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A new distributed protocol for consensus of discrete-time systems

Filippo Cacace¹, Mattia Mattioni², Salvatore Monaco² and Dorothée Normand-Cyrot³

Abstract—In this paper, a new distributed protocol is proposed to force consensus in a discrete-time network of scalar agents with an arbitrarily assignable convergence rate. Several simulations validate the performances and the improvements with respect to more standard protocols.

Index Terms—Linear systems; Consensus control and estimation; Network analysis and control.

I. INTRODUCTION

Numerous problems in engineering and beyond can be lead to enforcing consensus in a network of discrete-time dynamics as, for instance, controlling multi-agent systems under sampled-data communication [1], modeling opinion dynamics and social networks [2], [3], non-cooperative or selfish routing in network systems [4], identification and filtering in, among many, sensor networks [5]. In all those cases, the problem consists in defining a suitable distributed interconnection protocol driving the dynamics of all agents of the network to a shared behavior, commonly referred to as consensus [6]. In general, such an interconnection is performed by emulating the continuous-time counterpart. Despite such a rule is simple to implement, it guarantees consensus of the network only under very restrictive conditions on the coupling strength (the gain weighting the influence of the network on each agent). As a matter of fact, even for networks of discrete-time scalar agents, the standard connection protocol guarantees convergence to an agreement state only if the coupling is small enough with respect to the network size [7], [8]. Accordingly, these results provides conservative values for the gain that are generally inversely proportional to either the smallest non zero eigenvalue of the Laplacian (which must be known to all agents) or, alternatively, the number of agents involved in the network. As a consequence the coupling gain decreases with the size of the network affecting both the convergence rate to consensus, slowed down significantly for large networks, and the amount of information the agents need to exchange before reaching an agreement.

In such a context, this paper aims at designing a new distributed protocol overcoming the aforementioned issues for a network of discrete-time scalar dynamical agents.

A first step toward this goal is in [9], where a new coupling control is provided for discrete-time networks forcing consensus for all the coupling strength values that can be then arbitrarily set. However, despite well-performing in the nominal cases, the proposed protocol suffered from two main issues: (i) it cannot be implemented in a distributed manner as the coupling protocol is implicitly defined; (ii) the convergence rate cannot be fixed arbitrarily as directly proportional to the coupling gain.

Starting from this result, the contribution of this paper is twofold. First, the centralized protocol in [9] is improved by allowing to fix the convergence rate to consensus arbitrary fast as directly proportional to the coupling gain. However, such a protocol is implicitly defined by a linear equality whose solution, defining the control action, cannot be instantaneously and independently computed by each node. Accordingly, as second and major contribution, we propose a distributed version of the aforementioned protocol allowing to approximately solve, in an arbitrary number of steps, the linear equation that defines it. The resulting design approach is reminiscent of a time-scale separation procedure typically employed in distributed filtering [10]–[12]. Roughly speaking, two consensus processes are nested over a time window of length γ , so resorting to a multi-rate consensus controller. The local control at step t , $u_i(t)$, is computed via an intra-consensus forward computation over a time window of length γ . At all steps $t + \tau$ (with $\tau = 1, \dots, \gamma$), each agent computes an approximate solution (say $v_i(t, \tau)$) to the equation defining the consensus protocol only based on the available (local) information and then sends it to the corresponding neighbors. At step $t + \gamma$, the actual control is deduced as the result of the approximating consensus phase after γ steps, i.e., $u_i(t) = v_i(t, \gamma)$. The so deduced controller is ensured to enforce consensus for all values of γ with performance of the centralized implementation recovered as γ increases.

The rest of the paper is organized as follows. In Section II the problem is settled with the centralized and decentralized solutions presented in Section III. Simulations are given in Section IV with concluding remarks and perspectives in Section V.

Notations. \mathbb{C} , \mathbb{R} and \mathbb{N} denote the set of complex, real and natural numbers including 0 respectively. I and 0 denote respectively the identity and zero matrices of suitable dimensions. $|\cdot| \in \mathbb{R}$ denotes, depending on the argument, either the cardinality of a set \mathcal{S} or the absolute value of a complex number $\lambda \in \mathbb{C}$. 0 denotes either the zero scalar or the zero matrix of suitable dimensions. $\mathbb{1}_c$ denotes the c -dimensional column vector whose elements are all ones while I is the identity matrix of suitable dimensions. Given

¹Università Campus-Biomedico of Rome; Rome, Italy. f.cacace@unicampus.it

²Dipartimento di Ingegneria Informatica, Automatica e Gestionale A. Ruberti (Sapienza University of Rome); Rome, Italy. {mattia.mattioni, salvatore.monaco}@uniroma1.it

³Laboratoire de Signaux et Systèmes (CNRS, Centrale-Supelec, Université Paris-Saclay); Gif-sur-Yvette, France dorothee.normand-cyrot@centralesupelec.fr

a matrix $A \in \mathbb{R}^{n \times n}$, $\sigma\{A\} \subset \mathbb{C}$ is its spectrum, $\|A\|$ its norm and $\rho(A)$ its spectral radius. A matrix is said non-negative if all its entries are non-negative. Given $x : t \mapsto x(t)$ with $t \in \mathbb{N}$ and $x(t) \in \mathbb{R}^n$, we denote for simplicity $x = x(t)$ and $x^+ = x(t+1)$. A scalar real-valued function $H : \mathbb{R}^n \rightarrow \mathbb{R}$ we denote by $\Delta H(x) = H(x^+) - H(x)$ the corresponding one step increment.

II. PROBLEM STATEMENT AND RECALLS

A. Recalls on graph

We consider an unweighted directed graph (or digraph for short) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of neighbors to a node $i \in \mathcal{V}$ is defined as $\mathcal{N}_i = \{j \in \mathcal{V} \text{ s.t. } (j, i) \in \mathcal{E}\}$. For all pairs of distinct nodes $i, j \in \mathcal{V}$, a directed path from i to j is defined as $i \rightsquigarrow j := \{(r, r+1) \in \mathcal{E} \text{ s.t. } \cup_{r=0}^{\ell-1} (r, r+1) \subseteq \mathcal{E} \text{ with } 0 = i, \ell = j \text{ and } \ell > 0\}$. The reachable set from a node $i \in \mathcal{V}$ is defined as $R(i) := \{i\} \cup \{j \in \mathcal{V} \text{ s.t. } i \rightsquigarrow j\}$. A set \mathcal{R} is called a reach if it is a maximal reachable set, that is, $\mathcal{R} = R(i)$ for some $i \in \mathcal{V}$ and there is no $\mu \in \mathcal{V}$ such that $R(i) \subset R(\mu)$. Since \mathcal{G} possesses a finite number of vertices, such maximal sets exist and are uniquely determined by the graph itself. Denoting by \mathcal{R}_i for $i = 1, \dots, \mu$, the reaches of \mathcal{G} , the exclusive part of \mathcal{R}_i is defined as $\mathcal{H}_i = R(i) \setminus \cup_{\ell=1, \ell \neq i}^{\mu} R_\ell$ with cardinality $h_i = |\mathcal{H}_i|$. Finally, the common part of \mathcal{G} is given by $\mathcal{C} = \mathcal{V} \setminus \cup_{i=1}^{\mu} \mathcal{H}_i$ with cardinality $c = |\mathcal{C}|$.

The Laplacian matrix associated to \mathcal{G} is given by $L = D - A$ with $D \in \mathbb{R}^{N \times N}$ and $A \in \mathbb{R}^{N \times N}$ being respectively the in-degree and the adjacency matrices. As proved in [13], L possesses one eigenvalue $\lambda = 0$ with both algebraic and geometric multiplicities coinciding with μ , the number of reaches of \mathcal{G} . In the following, it is assumed that $\mu = 1$, that is the network only possesses one reach.

B. Motivation and problem formulation

Consider a multi-agent system exchanging information via a communication digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with each vertex $i \in \mathcal{V}$ being a dynamical unit of the form

$$x_i^+ = x_i + u_i \quad (1)$$

with $x_i, u_i \in \mathbb{R}$ and $i = 1, \dots, N$. When coupling all agents via the standard protocol [6], [7]

$$u_i = -\kappa_1 \sum_{\nu_j \in \mathcal{N}(v_i)} (x_i - x_j) \quad (2)$$

consensus is achieved only if the coupling gain $\kappa_1 a > 0$ is small enough. More precisely, denoting the agglomerate vectors

$$\begin{aligned} \mathbf{x} &= \text{col}\{x_i, i = 1, \dots, N\} \in \mathbb{R}^N \\ \mathbf{u} &= \text{col}\{u_i, i = 1, \dots, N\} \in \mathbb{R}^N. \end{aligned} \quad (3)$$

the network dynamics

$$\mathbf{x}(t+1) = (I - \kappa_1 L)\mathbf{x}(t) \quad (4)$$

possesses dynamic matrix $I - \kappa_1 L$ with one eigenvalue in $\lambda_d = 1$ with unitary geometric multiplicity; however, all

agents converge to a suitable consensus $x_s \in \mathbb{R}$ only if all other eigenvalues lie within the open unit circle, that is when

$$\kappa_1 \leq \frac{1}{\lambda_{\max}}, \quad \lambda_{\max} = \max_{\lambda > 0} \{\lambda \in \sigma\{L\}\}. \quad (5)$$

Accordingly, if κ_1 is not small enough consensus might be lost and the network might be unstable. In addition, even in the best case scenario (when consensus is preserved), because κ_1 should be small, the exchange of information is significantly filtered by all agents so notably affecting both the convergence rate to consensus and the amount of information that must be exchanged. Finally, λ_{\max} might not be known by all agents and, even if upper bounds can be computed, the transient performances might not be acceptable. In [9], a new consensus protocol solving part of those issues have been proposed. However, the proposed solution is centralized and cannot assign an arbitrary convergence rate to consensus. In this paper, we try to make a step farther investigating and solving the problem of designing, if any, a local and distributed control law of the form

$$u_i(t) = \kappa \varphi_i(x_i(t), \text{col}\{x_j(t), u_j(t-1) \text{ s.t. } j \in \mathcal{N}_i\})$$

making all agents asymptotically converge to some consensus $x_s \in \mathbb{R}$ for all $\kappa > 0$; namely, as $t \rightarrow \infty$

$$\mathbf{x}(t) \rightarrow \mathbf{1}_N x_s, \quad (6)$$

for suitably defined consensus $x_s \in \mathbb{R}$.

III. MAIN RESULT

A. A refined centralized consensus protocol

First, we refine and extend the centralized algorithm proposed in [9] where the necessity of a direct input term in the output all agents exchange through the network.

Theorem 3.1: Consider a network of N discrete-time agents of the form (1) with communication digraph \mathcal{G} with only one reach, *i.e.* the Laplacian L has a zero eigenvalue with multiplicity 1. Then, for all $i = 1, \dots, N$ the local control law

$$\mathbf{u} = -\kappa (I_N + \kappa g L)^{-1} L \mathbf{x} \quad (7)$$

whose components are solutions to

$$u_i = -\kappa \sum_{j \in \mathcal{N}_i} (y_i - y_j) \quad (8a)$$

$$y_i = x_i + g u_i \quad (8b)$$

guarantee consensus for all $\kappa > 0$ and $g \geq \frac{1}{2}$; namely, for the network dynamics

$$\mathbf{x}(t+1) = \Theta_c(\kappa, g)\mathbf{x}(t) \quad (9)$$

$$\Theta_c(\kappa, g) = (I_N + \kappa g L)^{-1} (I_N + \kappa(g-1)L) \quad (10)$$

as $t \rightarrow \infty$, one gets that (6) holds with

$$x_s = v_1^\top \mathbf{x}(0) \quad (11)$$

with $v_1^\top \in \mathbb{R}^N$ such that $v_1^\top L = 0$ and $v_1^\top \mathbf{1}_N = 1$.

Proof: The proof follows along the lines of [9, Theorem 4.1] noticing that for all choices of $\kappa, g \in \mathbb{R}$, the eigenvalues of $\Theta_c(\kappa, g)$ are given by

$$\lambda_d^i(\kappa, g) = \frac{1 + \kappa(g-1)\lambda^i}{1 + \kappa g \lambda^i}, \quad \text{for all } \lambda^i \in \sigma\{L\}. \quad (12)$$

Thus, Θ_c possesses an eigenvalue in $\lambda_d = 1$ with multiplicity 1, corresponding to the eigenvalue $\lambda = 0$ of the Laplacian L . In addition, all other eigenvalues are in the open unit circle if and only if, fixing $\kappa > 0$, $g > \frac{1}{2}$. As a consequence, the center subspace associated to the one eigenvalue

$$\mathcal{V}_c = \ker\{L\} = \text{span}\{\mathbf{1}_N\}$$

is attractive and coincides with the consensus subspace. Introducing now

$$\begin{pmatrix} x_s \\ x_r \end{pmatrix} = v_1^\top \mathbf{x} = \begin{pmatrix} V_0^\top \\ V_r^\top \end{pmatrix} \mathbf{x} \quad (13)$$

with

$$V^{-\top} = Z = \begin{pmatrix} \mathbf{1}_N & Z_r \end{pmatrix}$$

$$V^\top LZ = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda^r \end{pmatrix}, \quad \Lambda^r = \text{diag}\{\sigma\{L\} \setminus \{0\}\}$$

one gets that

$$V^\top \Theta_c(\kappa, g) Z = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda_d^r \end{pmatrix}$$

$$\Lambda_d^r(\kappa, g) = \text{diag}\{\sigma\{\Theta_c(\kappa, g)\} \setminus \{1\}\}.$$

In detail, $x_s \in \mathbb{R}$ is the projection of the trajectories onto \mathcal{V}_c , whereas $x_r \in \mathbb{R}^{N-1}$ is the orthogonal component that converges to zero, by construction of κ and g . When consensus is achieved (i.e., $\mathbf{x} \in \mathcal{V}_c$), one gets $x_r \equiv 0$ and thus

$$\mathbf{x} = \mathbf{1}_N x_s$$

so that the proof follows. \blacksquare

Remark 3.1: The output (8b) agents exchange through the network is the one making all agents Input-Feedforward Passive [14]; that is, fixing the storage $S(x_i) = \frac{1}{2}x_i^2$ one gets the dissipation inequality

$$\Delta S(x_i) \leq u_i y_i - (g - \frac{1}{2})u_i^2. \quad (14)$$

Accordingly, when $g > \frac{1}{2}$, $u_i \mapsto y_i$ is strictly passive whereas it is passive for $g = \frac{1}{2}$.

Remark 3.2: When fixing $g = \frac{1}{2}$, the consensus protocol proposed in [9] is recovered. However, in this case, one cannot fix the convergence rate to consensus arbitrarily small via $\kappa > 0$. As a matter of fact, one cannot compute the gain κ to make all eigenvalues of $\Theta_c(\kappa, \frac{1}{2})$

$$\lambda_d^i(\kappa, \frac{1}{2}) = \frac{1 - \frac{\kappa}{2}\lambda^i}{1 + \frac{\kappa}{2}\lambda^i}, \quad \text{for all } \lambda^i \in \sigma\{L\}.$$

arbitrarily close to 0. On the other side, the value of the largest eigenvalue (in module) can be minimized depending on the particular spectrum of L , which must be thus known to all nodes apriori.

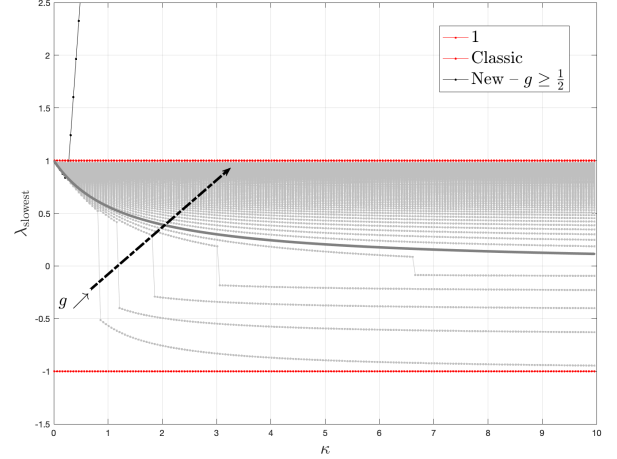


Fig. 1. Plot of the farthest eigenvalue of Θ_c from 0 for increasing values of $\kappa > 0$ and $g \geq \frac{1}{2}$.

Remark 3.3: When fixing $g = 1$ in the consensus protocol (8), one can modulate the convergence rate to consensus arbitrarily via $\kappa > 0$. More precisely, the eigenvalues of $\Theta_c(\kappa, 1)$

$$\lambda_d^i(\kappa, 1) = \frac{1}{1 + \kappa \lambda^i}, \quad \text{for all } \lambda^i \in \sigma\{L\}$$

are as close to 0 (in module) as κ increases; namely, trajectories of the network (9) converge to the multi-consensus with a velocity that is directly proportional to $\kappa > 0$. With this in mind, contrarily to the case [6], [9], one can assign the convergence rate arbitrarily small picking $\kappa \rightarrow \infty$, with no knowledge of the spectrum of the Laplacian. For a randomly generated graph of 10 nodes, Fig. 1 depicts the location of the slowest eigenvalue of $\Theta_c(\kappa, g)$ for increasing values of both κ and g .

The coupling rule (8) defining the consensus control (7) is implicitly defined and, in general, cannot be computed (at least statically) in a fully distributed manner. As a matter of fact, at each step $k \in \mathbb{N}$, the i^{th} agent needs the input u_j of all its neighbors for computing the corresponding u_i so creating a bottleneck that cannot be solved locally.

The feedback (7) can be rewritten as

$$\mathbf{u} = -\kappa W(\kappa, g) L \mathbf{x} \quad (15)$$

$$W(\kappa, g) := (I_N + \kappa g L)^{-1} \quad (16)$$

with the term $W(\kappa, g)L$ being a new weighted Laplacian. This new Laplacian is associated to a new dummy graph in which new (and weighted) edges appear within connected components due to the presence of the weighting matrix $W(\kappa, g)$ which depends on the original Laplacian L . Such a matrix cannot be computed locally by each agent unless all of them possess exact knowledge on the original topology of \mathcal{G} .

B. A distributed implementation of the new protocol

In this section we study the problem of a distributed (dynamic) implementation of the coupling rule defined by

Approximate γ steps implementation of (8) at node i .

- 1: At each time $t \geq 0$, send $x_i(t)$ to the neighbors.
- 2: Receive $x_j(t)$ from the neighbors, $j \in \mathcal{N}_i$, and compute, with $d_i = |\mathcal{N}_i|$,

$$v_i(t, 0) = \frac{-\kappa}{1 + g\kappa d_i} \sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)) \quad (17)$$

- 3: For $h = 0, \dots, \gamma - 1$ do:
 - 3.1: Send $v_i(t, h)$ to the neighbors
 - 3.2: Compute

$$v_i(t, h+1) = v_i(t, 0) + \frac{g\kappa}{1 + g\kappa d_i} \sum_{j \in \mathcal{N}_i} v_j(t, h) \quad (18)$$

- 4: Set $u_i(t) = v_i(t, \gamma)$.

Fig. 2. Distributed approximate multi-step implementation of (8)

(8). The problem is not trivial, as it is evident for general reasons. Since, as remarked in Remark 3.3, (8) allows for an arbitrary rate of convergence to consensus, it follows that its distributed implementation would yield distributed practical convergence in any finite number of steps, including 1. However, this is not possible, since the speed of the information trough a graph with discrete-time communications is obviously limited.

One possible way to overcome this conceptual limitation is to separate the time-scale of the information exchange and of the system evolution by resorting to a multi-rate controller. The idea of multiple information exchanges per time unit is not new and it has been used extensively in the field of distributed filtering [10], [11]. The coupling rule defined by (8) can be written

$$u_i = -\kappa \sum_{j \in \mathcal{N}_i} (x_i - x_j) - g\kappa \sum_{j \in \mathcal{N}_i} (u_i - u_j). \quad (19)$$

By solving with respect to u_i one gets, with $d_i = |\mathcal{N}_i|$,

$$u_i = -\frac{\kappa}{1 + g\kappa d_i} \sum_{j \in \mathcal{N}_i} (x_i - x_j) + \frac{g\kappa}{1 + g\kappa d_i} \sum_{j \in \mathcal{N}_i} u_j, \quad (20)$$

where the first term of the right-hand is immediately available at each time t , whereas the second one can be approximated by a truncated fixed-point iteration with $\gamma \in \mathbb{N}$ steps. The resulting approximate distributed implementation of (8) is summarized in Fig. 2.

Theorem 3.2: If the graph \mathcal{G} has exactly one reach, then at each node i and time t the sequence $v_i(t, \gamma)$ generated by (18) is such that, for all $\kappa, g > 0$ and $\gamma \in \mathbb{N}$,

$$\lim_{\gamma \rightarrow \infty} v_i(t, \gamma) = u_i(t), \quad (21)$$

where $u_i(t)$ is the i -th component of $\mathbf{u} = -\kappa(I_N + \kappa g L)^{-1} L \mathbf{x}$.

Proof: Introduce the matrices

$$\tilde{D} = (I_N + \kappa g D)^{-1} \quad (22)$$

$$G = (I_N + \kappa g D)^{-1} \kappa g A = \kappa g \tilde{D} A. \quad (23)$$

G is non-negative with $\rho(G) < 1$, i.e. G is Schur. G is non-negative because \tilde{D}, A are non-negative. Since each row i

of G is obtained from the corresponding row of A divided by $1 + \kappa g d_i$, an application of the Gerschgorin criterion [15] to the rows of G allows to conclude that $\rho(G) < 1$ for any positive values of κ and g . In fact, $G_{ii} = 0$ is the center of the circles and the radius is $\sum_{j \neq i} G_{ij} = \kappa g d_i / (1 + \kappa g d_i) < 1$, that implies $\rho(G) < 1$. In vector form (18) reads

$$V(t, \gamma) = -\kappa (I_N + G + \dots + G^\gamma) (I + \kappa g D)^{-1} L \mathbf{x}(t). \quad (24)$$

From $L = D - A$ we obtain

$$\begin{aligned} (I_N + \kappa g L) &= I_N + \kappa g D - \kappa g A = \tilde{D}^{-1} (I_N - \kappa g \tilde{D} A) \\ &= (I_N + \kappa g D) (I_N - G) \end{aligned} \quad (25)$$

one gets

$$V(t, \gamma) = -\kappa \sum_{h=0}^{\gamma} G^h (I_N + \kappa g L)^{-1} L \mathbf{x}(t).$$

Since when $\rho(G) < 1$ it holds that $(I_N - G)^{-1} = \sum_{i=0}^{\infty} G^i$ one concludes, as $\gamma \rightarrow \infty$,

$$V(t, \gamma) \rightarrow -\kappa (I_N + \kappa g L)^{-1} L \mathbf{x}(t)$$

that is exactly the centralized control (7). ■

Theorem 3.2 shows that with a sufficient number of consensus steps for time unit, we can approximate the performance of the centralized controller (7) as much as desired. In particular with $g = 1$ it is possible to choose any rate of convergence to the consensus value. In practice it is also of interest to investigate when the consensus is reached for a finite number of consensus steps γ . We therefore investigate the consensus properties of the control $u_i(t) = v_i(t, \gamma)$. The following result is instrumental to this goal.

Lemma 3.1: With the control law $u_i(t) = v_i(t, \gamma)$ the network dynamics takes the form

$$\mathbf{x}(t+1) = \Theta_d(\kappa, g, \gamma) \mathbf{x}(t) \quad (26)$$

with

$$\Theta_d(\kappa, g, \gamma) = I_N - \kappa (I_N - G^{\gamma+1}) W(\kappa, g) L \quad (27)$$

and where $W(\kappa, g) = (I_N + \kappa g L)^{-1}$ in (16) is non-negative.

Proof: The proof follows from (24), (25) and $\sum_{h=0}^{\gamma} G^h = (I_N - G^{\gamma+1})(I_N - G)^{-1}$. We get

$$V(t, \gamma) = -\kappa (I - G^{\gamma+1}) W(\kappa, g) L \mathbf{x}(t) \quad (28)$$

with G as in (23) and $W(\kappa, g)$ as in (16). Thus, $\mathbf{x}(t+1) = \mathbf{x}(t) + \mathbf{u}(t)$ can be written

$$\begin{aligned} \mathbf{x}(t+1) &= (I_N - \kappa (I_N - G^{\gamma+1}) W(\kappa, g) L) \mathbf{x}(t) \\ &= \Theta_d(\kappa, g, \gamma) \mathbf{x}(t). \end{aligned} \quad (29)$$

Finally, (25) implies $W = (I_N - G)^{-1} \tilde{D}$ and since both $(I_N - G)^{-1} = \sum_{h=0}^{\infty} G^h$ and \tilde{D} are non-negative, W is non-negative too. ■

Notice that by using (15) the matrix $\Theta_c(\kappa, g)$ in (10) can be written $\Theta_c = I_N - \kappa W L$ and, since G is Schur, $\lim_{\gamma \rightarrow \infty} \Theta_d(\kappa, g, \gamma) = \Theta_c(\kappa, g)$, in accordance with Theorem 3.2.

The following result specifies conditions allowing to enforce consensus under the distributed approximation in (18) with no consensus iteration (i.e., when $\gamma = 0$).

Lemma 3.2: When $\kappa g \geq 1$ and the graph contains only one reach, the control law $u_i = v_i(t, 0)$ (i.e. $\gamma = 0$) makes the agents converge to the same consensus value.

Proof: From (24), $V(t, 0) = -\kappa \tilde{D}Lx(t)$ and (27) reads

$$\Theta_d(\kappa, g, 0) = I_N - \kappa \tilde{D}L. \quad (30)$$

$\Theta_d(\kappa, g, 0)$ is row-stochastic, i.e. $\Theta_d(\kappa, g, 0)\mathbf{1}_N = \mathbf{1}_N$. Moreover, if $\kappa g \geq 1$ then $\Theta_d(\kappa, g, 0)$ is non-negative. An application of the Gerschgorin circles to the rows of $\Theta_d(\kappa, g, 0)$ yields $\rho(\Theta_d(\kappa, g, 0)) \leq 1$. Since it is immediate to verify that $\lambda_1 = 1 \in \sigma\{\Theta_d(\kappa, g, 0)\}$ with right eigenvector $u_1 = \mathbf{1}_N$ we have $\rho(\Theta_d(\kappa, g, 0)) = 1$. When the graph is strongly connected, $\Theta_d(\kappa, g, 0)$ is irreducible and, by the presence of at least a non zero element on the diagonal, primitive [15]. Consequently, $\lambda_1 = 1$ is the only eigenvalue on the unit circle and it has algebraic and geometric multiplicity 1 by the Frobenius-Perron theorem. Thus, $x(t)$ tends to the eigenspace of λ_1 , that is, to a vector of the form $\mathbf{1}_N x_s$, $x_s = v_1^\top x(0) \in \mathbb{R}$, where v_1 is the corresponding left eigenvector of $\Theta_d(\kappa, g, 0)$ for $\lambda_1 = 1$ satisfying $v_1^\top \mathbf{1}_N = 1$. When the graph is weakly connected and with one reach only, the multiplicity of $\lambda_1 = 1$ is still 1 (see [13, Theorem 3.2]). In this situation, the set of roots of the graph form a strongly connected sub-graph and therefore they reach consensus. The consensus of the remaining nodes of the graph follows from a simple partitioning of L between root and non-root nodes (see [16, Section 2]). ■

Theorem 3.1 and Lemma 3.2 show that the distributed control (18) on weakly connected digraphs with only one reach (i.e. $\mu = 1$) guarantees consensus for, respectively, $\gamma = \infty$ and $\gamma = 0$ consensus iterations. Our main result, here below, states that this property holds for *any* number of consensus iterations and for all $\gamma \geq 0$, $\kappa > 0$, $g \geq 1$.

Theorem 3.3: If the graph \mathcal{G} has exactly one reach, then the control $u_i(t) = v_i(t, \gamma)$ generated by (18) makes the agents converge to the same consensus value $x_s \in \mathbb{R}$ for all $\gamma \geq 0$, $\kappa > 0$, $g \geq 1$.

Proof: We know from Lemma 3.1 that the collective dynamics of the nodes is $x(t+1) = \Theta_d(\kappa, g, \gamma)x(t)$ with $\Theta_d(\kappa, g, \gamma)$ given in (27), that can be re-written as

$$\begin{aligned} \Theta_d(\kappa, g, \gamma) &= I_N - \kappa W(\kappa, g)L + \kappa G^{\gamma+1}W(\kappa, g)L \\ &= \Theta_c(\kappa, g) + \kappa G^{\gamma+1}W(\kappa, g)L. \end{aligned} \quad (31)$$

Let $W = W(\kappa, g)$ for concision. We now prove that $W = I_N - \kappa gWL$. In fact,

$$I_N - \kappa gWL = W(W^{-1} - \kappa gL) = W \cdot I_N = W.$$

Consequently, $\kappa WL = (I_N - W)/g$ and

$$\Theta_c(\kappa, g) = I_N - \kappa WL = I_N - \frac{1}{g}(I_N - W)$$

Replacing into (31) yields

$$\Theta_d(\kappa, g) = I_N - \frac{1}{g}I_N + \frac{1}{g}G^{\gamma+1} + \frac{1}{g}(I_N - G^{\gamma+1})W. \quad (32)$$

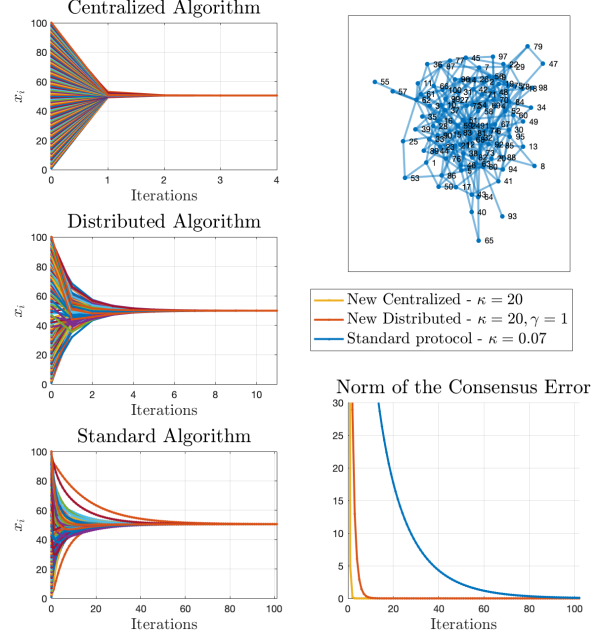


Fig. 3. Network of $N = 100$ scalar agents over an undirected graph.

We notice that, for $g \geq 1$, $I_N - \frac{1}{g}I_N > 0$ and $G^{\gamma+1} > 0$ because G is positive. Finally,

$$(I_N - G^{\gamma+1})W = \sum_{h=0}^{\gamma} G^h(I_N - G)W = \sum_{h=0}^{\gamma} G^h \tilde{D} \geq 0, \quad (33)$$

where we have used $(I_N - G)W = \tilde{D}$ that descends from (25), and $\tilde{D} \geq 0$. We can now conclude $\Theta(\kappa, g, \gamma) \geq 0$ whenever $g \geq 1$, $\kappa > 0$, $\gamma \geq 0$. Since $\Theta(\kappa, g, \gamma)\mathbf{1}_N = \mathbf{1}_N$ we can repeat the same steps as in Lemma 3.2 to conclude that $\rho(\Theta(\kappa, g, \gamma)) = 1$, with exactly one eigenvalue $\lambda_1 = 1$ on the unit circle, and this guarantees consensus. ■

Remark 3.4: Notice that in general the consensus value will depend on κ , g and γ . If $v_1^\top(\kappa, g, \gamma)$ is the left-eigenvector for $\lambda_1 = 1$ of $\Theta_d(\kappa, g, \gamma)$, then $x(t) \rightarrow \mathbf{1}_N v_1^\top(\kappa, g, \gamma)x(0)$, or $x_i(t) \rightarrow v_1^\top(\kappa, g, \gamma)x(0)$.

IV. SIMULATIONS

In this section, we report the results of the proposed algorithm (in both the centralized and distributed implementation) when applied to different networks. Performances are compared with respect to the standard discrete-time protocol (2) when fixing the coupling gain as the largest value guaranteeing that (5) holds. For evaluating performances, we use the following parameters: the $M\%$ -consensus settling time t_s^M defined as the minimum amount of steps that is required for the trajectories of the network to reach $M\%$ of the corresponding consensus value. For (2), and the proposed centralized algorithm in Proposition 3.1, t_s^M provides an estimate of the minimum number of iterations that are required for consensus to be achieved; for the distributed implementation in Theorem 3.3, such a quantity is given by $(\gamma + 1)t_s^M$, with $\gamma \in \mathbb{N}$ as in (28) and $M = 10^{-1}$.

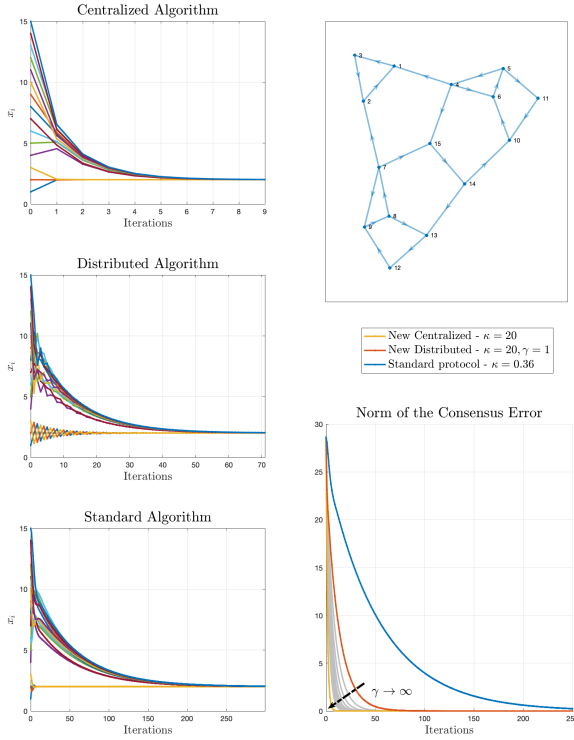


Fig. 4. Network of $N = 15$ agents over a digraph.

In Fig. 3, an undirect network of $N = 100$ is simulated when fixing, for the proposed algorithm, $\kappa = 10$ and $\gamma = 1$. Despite the amount of agents, the network controlled via the centralized algorithm converges to consensus in three iterations (and $t_s^{10^{-1}} = 3$); for the distributed implementation one gets that $t_s^{10^{-1}} = 10$ so getting convergence with good performances even when $\gamma = 1$ and, in this case, in exactly 20 time steps. Those performances are far better than the ones exhibited by the standard consensus algorithm for which $t_s^{10^{-1}} = 100$.

In Fig. 4, the simulations are performed for a directed network of $N = 15$ agents. Similar considerations as in the previous case hold. In particular, fixing $\kappa = 1$ consensus is achieved in 14 iterations in exactly 20 time steps; the corresponding distributed implementation with $\gamma = 1$ requires 148 iterations to converge to consensus (with $t_s^{10^{-1}} = 74$). Such a result is notably better than the one provided by the standard algorithm for which approximately $t_s^{10^{-1}} = 298$. We underline that, as highlighted in Fig. 4, performances of the distributed algorithm improve (over the big-consensus steps) as γ increases. In that case, the distributed algorithm approaches the nominal performances (i.e., the ones under (7)) for finite and reasonable values of γ at the price of larger computational delays introduced by the consensus steps. Summarizing, performances achieved by the distributed algorithm are always far better than the usual one even when γ is small and fixed to 1, the worst case scenario.

Remark 4.1: κ_1 is in general fixed smaller and inversely

proportional to the size of the network [7]. However, in our specific case, we have fixed it as the minimal value guaranteeing that network dynamics (2) is critically stable with, thus, an attractive consensus. This choice has been performed to compare the the solution we propose with the most favorable implementation of algorithm in the literature.

V. CONCLUSIONS AND PERSPECTIVES

The centralized consensus protocol proposed in [9] has been generalized to assign the convergence rate to consensus arbitrarily fast. Then, a distributed implementation of such an algorithm is proposed. It is based on a multi-rate forward computation over an arbitrary number of consensus steps. Future works include the extension of this protocol to deal with multi-consensus of heterogeneous networks in discrete time and the presence of delays too [17].

REFERENCES

- [1] M. Di Ferdinando, D. Bianchi, S. Di Gennaro, and P. Pepe, "On the robust quantized sampled-data leaderless consensus tracking of nonlinear multi-agent systems," in *60th IEEE Conference on Decision and Control (CDC)*, 2021.
- [2] A. V. Proskurnikov, C. Ravazzi, and F. Dabbene, "Dynamics and structure of social networks from a systems and control viewpoint: A survey of roberto tempo's contributions," *Online Social Networks and Media*, 2018.
- [3] M. Bini, P. Frasca, C. Ravazzi, and F. Dabbene, "Graph structure-based heuristics for optimal targeting in social networks," *IEEE Transactions on Control of Network Systems*, 2022.
- [4] A. Giuseppe and A. Pietrabissa, "Stability and wardrop equilibria of non-cooperative routing with time-varying load," *IEEE Transactions on Automatic Control*, pp. 1–8, 2022.
- [5] D. Deplano, M. Franceschelli, and A. Giua, "Discrete-time dynamic consensus on the max value," in *15th European Workshop on Advanced Control and Diagnosis*. Springer, 2022, pp. 367–383.
- [6] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.
- [7] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.
- [8] Y. Chen, J. Lü, F. Han, and X. Yu, "On the cluster consensus of discrete-time multi-agent systems," *Systems & Control Letters*, vol. 60, no. 7, pp. 517–523, 2011.
- [9] M. Mattioni, S. Monaco, and D. Normand-Cyrot, "A new connection protocol for multi-consensus of discrete-time systems," in *2022 American Control Conference (ACC)*. IEEE, 2022, pp. 5179–5184.
- [10] G. Battistelli, L. Chisci, G. Mugnai, A. Farina, and A. Graziano, "Consensus-based linear and nonlinear filtering," *IEEE Transactions on Automatic Control*, vol. 60, no. 5, pp. 1410–1415, 2014.
- [11] A. T. Kamal, J. A. Farrell, and A. K. Roy-Chowdhury, "Information weighted consensus filters and their application in distributed camera networks," *IEEE Trans. on Automatic Control*, vol. 58, no. 12, 2013.
- [12] S. Battilotti, F. Cacace, M. D'Angelo, and A. Germani, "Cooperative filtering with absolute and relative measurements," in *2018 IEEE Conference on Decision and Control (CDC)*, 2018, pp. 7182–7187.
- [13] J. S. Caughman and J. Veerman, "Kernels of directed graph Laplacians," *The Electronic Journal of Combinatorics*, vol. 13, no. 1, 2006.
- [14] A. Van der Schaft, *L2-gain and passivity techniques in nonlinear control*. Springer, 2017.
- [15] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge university press, 2012.
- [16] F. Cacace, M. Mattioni, S. Monaco, and L. Ricciardi Celsi, "Topology-induced containment for general linear systems on weakly connected digraphs," *Automatica*, vol. 131, p. 109734, 2021.
- [17] S. Battilotti and M. d'Angelo, "Stochastic output delay identification of discrete-time gaussian systems," *Automatica*, vol. 109, p. 108499, 2019.