



**HAL**  
open science

## Monotone-Cevian and finitely separable lattices

Miroslav Ploscica, Friedrich Wehrung

► **To cite this version:**

Miroslav Ploscica, Friedrich Wehrung. Monotone-Cevian and finitely separable lattices. 2023. hal-04228820

**HAL Id: hal-04228820**

**<https://hal.science/hal-04228820>**

Preprint submitted on 4 Oct 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MONOTONE-CEVIAN AND FINITELY SEPARABLE LATTICES

MIROSLAV PLOŠČICA AND FRIEDRICH WEHRUNG

ABSTRACT. A distributive lattice with zero is *completely normal* if its prime ideals form a root system under set inclusion. Every such lattice admits a binary operation  $(x, y) \mapsto x \setminus y$  satisfying the rules  $x \leq y \vee (x \setminus y)$  and  $(x \setminus y) \wedge (y \setminus x) = 0$  — in short a *deviation*. In this paper we study the following additional properties of deviations: *monotone* (i.e., isotone in  $x$  and antitone in  $y$ ) and *Cevian* (i.e.,  $x \setminus z \leq (x \setminus y) \vee (y \setminus z)$ ). We relate those matters to *finite separability* as defined by Freese and Nation. We prove that every finitely separable completely normal lattice has a monotone deviation.

We pay special attention to lattices of principal  $\ell$ -ideals of Abelian  $\ell$ -groups (which are always completely normal). We prove that for free Abelian  $\ell$ -groups (and also free  $\mathbb{k}$ -vector lattices) those lattices admit monotone Cevian deviations. On the other hand, we construct an Archimedean  $\ell$ -group with strong unit whose principal  $\ell$ -ideal lattice does not have a monotone deviation.

## 1. INTRODUCTION

This paper is motivated by the investigation of principal  $\ell$ -ideal lattices of Abelian  $\ell$ -groups. It has been known for a long time that those lattices are distributive with 0 and are completely normal. Recall (cf. Wehrung [14]) that a lattice  $D$  is *completely normal* if it is distributive, has a least element (usually denoted by 0), and for all  $a, b \in D$  there are  $x, y \in D$  such that  $a \vee b = a \vee y = x \vee b$  whereas  $x \wedge y = 0$ . Equivalently, the prime ideals of  $D$  form a root system under set inclusion (cf. Monteiro [10]). It is an easy exercise to verify that  $D$  is completely normal iff it admits a deviation in the following sense.

**Definition 1.1.** A binary operation  $\setminus$ , on a distributive 0-lattice  $D$ , is a *deviation on  $D$*  if the relations  $x \leq y \vee (x \setminus y)$  and  $(x \setminus y) \wedge (y \setminus x) = 0$  both hold whenever  $x, y \in D$ . The deviation  $\setminus$  is

- *left isotone* if  $x \leq x'$  implies that  $x \setminus y \leq x' \setminus y$ ,
- *right antitone* if  $y \leq y'$  implies that  $x \setminus y' \leq x \setminus y$ ,
- *monotone* if it is both left isotone and right antitone;
- *Cevian* if  $x \setminus z \leq (x \setminus y) \vee (y \setminus z)$  whenever  $x, y, z \in D$ ;
- *monotone-Cevian* if it is both monotone and Cevian.

We say that the lattice  $D$  is *Cevian* (resp., *monotone-Cevian*) if it has a Cevian (resp., monotone-Cevian) deviation.

Any homomorphic image of the principal  $\ell$ -ideal lattice of an Abelian  $\ell$ -group is Cevian (cf. Wehrung [15]). It is easy to find small completely normal lattices

---

*Date:* October 4, 2023.

*2020 Mathematics Subject Classification.* 06A06; 06B25; 06D05; 06F20; 08A30.

*Key words and phrases.* Lattice; distributive; completely normal; finitely separable; deviation; monotone; Cevian; lattice-ordered group; vector lattice; spectrum.

with a non-monotone or non-Cevian deviation. However, in many cases the deviation can be “adjusted” to become monotone and Cevian. By Ploščica [11], every completely normal lattice with at most  $\aleph_1$  elements is Cevian (see also Ploščica and Wehrung [12]). This does not extend to the cardinality  $\aleph_2$ : by [15], not every completely normal lattice with  $\aleph_2$  elements is Cevian.

The existence of *monotone* deviations on completely normal lattices has not been investigated so far. It is not very difficult to prove that a deviation on a (at most) countable lattice can be adjusted to become monotone. In this paper we use the idea of an adjustment to prove the much stronger result that *every finitely separable completely normal lattice has a monotone deviation*. Finite separability is here meant in the sense of Freese and Nation [5]. As an application, we prove that *principal  $\ell$ -ideal lattices of free Abelian  $\ell$ -groups are monotone-Cevian* (cf. Corollary 5.9).

The concept of finite separability, originally invented as one of the conditions that characterize projective lattices, seems to have potential for more applications. In that direction, we shall establish two equivalent conditions for finite separability in lattices and, more generally, posets. One of them (viz. Proposition 3.3) states that a poset is finitely separable iff it has a *finitely shadowing well-ordering*. The other one (viz. Theorem 3.6) states that a poset is finitely separable iff it is a “strong amalgam” of finite posets over a lower finite poset.

In Section 4 we prove that any deviation on a distributive 0-lattice with a finitely shadowing well-ordering can be adjusted to become monotone. This implies the above-mentioned result (cf. Theorem 4.8). In Section 5 we turn our attention to (necessarily completely normal) lattices that arise as the lattices of all principal  $\ell$ -ideals of Abelian  $\ell$ -groups. We verify that for free Abelian  $\ell$ -groups (more generally, for free  $\mathbb{k}$ -vector lattices), those lattices are finitely separable (cf. Proposition 5.6), which implies, with the help of the Belluce map, that they are monotone-Cevian (cf. Corollaries 5.8 and 5.9). We also verify, invoking the main result of Ploščica and Wehrung [12], that *every finitely separable completely normal lattice with at most  $\aleph_1$  elements is monotone-Cevian* (cf. Corollary 5.3).

Our final achievement (cf. Section 6) is a construction of *an Archimedean  $\ell$ -group with strong unit, of cardinality  $\aleph_1$ , whose principal  $\ell$ -ideal lattice does not have any left isotone or right antitone deviation*. Hence, not only there are completely normal lattices without a monotone deviation; they can be constructed as principal  $\ell$ -ideal lattices of Abelian  $\ell$ -groups. This is rather surprising, in view of the above-mentioned result for free Abelian  $\ell$ -groups.

Let us introduce some notation. Order-preserving maps between posets will be called *isotone*, order-reversing ones are *antitone*. We will denote by  $\text{Min } X$  (resp.,  $\text{Max } X$ ) the set of all minimal (resp., maximal) elements of a subset  $X$  in a poset  $P$ .

**Notation 1.2.** For any set  $P$ , any  $A \subseteq P$ , and any binary relation  $\triangleleft$  on  $P$ ,

$$\begin{aligned} \downarrow_{\triangleleft} A &\stackrel{\text{def}}{=} \{x \in P \mid (\exists a \in A)(x \triangleleft a)\}, \\ \uparrow_{\triangleleft} A &\stackrel{\text{def}}{=} \{x \in P \mid (\exists a \in A)(a \triangleleft x)\}. \end{aligned}$$

For  $a \in P$ , we will also write  $\downarrow_{\triangleleft} a$  and  $\uparrow_{\triangleleft} a$  instead of  $\downarrow_{\triangleleft} \{a\}$  and  $\uparrow_{\triangleleft} \{a\}$ , respectively.

A poset  $(P, \leq)$  is *lower finite* if  $\downarrow_{\leq} a$  is finite whenever  $a \in P$ .

For any set  $X$ ,  $\text{Pow } X$  denotes the powerset of  $X$ . “Countable” will always mean “at most countable”. We will denote by  $[X]^{<\omega}$  the set of all finite subsets of a set  $X$ .

## 2. FINITE SEPARABILITY, STRONG AMALGAMS, AND SHADOWS

The following definition is stated in Freese and Nation [5]. However, the condition given in Definition 2.1 already appeared in Freese and Nation [4], in connection with projective lattices. It was then used in several other works on projectivity. In Heindorf and Shapiro [7] the condition was given the name *Freese/Nation property*. In Fuchino, Koppelberg, and Shelah [6], it is studied from a set-theoretical point of view.

**Definition 2.1.** A poset  $M$  is *finitely separable* if there are functions  $A$  and  $B$  with domain  $M$  such that each  $A(z)$  is a finite set of upper bounds of  $z$ , each  $B(z)$  is a finite set of lower bounds of  $z$ , and for all  $x, y \in M$ ,  $x \leq y$  implies  $A(x) \cap B(y) \neq \emptyset$ . Such a pair  $(A, B)$  will be called a *separability witness* for  $M$ .

The following deep result is contained in Freese and Nation [5, Theorem 1].

**Theorem 2.2.** *A lattice  $L$  is finitely separable iff every lattice homomorphism onto  $L$  has an isotone section.*

It follows from Freese and Nation [5, Theorem 6] that *every projective member* (thus, in particular, any free member) *of any variety of lattices is finitely separable*. An example of non-finitely separable lattice is the chain  $\omega_1$  of all countable ordinals. The following result collects a few elementary observations on finite separability.

### Proposition 2.3.

- (1) *A poset is finitely separable iff its dual poset is finitely separable.*
- (2) *Every countable poset is finitely separable.*
- (3) *Every order-retract of a finitely separable poset is finitely separable.*
- (4) *Every order-convex subset of a finitely separable poset is finitely separable.*
- (5) *Let  $M \sqcup \{1\}$  be the poset obtained by adding a new upper bound 1 atop all elements of a poset  $M$ . Then  $M$  is finitely separable iff  $M \sqcup \{1\}$  is finitely separable. A similar result holds for the poset  $M \sqcup \{0\}$  obtained by adding a new lower bound 0.*
- (6) *Any finite product of finitely separable posets is finitely separable.*

*Proof.* (1) is trivial. Moreover, as observed on [5, page 246], (2) is easy. The argument for (3) is established in the course of the proof of the direction (3) $\Rightarrow$ (1) of [5, Theorem 1]. For any separability witness  $(A, B)$  for a poset  $N$  and any order-convex subset  $M$  of  $N$ ,  $x \mapsto A(x) \cap M$  and  $x \mapsto B(x) \cap M$  form a separability witness for  $M$ ; (4) follows. If  $M \sqcup \{1\}$  is finitely separable, then, since  $M$  is an order-convex subset of  $M \sqcup \{1\}$ , so is  $M$ . Conversely, for any separability witness  $(A, B)$  for  $M$ ,  $x \mapsto A(x) \cup \{1\}$  and  $x \mapsto B(x)$  form a separability witness for  $M \sqcup \{1\}$ ; (5) follows (*see also* [5, Theorem 10] *for a more general fact*). Finally, if  $(A_i, B_i)$  is a separability witness for a poset  $M_i$  whenever  $i \in \{1, 2\}$ , then the maps  $(x_1, x_2) \mapsto A_1(x_1) \times A_2(x_2)$  and  $(x_1, x_2) \mapsto B_1(x_1) \times B_2(x_2)$  form a separability witness for  $M_1 \times M_2$ ; (6) follows.  $\square$

As we will see in Example 2.8, the “order-convex subset” assumption cannot be replaced by “subset” in the statement of Proposition 2.3(4): that is, a sublattice of a finitely separable lattice need not be finitely separable (cf. Example 2.8).

**Definition 2.4.** Let  $M$  be a poset, let  $A \subseteq M$ , and let  $x \in M$ . A subset  $U$  of  $A$  is a *lower shadow* of  $x$  on  $A$  if  $A \cap \downarrow_{\leq} x = A \cap \downarrow_{\leq} U$ ; *upper shadows* are defined dually.

**Definition 2.5.** We say that a subset  $A$  is *finitely shadowing in a poset  $M$*  if every element of  $M$  has both a finite lower shadow and a finite upper shadow on  $A$ .

Of course, if  $A$  is finitely shadowing in  $M$ , then for every  $x \in M$ , the smallest lower shadow (resp., upper shadow) of  $x$ , with respect to set inclusion, on  $A$  is  $\text{Max}(A \cap \downarrow_{\leq} x)$  (resp.,  $\text{Min}(A \cap \uparrow_{\leq} x)$ ).

**Definition 2.6.** Let  $P$  be a poset. We say that a poset  $M$  is the *strong amalgam* of a family  $(M_p \mid p \in P)$  of subsets of  $M$  if the following statements hold:

- (1)  $M = \bigcup_{p \in P} M_p$ ;
- (2) for all  $p \leq q$  in  $P$ ,  $M_p$  is a finitely shadowing subset of  $M_q$ ;
- (3) (Interpolation Property) for all  $p, q \in P$ , all  $x \in M_p$ , and  $y \in M_q$ , if  $x \leq y$ , then there are  $r \leq p, q$  in  $P$  and  $z \in M_r$  such that  $x \leq z \leq y$ .

We say that the strong amalgam above is *lower finite* if the poset  $P$  is lower finite.

Items (2) and (3) together obviously entail the following:

$$M_p \cap M_q = \bigcup \{M_r \mid r \leq p, q\}, \quad \text{whenever } p, q \in P.$$

**Proposition 2.7.** *The following statements hold, for any lower finite strong amalgam  $M$  of a family  $(M_p \mid p \in P)$  of subsets:*

- (1) *Each  $M_p$  is finitely shadowing in  $M$ .*
- (2) *If each  $M_p$  is finitely separable, then so is  $M$ .*

*Proof.* Ad (1) We must prove that every  $x \in M$  has (say) an upper shadow on each  $M_q$ . Pick  $p \in P$  such that  $x \in M_p$ . Since  $P$  is lower finite,  $R \stackrel{\text{def}}{=} \downarrow_{\leq} p \cap \downarrow_{\leq} q$  is finite. For each  $r \in R$ , it follows from Definition 2.6(2) that  $M_r$  is finitely shadowing in  $M_p$ ; thus  $x$  has a finite upper shadow  $U_r$  on  $M_r$ . A direct application of the Interpolation Property (cf. Definition 2.6(3)) then shows that  $\bigcup_{r \in R} U_r$  is a finite upper shadow of  $x$  on  $M_q$ .

Ad (2). For each  $x \in M$ , pick  $\nu(x) \in P$  such that  $x \in M_{\nu(x)}$ . For every  $p \in P$ , pick a separability witness  $(A_p, B_p)$  for  $M_p$ . For all  $x \in M$  and all  $p \leq \nu(x)$ , it follows from Definition 2.6(2) that  $x$  has a finite upper shadow  $U_{x,p}$  and a finite lower shadow  $V_{x,p}$  on  $M_p$ . The sets

$$A(x) \stackrel{\text{def}}{=} \bigcup \{A_p(u) \mid p \leq \nu(x), u \in U_{x,p}\},$$

$$B(x) \stackrel{\text{def}}{=} \bigcup \{B_p(v) \mid p \leq \nu(x), v \in V_{x,p}\}$$

are, respectively, a finite set of upper bounds of  $x$  and a finite set of lower bounds of  $x$  in  $M$ . Let  $x \leq y$  in  $M$ ; set  $p \stackrel{\text{def}}{=} \nu(x)$  and  $q \stackrel{\text{def}}{=} \nu(y)$ . By the Interpolation Property (cf. Definition 2.6(3)), there are  $r \in \downarrow_{\leq} p \cap \downarrow_{\leq} q$  and  $z \in M_r$  such that  $x \leq z \leq y$ . By definition, there are  $u \in U_{x,r}$  and  $v \in V_{y,r}$  such that  $x \leq u \leq z \leq v \leq y$ . Since  $u \leq v$  within  $M_r$ , there exists  $w \in A_r(u) \cap B_r(v)$ ; so  $x \leq w \leq y$  whereas  $w \in A(x) \cap B(y)$ . Therefore,  $(A, B)$  is a separability witness for  $M$ .  $\square$

**Example 2.8.** *A finitely separable lattice with a non-finitely separable sublattice.*

*Proof.* The lattice  $P \stackrel{\text{def}}{=} [\omega_1]^{<\omega}$  is the strong amalgam of its finite sublattices  $[X]^{<\omega}$ , for finite  $X \subset \omega_1$ ; thus, by Proposition 2.7, it is finitely separable, and thus so is

its opposite lattice  $P^{\text{op}}$ . It follows from Freese and Nation [5, Lemma 9] that the ordinal sum  $P \dot{+} P^{\text{op}}$  is not finitely separable.

Denote by  $u$  a new top element for  $P$  (thus also a new bottom element for  $P^{\text{op}}$ ). It follows from Proposition 2.3(5) that  $P \cup \{u\}$  and  $P^{\text{op}} \cup \{u\}$  are both finitely separable, thus so is their product  $(P \cup \{u\}) \times (P^{\text{op}} \cup \{u\})$ . Moreover,  $P \dot{+} P^{\text{op}}$  embeds into  $(P \cup \{u\}) \times (P^{\text{op}} \cup \{u\})$ , by mapping each  $x \in P$  to  $(x, u)$  and each  $y \in P^{\text{op}}$  to  $(u, y)$ .  $\square$

### 3. FINITELY SHADOWING WELL-ORDERINGS

A typical situation that will arise in this section will involve two distinct orderings  $\leq$  and  $\sqsubseteq$  on the same universe  $M$ , occasionally prompting the need to spell out which one is in question. For example, Definition 3.1 will begin with “Let  $(M, \leq)$  be a poset” instead of “Let  $M$  be a poset”.

**Definition 3.1.** Let  $(M, \leq)$  be a poset. A binary relation  $\sqsubseteq$  of  $M$  is *finitely shadowing* on  $(M, \leq)$  if  $\downarrow_{\sqsubseteq} x$  is finitely shadowing in  $(M, \leq)$  whenever  $x \in M$ .

If  $\sqsubseteq$  is a partial ordering, with associated strict ordering  $\sqsubset$ , then  $\downarrow_{\sqsubseteq} x$  is finitely shadowing iff  $\downarrow_{\sqsubset} x$  is finitely shadowing (for these two sets differ by the singleton  $\{x\}$ ). In particular,  $\sqsubseteq$  is finitely shadowing iff  $\sqsubset$  is finitely shadowing.

**Proposition 3.2.** Let  $(M, \leq)$  be a poset. Then a well-ordering  $\sqsubseteq$  on  $M$  is finitely shadowing in  $(M, \leq)$  iff every  $a \in M$  has both a finite upper shadow and a finite lower shadow on  $\downarrow_{\sqsubseteq} a$ .

*Proof.* We verify the nontrivial direction. Suppose that the given condition holds and let  $a, b \in M$ ; we must verify that  $b$  has (say) a finite upper shadow on  $\downarrow_{\sqsubseteq} a$ . We argue by  $\sqsubseteq$ -induction on  $b$ . The result is trivial if  $b \sqsubset a$  (for then  $b \in \downarrow_{\sqsubseteq} a$ ); we may thus suppose that  $a \sqsubseteq b$ . By assumption,  $b$  has a finite upper shadow  $A$  on  $\downarrow_{\sqsubseteq} b$ . By induction hypothesis, every  $x \in \downarrow_{\sqsubseteq} b$  has a finite upper shadow  $U_x$  on  $\downarrow_{\sqsubseteq} a$ . Then  $\bigcup_{x \in A} U_x$  is a finite upper shadow of  $b$  on  $\downarrow_{\sqsubseteq} a$ .  $\square$

**Proposition 3.3.** A poset  $(M, \leq)$  is finitely separable iff it has a finitely shadowing well-ordering. Furthermore, for every finitely shadowing well-ordering  $\sqsubseteq$  on  $M$ , there exists a separability witness  $(A, B)$  of  $M$  such that

$$x \in A(y) \cup B(y) \text{ implies that } x \sqsubseteq y, \quad \text{for all } x, y \in M. \quad (3.1)$$

*Proof.* The argument of the proof that every finitely separable poset has a finitely shadowing well-ordering is mostly contained in the proof of Freese and Nation [5, Theorem 1]. For convenience, we provide a description of the well-ordering. Let  $(A, B)$  be a separability witness for  $M$ . We define inductively an ordinal  $\delta$  and a partition  $(M_\xi \mid \xi < \delta)$  of  $M$  into countable blocks, as follows. Suppose  $(M_\xi \mid \xi < \alpha)$  already defined and set  $M_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\xi < \alpha} M_\xi$ . If  $M_{<\alpha} = M$  then set  $\delta \stackrel{\text{def}}{=} \alpha$  and stop. Suppose that  $M_{<\alpha} \neq M$  and pick  $c \in M \setminus M_{<\alpha}$ . The smallest subset  $M_\alpha$  of  $M$  such that  $c \in M_\alpha$  and  $(A(x) \cup B(x)) \setminus M_{<\alpha} \subseteq M_\alpha$  whenever  $x \in M_\alpha$  is countable, and disjoint from  $M_{<\alpha}$ . This completes the induction step.

For any  $x \in M$ , denote by  $\nu(x)$  the unique  $\xi < \delta$  such that  $x \in M_\xi$ . Pick a well-ordering  $\sqsubseteq_\xi$  of  $M_\xi$  of type at most  $\omega$ , for each  $\xi < \alpha$ , and let

$$x \sqsubseteq y \quad \text{if} \quad (\nu(x) < \nu(y) \text{ or } (\nu(x) = \nu(y) \text{ and } x \sqsubseteq_{\nu(x)} y)), \quad \text{for all } x, y \in M.$$

Then  $\sqsubseteq$  is a well-ordering of  $M$  and for any  $c \in M$ , with  $\gamma \stackrel{\text{def}}{=} \nu(c)$ ,  $A(c) \cap M_{<\gamma}$  is a finite upper shadow and  $B(c) \cap M_{<\gamma}$  is a finite lower shadow of  $c$  on  $M_{<\gamma}$ . Since  $\downarrow_{\sqsubseteq} c = M_{<\gamma} \cup F$  for the finite set  $F \stackrel{\text{def}}{=} \{x \in M_{\gamma} \mid x \sqsubset_{\gamma} c\}$ ,  $c$  also has a finite upper shadow and a finite lower shadow on  $\downarrow_{\sqsubseteq} c$ . Hence, the well-ordering  $\sqsubseteq$  is finitely shadowing in  $(M, \leq)$ .

Conversely, let  $\sqsubseteq$  be a finitely shadowing well-ordering on  $M$ . For  $c \in M$ , we shall define  $A(c)$  and  $B(c)$  by  $\sqsubseteq$ -induction, in such a way that  $A(c) \cup B(c) \subseteq \downarrow_{\sqsubseteq} c$ ; this will ensure (3.1). By induction hypothesis,  $c$  has a finite upper shadow  $U_c$  and a finite lower shadow  $V_c$  on  $\downarrow_{\sqsubseteq} c$ . Then  $A(c) \stackrel{\text{def}}{=} \{c\} \cup \bigcup_{u \in U_c} A(u)$  is a finite set of upper bounds of  $c$ , and for each  $u \in U_c$ ,  $A(u) \subseteq \downarrow_{\sqsubseteq} u$  by induction hypothesis, so  $A(u) \subseteq \downarrow_{\sqsubseteq} c$ ; whence  $A(c) \subseteq \downarrow_{\sqsubseteq} c$ . Symmetrically,  $B(c) \stackrel{\text{def}}{=} \{c\} \cup \bigcup_{v \in V_c} B(v)$  is a finite set of lower bounds of  $c$  contained in  $\downarrow_{\sqsubseteq} c$ .

We claim that  $(A, B)$  is a separability witness for  $(M, \leq)$ . Let  $a, b \in M$  such that  $a \leq b$ , we verify that  $A(a) \cap B(b) \neq \emptyset$ . We argue by  $\sqsubseteq$ -induction with respect to  $\max_{\sqsubseteq} \{a, b\}$  (i.e., the maximum of  $\{a, b\}$  with respect to  $\sqsubseteq$ ). Since each  $c \in A(c) \cap B(c)$ , we may assume that  $a < b$ . If  $a \sqsubset b$ , then  $a \leq v$  for some  $v \in V_b$ , so, since  $\{a, v\} \subseteq \downarrow_{\sqsubseteq} b$  and by our induction hypothesis,  $A(a) \cap B(v) \neq \emptyset$ , and so, since  $B(v) \subseteq B(b)$ , we get  $A(a) \cap B(b) \neq \emptyset$ . The argument for  $b \sqsubset a$  is symmetric. This completes the proof of our claim.  $\square$

**Definition 3.4.** For a map  $C: M \rightarrow \text{Pow } M$ , we set  $C^0(x) = \{x\}$  and  $C^{n+1}(x) \stackrel{\text{def}}{=} \bigcup_{y \in C(x)} C^n(y)$  whenever  $n < \omega$ . We say that the map  $C$  is *locally finite* if  $C^\omega(x) \stackrel{\text{def}}{=} \bigcup_{n < \omega} C^n(x)$  is finite whenever  $x \in M$ .

**Proposition 3.5.** *Every finitely separable poset  $M$  has a separability witness  $(A, B)$  such that the set map  $(A \cup B: x \mapsto A(x) \cup B(x))$  is locally finite.*

*Proof.* By Proposition 3.3,  $M$  has a finitely shadowing well-ordering  $\sqsubseteq$  and a separability witness  $(A, B)$  satisfying (3.1). It is straightforward to verify, by  $\sqsubseteq$ -induction on  $x$ , that  $(A \cup B)^\omega(x)$  is finite whenever  $x \in M$ .  $\square$

**Theorem 3.6.** *A poset  $M$  is finitely separable iff it is the strong amalgam, over a lower finite poset (resp., a sublattice of  $[M]^{<\omega}$ ), of a family of nonempty finite subsets.*

*Proof.* One direction, that any lower finite strong amalgam of finite posets is finitely separable, is provided by Proposition 2.7.

Let, conversely,  $M$  be a finitely separable poset. By Proposition 3.5,  $M$  has a separability witness  $(A, B)$  such that the set map  $A \cup B$  is locally finite. We say that a subset  $X$  of  $M$  is *closed* if  $A(x) \cup B(x) \subseteq X$  whenever  $x \in X$ . Fix  $o \in M$ . Since the map  $A \cup B$  is locally finite, every finite subset of  $M$  is contained in some finite closed subset of  $M$ , thus the collection  $\Lambda$  of all finite closed subsets of  $M$  containing  $\{o\}$  is a sublattice of  $([M]^{<\omega}, \cup, \cap)$  with  $M = \bigcup \Lambda$ . Now let  $P, Q \in \Lambda$  and let  $x \in P$ ,  $y \in Q$  such that  $x \leq y$ . Since  $(A, B)$  is a separability witness for  $M$ , there exists  $z \in A(x) \cap B(y)$ . Since  $P$  and  $Q$  are both closed,  $z$  belongs to  $P \cap Q$ . Since  $x \leq z \leq y$ , this yields the Interpolation Property (cf. Definition 2.6(3)) for  $\Lambda$ . The condition 2.6(2) follows from the finiteness of all members of  $\Lambda$ .  $\square$

#### 4. THE MONOTONE ADJUSTMENT OF A MAP

Now we turn our attention to monotone deviations, the original motivation.



**Definition 4.1.** Let  $M$  and  $L$  be posets. A map  $d: M \times M \rightarrow L$  is *monotone* on a subset  $Z$  of  $M \times M$  if  $x \leq x'$  and  $y' \leq y$  implies that  $d(x, y) \leq d(x', y')$  whenever  $(x, y), (x', y') \in Z$ .

The proof of the following lemma is routine and we omit it.

**Lemma 4.2.** For any elements  $x, y, x', y'$  in a chain  $(M, \sqsubseteq)$ , let  $\{x, y\} \trianglelefteq \{x', y'\}$  hold if either  $\max_{\sqsubseteq} \{x, y\} \sqsubset \max_{\sqsubseteq} \{x', y'\}$  or  $(\max_{\sqsubseteq} \{x, y\} = \max_{\sqsubseteq} \{x', y'\} \text{ and } \min_{\sqsubseteq} \{x, y\} \sqsubseteq \min_{\sqsubseteq} \{x', y'\})$ . Then  $\trianglelefteq$  is a total ordering on  $M_{1,2} = \{N \subseteq M \mid |N| \in \{1, 2\}\}$  and the following statements hold whenever  $x, y, x', y', z \in M$ :

- (1)  $x \sqsubseteq y$  iff  $\{x, z\} \trianglelefteq \{y, z\}$  and  $x \sqsupset y$  iff  $\{x, z\} \triangleleft \{y, z\}$ .
- (2)  $(x \sqsubseteq x' \text{ and } y \sqsubseteq y')$  implies that  $\{x, y\} \trianglelefteq \{x', y'\}$ .
- (3)  $\{x, y\} \triangleleft \{x', y'\}$  implies that  $x \sqsupset x'$  or  $y \sqsupset y'$ .
- (4) If  $(M, \sqsubseteq)$  is well-ordered then so is  $(M_{1,2}, \trianglelefteq)$ .

The following technical lemma is the key point to our forthcoming definition of the monotone adjustment of a map.

**Lemma 4.3.** Let  $M$  and  $L$  be posets, let  $d: M \times M \rightarrow L$ , and let  $\sqsubseteq$  be a finitely shadowing total ordering on  $M$ . Denote by  $\trianglelefteq$  the total ordering on  $M_{1,2}$  introduced in Lemma 4.2. Let  $a, b \in M$  and suppose that  $d$  is monotone on  $D_{a,b} \stackrel{\text{def}}{=} \{(x, y) \in M \times M \mid \{x, y\} \triangleleft \{a, b\}\}$ . For each  $x \in \{a, b\}$ , let  $U_x$  (resp.,  $V_x$ ) be an upper shadow (resp., lower shadow) of  $x$  on  $\downarrow_{\sqsubseteq} x$ . We set

$$\begin{aligned} \mathcal{A} &\stackrel{\text{def}}{=} \{d(x, y) \mid \{x, y\} \triangleleft \{a, b\}, a \leq x, y \leq b\}, \\ \mathcal{A}' &\stackrel{\text{def}}{=} \{d(x, b) \mid x \in U_a\} \cup \{d(a, y) \mid y \in V_b\}, \\ \mathcal{B} &\stackrel{\text{def}}{=} \{d(x, y) \mid \{x, y\} \triangleleft \{a, b\}, x \leq a, b \leq y\}, \\ \mathcal{B}' &\stackrel{\text{def}}{=} \{d(x, b) \mid x \in V_a\} \cup \{d(a, y) \mid y \in U_b\}. \end{aligned}$$

Then  $\mathcal{A}'$  is a coinital subset of  $\mathcal{A}$  and  $\mathcal{B}'$  is a cofinal subset of  $\mathcal{B}$ , both finite if  $U_a, V_a, U_b, V_b$  are finite.

*Proof.* We establish for example the part about  $\mathcal{A}$  and  $\mathcal{A}'$ ; the proof for  $\mathcal{B}$  and  $\mathcal{B}'$  is similar. The containment  $\mathcal{A}' \subseteq \mathcal{A}$  follows immediately from Lemma 4.2. Now let  $\{x, y\} \triangleleft \{a, b\}$ ,  $a \leq x, y \leq b$ . By Lemma 4.2, either  $x \sqsupset a$  or  $y \sqsupset b$ . In the first case, there exists  $u \in U_a$  such that  $u \leq x$ . Since  $\{u, b\} \triangleleft \{a, b\}$  (cf. Lemma 4.2),  $\{x, y\} \triangleleft \{a, b\}$ ,  $u \leq x$ , and  $y \leq b$ , it follows that  $d(x, y) \geq d(u, b)$ , with  $d(u, b) \in \mathcal{A}'$ . In the second case, there exists  $v \in V_b$  such that  $y \leq v$ . By a similar argument to the above,  $d(x, y) \geq d(a, v)$  with  $d(a, v) \in \mathcal{A}'$ .  $\square$

**Definition 4.4.** For any poset  $L$  and any  $X \subseteq L$ , say that an element  $a$  is the *finitary join* of  $X$  if there exists a finite cofinal subset  $X'$  of  $X$  such that  $a = \bigvee X'$  (of course in such a case  $a = \bigvee X$  as well). Finitary meets are defined dually.

**Proposition 4.5.** Let  $M$  be a poset, let  $D$  be a lattice, let  $d: M \times M \rightarrow D$ , and let  $\sqsubseteq$  be a finitely shadowing well-ordering on  $M$ . Denote by  $\trianglelefteq$  the well-ordering on  $M_{1,2}$  introduced in Lemma 4.2. Then there exists a (necessarily unique) monotone map  $d': M \times M \rightarrow D$  such that for all  $a, b \in M$ ,  $d'(a, b) = d'_\wedge(a, b) \vee d'_\vee(a, b)$  where

$$d'_\wedge(a, b) \stackrel{\text{def}}{=} d(a, b) \wedge \bigwedge \{d'(x, y) \mid \{x, y\} \triangleleft \{a, b\}, a \leq x, y \leq b\}, \quad (4.1)$$

$$d'_\vee(a, b) \stackrel{\text{def}}{=} \bigvee \{d'(x, y) \mid \{x, y\} \triangleleft \{a, b\}, x \leq a, b \leq y\} \quad (4.2)$$



(where as usual, the empty meet is a new top element and the empty join is a new bottom element) are a finitary meet and a finitary join, respectively.

*Proof.* We argue by  $\leq$ -induction. We prove that, for every  $a, b \in M$ ,  $d'$  is correctly defined and monotone on  $D_{a,b} \cup \{(a,b), (b,a)\}$ . Our induction hypothesis implies that  $d'$  is correctly defined and monotone on  $D_{a,b}$ . By Lemma 4.3 applied to the restriction of  $d'$  to  $D_{a,b}$ , the meet  $\bigwedge \{d'(x,y) \mid \{x,y\} \triangleleft \{a,b\}, a \leq x, y \leq b\}$  (i.e., the second meetand of (4.1)) and the join (4.2) are both finitary, so  $d'_\wedge(a,b)$ ,  $d'_\vee(a,b)$ ,  $d'(a,b)$ , as well as  $d'_\wedge(b,a)$ ,  $d'_\vee(b,a)$ ,  $d'(b,a)$  are all well defined.

It remains to verify that  $d'$  is monotone on  $D_{a,b} \cup \{(a,b), (b,a)\}$ . This verification splits into several cases.

First, we need to argue that for all  $(x,y) \in D_{a,b}$ ,  $x \leq a$  and  $b \leq y$  implies  $d'(x,y) \leq d'(a,b)$ . This is obvious, because  $d'(x,y)$  is then a joinand of  $d'_\vee(a,b)$ . The same argument applies when  $a$  and  $b$  are interchanged.

The second case consists of verifying that for all  $(x,y) \in D_{a,b}$ ,  $a \leq x$  and  $y \leq b$  implies that  $d'(a,b) \leq d'(x,y)$ ; that is,  $d'_\wedge(a,b) \leq d'(x,y)$  and  $d'_\vee(a,b) \leq d'(x,y)$ . The first inequality is obvious, because  $d'(x,y)$  is then a meetand of  $d'_\wedge(a,b)$ . In order to prove the second inequality, we must verify that for any  $(u,v) \in D_{a,b}$ ,  $u \leq a$  and  $b \leq v$  implies that  $d'(u,v) \leq d'(x,y)$ . This follows from our induction hypothesis, because  $u \leq a \leq x$  and  $y \leq b \leq v$ . The same argument applies when  $a$  and  $b$  are interchanged.

Finally, we need to prove that  $d'(a,b) \leq d'(b,a)$  when  $a < b$ . If  $a \sqsubseteq b$  then  $(a,a) \in D_{a,b}$  and we have already proved that  $d'(a,b) \leq d'(a,a) \leq d'(b,a)$ . If  $b \sqsubseteq a$  then  $(b,b) \in D_{a,b}$  and we have already proved that  $d'(a,b) \leq d'(b,b) \leq d'(b,a)$ .  $\square$

Owing to Proposition 4.5, the map  $d'$  will be called the *monotone adjustment* of  $d$ . It depends not only of  $d$ , but also of the chosen finitely shadowing well-ordering  $\sqsubseteq$  of  $M$ . Note that  $d$  is monotone iff  $d = d'$ .

Our next two propositions will entail that if  $d$  is a deviation, then so is  $d'$ . Because of possible future applications, we prove them under slightly more general assumptions.

**Proposition 4.6.** *Let  $M$  be a poset, let  $D$  be a distributive lattice, let  $d: M \times M \rightarrow D$ , and let  $f: M \rightarrow D$  be an isotone map. We denote by  $d'$  the monotone adjustment of  $d$  with respect to a finitely shadowing well-ordering  $\sqsubseteq$  of  $M$ . If  $f(x) \leq f(y) \vee d(x,y)$  whenever  $x, y \in M$ , then  $f(a) \leq f(b) \vee d'(a,b)$  whenever  $a, b \in M$ .*

*Proof.* We argue by  $\leq$ -induction. Let  $a, b \in M$  and suppose that the inequality  $f(x) \leq f(y) \vee d'(x,y)$  holds whenever  $\{x,y\} \triangleleft \{a,b\}$ . In order to prove that  $f(a) \leq f(b) \vee d'(a,b)$  it suffices to verify that  $f(a) \leq f(b) \vee d'_\wedge(a,b)$ . Since the meet defining  $d'_\wedge(a,b)$  is finitary, it follows from the distributivity of  $D$  that it suffices to verify that for all  $(x,y) \in D_{a,b}$ ,  $a \leq x$  and  $y \leq b$  implies that  $f(a) \leq f(b) \vee d'(x,y)$ . Since  $f$  is isotone and by our induction hypothesis, we get the inequalities

$$f(a) \leq f(x) \leq f(y) \vee d'(x,y) \leq f(b) \vee d'(x,y). \quad \square$$

Say that a 0-lattice  $D$  is 0-distributive if  $x \wedge z = y \wedge z = 0$  implies  $(x \vee y) \wedge z = 0$  whenever  $x, y, z \in D$ .

**Proposition 4.7.** *Let  $M$  be a poset, let  $D$  be a 0-distributive 0-lattice, and let  $d: M \times M \rightarrow D$ . We denote by  $d'$  the monotone adjustment of  $d$  with respect to a*

finitely shadowing well-ordering  $\sqsubseteq$  of  $M$ . If  $d(x, y) \wedge d(y, x) = 0$  whenever  $x, y \in M$ , then  $d'(a, b) \wedge d'(b, a) = 0$  whenever  $a, b \in M$ .

*Proof.* We argue by  $\leq$ -induction. Let  $a, b \in M$  and suppose that  $d'(x, y) \wedge d'(y, x) = 0$  whenever  $\{x, y\} \triangleleft \{a, b\}$ . Since  $D$  is 0-distributive, the proof breaks up into four statements.

- $d'_\wedge(a, b) \wedge d'_\wedge(b, a) = 0$ . Since  $d'_\wedge(a, b) \leq d(a, b)$ ,  $d'_\wedge(b, a) \leq d(b, a)$ , and  $d(a, b) \wedge d(b, a) = 0$ , this is obvious.
- $d'_\wedge(a, b) \wedge d'_\vee(b, a) = 0$ . Since  $D$  is 0-distributive, it suffices to prove that for any  $(x, y) \in D_{a, b}$ ,  $a \leq x$  and  $y \leq b$  implies that  $d'_\wedge(a, b) \wedge d'_\vee(y, x) = 0$ . Since  $d'(x, y)$  is a meetand of  $d'_\wedge(a, b)$ , we get  $d'(a, b) \leq d'(x, y)$ . By our induction hypothesis,  $d'(x, y) \wedge d'(y, x) = 0$ , so we are done.
- $d'_\vee(a, b) \wedge d'_\wedge(b, a) = 0$ . This case is symmetric to the case above.
- $d'_\vee(a, b) \wedge d'_\vee(b, a) = 0$ . Since  $D$  is 0-distributive, it suffices to prove that for any  $(x, y), (u, v) \in D_{a, b}$ ,  $x \leq a \leq u$  and  $v \leq b \leq y$  implies that  $d'(x, y) \wedge d'(v, u) = 0$ . Since  $\sqsubseteq$  is a total ordering, either  $v \sqsubseteq y$  or  $y \sqsubseteq v$ . In the first case, then (cf. Lemma 4.2)  $\{x, v\} \sqsubseteq \{x, y\} \triangleleft \{a, b\}$ , thus, by our induction hypothesis,  $d'(x, v) \wedge d'(v, x) = 0$ . Since  $d'(x, y) \leq d'(x, v)$  and  $d'(v, u) \leq d'(v, x)$ , the desired conclusion follows. In the second case, then (cf. Lemma 4.2)  $\{u, y\} \sqsubseteq \{u, v\} \triangleleft \{a, b\}$ , thus, by our induction hypothesis,  $d'(u, y) \wedge d'(y, u) = 0$ . Since  $d'(x, y) \leq d'(u, y)$  and  $d'(v, u) \leq d'(y, u)$ , the desired conclusion follows.  $\square$

By taking  $M = D$ ,  $f = \text{id}_D$ , and  $d$  any deviation on  $D$  in Propositions 4.5, 4.6, and 4.7, we obtain immediately the following.

**Theorem 4.8.** *Every finitely separable completely normal lattice has a monotone deviation.*

## 5. LATTICES OF PRINCIPAL $\ell$ -IDEALS IN ABELIAN $\ell$ -GROUPS

We will always denote  $\ell$ -groups additively and set  $|a| \stackrel{\text{def}}{=} a \vee (-a)$  whenever  $a \in G$ . The convex  $\ell$ -subgroup generated by  $a$  is

$$\langle a \rangle \stackrel{\text{def}}{=} \{x \in G \mid -n|a| \leq x \leq n|a| \text{ for some } n \in \omega\}.$$

Clearly, for any  $a, b \in G^+$ ,  $\langle a \rangle \subseteq \langle b \rangle$  iff  $a \leq mb$  for some  $m \in \omega$ . All these convex  $\ell$ -subgroups form a completely normal lattice  $\text{Cs}_c G$ , often denoted  $\text{Id}_c G$  if  $G$  is Abelian (for in that case,  $\ell$ -ideals and convex  $\ell$ -subgroups are the same). A deviation on  $\text{Cs}_c G$  can be defined by choosing a positive generator for each convex  $\ell$ -subgroup and setting

$$\langle a \rangle \setminus \langle b \rangle \stackrel{\text{def}}{=} \langle (a - b)^+ \rangle,$$

where  $x^+ \stackrel{\text{def}}{=} x \vee 0$ . Of course, this operation depends on the choice of the generators  $a, b$ . It is always Cevian (cf. [15, Proposition 5.5]). It will be monotone if we could choose the generators in an isotone way, that is  $\langle a \rangle \subseteq \langle b \rangle$  implies  $a \leq b$ . This argument yields the following observation.

**Theorem 5.1.** *Let  $D$  be a distributive 0-lattice and suppose that there exists a surjective lattice homomorphism  $f: \text{Cs}_c G \rightarrow D$  for some  $\ell$ -group  $G$ . If  $D$  is finitely separable, then it is monotone-Cevian.*

*Note.* Since every homomorphic image of a completely normal lattice is completely normal, our assumptions imply that  $D$  is completely normal.

*Proof.* The map  $\beta: G^+ \rightarrow \text{Cs}_c G$ ,  $x \mapsto \langle x \rangle$  ( $\beta$  is sometimes called the *Belluce map*), is a surjective lattice homomorphism (cf. Bigard *et al.* [3, Proposition 2.2.11]). Since  $D$  is finitely separable, by Theorem 2.2 the composite map  $f \circ \beta$  has an isotone section  $\sigma: D \hookrightarrow G^+$ . The assignment  $(x, y) \mapsto f\langle(\sigma(x) - \sigma(y))^+\rangle$  defines a monotone-Cevian operation on  $D$ .  $\square$

*Remark 5.2.* Theorem 5.1 trivially extends to finitely separable homomorphic images of  $\ell$ -ideal lattices of vector lattices. Indeed, for any vector lattice  $E$  over a totally ordered division ring  $\mathbb{k}$ , with underlying Abelian  $\ell$ -group  $G$ , the assignment  $\langle x \rangle \mapsto \mathbb{k} \cdot \langle x \rangle$  defines a surjective lattice homomorphism  $\text{Id}_c G \rightarrow \text{Id}_c E$  (which is an isomorphism if  $\mathbb{k}$  is Archimedean).

The main result of Ploščica and Wehrung [12] states that *every completely normal lattice with at most  $\aleph_1$  elements is a homomorphic image of  $\text{Id}_c G$  for some Abelian  $\ell$ -group  $G$* . Hence, we get the following variant of Ploščica [11, Theorem 3.2], obtained by strengthening there both the assumption (the lattice is now assumed to be finitely separable) and the conclusion (“Cevian” becomes “monotone-Cevian”).

**Corollary 5.3.** *Every finitely separable completely normal lattice with at most  $\aleph_1$  elements is monotone-Cevian.*

By Theorem 5.1, if  $\text{Id}_c G$  is finitely separable, then it is monotone-Cevian. In the sequel we shall verify that this is the case for free Abelian  $\ell$ -groups. We find it convenient to work in the slightly more general setting of vector lattices. Let  $\mathbb{k}$  be a countable totally ordered division ring and let  $E = \mathbb{k}^{(I)}$  be the left vector space over  $\mathbb{k}$  with basis  $I$ . For  $J \subseteq I$  we identify  $\mathbb{k}^{(J)}$  with its canonical copy in  $E$ . For every  $a \in E$  we set

$$[a] \stackrel{\text{def}}{=} \{x \in E \mid (a \mid x) > 0\},$$

(where  $(a \mid x) \stackrel{\text{def}}{=} \sum_{i \in I} a_i x_i$ ). Following Wehrung [14, 16], we denote by  $\text{Op}^- \mathbb{k}^{(J)}$  the 0-sublattice of the powerset of  $E$  generated by  $\{[x] \mid x \in \mathbb{k}^{(J)}\}$ . In other words,  $\text{Op}^- \mathbb{k}^{(J)}$  is the lattice of all subsets of  $E$  defined as disjunctions of conjunctions of strict linear inequalities with variables in  $J$ . Further, let  $\text{Op} \mathbb{k}^{(J)} = \text{Op}^- \mathbb{k}^{(J)} \cup \{E\}$ . (Notice that  $E \notin \text{Op}^- \mathbb{k}^{(J)}$ , so we are adding a new top element.) With the above conventions,  $\text{Op} \mathbb{k}^{(J)}$  is thus identified with a 0-sublattice of  $\text{Op} \mathbb{k}^{(I)}$ .

**Lemma 5.4.** *For every  $X \in [I]^{<\omega}$ , the lattice  $\text{Op} \mathbb{k}^{(X)}$  is a finitely shadowing subset of  $\text{Op} E$ .*

*Proof.* (See also the proof of Ploščica and Wehrung [12, Lemma 12].) We claim that the upper and lower shadow of any  $U \in \text{Op} E$  are, respectively,  $\{U^*\}$  and  $\{U_*\}$ , where

$$\begin{aligned} U^* &\stackrel{\text{def}}{=} \{v \in E \mid v \upharpoonright_X = u \upharpoonright_X \text{ for some } u \in U\}, \\ U_* &\stackrel{\text{def}}{=} \{v \in E \mid u \in U \text{ whenever } v \upharpoonright_X = u \upharpoonright_X\}. \end{aligned}$$

First, the above defined sets are both elements of  $\text{Op} \mathbb{k}^{(X)}$ : this follows from quantifier elimination for the theory of all nontrivial totally ordered  $\mathbb{k}$ -vector spaces (cf. van den Dries [13, Corollary I.7.8]).

Clearly,  $U_* \subseteq U \subseteq U^*$ . Now let  $Z \in \text{Op}\mathbb{k}^{(X)}$ ,  $U \subseteq Z$ . We verify that  $U^* \subseteq Z$ . It suffices to consider the case when  $Z = \bigcup_{a \in M} [a]$  for a finite  $M \subseteq \mathbb{k}^{(X)}$ . Now, if  $v \in U^*$ , then  $v|_X = u|_X$  for some  $u \in U \subseteq Z$ . Hence,  $(a | u) > 0$  for some  $a \in M$ . Since  $a_i = 0$  for  $i \notin X$ , we also have  $(a | v) = (a | u) > 0$ , hence  $v \in Z$ .

Similarly, let  $Z \in \text{Op}\mathbb{k}^{(X)}$ ,  $Z \subseteq U$ . We verify that  $Z \subseteq U_*$ . It suffices to consider the case when  $Z = \bigcap_{a \in M} [a]$  for a finite  $M \subseteq \mathbb{k}^{(X)}$ . Now, let  $v \in Z \subseteq U$  and  $u \in E$  with  $v|_X = u|_X$ . Then  $(a | v) = (a | u) > 0$  for every  $a \in M$ , hence  $u \in U$ . By the definition,  $v \in U_*$ .  $\square$

**Lemma 5.5.** *Let  $X, Y \in [I]^{<\omega}$ ,  $U \in \text{Op}\mathbb{k}^{(X)}$ ,  $V \in \text{Op}\mathbb{k}^{(Y)}$ , and  $U \subseteq V$ . Then  $U \subseteq W \subseteq V$  for some  $W \in \text{Op}\mathbb{k}^{(X \cap Y)}$ .*

*Proof.* Since quantifier elimination does not add new variables, the upper shadow  $U^*$  on  $\mathbb{k}^{(Y)}$  belongs to  $\mathbb{k}^{(X \cap Y)}$ . From  $U \subseteq V$  we obtain  $U^* \subseteq V$ , so we can set  $W = U^*$ .  $\square$

**Proposition 5.6.** *The distributive 0-lattices  $\text{Op}E$  and  $\text{Op}^-E$  are both finitely separable.*

*Proof.* By Lemmas 5.4 and 5.5,  $\text{Op}\mathbb{k}^{(I)}$  is the strong amalgam (cf. Definition 2.6) of all  $\text{Op}\mathbb{k}^{(X)}$  for  $X \in [I]^{<\omega}$ . Since  $\mathbb{k}$  is countable, so are all  $\text{Op}\mathbb{k}^{(X)}$ , which are therefore finitely separable (cf. Proposition 2.3(2)). By Proposition 2.7,  $\text{Op}\mathbb{k}^{(I)}$  is finitely separable. By Proposition 2.3(4), so is thus  $\text{Op}^- \mathbb{k}^{(I)} = (\text{Op}\mathbb{k}^{(I)}) \setminus \{E\}$ .  $\square$

Note that by the Baker-Bernau-Madden duality (cf. Baker [1], Bernau [2], Madden [9, Ch. III], and also Wehrung [16, page 13] for a summary),  $\text{Op}^- \mathbb{k}^{(I)}$  is isomorphic to the lattice of all principal  $\ell$ -ideals of the free  $\mathbb{k}$ -vector lattice on  $I$ . Hence, by applying Theorem 5.1 and Remark 5.2, we obtain the following.

**Corollary 5.7.** *Let  $E$  be a left vector space over a countable totally ordered division ring  $\mathbb{k}$ . Then the distributive 0-lattices  $\text{Op}E$  and  $\text{Op}^-E$  are both monotone-Cevian.*

If we wanted only “existence of a monotone deviation” instead of “monotone-Cevian” in Corollary 5.7, then it would be sufficient to use Theorem 4.8 instead of Theorem 5.1 (the latter itself following from Freese and Nation [5, Theorem 1]).

**Corollary 5.8.** *For any set  $I$  and any countable totally ordered division ring  $\mathbb{k}$ , the lattice of all principal  $\ell$ -ideals of the free  $\mathbb{k}$ -vector lattice on  $I$  is monotone-Cevian.*

If  $\mathbb{k} = \mathbb{Q}$  is the ordered field of rationals, then the free  $\mathbb{Q}$ -vector lattice  $F$  on  $I$  is the divisible hull of the free Abelian  $\ell$ -group  $G$  on  $I$ , thus the lattices of principal  $\ell$ -ideals of  $F$  and  $G$  are both isomorphic to  $\text{Op}^- \mathbb{Q}^{(I)}$ .

**Corollary 5.9.** *For any set  $I$ , the lattice of all principal  $\ell$ -ideals of the free Abelian  $\ell$ -group on  $I$  is monotone-Cevian.*

## 6. AN ARCHIMEDEAN VECTOR LATTICE COUNTEREXAMPLE

Since the lattice of all principal  $\ell$ -ideals of any countable Abelian  $\ell$ -group  $G$  is countable, thus finitely separable, the Belluce map of  $G$  has in that case an isotone section; hence  $\text{Id}_c G$  is monotone-Cevian (cf. Corollary 5.3). In this section we will verify that that observation does not extend to the uncountable case, by constructing an Archimedean vector lattice  $G$  with strong unit, of cardinality  $\aleph_1$ ,

whose lattice of all principal  $\ell$ -ideals does not have any monotone deviation (thus, *a fortiori*, no isotone section for the Belluce map; see Theorem 5.1).

We begin with a refinement of Shanin's classical  $\Delta$ -Lemma (see for example Jech [8, Theorem 9.18]).

**Lemma 6.1.** *Let  $\Phi: \omega_1 \rightarrow [\omega_1]^{<\omega}$  and let  $X$  be a cofinal subset of  $\omega_1$ . Then there are ascending  $\omega_1$ -sequences  $(\tau_i \mid i < \omega_1)$  and  $(\alpha_i \mid i < \omega_1)$  of ordinals in  $\omega_1$  such that for every  $k < \omega_1$ ,*

- (1)  $\alpha_k \in X$  and  $\tau_k < \alpha_k < \tau_{k+1}$ ;
- (2)  $\tau_k = \sup_{j < k} \tau_j$  if  $k$  is a limit ordinal;
- (3)  $\Phi(\alpha_k) \subseteq \tau_0 \cup (\tau_{k+1} \setminus \tau_k)$ .

*Proof.* By the  $\Delta$ -Lemma, there are a cofinal subset  $Y$  of  $X$  and a set  $K$  such that

$$\Phi(\alpha) \cap \Phi(\beta) = K \text{ whenever } \alpha, \beta \in Y \text{ and } \alpha \neq \beta. \quad (6.1)$$

Pick  $\tau_0 < \omega_1$  such that  $K \subseteq \tau_0$ . Let  $i < \omega_1$  and set

$$i^* \stackrel{\text{def}}{=} \begin{cases} i, & \text{if } i \text{ is a limit ordinal,} \\ i + 1, & \text{otherwise.} \end{cases}$$

We argue by induction on  $i$ . Suppose having constructed ascending sequences  $(\tau_k \mid k < i^*)$  and  $(\alpha_k \mid k < i)$  satisfying (1)–(3) whenever  $k < i$ . If  $i$  is either 0 or a successor, then  $\tau_i$  is already defined. If  $i$  is a limit, set  $\tau_i \stackrel{\text{def}}{=} \sup_{j < i} \tau_j$ ; this takes care of (2) at level  $i$ . By the cofinality statement on  $Y$ , the countability of  $\tau_i$ , and (6.1), there exists  $\alpha_i \in Y$  such that  $\alpha_i > \tau_i$  and  $\Phi(\alpha_i) \cap \tau_i \subseteq K$ . Pick  $\tau_{i+1} < \omega_1$  such that  $\tau_{i+1} > \alpha_i$  and  $\Phi(\alpha_i) \subseteq \tau_{i+1}$ . Since  $K \subseteq \tau_0$ , (1) and (3) at level  $i$  follow.  $\square$

By the Baker-Bernau-Madden duality, the free  $\mathbb{Q}$ -vector lattice  $F(\omega_1)$  on  $\omega_1 + 1$  is isomorphic to the vector sublattice  $F$  of  $\mathbb{Q}^{\mathbb{Q}^{\omega_1+1}}$  generated by all canonical projections  $x_\alpha: \mathbb{Q}^{\omega_1+1} \rightarrow \mathbb{Q}$  for  $\alpha \in \omega_1 + 1$ . (Here, unlike in Section 5, we use  $\mathbb{Q}^{\omega_1+1}$  instead of  $\mathbb{Q}^{(\omega_1+1)}$ .) Every element of  $F$  has the form  $\dot{f}(x_\alpha \mid \alpha \in \omega_1 + 1)$  for some vector lattice term  $\dot{f}$ . By the above-cited duality,  $\text{Id}_c F$  is isomorphic to the 0-sublattice of the powerset lattice of  $\mathbb{Q}^{\omega_1}$  generated by the cozero sets  $\{u \in \mathbb{Q}^{\omega_1+1} \mid g(u) \neq 0\}$ , the latter corresponding to the principal  $\ell$ -ideal  $\langle g \rangle$ . This leads immediately to the following assertion.

**Lemma 6.2.** *For any  $g, h \in F^+$ , the inequality  $\langle g \rangle \leq \langle h \rangle$  holds (within  $\text{Id}_c F$ ) iff  $h^{-1}\{0\} \subseteq g^{-1}\{0\}$ .*

Now we introduce our counterexample. Let us denote

$$\begin{aligned} \Omega_M &\stackrel{\text{def}}{=} \{u \in \mathbb{Q}^{\omega_1} \mid 0 \leq u_\alpha \leq 1 \text{ whenever } \alpha \in M \\ &\quad \text{and } u_\gamma \leq 2u_\beta \text{ whenever } \beta, \gamma \in M \text{ and } 0 < \gamma < \beta\} \end{aligned}$$

whenever  $M \subseteq \omega_1$ , and set  $\Omega \stackrel{\text{def}}{=} \Omega_{\omega_1}$ . In the sequel, let  $G$  be the vector sublattice of  $\mathbb{Q}^\Omega$  generated by all canonical projections  $p_\alpha: \Omega \rightarrow \mathbb{Q}$  for  $\alpha \in \omega_1$  together with the constant function  $\Omega \rightarrow \mathbb{Q}$ ,  $u \mapsto 1$ , which we shall conveniently denote  $p_{\omega_1}$  even though it is not a “projection”.

**Lemma 6.3.** *Let  $f_0: \{p_\alpha \mid \alpha \in \omega_1\} \rightarrow \mathbb{Q}$  satisfy  $0 \leq f_0(p_\alpha) \leq 1$  whenever  $\alpha < \omega_1$  and  $f_0(p_\gamma) \leq 2f_0(p_\beta)$  whenever  $0 < \gamma < \beta < \omega_1$ . Then  $f_0$  can be extended to a unique  $\ell$ -homomorphism  $G \rightarrow \mathbb{Q}$  mapping  $p_{\omega_1}$  to 1.*

*Proof.* We only need to verify the existence part. The vector  $w \stackrel{\text{def}}{=} (f_0(p_\alpha) \mid \alpha \in \omega_1)$  belongs to  $\Omega$ . The evaluation morphism  $e_w: g \mapsto g(w)$  is the desired extension.  $\square$

**Lemma 6.4.** *Let  $f_0: \{p_\alpha \mid \alpha \in \omega_1\} \rightarrow G$  satisfy  $0 \leq f_0(p_\alpha) \leq p_{\omega_1}$  whenever  $\alpha < \omega_1$  and  $f_0(p_\gamma) \leq 2f_0(p_\beta)$  whenever  $0 < \gamma < \beta < \omega_1$ . Then  $f_0$  can be extended to a unique  $\ell$ -endomorphism of  $G$  mapping  $p_{\omega_1}$  to itself.*

*Proof.* We use the fact that  $G$  is a vector sublattice of  $\mathbb{Q}^\Omega$  via the evaluation morphisms  $e_u$  for  $u \in \Omega$ . By Lemma 6.3, each map  $f_0^u \stackrel{\text{def}}{=} e_u f_0$  can be extended to an  $\ell$ -homomorphism  $f_u: G \rightarrow \mathbb{Q}$  such that  $f_u(p_{\omega_1}) = 1$ . Then the product homomorphism  $f \stackrel{\text{def}}{=} \prod_{u \in \Omega} f_u$  extends  $f_0$  and  $f(p_{\omega_1}) = p_{\omega_1}$ .  $\square$

**Lemma 6.5.** *Let  $M$  be a subset of  $\omega_1$ . Then for every  $u \in \Omega_M$  there exists  $v \in \Omega$  such that  $u \upharpoonright_M = v \upharpoonright_M$ .*

*Proof.* Set  $M \uparrow \xi \stackrel{\text{def}}{=} \{\alpha \in M \mid \alpha \geq \xi\}$  whenever  $\xi < \omega_1$ , and define

$$v_\xi \stackrel{\text{def}}{=} \begin{cases} u_\tau, & \text{if } M \uparrow \xi \neq \emptyset \text{ and } \tau = \min M \uparrow \xi, \\ 1, & \text{otherwise,} \end{cases} \quad \text{whenever } \xi < \omega_1.$$

It is easy to see that  $v \in \Omega$ .  $\square$

**Lemma 6.6.** *For any  $g, h \in G^+$ , the inequality  $\langle g \rangle \leq \langle h \rangle$  holds (within  $\text{Id}_c G$ ) iff  $h^{-1}\{0\} \subseteq g^{-1}\{0\}$ .*

*Proof.* We verify the nontrivial direction. Suppose that  $h^{-1}\{0\} \subseteq g^{-1}\{0\}$ . Since  $\{g, h\} \subseteq G$  there are a finite subset  $M$  of  $\omega_1$  and vector lattice terms  $\dot{g}$  and  $\dot{h}$  such that

$$g = \dot{g}(p_\alpha \mid \alpha \in M \cup \{\omega_1\}) \text{ and } h = \dot{h}(p_\alpha \mid \alpha \in M \cup \{\omega_1\}).$$

We can assume  $M \neq \emptyset$ . Define elements  $d, g_1, h_1$  of  $F$  as

$$d \stackrel{\text{def}}{=} \bigvee_{\alpha \in M} (-x_\alpha) \vee \bigvee_{\alpha \in M} (x_\alpha - x_{\omega_1}) \vee \bigvee_{\gamma, \beta \in M, 0 < \gamma < \beta} (x_\gamma - 2x_\beta),$$

then  $g_1 \stackrel{\text{def}}{=} \dot{g}(x_\alpha \mid \alpha \in M \cup \{\omega_1\}) \vee d$  and  $h_1 \stackrel{\text{def}}{=} \dot{h}(x_\alpha \mid \alpha \in M \cup \{\omega_1\}) \vee d$ . We denote

$$\begin{aligned} \overline{\Omega}_M &\stackrel{\text{def}}{=} \{u \in \mathbb{Q}^{\omega_1+1} \mid 0 \leq u_\alpha \leq u_{\omega_1} \text{ whenever } \alpha \in M \\ &\quad \text{and } u_\gamma \leq 2u_\beta \text{ whenever } \beta, \gamma \in M \text{ and } 0 < \gamma < \beta\}. \end{aligned}$$

**Claim.** *Let  $u \in \overline{\Omega}_M$ . Then  $g_1(u) \geq 0$  and  $h_1(u) \geq 0$ , and, further,  $h_1(u) = 0$  implies that  $g_1(u) = 0$ .*

*Proof of Claim.* The assumption  $u \in \overline{\Omega}_M$  implies  $d(u) \leq 0$ . The case  $u_{\omega_1} < 0$  is excluded as  $0 \leq u_\alpha \leq u_{\omega_1}$  for  $\alpha \in M \neq \emptyset$ . If  $u_{\omega_1} = 0$  then  $u_\alpha = 0$  for every  $\alpha \in M$ , so  $g_1(u) = h_1(u) = 0$  and we are done. Suppose now that  $u_{\omega_1} > 0$ . Consider the element  $u' \stackrel{\text{def}}{=} u_{\omega_1}^{-1} \cdot u$ . Then  $u' \upharpoonright_{\omega_1}$  belongs to  $\Omega_M$  and it follows from Lemma 6.5 that  $u' \upharpoonright_M = v \upharpoonright_M$  for some  $v \in \Omega$ . We obtain

$$\dot{g}(u_\alpha \mid \alpha \in M \cup \{\omega_1\}) = u_{\omega_1} \dot{g}(u'_\alpha \mid \alpha \in M \cup \{\omega_1\}) = u_{\omega_1} g(v).$$

(For the last equality notice that  $u'_{\omega_1} = 1 = p_{\omega_1}(v)$ .) Since  $g \in G^+$  we get  $g(v) \geq 0$ . Since  $d(u) \leq 0$ , we obtain

$$g_1(u) = u_{\omega_1} g(v) \vee d(u) = u_{\omega_1} g(v) \geq 0,$$

and similarly,

$$h_1(u) = u_{\omega_1} h(v) \geq 0.$$

Hence  $h_1(u) = 0$  implies  $h(v) = 0$ , thus (since  $v \in \Omega$ )  $g(v) = 0$ , and thus  $g_1(u) = 0$ .  $\square$  Claim.

On the other hand, for any  $u \in \mathbb{Q}^{\omega_1+1} \setminus \overline{\Omega}_M$ ,  $d(u) > 0$  and hence  $g_1(u), h_1(u) > 0$ . Therefore, it follows from the Claim above that  $\{g_1, h_1\} \subseteq F^+$  and  $h_1^{-1}\{0\} \subseteq g_1^{-1}\{0\} \subseteq \Omega_M$ . By Lemma 6.2, the latter relation entails  $\langle g_1 \rangle \leq \langle h_1 \rangle$  (within  $\text{Id}_c F$ ), that is,  $g_1 \leq mh_1$  for some  $m > 0$ . Then for every  $u \in \Omega$  (so  $u \wedge (1) \in \Omega_M$ ),  $g(u) = g_1(u \wedge (1)) \leq mh_1(u \wedge (1)) = mh(u)$ , thus  $g \leq mh$ , and thus  $\langle g \rangle \leq \langle h \rangle$  (within  $\text{Id}_c G$ ).  $\square$

**Lemma 6.7.** *For every  $\alpha \in \omega_1 \setminus \{0\}$  and every positive  $c \in \mathbb{Q}$ ,  $\langle (p_0 - cp_\alpha)^+ \rangle$  and  $\langle (cp_\alpha - p_0)^+ \rangle$  are pseudocomplements of each other within  $\text{Id}_c G$ .*

*Proof.* Trivially  $\langle (p_0 - cp_\alpha)^+ \rangle \wedge \langle (cp_\alpha - p_0)^+ \rangle = 0$ .

Now let  $t \in G^+$ . Assume first that  $\langle t \rangle \wedge \langle (p_0 - cp_\alpha)^+ \rangle = 0$ . By way of contradiction, suppose that  $\langle t \rangle \not\leq \langle (cp_\alpha - p_0)^+ \rangle$ . By Lemma 6.6, there exists  $z \in \Omega$  such that  $t(z) > 0$  whereas  $cz_\alpha \leq z_0$ . From  $\langle t \rangle \wedge \langle (p_0 - cp_\alpha)^+ \rangle = 0$  it thus follows that  $z_0 \leq cz_\alpha$ , thus  $z_0 = cz_\alpha$ . We separate cases.

- If  $z_0 < 1$ , let  $y \in \Omega$  be defined by  $y_0 = z_0 + \varepsilon$  whereas  $y_\beta = z_\beta$  whenever  $0 < \beta < \omega_1$ , with  $\varepsilon > 0$  chosen small enough to ensure  $z_0 + \varepsilon \leq 1$  and  $t(y) > 0$ . This is possible since  $t(y)$  depends only on a finite number of components of  $y$ . Then  $y_0 - cy_\alpha = (z_0 - cz_\alpha) + \varepsilon = \varepsilon > 0$ , in contradiction with  $\langle t \rangle \wedge \langle (p_0 - cp_\alpha)^+ \rangle = 0$ .
- If  $z_0 = 1$ , so  $z_\alpha > 0$ , let  $y \in \Omega$  be defined by  $y_0 = z_0$  whereas  $y_\beta = (1 - \varepsilon)z_\beta$  whenever  $0 < \beta < \omega_1$ , with  $\varepsilon > 0$  chosen small enough to ensure  $t(y) > 0$ . Then  $y_0 - cy_\alpha = (z_0 - cz_\alpha) + \varepsilon z_\alpha = \varepsilon z_\alpha > 0$ , in contradiction with  $\langle t \rangle \wedge \langle (p_0 - cp_\alpha)^+ \rangle = 0$ .

Assume now that  $\langle t \rangle \wedge \langle (cp_\alpha - p_0)^+ \rangle = 0$ . By way of contradiction, suppose that  $\langle t \rangle \not\leq \langle (p_0 - cp_\alpha)^+ \rangle$ . By Lemma 6.6, there exists  $z \in \Omega$  such that  $t(z) > 0$  whereas  $cz_\alpha \geq z_0$ . From  $\langle t \rangle \wedge \langle (cp_\alpha - p_0)^+ \rangle = 0$  it thus follows that  $cz_\alpha \leq z_0$ , thus  $z_0 = cz_\alpha$ . We separate cases.

- If  $z_0 > 0$ , let  $y \in \Omega$  be defined by  $y_0 = (1 - \varepsilon)z_0$  whereas  $y_\beta = z_\beta$  whenever  $0 < \beta < \omega_1$ , with  $\varepsilon > 0$  chosen small enough to ensure  $t(y) > 0$ . Then  $cy_\alpha - y_0 = (cz_\alpha - z_0) + \varepsilon z_0 = \varepsilon z_0 > 0$ , in contradiction with  $\langle t \rangle \wedge \langle (cp_\alpha - p_0)^+ \rangle = 0$ .
- If  $z_0 = 0$ , so  $z_\alpha = 0$ , let  $y \in \Omega$  be defined by  $y_0 = z_0$  whereas  $y_\beta = \min\{z_\beta + \varepsilon, 1\}$  whenever  $0 < \beta < \omega_1$ , with  $\varepsilon > 0$  chosen small enough to ensure  $\varepsilon < 1$  and  $t(y) > 0$ . Then  $cy_\alpha - y_0 = (cz_\alpha - z_0) + c\varepsilon = c\varepsilon > 0$ , in contradiction with  $\langle t \rangle \wedge \langle (cp_\alpha - p_0)^+ \rangle = 0$ .  $\square$

**Theorem 6.8.** *The vector lattice  $G$  is Archimedean with strong unit. Furthermore, no deviation on  $\text{Id}_c G$  can be either left isotone or right antitone. In fact, for every deviation  $\setminus$  on  $\text{Id}_c G$  there are ordinals  $\alpha, \beta, \alpha', \beta'$  such that  $0 < \alpha < \beta < \omega_1$ ,  $0 < \alpha' < \beta' < \omega_1$ ,  $\langle p_0 \rangle \setminus \langle p_\beta \rangle \not\leq \langle p_0 \rangle \setminus \langle p_\alpha \rangle$ , and  $\langle p_{\alpha'} \rangle \setminus \langle p_0 \rangle \not\leq \langle p_{\beta'} \rangle \setminus \langle p_0 \rangle$ .*

*Proof.* Since  $G$  is an  $\ell$ -subgroup of a power of  $\mathbb{Q}$  it is Archimedean. Moreover,  $p_{\omega_1}$  is a strong unit of  $G$ .



Let  $\setminus$  be a deviation on  $\text{Id}_c G$  and suppose first that

$$\langle p_0 \rangle \setminus \langle p_\beta \rangle \leq \langle p_0 \rangle \setminus \langle p_\alpha \rangle \quad \text{whenever } 0 < \alpha < \beta < \omega_1. \quad (6.2)$$

Whenever  $0 < \alpha < \omega_1$  let  $s_\alpha \in G^+$  with  $\langle s_\alpha \rangle = \langle p_0 \rangle \setminus \langle p_\alpha \rangle$ . There exists a finite subset  $\Phi(\alpha)$  of  $\omega_1$  such that  $s_\alpha$  belongs to the vector sublattice of  $G$  generated by  $\{p_\beta \mid \beta \in \Phi(\alpha) \cup \{\omega_1\}\}$ .

It is well known that the elements  $\langle (p_0 - mp_\alpha)^+ \rangle$  (with  $m \in \omega$ ) form a cointial subset in the set  $\{\mathbf{x} \in \text{Id}_c G \mid \langle p_0 \rangle \leq \langle p_\alpha \rangle \vee \mathbf{x}\}$ . Hence,  $\langle (p_0 - mp_\alpha)^+ \rangle \leq \langle s_\alpha \rangle$  for some positive  $m$ . Similarly,  $\langle (p_\alpha - np_0)^+ \rangle \leq \langle p_\alpha \rangle \setminus \langle p_0 \rangle$  for some positive  $n$ . Hence,

$$\langle s_\alpha \rangle \wedge \langle (p_\alpha - np_0)^+ \rangle \leq (\langle p_0 \rangle \setminus \langle p_\alpha \rangle) \wedge (\langle p_\alpha \rangle \setminus \langle p_0 \rangle) = 0.$$

By Lemma 6.7,  $\langle s_\alpha \rangle \leq \langle (np_0 - p_\alpha)^+ \rangle$ . We have thus obtained the following.

**Claim 1.** *Whenever  $0 < \alpha < \omega_1$  there are positive integers  $m, n$  such that*

$$\langle (p_0 - mp_\alpha)^+ \rangle \leq \langle s_\alpha \rangle \leq \langle (np_0 - p_\alpha)^+ \rangle.$$

For all positive integers  $m, n$  denote

$$X_{m,n} \stackrel{\text{def}}{=} \{\alpha \in \omega_1 \setminus \{0\} \mid \langle (p_0 - mp_\alpha)^+ \rangle \leq \langle s_\alpha \rangle \leq \langle (np_0 - p_\alpha)^+ \rangle\}.$$

By Claim 1, at least one of those sets must be cofinal in  $\omega_1$ ; we choose such  $(m, n)$  and set  $X \stackrel{\text{def}}{=} X_{m,n}$ . Further, we pick a positive integer  $k$  such that  $2^{k-1} > mn$ .

Applying Lemma 6.1 with  $X$  and  $\Phi$  defined above, we find corresponding ordinals  $\tau_i$  and  $\alpha_i$  (for  $i < \omega_1$ ). We may assume that  $\tau_0 > 0$ . Let us keep only those  $\tau_i, \alpha_i$  with  $0 \leq i < k$ , and reset  $\tau_k \stackrel{\text{def}}{=} \omega_1$ . Let  $f: \{p_\gamma \mid \gamma \in \omega_1\} \rightarrow G$  be defined as follows:

$$f(p_\gamma) \stackrel{\text{def}}{=} \begin{cases} p_0, & \text{if } \gamma \in \tau_0, \\ p_{\alpha_i}, & \text{if } \gamma \in \tau_{i+1} \setminus \tau_i \end{cases} \quad \text{whenever } \gamma \in \omega_1. \quad (6.3)$$

Since  $0 \leq f(p_\alpha) \leq p_{\omega_1}$  for any  $\alpha \in \omega_1$  and since  $f(p_\gamma) \leq 2f(p_\delta)$  whenever  $0 < \gamma \leq \delta$ , the map  $f$  can, by Lemma 6.4, be extended to an  $\ell$ -endomorphism of  $G$ , which we shall also denote by  $f$ , such that  $f(p_{\omega_1}) = p_{\omega_1}$ . The definition of  $\Phi$  ensures that each  $f(s_{\alpha_i})$  belongs to the vector sublattice of  $G$  generated by  $\{p_0, p_{\alpha_i}, p_{\omega_1}\}$ .

**Claim 2.** *The inequality  $\langle f(s_{\alpha_{k-i}}) \rangle \geq \langle (2^{i-1}p_0 - mp_{\alpha_{k-i}})^+ \rangle$  holds whenever  $1 \leq i \leq k$ .*

*Proof of Claim.* We proceed by induction. For  $i = 1$  our statement follows from the inequality  $\langle (p_0 - mp_{\alpha_{k-1}})^+ \rangle \leq \langle s_{\alpha_{k-1}} \rangle$ . (Notice that  $f(p_0) = p_0$  and  $f(p_{\alpha_j}) = p_{\alpha_j}$  for every  $j$ , thus  $f((p_0 - mp_{\alpha_{k-1}})^+) = (p_0 - mp_{\alpha_{k-1}})^+.$ )

Now let  $i > 1$  and suppose for contradiction that  $\langle f(s_{\alpha_{k-i}}) \rangle \not\geq \langle (2^{i-1}p_0 - mp_{\alpha_{k-i}})^+ \rangle$ . By Lemma 6.6, there is  $z \in \Omega$  such that  $f(s_{\alpha_{k-i}})(z) \leq 0$  and  $2^{i-1}z_0 > mz_{\alpha_{k-i}}$ . We define  $y \in \Omega$  by

$$y_\beta \stackrel{\text{def}}{=} \begin{cases} z_0, & \text{if } \beta = 0, \\ z_{\alpha_{k-i}}, & \text{if } 0 < \beta \leq \alpha_{k-i}, \\ \frac{1}{2}z_{\alpha_{k-i}}, & \text{if } \beta > \alpha_{k-i}, \end{cases} \quad \text{whenever } \beta < \omega_1. \quad (6.4)$$

Since  $f(s_{\alpha_{k-i}})$  belongs to the vector sublattice of  $G$  generated by  $\{p_0, p_{\alpha_{k-i}}, p_{\omega_1}\}$ , we get  $f(s_{\alpha_{k-i}})(y) = f(s_{\alpha_{k-i}})(z) \leq 0$ . Our assumption (6.2) entails the inequality

$\langle s_{\alpha_{k-i}} \rangle \geq \langle s_{\alpha_{k-i+1}} \rangle$ , hence  $\langle f(s_{\alpha_{k-i}}) \rangle \geq \langle f(s_{\alpha_{k-i+1}}) \rangle$  and therefore  $f(s_{\alpha_{k-i+1}})(y) \leq 0$ . Further,

$$2^{i-2}y_0 - my_{\alpha_{k-i+1}} = \frac{1}{2}(2^{i-1}z_0 - mz_{\alpha_{k-i}}) > 0,$$

which contradicts the induction hypothesis  $\langle f(s_{\alpha_{k-i+1}}) \rangle \geq \langle (2^{i-2}p_0 - mp_{\alpha_{k-i+1}})^+ \rangle$ .  $\square$  Claim 2.

For  $i = k$ , Claim 2 yields  $\langle f(s_{\alpha_0}) \rangle \geq \langle (2^{k-1}p_0 - mp_{\alpha_0})^+ \rangle$ . On the other hand,  $\alpha_0 \in X_{m,n}$  implies  $\langle s_{\alpha_0} \rangle \leq \langle (np_0 - p_{\alpha_0})^+ \rangle$ , hence  $\langle f(s_{\alpha_0}) \rangle \leq \langle f((np_0 - p_{\alpha_0})^+) \rangle = \langle (np_0 - p_{\alpha_0})^+ \rangle$ . Therefore,

$$\langle (2^{k-1}p_0 - mp_{\alpha_0})^+ \rangle \leq \langle (np_0 - p_{\alpha_0})^+ \rangle.$$

However, this inequality does not hold. Indeed, by Lemma 6.5 there exists  $z \in \Omega$  with  $z_0 = 1/n$  and  $z_{\alpha_0} = 1$ . Then  $nz_0 - z_{\alpha_0} = 0$ , while

$$2^{k-1}z_0 - mz_{\alpha_0} = 2^{k-1}/n - m > 0$$

according to the choice of  $k$ . This contradiction shows that (6.2) cannot hold.

The ‘‘left isotone’’ case is handled similarly. Suppose that

$$\langle p_\alpha \rangle \setminus \langle p_0 \rangle \leq \langle p_\beta \rangle \setminus \langle p_0 \rangle \quad \text{whenever } 0 < \alpha < \beta < \omega_1. \quad (6.5)$$

Whenever  $0 < \alpha < \omega_1$  let  $t_\alpha \in G^+$  with  $\langle t_\alpha \rangle = \langle p_\alpha \rangle \setminus \langle p_0 \rangle$ . There exists a finite subset  $\Phi(\alpha)$  of  $\omega_1$  such that  $t_\alpha$  belongs to the vector sublattice of  $G$  generated by  $\{p_\beta \mid \beta \in \Phi(\alpha) \cup \{\omega_1\}\}$ .

The proof of the following claim is then, *mutatis mutandis*, identical to the one of Claim 1, thus we shall omit it.

**Claim 3.** *Whenever  $0 < \alpha < \omega_1$  there are positive integers  $m, n$  such that*

$$\langle (p_\alpha - mp_0)^+ \rangle \leq \langle t_\alpha \rangle \leq \langle (np_\alpha - p_0)^+ \rangle.$$

For all positive integers  $m, n$  we now denote

$$Y_{m,n} \stackrel{\text{def}}{=} \{ \alpha \in \omega_1 \setminus \{0\} \mid \langle (p_\alpha - mp_0)^+ \rangle \leq \langle t_\alpha \rangle \leq \langle (np_\alpha - p_0)^+ \rangle \}.$$

By Claim 3, at least one of those sets must be cofinal in  $\omega_1$ ; we choose such  $(m, n)$  and set  $Y \stackrel{\text{def}}{=} Y_{m,n}$ . Again, we pick a positive integer  $k$  such that  $2^{k-1} > mn$ .

Applying Lemma 6.1 with  $Y$  and  $\Phi$  defined above, we find corresponding ordinals  $\tau_i$  and  $\alpha_i$  (for  $i < \omega_1$ ) with  $\tau_0 > 0$ . Again, we keep only those  $\tau_i, \alpha_i$  with  $0 \leq i < k$ , and reset  $\tau_k \stackrel{\text{def}}{=} \omega_1$ . The map  $f: \{p_\gamma \mid \gamma \in \omega_1\} \rightarrow G$  defined as in (6.3) can again be extended to a unique  $\ell$ -endomorphism of  $G$  such that  $f(p_{\omega_1}) = p_{\omega_1}$ , and the definition of  $\Phi$  ensures that each  $f(t_{\alpha_i})$  belongs to the vector sublattice of  $G$  generated by  $\{p_0, p_{\alpha_i}, p_{\omega_1}\}$ .

**Claim 4.** *The inequality  $\langle f(t_{\alpha_{k-i}}) \rangle \leq \langle (np_{\alpha_{k-i}} - 2^{i-1}p_0)^+ \rangle$  holds whenever  $1 \leq i \leq k$ .*

*Proof of Claim.* We proceed by induction, following the lines of the proof of Claim 2. For  $i = 1$  our statement follows from the inequality  $\langle t_{\alpha_{k-1}} \rangle \leq \langle (np_{\alpha_{k-1}} - p_0)^+ \rangle$ .

Now let  $i > 1$  and suppose for contradiction that  $\langle f(t_{\alpha_{k-i}}) \rangle \not\leq \langle (np_{\alpha_{k-i}} - 2^{i-1}p_0)^+ \rangle$ . By Lemma 6.6, there is  $z \in \Omega$  such that  $f(t_{\alpha_{k-i}})(z) > 0$  whereas  $nz_{\alpha_{k-i}} \leq 2^{i-1}z_0$ . We define  $y \in \Omega$  as in (6.4) and observe again that  $f(t_{\alpha_{k-i}})(y) = f(t_{\alpha_{k-i}})(z) > 0$ .

Our assumption (6.5) entails  $\langle t_{\alpha_{k-i}} \rangle \leq \langle t_{\alpha_{k-i+1}} \rangle$ , hence  $\langle f(t_{\alpha_{k-i}}) \rangle \leq \langle f(t_{\alpha_{k-i+1}}) \rangle$  and therefore  $f(t_{\alpha_{k-i+1}})(y) > 0$ . Further,

$$ny_{\alpha_{k-i+1}} - 2^{i-2}y_0 = \frac{1}{2}(nz_{\alpha_{k-i}} - 2^{i-1}z_0) \leq 0,$$

which contradicts the induction hypothesis  $\langle f(t_{\alpha_{k-i+1}}) \rangle \leq \langle (np_{\alpha_{k-i+1}} - 2^{i-2}p_0)^+ \rangle$ .  
 $\square$  Claim 4.

For  $i = k$ , Claim 4 yields  $\langle f(t_{\alpha_0}) \rangle \leq \langle (np_{\alpha_0} - 2^{k-1}p_0)^+ \rangle$ . On the other hand,  $\alpha_0 \in Y_{m,n}$  implies  $\langle (p_{\alpha_0} - mp_0)^+ \rangle \leq \langle t_{\alpha_0} \rangle$ , hence also  $\langle (p_{\alpha_0} - mp_0)^+ \rangle \leq \langle f(t_{\alpha_0}) \rangle$ . Therefore,

$$\langle (p_{\alpha_0} - mp_0)^+ \rangle \leq \langle (np_{\alpha_0} - 2^{k-1}p_0)^+ \rangle.$$

However, this inequality does not hold. Indeed, by Lemma 6.5 there exists  $z \in \Omega$  with  $z_0 = 2^{1-k}$  and  $z_{\alpha_0} = 1/n$ . Then  $nz_{\alpha_0} - 2^{k-1}z_0 = 0$ , while

$$z_{\alpha_0} - mz_0 = 1/n - m2^{1-k} > 0$$

according to the choice of  $k$ . This contradiction shows that (6.5) cannot hold, thus concluding the proof of Theorem 6.8.  $\square$

Finally, we would like to state some open problems.

**Problem 1.** Is every finitely separable completely normal lattice Cevian?

By Theorem 4.8, every finitely separable completely normal lattice has a monotone deviation; we do not know whether that deviation can be made Cevian. So far, the only known examples of non-Cevian completely normal lattices are due to Wehrung [15, § 7] and [17, § 7], the former lattice satisfying the additional property of having *countably based differences*. We do not know whether any of those lattices is finitely separable.

**Problem 2.** Is it possible that a completely normal lattice has a deviation monotone in one variable, but none in both variables?

Recall that for the  $\ell$ -group  $G$  constructed above, the lattice  $\text{Id}_c G$  does not have a deviation monotone in either variable.

**Problem 3.** Is it possible, for an Abelian  $\ell$ -group  $G$ , that  $\text{Id}_c G$  has a monotone deviation but no isotone Belluce section?

Notice that in the proof of Corollary 5.9 we actually constructed an isotone Belluce section for any free Abelian  $\ell$ -group.

## 7. DECLARATIONS

### Authors' Contribution:

Both authors contributed to the results in this paper.

### Conflict of Interest:

On behalf of all authors, the corresponding author states that there is no conflict of interest.

### Availability of Data and Materials:

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

**Funding:**

The first author was supported by Slovak VEGA grant 1/0152/22.

## REFERENCES

- [1] Kirby A. Baker, *Free vector lattices*, Canad. J. Math. **20** (1968), 58–66. MR 0224524
- [2] Simon J. Bernau, *Free abelian lattice groups*, Math. Ann. **180** (1969), 48–59. MR 0241340
- [3] Alain Bigard, Klaus Keimel, and Samuel Wolfenstein, *Groupes et Anneaux Réticulés*, Lecture Notes in Mathematics, Vol. 608, Springer-Verlag, Berlin-New York, 1977. MR 0552653 (58 #27688)
- [4] Ralph Freese and James B. Nation, *Projective lattices*, Pacific J. Math. **75** (1978), no. 1, 93–106. MR 500031 (80c:06012)
- [5] ———, *Projective ordinal sums of lattices and isotone sections*, Order **32** (2015), no. 2, 245–254. MR 3353951
- [6] Sakaé Fuchino, Sabine Koppelberg, and Saharon Shelah, *Partial orderings with the weak Freese-Nation property*, Ann. Pure Appl. Logic **80** (1996), no. 1, 35–54. MR 1395682
- [7] Lutz Heindorf and Leonid B. Shapiro, *Nearly Projective Boolean Algebras*, Lecture Notes in Mathematics, vol. 1596, Springer-Verlag, Berlin, 1994, with an appendix by Sakaé Fuchino. MR 1329090
- [8] Thomas Jech, *Set Theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513 (2004g:03071)
- [9] James J. Madden, *Two methods in the study of  $k$ -vector lattices*, Ph.D. thesis, Wesleyan University, 1984. MR 2633496
- [10] António A. Monteiro, *L'arithmétique des filtres et les espaces topologiques*, Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954, pp. 129–162. MR 0074805
- [11] Miroslav Ploščica, *Cevian properties in ideal lattices of Abelian  $\ell$ -groups*, Forum Math. **33** (2021), no. 6, 1651–1658. MR 4333993
- [12] Miroslav Ploščica and Friedrich Wehrung, *Spectral subspaces of spectra of Abelian lattice-ordered groups in size aleph one*, Acta Sci. Math. (Szeged) (2023), DOI 10.1007/s44146-023-00080-z.
- [13] Lou van den Dries, *Tame Topology and  $o$ -Minimal Structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, Cambridge, 1998. MR 1633348
- [14] Friedrich Wehrung, *Spectral spaces of countable Abelian lattice-ordered groups*, Trans. Amer. Math. Soc. **371** (2019), no. 3, 2133–2158. MR 3894048
- [15] ———, *Cevian operations on distributive lattices*, J. Pure Appl. Algebra **224** (2020), no. 4, 106202. MR 4021916
- [16] ———, *Real spectra and  $\ell$ -spectra of algebras and vector lattices over countable fields*, J. Pure Appl. Algebra **226** (2022), no. 4, Paper No. 106861, 25 p. MR 4301012
- [17] ———, *Real spectrum versus  $\ell$ -spectrum via Brumfiel spectrum*, Algebr. Represent. Theory **26** (2023), no. 1, 137–158. MR 4546136

FACULTY OF NATURAL SCIENCES, ŠAFÁRIK'S UNIVERSITY, JESENNÁ 5, 04154 KOŠICE, SLOVAKIA  
 Email address: [miroslav.ploscica@upjs.sk](mailto:miroslav.ploscica@upjs.sk)  
 URL: <https://ploscica.science.upjs.sk>

NORMANDIE UNIVERSITÉ, UNICAEN, CNRS UMR 6139, LMNO, 14000 CAEN, FRANCE  
 Email address: [friedrich.wehrung01@unicaen.fr](mailto:friedrich.wehrung01@unicaen.fr)  
 URL: <https://wehrungf.users.lmno.cnrs.fr>