



HAL
open science

Stackelberg exact controllability for the Boussinesq system

Takéo Takahashi, Luz de Teresa, Yingying Wu-Zhang

► **To cite this version:**

Takéo Takahashi, Luz de Teresa, Yingying Wu-Zhang. Stackelberg exact controllability for the Boussinesq system. 2023. hal-04228391

HAL Id: hal-04228391

<https://hal.science/hal-04228391>

Preprint submitted on 4 Oct 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Stackelberg exact controllability for the Boussinesq system

Takéo Takahashi ^{*1}, Luz de Teresa ^{†2}, and Yingying Wu-Zhang^{‡2}

¹Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France

²Instituto de Matematicas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U.,
C. P. 04510 Ciudad de México, México

October 3, 2023

Abstract

We consider a Stackelberg control strategy applied to the Boussinesq system. More precisely, we act on this system with a hierarchy of two controls. The aim of the “leader” control is the null controllability property whereas the objective of “follower” control is to keep the state close to a given trajectory. By solving first the optimal control problem associated with the follower control, we are lead to show the null-controllability property of a system coupling a forward with a backward Boussinesq type systems. Our main result states that for an adequate weighted functional for the optimal control problem, this coupled system is locally null-controllable. To show this result, we first study the adjoint system of the linearized system and obtain a weighted observability estimate by combining several Carleman estimates and an adequate decomposition for the heat and the Stokes system.

Keywords: Hierarchical controls, Null controllability, Boussinesq systems, Carleman estimates

2020 Mathematics Subject Classification. 76D05, 35Q30, 49J20, 93B05, 93B07, 93C10, 91A65

1 Introduction

In the last years, the PDE community has intensively studied the controllability properties of different equations when acting on them with two or more controls having different objectives. Game theory originally studied this kind of problem. The idea is that a leader player (or control) acts, and a follower player (or control) reacts to this action trying to fulfill its own objective. Following this structure, J. L. Lions [21] proposed in 1994 a Stackelberg control strategy for the heat equation. In his formulation, the leader control had an approximate controllability objective, and the follower control, reacting to the leader control, had an optimization objective. Since then, several works have explored different models and actions. See e.g. [1–4, 7, 18].

In most of these papers, the leader control has a controllability objective. That is, its target is a null, exact or approximate one, and the follower aims to minimize a given functional. Classically, this functional adds the square of the proximity of the solution to a given function in some domain plus the square of the L^2 norm of the control. This approach gives a characterization of the follower control that introduces a coupled equation to the original one. This characterization implies that achieving the leader control objective implies controlling the first component of a strongly coupled forward-backward system. For some classical equations dealing with an approximate controllability

*takeo.takahashi@inria.fr

†ldeteresa@im.unam.mx. Partially supported by Conacyt, project A1-S-17475

‡yingwu@im.unam.mx. Partially supported by Conacyt grant 849458

objective for the leader has already been done using unique continuation techniques. This is the case in [21] for the heat equation, in [17] for the Stokes system or in [4] for micropolar fluids. However, when the leader objective is a null controllability one, the problem is much more complex. In [3] for example, this null controllability objective is proved using very precise Carleman inequalities for a strongly coupled forward-backward heat equations. In this paper we explore a Stackelberg control strategy for the two dimensional Boussinesq system. In order to handle the null controllability problem of the strongly coupled system that arises in the characterization of the optimal (follower) control, we work in a weighted L^2 space norm of the follower control. The weight comes precisely from a function that appears in the Carleman inequalities. In this sense we used the strategy followed in [19].

The problem we are considering is the following: let Ω be a bounded smooth domain of \mathbb{R}^2 and $T > 0$. Let ω and \mathcal{O} be two nonempty open subsets of Ω with

$$\omega \cap \mathcal{O} = \emptyset.$$

We consider the Boussinesq system with two controls f and f^* acting on the temperature:

$$\begin{cases} \partial_t y - \Delta y + \nabla \pi_y + (y \cdot \nabla) y = \theta e_2 & \text{in } (0, T) \times \Omega, \\ \partial_t \theta - \Delta \theta + y \cdot \nabla \theta = f 1_\omega + f^* 1_{\mathcal{O}} & \text{in } (0, T) \times \Omega, \\ \nabla \cdot y = 0 & \text{in } (0, T) \times \Omega, \\ y = 0, \quad \theta = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0, \quad \theta(0, \cdot) = \theta^0 & \text{in } \Omega. \end{cases} \quad (1.1)$$

In the above system $y = y(t, x) \in \mathbb{R}^2$, $\pi_y = \pi_y(t, x) \in \mathbb{R}$ and $\theta = \theta(t, x) \in \mathbb{R}$ are respectively the velocity, the pressure and the temperature of the fluid, whereas (e_1, e_2) is the canonical basis of \mathbb{R}^2 . Note that, to simplify the presentation, we have taken several physical constants equal to 1: the viscosity, the density and the thermal conductivity of the fluid and the gravitational acceleration. The functions f and f^* are two controls that have two different objectives:

1. the aim of f^* is that (y, θ) remains close to a given target $(y^*, \theta^*) \in L^2(0, T; L^2(\Omega)^3)$;
2. the aim of f is that the state is at rest at the final time T : $(y, \theta)(T, \cdot) = 0$ in Ω .

We write the first objective as an optimal control problem, that is, we are looking for a control f^* that minimizes a functional involving the “distance” between (y, θ) and the desired state (y^*, θ^*) . For the second objective, f will solve a classical null-controllability problem.

Let us describe the problem framework. First, let us recall the definition of some standard spaces for the study of the Stokes system:

$$\mathbb{H} := \{y \in L^2(\Omega)^2 : \nabla \cdot y = 0 \text{ in } \Omega, \quad y \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad (1.2)$$

$$\mathbb{V} := \{y \in H_0^1(\Omega)^2 : \nabla \cdot y = 0 \text{ in } \Omega\}, \quad (1.3)$$

where we denote by ν the unit outward normal vector field on the boundary $\partial\Omega$. For the first objective, we introduce an optimal control problem for f^* where we fix

$$f \in L^2(0, T; L^2(\omega)), \quad y^0 \in \mathbb{V}, \quad \theta^0 \in H_0^1(\Omega). \quad (1.4)$$

Then, for any $f^* \in L^2(0, T; L^2(\mathcal{O}))$, one can check that system (1.1) admits a unique strong solution $(y(f^*), \theta(f^*))$ and we can thus consider the weighted functional

$$J(f^*) := \frac{1}{2} \iint_{(0, T) \times \mathcal{O}^*} \left[|y(f^*) - y^*|^2 + |\theta(f^*) - \theta^*|^2 \right] dx dt + \frac{1}{2} \iint_{(0, T) \times \mathcal{O}} \frac{|f^*|^2}{\mu} dx dt \quad (1.5)$$

where \mathcal{O}^* is a nonempty open set and represents the observability domain for f^* and μ is a weight function satisfying

$$\mu \in C^0([0, T]; \mathbb{R}_+), \quad \mu(0) = \mu(T) = 0, \quad \mu > 0 \text{ in } (0, T),$$

that will be described later (see (1.10) and(1.15)). For this functional, we aim to solve the optimal control problem

$$\inf_{f^* \in L^2(0,T;L^2(\mathcal{O}))} J(f^*). \quad (1.6)$$

In relation to the above, we have the following result:

Lemma 1.1. *Assume (1.4) and that $(y^*, \theta^*) \in L^2(0, T; L^2(\Omega)^3)$. There exists $C > 0$ such that if*

$$\|\mu\|_{L^\infty(0,T)} \leq C, \quad (1.7)$$

then the optimal control problem (1.6) admits a unique solution given by

$$f^* := -\mu\sigma, \quad (1.8)$$

where (y, θ, u, σ) is the solution of the coupled system

$$\begin{cases} \partial_t y - \Delta y + \nabla \pi_y + (y \cdot \nabla) y = \theta e_2 & \text{in } (0, T) \times \Omega, \\ \partial_t \theta - \Delta \theta + y \cdot \nabla \theta = f 1_\omega - \mu \sigma 1_{\mathcal{O}} & \text{in } (0, T) \times \Omega, \\ -\partial_t u - \Delta u + \nabla \pi_u + (\nabla y)^\top u - (y \cdot \nabla) u + \sigma \nabla \theta = (y - y^*) 1_{\mathcal{O}^*} & \text{in } (0, T) \times \Omega, \\ -\partial_t \sigma - \Delta \sigma - y \cdot \nabla \sigma = u_2 + (\theta - \theta^*) 1_{\mathcal{O}^*} & \text{in } (0, T) \times \Omega, \\ \nabla \cdot y = \nabla \cdot u = 0 & \text{in } (0, T) \times \Omega, \\ y = u = 0, \quad \theta = \sigma = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0, \quad \theta(0, \cdot) = \theta^0, \quad u(T, \cdot) = 0, \quad \sigma(T, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (1.9)$$

Here, we have denoted by \cdot^\top the transpose of a matrix. The proof of Lemma 1.1 is standard but for sake of completeness, we give a sketch of its proof in Section 2.3.

Our first main result states the partial null-controllability of the above coupled system:

Theorem 1.2. *Assume that $\omega \cap \mathcal{O} = \emptyset$ and that $\omega \cap \mathcal{O}^* \neq \emptyset$. There exist $C > 0$ and $\rho_1, \rho_2 \in C^0([0, T])$ such that*

$$\rho_1(0) = \rho_1(T) = \rho_2(T) = 0, \quad \rho_1 > 0 \text{ in } (0, T), \quad \rho_2 > 0 \text{ in } [0, T]$$

with the following property: if

$$\frac{\mu}{\rho_1} \in L^\infty(0, T), \quad \frac{(y^*, \theta^*)}{\rho_2} \in L^2(0, T; L^2(\Omega)^3), \quad (y^0, \theta^0) \in \mathbb{V} \times H_0^1(\Omega), \quad (1.10)$$

with

$$\left\| \frac{(y^*, \theta^*)}{\rho_2} \right\|_{L^2(0,T;L^2(\Omega)^3)} + \|(y^0, \theta^0)\|_{H^1(\Omega)^3} \leq C,$$

then, there exist a control $f \in L^2(0, T; L^2(\omega))$ and a solution

$$(y, \theta, u, \sigma) \in L^2(0, T; H^2(\Omega)^6) \cap C^0([0, T]; H^1(\Omega)^6) \cap H^1(0, T; L^2(\Omega)^6), \quad (\pi_y, \pi_u) \in L^2(0, T; H^1(\Omega)^2)$$

of (1.9) satisfying

$$y(T, \cdot) = 0, \quad \theta(T, \cdot) = 0 \quad \text{in } \Omega.$$

Remark 1.3. We will take ρ_1 such that the first condition in (1.10) implies (1.7). The precise definition of ρ_1 and ρ_2 are given in (4.1).

In order to show this result, we linearize system (1.9) and consider

$$\begin{cases} \partial_t y - \Delta y + \nabla \pi_y = \theta e_2 + h^{(1)} & \text{in } (0, T) \times \Omega, \\ \partial_t \theta - \Delta \theta = f 1_\omega - \mu \sigma 1_{\mathcal{O}} + h^{(2)} & \text{in } (0, T) \times \Omega, \\ -\partial_t u - \Delta u + \nabla \pi_u = y 1_{\mathcal{O}^*} + h^{(3)} & \text{in } (0, T) \times \Omega, \\ -\partial_t \sigma - \Delta \sigma = u_2 + \theta 1_{\mathcal{O}^*} + h^{(4)} & \text{in } (0, T) \times \Omega, \\ \nabla \cdot y = \nabla \cdot u = 0 & \text{in } (0, T) \times \Omega, \\ y = u = 0, \quad \theta = \sigma = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0, \quad \theta(0, \cdot) = \theta^0, \quad u(T, \cdot) = 0, \quad \sigma(T, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (1.11)$$

where $h^{(1)}, h^{(2)}, h^{(3)}$ and $h^{(4)}$ are given source terms. Then, we consider the partial null-controllability of (1.11): we look for a control $f \in L^2(0, T; L^2(\omega))$ such that $y(T, \cdot) = 0$ and $\theta(T, \cdot) = 0$. Finally, we deduce Theorem 1.2 by a fixed-point argument where we replace $h^{(1)}, h^{(2)}, h^{(3)}$ and $h^{(4)}$ by the nonlinearities

$$h^{(1)} = -(y \cdot \nabla) y, \quad h^{(2)} = -y \cdot \nabla \theta, \quad (1.12)$$

$$h^{(3)} = -(\nabla y)^\top u + (y \cdot \nabla) u - \sigma \nabla \theta - y^* 1_{\mathcal{O}^*}, \quad h^{(4)} = y \cdot \nabla \sigma - \theta^* 1_{\mathcal{O}^*}. \quad (1.13)$$

In order to prove the partial null-controllability of (1.11), we use a standard duality argument and need to prove an observability inequality for the adjoint system

$$\begin{cases} -\partial_t v - \Delta v + \nabla \pi_v = w 1_{\mathcal{O}^*} + g^{(1)} & \text{in } (0, T) \times \Omega, \\ -\partial_t \varphi - \Delta \varphi = v_2 + \psi 1_{\mathcal{O}^*} + g^{(2)} & \text{in } (0, T) \times \Omega, \\ \partial_t w - \Delta w + \nabla \pi_w = \psi e_2 + g^{(3)} & \text{in } (0, T) \times \Omega, \\ \partial_t \psi - \Delta \psi = -\mu \varphi 1_{\mathcal{O}} + g^{(4)} & \text{in } (0, T) \times \Omega, \\ \nabla \cdot v = \nabla \cdot w = 0 & \text{in } (0, T) \times \Omega, \\ v = w = 0, \quad \varphi = \psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ v(T, \cdot) = v^0, \quad \varphi(T, \cdot) = \varphi^0, \quad w(0, \cdot) = 0, \quad \psi(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (1.14)$$

We show in the next section (see Proposition 2.2) that the above linear system is well-posed for $\|\mu\|_{L^\infty(0, T)}$ small enough. Our second main result states as follows:

Theorem 1.4. *Assume that $\omega \cap \mathcal{O} = \emptyset$ and that $\omega \cap \mathcal{O}^* \neq \emptyset$. There exists $\rho \in C^0([0, T])$ such that*

$$\rho(0) = \rho(T) = 0, \quad \rho > 0 \text{ in } (0, T)$$

with the following property: for any μ satisfying

$$\frac{\mu}{\rho^{11}} \in L^\infty(0, T), \quad (1.15)$$

there exists $C > 0$ such that any solution (v, φ, w, ψ) of (1.14) satisfies

$$\begin{aligned} & \left\| \rho^{22}(v, \varphi, w, \psi) \right\|_{L^2(0, T; L^2(\Omega)^6)}^2 \\ & \leq C \left(\left\| \rho^{12}(g^{(1)}, g^{(2)}, g^{(3)}, g^{(4)}) \right\|_{L^2(0, T; L^2(\Omega)^6)}^2 + \iint_{(0, T) \times \omega} \rho^{26} |\varphi|^2 dx dt \right). \end{aligned} \quad (1.16)$$

The precise definition of ρ is given in (2.19) and is such that the condition (1.15) implies (1.7). Note that the powers in (1.16) are not optimal but are sufficient to prove Theorem 1.2.

As far as we know, this is one of the first results for the hierarchical control for the Boussinesq system and in the case where the number of scalar controls is less than the number of the dimension of the states. Here we only use one control acting on the heat equation to control both the fluid velocity and temperature. Note also that using the techniques developed in [18] and [22], we could extend our work in the case of the robust Stackelberg controllability for the Boussinesq system. We could also extend this result to the Boussinesq equations in 3D, but in that case, we would need an additional scalar control in the Navier-Stokes system.

Our method to prove Theorem 1.4 is based on a (*global*) *Carleman inequality* that is a standard tool used in the proof of the null controllability of parabolic equations and were introduced in [16]. Several works have used such kind of estimates to study the controllability of Stokes or Navier-Stokes systems (for instance, [20] or [14]). The case of controls with some vanishing components as in (1.1) can be also studied with this approach, see for instance [15], [11] and [9]. Note that, another method, based on results of Gromov, has been applied to obtain the local null controllability of the Navier-Stokes system in dimension 3 with a control having two vanishing components.

Here, we follow the method based on the Carleman estimates and more precisely the strategy introduced in [11] that consists in applying appropriate differential operators to get rid of the pressure. Such a strategy was also used

in several articles devoted to the insensitizing controllability of the Navier-Stokes system [9] or to the Boussinesq system [8]. The linear system associated with the Boussinesq system in [8] is similar to (1.14) and we thus consider the same steps and in particular the same method to deal with the local terms in ψ that appear in the Carleman estimates. However, one of the main differences comes from the fact that in [8], the authors need to impose some regularity on the source terms $g^{(i)}$. Here we avoid such a restriction and use for v , w and ψ a decomposition introduced in [10]. This decomposition generalizes a standard decomposition in two terms, used for instance for the Navier-Stokes system or the Boussinesq system. A similar decomposition is also introduced in [5] but it can not be used for systems of the form (1.14) where some equations are forward in time and other are backwards in time.

The outline of the article is as follows: in next section, we present the functional framework to study (1.11) and (1.14), and we state some well-posedness results. We also show Lemma 1.1 and introduce the weight functions for the Carleman estimate. Section 3 is devoted to the Carleman estimates that lead to the proof of Theorem 1.4. Finally in Section 4, we show Theorem 1.2 with standard arguments and in particular a fixed point procedure.

Notation. In the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from line to line. We also use the notation $X \lesssim Y$ if there exists a constant $C > 0$ such that we have the inequality $X \leq CY$. The notation $X \lesssim_k Y$ stands for $X \leq CY$, where C is a positive constant depending on k .

2 Preliminaries

In this section, we introduce the operators associated with (1.11) and (1.14). We also state some results for the well-posedness and regularity of Stokes type systems. We sketch the proof of Lemma 1.1 and we give the weight functions needed in the Carleman estimates for the next section.

2.1 Functional framework and first regularity results

Recall the spaces (1.2) and (1.3), we introduce the Laplace operator and the Stokes operator:

$$\mathcal{D}(\mathbf{A}) := H^2(\Omega) \cap H_0^1(\Omega), \quad \mathbf{A} := -\Delta : \mathcal{D}(\mathbf{A}) \rightarrow L^2(\Omega),$$

and

$$\mathcal{D}(\mathbb{A}) := \left\{ y \in [H^2(\Omega) \cap H_0^1(\Omega)]^2 : \nabla \cdot y = 0 \text{ in } \Omega \right\}, \quad \mathbb{A} := -\mathbb{P}\Delta : \mathcal{D}(\mathbb{A}) \rightarrow \mathbb{H}, \quad (2.1)$$

where $\mathbb{P} : L^2(\Omega)^2 \rightarrow \mathbb{H}$ is the orthogonal projection (Leray projector). It is well-known that both operators are self-adjoint and positive and since $\partial\Omega$ is smooth, the elliptic regularity yields that for $k \in \mathbb{N}^*$,

$$\mathcal{D}(\mathbf{A}^k) \subset H^{2k}(\Omega), \quad \mathcal{D}(\mathbb{A}^k) \subset H^{2k}(\Omega)^2.$$

For $T > 0$, we now set

$$\mathbf{X}_k := \bigcap_{j=0}^k H^j(0, T; \mathcal{D}(\mathbf{A}^{k-j})), \quad \mathbb{X}_k := \bigcap_{j=0}^k H^j(0, T; \mathcal{D}(\mathbb{A}^{k-j})).$$

In particular,

$$\mathbf{X}_0 = L^2(0, T; L^2(\Omega)), \quad \mathbf{X}_1 = L^2(0, T; \mathcal{D}(\mathbf{A})) \cap H^1(0, T; L^2(\Omega)), \quad \mathbb{X}_1 = L^2(0, T; \mathcal{D}(\mathbb{A})) \cap H^1(0, T; \mathbb{H}). \quad (2.2)$$

Let us assume that ψ and w are the strong solutions of the systems

$$\begin{cases} \psi' + \mathbf{A}\psi = \kappa\psi + \kappa\mathbf{F} & \text{in } (0, T) \\ \psi(0) = 0 \end{cases}, \quad \begin{cases} w' + \mathbb{A}w = \kappa w + \kappa\mathbb{F} & \text{in } (0, T) \\ w(0) = 0 \end{cases} \quad (2.3)$$

where $\kappa \in C^\infty((0, T))$. Assume $j_0 \geq 1$ and that for $j \in \{0, \dots, j_0\}$,

$$\tau_j \in C^\infty([0, T]; \mathbb{R}_+), \quad \tau_j(0) = 0, \quad (2.4)$$

$$\left| \frac{d^k \tau_j}{dt^k} \right| \lesssim \tau_{j-k} \quad (k \in \{0, \dots, j\}), \quad \left| \frac{d^{k-1} (\kappa \tau_j)}{dt^{k-1}} \right| \lesssim \tau_{j-k} \quad (k \in \{1, \dots, j\}). \quad (2.5)$$

Following the proof of [23, Lemma 2.4], we have the following result:

Lemma 2.1. *Assume that ψ and w are the strong solutions of the systems of (2.3), that $j_0 \geq 1$, and that $\kappa, \tau_j, j \in \{0, \dots, j_0\}$ satisfy the above hypotheses. If $\mathbb{F} \in \mathbb{X}_{j_0-1}$, then for any $j \in \{1, \dots, j_0\}$,*

$$\tau_j w \in \mathbb{X}_j, \quad \tau_j \partial_t^k w \in L^2(0, T; \mathcal{D}(\mathbb{A}^{j-k})) \quad (k \in \{1, \dots, j\}),$$

and we have the estimate

$$\sum_{j=1}^{j_0} \sum_{k=0}^j \|\tau_j \partial_t^k w\|_{L^2(0, T; \mathcal{D}(\mathbb{A}^{j-k}))} \lesssim \|\tau_0 w\|_{\mathbb{X}_0} + \|\mathbb{F}\|_{\mathbb{X}_{j_0-1}}.$$

Similarly, if $\mathbf{F} \in \mathbf{X}_{j_0-1}$, then for any $j \in \{1, \dots, j_0\}$,

$$\tau_j \psi \in \mathbf{X}_j, \quad \tau_j \partial_t^k \psi \in L^2(0, T; \mathcal{D}(\mathbf{A}^{j-k})) \quad (k \in \{1, \dots, j\}),$$

and we have the estimate

$$\sum_{j=1}^{j_0} \sum_{k=0}^j \|\tau_j \partial_t^k \psi\|_{L^2(0, T; \mathcal{D}(\mathbf{A}^{j-k}))} \lesssim \|\tau_0 \psi\|_{\mathbf{X}_0} + \|\mathbf{F}\|_{\mathbf{X}_{j_0-1}}.$$

2.2 Well-posedness properties for the direct and the adjoint system

We first show the existence and uniqueness of strong solutions for the adjoint system (1.14) provided that $\|\mu\|_{L^\infty(0, T)}$ is small enough. To simplify the notation, in what follows, we write

$$h := (h^{(1)}, h^{(2)}, h^{(3)}, h^{(4)}) \quad \text{and} \quad g := (g^{(1)}, g^{(2)}, g^{(3)}, g^{(4)}). \quad (2.6)$$

We recall that $\mathbf{X}_0, \mathbf{X}_1, \mathbb{X}_1$ are given by (2.2) and that \mathbb{V} is given by (1.3).

Proposition 2.2. *Let $T > 0$. There exists a constant $C > 0$ such that if*

$$\|\mu\|_{L^\infty(0, T)} \leq C, \quad (2.7)$$

then (1.14) is well-posed: for any $g \in [\mathbf{X}_0]^6, v^0 \in \mathbb{V}$ and $\varphi^0 \in H^1_0(\Omega)$, there exists a unique strong solution of (1.14) with

$$\|(v, \varphi, w, \psi)\|_{\mathbb{X}_1 \times \mathbf{X}_1 \times \mathbb{X}_1 \times \mathbf{X}_1} \lesssim \|g\|_{[\mathbf{X}_0]^6} + \|(v^0, \varphi^0)\|_{H^1(\Omega)^3}.$$

Proof. We use a fixed-point argument: for any $\tilde{\varphi} \in \mathbf{X}_0$, we can solve the system

$$\begin{cases} -\partial_t v - \Delta v + \nabla \pi_v = w \mathbf{1}_{\mathcal{O}^*} + g^{(1)} & \text{in } (0, T) \times \Omega, \\ -\partial_t \varphi - \Delta \varphi = v_2 + \psi \mathbf{1}_{\mathcal{O}^*} + g^{(2)} & \text{in } (0, T) \times \Omega, \\ \partial_t w - \Delta w + \nabla \pi_w = \psi e_2 + g^{(3)} & \text{in } (0, T) \times \Omega, \\ \partial_t \psi - \Delta \psi = -\mu \tilde{\varphi} \mathbf{1}_{\mathcal{O}} + g^{(4)} & \text{in } (0, T) \times \Omega, \\ \nabla \cdot v = \nabla \cdot w = 0 & \text{in } (0, T) \times \Omega, \\ v = w = 0, \quad \varphi = \psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ v(T, \cdot) = v^0, \quad \varphi(T, \cdot) = \varphi^0, \quad w(0, \cdot) = 0, \quad \psi(0, \cdot) = 0 & \text{in } \Omega, \end{cases} \quad (2.8)$$

More precisely, the system is in ‘‘cascade’’: we can obtain successively ψ, w, v and φ with the estimate

$$\|(v, \varphi, w, \psi)\|_{\mathbb{X}_1 \times \mathbf{X}_1 \times \mathbb{X}_1 \times \mathbf{X}_1} \lesssim \|g\|_{[\mathbf{X}_0]^6} + \|(v^0, \varphi^0)\|_{H^1(\Omega)^3} + \|\mu\|_{L^\infty(0, T)} \|\tilde{\varphi}\|_{\mathbf{X}_0}.$$

In particular, we can consider the mapping $\mathcal{Z} : \mathbf{X}_0 \rightarrow \mathbf{X}_0, \tilde{\varphi} \mapsto \varphi$, where (v, φ, w, ψ) is the corresponding solution of (2.8). We can see from the above estimate that if $\|\mu\|_{L^\infty(0, T)}$ is small enough, then \mathcal{Z} is a strict contraction. Applying the Banach fixed point theorem, we deduce the existence and the uniqueness of a fixed point for \mathcal{Z} and this yields the results of the proposition. \square

We can obtain the same result for (1.11) with a similar proof.

Proposition 2.3. *Let $T > 0$. There exists a constant $C > 0$ such that if (2.7) holds, then (1.11) is well-posed: for any $h \in [\mathbf{X}_0]^6$, $f \in L^2(0, T; L^2(\omega))$, $y^0 \in \mathbb{V}$ and $\theta^0 \in H_0^1(\Omega)$, there exists a unique strong solution of (1.11) with*

$$\|(y, \theta, u, \sigma)\|_{\mathbb{X}_1 \times \mathbf{X}_1 \times \mathbb{X}_1 \times \mathbf{X}_1} \lesssim \|h\|_{[\mathbf{X}_0]^6} + \|f\|_{L^2(0, T; L^2(\omega))} + \|(y^0, \theta^0)\|_{H^1(\Omega)^3}.$$

We now define the following operators

$$L(y, \pi_y, \theta, u, \pi_u, \sigma) := \begin{bmatrix} \partial_t y - \Delta y + \nabla \pi_y - \theta e_2 \\ \partial_t \theta - \Delta \theta + \mu \sigma 1_{\mathcal{O}} \\ -\partial_t u - \Delta u + \nabla \pi_u - y 1_{\mathcal{O}^*} \\ -\partial_t \sigma - \Delta \sigma - u_2 - \theta 1_{\mathcal{O}^*} \end{bmatrix},$$

and

$$L^*(v, \pi_v, \varphi, w, \pi_w, \psi) := \begin{bmatrix} -\partial_t v - \Delta v + \nabla \pi_v - w 1_{\mathcal{O}^*} \\ -\partial_t \varphi - \Delta \varphi - v_2 - \psi 1_{\mathcal{O}^*} \\ \partial_t w - \Delta w + \nabla \pi_w - \psi e_2 \\ \partial_t \psi - \Delta \psi + \mu \varphi 1_{\mathcal{O}} \end{bmatrix}. \quad (2.9)$$

This allows us to write (1.11) and (1.14) as

$$\begin{cases} L(y, \pi_y, \theta, u, \pi_u, \sigma) = h + (0, 0, f 1_{\omega}, 0, 0, 0) & \text{in } (0, T) \times \Omega, \\ \nabla \cdot y = \nabla \cdot u = 0 & \text{in } (0, T) \times \Omega, \\ y = u = 0, \quad \theta = \sigma = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y^0, \quad \theta(0, \cdot) = \theta^0, \quad u(T, \cdot) = 0, \quad \sigma(T, \cdot) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\begin{cases} L^*(v, \pi_v, \varphi, w, \pi_w, \psi) = g & \text{in } (0, T) \times \Omega, \\ \nabla \cdot v = \nabla \cdot w = 0 & \text{in } (0, T) \times \Omega, \\ v = w = 0, \quad \varphi = \psi = 0 & \text{on } (0, T) \times \partial\Omega, \\ v(T, \cdot) = v^0, \quad \varphi(T, \cdot) = \varphi^0, \quad w(0, \cdot) = 0, \quad \psi(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (2.10)$$

We now define the solutions by transposition for (1.11). In order to do this we first consider a strong solution $(y, \pi_y, \theta, u, \pi_u, \sigma)$ of (1.11) and

$$(v, \pi_v, \varphi, w, \pi_w, \psi) \in [\mathbb{X}_1 \times L^2(0, T; H^1(\Omega)/\mathbb{R}) \times \mathbf{X}_1]^2 \quad (2.11)$$

such that

$$\begin{cases} v(T, \cdot) = 0, \quad \varphi(T, \cdot) = 0 & \text{in } \Omega, \\ w(0, \cdot) = 0, \quad \psi(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (2.12)$$

We multiply the first equation of (1.11) by v , the second equation of (1.11) by φ , the third equation of (1.11) by w and the fourth equation of (1.11) by ψ . After some integration by parts, we obtain

$$\begin{aligned} & \iint_{(0, T) \times \Omega} (y, \theta, u, \sigma) \cdot L^*(v, \pi_v, \varphi, w, \pi_w, \psi) \, dx \, dt + \iint_{(0, T) \times \omega} f \varphi \, dx \, dt \\ & = \int_{\Omega} (y^0, \theta^0) \cdot (v(0, \cdot), \varphi(0, \cdot)) \, dx + \iint_{(0, T) \times \Omega} h \cdot (v, \varphi, w, \psi) \, dx \, dt. \end{aligned} \quad (2.13)$$

This leads to the following definition:

Definition 2.4. Assume $f \in L^2(0, T; L^2(\omega))$, $(y^0, \theta^0) \in \mathbb{H} \times L^2(\Omega)$ and $h \in [\mathbf{X}_0]^6$. We say that $(y, \theta, u, \sigma) \in [\mathbb{X}_0 \times \mathbf{X}_0]^2$ is a solution by transposition of (1.11) if (2.13) holds for any $(v, \pi_v, \varphi, w, \pi_w, \psi)$ satisfying (2.11) and (2.12).

We deduce from Proposition 2.2 the uniqueness of solutions by transposition of (1.11). In particular if (y, θ, u, σ) is a strong solution of (1.11), it is the solution by transposition of (1.11).

2.3 Characterization of the optimal control

We give here a sketch of the proof of Lemma 1.1.

Proof of Lemma 1.1. The existence and uniqueness of the optimal control can be obtained by following the arguments in [6]. The idea is to show that the convexity of J under the condition (1.7) and the Gateaux-differentiability of the mapping

$$G : L^2(0, T; L^2(\mathcal{O})) \rightarrow \mathbb{X}_1 \times \mathbf{X}_1, \quad f^* \mapsto (y(f^*), \theta(f^*)),$$

where $(y(f^*), \theta(f^*))$ is the unique strong solution of (1.1) associated with f^* and with (y^0, θ^0, f) satisfying (1.4). Here, we only give some ideas about how to characterize the optimal control. Assume $f^*, h^* \in L^2(0, T; L^2(\mathcal{O}))$ and let us write

$$(y, \theta) = G(f^*), \quad (\tilde{y}, \tilde{\theta}) = DG_{f^*}(h^*),$$

where $DG_{f^*}(h^*)$ is the Gateaux derivative of G at f^* in the direction h^* . Then, we deduce from (1.1) that $(\tilde{y}, \tilde{\theta})$ satisfies the system

$$\begin{cases} \partial_t \tilde{y} - \Delta \tilde{y} + \nabla \pi_{\tilde{y}} + (\tilde{y} \cdot \nabla) y + (y \cdot \nabla) \tilde{y} = \tilde{\theta} e_2 & \text{in } (0, T) \times \Omega, \\ \partial_t \tilde{\theta} - \Delta \tilde{\theta} + \tilde{y} \cdot \nabla \theta + y \cdot \nabla \tilde{\theta} = h^* \mathbf{1}_{\mathcal{O}} & \text{in } (0, T) \times \Omega, \\ \nabla \cdot \tilde{y} = 0 & \text{in } (0, T) \times \Omega, \\ \tilde{y} = 0, \quad \tilde{\theta} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \tilde{y}(0, \cdot) = 0, \quad \tilde{\theta}(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (2.14)$$

Moreover, some standard computation yields that the Gateaux derivative of J , given by (1.5), at f^* in the direction h^* , is given by

$$DJ_{f^*}(h^*) = \iint_{(0, T) \times \mathcal{O}^*} \left[(y - y^*) \cdot \tilde{y} + (\theta - \theta^*) \cdot \tilde{\theta} \right] dx dt + \iint_{(0, T) \times \mathcal{O}} \frac{f^* h^*}{\mu} dx dt. \quad (2.15)$$

From the standard theory on the Navier-Stokes system (see, for instance, [24, Chapter III §1 and §3]) and using that $(y, \theta) \in \mathbb{X}_1 \times \mathbf{X}_1$, one can show the existence and uniqueness of a strong solution $(u, \sigma) \in \mathbb{X}_1 \times \mathbf{X}_1$ of the following system

$$\begin{cases} -\partial_t u - \Delta u + \nabla \pi_u + (\nabla y)^\top u - (y \cdot \nabla) u + \sigma \nabla \theta = (y - y^*) \mathbf{1}_{\mathcal{O}^*} & \text{in } (0, T) \times \Omega, \\ -\partial_t \sigma - \Delta \sigma - y \cdot \nabla \sigma = u_2 + (\theta - \theta^*) \mathbf{1}_{\mathcal{O}^*} & \text{in } (0, T) \times \Omega, \\ \nabla \cdot u = 0 & \text{in } (0, T) \times \Omega, \\ u = 0, \quad \theta = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(T, \cdot) = 0, \quad \sigma(T, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

Multiplying the first equation of the above system by \tilde{y} and the second equation by $\tilde{\theta}$ and combining it with (2.14), we obtain after some integration by parts that

$$\iint_{(0, T) \times \mathcal{O}} h^* \sigma dx dt = \iint_{(0, T) \times \mathcal{O}^*} \left[(y - y^*) \cdot \tilde{y} + (\theta - \theta^*) \cdot \tilde{\theta} \right] dx dt. \quad (2.16)$$

Comparing (2.15) and (2.16) we deduce that if f^* is a critical point of J , then (1.8) holds. \square

2.4 Definition and properties of the weight functions

In the next section, we are going to show some Carleman estimates in order to prove Theorem 1.4. Here, we define the corresponding weight functions. First, let us consider a nonempty domain ω_0 such that

$$\overline{\omega_0} \subset \omega \cap \mathcal{O}^*.$$

Using [16] (see also [25, Theorem 9.4.3, p.299]), we can construct $\eta^0 \in C^2(\overline{\Omega})$ satisfying

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \partial\Omega, \quad \max_{\Omega} \eta^0 = 1, \quad \nabla \eta^0 \neq 0 \text{ in } \overline{\Omega} \setminus \overline{\omega_0}.$$

Then, we define the following functions:

$$\begin{aligned}\alpha(t, x) &= \frac{\exp(24\lambda) - \exp[\lambda(22 + \eta^0(x))]}{t^{11}(T-t)^{11}}, & \xi(t, x) &= \frac{\exp[\lambda(22 + \eta^0(x))]}{t^{11}(T-t)^{11}}, \\ \alpha_{\sharp}(t) &= \max_{x \in \overline{\Omega}} \alpha(t, x) = \frac{\exp(24\lambda) - \exp(22\lambda)}{t^{11}(T-t)^{11}}, & \xi_{\sharp}(t) &= \min_{x \in \overline{\Omega}} \xi(t, x) = \frac{\exp(22\lambda)}{t^{11}(T-t)^{11}}, \\ \alpha_{\flat}(t) &= \min_{x \in \overline{\Omega}} \alpha(t, x) = \frac{\exp(24\lambda) - \exp(23\lambda)}{t^{11}(T-t)^{11}}, & \xi_{\flat}(t) &= \max_{x \in \overline{\Omega}} \xi(t, x) = \frac{\exp(23\lambda)}{t^{11}(T-t)^{11}},\end{aligned}$$

where $\lambda > 1$. For the previous functions, we have the following relations: there exists $C > 0$ depending on the geometry such that

$$|(\alpha_{\sharp})'| + |(\xi_{\sharp})'| \leq CT(\xi_{\sharp})^{1+1/11}, \quad |(\alpha_{\sharp})''| + |(\xi_{\sharp})''| \leq CT^2(\xi_{\sharp})^{1+2/11}, \quad (2.17)$$

$$|\nabla\alpha| = |\nabla\xi| \leq C\lambda\xi, \quad |\Delta\alpha| = |\Delta\xi| \leq C\lambda^2\xi. \quad (2.18)$$

Now, let us recall two standard results, both of them are stated and proved in [11]. The first one is a Carleman estimate for the gradient:

Lemma 2.5. *Let $T > 0$ and $r \in \mathbb{R}$. There exists $C > 0$ depending only on r, Ω and ω_0 such that, for every $u \in L^2(0, T; H^1(\Omega))$,*

$$\begin{aligned}\iint_{(0, T) \times \Omega} e^{-2s\alpha} s^{r+2} \lambda^{r+1} \xi^{r+2} |u|^2 \, dx \, dt \\ \leq C \left(\iint_{(0, T) \times \Omega} e^{-2s\alpha} s^r \lambda^{r+1} \xi^r |\nabla u|^2 \, dx \, dt + \iint_{(0, T) \times \omega_0} e^{-2s\alpha} s^{r+2} \lambda^{r+1} \xi^{r+2} |u|^2 \, dx \, dt \right),\end{aligned}$$

for any $\lambda \geq \lambda_0$ and $s \geq s_0 T^{11}$.

The second result is a Carleman estimate for the Laplace operator:

Lemma 2.6. *Let $T > 0$ and $r \in \mathbb{R}$. There exists $C > 0$ depending only on r, Ω and ω_0 such that, for every $u \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$,*

$$\begin{aligned}\iint_{(0, T) \times \Omega} e^{-2s\alpha} \left(s^{r+3} \lambda^{r+4} \xi^{r+3} |u|^2 + s^{r+1} \lambda^{r+2} \xi^{r+1} |\nabla u|^2 \right) \, dx \, dt \\ \leq C \left(\iint_{(0, T) \times \Omega} e^{-2s\alpha} s^r \lambda^{r+1} \xi^r |\Delta u|^2 \, dx \, dt + \iint_{(0, T) \times \omega_0} e^{-2s\alpha} s^{r+3} \lambda^{r+4} \xi^{r+3} |u|^2 \, dx \, dt \right),\end{aligned}$$

for any $\lambda \geq \lambda_0$ and $s \geq s_0 (T^{11} + T^{22})$.

We also introduce the following weight function that will be used multiple times in what follows:

$$\rho := e^{-s\alpha_{\sharp}}. \quad (2.19)$$

For this function, we have the following result:

Lemma 2.7. *Let $T > 0$. For any $\varepsilon > 0$, there exist $s_0, \lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ and $s \geq s_0 T^{22}$,*

$$\rho \leq \varepsilon \quad \text{in } (0, T).$$

Proof. There exists $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$

$$e^{24\lambda} - e^{22\lambda} \geq \frac{1}{2}e^{24\lambda_0}$$

and thus for $\lambda \geq \lambda_0$ and $s \geq s_0 T^{22}$

$$s\alpha_{\sharp} \geq s_0 e^{24\lambda_0} 2^{21}$$

and by taking s_0 and λ_0 large enough, we deduce

$$s\alpha_{\sharp} \geq -\ln \varepsilon.$$

□

With a similar proof, we can also show the following result:

Lemma 2.8. *Let $T > 0$, $p, q > 0$ and $N > M$. For any $\varepsilon > 0$, there exist $s_0, \lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ and $s \geq s_0 T^{22}$,*

$$\lambda^p (s\xi)^q e^{-Ns\alpha} \leq \varepsilon \rho^M \quad \text{in } (0, T) \times \Omega.$$

In the statement of Theorem 1.4, the weight function ρ is defined by (2.19), with $\lambda \geq \lambda_0$ and $s \geq s_0 (T^{11} + T^{22})$ for λ_0 and s_0 large enough. From the first condition in (1.15), we have $\mu \lesssim \rho^{11}$. From Lemma 2.7, there exist $s_0, \lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ and $s \geq s_0 T^{22}$, the smallness condition (2.7) in Proposition 2.2 and Proposition 2.3 is satisfied together with condition (1.7).

3 Carleman estimates

This section is devoted to the proof of Theorem 1.4. We are going to consider Carleman estimates for the first four equations in (1.14). For the two Stokes systems (for v and w) and for the heat equation satisfied by ψ , we introduce a decomposition in three terms where the third term is regular enough to apply the differential operator $\nabla^2 \Delta$ or $\nabla^2 \partial_{x_1}^2$. This decomposition, introduced in [10], allows us to consider source terms with lower regularity (L^2 in this case), instead of the one needed in [8], where some of the terms $g^{(i)}$ required to be in $L^2(0, T; H^4(\Omega)^2) \cap H^2(0, T; L^2(\Omega)^2)$ or in $L^2(0, T; H^8(\Omega)) \cap H^4(0, T; L^2(\Omega))$.

In what follows, we consider a sequence of nonempty domains ω_i such that

$$\overline{\omega_i} \subset \omega_{i+1}, \quad \overline{\omega_{i+1}} \subset \omega \cap \mathcal{O}^* \quad (i \geq 0). \quad (3.1)$$

We also consider a corresponding sequence of smooth functions such that

$$\chi_i \in C^\infty(\mathbb{R}^2; \mathbb{R}_+), \quad \chi_i \equiv 1 \text{ in } \omega_i, \quad \text{with compact support in } \omega_{i+1}. \quad (3.2)$$

3.1 Decomposition and Carleman estimate for v

We recall that ρ is defined by (2.19) and that \mathbb{A} is defined by (2.1). In the system (1.14), we consider the following decomposition of v :

$$\rho^{20} v = \rho v^{(1)} + v^{(2)} + v^{(3)}, \quad (3.3)$$

where

$$\begin{cases} -\partial_t v^{(1)} + \mathbb{A}v^{(1)} = \rho^{19} \mathbb{P} \left(w_{1\mathcal{O}^*} + g^{(1)} \right) & \text{in } (0, T), \quad v^{(1)}(T, \cdot) = 0 \quad \text{in } \Omega, & (3.4a) \\ -\partial_t v^{(2)} + \mathbb{A}v^{(2)} = -19\rho' v^{(1)} & \text{in } (0, T), \quad v^{(2)}(T, \cdot) = 0 \quad \text{in } \Omega, & (3.4b) \\ -\partial_t v^{(3)} + \mathbb{A}v^{(3)} = -20 \frac{\rho'}{\rho} \left(v^{(2)} + v^{(3)} \right) & \text{in } (0, T), \quad v^{(3)}(T, \cdot) = 0 \quad \text{in } \Omega. & (3.4c) \end{cases}$$

Using the maximal regularity for the Stokes system, we have

$$\|v^{(1)}\|_{\mathbb{X}_1} + \|v^{(2)}\|_{\mathbb{X}_2} \lesssim \|\rho^{19} (w1_{\mathcal{O}^*} + g^{(1)})\|_{\mathbb{X}_0}. \quad (3.5)$$

Let us define

$$\begin{aligned} I_1(v^{(3)}) := & \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s^{10} \lambda^{11} \xi^{10} |v_2^{(3)}|^2 + s^8 \lambda^9 \xi^8 |\nabla v_2^{(3)}|^2 \right) dx dt \\ & + \sum_{j=0}^3 \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^{1+2j} \lambda^{2+2j} \xi^{1+2j} |\nabla^{3-j} \Delta v_2^{(3)}|^2 dx dt + \iint_{(0,T) \times \Omega} e^{-2s\alpha_{\sharp}} s^8 \lambda^9 \xi_{\sharp}^8 |v^{(3)}|^2 dx dt \end{aligned} \quad (3.6)$$

and

$$J_1(v^{(3)}) := \sum_{j=1}^3 \sum_{k=0}^j \left\| \tau_j \partial_t^k v^{(3)} \right\|_{L^2(0,T; H^{2(j-k)}(\Omega)^2)}^2, \quad (3.7)$$

where

$$\tau_j := e^{-s\alpha_{\sharp}} \lambda^{\frac{9}{2}} (s\xi_{\sharp})^{4-j\frac{12}{11}} \quad (j = 0, \dots, 3). \quad (3.8)$$

The aim of this section is to show the following result:

Lemma 3.1. *There exist λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$, the solution (v, φ, w, ψ) of (1.14) and $v^{(3)}$ defined as above satisfy*

$$I_1(v^{(3)}) + J_1(v^{(3)}) + \|\rho^{21} v\|_{\mathbb{X}_0}^2 \lesssim \|\rho^{19} (w1_{\mathcal{O}^*} + g^{(1)})\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \omega_4} e^{-2s\alpha} s^{12} \lambda^{13} \xi^{12} |v_2^{(3)}|^2 dx dt. \quad (3.9)$$

Proof. The equation (3.4c) can be written as

$$\begin{cases} -\partial_t v^{(3)} - \Delta v^{(3)} + \nabla \pi_v^{(3)} = 20s\alpha'_{\sharp} (v^{(2)} + v^{(3)}) & \text{in } (0, T) \times \Omega, \\ \nabla \cdot v^{(3)} = 0 & \text{in } (0, T) \times \Omega, \\ v^{(3)} = 0 & \text{on } (0, T) \times \partial\Omega, \\ v^{(3)}(T, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

Following the method introduced in [11] (see also [9, 23]), we apply the operator $\nabla^2 \Delta$ on the second component of the previous system, and we obtain

$$-\partial_t \nabla^2 \Delta v_2^{(3)} - \Delta \nabla^2 \Delta v_2^{(3)} = 20s\alpha'_{\sharp} \left(\nabla^2 \Delta v_2^{(2)} + \nabla^2 \Delta v_2^{(3)} \right) \quad \text{in } (0, T) \times \Omega.$$

Applying the Carleman estimate for the heat equation with non-homogeneous Neumann boundary conditions (see [12, Theorem 1]) on the above equation, we obtain the existence of λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$\begin{aligned} & \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s\lambda^2 \xi |\nabla^3 \Delta v_2^{(3)}|^2 + s^3 \lambda^4 \xi^3 |\nabla^2 \Delta v_2^{(3)}|^2 \right) dx dt \\ & \lesssim \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^2 (\alpha'_{\sharp})^2 \left(|\nabla^2 \Delta v_2^{(2)}|^2 + |\nabla^2 \Delta v_2^{(3)}|^2 \right) dx dt \\ & + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_{\sharp}} s\lambda \xi_{\sharp} \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta v_2^{(3)} \right|^2 d\gamma dt + \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^3 \lambda^4 \xi^3 |\nabla^2 \Delta v_2^{(3)}|^2 dx dt. \end{aligned}$$

Then, using (2.17) and (3.5), we deduce from the above relation

$$\begin{aligned}
& \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s\lambda^2 \xi \left| \nabla^3 \Delta v_2^{(3)} \right|^2 + s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta v_2^{(3)} \right|^2 \right) dx dt \\
& \lesssim \left\| \rho^{19} \left(w1_{\mathcal{O}^*} + g^{(1)} \right) \right\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha} s \lambda \xi_{\#} \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta v_2^{(3)} \right|^2 d\gamma dt \\
& \quad + \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta v_2^{(3)} \right|^2 dx dt. \quad (3.10)
\end{aligned}$$

On the other hand, using Lemma 2.5 twice, the first time with $u = \nabla \Delta v_2^{(3)}$ and $r = 3$, the second time with $u = \Delta v_2^{(3)}$ and $r = 5$ and adding them together, we have

$$\begin{aligned}
& \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s^5 \lambda^6 \xi^5 \left| \nabla \Delta v_2^{(3)} \right|^2 + s^7 \lambda^8 \xi^7 \left| \Delta v_2^{(3)} \right|^2 \right) dx dt \\
& \lesssim \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \left(s^5 \lambda^6 \xi^5 \left| \nabla \Delta v_2^{(3)} \right|^2 + s^7 \lambda^8 \xi^7 \left| \Delta v_2^{(3)} \right|^2 \right) dx dt \\
& \quad + \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta v_2^{(3)} \right|^2 dx dt. \quad (3.11)
\end{aligned}$$

Next, using Lemma 2.6 with $u = v_2^{(3)}$ and $r = 7$, we also have

$$\begin{aligned}
& \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s^8 \lambda^9 \xi^8 \left| \nabla v_2^{(3)} \right|^2 + s^{10} \lambda^{11} \xi^{10} \left| v_2^{(3)} \right|^2 \right) dx dt \\
& \lesssim \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{10} \lambda^{11} \xi^{10} \left| v_2^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^7 \lambda^8 \xi^7 \left| \Delta v_2^{(3)} \right|^2 dx dt. \quad (3.12)
\end{aligned}$$

Moreover, using the Poincaré inequality and the divergence-free condition, we also obtain

$$\iint_{(0,T) \times \Omega} e^{-2s\alpha} s^8 \lambda^9 \xi^8 \left| v_1^{(3)} \right|^2 dx dt \lesssim \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^8 \lambda^9 \xi^8 \left| \nabla v_2^{(3)} \right|^2 dx dt. \quad (3.13)$$

Gathering (3.10)–(3.13) and recalling (3.6), we deduce the existence of λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$\begin{aligned}
I_1(v^{(3)}) & \lesssim \left\| \rho^{19} \left(w1_{\mathcal{O}^*} + g^{(1)} \right) \right\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha} s \lambda \xi_{\#} \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta v_2^{(3)} \right|^2 d\gamma dt \\
& \quad + \sum_{j=0}^2 \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{3+2j} \lambda^{4+2j} \xi^{3+2j} \left| \nabla^{2-j} \Delta v_2^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{10} \lambda^{11} \xi^{10} \left| v_2^{(3)} \right|^2 dx dt. \quad (3.14)
\end{aligned}$$

We can check that the weight functions defined by (3.8) satisfy (2.4) and (2.5) with $\kappa = 20s\alpha'_{\#}$, so that we can apply Lemma 2.1 and using (3.14) yield

$$\begin{aligned}
I_1(v^{(3)}) + J_1(v^{(3)}) & \lesssim \left\| \rho^{19} \left(w1_{\mathcal{O}^*} + g^{(1)} \right) \right\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha} s \lambda \xi_{\#} \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta v_2^{(3)} \right|^2 d\gamma dt \\
& \quad + \sum_{j=0}^2 \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{3+2j} \lambda^{4+2j} \xi^{3+2j} \left| \nabla^{2-j} \Delta v_2^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{10} \lambda^{11} \xi^{10} \left| v_2^{(3)} \right|^2 dx dt. \quad (3.15)
\end{aligned}$$

Using a trace inequality and an interpolation inequality, we have

$$\iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_\# s \lambda \xi_\#} \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta v_2^{(3)} \right|^2 d\gamma dt \lesssim \lambda^{-8} \int_0^T \left(\tau_2 \|v_2^{(3)}\|_{H^4(\Omega)} \right)^{1/2} \left(\tau_3 \|v_2^{(3)}\|_{H^6(\Omega)} \right)^{3/2} dt \lesssim \lambda^{-8} J_1(v^{(3)})$$

and thus we can absorb the boundary term in (3.15) by taking λ_0 large enough. We thus deduce the existence of λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$\begin{aligned} I_1(v^{(3)}) + J_1(v^{(3)}) &\lesssim \left\| \rho^{19} (w_{1\mathcal{O}^*} + g^{(1)}) \right\|_{\mathbb{X}_0}^2 + \sum_{j=0}^2 \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{3+2j} \lambda^{4+2j} \xi^{3+2j} \left| \nabla^{2-j} \Delta v_2^{(3)} \right|^2 dx dt \\ &\quad + \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{10} \lambda^{11} \xi^{10} \left| v_2^{(3)} \right|^2 dx dt. \end{aligned} \quad (3.16)$$

Then, we remove some of the above local terms by using standard techniques: using (3.1) and (3.2) and integration by parts, we have

$$\begin{aligned} \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left(\frac{\partial^2}{\partial x_k \partial x_q} \Delta v_2^{(3)} \right)^2 dx dt &\leq \iint_{(0,T) \times \omega_1} \chi_0 e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left(\frac{\partial^2}{\partial x_k \partial x_q} \Delta v_2^{(3)} \right)^2 dx dt \\ &= - \iint_{(0,T) \times \omega_1} \frac{\partial}{\partial x_k} (\chi_0 e^{-2s\alpha} s^3 \lambda^4 \xi^3) \frac{\partial^2}{\partial x_k \partial x_q} \Delta v_2^{(3)} \frac{\partial}{\partial x_q} \Delta v_2^{(3)} dx dt \\ &\quad - \iint_{(0,T) \times \omega_1} \chi_0 e^{-2s\alpha} s^3 \lambda^4 \xi^3 \frac{\partial^3}{\partial x_k^2 \partial x_q} \Delta v_2^{(3)} \frac{\partial}{\partial x_q} \Delta v_2^{(3)} dx dt. \end{aligned}$$

Using (2.18) and the Young's inequality, we can check that for any $\varepsilon > 0$,

$$\iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta v_2^{(3)} \right|^2 dx dt \lesssim \varepsilon I_1(v^{(3)}) + \frac{1}{\varepsilon} \iint_{(0,T) \times \omega_1} e^{-2s\alpha} s^5 \lambda^6 \xi^5 \left| \nabla \Delta v_2^{(3)} \right|^2 dx dt.$$

We have similar estimates for the rest of the local terms, thus we deduce from (3.16) that

$$I_2(v^{(3)}) + J_2(v^{(3)}) \lesssim \left\| \rho^{19} (w_{1\mathcal{O}^*} + g^{(1)}) \right\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \omega_4} e^{-2s\alpha} s^{12} \lambda^{13} \xi^{12} \left| v_2^{(3)} \right|^2 dx dt.$$

Combining the above relation with (3.3) and (3.5), we finally find (3.9). \square

3.2 Decomposition and Carleman estimates for w

We perform a similar procedure on w than the one we did above on v . We first consider the following decomposition of w :

$$\rho^{17} w = \rho w^{(1)} + w^{(2)} + w^{(3)}, \quad (3.17)$$

where

$$\begin{cases} \partial_t w^{(1)} + \mathbb{A} w^{(1)} = \rho^{16} \mathbb{P} (\psi e_2 + g^{(3)}) & \text{in } (0, T), \quad w^{(1)}(0, \cdot) = 0 \quad \text{in } \Omega, & (3.18a) \\ \partial_t w^{(2)} + \mathbb{A} w^{(2)} = 16 \rho' w^{(1)} & \text{in } (0, T), \quad w^{(2)}(0, \cdot) = 0 \quad \text{in } \Omega, & (3.18b) \\ \partial_t w^{(3)} + \mathbb{A} w^{(3)} = 17 \frac{\rho'}{\rho} (w^{(2)} + w^{(3)}) & \text{in } (0, T), \quad w^{(3)}(0, \cdot) = 0 \quad \text{in } \Omega. & (3.18c) \end{cases}$$

Using the maximal regularity of the Stokes system, we have

$$\left\| w^{(1)} \right\|_{\mathbb{X}_1} + \left\| w^{(2)} \right\|_{\mathbb{X}_2} \lesssim \left\| \rho^{16} (\psi e_2 + g^{(3)}) \right\|_{\mathbb{X}_0}. \quad (3.19)$$

Then, we set

$$I_2(w^{(3)}) := \sum_{j=0}^3 \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^{1+2j} \lambda^{2+2j} \xi^{1+2j} \left| \nabla^{3-j} \Delta w_2^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^7 \lambda^8 \xi^7 \left| w^{(3)} \right|^2 dx dt \quad (3.20)$$

and

$$J_2(w^{(3)}) := \sum_{j=1}^3 \sum_{k=0}^j \left\| \tilde{\tau}_j \partial_t^k w^{(3)} \right\|_{L^2(0,T; H^{2(j-k)}(\Omega)^2)}^2, \quad (3.21)$$

where

$$\tilde{\tau}_j := e^{-s\alpha} \lambda^4 (s\xi)_{\sharp}^{\frac{7}{2}-j\frac{12}{11}} \quad (j = 0, \dots, 3). \quad (3.22)$$

The aim of this section is to show the following result:

Lemma 3.2. *There exist λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$, the solution (v, φ, w, ψ) of (1.14) satisfies*

$$\begin{aligned} I_1(v^{(3)}) + J_1(v^{(3)}) + \|\rho^{21}v\|_{\mathbb{X}_0}^2 + I_2(w^{(3)}) + J_2(w^{(3)}) + \|\rho^{18}w\|_{\mathbb{X}_0}^2 \\ \lesssim \|\rho^{19}g^{(1)}\|_{\mathbb{X}_0}^2 + \|\rho^{16}g^{(3)}\|_{\mathbb{X}_0}^2 + \|\rho^{16}\psi\|_{\mathbb{X}_0}^2 \\ + \iint_{(0,T) \times \omega_4} e^{-2s\alpha} \left(s^{12} \lambda^{13} \xi^{12} \left| v_2^{(3)} \right|^2 + s^7 \lambda^8 \xi^7 \left| \Delta w_2^{(3)} \right|^2 \right) dx dt, \end{aligned} \quad (3.23)$$

where $v^{(3)}$ and $w^{(3)}$ are defined by (3.3)-(3.4c) and (3.17)-(3.18c).

Proof. The equation (3.18c) can be written as

$$\begin{cases} \partial_t w^{(3)} - \Delta w^{(3)} + \nabla \pi_w^{(3)} = -17s\alpha'_{\sharp} (w^{(2)} + w^{(3)}) & \text{in } (0, T) \times \Omega, \\ \nabla \cdot w^{(3)} = 0 & \text{in } (0, T) \times \Omega, \\ w^{(3)} = 0 & \text{on } (0, T) \times \partial\Omega, \\ w^{(3)}(0, \cdot) = 0 & \text{in } \Omega. \end{cases} \quad (3.24)$$

Applying the operator $\nabla^2 \Delta$ on the second component of the first equation of (3.24), we obtain

$$\partial_t \nabla^2 \Delta w_2^{(3)} - \Delta \nabla^2 \Delta w_2^{(3)} = -17s\alpha'_{\sharp} \left(\nabla^2 \Delta w_2^{(2)} + \nabla^2 \Delta w_2^{(3)} \right) \quad \text{in } (0, T) \times \Omega.$$

Applying the Carleman estimate for the heat equation with non-homogeneous Neumann boundary conditions (see [12, Theorem 1]) on the above equation, we deduce the existence of λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$\begin{aligned} \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s\lambda^2 \xi \left| \nabla^3 \Delta w_2^{(3)} \right|^2 + s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta w_2^{(3)} \right|^2 \right) dx dt \\ \lesssim \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^2 (\alpha'_{\sharp})^2 \left(\left| \nabla^2 \Delta w_2^{(2)} \right|^2 + \left| \nabla^2 \Delta w_2^{(3)} \right|^2 \right) dx dt \\ + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha} s \lambda \xi_{\sharp} \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta w_2^{(3)} \right|^2 d\gamma dt + \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta w_2^{(3)} \right|^2 dx dt. \end{aligned} \quad (3.25)$$

Then, using (2.17) and (3.19), we deduce from the above relation

$$\begin{aligned} & \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s\lambda^2 \xi \left| \nabla^3 \Delta w_2^{(3)} \right|^2 + s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta w_2^{(3)} \right|^2 \right) dx dt \\ & \lesssim \left\| \rho^{16} \left(\psi e_2 + g^{(3)} \right) \right\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_\#} s \lambda \xi_\# \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta w_2^{(3)} \right|^2 d\gamma dt \\ & \quad + \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta w_2^{(3)} \right|^2 dx dt, \end{aligned} \quad (3.26)$$

for $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$. Again, using Lemma 2.5 twice, the first time with $u = \nabla \Delta w_2^{(3)}$ and $r = 3$, the second time with $u = \Delta w_2^{(3)}$ and $r = 5$ and adding them together, we have

$$\begin{aligned} & \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s^5 \lambda^6 \xi^5 \left| \nabla \Delta w_2^{(3)} \right|^2 + s^7 \lambda^8 \xi^7 \left| \Delta w_2^{(3)} \right|^2 \right) dx dt \\ & \lesssim \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \left(s^5 \lambda^6 \xi^5 \left| \nabla \Delta w_2^{(3)} \right|^2 + s^7 \lambda^8 \xi^7 \left| \Delta w_2^{(3)} \right|^2 \right) dx dt \\ & \quad + \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left| \nabla^2 \Delta w_2^{(3)} \right|^2 dx dt, \end{aligned} \quad (3.27)$$

for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$. Next, using the ellipticity of the Laplace operator,

$$\iint_{(0,T) \times \Omega} e^{-2s\alpha_\#} s^7 \lambda^8 \xi_\#^7 \left(\left| w_2^{(3)} \right|^2 + \left| \nabla w_2^{(3)} \right|^2 \right) dx dt \lesssim \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^7 \lambda^8 \xi^7 \left| \Delta w_2^{(3)} \right|^2 dx dt.$$

Moreover, using the Poincaré inequality and the divergence-free condition, we also obtain

$$\iint_{(0,T) \times \Omega} e^{-2s\alpha_\#} s^7 \lambda^8 \xi_\#^7 \left| w_1^{(3)} \right|^2 dx dt \lesssim \iint_{(0,T) \times \Omega} e^{-2s\alpha_\#} s^7 \lambda^8 \xi_\#^7 \left| \nabla w_2^{(3)} \right|^2 dx dt. \quad (3.28)$$

Combining (3.26)-(3.28) and recalling (3.20), we deduce that for $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$\begin{aligned} I_2(w^{(3)}) & \lesssim \left\| \rho^{16} \left(\psi e_2 + g^{(3)} \right) \right\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_\#} s \lambda \xi_\# \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta w_2^{(3)} \right|^2 d\gamma dt \\ & \quad + \sum_{j=0}^2 \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{3+2j} \lambda^{4+2j} \xi^{3+2j} \left| \nabla^{2-j} \Delta w_2^{(3)} \right|^2 dx dt. \end{aligned} \quad (3.29)$$

Then, we can check that the weight functions defined by (3.22) satisfy (2.4) and (2.5) with $\kappa := -17s\alpha'_\#$, so that we can apply Lemma 2.1 and we deduce from (3.29)

$$\begin{aligned} I_2(w^{(3)}) + J_2(w^{(3)}) & \lesssim \left\| \rho^{16} \left(\psi e_2 + g^{(3)} \right) \right\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_\#} s \lambda \xi_\# \left| \frac{\partial}{\partial \nu} \nabla^2 \Delta w_2^{(3)} \right|^2 d\gamma dt \\ & \quad + \sum_{j=0}^2 \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{3+2j} \lambda^{4+2j} \xi^{3+2j} \left| \nabla^{2-j} \Delta w_2^{(3)} \right|^2 dx dt. \end{aligned}$$

As in the proof of Lemma 3.1, by taking λ_0 and s_0 large enough, we can absorb the boundary term and the local terms in $\nabla^{2-j} \Delta w_2^{(3)}$ ($j = 0, 1, 2$) to obtain

$$I_2(w^{(3)}) + J_2(w^{(3)}) \lesssim \left\| \rho^{16} \left(\psi e_2 + g^{(3)} \right) \right\|_{\mathbb{X}_0}^2 + \iint_{(0,T) \times \omega_2} e^{-2s\alpha} s^7 \lambda^8 \xi^7 \left| \Delta w_2^{(3)} \right|^2 dx dt.$$

Putting together the above relation with (3.9) and combining it with (3.17) and (3.19), we finally obtain (3.23). \square

3.3 Removing the local terms in v and w

The aim of this section is to remove in (3.23) the local terms in $v_2^{(3)}$ and in $\Delta w_2^{(3)}$. More precisely we show the following result:

Lemma 3.3. *There exist λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$, the solution (v, φ, w, ψ) of (1.14) satisfies*

$$\begin{aligned} I_1(v^{(3)}) + J_1(v^{(3)}) + \|\rho^{21}v\|_{\mathbb{X}_0}^2 + I_2(w^{(3)}) + J_2(w^{(3)}) + \|\rho^{18}w\|_{\mathbb{X}_0}^2 \\ \lesssim \|\rho^{18-\frac{1}{7}}g^{(1)}\|_{\mathbb{X}_0}^2 + \|\rho^{15-\frac{5}{7}}g^{(2)}\|_{\mathbb{X}_0}^2 + \|\rho^{16}g^{(3)}\|_{\mathbb{X}_0}^2 + \|\rho^{16}\psi\|_{\mathbb{X}_0}^2 \\ + \iint_{(0,T) \times \omega_6} \rho^{20-\frac{10}{7}} (|\varphi|^2 + |\psi|^2) dx dt, \end{aligned} \quad (3.30)$$

where $v^{(3)}$ and $w^{(3)}$ are defined by (3.3)-(3.4c) and (3.17)-(3.18c).

Proof. Using the first equation of (1.14), we have

$$\Delta w_2 = -\partial_t \Delta v_2 - \Delta^2 v_2 + \partial_{x_2} \nabla \cdot g^{(1)} - \Delta g_2^{(1)} \quad \text{in } (0, T) \times \mathcal{O}^*$$

and from (3.17)

$$w_2^{(3)} = \rho^{17}w_2 - \rho w_2^{(1)} - w_2^{(2)},$$

so that

$$\Delta w_2^{(3)} = \rho^{17} \left(-\partial_t \Delta v_2 - \Delta^2 v_2 + \partial_{x_2} \nabla \cdot g^{(1)} - \Delta g_2^{(1)} \right) - \rho \Delta w_2^{(1)} - \Delta w_2^{(2)}.$$

Using (3.1), (3.2) and Lemma 2.8, we can thus write

$$\begin{aligned} \iint_{(0,T) \times \omega_2} e^{-2s\alpha} s^7 \lambda^8 \xi^7 \left| \Delta w_2^{(3)} \right|^2 dx dt &\leq \iint_{(0,T) \times \omega_5} \chi_4 e^{-(2-\frac{1}{7})s\alpha} \left| \Delta w_2^{(3)} \right|^2 dx dt \\ &= \iint_{(0,T) \times \omega_5} \chi_4 \rho^{(19-\frac{1}{7})} \Delta w_2^{(3)} \left(-\partial_t \Delta v_2 - \Delta^2 v_2 + \partial_{x_2} \nabla \cdot g^{(1)} - \Delta g_2^{(1)} \right) dx dt \\ &\quad - \iint_{(0,T) \times \omega_5} \chi_4 \rho^{(2-\frac{1}{7})} \Delta w_2^{(3)} \left(\rho \Delta w_2^{(1)} + \Delta w_2^{(2)} \right) dx dt. \end{aligned} \quad (3.31)$$

Let us estimate the first term on the right-hand side of (3.31), the other terms can be estimated similarly. After some integrations by parts, we find

$$\begin{aligned} - \iint_{(0,T) \times \omega_5} \chi_4 \rho^{(19-\frac{1}{7})} \Delta w_2^{(3)} \partial_t \Delta v_2 dx dt \\ = \iint_{(0,T) \times \omega_5} \left(19 - \frac{1}{7} \right) \rho^{(18-\frac{1}{7})} \rho' \Delta \left(\chi_4 \Delta w_2^{(3)} \right) v_2 dx dt \\ + \iint_{(0,T) \times \omega_5} \rho^{(19-\frac{1}{7})} \Delta \left(\chi_4 \partial_t \Delta w_2^{(3)} \right) v_2 dx dt. \end{aligned}$$

Using Lemma 2.8, we have

$$|\rho'| \lesssim \rho^{1-\frac{1}{7}}$$

and therefore, using (3.19), (3.21), (3.22) and Young's inequality we obtain that for any $\varepsilon > 0$,

$$\begin{aligned} \iint_{(0,T) \times \omega_2} e^{-2s\alpha} s^7 \lambda^8 \xi^7 \left| \Delta w_2^{(3)} \right|^2 dx dt &\lesssim \varepsilon J_1(w^{(3)}) + \frac{1}{\varepsilon} \iint_{(0,T) \times \omega_5} \rho^{2(18-\frac{2}{7})} |v_2|^2 dx dt \\ &\quad + \|\rho^{18-\frac{1}{7}}g^{(1)}\|_{\mathbb{X}_0}^2 + \|\rho^{16}g^{(3)}\|_{\mathbb{X}_0}^2 + \|\rho^{16}\psi\|_{\mathbb{X}_0}^2. \end{aligned} \quad (3.32)$$

Now using (3.3) and (3.5),

$$\iint_{(0,T) \times \omega_4} e^{-2s\alpha} s^{12} \lambda^{13} \xi^{12} |v_2^{(3)}|^2 dx dt \lesssim \iint_{(0,T) \times \omega_4} \rho^{40} |v_2|^2 dx dt + \left\| \rho^{19} (w1_{\mathcal{O}^*} + g^{(1)}) \right\|_{\mathbb{X}_0}^2$$

Combining the above estimate with Lemma 2.7, (3.9) and (3.32) yields

$$\begin{aligned} I_1(v^{(3)}) + J_1(v^{(3)}) + \left\| \rho^{21} v \right\|_{\mathbb{X}_0}^2 + I_2(w^{(3)}) + J_2(w^{(3)}) + \left\| \rho^{18} w \right\|_{\mathbb{X}_0}^2 \\ \lesssim \left\| \rho^{18-\frac{1}{7}} g^{(1)} \right\|_{\mathbb{X}_0}^2 + \left\| \rho^{16} g^{(3)} \right\|_{\mathbb{X}_0}^2 + \left\| \rho^{16} \psi \right\|_{\mathbf{X}_0}^2 + \iint_{(0,T) \times \omega_5} \rho^{2(18-\frac{2}{7})} |v_2|^2 dx dt. \end{aligned} \quad (3.33)$$

Let us now remove the local term in v_2 . The second equation of (1.14) implies

$$-\partial_t \varphi - \Delta \varphi = v_2 + \psi + g^{(2)} \quad \text{in } (0, T) \times \mathcal{O}^*$$

so that with (3.3)

$$\iint_{(0,T) \times \omega_5} \rho^{2(18-\frac{2}{7})} |v_2|^2 dx dt \lesssim \iint_{(0,T) \times \omega_6} \chi_5 \rho^{36-\frac{4}{7}} \left(\rho v_2^{(1)} + v_2^{(2)} + v_2^{(3)} \right) \left(-\partial_t \varphi - \Delta \varphi - \psi - g^{(2)} \right) dx dt.$$

After some integrations by parts and using (3.5), (3.7), (3.8) and Young's inequality, we find that for any $\varepsilon > 0$

$$\begin{aligned} \iint_{(0,T) \times \omega_5} \rho^{2(18-\frac{2}{7})} |v_2|^2 dx dt \\ \lesssim \varepsilon J_2(v^{(3)}) + \frac{1}{\varepsilon} \left\| \rho^{19} (w1_{\mathcal{O}^*} + g^{(1)}) \right\|_{\mathbb{X}_0}^2 + \frac{1}{\varepsilon} \left\| \rho^{15-\frac{5}{7}} g^{(2)} \right\|_{\mathbf{X}_0}^2 \\ + \frac{1}{\varepsilon} \iint_{(0,T) \times \omega_6} \rho^{20-\frac{10}{7}} \left(|\varphi|^2 + |\psi|^2 \right) dx dt. \end{aligned}$$

Gathering this with (3.33) implies (3.30). \square

3.4 Carleman estimates on φ and a new equation for φ

We now apply a standard Carleman estimate for the heat equation on the second equation of (1.14) and combine it with Lemma 3.3. Let us set

$$I_3(\varphi) := \iint_{(0,T) \times \Omega} e^{-44s\alpha} \left(s^3 \lambda^4 \xi^3 |\varphi|^2 + s \lambda^2 \xi |\nabla \varphi|^2 + (s\xi)^{-1} \left(|\Delta \varphi|^2 + |\partial_t \varphi|^2 \right) \right) dx dt$$

and let us show the following result:

Lemma 3.4. *There exist λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$, the solution (v, φ, w, ψ) of (1.14) satisfies*

$$\begin{aligned} I_1(v^{(3)}) + J_1(v^{(3)}) + \left\| \rho^{21} v \right\|_{\mathbb{X}_0}^2 + I_2(w^{(3)}) + J_2(w^{(3)}) + \left\| \rho^{18} w \right\|_{\mathbb{X}_0}^2 + I_3(\varphi) + \left\| \rho^{22} \varphi \right\|_{\mathbf{X}_0}^2 \\ \lesssim \left\| \rho^{18-\frac{1}{7}} g^{(1)} \right\|_{\mathbb{X}_0}^2 + \left\| \rho^{15-\frac{5}{7}} g^{(2)} \right\|_{\mathbf{X}_0}^2 + \left\| \rho^{16} g^{(3)} \right\|_{\mathbb{X}_0}^2 + \left\| \rho^{16} \psi \right\|_{\mathbf{X}_0}^2 \\ + \iint_{(0,T) \times \omega_6} \rho^{20-\frac{10}{7}} \left(|\varphi|^2 + |\psi|^2 \right) dx dt, \end{aligned} \quad (3.34)$$

where $v^{(3)}$ and $w^{(3)}$ are defined by (3.3)-(3.4c) and (3.17)-(3.18c).

Proof. Recall that φ satisfies

$$\begin{cases} -\partial_t \varphi - \Delta \varphi = v_2 + \psi \mathbf{1}_{\mathcal{O}^*} + g^{(2)} & \text{in } (0, T) \times \Omega, \\ \varphi = 0 & \text{on } (0, T) \times \partial\Omega, \\ \varphi(T, \cdot) = \varphi^0 & \text{in } \Omega. \end{cases}$$

Thus, applying a standard Carleman estimate for the heat equation (see, for instance, [13]), there exist λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$I_3(\varphi) \lesssim \iint_{(0, T) \times \omega_0} e^{-44s\alpha} s^3 \lambda^4 \xi^3 |\varphi|^2 dx dt + \iint_{(0, T) \times \Omega} e^{-44s\alpha} \left(|v_2|^2 + |\psi|^2 + |g^{(2)}|^2 \right) dx dt.$$

Combining the above relation with (3.30) and using Lemma 2.7, we find (3.34). \square

Now, before working on ψ , we first combine the equations in (1.14) to obtain a new equation for φ :

Lemma 3.5. *Assume (v, φ, w, ψ) is a smooth solution of (1.14). Then*

$$\begin{aligned} (\partial_t - \Delta)(-\partial_t - \Delta)^2 \Delta \varphi &= \partial_{x_1}^2 \psi + \Delta g_2^{(3)} - \partial_{x_2} \nabla \cdot g^{(3)} + (\partial_t - \Delta) \left(\Delta g_2^{(1)} - \partial_{x_2} \nabla \cdot g^{(1)} \right) \\ &\quad + (-\partial_t - \Delta) \Delta g^{(4)} + (\partial_t - \Delta)(-\partial_t - \Delta) \Delta g^{(2)} \quad \text{in } (0, T) \times [\mathcal{O}^* \cap \omega]. \end{aligned} \quad (3.35)$$

Proof. From (1.14), we have first

$$\Delta \pi_w = \partial_{x_2} \psi + \nabla \cdot g^{(3)} \quad \text{in } (0, T) \times \Omega, \quad \Delta \pi_v = \nabla \cdot g^{(1)} \quad \text{in } (0, T) \times \mathcal{O}^*,$$

and we deduce

$$\begin{aligned} \partial_t \Delta w_2 - \Delta^2 w_2 &= \partial_{x_1}^2 \psi + \Delta g_2^{(3)} - \partial_{x_2} \nabla \cdot g^{(3)} \quad \text{in } (0, T) \times \Omega, \\ -\partial_t \Delta v_2 - \Delta^2 v_2 &= \Delta w_2 + \Delta g_2^{(1)} - \partial_{x_2} \nabla \cdot g^{(1)} \quad \text{in } (0, T) \times \mathcal{O}^*. \end{aligned}$$

The two above relations imply

$$(\partial_t - \Delta)(-\partial_t - \Delta) \Delta v_2 = \partial_{x_1}^2 \psi + \Delta g_2^{(3)} - \partial_{x_2} \nabla \cdot g^{(3)} + (\partial_t - \Delta) \left(\Delta g_2^{(1)} - \partial_{x_2} \nabla \cdot g^{(1)} \right) \quad \text{in } (0, T) \times \mathcal{O}^*.$$

Combining this relation with the second equation of (1.14) yields

$$\begin{aligned} (\partial_t - \Delta)(-\partial_t - \Delta)^2 \Delta \varphi &= \partial_{x_1}^2 \psi + \Delta g_2^{(3)} - \partial_{x_2} \nabla \cdot g^{(3)} + (\partial_t - \Delta) \left(\Delta g_2^{(1)} - \partial_{x_2} \nabla \cdot g^{(1)} \right) \\ &\quad + (\partial_t - \Delta)(-\partial_t - \Delta) \Delta \left(\psi + g^{(2)} \right) \quad \text{in } (0, T) \times \mathcal{O}^*. \end{aligned}$$

Finally, using the fourth equation in (1.14) and recalling $\omega \cap \mathcal{O} = \emptyset$, we deduce (3.35). \square

3.5 Decomposition and Carleman estimates for ψ

We now deal with ψ that satisfies the fourth equation in (1.14). We follow the same approach as what we did for v and w . First we consider the following decomposition of ψ :

$$\rho^{13} \psi = \rho \psi^{(1)} + \psi^{(2)} + \psi^{(3)}, \quad (3.36)$$

where

$$\begin{cases} \partial_t \psi^{(1)} + \mathbf{A} \psi^{(1)} = \rho^{12} \left(-\mu \varphi \mathbf{1}_{\mathcal{O}} + g^{(4)} \right) & \text{in } (0, T), \quad \psi^{(1)}(0, \cdot) = 0 \quad \text{in } \Omega, & (3.37a) \\ \partial_t \psi^{(2)} + \mathbf{A} \psi^{(2)} = 12 \rho' \psi^{(1)} & \text{in } (0, T), \quad \psi^{(2)}(0, \cdot) = 0 \quad \text{in } \Omega, & (3.37b) \\ \partial_t \psi^{(3)} + \mathbf{A} \psi^{(3)} = 13 \frac{\rho'}{\rho} \left(\psi^{(2)} + \psi^{(3)} \right) & \text{in } (0, T), \quad \psi^{(3)}(0, \cdot) = 0 \quad \text{in } \Omega. & (3.37c) \end{cases}$$

Using the maximal regularity of the heat equation, we have

$$\left\| \psi^{(1)} \right\|_{\mathbf{x}_1} + \left\| \psi^{(2)} \right\|_{\mathbf{x}_2} \lesssim \left\| \rho^{12} \left(-\mu\varphi 1_{\mathcal{O}} + g^{(4)} \right) \right\|_{\mathbf{x}_0}. \quad (3.38)$$

Let us set

$$I_4(\psi^{(3)}) := \sum_{j=0}^3 \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^{1+2j} \lambda^{2+2j} \xi^{1+2j} \left| \nabla^{3-j} \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt + \iint_{(0,T) \times \Omega} e^{-2s\alpha_{\#}} s^7 \lambda^8 \xi_{\#}^7 \left| \psi^{(3)} \right|^2 dx dt \quad (3.39)$$

and

$$J_4(\psi) := \sum_{j=1}^5 \sum_{k=0}^j \left\| \widehat{\tau}_j \partial_t^k \psi \right\|_{L^2(0,T; H^{2(j-k)}(\Omega))}^2, \quad (3.40)$$

where

$$\widehat{\tau}_j := e^{-s\alpha_{\#}} \lambda^4 (s\xi_{\#})^{\frac{7}{2}-j\frac{12}{11}} \quad (j = 0, \dots, 5). \quad (3.41)$$

We then show the following result:

Lemma 3.6. *There exist λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$, the solution (v, φ, w, ψ) of (1.14) satisfies*

$$I_4(\psi^{(3)}) + J_4(\psi^{(3)}) + \left\| \rho^{14} \psi \right\|_{\mathbf{x}_0}^2 \lesssim \left\| \rho^{12} \left(-\mu\varphi 1_{\mathcal{O}} + g^{(4)} \right) \right\|_{\mathbf{x}_0}^2 + \iint_{(0,T) \times \omega_2} e^{-2s\alpha} s^7 \lambda^8 \xi^7 \left| \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt. \quad (3.42)$$

where $\psi^{(3)}$ is defined by (3.36)-(3.37c).

Proof. Applying the operator $\nabla^2 \partial_{x_1}^2$ on the equation of $\psi^{(3)}$, we obtain

$$\partial_t \nabla^2 \partial_{x_1}^2 \psi^{(3)} - \Delta \nabla^2 \partial_{x_1}^2 \psi^{(3)} = -13s\alpha'_{\#} \left(\nabla^2 \partial_{x_1}^2 \psi^{(2)} + \nabla^2 \partial_{x_1}^2 \psi^{(3)} \right) \quad \text{in } (0, T) \times \Omega.$$

We apply on the above equation the Carleman on the heat equation with non-homogeneous Neumann boundary conditions (see [12, Theorem 1]): there exist λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$\begin{aligned} & \iint_{(0,T) \times \Omega} e^{-2s\alpha} \left(s\lambda^2 \xi \left| \nabla^3 \partial_{x_1}^2 \psi^{(3)} \right|^2 + s^3 \lambda^4 \xi^3 \left| \nabla^2 \partial_{x_1}^2 \psi^{(3)} \right|^2 \right) dx dt \\ & \lesssim \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^2 (\alpha'_{\#})^2 \left(\left| \nabla^2 \partial_{x_1}^2 \psi^{(2)} \right|^2 + \left| \nabla^2 \partial_{x_1}^2 \psi^{(3)} \right|^2 \right) dx dt \\ & + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_{\#}} s \lambda \xi_{\#} \left| \frac{\partial}{\partial \nu} \nabla^2 \partial_{x_1}^2 \psi^{(3)} \right|^2 d\gamma dt + \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^3 \lambda^4 \xi^3 \left| \nabla^2 \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt. \end{aligned}$$

Next, we combine the above relation with (3.38) and with a Carleman estimate for the gradient operator (similar to those used for v and w): there exist λ_0 and s_0 such that for any $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$\begin{aligned} & \sum_{j=0}^3 \iint_{(0,T) \times \Omega} e^{-2s\alpha} s^{1+2j} \lambda^{2+2j} \xi^{1+2j} \left| \nabla^{3-j} \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt \\ & \lesssim \left\| \rho^{12} \left(-\mu\varphi 1_{\mathcal{O}} + g^{(4)} \right) \right\|_{\mathbf{x}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_{\#}} s \lambda \xi_{\#} \left| \frac{\partial}{\partial \nu} \nabla^2 \partial_{x_1}^2 \psi^{(3)} \right|^2 d\gamma dt \\ & \quad + \sum_{j=0}^2 \iint_{(0,T) \times \omega_0} e^{-2s\alpha} s^{3+2j} \lambda^{4+2j} \xi^{3+2j} \left| \nabla^{2-j} \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt. \end{aligned}$$

Next, we use the following property of the operator $\partial_{x_1}^2$ on $H_0^1(\Omega)$ (see [8, relation (3.15)])

$$\iint_{(0,T) \times \Omega} e^{-2s\alpha_\# s^7 \lambda^8 \xi_\#^7} \left| \psi^{(3)} \right|^2 dx dt \lesssim \iint_{(0,T) \times \Omega} e^{-2s\alpha_\# s^7 \lambda^8 \xi_\#^7} \left| \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt.$$

Combining the above relations, we deduce that $I_4(\psi^{(3)})$ defined by (3.39) satisfies for $\lambda \geq \lambda_0$ and $s \geq s_0(T^{11} + T^{22})$,

$$\begin{aligned} I_4(\psi^{(3)}) &\lesssim \left\| \rho^{12} \left(-\mu\varphi 1_{\mathcal{O}} + g^{(4)} \right) \right\|_{\mathbf{X}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_\# s \lambda \xi_\#} \left| \frac{\partial}{\partial \nu} \nabla^2 \partial_{x_1}^2 \psi^{(3)} \right|^2 d\gamma dt \\ &\quad + \sum_{j=0}^2 \iint_{(0,T) \times \omega_0} e^{-2s\alpha_\# s^{3+2j} \lambda^{4+2j} \xi_\#^{3+2j}} \left| \nabla^{2-j} \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt. \end{aligned} \quad (3.43)$$

We can check that the weight functions defined by (3.41) satisfy (2.4) and (2.5) with $\kappa := -13s\alpha'_\#$. Applying Lemma 2.1 and using (3.43) yield

$$\begin{aligned} I_4(\psi^{(3)}) + J_4(\psi) &\lesssim \left\| \rho^{12} \left(-\mu\varphi 1_{\mathcal{O}} + g^{(4)} \right) \right\|_{\mathbf{X}_0}^2 + \iint_{(0,T) \times \partial\Omega} e^{-2s\alpha_\# s \lambda \xi_\#} \left| \frac{\partial}{\partial \nu} \nabla^2 \partial_{x_1}^2 \psi^{(3)} \right|^2 d\gamma dt \\ &\quad + \sum_{j=0}^2 \iint_{(0,T) \times \omega_0} e^{-2s\alpha_\# s^{3+2j} \lambda^{4+2j} \xi_\#^{3+2j}} \left| \nabla^{2-j} \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt. \end{aligned}$$

Taking λ_0 and s_0 large enough, we can absorb the boundary term and estimate the local terms in $\nabla^2 \partial_{x_1}^2 \psi^{(3)}$ and in $\nabla \partial_{x_1}^2 \psi^{(3)}$ as in the proof of Lemma 3.2 to obtain (3.42). \square

3.6 Proof of Theorem 1.4

We now combine Lemma 3.4, Lemma 3.5 and Lemma 3.6 to prove Theorem 1.4:

Proof of Theorem 1.4. We estimate the local term appearing in (3.42): using (3.36) we can write

$$\begin{aligned} \iint_{(0,T) \times \omega_2} e^{-2s\alpha_\# s^7 \lambda^8 \xi_\#^7} \left| \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt \\ \lesssim \iint_{(0,T) \times \omega_3} \chi_2 \rho^{(2-\frac{1}{7})} \partial_{x_1}^2 \psi^{(3)} \left(\rho^{13} \partial_{x_1}^2 \psi - \rho \partial_{x_1}^2 \psi^{(1)} - \partial_{x_1}^2 \psi^{(2)} \right) dx dt. \end{aligned}$$

Then, using (3.35), we deduce

$$\begin{aligned} \iint_{(0,T) \times \omega_3} \chi_2 \rho^{(15-\frac{1}{7})} \partial_{x_1}^2 \psi^{(3)} \partial_{x_1}^2 \psi dx dt &= \iint_{(0,T) \times \omega_3} \chi_2 \rho^{(15-\frac{1}{7})} \partial_{x_1}^2 \psi^{(3)} \left[(\partial_t - \Delta)(-\partial_t - \Delta)^2 \Delta \varphi \right. \\ &\quad \left. - \Delta g_2^{(3)} + \partial_{x_2} \nabla \cdot g^{(3)} - (\partial_t - \Delta) \left(\Delta g_2^{(1)} - \partial_{x_2} \nabla \cdot g^{(1)} \right) - (-\partial_t - \Delta) \Delta g^{(4)} - (\partial_t - \Delta)(-\partial_t - \Delta) \Delta g^{(2)} \right] dx dt. \end{aligned}$$

Using (3.40) and (3.41), several integration by parts and Young's inequality, we obtain that for any $\varepsilon > 0$,

$$\begin{aligned} \iint_{(0,T) \times \omega_2} e^{-2s\alpha_\# s^7 \lambda^8 \xi_\#^7} \left| \partial_{x_1}^2 \psi^{(3)} \right|^2 dx dt &\lesssim \varepsilon J_4(\psi^{(3)}) + \frac{1}{\varepsilon} \left\| \rho^{12} \left(-\mu\varphi 1_{\mathcal{O}} + g^{(4)} \right) \right\|_{\mathbf{X}_0}^2 \\ &\quad + \frac{1}{\varepsilon} \iint_{(0,T) \times \omega_3} \rho^{18-\frac{2}{7}} |\varphi|^2 dx dt + \sum_{i=1}^4 \frac{1}{\varepsilon} \iint_{(0,T) \times \Omega} \rho^{18-\frac{2}{7}} \left| g^{(i)} \right|^2 dx dt. \end{aligned}$$

Gathering the above estimate with (3.42) and (3.34), we finally find

$$\begin{aligned}
& I_1(v^{(3)}) + J_1(v^{(3)}) + \|\rho^{21}v\|_{\mathbf{x}_0}^2 + I_2(w^{(3)}) + J_2(w^{(3)}) + \|\rho^{18}w\|_{\mathbf{x}_0}^2 \\
& \quad + I_3(\varphi) + \|\rho^{22}\varphi\|_{\mathbf{x}_0}^2 + I_4(\psi^{(3)}) + J_4(\psi^{(3)}) + \|\rho^{14}\psi\|_{\mathbf{x}_0}^2 \\
& \lesssim \|\rho^{12}\mu\varphi 1_{\mathcal{O}}\|_{\mathbf{x}_0}^2 + \sum_{i=1}^3 \iint_{(0,T) \times \Omega} \rho^{18-\frac{2}{7}} |g^{(i)}|^2 dx dt + \|\rho^{12}g^{(4)}\|_{\mathbf{x}_0}^2 \\
& \quad + \iint_{(0,T) \times \omega_6} \rho^{18-\frac{2}{7}} |\varphi|^2 dx dt.
\end{aligned}$$

Recalling (1.15), we see that we can absorb the term $\rho^{12}\mu\varphi 1_{\mathcal{O}}$ by the left-hand side and we thus deduce (1.16). This concludes the proof of Theorem 1.4. \square

4 Proof of the exact hierarchical controllability

In this section, we show how to deduce Theorem 1.2 from Theorem 1.4. The arguments are quite standard but we recall them for sake of completeness. Recall that h and g are defined in (2.6). First, we modify the weight functions in (1.16) by replacing ρ defined in (2.19) by ρ_{\natural} given by

$$\rho_{\natural}(t) := \begin{cases} \rho\left(\frac{T}{2}\right) & t \in \left[0, \frac{T}{2}\right] \\ \rho(t) & t \in \left[\frac{T}{2}, T\right] \end{cases}.$$

Hence, the functions ρ_1 and ρ_2 mentioned in Theorem 1.2 are precisely

$$\rho_1 := \rho^{11}, \quad \rho_2 := \rho_{\natural}^{22}. \tag{4.1}$$

Therefore, we have the following:

Corollary 4.1. *With the hypotheses of Theorem 1.4, the solution (v, φ, w, ψ) of (1.14) satisfies*

$$\|\rho_{\natural}^{22}(v, \varphi, w, \psi)\|_{L^2(0,T;L^2(\Omega)^6)} + \|(v(0, \cdot), \varphi(0, \cdot))\|_{L^2(\Omega)^3} \lesssim \|\rho_{\natural}^{12}g\|_{L^2(0,T;L^2(\Omega)^6)} + \|\rho_{\natural}^{13}\varphi\|_{L^2(0,T;L^2(\omega))}. \tag{4.2}$$

Proof. From (1.16), we first deduce that

$$\|\rho_{\natural}^{22}(v, \varphi, w, \psi)\|_{L^2(T/2,T;L^2(\Omega)^6)} \lesssim \|\rho_{\natural}^{12}g\|_{L^2(0,T;L^2(\Omega)^6)} + \|\rho_{\natural}^{13}\varphi\|_{L^2(0,T;L^2(\omega))}. \tag{4.3}$$

We consider $\chi \in C^\infty(\mathbb{R}; [0, 1])$, $\chi \equiv 1$ in $(-\infty, T/2]$, $\chi \equiv 0$ in $[3T/4, \infty)$. We deduce from (1.14) and (2.9) that

$$\begin{cases} L^*(\chi(v, \pi_v, \varphi, w, \pi_w, \psi)) = \chi g + \chi'(v, \varphi, w, \psi) & \text{in } (0, T) \times \Omega, \\ \nabla \cdot (\chi v) = \nabla \cdot (\chi w) = 0 & \text{in } (0, T) \times \Omega, \\ (\chi v) = (\chi w) = 0, \quad (\chi \varphi) = (\chi \psi) = 0 & \text{on } (0, T) \times \partial\Omega, \\ (\chi v)(T, \cdot) = 0, \quad (\chi \varphi)(T, \cdot) = 0, \quad (\chi w)(0, \cdot) = 0, \quad (\chi \psi)(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

Applying Proposition 2.2, we deduce that

$$\|\chi(v, \varphi, w, \psi)\|_{L^2(0,T;H^2(\Omega)^6) \cap H^1(0,T;L^2(\Omega)^6)} \lesssim \|\chi g\|_{L^2(0,T;L^2(\Omega)^6)} + \|\chi'(v, \varphi, w, \psi)\|_{L^2(0,T;L^2(\Omega)^6)}. \tag{4.4}$$

We note that the support of χ' is included in $(T/2, 3T/4)$ and we have

$$\rho_{\natural}(t) \geq \rho_{\natural}\left(\frac{3T}{4}\right) > 0 \quad (t \in (T/2, 3T/4)).$$

Combining this with (4.4), we obtain

$$\begin{aligned} & \left\| \rho_{\natural}^{22} (v, \varphi, w, \psi) \right\|_{L^2(0, T/2; L^2(\Omega)^6)} + \|(v(0, \cdot), \varphi(0, \cdot))\|_{L^2(\Omega)^3} \\ & \lesssim \left\| \rho_{\natural}^{12} g \right\|_{L^2(0, 3T/4; L^2(\Omega)^6)} + \left\| \rho_{\natural}^{22} (v, \varphi, w, \psi) \right\|_{L^2(T/2, 3T/4; L^2(\Omega)^6)}. \end{aligned}$$

The above equation and (4.3) imply (4.2). \square

We now consider the Banach space

$$\mathcal{F} := \left\{ h : \frac{h}{\rho_{\natural}^{22}} \in L^2(0, T; L^2(\Omega)^6) \right\}$$

endowed with the norm

$$\|h\|_{\mathcal{F}} := \left\| \frac{h}{\rho_{\natural}^{22}} \right\|_{L^2(0, T; L^2(\Omega)^6)}.$$

We deduce from Corollary 4.1 the partial null-controllability of the linear system (1.11).

Proposition 4.2. *Assume that $\omega \cap \mathcal{O} = \emptyset$, that $\omega \cap \mathcal{O}^* \neq \emptyset$ and (1.15). Then, there exists a linear continuous operator*

$$\mathcal{M} : \mathbb{V} \times H_0^1(\Omega) \times \mathcal{F} \rightarrow [\mathbb{X}_1 \times \mathbf{X}_1]^2 \times L^2(0, T; L^2(\omega))$$

such that for any $(y^0, \theta^0) \in \mathbb{V} \times H_0^1(\Omega)$ and $h \in \mathcal{F}$,

$$((y, \theta, u, \sigma), f) := \mathcal{M}((y^0, \theta^0), h)$$

corresponds to the strong solution of (1.11) associated with h , (y^0, θ^0) and with a control f such that we have the estimate

$$\left\| \frac{(y, \theta, u, \sigma)}{\rho_{\natural}^{11}} \right\|_{L^2(0, T; H^2(\Omega)^6) \cap H^1(0, T; L^2(\Omega)^6)} + \left\| \frac{f}{\rho_{\natural}^{13}} \right\|_{L^2(0, T; L^2(\omega))} \lesssim \|h\|_{\mathcal{F}} + \|(y^0, \theta^0)\|_{H^1(\Omega)^3}. \quad (4.5)$$

In particular

$$y(T, \cdot) = 0, \quad \theta(T, \cdot) = 0 \quad \text{in } \Omega.$$

Proof. We define

$$\begin{aligned} \mathcal{X}_0 := & \left\{ (v, \pi_v, \varphi, w, \pi_w, \psi) \in C^\infty([0, T] \times \bar{\Omega})^8 : \nabla \cdot v = \nabla \cdot w = 0 \quad \text{in } [0, T] \times \bar{\Omega}, \right. \\ & \left. v = w = 0 \quad \text{on } [0, T] \times \partial\Omega, \quad \varphi = \psi = 0 \quad \text{on } [0, T] \times \partial\Omega, \quad \int_{\Omega} \pi_v \, dx = \int_{\Omega} \pi_w \, dx = 0 \quad \text{in } [0, T] \right\} \end{aligned}$$

and we recall that L^* is defined by (2.9) and permits to write (1.14) as (2.10). Then, we consider the bilinear form

$$\begin{aligned} & \left\langle (v, \pi_v, \varphi, w, \pi_w, \psi), (\widehat{v}, \widehat{\pi}_v, \widehat{\varphi}, \widehat{w}, \widehat{\pi}_w, \widehat{\psi}) \right\rangle_{\mathcal{X}} \\ & := \iint_{(0, T) \times \Omega} \rho_{\natural}^{24} L^*(v, \pi_v, \varphi, w, \pi_w, \psi) \cdot L^*(\widehat{v}, \widehat{\pi}_v, \widehat{\varphi}, \widehat{w}, \widehat{\pi}_w, \widehat{\psi}) \, dx \, dt + \iint_{(0, T) \times \Omega} \rho_{\natural}^{26} \varphi \widehat{\varphi} \, dx \, dt \quad (4.6) \end{aligned}$$

and the linear form

$$\ell \left((\widehat{v}, \widehat{\pi}_v, \widehat{\varphi}, \widehat{w}, \widehat{\pi}_w, \widehat{\psi}) \right) := \iint_{(0, T) \times \Omega} (\widehat{v}, \widehat{\varphi}, \widehat{w}, \widehat{\psi}) \cdot h \, dx \, dt + \int_{\Omega} (\widehat{v}(0, \cdot) \cdot y^0 + \widehat{\varphi}(0, \cdot) \theta^0) \, dx.$$

Using (4.2), we see that the bilinear map $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ defined by (4.6) is a scalar product on \mathcal{X}_0 . We denote by $\|\cdot\|_{\mathcal{X}}$ the corresponding norm and by \mathcal{X} the completion of \mathcal{X}_0 for this norm. We deduce from (4.2) that

$$\|\rho_{\natural}^{22}(v, \varphi, w, \psi)\|_{L^2(0,T;L^2(\Omega)^6)} + \|(v(0, \cdot), \varphi(0, \cdot))\|_{L^2(\Omega)^3} \lesssim \|(v, \pi_v, \varphi, w, \pi_w, \psi)\|_{\mathcal{X}}.$$

In particular, $\ell \in \mathcal{X}'$ with

$$\|\ell\|_{\mathcal{X}'} \lesssim \left\| \frac{h}{\rho_{\natural}^{22}} \right\|_{L^2(0,T;L^2(\Omega)^6)} + \|(y^0, \theta^0)\|_{L^2(\Omega)^3}.$$

We can thus apply the Riesz theorem and obtain the existence and uniqueness of $(v, \pi_v, \varphi, w, \pi_w, \psi) \in \mathcal{X}$ such that for any $(\widehat{v}, \widehat{\pi}_v, \widehat{\varphi}, \widehat{w}, \widehat{\pi}_w, \widehat{\psi}) \in \mathcal{X}$,

$$\left\langle (v, \pi_v, \varphi, w, \pi_w, \psi), (\widehat{v}, \widehat{\pi}_v, \widehat{\varphi}, \widehat{w}, \widehat{\pi}_w, \widehat{\psi}) \right\rangle_{\mathcal{X}} = \ell(\widehat{v}, \widehat{\pi}_v, \widehat{\varphi}, \widehat{w}, \widehat{\pi}_w, \widehat{\psi}). \quad (4.7)$$

We set

$$(y, \theta, u, \sigma) := \rho_{\natural}^{24} L^*(v, \pi_v, \varphi, w, \pi_w, \psi), \quad f := \rho_{\natural}^{26} \varphi$$

and we deduce from (4.7) that (y, θ, u, σ) is a solution by transposition of (1.11) associated with f and h (in the sense of Definition 2.4). Moreover, we also obtain the estimate

$$\left\| \frac{(y, \theta, u, \sigma)}{\rho_{\natural}^{12}} \right\|_{L^2(0,T;L^2(\Omega)^6)} + \left\| \frac{f}{\rho_{\natural}^{13}} \right\|_{L^2(0,T;L^2(\omega))} \lesssim \left\| \frac{h}{\rho_{\natural}^{22}} \right\|_{L^2(0,T;L^2(\Omega)^6)} + \|(y^0, \theta^0)\|_{L^2(\Omega)^3}.$$

Then, one can check that $(y, \pi_y, \theta, u, \pi_u, \sigma)$ is a solution of

$$L \left(\frac{(y, \pi_y, \theta, u, \pi_u, \sigma)}{\rho_{\natural}^{11}} \right) = \frac{(h^{(1)}, h^{(2)} + f1_{\omega}, h^{(3)}, h^{(4)})}{\rho_{\natural}^{11}} + \frac{11\rho_{\natural}'}{\rho_{\natural}^{12}} (-y, -\theta, u, \sigma)$$

with

$$\nabla \cdot \left(\frac{y}{\rho_{\natural}^{11}} \right) = \nabla \cdot \left(\frac{u}{\rho_{\natural}^{11}} \right) = 0 \quad \text{in } (0, T) \times \Omega, \quad \frac{(y, \theta, u, \sigma)}{\rho_{\natural}^{11}} = 0 \quad \text{on } (0, T) \times \partial\Omega$$

and

$$\left(\frac{(y, \theta)}{\rho_{\natural}^{11}} \right) (0, \cdot) = \left(\frac{(y^0, \theta^0)}{\rho_{\natural}^{11}} \right), \quad \left(\frac{(u, \sigma)}{\rho_{\natural}^{11}} \right) (T, \cdot) = 0 \quad \text{in } \Omega.$$

Applying Proposition 2.2, we obtain (4.5). □

We are now in a position to prove the null-controllability result:

Proof of Theorem 1.2. The above proposition allows us to define the mapping

$$\mathcal{N}(h) := \left(-(y \cdot \nabla) y, -y \cdot \nabla \theta, -(\nabla y)^{\top} u + (y \cdot \nabla) u - \sigma \nabla \theta - y^* 1_{\mathcal{O}^*}, y \cdot \nabla \sigma - \theta^* 1_{\mathcal{O}^*} \right), \quad (4.8)$$

where $((y, \theta, u, \sigma), f) := \mathcal{M}((y^0, \theta^0), h)$ for $h \in \mathcal{F}$. We notice that if h is a fixed point of \mathcal{N} , then the corresponding solution (y, θ, u, σ) is a solution of (1.9) (see (1.12) and (1.13)). Using Sobolev embeddings and Hölder inequalities, we have

$$\left\| \frac{(y \cdot \nabla) y}{\rho_{\natural}^{22}} \right\|_{L^2(0,T;L^2(\Omega)^2)} \lesssim \left\| \left(\left(\frac{y}{\rho_{\natural}^{11}} \right) \cdot \nabla \right) \left(\frac{y}{\rho_{\natural}^{11}} \right) \right\|_{L^2(0,T;L^2(\Omega)^2)} \lesssim \left\| \frac{y}{\rho_{\natural}^{11}} \right\|_{L^2(0,T;H^2(\Omega)^2) \cap H^1(0,T;L^2(\Omega)^2)}^2.$$

Combining this with (4.5), (4.8) and

$$\frac{(y^*, \theta^*)}{\rho_{\mathfrak{h}}^{22}} \in L^2(0, T; L^2(\Omega)^3),$$

we deduce the existence of a constant $C_0 > 0$ such that for any $h, \widehat{h} \in \mathcal{F}$,

$$\|\mathcal{N}(h)\|_{\mathcal{F}} \leq C_0 \left(\left\| \frac{h}{\rho_{\mathfrak{h}}^{22}} \right\|_{L^2(0, T; L^2(\Omega)^6)} + \|(y^0, \theta^0)\|_{H^1(\Omega)^3} \right)^2 + \left\| \frac{(y^*, \theta^*)}{\rho_{\mathfrak{h}}^{22}} \right\|_{L^2(0, T; L^2(\Omega)^3)} \quad (4.9)$$

and

$$\|\mathcal{N}(h) - \mathcal{N}(\widehat{h})\|_{\mathcal{F}} \leq C_0 \left(\|h\|_{\mathcal{F}} + \|\widehat{h}\|_{\mathcal{F}} + \|(y^0, \theta^0)\|_{H^1(\Omega)^3} \right) \|h - \widehat{h}\|_{\mathcal{F}}. \quad (4.10)$$

We now consider

$$R := \frac{1}{8C_0}$$

and we assume

$$\|(y^0, \theta^0)\|_{H^1(\Omega)^3} \leq R, \quad \left\| \frac{(y^*, \theta^*)}{\rho_{\mathfrak{h}}^{22}} \right\|_{L^2(0, T; L^2(\Omega)^3)} \leq \frac{R}{2}.$$

Then, we deduce from (4.9) and (4.10) that the closed ball of the Banach space \mathcal{F} defined by

$$\mathcal{B}_{\mathcal{F}}(0, R) := \{h \in \mathcal{F} : \|h\|_{\mathcal{F}} \leq R\}$$

is invariant by \mathcal{N} and on this ball, \mathcal{N} is a strict contraction. This implies the existence of a fixed point for \mathcal{N} and this concludes the theorem. \square

References

- [1] Fágner D. Araruna, Enrique Fernández-Cara, and Luciano Cipriano da Silva. Hierarchic control for the wave equation. *J. Optim. Theory Appl.*, 178(1):264–288, 2018.
- [2] Fágner D. Araruna, Enrique Fernández-Cara, Sergio Guerrero, and Maurício C. Santos. New results on the Stackelberg-Nash exact control of linear parabolic equations. *Syst. Control Lett.*, 104:78–85, 2017.
- [3] Fágner D. Araruna, Enrique Fernández-Cara, and Maurício C. Santos. Stackelberg-Nash exact controllability for linear and semilinear parabolic equations. *ESAIM, Control Optim. Calc. Var.*, 21(3):835–856, 2015.
- [4] Fágner Dias Araruna, Silvano D. B. De Menezes, and Marko A. Rojas-Medar. On the approximate controllability of Stackelberg-Nash strategies for linearized micropolar fluids. *Appl. Math. Optim.*, 70(3):373–393, 2014.
- [5] Jon Asier Bárcena-Petisco, Sergio Guerrero, and Ademir F. Pazoto. Local null controllability of a model system for strong interaction between internal solitary waves. *Commun. Contemp. Math.*, 24(2):30, 2022. Id/No 2150003.
- [6] Thomas R. Bewley, Roger Temam, and Mohammed Ziane. A general framework for robust control in fluid mechanics. *Physica D*, 138(3-4):360–392, 2000.
- [7] Bianca M. R. Calsavara, Enrique Fernández-Cara, Luz de Teresa, and José Antonio Villa. New results concerning the hierarchical control of linear and semilinear parabolic equations. *ESAIM Control Optim. Calc. Var.*, 28:Paper No. 14, 26, 2022.
- [8] Nicolás Carreño, Sergio Guerrero, and Mamadou Gueye. Insensitizing controls with two vanishing components for the three-dimensional Boussinesq system. *ESAIM Control Optim. Calc. Var.*, 21(1):73–100, 2015.

- [9] Nicolás Carreño and Mamadou Gueye. Insensitizing controls with one vanishing component for the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 101(1):27–53, 2014.
- [10] Nicolás Carreño and Takéo Takahashi. Control problems for the Navier-Stokes system with nonlocal spatial terms. <https://hal.science/hal-03819799>, 2022.
- [11] Jean-Michel Coron and Sergio Guerrero. Null controllability of the N -dimensional Stokes system with $N - 1$ scalar controls. *J. Differential Equations*, 246(7):2908–2921, 2009.
- [12] Enrique Fernández-Cara, Manuel González-Burgos, Sergio Guerrero, and Jean-Pierre Puel. Null controllability of the heat equation with boundary Fourier conditions: the linear case. *ESAIM Control Optim. Calc. Var.*, 12(3):442–465, 2006.
- [13] Enrique Fernández-Cara and Sergio Guerrero. Global Carleman Inequalities for Parabolic Systems and Applications to Controllability. *SIAM Journal on Control and Optimization*, 45(4):1395–1446, 2006.
- [14] Enrique Fernández-Cara, Sergio Guerrero, Oleg Yu. Imanuvilov, and Jean-Pierre Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl. (9)*, 83(12):1501–1542, 2004.
- [15] Enrique Fernández-Cara, Sergio Guerrero, Oleg Yu. Imanuvilov, and Jean-Pierre Puel. Some controllability results for the N -dimensional Navier-Stokes and Boussinesq systems with $N - 1$ scalar controls. *SIAM J. Control Optim.*, 45(1):146–173, 2006.
- [16] Andrei V. Fursikov and Oleg Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [17] Francisco Guillén-González, Francisco Marques-Lopes, and Marko A. Rojas-Medar. On the approximate controllability of Stackelberg-Nash strategies for Stokes equations. *Proc. Am. Math. Soc.*, 141(5):1759–1773, 2013.
- [18] Víctor Hernández-Santamaría and Luz de Teresa. Robust Stackelberg controllability for linear and semilinear heat equations. *Evol. Equ. Control Theory*, 7(2):247–273, 2018.
- [19] Víctor Hernández-Santamaría, Luz de Teresa, and Alexander Poznyak. Corrigendum and addendum to “Hierarchic control for a coupled parabolic system”, *portugaliae math.* 73 (2016), 2: 115–137 [MR3500826]. *Port. Math.*, 74(2):161–168, 2017.
- [20] Oleg Yu. Imanuvilov. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6:39–72, 2001.
- [21] Jacques-Louis Lions. Some remarks on Stackelberg’s optimization. *Math. Models Methods Appl. Sci.*, 4(4):477–487, 1994.
- [22] Cristhian Montoya and Luz de Teresa. Robust Stackelberg controllability for the Navier-Stokes equations. *NoDEA, Nonlinear Differ. Equ. Appl.*, 25(5):33, 2018. Id/No 46.
- [23] Takéo Takahashi, Luz de Teresa, and Yingying Wu-Zhang. Controllability results for cascade systems of m coupled N -dimensional Stokes and Navier-Stokes systems by $N - 1$ scalar controls. *ESAIM Control Optim. Calc. Var.*, 29:Paper No. 31, 24, 2023.
- [24] Roger Temam. *Navier-Stokes equations*, volume 2 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York, revised edition, 1979. Theory and numerical analysis, With an appendix by F. Thomasset.
- [25] Marius Tucsnak and George Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.