event \mathcal{T} as the event on which for all edge $e \in B_\infty(0, r_P)$,

- $T(e) < t_{\min} + 1$ if $e \in \partial B_\infty(0, r_P)$,
- $T(e) > \nu$ else.

Since the support of F is unbounded, the event \mathcal{T} has a positive probability. Assume that \mathcal{T} occurs. Then, let $x, y \in \partial B_\infty(0, r_P)$. Let π be a self-avoiding path going from x to y and using only edges in $\partial B_\infty(0, r_P)$ and $\tilde{\pi}$ be a path going from x to y using edges of $B_\infty(0, r_P)$ and at least one edge e' which is not in $\partial B_\infty(0, r_P)$. We get

$$T(\pi) \leq |\partial B_\infty(0, r_P)|_e(t_{\min} + 1),$$

and

$$T(\tilde{\pi}) \geq T(e') \geq \nu > |\partial B_\infty(0, r_P)|_e(t_{\min} + 1).$$

Hence, $T(\pi) < T(\tilde{\pi})$. □

Boxes. Recall that, when $t_{\min} = 0$, ρ is fixed at (1.23). Fix

$$\begin{aligned} r_1 &= 1, r_2 > 4dr_P \text{ and} \\ \text{when } t_{\min} > 0, r_3 &> 2d(r_2 + 1) \\ \text{when } t_{\min} = 0, r_3 &> \max\left(2d(r_2 + 1), \frac{2\nu_0}{\rho\delta}\right). \end{aligned} \tag{2.4}$$

Note that, in particular, we have the following inequalities:

- $r_2 > 4r_1 > 2r_1$ since $r_1 = 1$ and $r_P \geq 1$,
- $r_3 > 2r_2$ and $r_3 > r_2 + 1$.

Then, for all $N \geq 1$, we define

$$B_{3,s,N} = \{v \in \mathbb{Z}^d : (s - r_3)N \leq z < (s + r_3)N\}.$$

and for $i \in \{1, 2\}$, we define

$$B_{i,s,N} = \{v \in \mathbb{Z}^d : (s - r_i)N \leq z \leq (s + r_i)N\}.$$

We use the word "box" to talk about $B_{3,s,N}$. For $i \in \{1, 2, 3\}$, $\partial B_{i,s,N}$ is the set of vertices of $B_{i,s,N}$ having an adjacent vertex not contained in $B_{i,s,N}$.

Definition 2.6 (Directed path and its selected straight segment). *Let $u \in \partial B_{2,s,N}$ and $v \in \partial B_{1,s,N}$.*

- *We fix in an arbitrary way $\vec{\pi}(u, v)$ an oriented⁵ path from u to v with a subpath $\vec{\pi}[u, v]$ between $\partial B_{2,s,N}$ and v using only edges in the same direction. We say that $\vec{\pi}(u, v)$ is the directed path between u and v . The subpath $\vec{\pi}[u, v]$ is called the straight segment between u and v and its length is greater than or equal to $(r_2 - r_1)N$.*
- *We define the selected straight segment between u and v as the set of vertices c belonging to the straight segment between u and v and such that the distance for the norm $\|\cdot\|_1$ between c and $(B_{2,s,N})^c$ is at least $\frac{(r_2 - r_1)N}{2} + dr_P$.*

Remark 2.7. *The selected straight segment is not the empty set since v belongs to it. Indeed, the distance between v and $(B_{2,s,N})^c$ is equal to $(r_2 - r_1)N$ and $(r_2 - r_1)N > \frac{(r_2 - r_1)N}{2} + dr_P$ since $r_2 > 2r_1$ and $r_2 > 4dr_P$ by (2.4).*

⁵Recall that, as it is defined in Section 1.6, a path, whose endpoints are denoted by u and v , is oriented if this path has exactly $\|u - v\|_1$ edges.

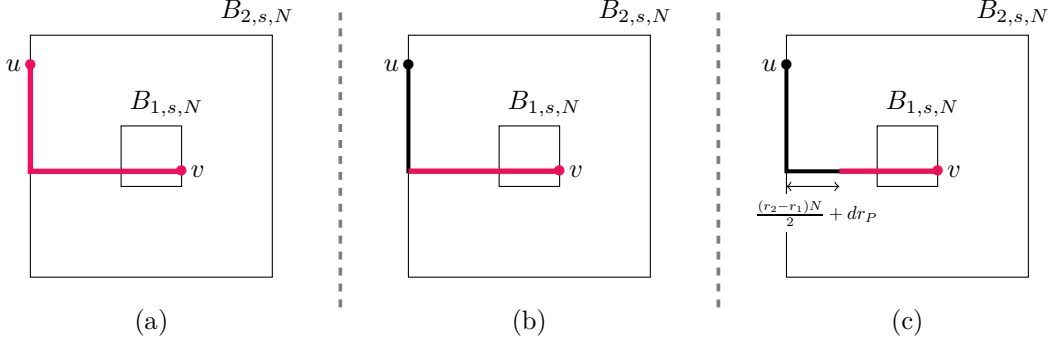


Figure 2: Representation of the objects defined in Definition 2.6 in two dimensions. (a) The directed path $\bar{\pi}(u, v)$ is in red. (b) The straight segment $\bar{\pi}[u, v]$ is in red. (c) The selected straight segment between u and v is in red.

Typical boxes in the case (INF). In these cases, a box $B_{3,s,N}$ is typical if it verifies the following properties:

- (i) if $t_{\min} = 0$, every path π entirely contained in $B_{3,s,N}$ from u_π to v_π with $\|u_\pi - v_\pi\|_1 \geq N$ has at least $\rho\|u_\pi - v_\pi\|_1$ edges whose passage time is greater than δ ,
- (ii) every path π entirely contained in $B_{3,s,N}$ from u_π to v_π with $\|u_\pi - v_\pi\|_1 \geq N$ has a passage time verifying:

$$T(\pi) \geq (t_{\min} + \delta)\|u_\pi - v_\pi\|_1, \quad (2.5)$$

- (iii) for all vertices $u \in \partial B_{2,s,N}$ and $v \in \partial B_{1,s,N}$, there exists a vertex c belonging to the selected straight segment between u and v such that for every edge $e \in B_\infty(c, r_P)$, $T(e) = \infty$.

Typical boxes in the case (FU). Fix \mathcal{T} the event given by Lemma 2.5. We define a sequence $(\nu_1(N))_{N \in \mathbb{N}^*}$ such that:

- for all $N \in \mathbb{N}^*$, $\nu_1(N) > T^\Lambda$ if $t_{\min} > 0$ and $\nu_1(N) > \max(\nu_0, T^\Lambda)$ if $t_{\min} = 0$,
- we have

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\sum_{e \in B_{3,0,N}} T(e) \geq \nu_1(N) \right) = 0. \quad (2.6)$$

Note that by (2.2) and by the first item above, for all $N \in \mathbb{N}^*$, when \mathcal{A}^Λ occurs, $\nu_1(N)$ is strictly greater than the passage time of every edge belonging to Λ but not to $\partial\Lambda$. Note also that $\mathcal{L}((\nu_1(N), \infty)) > 0$ for all $N \in \mathbb{N}^*$ since the support of \mathcal{L} is unbounded.

In this case, a box $B_{3,s,N}$ is typical if it verifies the following properties:

- (i) if $t_{\min} = 0$, every path π entirely contained in $B_{3,s,N}$ from u_π to v_π with $\|u_\pi - v_\pi\|_1 \geq N$ has at least $\rho\|u_\pi - v_\pi\|_1$ edges whose passage time is greater than δ ,
- (ii) every path π entirely contained in $B_{3,s,N}$ from u_π to v_π with $\|u_\pi - v_\pi\|_1 \geq N$ has a passage time verifying:

$$T(\pi) \geq (t_{\min} + \delta)\|u_\pi - v_\pi\|_1, \quad (2.5)$$

- (iii) for all vertices $u \in \partial B_{2,s,N}$ and $v \in \partial B_{1,s,N}$, there exists a vertex c belonging to the selected straight segment between u and v such that $\theta_c T \in \mathcal{T}$,

- (iv) $\sum_{e \in B_{3,s,N}} T(e) < \nu_1(N)$.

Properties of typical boxes in both cases.

Lemma 2.8. *We have these two properties about typical boxes.*

1. Let $s \in \mathbb{Z}^d$ and $N \in \mathbb{N}^*$. The typical box property only depends on the passage times of the edges in $B_{3,s,N}$.
2. We have

$$\lim_{N \rightarrow \infty} \mathbb{P}(B_{3,0,N} \text{ is a typical box}) = 1.$$

Proof.

1. Properties (i) and (ii) in the two cases and property (iv) in the case (FU) only depend on the edges of $B_{3,s,N}$. Then, for all vertices $u \in \partial B_{2,s,N}$ and $v \in \partial B_{1,s,N}$, every vertex c belonging to the selected straight segment between u and v has a distance with $(B_{2,s,N})^c$ greater than or equal to $\frac{(r_2 - r_1)N}{2} + dr_P$. Hence, property (iii) only depends on the edges of $B_{2,s,N}$ and $B_{2,s,N} \subset B_{3,s,N}$.
2. First, let us prove that, in the two cases, the probability that (i) is satisfied by $B_{3,0,N}$ goes to 1. For this item, assume that $t_{\min} = 0$. Let Π denote the set of self-avoiding paths entirely contained in $B_{3,0,N}$. For a path π going from a vertex u_π to a vertex v_π , we say that π satisfies the property \mathcal{P}_δ if π takes at least $\rho \|u_\pi - v_\pi\|_1$ edges e such that $T(e) > \delta$. Then, using Lemma 1.9,

$$\begin{aligned} & \mathbb{P}(B_{3,0,N} \text{ does not satisfy (i)}) \\ & \leq \sum_{\substack{u_\pi, v_\pi \in B_{3,0,N} \\ \|u_\pi - v_\pi\|_1 \geq N}} \mathbb{P}(\mathcal{P}_\delta \text{ is not satisfied by a path of } \Pi \text{ whose endpoints are } u_\pi \text{ and } v_\pi) \\ & \leq \sum_{\substack{u_\pi, v_\pi \in B_{3,0,N} \\ \|u_\pi - v_\pi\|_1 \geq N}} \mathbb{P}(\mathcal{P}_\delta \text{ is not satisfied by a path whose endpoints are } u_\pi \text{ and } v_\pi) \\ & \leq |B_{3,0,N}|_v^2 \beta' e^{-\beta N} \xrightarrow{N \rightarrow \infty} 0, \end{aligned}$$

since $|B_{3,0,N}|_v$ is bounded by a polynomial in N .

Now, for the remaining of this proof, t_{\min} can be positive. Using (1.21) and a similar computation as above, we get that

$$\mathbb{P}(B_{3,0,N} \text{ does not satisfy (ii)}) \xrightarrow{N \rightarrow \infty} 0.$$

Recall Definition 2.6. To prove that the probability that (iii) is satisfied by $B_{3,0,N}$ goes to 1, we begin by associating in a deterministic way to each couple of vertices $(u, v) \in \partial B_{2,0,N} \times \partial B_{1,0,N}$ a set of vertices, denoted by $V(\vec{\pi}(u, v))$ such that:

- every vertex of $V(\vec{\pi}(u, v))$ belongs to the selected straight segment between u and v ,
- for all $z_1, z_2 \in V(\vec{\pi}(u, v))$, $B_\infty(z_1, r_P) \cap B_\infty(z_2, r_P) = \emptyset$,
- there can be no other set satisfying the two conditions above containing strictly more vertices than $V(\vec{\pi}(u, v))$.

Note that there exists a constant K_1 only depending on r_1, r_2 and r_P such that $|V(\vec{\pi}(u, v))| \geq K_1 N$. In the case (INF), we denote by \mathcal{T}_∞ the event on which for all $e \in B_\infty(0, r_P)$, $T(e) = \infty$. We use the notation $\overline{\mathcal{T}}$ to designate the event \mathcal{T} in the case (FU) and to designate \mathcal{T}_∞ in the case (INF), which allows us to conclude this part of proof in the two cases. We have

$$\begin{aligned} \mathbb{P}(B_{3,0,N} \text{ does not satisfy (iii)}) & \leq \sum_{\substack{u \in \partial B_{2,0,N} \\ v \in \partial B_{1,0,N}}} \mathbb{P}(\forall c \in \vec{\pi}(u, v), \theta_c \overline{\mathcal{T}} \text{ does not occur}) \\ & \leq \sum_{\substack{u \in \partial B_{2,0,N} \\ v \in \partial B_{1,0,N}}} \mathbb{P}(\forall c \in V(\vec{\pi}(u, v)), \theta_c \overline{\mathcal{T}} \text{ does not occur}). \end{aligned}$$

Fix $u \in \partial B_{2,0,N}$ and $v \in \partial B_{1,0,N}$. Since for all $z_1, z_2 \in V(\bar{\pi}(u, v))$, $B_\infty(z_1, r_P) \cap B_\infty(z_2, r_P) = \emptyset$, the family of events $(\{\theta_c \bar{\mathcal{T}} \text{ does not occur}\})_{c \in V(\bar{\pi}(u, v))}$ are independent and thus

$$\mathbb{P}(\forall c \in V(\bar{\pi}(u, v)), \theta_c \bar{\mathcal{T}} \text{ does not occur}) \leq (1 - \mathbb{P}(\bar{\mathcal{T}}))^{|V(\bar{\pi}(u, v))|} \leq (1 - \mathbb{P}(\bar{\mathcal{T}}))^{K_1 N}.$$

Since $\mathbb{P}(\bar{\mathcal{T}}) > 0$, we get the existence of a constant K_2 not depending on u, v and N such that:

$$\mathbb{P}(\forall c \in V(\bar{\pi}(u, v)), \theta_c \bar{\mathcal{T}} \text{ does not occur}) \leq e^{-K_2 N}.$$

Hence,

$$\mathbb{P}(B_{3,0,N} \text{ does not satisfy (iii)}) \leq |\partial B_{2,0,N}|_v |\partial B_{1,0,N}|_v e^{-K_2 N} \xrightarrow{N \rightarrow \infty} 0,$$

since $|\partial B_{2,0,N}|_v |\partial B_{1,0,N}|_v$ is bounded by a polynomial in N .

Finally, in the case (FU), we get that the probability that (iv) is satisfied by $B_{3,0,N}$ goes to 1 by (2.6). □

Crossing a box. A self-avoiding path crosses a box $B_{3,s,N}$ if it visits one vertex of $\partial B_{3,s,N}$, then one of $B_{1,s,N}$ and then another one of $\partial B_{3,s,N}$.

The following lemma is a consequence of Lemma 5.2 in [9] which applies using Lemma 2.8.

Lemma 2.9. *For any N sufficiently large, we can take $D_1 > 0$ and $\alpha > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}(\exists z \in \Gamma_n \text{ such that } (0, z) \in \mathfrak{C} \text{ and } \exists \text{ a path from } 0 \text{ to } z \text{ that crosses at most } \lfloor \alpha n \rfloor \text{ typical boxes}) \leq e^{-D_1 n}. \quad (2.7)$$

Using Lemma 2.9 above, we fix

$$N \geq 1 \text{ large enough and } D_1 > 0, \alpha > 0 \text{ such that (2.7) holds.} \quad (2.8)$$

For the remaining of the proof, since N is fixed, we write ν_1 instead of $\nu_1(N)$ in the case (FU). We fix $\delta' > 0$ such that

$$\delta' < \min\left(\frac{\delta}{2}, \frac{1}{N}\right). \quad (2.9)$$

Note that, in particular, since $N \geq 1$, we have $\delta' < 1$.

2.1.3 Boundaries of the patterns in the case (FU)

In the case (FU), we fix $\nu_2 > \nu_1$ such that

$$\mathcal{A}^\Lambda \cap \{\forall e \in \partial\Lambda, T(e) \in (\nu_1, \nu_2)\} \text{ has a positive probability.} \quad (2.10)$$

It is possible since by the condition (AF-4) in Section 2.1.1, for all $M >$, $\mathcal{A}^\Lambda \cap \{\forall e \in \partial\Lambda, T(e) > M\}$ has a positive probability.

(AF-4') For the remaining of the proof, we now replace \mathcal{A}^Λ by

$$\mathcal{A}^\Lambda \cap \{\forall e \in \partial\Lambda, T(e) \in (\nu_1, \nu_2)\}.$$

As announced in Remark 2.3, from now on, the event \mathcal{A}^Λ of the pattern \mathfrak{P} has been modified. The assumption (AF-4) is not satisfied by this new event but it satisfies the assumption (AF-4') above.

Remark 2.10. *In the case (FU), since there is no edge whose passage time is greater than ν_1 in a typical box, there can be no pattern in a typical box.*

2.1.4 k -penalized paths, shortcuts and successful boxes

Recall that r_3 is fixed at (2.4) and that N is fixed at (2.8). We partition \mathbb{Z}^d with boxes $B_{3,s,N}$ in $K = K(r_3) = \lfloor B_{3,0,1} \rfloor_v$ ways as follows. For each $z \in B_{3,0,1}$, the partition associated with z is

$$\{B_{3,s,N}, s - z \in 2r_3\mathbb{Z}^d\}.$$

For convenience, we index these different partitions from 1 to K . For $k \in \{1, \dots, K\}$, the boxes belonging to the k -th partition are called k -boxes.

Recall that we say that a self-avoiding path π takes the pattern if there exists $z \in \mathbb{Z}^d$ satisfying the condition $(\pi; \mathfrak{P})$. For $k \in \{1, \dots, K\}$, we say that a self-avoiding path π takes a pattern entirely contained in a k -box if there exists $z \in \mathbb{Z}^d$ such that:

- z satisfies the condition (π, \mathfrak{P}) ,
- there exists a k -box containing $B_\infty(z, \ell^\Lambda)$.

k -penalized paths. For $k \in \{1, \dots, K\}$, a k -penalized path is a self-avoiding path which takes no pattern entirely contained in a k -box.

Penalized passage time. For all $x \in \mathbb{Z}^d$, and for $k \in \{1, \dots, K\}$, we define

$$t_k(0, x) = \inf\{T(\pi) : \pi \text{ is a } k\text{-penalized path from } 0 \text{ to } x\},$$

with the convention $\inf \emptyset = \infty$.

k -geodesics. For all $x \in \mathbb{Z}^d$, for all $k \in \{1, \dots, K\}$, a k -geodesic from 0 to x is a k -penalized path γ from 0 to x such that $T(\gamma) = t_k(0, x)$.

Shortcuts. For all boxes $B_{3,s,N}$, we say that a path π has a shortcut in $B_{3,s,N}$ if π crosses $B_{3,s,N}$ and if there exist two vertices u and v of π and a path π' going from u to v such that:

- π' is entirely contained in $B_{3,s,N}$,
- $\pi_{u,v}$ and π' have only u and v as vertices in common,
- π' takes a pattern entirely contained in $B_{3,s,N}$,
- $T(\pi') < T(\pi_{u,v} \cap B_{3,s,N})$.

Successful boxes. Let $B_{3,s,N}$ be a box and π a self-avoiding path. We say that $B_{3,s,N}$ is successful for the path π if the following two conditions hold:

- π crosses $B_{3,s,N}$,
- $B_{3,s,N}$ is a typical box or π has a shortcut in $B_{3,s,N}$.

S^k -sequences. For every $k \in \{1, \dots, K\}$, for every k -geodesic γ between two vertices, the S^k -sequence of γ is the sequence of different k -boxes successful for γ by order of first visit by γ . Note that the boxes of this S^k -sequence are pairwise disjoint by the definition of k -boxes.

2.2 Proof

2.2.1 Reduction

We begin the proof with some definitions. Recall that α is fixed at (2.8) and that K is fixed at the beginning of Section 2.1.4. For all $n \geq 1$, write

$$Q_n = \left\lfloor \frac{\alpha n}{K} \right\rfloor.$$

Fix

$$n \geq 1 \text{ and } x \in \Gamma_n \tag{2.11}$$

From now on, when we talk about a path, a geodesic or a k -geodesic without specifying its extremities, we mean that it is from 0 to x . For all $k \in \{1, \dots, K\}$, we define

$$\mathcal{G}^k = \{(0, x) \in \mathfrak{C} \text{ and there exists a } k\text{-geodesic whose passage time is finite having at least } Q_n \text{ boxes in its } S^k\text{-sequence}\}.$$

Selected k -geodesic and S^k -variables. For all $k \in \{1, \dots, K\}$, on the event \mathcal{G}^k ,

- we define the selected k -geodesic as the first k -geodesic in the lexicographical order⁶ among those having at least Q_n boxes in their S^k -sequences,
- for all $j \in \{1, \dots, Q_n\}$, we define the random variable S_j^k as the vertex s such that $B_{3,s,N}$ is the j -th box of the S^k -sequence of the selected k -geodesic.

Events \mathcal{M}^k . For all $k \in \{1, \dots, K\}$ and all $j \in \{1, \dots, Q_n\}$, we define

$$\mathcal{M}^k(j) = \mathcal{G}^k \cap \{\text{the selected } k\text{-geodesic does not have a shortcut in any of the first } j \text{ boxes of its } S^k\text{-sequence}\}.$$

To make the end of this proof easier to read, we define the events

$$\begin{aligned} A &= \{(0, x) \in \mathfrak{C} \text{ and there exists a geodesic from } 0 \text{ to } x \text{ which does not take the pattern}\}, \\ B &= \{\text{every path from } 0 \text{ to } \Gamma_n \text{ crosses at least } \lfloor \alpha n \rfloor + 1 \text{ typical boxes}\}, \end{aligned}$$

Note that A is the event considered in Proposition 1.8 and B the complementary event to the one considered in Lemma 2.9. The proof of Proposition 1.8 is based on the following two lemmas.

Lemma 2.11. We have $A \cap B \subset \bigcup_{k=1}^K \mathcal{M}^k(Q_n)$.

Lemma 2.12. There exists $\lambda \in (0, 1)$ which does not depend on x and n such that for all $k \in \{1, \dots, K\}$,

$$\mathbb{P}(\mathcal{M}^k(Q_n)) \leq \lambda^{Q_n}.$$

Proof of Proposition 1.8 using Lemma 2.11 and 2.12. Recall that N is fixed at (2.8) and that n and x are fixed at (2.11) but that D_1 and λ do not depend on x and n . We have

$$\begin{aligned} \mathbb{P}(A) &\leq \mathbb{P}(A \cap B) + \mathbb{P}(B^c) \\ &\leq \sum_{k=1}^K \mathbb{P}(\mathcal{M}^k(Q_n)) + \mathbb{P}(B^c) \text{ by Lemma 2.11,} \\ &\leq K\lambda^{Q_n} + \mathbb{P}(B^c) \text{ by Lemma 2.12,} \\ &\leq K\lambda^{Q_n} + e^{-D_1 n} \text{ by Lemma 2.9.} \end{aligned}$$

As $D_1 > 0$ and $\lambda \in (0, 1)$, and as this inequality holds for any $n \geq 1$ and any $x \in \Gamma_n$, we get the existence of two constants $C > 0$ and $D > 0$ such that for all n , for all $x \in \Gamma_n$,

$$\mathbb{P}(A) \leq D \exp(-Cn).$$

□

Proof of Lemma 2.11. Assume that A occurs. Then there exists a self-avoiding path γ from 0 to x such that:

- γ does not take the pattern,

⁶The lexicographical order is based on the directions of the consecutive edges of the geodesics.

- $T(\gamma) = t(0, x) < \infty$.

For all $k \in \{1, \dots, K\}$, we get that:

- γ is a k -penalized path,
- $T(\gamma) = t(0, x) = t_k(0, x) < \infty$.

Thus, for all $k \in \{1, \dots, K\}$, γ is a k -geodesic from 0 to x and no k -geodesic has a shortcut in any box. Assume that B also occurs. Then γ crosses at least $\lfloor \alpha n \rfloor + 1$ typical boxes. Hence, there exists $k \in \{1, \dots, K\}$ such that γ crosses at least $Q_n = \lfloor \frac{\alpha n}{K} \rfloor$ typical boxes. Since every typical k -box crossed by γ is a successful box for γ , γ is a k -geodesic having at least Q_n boxes in its S^k -sequence. Hence the event \mathcal{G}^k occurs and the selected k -geodesic (which is not necessarily γ) does not have a shortcut in any of the first Q_n boxes of its S^k -sequence since it does not have a shortcut in any box. So the event $\mathcal{M}^k(Q_n)$ occurs. \square

Now, for the remaining of the proof, the aim is to prove Lemma 2.12.

Modification argument. We introduce an independent copy T' of the environment T , the two being defined on the same probability space. It is thus convenient to refer to the considered environment when dealing with the objects defined above. To this aim, we shall use the notation $\{T \in \mathcal{M}^k(j)\}$ to denote that the event $\mathcal{M}^k(j)$ holds with respect to the environment T . In other words, $\mathcal{M}^k(j)$ is now seen as a subset of $[0, \infty]^\mathcal{E}$ where \mathcal{E} is the set of all the edges. Similarly, we denote by $\mathcal{S}_j^k(T')$ the random variables defined above but in the environment T' .

Fix $k \in \{1, \dots, K\}$ and $\ell \in \{1, \dots, Q_n\}$. On $\{T \in \mathcal{M}^k(\ell)\}$, the event \mathcal{G}^k occurs and $B_{3, S_\ell^k(T), N}$ is the ℓ -th box of the S^k -sequence of the selected k -geodesic. From this new environment, we associate a set of edges $E_{\text{modif}}^*(T)$ which is contained in $B_{3, S_\ell^k(T), N}$. It corresponds to the edges for which we want to modify the time. We get a new environment T^* defined for all edges e by:

$$T^*(e) = \begin{cases} T(e) & \text{if } e \notin E_{\text{modif}}^*(T), \\ T'(e) & \text{else.} \end{cases}$$

For y and z in \mathbb{Z}^d , we denote by $t^*(y, z)$ the geodesic time between y and z in the environment T^* . Note that T and T^* do not have the same distribution as the set $E_{\text{modif}}^*(T)$ depends on T .

The proof of Lemma 2.12 relies on the following lemma whose proof is given in the next subsection. Recall that, in the case (FU), ν_2 is fixed at (2.10) and that, if $t_{\min} = 0$ in the cases= (INF), ν_0 is fixed at (1.22).

Lemma 2.13. *There exists $\eta = \eta(N) > 0$ such that for all ℓ in $\{1, \dots, Q_n\}$, for all $k \in \{1, \dots, K\}$, there exist measurable functions E_-^* , E_{mid}^* , E_+^* , E_∞^* all from $[0, \infty]^\mathcal{E}$ to $\mathcal{P}(\mathcal{E})$ and a measurable function $\mathcal{C} : [0, \infty]^\mathcal{E} \mapsto \mathbb{Z}^d$ such that:*

(i) *on the event $\{T \in \mathcal{M}^k(\ell)\}$, $E_-^*(T)$, $E_{\text{mid}}^*(T)$, $E_+^*(T)$, $E_\infty^*(T)$ and $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ are pairwise disjoint and are contained in $B_{3, S_\ell^k(T), N}$,*

(ii) *on the event $\{T \in \mathcal{M}^k(\ell)\}$, we have $\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \eta$, where $\{T' \in \mathcal{B}^*(T)\}$ is a shorthand for the event on which:*

- $\forall e \in E_-^*(T), T'(e) \leq t_{\min} + \delta'$,
- $\forall e \in E_{\text{mid}}^*(T), T'(e) \in (t_{\min} + \delta, \nu_0)$,
- $\forall e \in E_+^*(T), T'(e) > \nu_2$,
- $\forall e \in E_\infty^*(T), T'(e) = \infty$,
- $\theta_{\mathcal{C}(T)} T' \in \mathcal{A}^\Lambda$.

(iii) $\{T \in \mathcal{M}^k(\ell)\} \cap \{T' \in \mathcal{B}^*(T)\} \subset \{T^* \in \mathcal{M}^k(\ell - 1) \setminus \mathcal{M}^k(\ell)\}$ and $S_\ell^k(T^*) = S_\ell^k(T)$.

Remark 2.14. *Several of the functions of the previous lemma can be equal to the empty set depending on the cases. Thus this does not prevent us from having $\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \eta$. In particular, for every environment T :*

- in the case (INF), $E_+^*(T) = \emptyset$,
- in the case (FU), $E_\infty^*(T) = \emptyset$,
- in the two cases, if $t_{\min} > 0$, $E_{\text{mid}}^*(T) = \emptyset$.

Proof of Lemma 2.12 using Lemma 2.13. Let $\ell \in \{1, \dots, Q_n\}$ and $k \in \{1, \dots, K\}$. For every $s \in \mathbb{Z}^d$ and \mathcal{E}^* subset of edges of $B_{3,s,N}$, let us consider the environment T_{s,\mathcal{E}^*}^* defined for all edges e by:

$$T_{s,\mathcal{E}^*}^*(e) = \begin{cases} T'(e) & \text{if } e \in \mathcal{E}^*, \\ T(e) & \text{else.} \end{cases}$$

We define $E_{\text{modif}}^*(T) = E_-^*(T) \cup E_{\text{mid}}^*(T) \cup E_+^*(T) \cup E_\infty^*(T) \cup B_\infty(\mathcal{C}(T), \ell^\Lambda)$. Thus, for every s and \mathcal{E}^* , T_{s,\mathcal{E}^*}^* and T have the same distribution and on the event $\{T \in \mathcal{M}^k(\ell)\} \cap \{S_\ell^k(T) = s\} \cap \{E_{\text{modif}}^*(T) = \mathcal{E}^*\}$, $T^* = T_{s,\mathcal{E}^*}^*$. So, using this environment and writing with indicator functions the result of Lemma 2.13, we get:

$$\mathbb{1}_{\{T \in \mathcal{M}^k(\ell)\}} \mathbb{1}_{\{S_\ell^k(T) = s\}} \mathbb{1}_{\{E_{\text{modif}}^*(T) = \mathcal{E}^*\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \leq \mathbb{1}_{\{T_{s,\mathcal{E}^*}^* \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell)\}} \mathbb{1}_{\{S_\ell^k(T_{s,\mathcal{E}^*}^*) = s\}}, \quad (2.12)$$

We compute the expectation on both sides. For the left side, we have

$$\begin{aligned} & \mathbb{E} \left[\mathbb{1}_{\{T \in \mathcal{M}^k(\ell)\}} \mathbb{1}_{\{S_\ell^k(T) = s\}} \mathbb{1}_{\{E_{\text{modif}}^*(T) = \mathcal{E}^*\}} \mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \right] \\ &= \mathbb{E} \left[\mathbb{1}_{\{T \in \mathcal{M}^k(\ell)\}} \mathbb{1}_{\{S_\ell^k(T) = s\}} \mathbb{1}_{\{E_{\text{modif}}^*(T) = \mathcal{E}^*\}} \mathbb{E} \left[\mathbb{1}_{\{T' \in \mathcal{B}^*(T)\}} \mid T \right] \right]. \end{aligned}$$

Since on the event $\{T \in \mathcal{M}^k(\ell)\} \cap \{S_\ell^k(T) = s\}$, we have $\mathbb{P}(T' \in \mathcal{B}^*(T) \mid T) \geq \eta$, the left side is bounded from below by $\eta \mathbb{P}(T \in \mathcal{M}^k(\ell), S_\ell^k(T) = s, E_{\text{modif}}^*(T) = \mathcal{E}^*)$. Since T_{s,\mathcal{E}^*}^* and T have the same distribution, using (2.12), we get:

$$\eta \mathbb{P}(T \in \mathcal{M}^k(\ell), S_\ell^k(T) = s, E_{\text{modif}}^*(T) = \mathcal{E}^*) \leq \mathbb{P}(T \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell), S_\ell^k(T) = s).$$

Then, by writing K' the number of subsets of edges of $B_{3,0,N}$ and by summing on all subsets \mathcal{E}^* of edges of $B_{3,s,N}$, we get for all $s \in \mathbb{Z}^d$,

$$\frac{\eta}{K'} \mathbb{P}(T \in \mathcal{M}^k(\ell), S_\ell^k(T) = s) \leq \mathbb{P}(T \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell), S_\ell^k(T) = s).$$

Finally, by summing⁷ on all $s \in \mathbb{Z}^d$, we get

$$\frac{\eta}{K'} \mathbb{P}(T \in \mathcal{M}^k(\ell)) \leq \mathbb{P}(T \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell)).$$

Now, since $\mathcal{M}^k(\ell) \subset \mathcal{M}^k(\ell-1)$,

$$\mathbb{P}(T \in \mathcal{M}^k(\ell-1) \setminus \mathcal{M}^k(\ell)) = \mathbb{P}(T \in \mathcal{M}^k(\ell-1)) - \mathbb{P}(T \in \mathcal{M}^k(\ell)).$$

Thus,

$$\mathbb{P}(T \in \mathcal{M}^k(\ell)) \leq \lambda \mathbb{P}(T \in \mathcal{M}^k(\ell-1)),$$

where $\lambda = \frac{1}{1 + \frac{\eta}{K'}} \in (0, 1)$ does not depend on x and n . Hence, using $\mathbb{P}(T \in \mathcal{M}^k(0)) = 1$, we get by induction

$$\mathbb{P}(T \in \mathcal{M}^k(Q_n)) \leq \lambda_n^Q.$$

□

⁷Note that here, we must have the event $\{S_\ell^k(T_{s,\mathcal{E}^*}^*) = s\}$ on the right side of the inequality (2.12) to sum on all $s \in \mathbb{Z}^d$.

2.2.2 Proof of Lemma 2.13: modification

Let $\ell \in \{1, \dots, Q_n\}$, $k \in \{1, \dots, K\}$ and $s \in \mathbb{Z}^d$ such that $B_{3,s,N}$ is a k -box. Assume that the event $\{T \in \mathcal{M}^k(\ell)\} \cap \{S_\ell^k(T) = s\}$ occurs. Note that $(0, x) \in \mathfrak{C}$. We denote by γ the selected k -geodesic. We know that:

- (H1) γ has at least Q_n boxes in its S^k -sequence,
- (H2) γ does not have a shortcut in any of the first ℓ boxes of its S^k -sequence,
- (H3) $B_{3,s,N}$ is the ℓ -th box of the S^k -sequence of γ and is a typical box.

Construction of the forbidden zone and definition of $\mathcal{C}(T)$. Let u_2 (resp. u_1) be the entry point of γ in $B_{2,s,N}$ (resp. $B_{1,s,N}$). Since $B_{3,s,N}$ is a typical box, we can define:

- π as the path $\bar{\pi}(u_2, u_1)$ defined in Definition 2.6,
- $\mathcal{C}(T)$ the first vertex belonging to the selected straight segment between u_2 and u_1 and satisfying property (iii) of a typical box. Such a vertex exists by Remark 2.7 and since $B_{3,s,N}$ is a typical box in the environment T . Recall that in the case (INF), for every edge $e \in B_\infty(\mathcal{C}(T), r_P)$, $T(e) = \infty$ and in the case (FU), $\theta_{-\mathcal{C}(T)}T \in \mathcal{T}$.

We use the expression "forbidden zone" to refer to $B_\infty(\mathcal{C}(T), r_P) \setminus \partial B_\infty(\mathcal{C}(T), r_P)$. This is the place where we want to place the pattern taken by the shortcut in the modified environment.

Properties of the forbidden zone.

Lemma 2.15. 1. *The path γ does not visit any vertex of the forbidden zone.*

- 2. *The ball $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ is contained in the forbidden zone.*
- 3. *The forbidden zone is contained in $B_{2,s,N}$ and for every c in the forbidden zone and every $z \in \partial B_{2,s,N}$,*

$$\|z - c\|_1 \geq \frac{(r_2 - r_1)N}{2}.$$

Proof. 1. In the case (INF), every edge of the forbidden zone has an infinite passage time although γ has a finite passage time. In the case (FU), it follows from Lemma 2.5 and the fact that γ crosses this box and is a geodesic in the environment T .

- 2. It comes from the inequality $r_P > \ell^\Lambda + 1$ by (2.3).
- 3. By Definition 2.6, $\mathcal{C}(T)$ belongs to $B_{2,s,N}$ and the distance between $\mathcal{C}(T)$ and $(B_{2,s,N})^c$ is at least $\|z - c\|_1 \geq \frac{(r_2 - r_1)N}{2}$. We get the result using that, for every c in the forbidden zone, $\|c - \mathcal{C}(T)\|_1 \leq dr_P$.

□

Construction of the shortcut π' . Let u_π be the last vertex of π belonging to γ before π visits the forbidden zone and v_π be the first vertex of π belonging to γ after the forbidden zone. One can check that we can build a path, denoted by π' for the remaining of the proof, such that:

- π' is a self-avoiding path from u_π to v_π ,
- π' is the concatenation of the subpath of π between u_π and the forbidden zone, then of a path entirely contained in the forbidden zone and then of the subpath of π between the forbidden zone and v_π ,
- π' visits $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ for the first time in $\theta_{-\mathcal{C}(T)}u_\pi$ and for the last time in $\theta_{-\mathcal{C}(T)}v_\pi$, and between these two vertices, π' is entirely contained in $B_\infty(\mathcal{C}(T), \ell^\Lambda)$. Furthermore, in the case (INF), between these two vertices, π' is equal to $\theta_{\mathcal{C}(T)}\pi_\infty$ where π_∞ is defined in Assumption (AI-3). Note that, if $T' \in \mathcal{B}^*(T)$ (where $\mathcal{B}^*(T)$ is defined in Lemma 2.13), then $\theta_{\mathcal{C}(T)}\pi_\infty$ has finite passage time in the environment T^* ,

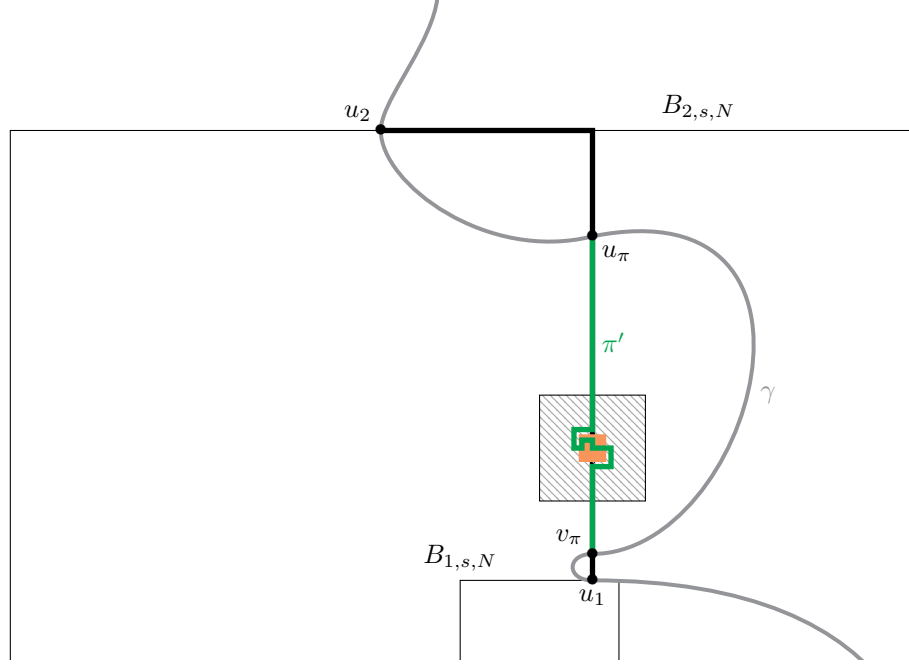


Figure 3: Example of construction of the shortcut π' in two dimensions. The shortcut π' is represented in green, the path γ in gray, the ball $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ in orange and the forbidden zone by the hatched area

- we have an upper bound for the number of edges in π' :

$$|\pi'|_e \leq \|u_\pi - v_\pi\|_1 + |B_\infty(0, \ell^\Lambda + 1)|_e. \quad (2.13)$$

Note that, by the second item above and the definition of u_π and v_π , π' only has u_π and v_π in common with γ . See Figure 3 for an example of construction of the shortcut π' .

2.2.2.1 Beginning of the modification. There are two cases for the beginning of the modification depending on whether $t_{\min} = 0$ or $t_{\min} > 0$ and on the number of edges in γ_{u_π, v_π} , denoted by $|\gamma_{u_\pi, v_\pi}|_e$. We have to distinguish two cases because we must be able to have a lower bound on the passage time of γ_{u_π, v_π} . To this aim, if γ_{u_π, v_π} takes enough edges, we can use the second property of a typical box and if it is not the case, we can have a lower bound using the number of edges of γ_{u_π, v_π} and t_{\min} if $t_{\min} > 0$. If $t_{\min} = 0$ and if we can not use the second property of a typical box, then we use the modification to increase the passage times of γ_{u_π, v_π} . We describe the modification in each case. See Figure 4 for a representation of the objects involved in the modification.

Case A: assume $t_{\min} = 0$ and $|\gamma_{u_\pi, v_\pi}|_e < N$. The beginning of the modification is the following.

- The edges of $E_{\text{mid}}^*(T)$ are the edges e belonging to γ_{u_π, v_π} and such that $T(e) < t_{\min} + \delta$.
- Recall that $\mathcal{C}(T)$ is defined at the beginning of Section 2.2.2.
- The edges of $E_-^*(T)$ are the edges e of $B_{3,s,N}$ satisfying the following two conditions:
 - e belongs to $(\gamma \setminus \gamma_{u_\pi, v_\pi}$ or to $\pi')$ but not to $B_\infty(\mathcal{C}(T), \ell^\Lambda)$,
 - $T(e) \geq t_{\min} + \delta'$.

Case B: assume $t_{\min} > 0$ or $|\gamma_{u_\pi, v_\pi}|_e \geq N$. The beginning of the modification is the following.

- $E_{\text{mid}}^*(T) = \emptyset$.

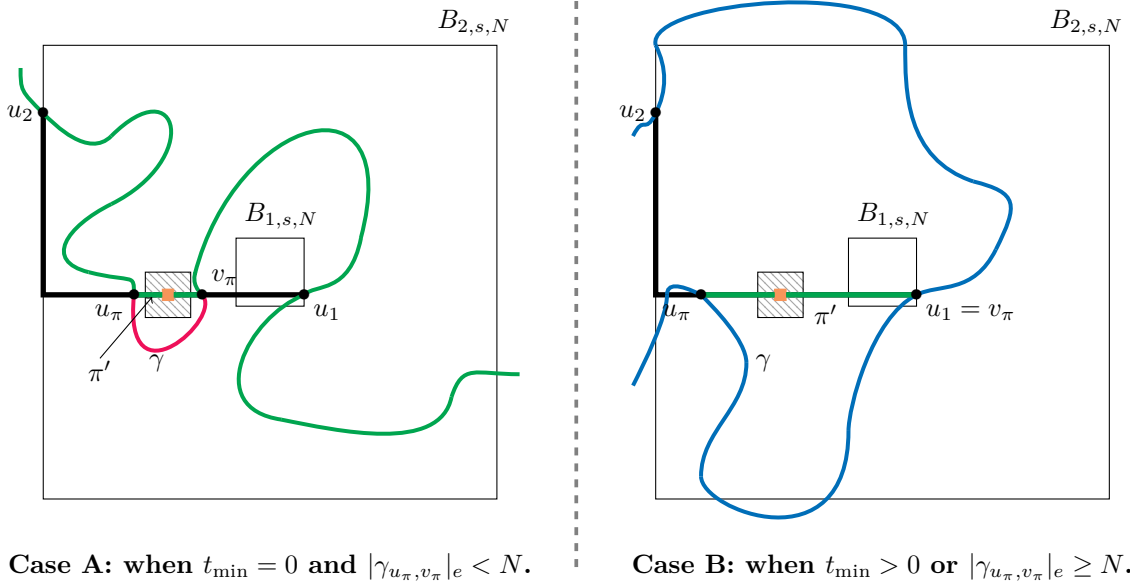


Figure 4: The modification. In the cases A and B, the pattern centered in $\mathcal{C}(T)$ is represented in orange and the forbidden zone by the hatched area. In the case A, γ is the path composed by the edges in green up to u_π , then the edges in red and then the edges in green from v_π . In the case B, γ is the path composed by the edges in blue. In the cases A and B, every edge of $E_-^*(T)$ belongs to the green part of the figure. When the modification is successful, the passage times of the edges in green which are greater than or equal to $t_{\min} + \delta'$ are replaced by passage times smaller than $t_{\min} + \delta'$. In the case A, every edge of $E_{\text{mid}}^*(T)$ belongs to the red part. When the modification is successful, the passage times of the edges in red which are smaller than $t_{\min} + \delta$ are replaced by passage times belonging to $(t_{\min} + \delta, \nu_0)$. In the case B, the passage times of the edges in blue are not modified. The boundary of $B_{3,s,N}$ is not represented even if $B_{2,s,N}$ is included in $B_{3,s,N}$. The edges of $E_\infty^*(T)$ or $E_+^*(T)$ (depending on the case (INF) or (FU)) are all edges which are not in green, red, orange and blue.

- Recall that $\mathcal{C}(T)$ is defined at the beginning of Section 2.2.2.
- The edges of $E_-^*(T)$ are the edges e of $B_{3,s,N}$ satisfying the following two conditions:
 - e belongs to π' but not to $B_\infty(\mathcal{C}(T), \ell^\Lambda)$,
 - $T(e) \geq t_{\min} + \delta'$.

2.2.2.2 End of the modification. Up to now, we have defined $E_-^*(T)$, $E_{\text{mid}}^*(T)$ and $\mathcal{C}(T)$. Note that $E_-^*(T)$ and $E_{\text{mid}}^*(T)$ disjoint sets included in $B_{3,s,N} \cap (\gamma \cup \pi')$. It remains to define $E_+^*(T)$ and $E_\infty^*(T)$. There are two cases depending on whether there can be edges with infinite passage times or not.

In the case (INF). In this case, $E_+^*(T) = \emptyset$ and the edges of $E_\infty^*(T)$ are the edges of $B_{3,s,N}$ which does not belong to $B_\infty(\mathcal{C}(T), \ell^\Lambda)$, to π' or to γ .

In the case (FU). In this case, $E_\infty^*(T) = \emptyset$ and the edges of $E_+^*(T)$ are the edges of $B_{3,s,N}$ which does not belong to $B_\infty(\mathcal{C}(T), \ell^\Lambda)$, to π' or to γ .

2.2.3 Proof of Lemma 2.13: consequences of the modification

Assume for the remaining of the proof that the event

$$\{T \in \mathcal{M}^k(\ell)\} \cap \{S_\ell^k(T) = s\} \cap \{T' \in \mathcal{B}^*(T)\} \text{ occurs,}$$

where $\mathcal{B}^*(T)$ is defined in (ii) of Lemma 2.13. We now state some consequences of the modification useful for the following.

- (a) If $|\gamma_{u_\pi, v_\pi}|_e < N$, every edge in γ_{u_π, v_π} belongs to $B_{3,s,N}$. Thus, every edge belonging to $E_{\text{mid}}^*(T)$ belongs to $B_{3,s,N}$.
- (b) $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ is entirely contained in $B_{3,s,N}$.
- (c) We have $T^*(\gamma) \leq T(\gamma)$. Furthermore, for all vertices u and v in $\partial B_{3,s,N}$ visited by γ , $T^*(\gamma_{u,v}) \leq T(\gamma_{u,v})$.
- (d) $T^*(\pi') < T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N})$.
- (e) There is only one pattern entirely contained in $B_{3,s,N}$ in the environment T^* , which is the one centered in $\mathcal{C}(T)$.
- (f) Let π_0 be a self-avoiding path from $\partial B_{3,s,N}$ to $\partial B_{3,s,N}$ entirely contained in $B_{3,s,N}$ such that in the environment T^* :
 - in the case (INF), it has a finite passage time,
 - in the case (FU), it does not take any edge whose passage time is greater than ν_1 ,
 - in the cases (INF) and (FU), it does not take any pattern contained in $B_{3,s,N}$.

Then every edge of $B_{3,s,N}$ belonging to π_0 belongs to γ .

- (g) Let $\tilde{\pi}$ be a k -penalized path from 0 to x in the environment T^* with $T^*(\tilde{\pi})$ finite. Then

$$T(\tilde{\pi}) - T^*(\tilde{\pi}) \leq T(\gamma) - T^*(\gamma).$$

In other words, no k -penalized path can save more time than γ during the modification.

Proof of (a). Since π is included in $B_{2,s,N}$, $u_\pi \in B_{2,s,N}$. Thus

$$\|u_\pi - sN\|_1 \leq r_2 N. \quad (2.14)$$

Assume that $|\gamma_{u_\pi, v_\pi}|_e < N$ and let z be a vertex visited by γ_{u_π, v_π} . We have

$$\|z - u_\pi\|_1 \leq N. \quad (2.15)$$

Combining (2.14) and (2.15) gives

$$\|z - sN\|_1 \leq (r_2 + 1)N < r_3 N,$$

since $r_3 > r_2 + 1$ by (2.4). Hence z belongs to $B_{3,s,N}$. For the second part of the property, in the case B of the modification, there is no edge in $E_{\text{mid}}^*(T)$ and in the case A of the modification all the edges of $E_{\text{mid}}^*(T)$ belong to γ_{u_π, v_π} . \square

Proof of (b). By Lemma 2.15, $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ is contained in the forbidden zone and the forbidden zone is contained in $B_{2,s,N}$. Since $r_3 > r_2$, $B_{2,s,N}$ is contained in $B_{3,s,N}$. \square

Proof of (c). In the case B of the modification, the passage time of every edge of γ in T^* is equal to its passage time in T . Thus, property (c) holds. Now, assume the case A of the modification and recall that in this case $t_{\min} = 0$. The only edges of γ whose passage times in T^* are strictly greater than their passage times in T are those in γ_{u_π, v_π} and all of these edges are contained in $B_{3,s,N}$ by property (a) above. Hence, to prove property (c), it is sufficient to prove that $T^*(\gamma_{u,v}) \leq T(\gamma_{u,v})$ when u is the last vertex in $\partial B_{3,s,N}$ visited by γ before it visits u_π and v is the first vertex in $\partial B_{3,s,N}$ visited by γ after it visits v_π . So, let u and v be these vertices. First, since u_π belongs to $B_{2,s,N}$ and u to $\partial B_{3,s,N}$, we have

$$\|u - u_\pi\|_1 \geq (r_3 - r_2)N \geq N,$$

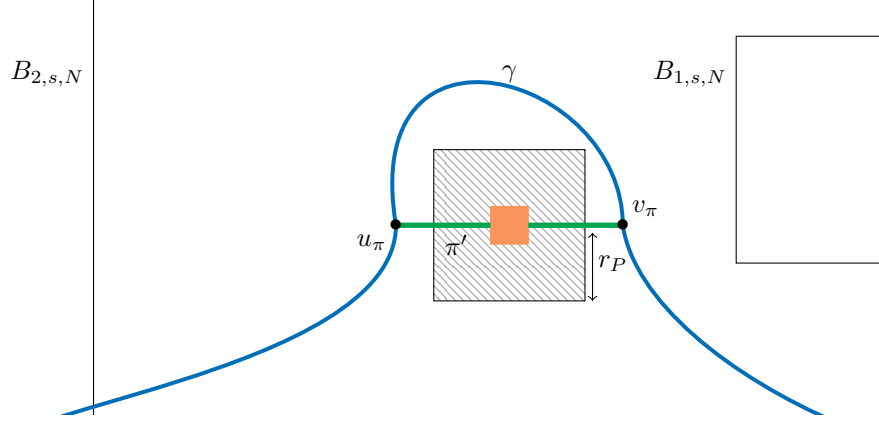


Figure 5: A picture to illustrate (2.22). The legend is the same as in Figure 4. In this example in two dimensions, when $t_{\min} > 0$ and $|\gamma_{u_\pi, v_\pi}|_e < N$ (a special case of case B of the modification), γ_{u_π, v_π} has to take $\|u_\pi - v_\pi\|_1$ edges in the direction ε_1 and also at least $2r_P$ edges in the direction ε_2 to avoid the forbidden zone.

since $r_3 > r_2 + 1$ by (2.4). Hence, by the first property of a typical box, γ_{u, u_π} takes at least $\rho(r_3 - r_2)N$ edges whose passage times is greater than δ . Since $\delta > \delta'$ by (2.9), all of these edges belongs to $E_-^*(T)$ and there are no edges of γ_{u, u_π} whose passage times have been increased. Thus

$$T^*(\gamma_{u, u_\pi}) - T(\gamma_{u, u_\pi}) \leq -\rho(r_3 - r_2)N(\delta - \delta'). \quad (2.16)$$

The same arguments give

$$T^*(\gamma_{v_\pi, v}) - T(\gamma_{v_\pi, v}) \leq -\rho(r_3 - r_2)N(\delta - \delta'). \quad (2.17)$$

Furthermore, since the only edges of γ_{u_π, v_π} whose passage times have been modified belong to $E_{\text{mid}}^*(T)$,

$$T^*(\gamma_{u_\pi, v_\pi}) - T(\gamma_{u_\pi, v_\pi}) \leq N\nu_0. \quad (2.18)$$

Thus, we get

$$\begin{aligned} T^*(\gamma_{u, v}) - T(\gamma_{u, v}) &= T^*(\gamma_{u, u_\pi}) - T(\gamma_{u, u_\pi}) + T^*(\gamma_{u_\pi, v_\pi}) - T(\gamma_{u_\pi, v_\pi}) + T^*(\gamma_{v_\pi, v}) - T(\gamma_{v_\pi, v}) \\ &\leq N(\nu_0 - 2\rho(\delta - \delta')(r_3 - r_2)) \quad \text{by (2.16), (2.17) and (2.18),} \\ &\leq 0, \end{aligned}$$

since $r_3 > 2r_2$, $\delta > 2\delta'$, r_3 is large enough compared to ν_0 by (2.4) and $N \geq 1$ by (2.8). \square

Proof of (d). First, in all cases, by (2.13), and by (2.1) and (2.2),

$$T^*(\pi') \leq (\|u_\pi - v_\pi\|_1 + |B_\infty(0, \ell^\Lambda + 1)|_e)(t_{\min} + \delta') + T^\Lambda. \quad (2.19)$$

To conclude this proof, we distinguish three cases.

If $|\gamma_{u_\pi, v_\pi}|_e \geq N$ (case B of the modification). First, let us prove that

$$T^*(\gamma_{u_\pi, v_\pi} \cap B_{3, s, N}) \geq \|u_\pi - v_\pi\|_1(t_{\min} + \delta). \quad (2.20)$$

Since this is the case B of the modification, the edges of γ_{u_π, v_π} have not been modified. Thus, if γ_{u_π, v_π} is entirely contained in $B_{3, s, N}$ the second property of the typical boxes gives (2.20). If γ_{u_π, v_π} is not entirely contained in $B_{3, s, N}$, let u' be the first vertex of $\partial B_{3, s, N}$ visited by γ_{u_π, v_π} . The vertex u_π is in $B_{2, s, N}$, so $\|u' - u_\pi\|_1 \geq (r_3 - r_2)N \geq N$ since $r_3 > r_2$ by (2.4). Using that $\|u_\pi - v_\pi\|_1 \leq d(r_2 + r_1)N$, the fact that $r_3 - r_2 > d(r_2 + r_1)$ by (2.4) and the second property of a typical box gives:

$$T^*(\gamma_{u_\pi, v_\pi} \cap B_{3, s, N}) \geq \underbrace{\|u' - u_\pi\|_1}_{\geq (r_3 - r_2)N} (t_{\min} + \delta) \geq (r_3 - r_2)N(t_{\min} + \delta) \geq \|u_\pi - v_\pi\|_1(t_{\min} + \delta).$$

This concludes the proof of (2.20). Then, combining (2.19) and (2.20), and using that $\|u_\pi - v_\pi\|_1 \geq r_P$, we have

$$\begin{aligned} T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) - T^*(\pi') &\geq \|u_\pi - v_\pi\|_1(\delta - \delta') - |B_\infty(0, \ell^\Lambda + 1)|_e(t_{\min} + \delta') - T^\Lambda \\ &\geq r_P(\delta - \delta') - |B_\infty(0, \ell^\Lambda + 1)|_e(t_{\min} + \delta') - T^\Lambda > 0, \end{aligned}$$

by (2.3) and since $\delta > 2\delta'$.

If $t_{\min} = 0$ and $|\gamma_{u_\pi, v_\pi}|_e < N$ (case A of the modification). In the environment T^* , all the edges e belonging to γ_{u_π, v_π} have a time greater than $t_{\min} + \delta$ and the property (a), $\gamma_{u_\pi, v_\pi} \cap B_{3,s,N} = \gamma_{u_\pi, v_\pi}$. Hence

$$T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) \geq \|u_\pi - v_\pi\|_1(t_{\min} + \delta). \quad (2.21)$$

We conclude the proof of this case as the previous one combining (2.19) and (2.21).

If $t_{\min} > 0$ and $|\gamma_{u_\pi, v_\pi}|_e < N$ (case B of the modification). In this case, since $\mathcal{C}(T)$ belongs to the selected straight segment between u_2 and u_1 and since v_π is visited by π after the forbidden zone, v_π belongs to the selected straight segment. Thus, the distance between v_π and $(B_{2,s,N})^c$ is greater than or equal to $\frac{(r_2 - r_1)N}{2}$. Then, since $|\gamma_{u_\pi, v_\pi}|_e < N$, we have $\|u_\pi - v_\pi\|_1 < N$. Since $r_2 > 4r_1$ by (2.4), u_π does not belong to $\partial B_{2,s,N} \cap \pi$ and thus π_{u_π, v_π} takes edges in only one direction: the direction of the selected straight segment. Denote this direction by ε_i . Then γ_{u_π, v_π} has to take $\|u_\pi - v_\pi\|_1$ edges in the direction ε_i but it can not take edges of the forbidden zone (see Figure 5). Hence

$$|\gamma_{u_\pi, v_\pi}|_e \geq \|u_\pi - v_\pi\|_1 + 2r_P. \quad (2.22)$$

By the property (a), we have $|\gamma_{u_\pi, v_\pi}|_e = |\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}|_e$ and thus

$$T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) \geq (\|u_\pi - v_\pi\|_1 + 2r_P)t_{\min}. \quad (2.23)$$

Combining (2.19) and (2.23) gives

$$\begin{aligned} T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) - T^*(\pi') &\geq 2r_P t_{\min} - \|u_\pi - v_\pi\|_1 \delta' - |B_\infty(0, \ell^\Lambda + 1)|_e(t_{\min} + \delta') - T^\Lambda \\ &\geq 2r_P t_{\min} - N\delta' - |B_\infty(0, \ell^\Lambda + 1)|_e(t_{\min} + \delta') - T^\Lambda > 0, \end{aligned}$$

since $N\delta' \leq 1$ by (2.9) and since r_P is large enough by (2.3). \square

Proof of (e). In the case (INF). Recall that the pattern satisfies the boundary condition (see Definition 2.1). Thus, if there is a pattern entirely contained in $B_{3,s,N}$ centered in a vertex z , it implies that there exists a path of length $2(\ell^\Lambda - 1)$ such that:

- it goes from $z - (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$ to $z + (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$ only using edges in the direction ε_1 ,
- its passage time is finite,
- there exists no path with finite passage time from $\partial B_{3,s,N}$ to one of its vertices.

In the environment T^* , the only edges with finite passage times are edges belonging to γ , π' , $\mathcal{S}_{\mathcal{C}(T), \ell^\Lambda}$ and $B_\infty(\mathcal{C}(T), \ell^\Lambda - 3)$. For every vertex z belonging to γ and π' , there exists a path from $\partial B_{3,s,N}$ to z with finite passage time. Furthermore, there is no path of length $2(\ell^\Lambda - 1)$ with finite passage time using only edges in the direction ε_1 having at least one edge in $B_\infty(\mathcal{C}(T), \ell^\Lambda - 3)$ and which does not visit any vertex of π' . Thus, the only path of length $2(\ell^\Lambda - 1)$ satisfying the three conditions above is the one from $\mathcal{C}(T) - (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$ to $\mathcal{C}(T) + (\ell^\Lambda - 1)\varepsilon_1 + (\ell^\Lambda - 1)\varepsilon_2$ and the only pattern entirely contained in $B_{3,s,N}$ is the one centered in $\mathcal{C}(T)$.

In the case (FU). If, in the environment T^* , there is a pattern entirely contained in $B_{3,s,N}$ centered in a vertex z , then for every edge $e \in \partial B_\infty(z, \ell^\Lambda)$, $T(e) \in (\nu_1, \nu_2)$ by the assumption (AF-4') in Section 2.1.3. Since $B_{3,s,N}$ is a typical box in the environment T , for every edge $e \in B_{3,s,N}$, $T(e) \leq \nu_1$. The only edges e such that $T^*(e) > T(e)$ are:

- the edges of $E_{\text{mid}}^*(T)$ when $t_{\min} = 0$ but for every edge $e \in E_{\text{mid}}^*(T)$, $T^*(e) < \nu_0 < \nu_1$ since $\nu_1 > \nu_0$,

- the edges of $E_+^*(T)$ but for every edge $e \in E_+^*(T)$, $T^*(e) > \nu_2$,
- some edges in $B_\infty(\mathcal{C}(T), \ell^\Lambda)$.

Thus, if an edge $e \in B_{3,s,N}$ is such that $T^*(e) \in (\nu_1, \nu_2)$, this edge belongs to $B_\infty(\mathcal{C}(T), \ell^\Lambda)$. Thus there is only one pattern entirely contained in $B_{3,s,N}$ which is the one centered in $\mathcal{C}(T)$. \square

Proof of (f). Let π_0 be a path from $\partial B_{3,s,N}$ to $\partial B_{3,s,N}$ entirely contained in $B_{3,s,N}$ such that in the environment T^* , it has a finite passage time in the case (INF) and it does not take any edge whose passage time is greater than ν_1 in the case (FU). Then the only edges of $B_{3,s,N}$ that π_0 can take are edges of γ , π' and some edges of $B_\infty(\mathcal{C}(T), \ell^\Lambda)$. Furthermore, since by Lemma 2.15, γ does not take any edge of the forbidden zone, if π_0 links two vertices of γ without taking edges of π' , then π_0 is exactly γ between these two vertices.

Now, assume that π_0 does not take any pattern entirely contained in $B_{3,s,N}$. Since π_0 can not take edges of $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ without taking the pattern centered in $\mathcal{C}(T)$, it remains to prove that π_0 does not take any edge of π' . But since π' is a self-avoiding path entirely contained in $B_{2,s,N}$ which takes the pattern centered in $\mathcal{C}(T)$, and which has only two vertices in common with γ , if π_0 takes an edge of π' , π_0 takes the pattern centered in $\mathcal{C}(T)$, which is impossible. \square

Proof of (g). Let $\tilde{\pi}$ be a k -penalized path from 0 to x in the environment T^* with finite passage time in the environment T^* . There are three cases.

First case. If $\tilde{\pi}$ does not take edges of $B_{3,s,N}$, $T^*(\tilde{\pi}) = T(\tilde{\pi})$ since the only edges whose passage time have been modified are edges of $B_{3,s,N}$. Property (g) follows from (c).

Second case. Assume that we are in the case (FU) and that $\tilde{\pi}$ takes an edge $e' \in B_{3,s,N}$ such that $T^*(e') \geq \nu_1$. Then,

$$T^*(\tilde{\pi}) = \sum_{e \in \tilde{\pi}} T^*(e) = \sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T^*(e) + \sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T^*(e).$$

Since $e' \in \tilde{\pi} \cap B_{3,s,N}$ and since $B_{3,s,N}$ is a typical box, we have using the fourth property of a typical box

$$\sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T^*(e) \geq \nu_1 > \sum_{e \in B_{3,s,N}} T(e) \geq \sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T(e).$$

Furthermore, the passage times of the edges outside $B_{3,s,N}$ have not been modified. Hence,

$$\sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T^*(e) = \sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T(e).$$

Thus,

$$T^*(\tilde{\pi}) = \sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T^*(e) + \sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T^*(e) > \sum_{e \in \tilde{\pi} \cap B_{3,s,N}} T(e) + \sum_{e \in \tilde{\pi} \cap B_{3,s,N}^c} T(e) = T(\tilde{\pi}). \quad (2.24)$$

In this case, property (g) follows from (2.24) and from property (c).

Third case. Now assume that $\tilde{\pi}$ takes at least one edge in $B_{3,s,N}$ and that in the case (FU), $\tilde{\pi}$ does not take any edge in $B_{3,s,N}$ having a passage time greater than or equal to ν_1 . Since $\tilde{\pi}$ has a finite passage time in the environment T^* , in the case (INF), $\tilde{\pi}$ does not take any edge $e' \in B_{3,s,N}$ such that $T^*(e') = \infty$. Since $\tilde{\pi}$ is a k -penalized path, it does not take any pattern entirely contained in $B_{3,s,N}$. Hence, using (f), the only edges in $B_{3,s,N}$ that $\tilde{\pi}$ can take are edges of γ . So, let $u_1, v_1, \dots, u_\kappa, v_\kappa$ be the successive entry and exit points of $\tilde{\pi}$ in $B_{3,s,N}$, we get for all $i \in \{1, \dots, \kappa\}$, $\tilde{\pi}_{u_i, v_i} = \gamma_{u_i, v_i}$. Furthermore, we also get that $T(\tilde{\pi})$ is finite. Indeed, since the only edges whose passage time have been modified are the edges in $B_{3,s,N}$ and since $T^*(\tilde{\pi})$ is finite, the only edges with infinite passage time in the environment T that $\tilde{\pi}$ can take are edges in $B_{3,s,N}$. But these edges being edges of γ which has a finite passage time in the environment T , $T(\tilde{\pi})$ is finite.

Thus, using again that the only edges whose passage time have been modified are the edges of $B_{3,s,N}$, we have

$$T(\tilde{\pi}) - T^*(\tilde{\pi}) = \sum_{i \in \{1, \dots, \kappa\}} T(\tilde{\pi}_{u_i, v_i}) - T^*(\tilde{\pi}_{u_i, v_i}) = \sum_{i \in \{1, \dots, \kappa\}} T(\gamma_{u_i, v_i}) - T^*(\gamma_{u_i, v_i}). \quad (2.25)$$

Now, using (c),

$$\sum_{i \in \{1, \dots, \kappa\}} T(\gamma_{u_i, v_i}) - T^*(\gamma_{u_i, v_i}) \leq T(\gamma) - T^*(\gamma). \quad (2.26)$$

Thus, combining (2.25) and (2.26), we also get in this last case that

$$T(\tilde{\pi}) - T^*(\tilde{\pi}) \leq T(\gamma) - T^*(\gamma).$$

□

2.2.4 End of the proof of Lemma 2.13

We prove Lemma 2.13 with the sets $E_-^*(T)$, $E_{\text{mid}}^*(T)$, $E_+^*(T)$ and $E_\infty^*(T)$ and the vertex $\mathcal{C}(T)$ defined in Section 2.2.2. Let us first prove item (i) of this lemma. $E_{\text{mid}}^*(T)$ is contained in $B_{3,s,N}$ by property (a) in Section 2.2.3. Using property (b) of Section 2.2.3, we get that $B_\infty(\mathcal{C}(T), \ell^\Lambda)$ is contained in $B_{3,s,N}$. $E_-^*(T)$, $E_\infty^*(T)$ and $E_+^*(T)$ are contained in $B_{3,s,N}$ by their definitions. To get that these sets are pairwise disjoint in both cases, we only have to prove that:

- γ_{u_π, v_π} does not visit any edge of $B_\infty(\mathcal{C}(T), \ell^\Lambda)$. This comes from the fact that $B_\infty(\mathcal{C}(T), \ell^\Lambda) \subset B_\infty(\mathcal{C}(T), r_P)$, which comes from the fact that $r_P > \ell^\Lambda + 1$ by (2.3). By property (iii) of a typical box, in the case (INF), in the environment T , for every edge $e \in B_\infty(\mathcal{C}(T), r_P)$, $T(e) = \infty$ but $T(\gamma_{u_\pi, v_\pi}) < \infty$. Hence, γ_{u_π, v_π} does not take any edge in $B_\infty(\mathcal{C}(T), r_P)$. In the case (FU), by property (iii) of a typical box, the event $\theta_{\mathcal{C}(T)}\mathcal{T}$ holds with respect to the environment T . Recall Remark 2.10: there is no pattern in $B_{3,s,N}$. Since \mathcal{T} satisfies the second condition in Lemma 2.5 and since γ_{u_π, v_π} is a k -geodesic, γ_{u_π, v_π} does not visit any edge of the forbidden zone. So γ_{u_π, v_π} does not visit any edge of $B_\infty(\mathcal{C}(T), \ell^\Lambda)$.
- γ_{u_π, v_π} and π' do not have any edge in common by the definition of u_π , v_π and π' .

To get item (ii), fix $\eta = \mathbb{P}(T \in \mathcal{A}^\Lambda) \tilde{p}^{|B_{3,s,N}|}$, where, in the case (INF) and if $t_{\min} = 0$,

$$\tilde{p} = \min(\mathcal{L}([t_{\min}, t_{\min} + \delta']), \mathcal{L}((t_{\min} + \delta, \nu_0)), \mathcal{L}(\infty)),$$

in the case (INF) and if $t_{\min} > 0$,

$$\tilde{p} = \min(\mathcal{L}([t_{\min}, t_{\min} + \delta']), \mathcal{L}(\infty)),$$

in the case (FU) and if $t_{\min} = 0$,

$$\tilde{p} = \min(\mathcal{L}([t_{\min}, t_{\min} + \delta']), \mathcal{L}((t_{\min} + \delta, \nu_0)), \mathcal{L}((\nu_2, \infty))),$$

and in the case (FU) and if $t_{\min} > 0$,

$$\tilde{p} = \min(\mathcal{L}([t_{\min}, t_{\min} + \delta']), \mathcal{L}((\nu_2, \infty))),$$

Thus, η only depends on \mathcal{L} , the pattern and N and we have that

$$\mathbb{P}(T' \in \mathcal{B}^*(T) | T) \geq \mathbb{P}(T \in \mathcal{A}^\Lambda) \tilde{p}^{|B_{3,s,N}|} = \eta.$$

Now, let us prove item (iii) of Lemma 2.13. Let γ^* be the selected k -geodesic in the environment T^* if it exists. The aim is to prove the following properties in the environment T^* :

- (C1) γ^* exists, i.e. there exists a k -geodesic having at least Q_n boxes in its S^k -sequence,
- (C2) γ^* does not have a shortcut in the first $\ell - 1$ boxes of its S^k -sequence,
- (C3) γ^* has a shortcut in the ℓ -th box of its S^k -sequence,
- (C4) $B_{3,s,N}$ is the ℓ -th box of the S^k -sequence of γ^* .

To get these four properties, we use the following ones:

- (P1) a k -box different from $B_{3,s,N}$ is a typical box in the environment T if and only if it is a typical box in the environment T^* ,
- (P2) a path has a shortcut in a k -box different from $B_{3,s,N}$ in the environment T if and only if it has a shortcut in this box in the environment T^* ,
- (P3) γ has a shortcut in $B_{3,s,N}$ in the environment T^* and then, $B_{3,s,N}$ is successful in the environment T^* for γ ,
- (P4) γ^* exists and $\gamma^* = \gamma$.

To conclude the proof, we have to prove (P1), (P2), (P3) and (P4). Indeed, by (P1) and (P2), a k -box different from $B_{3,s,N}$ is successful for a path $\tilde{\pi}$ in the environment T if and only if it is successful for $\tilde{\pi}$ in the environment T^* . Thus, since $B_{3,s,N}$ is a typical box for γ in the environment T , using (P3), the successful boxes crossed by γ in the environments T et T^* are the same. Furthermore, if we have (P4), then γ is a k -geodesic in the environment T^* and we can define its S^k -sequence in this environment. We get that the S^k -sequence of γ is the same in the environments T and T^* . Hence, using again (P4), we get (C1), we get (C2) by (H2) and we get (C3) and (C4) by (H3) using again (P3).

At this stage of the proof, we easily get (P1), (P2) and (P3) (which is the aim of the following paragraph) but the proof of (P4) is a bit longer (this is the aim of Section 2.2.4.2).

2.2.4.1 Proof of properties (P1), (P2) and (P3). We get (P1) using that the fact that a box is typical only depends on the edges of the box (by Lemma 2.8), that every k -box different from $B_{3,s,N}$ does not have edges in common with $B_{3,s,N}$ and that the only edges whose time has been modified are edges belonging to $B_{3,s,N}$.

(P2) uses the same arguments than above. The fact that a path has a shortcut in a k -box only depends on the edges of the box.

We get (P3) by considering the path π' defined at the beginning of Section 2.2.2. By construction π' is entirely contained in $B_{3,s,N}$, π' and γ only have u_π and v_π in common, π' takes the pattern and $T^*(\gamma_{u_\pi, v_\pi} \cap B_{3,s,N}) > T^*(\pi')$ by (d).

2.2.4.2 Proof of (P4): γ is the selected k -geodesic in the environment T^* . To prove this property, we prove the following ones in the last four lemmas of this section:

- γ is a k -penalized path in the environment T^* ,
- every k -penalized path from 0 to x in the environment T^* has a passage time greater than or equal to the passage time of γ ,
- if a path is a k -geodesic from 0 to x in the environment T^* , it is also a k -geodesic in the environment T ,
- if a k -geodesic from 0 to x in the environment T^* has at least Q_n boxes in its S^k -sequence, it has also at least Q_n boxes in its S^k -sequence in the environment T .

We can conclude with these properties. Indeed, with the first two properties above, γ is a k -geodesic in the environment T^* . As a consequence of (P1), (P2) and (P3), it has the same S^k -sequence in the environments T and T^* . Thus γ has at least Q_n boxes in its S^k -sequence and it can be the selected k -geodesic. By the last two properties above, we have that the set of the k -geodesics having at least Q_n boxes in their S^k -sequences in the environment T^* is included in the set of the k -geodesics having at least Q_n boxes in their S^k -sequences in the environment T . Since γ is the first path in the lexicographical order among the paths of this last set, it is also the first path in the lexicographical order in the first set.

It remains to prove the four properties above. Before proving them, we begin by the following lemma.

Lemma 2.16. *A k -penalized path in the environment T^* with finite passage time in the environment T^* is also a k -penalized path in the environment T .*

Proof. Let $\tilde{\pi}$ be a k -penalized path in the environment T^* with $T^*(\tilde{\pi})$ finite. Since the edges outside $B_{3,s,N}$ have not been modified, $\tilde{\pi}$ takes a pattern entirely contained in a k -box different from $B_{3,s,N}$ in the environment T if and only if it takes a pattern entirely contained in this box in the environment T^* . It remains to prove that $\tilde{\pi}$ does not take a pattern entirely contained in $B_{3,s,N}$ in the environment T .

In the case (INF), since the time of $\tilde{\pi}$ is finite and since $\tilde{\pi}$ is a k -penalized path in the environment T^* , by property (f), the only edges of $B_{3,s,N}$ that $\tilde{\pi}$ can take are edges of γ . It implies that, in the environment T , if $\tilde{\pi}$ takes a pattern entirely contained in $B_{3,s,N}$, γ also takes this pattern, which is impossible since γ is a k -penalized path in the environment T .

In the case (FU), it is impossible since $B_{3,s,N}$ is a typical box in the environment T and there is no pattern in a typical box by Remark 2.10. \square

Lemma 2.17. γ is a k -penalized path in the environment T^* .

Proof. The fact that a path takes a pattern entirely contained in a k -box only depends on the passage times of the edges of this k -box. Since γ is a k -penalized path in the environment T , it does not take a pattern entirely contained in a k -box in this environment. Since the edges of the k -boxes different from $B_{3,s,N}$ have not been modified, γ does not take a pattern entirely contained in a k -box different from $B_{3,s,N}$ in the environment T^* . To conclude, it remains to prove that γ does not take a pattern entirely contained in $B_{3,s,N}$ in the environment T^* . By (e), there is only one pattern entirely contained in $B_{3,s,N}$ which is the one centered in $\mathcal{C}(T)$. By Lemma 2.15, γ does not take any edge of the forbidden zone and the pattern centered in $\mathcal{C}(T)$ is entirely contained in the forbidden zone, which gives the result. \square

Lemma 2.18. Every k -penalized path from 0 to x in the environment T^* has a passage time for T^* greater than or equal to the passage time of γ for T^* .

Proof. Let $\bar{\gamma}^*$ be a k -penalized path in the environment T^* . By Lemma 2.16, $\bar{\gamma}^*$ is also a k -penalized path in the environment T . Thus, since $\bar{\gamma}^*$ is a k -penalized path in the environment T and since γ is a k -geodesic, we get $T(\gamma) \leq T(\bar{\gamma}^*)$. Hence, using (g) in Section 2.2.3,

$$T^*(\bar{\gamma}^*) \geq T^*(\gamma) + \underbrace{T(\bar{\gamma}^*) - T(\gamma)}_{\geq 0} \geq T^*(\gamma).$$

\square

Lemma 2.19. If a path from 0 to x is a k -geodesic in the environment T^* , it is also a k -geodesic in the environment T .

Proof. Let $\bar{\gamma}^*$ be a k -geodesic in the environment T^* from 0 to x . By Lemma 2.17, γ is a k -penalized path in the environment T^* . Thus, since $T^*(\gamma)$ is finite, $T^*(\bar{\gamma}^*)$ is also finite and by Lemma 2.16, $\bar{\gamma}^*$ is a k -penalized path in the environment T . Moreover, using (g) in Section 2.2.3, we get

$$T(\bar{\gamma}^*) \leq T(\gamma) + T^*(\bar{\gamma}^*) - T^*(\gamma).$$

Since $\bar{\gamma}^*$ is a k -geodesic, by Lemma 2.17 and Lemma 2.18, $T^*(\bar{\gamma}^*) = T^*(\gamma)$. So $T(\bar{\gamma}^*) \leq T(\gamma)$ and $\bar{\gamma}^*$ is a k -geodesic in the environment T . \square

Lemma 2.20. If a k -geodesic from 0 to x in the environment T^* has at least Q_n boxes in its S^k -sequence, it has also at least Q_n boxes in its S^k -sequence in the environment T .

Proof. Let $\bar{\gamma}^*$ be a k -geodesic in the environment T^* from 0 to x having at least Q_n boxes in its S^k -sequence. Using (P2) and by the construction of the S^k -sequence, each box different from $B_{3,s,N}$ belonging to the S^k -sequence of $\bar{\gamma}^*$ in the environment T^* belongs to its S^k -sequence in the environment T . If $B_{3,s,N}$ belongs to its S^k -sequence in the environment T^* , since $B_{3,s,N}$ is a typical box in the environment T , then $B_{3,s,N}$ belongs to its S^k -sequence in the environment T , which allows us to conclude. \square

3 Extension of the van den Berg-Kesten comparison principle

This section is dedicated to the proof of Theorem 1.7. Let \mathcal{L} and $\tilde{\mathcal{L}}$ be two distributions taking values in $[0, \infty]$ such that:

(H1) \mathcal{L} is useful,

(H2) $\mathcal{L}([0, \infty)) > p_c$ and $\tilde{\mathcal{L}}([0, \infty)) > p_c$,

(H3) $\mathcal{L} \neq \tilde{\mathcal{L}}$,

(H4) there exists a couple of random variables $(\tau, \tilde{\tau})$ on some probability space, with marginal distributions \mathcal{L} and $\tilde{\mathcal{L}}$, respectively, and satisfying

$$\mathbb{E}[\tilde{\tau}|\tau] \leq \tau. \quad (3.1)$$

In what follows, $(\tau, \tilde{\tau})$ is a couple of random variables with marginal distributions \mathcal{L} and $\tilde{\mathcal{L}}$, and satisfying (3.1). Such a couple exists by (H4). Note that by (3.1), we have

$$\{\tau < \infty\} \subset \{\tilde{\tau} < \infty\} \text{ a.s.} \quad (3.2)$$

Then, we consider a family $(T, \tilde{T}) = \{(T(e), \tilde{T}(e)) : e \in \mathcal{E}\}$ of i.i.d. random variables defined on the same probability space such that for all $e \in \mathcal{E}$, $(T(e), \tilde{T}(e))$ has the same distribution as $(\tau, \tilde{\tau})$.

The proof of Theorem 1.7 is an application of Theorem 1.5. We begin by defining a valid pattern in Section 3.1 and then, we apply Theorem 1.5 with this pattern in Section 3.2.

3.1 Definition of the valid pattern

The fact that a pattern is valid or not depends on the distribution of the passage times of the environment. Here, we use Theorem 1.5 only in the environment T . Thus, when we define a pattern $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ below, we consider that the event \mathcal{A}^Λ only depends on the family $(T(e))_{e \in \Lambda}$.

Now, for a valid pattern $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$, denote by $\Pi^\mathfrak{P}$ the set of all self-avoiding paths going from u^Λ to v^Λ and which are contained in Λ . Denote by \mathcal{G} the σ -field generated by the family $(T(e))_{e \in \mathcal{E}}$. Section 3.1 is devoted to the proof of the following lemma.

Lemma 3.1. *There exist a valid pattern $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ and a constant $\eta > 0$ such that on the event \mathcal{A}^Λ ,*

$$\mathbb{E} \left[\min_{\pi \in \Pi^\mathfrak{P}} \tilde{T}(\pi) | \mathcal{G} \right] < \min_{\pi \in \Pi^\mathfrak{P}} T(\pi) - \eta. \quad (3.3)$$

To prove Lemma 3.1, there are three different cases to be considered. Noting that, if $\mathbb{P}(\tilde{\tau} < \infty \text{ and } \tau = \infty) = 0$, using (3.2), we get

$$\{\tau < \infty\} = \{\tilde{\tau} < \infty\} \text{ a.s.},$$

these three cases can be written as follows:

- $\mathbb{P}(\tilde{\tau} < \infty \text{ and } \tau = \infty) > 0$,
- $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$ a.s. and $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] < \tau) > 0$,
- $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$ a.s. and $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] = \tau) = 1$.

The most technical case is the third one.

3.1.1 First case: when $\mathbb{P}(\tilde{\tau} < \infty \text{ and } \tau = \infty) > 0$

Proof of Lemma 3.1 in the first case.

Assume that $\mathbb{P}(\tilde{\tau} < \infty \text{ and } \tau = \infty) > 0$. Let $\nu \in (0, \infty)$ such that

$$\mathcal{L}([\nu, \infty)) > 0. \quad (3.4)$$

Such a constant exists since we have $\mathcal{L}(0) + \mathcal{L}(\infty) < 1$ since \mathcal{L} is useful by Assumption (H1) and $\mathcal{L}([0, \infty)) > p_c$ by Assumption (H2). Let $M \in [0, \infty)$ such that

$$\mathbb{P}(\tilde{\tau} < M \text{ and } \tau = \infty) > 0. \quad (3.5)$$

Let $\eta_0 > 0$. Fix

$$m > \frac{M + \eta_0}{2\nu}. \quad (3.6)$$

Then, define the pattern $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ where $\Lambda = \{0, 1\} \times \{0, \dots, m\} \times \prod_{j=3}^d \{0\}$, $u^\Lambda = (0, \dots, 0)$, $v^\Lambda = (1, 0, \dots, 0)$ and \mathcal{A}^Λ is defined as follows. Denote by π^{fin} the path going from u^Λ to $m\varepsilon_2$ by m steps in the direction ε_2 , then to $\varepsilon_1 + m\varepsilon_2$ by one step in the direction ε_1 and finally to v_2^Λ by m steps in the direction ε_2 . The event \mathcal{A}^Λ is the event on which for every $e \in \pi^{\text{fin}}$, $\nu \leq T(e) < \infty$ and for every edge $e \in \Lambda$ but not in π^{fin} , $T(e) = \infty$. This pattern is valid since the event \mathcal{A}^Λ has a positive probability by (3.4) and (3.5), and since the path π^{fin} is a path between u^Λ and v^Λ with a finite passage time in the environment T when the event \mathcal{A}^Λ occurs.

Now, denote

$$\beta = \mathbb{P}(\tilde{\tau} < M | \tau = \infty).$$

By (3.5), $\beta > 0$. Then, on the event \mathcal{A}^Λ , we have

$$\mathbb{E} \left[\min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] \leq T(\pi^{\text{fin}}) - \beta(T(\pi^{\text{fin}}) - M). \quad (3.7)$$

Indeed, denote by e^∞ the edge $\{u^\Lambda, v^\Lambda\}$ and by π^∞ the path going from u^Λ to v^Λ by taking only this edge. Then, on the event \mathcal{A}^Λ , since $\mathcal{A}^\Lambda \subset \{T(e^\infty) = \infty\}$,

$$\begin{aligned} \mathbb{E} \left[\min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] &\leq \mathbb{E} [\min(\tilde{T}(\pi^{\text{fin}}), \tilde{T}(\pi^\infty)) | \mathcal{G}] \\ &\leq \mathbb{E} [M \mathbf{1}_{\{\tilde{T}(e^\infty) < M\}} + \tilde{T}(\pi^{\text{fin}}) \mathbf{1}_{\{\tilde{T}(e^\infty) \geq M\}} | \mathcal{G}] \\ &\leq M\beta + \sum_{e \in \pi^{\text{fin}}} \mathbb{E} [\tilde{T}(e) \mathbf{1}_{\tilde{T}(e^\infty) \geq M} | \mathcal{G}]. \end{aligned} \quad (3.8)$$

But, for every $e \in \pi^{\text{fin}}$,

$$\begin{aligned} \mathbb{E} [\tilde{T}(e) \mathbf{1}_{\tilde{T}(e^\infty) \geq M} | \mathcal{G}] &= \mathbb{E} [\tilde{T}(e) | T(e)] \mathbb{P}(\tilde{T}(e^\infty) \geq M | T(e^\infty)) \\ &\leq \mathbb{E} [\tilde{T}(e) | T(e)] (1 - \beta) \text{ since } \mathcal{A}^\Lambda \subset \{T(e^\infty) = \infty\}, \\ &\leq T(e)(1 - \beta), \end{aligned} \quad (3.9)$$

since $(T(e), \tilde{T}(e))$ has the same distribution as $(\tau, \tilde{\tau})$ which satisfies (3.1). Thus, combining (3.8) and (3.9), we get, on the event \mathcal{A}^Λ ,

$$\mathbb{E} \left[\min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] \leq M\beta + (1 - \beta)T(\pi^{\text{fin}}) = T(\pi^{\text{fin}}) - \beta(T(\pi^{\text{fin}}) - M),$$

and (3.7) is proved.

Now, by the definition of the pattern, on the one hand, $T(\pi^{\text{fin}}) = \min_{\pi \in \Pi^{\mathfrak{P}}} T(\pi)$ and on the other hand, $T(\pi^{\text{fin}}) \geq 2m\nu$. This gives, using (3.6),

$$T(\pi^{\text{fin}}) - M > \eta_0.$$

Hence,

$$\mathbb{E} \left[\min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] < \min_{\pi \in \Pi^{\mathfrak{P}}} T(\pi) - \beta\eta_0,$$

which allows us to conclude since $\beta\eta_0 > 0$. \square

3.1.2 Second case: when $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$ a.s. and $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] < \tau) > 0$

Proof of Lemma 3.1 in the second case.

Assume that $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] < \tau) > 0$ and that $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$ a.s. Then there exist $\eta > 0$ and a Borel set $I \subset [0, \infty)$ such that $\mathbb{P}(\tau \in I) > 0$ and on the event $\{\tau \in I\}$,

$$\mathbb{E}[\tilde{\tau}|\tau] < \tau - \eta. \quad (3.10)$$

Now, define the pattern $\mathfrak{P} = (\{u^\Lambda, v^\Lambda\}, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ where $u^\Lambda = (0, \dots, 0)$, $v^\Lambda = (1, 0, \dots, 0)$ and \mathcal{A}^Λ is the event on which the passage time of the only edge of the pattern, denoted by e , belongs to I . Then, this pattern is valid since the event \mathcal{A}^Λ has a positive probability since $\mathbb{P}(\tau \in I) > 0$ and since the passage time of the path (u^Λ, v^Λ) in the environment T is finite when \mathcal{A}^Λ occurs since $I \subset [0, \infty)$. Furthermore, on the event \mathcal{A}^Λ ,

$$\begin{aligned} \mathbb{E} \left[\min_{\pi \in \Pi^{\mathfrak{P}}} \tilde{T}(\pi) | \mathcal{G} \right] &= \mathbb{E} [\tilde{T}(e) | \mathcal{G}] = \mathbb{E} [\tilde{T}(e) | T(e)] < T(e) - \eta \text{ by (3.10),} \\ &= \min_{\pi \in \Pi^{\mathfrak{P}}} T(\pi) - \eta. \end{aligned}$$

□

3.1.3 Third case: when $\{\tau < \infty\} = \{\tilde{\tau} < \infty\}$ a.s. and $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] = \tau) = 1$

Assume that

$$\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] = \tau) = 1, \quad (3.11)$$

and that

$$\{\tau < \infty\} = \{\tilde{\tau} < \infty\} \text{ a.s.} \quad (3.12)$$

Lemma 3.2. *In this case, there exist $\beta > 0$ and $\delta > 0$ such that*

$$\mathbb{P} \left(\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} \leq \tau - 2\delta | \tau) \geq \beta \text{ and } \mathbb{P}(\tilde{\tau} \geq \tau | \tau) \geq \beta \right) > 0. \quad (3.13)$$

Proof. Denote $A = \{\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} < \tau | \tau) > 0\}$ and $B = \{\mathbb{P}(\tilde{\tau} \geq \tau | \tau) > 0\}$. To prove the lemma, it suffices to prove that

$$\mathbb{P}(A \cap B) > 0. \quad (3.14)$$

Let K be a transition probability kernel such that for every measurable function $\varphi : [0, \infty] \times [0, \infty] \rightarrow [0, \infty]$,

$$\mathbb{E}[\varphi(\tau, \tilde{\tau}) | \tau] = \int_{[0, \infty]} \varphi(\tau, \tilde{t}) K(\tau, d\tilde{t}). \quad (3.15)$$

The existence of such a K is given by Theorem 2.19 in Chapter 4 in [4].

First, let us prove that $\mathbb{P}(B) = 1$. We have

$$\begin{aligned} B^c &= \{\mathbb{E}[\mathbb{1}_{\tilde{\tau} < \tau} | \tau] = 1\} = \{K(\tau, [0, \tau)) = 1\} \text{ using (3.15),} \\ &= \{K(\tau, [0, \tau)) = 1 \text{ and } \tau < \infty\} \text{ using (3.12)} \\ &\subset \left\{ \int_{[0, \infty]} \tilde{t} K(\tau, d\tilde{t}) < \tau \right\} \\ &= \{\mathbb{E}[\tilde{\tau} | \tau] < \tau\}. \end{aligned}$$

Now, since $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] < \tau) = 0$ by (3.11), we get $\mathbb{P}(B^c) = 0$ and thus $\mathbb{P}(B) = 1$.

So, to get (3.14), it remains to prove that $\mathbb{P}(A) > 0$. To this aim, we shall prove that

$$A^c = \{\tau = \tilde{\tau}\} \text{ a.s.,} \quad (3.16)$$

which leads to $\mathbb{P}(A^c) < 1$ by $(\mathcal{H}3)$, and thus $\mathbb{P}(A) > 0$. To prove (3.16), observe that

$$\begin{aligned} A^c &= \{\tau = \infty \text{ or } \mathbb{P}(\tilde{\tau} < \tau|\tau) = 0\} \\ &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} < \tau|\tau) = 0\}, \\ &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } K(\tau, [\tau, \infty]) = 1\} \text{ using (3.15),} \\ &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } K(\tau, \{\tau\}) = 1\}, \end{aligned}$$

since

$$\{\tau < \infty, K(\tau, [\tau, \infty]) = 1 \text{ and } K(\tau, (\tau, \infty]) > 0\} \subset \{\mathbb{E}[\tilde{\tau}|\tau] > \tau\}$$

and $\mathbb{P}(\mathbb{E}[\tilde{\tau}|\tau] > \tau) = 0$ by (3.11). Thus, using again (3.15), we get

$$\begin{aligned} A^c &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} = \tau|\tau) = 1\} \\ &= \{\tau = \infty\} \cup \{\tau < \infty \text{ and } \tilde{\tau} = \tau\} \text{ a.s.} \end{aligned}$$

Now, since $\{\tau = \infty\} = \{\tau = \tilde{\tau} = \infty\}$ a.s. by (3.12), we get (3.16). Hence, (3.14) holds and the lemma is proved. \square

Lemma 3.3. *In this case, there exists a bounded Borel set $I \subset (0, \infty)$ and $\eta > 0$ such that*

- $\mathbb{P}(\tau \in I) > 0$,
- and

$$\mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4)|\mathcal{F}] < \min(\tau_1 + \tau_2, \tau_3 + \tau_4) - \eta$$

on the event $\mathcal{I} = \{\tau_1, \dots, \tau_4 \in I\}$, where $(\tau_1, \tilde{\tau}_1), \dots, (\tau_4, \tilde{\tau}_4)$ are independent copies of $(\tau, \tilde{\tau})$ and $\mathcal{F} = \sigma(\tau_1, \dots, \tau_4)$.

Proof. Let $(\tau_1, \tilde{\tau}_1), \dots, (\tau_4, \tilde{\tau}_4)$ be independent copies of $(\tau, \tilde{\tau})$. Denote $\mathcal{F} = \sigma(\tau_1, \dots, \tau_4)$. Fix $\beta > 0$ and $\delta > 0$ given by Lemma 3.2. Using (3.13), we can find a Borel set $I_0 \subset (0, \infty)$, fixed for the remaining of the proof, such that $\mathbb{P}(\tau \in I_0) > 0$ and on the event $\{\tau \in I_0\}$,

$$\tau < \infty \text{ and } \mathbb{P}(\tilde{\tau} \leq \tau - 2\delta|\tau) \geq \beta \text{ and } \mathbb{P}(\tilde{\tau} \geq \tau|\tau) \geq \beta. \quad (3.17)$$

Now, fix $\eta > 0$ such that $\eta < 2\delta\beta^4$ and then fix

$$0 < \delta_0 < \frac{2\delta\beta^4 - \eta}{4}. \quad (3.18)$$

Note that this gives $\delta_0 < \frac{\delta}{2}$ as $\beta \leq 1$. Let $y_0 \in I_0$ such that $\mathbb{P}(\tau \in I_0 \cap (y_0 - \delta_0, y_0 + \delta_0)) > 0$. Such a y_0 exists because $\mathbb{P}(\tau \in I_0) > 0$. Set $I = I_0 \cap (y_0 - \delta_0, y_0 + \delta_0)$.

Then, we have

$$\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4) + 2\delta\mathbf{1}_{\{\tilde{\tau}_3 \leq \tilde{\tau}_1 - \delta \text{ and } \tilde{\tau}_4 \leq \tilde{\tau}_2 - \delta\}} \leq \tilde{\tau}_1 + \tilde{\tau}_2.$$

Thus, on the event $\mathcal{I} = \{\tau_1, \dots, \tau_4 \in I\}$, since $(\tau_1, \tilde{\tau}_1)$ and $(\tau_2, \tilde{\tau}_2)$ satisfy (3.1),

$$\begin{aligned} \mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4)|\mathcal{F}] &\leq \mathbb{E}[\tilde{\tau}_1 + \tilde{\tau}_2|\mathcal{F}] - 2\delta\mathbb{P}(\tilde{\tau}_3 \leq \tilde{\tau}_1 - \delta|\mathcal{F})\mathbb{P}(\tilde{\tau}_4 \leq \tilde{\tau}_2 - \delta|\mathcal{F}) \\ &\leq \tau_1 + \tau_2 - 2\delta\mathbb{P}(\tilde{\tau}_3 \leq \tilde{\tau}_1 - \delta|\mathcal{F})\mathbb{P}(\tilde{\tau}_4 \leq \tilde{\tau}_2 - \delta|\mathcal{F}) \\ &\leq 2\sup I - 2\delta\mathbb{P}(\tilde{\tau}_3 \leq \tau_3 - 2\delta \text{ and } \tilde{\tau}_1 \geq \tau_1|\mathcal{F})\mathbb{P}(\tilde{\tau}_4 \leq \tau_4 - 2\delta \text{ and } \tilde{\tau}_2 \geq \tau_2|\mathcal{F}), \end{aligned}$$

since the diameter of I is lower than δ by (3.18). Then, by (3.17)

$$\begin{aligned} \mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4)|\mathcal{F}] &\leq 2\sup I - 2\delta\beta^4 \\ &< 2\inf I - \eta, \end{aligned}$$

since $2(\sup I - \inf I) \leq 4\delta_0 < 2\delta\beta^4 - \eta$ by (3.18). Hence, since on the event \mathcal{I} ,

$$\min(\tau_1 + \tau_2, \tau_3 + \tau_4) \geq 2\inf I,$$

we get, on the event \mathcal{I} ,

$$\mathbb{E}[\min(\tilde{\tau}_1 + \tilde{\tau}_2, \tilde{\tau}_3 + \tilde{\tau}_4)|\mathcal{F}] < \min(\tau_1 + \tau_2, \tau_3 + \tau_4) - \eta.$$

\square

Proof of Lemma 3.1 in the third case. Fix $I \subset (0, \infty)$ and $\eta > 0$ given by Lemma 3.3. Define the pattern $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ where Λ is the set containing 4 vertices defined by $\Lambda = \{0, 1\} \times \{0, 1\} \times \prod_{j=3}^d \{0\}$, $u_1^\Lambda = (0, \dots, 0)$, $v_1^\Lambda = (1, 1, 0, \dots, 0)$ and \mathcal{A}^Λ is the event on which for every edge $e \in \Lambda$, $T(e) \in I$. This pattern is valid since the event \mathcal{A}^Λ has a positive probability since $\mathbb{P}(\tau \in I) > 0$ by Lemma 3.3, and since on the event \mathcal{A}^Λ , every edge of the pattern has a finite passage time.

Now denote $e_1 = \{u^\Lambda, u^\Lambda + \varepsilon_1\}$, $e_2 = \{u^\Lambda + \varepsilon_1, v^\Lambda\}$, $e_3 = \{u^\Lambda, u^\Lambda + \varepsilon_2\}$ and $e_4 = \{u^\Lambda + \varepsilon_2, v^\Lambda\}$. Since there are only two paths in $\Pi^\mathfrak{P}$, the one taking e_1 and then e_2 and the one taking e_3 and then e_4 , we get

$$\begin{aligned} \mathbb{E} \left[\min_{\pi \in \Pi^\mathfrak{P}} \tilde{T}(\pi) | \mathcal{G} \right] &= \mathbb{E} \left[\min(\tilde{T}(e_1) + \tilde{T}(e_2), \tilde{T}(e_3) + \tilde{T}(e_4)) | \mathcal{G} \right] \\ &= \mathbb{E} \left[\min(\tilde{T}(e_1) + \tilde{T}(e_2), \tilde{T}(e_3) + \tilde{T}(e_4)) | \sigma(T(e_1), T(e_2), T(e_3), T(e_4)) \right] \\ &< \min(T(e_1) + T(e_2), T(e_3) + T(e_4)) - \eta \text{ by Lemma 3.3,} \\ &= \min_{\pi \in \Pi^\mathfrak{P}} T(\pi) - \eta. \end{aligned}$$

□

3.2 End of the proof of Theorem 1.7

By (H2) and (3.2), we can fix $M > 0$ such that

$$\mathbb{P}(\tau \leq M \text{ and } \tilde{\tau} \leq M) > p_c. \quad (3.19)$$

Let \mathcal{C} and $\tilde{\mathcal{C}}$ be the clusters defined respectively as the clusters \mathcal{C}_M and $\tilde{\mathcal{C}}_M$ in Section 1.3 for M fixed above. Recall the definitions of φ , $\tilde{\varphi}$, μ and $\tilde{\mu}$ also given in Section 1.3, and the convergence given at (1.10).

Let $x \in \mathbb{Z}^d \setminus \{0\}$. For any $n \in \mathbb{N}$, we define the random path γ_n as the first geodesic in the lexicographical order from $\varphi(0)$ to $\varphi(nx)$ in the environment T . As stated in Section 1.1, almost surely, there exists at least one geodesic from $\varphi(0)$ to $\varphi(nx)$. Recall that \mathcal{G} is the σ -field generated by the family $(T(e))_{e \in \mathcal{E}}$. Note that γ_n is \mathcal{G} -measurable.

Lemma 3.4. *We have $\mathbb{E}[\tilde{t}(\varphi(0), \varphi(x))] < \infty$.*

Proof. Since γ_1 is \mathcal{G} -measurable,

$$\begin{aligned} \mathbb{E}[\tilde{t}(\varphi(0), \varphi(x)) | \mathcal{G}] &\leq \mathbb{E}[\tilde{T}(\gamma_1) | \mathcal{G}] \\ &= \sum_{e \in \gamma_1} \mathbb{E}[\tilde{T}(e) | T(e)] \\ &\leq \sum_{e \in \gamma_1} T(e) = T(\gamma_1) = t(\varphi(0), \varphi(x)). \end{aligned}$$

Note that the last inequality comes from the fact that for every $e \in \mathcal{E}$, $\mathbb{E}[\tilde{T}(e) | T(e)] \leq T(e)$. Then, taking expectation, it gives

$$\mathbb{E}[\tilde{t}(\varphi(0), \varphi(x))] \leq \mathbb{E}[t(\varphi(0), \varphi(x))].$$

Since by Proposition 2 in [3], $\mathbb{E}[t(\varphi(0), \varphi(x))] < \infty$, we get $\mathbb{E}[\tilde{t}(\varphi(0), \varphi(x))] < \infty$. □

With Lemma 3.4, we can use the Subadditive Ergodic Theorem (see for example Theorem 2.2 in [2]), and thus we get the existence of a finite constant $\bar{\mu}(x)$ such that

$$\lim_{n \rightarrow \infty} \frac{\tilde{t}(\varphi(0), \varphi(nx))}{n} = \bar{\mu}(x) \text{ a.s. and in } L^1.$$

Remark 3.5. *There are two main differences between the definitions of $\mu(x)$ and $\tilde{\mu}(x)$:*

- $\mu(x)$ is defined with the passage times in the environment T although $\tilde{\mu}(x)$ is defined with the passage times in the environment \tilde{T} ,

- $\mu(x)$ is defined with geodesic times between vertices obtained with φ although $\tilde{\mu}(x)$ is defined with geodesic times between vertices obtained with $\tilde{\varphi}$.

In order to compare $\mu(x)$ and $\tilde{\mu}(x)$, it is therefore natural to introduce $\bar{\mu}(x)$, an intermediate object which has one difference with $\mu(x)$ and one difference with $\tilde{\mu}(x)$ in its definition.

Remark 3.6. The proof of Theorem 1.7 given in this section also holds in the case originally proven by van den Berg and Kesten in [9], i.e. when we assume that τ has a finite first moment. However, it is simpler in this case since we do not need the clusters \mathcal{C} and $\tilde{\mathcal{C}}$ to define $\mu(x)$ and $\tilde{\mu}(x)$. In this case, for every $y \in \mathbb{Z}^d$, we can take $\varphi(y) = \tilde{\varphi}(y) = y$ and $\bar{\mu}(x) = \tilde{\mu}(x)$, and thus in the sequel, we do not need Lemma 3.7 and the proof of Lemma 3.8 is much simpler.

The following lemma is based on elementary arguments of percolation.

Lemma 3.7. We have $\tilde{\mu}(x) \leq \bar{\mu}(x)$.

Proof. Let $\hat{\mathcal{C}}$ be the infinite cluster for the Bernoulli percolation $(\mathbb{1}_{\{T(e) \leq M \text{ and } \tilde{T}(e) \leq M\}}, e \in \mathcal{E})$ which exists and is unique a.s. by (3.19). Note that $\hat{\mathcal{C}}$ is included in \mathcal{C} and in $\tilde{\mathcal{C}}$. For any $y \in \mathbb{R}^d$, we define $\hat{\varphi}(y)$ the random point of $\hat{\mathcal{C}}$ such that $\|y - \hat{\varphi}(y)\|_1$ is minimal, with a deterministic rule to break ties. For any $n \in \mathbb{Z}$, we have

$$\begin{aligned} \tilde{t}(\hat{\varphi}(0), \hat{\varphi}(nx)) &\leq \tilde{t}(\varphi(0), \hat{\varphi}(0)) + \tilde{t}(\hat{\varphi}(0), \tilde{\varphi}(0)) + \tilde{t}(\varphi(0), \varphi(nx)) + \tilde{t}(\varphi(nx), \hat{\varphi}(nx)) + \tilde{t}(\hat{\varphi}(nx), \tilde{\varphi}(nx)) \\ &= A(0) + \tilde{t}(\varphi(0), \varphi(nx)) + A(nx), \end{aligned} \quad (3.20)$$

by writing for any $y \in \mathbb{Z}^d$, $A(y) = \tilde{t}(\varphi(y), \hat{\varphi}(y)) + \tilde{t}(\hat{\varphi}(y), \tilde{\varphi}(y))$.

Now, for any $y \in \mathbb{Z}^d$, there exists a path between $\varphi(y)$ and $\hat{\varphi}(y)$ contained in \mathcal{C} . Thus, it only takes edges with finite passage times in the environment T , and thus with finite passage times in the environment \tilde{T} by (3.2). Furthermore, there exists a path between $\hat{\varphi}(y)$ and $\tilde{\varphi}(y)$ contained in $\tilde{\mathcal{C}}$, which gives that $\tilde{t}(\hat{\varphi}(y), \tilde{\varphi}(y))$ is also finite. Hence $A(y)$ is a finite random variable. It gives that

$$\frac{A(0)}{n}$$

converges almost surely, and thus in probability, towards 0. Hence

$$\frac{A(nx)}{n}$$

also converges towards 0 in probability. We get that

$$\frac{A(0) + \tilde{t}(\varphi(0), \varphi(nx)) + A(nx)}{n}$$

converges towards $\bar{\mu}(x)$ in probability. Using (3.20), this gives $\tilde{\mu}(x) \leq \bar{\mu}(x)$. \square

Let $\eta > 0$ and \mathfrak{P} the pattern given by Lemma 3.1. For any path π we denote by $\mathcal{N}^{\mathfrak{P}}(\pi)$ the maximum number of disjoint translations of the pattern \mathfrak{P} crossed by π .

Lemma 3.8. There exists a constant $c > 0$ such that for any n sufficiently large,

$$\mathbb{E} [\mathcal{N}^{\mathfrak{P}}(\gamma_n)] \geq cn.$$

Proof. Recall the definition of $N^{\mathfrak{P}}(\pi)$ given for any path π at (1.4). Since the pattern \mathfrak{P} is valid by Lemma 3.1, \mathcal{L} is useful by $(\mathcal{H}1)$ and $\mathcal{L}([0, \infty)) > p_c$ by $(\mathcal{H}2)$, we can use Theorem 1.5 and thus there exist $\alpha > 0$, $\beta_1 > 0$ and $\beta_2 > 0$ such that for every $n \in \mathbb{N}$ and for every $y \in B_1\left(0, \frac{\|nx\|_1}{4}\right)$ and $z \in B_1\left(nx, \frac{\|nx\|_1}{4}\right)$,

$$\mathbb{P}((y, z) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } y \text{ to } z \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha n) \leq \beta_1 e^{-\beta_2 n}. \quad (3.21)$$

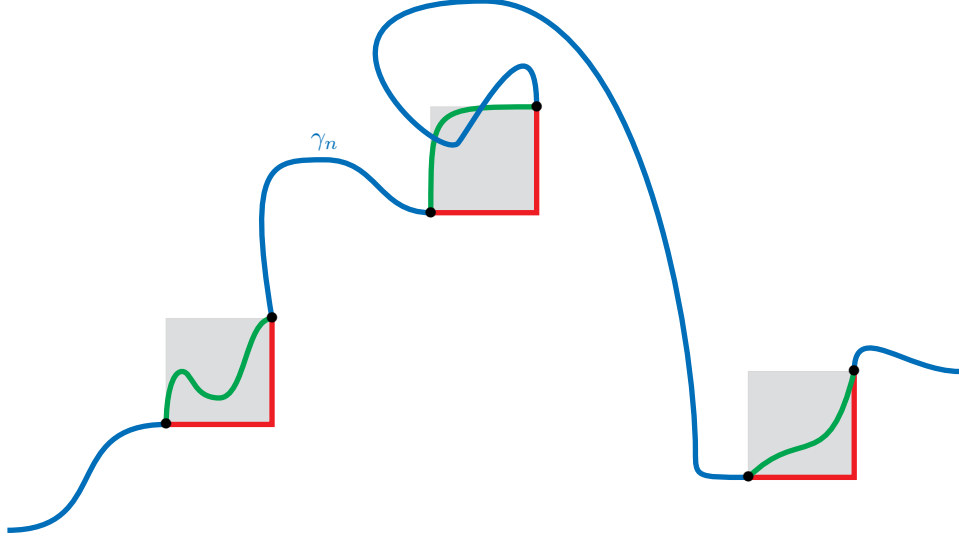


Figure 6: Illustration of the proof of Lemma 3.9. The translations of the pattern \mathfrak{P} are in gray. The path γ_n is the concatenation of the blue parts and of the red parts. In each pattern, the green part corresponds to the optimal path in the environment \tilde{T} among the paths between the endpoints of the pattern and entirely contained in the pattern. The idea is to bound from above the geodesic time between $\varphi(0)$ and $\varphi(nx)$ in the environment \tilde{T} by the passage time in the environment \tilde{T} of the concatenation of the blue parts and the green parts.

Denote by \mathcal{V}_n the set $B_1\left(0, \frac{\|nx\|_1}{4}\right) \times B_1\left(nx, \frac{\|nx\|_1}{4}\right)$. Then, for every $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}(N^{\mathfrak{P}}(\gamma_n) < \alpha n) &\leq \mathbb{P}((\varphi(0), \varphi(nx)) \notin \mathcal{V}_n) \\ &+ \sum_{(y,z) \in \mathcal{V}_n} \mathbb{P}((y, z) \in \mathfrak{C} \text{ and } \exists \text{ a geodesic } \gamma \text{ from } y \text{ to } z \text{ such that } N^{\mathfrak{P}}(\gamma) < \alpha n). \end{aligned} \quad (3.22)$$

Since $x \neq 0$, by Theorem 8 in [3], there exist $\beta_3 > 0$ and $\beta_4 > 0$ such that for every $n \in \mathbb{N}$,

$$\mathbb{P}((\varphi(0), \varphi(nx)) \notin \mathcal{V}_n) \leq \beta_3 e^{-\beta_4 n}.$$

Hence, using (3.21) and since $|\mathcal{V}_n|$ is bounded by a polynomial in n , there exist $\beta_5 > 0$ and $\beta_6 > 0$ such that for every $n \in \mathbb{N}$,

$$\mathbb{P}(N^{\mathfrak{P}}(\gamma_n) < \alpha n) \leq \beta_5 e^{-\beta_6 n}.$$

Thus, there exists $c' > 0$ such that for any n sufficiently large

$$\mathbb{E}[N^{\mathfrak{P}}(\gamma_n)] \geq c'n.$$

We conclude the proof by observing that simple geometric considerations provide a constant $c'' > 0$ such that for all path π ,

$$\mathcal{N}^{\mathfrak{P}}(\pi) \geq c'' N^{\mathfrak{P}}(\pi).$$

□

Lemma 3.9. *We have $\bar{\mu}(x) < \mu(x)$.*

Proof. For any $n \geq 1$, denote by $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$ the set, chosen according to a deterministic rule if there are several such sets, of $\mathcal{N}^{\mathfrak{P}}(\gamma_n)$ disjoint translations of \mathfrak{P} crossed by γ_n . Denote by $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$ the set of all edges of γ_n which are not in a subpath of γ_n between the endpoints of a pattern of $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$. Recall that we denote by $\Pi^{\mathfrak{P}}$ the set of all self-avoiding paths going from u^Λ to v^Λ and which are contained in Λ .

For a pattern $\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)$, we can associate a unique $s \in \mathbb{Z}^d$ such that s satisfies the condition $(\gamma_n; \mathfrak{P})$ and such that \mathfrak{P}' is located at $\theta_s \Lambda$. Then, we denote by $\Pi^{\mathfrak{P}'}$ the set of all self-avoiding paths π such that $\theta_{-s}\pi \in \Pi^{\mathfrak{P}}$. Denote by $\Gamma^{\mathfrak{P}}(\gamma_n)$ the set of all paths from $\varphi(0)$ to $\varphi(nx)$ following γ_n outside all the patterns of $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$ and following a path of $\Pi^{\mathfrak{P}'}$ for every pattern \mathfrak{P}' of $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$. With these definitions, we immediately get that

$$\min_{\pi \in \Gamma^{\mathfrak{P}}(\gamma_n)} \tilde{T}(\pi) = \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} \tilde{T}(e) + \sum_{\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)} \min_{\pi' \in \Pi^{\mathfrak{P}'}} \tilde{T}(\pi'). \quad (3.23)$$

Let n be sufficiently large such that Lemma 3.8 holds and let $c > 0$ be the constant given by this lemma. Recall that \mathcal{G} is the σ -field generated by the family $(T(e))_{e \in \mathcal{E}}$. Then, γ_n , $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$ and $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$ are \mathcal{G} -measurable and we get

$$\begin{aligned} \mathbb{E} [\tilde{t}(\varphi(0), \varphi(nx)) | \mathcal{G}] &\leq \mathbb{E} \left[\min_{\pi \in \Gamma^{\mathfrak{P}}(\gamma_n)} \tilde{T}(\pi) | \mathcal{G} \right] \text{ by the definition of the geodesic time in the environment } \tilde{T}, \\ &= \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} \mathbb{E} [\tilde{T}(e) | T(e)] + \sum_{\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)} \mathbb{E} \left[\min_{\pi' \in \Pi^{\mathfrak{P}'}} \tilde{T}(\pi') | \mathcal{G} \right] \text{ by (3.23)}, \\ &\leq \sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} T(e) + \sum_{\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)} \left(\min_{\pi' \in \Pi^{\mathfrak{P}'}} T(\pi') - \eta \right). \end{aligned}$$

For the first sum, the last inequality comes from the fact that for every edge $e \in \mathcal{E}$, $(T(e), \tilde{T}(e))$ has the same distribution as $(\tau, \tilde{\tau})$ and thus satisfies $E[\tilde{T}(e) | T(e)] \leq T(e)$. For the second sum, it comes from Lemma 3.1.

Then, by the definitions of $\mathcal{E}^{\mathfrak{P}}(\gamma_n)$ and $\mathcal{S}^{\mathfrak{P}}(\gamma_n)$,

$$\sum_{e \in \mathcal{E}^{\mathfrak{P}}(\gamma_n)} T(e) + \sum_{\mathfrak{P}' \in \mathcal{S}^{\mathfrak{P}}(\gamma_n)} \min_{\pi' \in \Pi^{\mathfrak{P}'}} T(\pi') = T(\gamma_n) = t(\varphi(0), \varphi(nx)).$$

Furthermore, recall that $\mathcal{N}^{\mathfrak{P}}(\gamma_n)$ is the number of elements of $\Pi^{\mathfrak{P}'}$. Thus, we get

$$\mathbb{E} [\tilde{t}(\varphi(0), \varphi(nx)) | \mathcal{G}] \leq t(\varphi(0), \varphi(nx)) - \eta \mathcal{N}^{\mathfrak{P}}(\gamma_n).$$

Now, taking expectation and dividing by n gives

$$\frac{\mathbb{E} [\tilde{t}(\varphi(0), \varphi(nx))]}{n} \leq \frac{\mathbb{E} [t(\varphi(0), \varphi(nx))]}{n} - \eta \frac{\mathbb{E} [\mathcal{N}^{\mathfrak{P}}(\gamma_n)]}{n} \leq \frac{\mathbb{E} [t(\varphi(0), \varphi(nx))]}{n} - \eta c,$$

by Lemma 3.8. We conclude the proof using that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} [\tilde{t}(\varphi(0), \varphi(nx))]}{n} = \bar{\mu}(x) \text{ and } \lim_{n \rightarrow \infty} \frac{\mathbb{E} [t(\varphi(0), \varphi(nx))]}{n} = \mu(x).$$

□

Now, we conclude the proof of Theorem 1.7 by combining Lemma 3.7 and Lemma 3.9.

A Existence of geodesics

Proposition A.1. *Assume that $\mathcal{L}(0) < p_c$. With probability one, for all x, y such that $(x, y) \in \mathfrak{C}$, there exists a geodesic between x and y .*

To prove the above proposition, we begin by the following lemma.

Lemma A.2. *Assume that $\mathcal{L}(0) < p_c$. There exists $\beta > 0$, $\beta' > 0$ and $\rho > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } 0 \text{ which contains at least } n \text{ edges but has } T(\pi) < \rho n) \leq \beta' e^{-\beta n}.$$

Proof of Lemma A.2. For each environment T , we define a new environment \tilde{T} defined for all edges e by

$$\tilde{T}(e) = \begin{cases} T(e) & \text{if } T(e) < \infty \\ 1 & \text{else.} \end{cases}$$

Since $\mathcal{L}(0) < p_c$, we have $\mathbb{P}(\tilde{T}(e) = 0) = \mathbb{P}(T(e) = 0) < p_c$. Thus, we can use Proposition (5.8) in [7] and we get $\beta > 0$, $\beta' > 0$ and $\rho > 0$ such that for all $n \geq 1$, for all $z \in \mathbb{Z}^d$,

$$\mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } z \text{ which contains at least } n \text{ edges but has } \tilde{T}(\pi) < \rho n) \leq \beta' e^{-\beta n}.$$

Now, for every edge e , $\tilde{T}(e) \leq T(e)$. Therefore,

$$\begin{aligned} & \mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } 0 \text{ which contains at least } n \text{ edges but has } T(\pi) < \rho n) \\ & \leq \mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } z \text{ which contains at least } n \text{ edges but has } \tilde{T}(\pi) < \rho n), \end{aligned}$$

which allows us to conclude. \square

Proof of Proposition A.1. It is sufficient to prove that for every x and y , with probability one, there exists a geodesic between x and y if $(x, y) \in \mathfrak{C}$. Fix x and y in \mathbb{Z}^d . Fix β, β' and ρ given by Lemma A.2. For every $n \geq 1$, denote by A_n the event on which every path π from x which contains at least n edges has $T(\pi) \geq \rho n$. By the Borel-Cantelli Lemma and by Lemma A.2, with probability one, for all n large enough, A_n occurs. We work on this probability one event. Assume that $(x, y) \in \mathfrak{C}$. Let $\pi_{x,y}$ be a path between x and y such that $T(\pi_{x,y}) < \infty$. Fix n large enough such that A_n occurs and

$$n > \frac{T(\pi_{x,y})}{\rho}. \tag{A.1}$$

Then, every path from x to the boundary of $B_1(x, n)$ has a passage time greater than or equal to ρn since the event A_n occurs, and thus a passage time strictly greater than $T(\pi_{x,y})$ by (A.1). Hence, the infimum in the definition of $t(x, y)$ is over the finite set of paths contained in $B_1(x, n)$, and there must be a geodesic between x and y . \square

B Edges with positive passage times taken by self-avoiding paths

Proof of Lemma 1.9. Assume that \mathcal{L} is useful and that $t_{\min} = 0$. Let $\tau > 0$ such that $\mathcal{L}([0, \tau]) < p_c$. For each environment T , we define a new environment \tilde{T} defined for all edges e by

$$\tilde{T}(e) = \begin{cases} 0 & \text{if } T(e) \leq \tau \\ 1 & \text{else.} \end{cases}$$

We have $\mathbb{P}(\tilde{T}(e) = 0) = \mathbb{P}(T(e) \leq \tau) < p_c$. By Proposition (5.8) in [7], we get $\beta > 0$, $\beta' > 0$ and $\rho > 0$ such that for all $n \geq 1$,

$$\mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } 0 \text{ which contains at least } n \text{ edges but has } \tilde{T}(\pi) < \rho n) \leq \beta' e^{-\beta n}.$$

Thus,

$$\begin{aligned} & \mathbb{P}(\exists \text{ a self-avoiding path } \pi \text{ from } 0 \text{ which contains at least } n \text{ edges but contains at most} \\ & \quad \rho n \text{ edges } e \text{ such that } T(e) > \tau) \leq \beta' e^{-\beta n}, \end{aligned}$$

and we get (1.24) for all $v, w \in \mathbb{Z}^d$. \square

C Overlapping patterns

Proof of Lemma 2.4. Let $\mathfrak{P}_0 = (\Lambda_0, u_0^\Lambda, v_0^\Lambda, \mathcal{A}_0^\Lambda)$ be a valid pattern. Denote by L_1, \dots, L_d the integers such that $\Lambda_0 = \prod_{i=1}^d \{0, \dots, L_i\}$. Fix

$$\ell^\Lambda = \max(L_1, \dots, L_d) + 4. \tag{C.1}$$

Let $M_0^\Lambda > 0$ such that

$$\mathbb{P}(\mathcal{A}_0^\Lambda \cap \{\forall e \in \Lambda_0, T(e) \leq M_0^\Lambda \text{ or } T(e) = \infty\}) > 0. \quad (\text{C.2})$$

In the case (INF). Consider the pattern $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ defined as follows:

- $\Lambda = B_\infty(0, \ell^\Lambda)$.
- $u^\Lambda = -\ell^\Lambda \varepsilon_1$ and $v^\Lambda = \ell^\Lambda \varepsilon_1$.
- Let π_∞ be a path from u^Λ to v^Λ such that:
 - π_∞ is a self-avoiding path.
 - In $\Lambda \setminus B_\infty(0, \ell^\Lambda - 3)$, π_∞ uses only 6 vertices, all in the set $\{k\varepsilon_1, -\ell^\Lambda \leq k \leq \ell^\Lambda\}$,
 - π_∞ visits u_0^Λ and v_0^Λ , and the portion of π_∞ between these two vertices, denoted by $\pi_{\infty,0}$ is entirely contained in Λ_0 . Furthermore, when \mathcal{A}_0^Λ occurs, $T(\pi_{\infty,0}) < \infty$. Note that this is possible since \mathfrak{P}_0 is valid.
 - $\pi_\infty \setminus \pi_{\infty,0}$ does not take any edge of Λ_0 .

Then, \mathcal{A}^Λ is the event such that:

- $\mathcal{A}_0^\Lambda \cap \{\forall e \in \Lambda_0, T(e) \leq M_0^\Lambda \text{ or } T(e) = \infty\}$ occurs,
- for all e belonging to $\mathcal{S}_{s,\ell^\Lambda}$, $T(e) < \infty$,
- for all e belonging to $\pi_\infty \setminus \pi_{\infty,0}$, $T(e) \leq M_0^\Lambda$,
- for all e which does not belong to $\Lambda_0 \cup \pi_\infty$, $T(e) = \infty$.

We get that \mathfrak{P} satisfies (AI-1), (AI-2) and (AI-4). Then, since π_∞ takes only edges whose passage time is smaller than or equal to M_0^Λ , \mathfrak{P} satisfies (AI-3) by taking $T^\Lambda > |B_\infty(0, \ell^\Lambda)|_e M_0^\Lambda$.

Furthermore, we have $\mathcal{A}^\Lambda \subset \mathcal{A}_0^\Lambda$ by the definition of \mathcal{A}^Λ , $\Lambda_0 \subset \Lambda$ by (C.1), \mathcal{A}^Λ has a positive probability by (C.2) and when \mathcal{A}^Λ occurs, every path from u^Λ to v^Λ whose passage time is finite is equal to $\pi_\infty \setminus \pi_{\infty,0}$ outside Λ_0 , visits u_0^Λ and v_0^Λ and is entirely contained in Λ_0 .

In the case (FU). Let $M^\Lambda > |B_\infty(0, \ell^\Lambda)|_e M_0^\Lambda + 1$ such that

$$\mathcal{L}((M^\Lambda - 1, M^\Lambda)) > 0. \quad (\text{C.3})$$

Consider the pattern $\mathfrak{P} = (\Lambda, u^\Lambda, v^\Lambda, \mathcal{A}^\Lambda)$ defined as follows:

- $\Lambda = B_\infty(0, \ell^\Lambda)$.
- $u^\Lambda = -\ell^\Lambda \varepsilon_1$ and $v^\Lambda = \ell^\Lambda \varepsilon_1$.
- Let π_f be a path from u^Λ to v^Λ such that:
 - π_f is a self-avoiding path.
 - π_f does not visit any vertex in $\partial\Lambda$ except u^Λ and v^Λ .
 - π_f visits u_0^Λ and v_0^Λ and the portion of π_f between these two vertices, denoted by $\pi_{f,0}$ is entirely contained in Λ_0 .
 - $\pi_f \setminus \pi_{f,0}$ does not take any edge of Λ_0 .

Then \mathcal{A}^Λ is the event such that:

- $\mathcal{A}_0^\Lambda \cap \{\forall e \in \Lambda_0, T(e) \leq M_0^\Lambda\}$ occurs,
- for all e belonging to $\pi_f \setminus \pi_{f,0}$, we have $T(e) \leq M_0^\Lambda$,
- for all e which does not belong to $\partial\Lambda \cup \Lambda_0 \cup \pi_f$, $T(e) \in (M^\Lambda - 1, M^\Lambda)$,
- for all $e \in \partial\Lambda$, $T(e) \geq M^\Lambda$.

We get that \mathfrak{P} satisfies (AF-1), (AF-2), (AF-3) and (AF-4). Furthermore,

- $\Lambda_0 \subset \Lambda$ by (C.1).

- $\mathbb{P}(\mathcal{A}^\Lambda)$ is positive by (C.2) and (C.3), and then \mathfrak{P} is a valid pattern.
- On \mathcal{A}^Λ , any path from u^Λ to v^Λ optimal for the passage time among the paths entirely inside Λ contains a subpath from u_0^Λ to v_0^Λ entirely inside Λ_0 . Indeed, let π be a path from u^Λ to v^Λ which does not contain a subpath from u^Λ to v^Λ entirely inside Λ_0 . Since π_f is a self-avoiding path, it implies that π takes an edge whose time is greater than $M^\Lambda - 1 > |\Lambda|_e M_0^\Lambda$. But we have $T(\pi_f) \leq |\Lambda|_e M_0^\Lambda < T(\pi)$ and thus π is not an optimal path. Hence, for every optimal path π , if a vertex $x \in \mathbb{Z}^d$ satisfies the condition $(\pi; \mathfrak{P}_0)$, x satisfies the condition $(\pi; \mathfrak{P})$.
- $\mathcal{A}^\Lambda \subset \mathcal{A}_0^\Lambda$ by the definition of \mathcal{A}^Λ .

□

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