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***Abstract*— This article discusses the Dirac structure and the state-space representation of a class of port-Hamiltonian systems that evolve in discrete time. The characterization of the underlying Dirac structure depends on separating the stored energy associated with the system into two distinct components. Moreover, it is shown that power-preserving interconnection and negative output feedback maintain the port-Hamiltonian structure while increasing the dimension of the Dirac structure. Finally, the proposed approach is illustrated by means of an approximated gravity pendulum model.**

***Index Terms*— Nonlinear systems, modeling, Energy Systems, Algebraic/geometric methods**

I. Introduction

The idea of modeling a dynamical system as a network composed of interconnected subsystems is rooted in [1] for the analysis of electromechanical systems. Interconnected systems, which appear in energy-conserving systems, were modeled in the context of Poisson structures in [2] and [3], and later in the general context of Dirac structures in [4]. A Dirac structure is a geometric object that generalizes Poisson structures, bond graphs, and presymplectic structures on manifolds [5] and forms the foundation for the *port-Hamiltonian* framework. The port-Hamiltonian framework in [6] and [7] emerges from the need to provide a systematic framework for the analysis and control of complex, possibly networked, physical systems. In the case of port-Hamiltonian systems, the key characteristic of a Dirac structure is its compositional closure, which results in the preservation of energy under power-conserving interconnections [8]. As a consequence, this interconnection-based modeling among subsystems has led to the definition of numerous passivity-based controllers, such as Interconnection and Damping Assignment (IDA-PBC), Energy Shaping, and Control by Interconnection, [9]–[11].

The majority of the energy-based approaches for modeling and controlling port-Hamiltonian systems are carried out in the continuous-time domain. Yet, computer-aided technologies operate in discrete time and hence continuous-time systems must be approximated with discrete-time models for implementation purposes. Nevertheless, the discretization procedure may thereby cause the loss of the structural properties of the continuous-time system, see [12]–[16]. In this respect, it is of utmost importance to define faithful discrete-time representations for numerical integration, simulation, and control. Various definitions of discrete-time port-Hamiltonian systems have been developed to ensure energy preservation. These include dynamics based on discrete manifolds [17], midpoint discretization [18], spatial-discretization [19], pseudo-spectral discretization [20], or first-order discrete gradient methods [21]. The disadvantage of these representations is that, unlike in the continuous-time domain, the power balance derived from the forward difference of the Hamiltonian function does not enable the identification of the power supplied by the stored energy. This limitation hinders the precise characterization of the power balancing equation, which is essential for discrete-time energy-based control approaches such as negative output feedback [22], energy balance [23], and Control by Interconnection [24].

The purpose of this work is to characterize the underlying Dirac structure of the discrete-time port-Hamiltonian system recently proposed by the authors in [25]. The proposed discrete-time port-Hamiltonian modeling, which overcomes the aforementioned difficulty in characterizing the power balance, relies upon the notions of Difference and Differential Representation (DDR) of discrete-time dynamics [26], discrete gradient function [27]. and average passivity [28]. The DDR is a well-established model for systems evolving in discrete time. It consists of a difference equation that represents the system's free evolution through jumps, and a differential equation that captures the influence of control. This representation has inspired numerous results in the study of the geometric properties of discrete-time systems, including Lie conditions for accessibility, controllability, and invariance [29]–[31].

In this article, we focus our attention on the construction of a Dirac structure associated with the class of systems proposed in [25]. In detail, we identify the flow and effort elements that characterize the Dirac structure of the input-affine port-Hamiltonian structure proposed in [25]. In the same vein, we discuss how the underlying Dirac structure is transformed and preserved under negative output feedback and feedback interconnection. Finally, we demonstrate that the power-preserving interconnection of two average passive systems, as proposed in [24], recovers the composition of Dirac structures.

The article is structured as follows. In Section II we recall the notion of discrete-time systems in a Difference and Differential Representation (DDR), the notion of u-average passivity, and the general theory of port-Hamiltonian systems. In Section III we first introduce the definition of discrete port-Hamiltonian system defined by a discrete Dirac structure, and then we reveal how the discrete-time system in [25] can be characterized in terms of an underlying Dirac structure. In Section IV we show how the proposed Dirac structure is transformed under negative output feedback and negative feedback interconnection. In particular, we emphasize how the port-Hamiltonian structure is preserved whilst the underlying Dirac structure grows in dimension. In Section V the proposed framework is illustrated by means of a discrete-time system which describes the approximated behaviour of the gravity pendulum. Finally, in Section VI we provide some concluding remarks.

Notation: Throughout the article all functions and vector fields are assumed smooth and complete over the respective definition spaces. For ease of notation, we drop the argument of the functions whenever

The work of Alessio Moreschini has been supported by the EPSRC grant "Model Reduction from Data", Grant No. EP/W005557.

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this does not cause confusion. We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real and natural numbers respectively (with $0 \in \mathbb{N}$). For any real-valued vector $v \in \mathbb{R}^n$, $v^\top$ defines the transpose of v. The inner product is denoted by $\langle \cdot, \cdot \rangle$, that is for any $v, w \in \mathbb{R}^n$ we let $\langle v, w \rangle = v^\top w$. Let S be a differentiable function $S : \mathbb{R}^n \to \mathbb{R}$, the gradient of S is denoted by ∇S, where ∇ is the vector of partial derivatives in $\mathbb{R}^n$.

II. Preliminaries

In this article, a discrete-time dynamical system evolving on $\mathbb{R}^n$ is a system in which the state variable $x(k) \in \mathbb{R}^n$ evolves in discrete instants $k \in \mathbb{N}$. The evolution of the state is described by a difference equation of the form

$$x(k+1) = F(x(k), u(k)), \tag{1}$$

where $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth map and $u(k) \in \mathbb{R}$ denotes the input (control) variable[1]. Throughout the article, we omit the k-dependency of the variables to simplify the notation. Without loss of generality, we decompose $F(x, u)$ as follows

$$F(x, u) = x + F_0(x) + g(x, u)u, \tag{2}$$

where $x + F_0(x) := F(x, 0)$ describes the free evolution of the dynamics (1) and $g(x, u)u$ describes the effect of the control.

A. Difference and Differential Representation (DDR)

As proposed in [26, Section 2], under mild conditions (*e.g.* submersivity of the map $F(x, u)$), the dynamics (1) can be equivalently described by a pair of difference and differential equations (DDR) of the form

$$x^+ = x + F_0(x) \tag{3a}$$

$$\frac{\mathrm{d}x^+(u)}{\mathrm{d}u} = G(x^+(u), u) \quad \text{with} \quad x^+(0) = x^+ \tag{3b}$$

where $x^+(u) := F(x, u)$ and $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is such that

$$\frac{\partial F(x, u)}{\partial u} = G(F(x, u), u). \tag{4}$$

In the DDR form of the dynamics, the free evolution is described by the discrete-time dynamics (3a), while the effect of the control results from the integration of the differential equation (3b) that models the rate of change of the dynamics under the action of u. Accordingly, integrating (3b) over the set $[0, u]$ with initial condition $x^+(0) = x + F_0(x)$, one recovers (1); *i.e.*

$$x^+(u) = F(x, u) = x + F_0(x) + \int_0^u G(x^+(s), s)\mathrm{d}s. \tag{5}$$

We recall that, $G(x, u)$ in (3b) takes the form of a vector field that is dependent on u. This type of vector field has been previously used in [32] to characterize accessibility properties around equilibria. An integral relationship in terms of u between the control map $g(x, u)$ in (2) and the u dependent vector field $G(x, u)$ in (3b) follows by construction. In particular, for all pair $(x, u) \in \mathbb{R}^n \times \mathbb{R}$ the equation

$$g(x, u)u := \int_0^u G(x^+(s), s)\mathrm{d}s \tag{6}$$

holds, where $G(x, u)$ admits the power series expansion

$$G(x, u) = G_1(x) + \sum_{i \geq 1} \frac{u^i}{i!} G_{i+1}(x). \tag{7}$$

The vector fields G_i and their Lie algebra have played a crucial role in characterizing geometric properties of discrete-time dynamics in [32] and [26]. Furthermore, they have been recently employed to characterize passivity in [28], as well as to define port-controlled Hamiltonian structures in [25] and [22]. One of the computational properties inherited from the DDR structure (3) is that, given a smooth real-valued map $\lambda : \mathbb{R}^n \to \mathbb{R}$, its evolution along the dynamics (3) can be specified by the equation

$$\lambda(x^+(u)) = \lambda(x^+) + \int_0^u G^\top(x^+(s), s)\nabla\lambda(x^+(s))\mathrm{d}s. \tag{8}$$

This property, which distinguishes the independent free evolution from the dependent controlled part, is crucial in the following steps for determining the rate of change of the Hamiltonian function (or any energy-like function) along the discrete-time dynamics (3).

Remark 2.1: This article focuses on single-input dynamics, but the same framework can be applied to the multi-input case by using a set of partial differential equations that describe the rate of change for each control component, see [28, Sec. 6].

B. Discrete-time average passivity

The concept of average passivity has been introduced in [28] to weaken the necessity of a throughput term in the given discrete-time system. The formal definition is here recalled.

Definition 2.1: The discrete-time system defined by the dynamics (1) with output map $h : \mathbb{R}^n \to \mathbb{R}$, is said *average passive*, if there exists a positive semi-definite function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (the storage function) such that, for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$

$$S(x^+(u)) - S(x) \leq \int_0^u h(x^+(s))\mathrm{d}s = uh^{av}(x, u), \tag{9}$$

with *u-average output* defined by

$$h^{av}(x, u) := \frac{1}{u}\int_0^u h(x^+(s))\mathrm{d}s. \tag{10}$$

The average passivity of the input-output behaviour associated with the output map $h(x)$ is equivalent to the usual notion of passivity of the input-output behaviour associated with the u-average output map $h^{av}(x, u)$. The notion of average passivity from some non-zero control value $\bar{u}$ is recalled below to deal with systems under state feedback [24].

Definition 2.2: The discrete-time system defined by the dynamics (1) and output map $h : \mathbb{R}^n \to \mathbb{R}$, is said *$u$-average passive from* $\bar{u} \in \mathbb{R}$, if there exists a positive semi-definite function $S : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ (the storage function) such that for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$

$$S(x^+(\bar{u} + u)) - S(x) \leq uh^{av}_{\bar{u}}(x, u) \tag{11}$$

with u-average output from $\bar{u}$ defined as

$$h^{av}_{\bar{u}}(x, u) = \frac{1}{u}\int_0^u h(x^+(\bar{u} + s), \bar{u} + s)\mathrm{d}s. \tag{12}$$

C. Discrete gradient function

The discrete gradient function (or simply discrete gradient) has been introduced by Gonzalez in [27] to establish a framework for designing time-integration schemes that conserve energy in Hamiltonian systems. The definition of discrete gradient is given below.

Definition 2.3: Given a smooth real-valued function $H : \mathbb{R}^n \to \mathbb{R}$, its *discrete gradient* is a function of two variables $\bar{\nabla}H|_v^w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$(w - v)^\top \bar{\nabla}H|_v^w = H(w) - H(v), \tag{13}$$

with $\lim_{w \to v} \bar{\nabla}H|_v^w = \nabla H(v)$.

The discrete gradient function satisfying the property (13) is generally not uniquely defined. Thus, several efficient computational

[1]For the sake of presentation we deal with single-input systems. However, the foregoing dynamics can be extended to systems with multiple inputs.

methods have been proposed to solve (13), see *e.g.* [27], [33], [34]. For instance, one of the possible discrete gradients can be obtained through component-wise integration, that is

$$\bar{\nabla} H|_v^w = \begin{bmatrix} \bar{\nabla} H|_{v_1}^{w_1} & \cdots & \bar{\nabla} H|_{v_n}^{w_n} \end{bmatrix}^\top,$$

with entries defined by

$$\bar{\nabla} H|_{v_i}^{w_i} = \frac{1}{w_i - v_i} \int_{v_i}^{w_i} \frac{\partial H(v_1, ..., v_{i-1}, s, w_{i+1}, ..., w_n)}{\partial s} \, \mathrm{d}s.$$

Nevertheless, the results presented in the upcoming sections hold for any discrete gradient that meets the criteria of Definition 2.3.

The following lemma relates the property (13) of the discrete gradient function with the integral property (8).

Lemma 2.1: Let $\lambda : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and assume the discrete-time dynamics (5). Then

$$\int_0^u G^\top(x^+(s), s) \nabla \lambda(x^+(s)) \mathrm{d}s = u g^\top(x, u) \bar{\nabla} \lambda|_{x^+}^{x^+(u)}. \quad (14)$$

Proof: The proof is constructive. In particular, from the definition of the discrete gradient function in Definition 2.3 easy algebra shows that

$$\begin{aligned} \lambda(x^+(u)) - \lambda(x^+) &= (x^+(u) - x^+)^\top \bar{\nabla} \lambda|_{x^+}^{x^+(u)} \\ &= u g^\top(x, u) \bar{\nabla} \lambda|_{x^+}^{x^+(u)}, \end{aligned}$$

and by property (8) the statement holds. ∎

D. Port-Hamiltonian systems in discrete time

A novel description of discrete-time port-Hamiltonian systems has been proposed in [25]. It relies on three key ingredients: i) the DDR form (5) of discrete-time dynamics; ii) the notion of average passivity (9); iii) the description of the average passivating output in terms of the discrete gradient function through property (14). The following definition is restated.

Definition 2.4: Given a real-valued Hamiltonian function $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a discrete-time input-state-output port-Hamiltonian system is described by the equations[2]

$$x^+(u) = x + (J(x) - R(x)) \bar{\nabla} H|_x^{x^+} + g(x, u) u, \quad (15a)$$

$$y(x, u) = g^\top(x, u) \bar{\nabla} H|_{x^+}^{x^+(u)}, \quad (15b)$$

with $J = -J^\top$, $R = R^\top \succeq 0$, and $g(x, u)u$ given by (6).

A discrete-time port-Hamiltonian system described by the equations (15) satisfies the following properties.

1) For $u = 0$, the time evolution of the Hamiltonian function H is decreasing since

$$H(x^+) - H(x) = -\bar{\nabla}^\top H|_x^{x^+} R(x) \bar{\nabla} H|_x^{x^+} \leq 0.$$

Accordingly, for $R = 0$ the free evolution of the system is conservative, and the Hamiltonian function H is an invariant for the dynamics since

$$H(x^+) - H(x) = \bar{\nabla}^\top H|_x^{x^+} J(x) \bar{\nabla} H|_x^{x^+} = 0.$$

2) For any $u \neq 0$, the system is average passive with respect to the output map

$$Y(x, u) = G^\top(x, u) \nabla H(x), \quad (16)$$

[2]The coordinates $x^+(u)$ and x^+ are defined by the DDR in (3).

i.e. it is passive with respect to the average output given by

$$Y^{av}(x, u) = \frac{1}{u} \int_0^u G^\top(x^+(s), s) \nabla H(x^+(s)) \mathrm{d}s. \quad (17)$$

Hence, by virtue of the property (14), $y(x, u) = Y^{av}(x, u)$.

Remark 2.2: It is worth mentioning that the output (15b) of the port-Hamiltonian system, by construction, yields a series expansion in u which can be determined iteratively. Hence, from (17) we compute the series expansion

$$y(x, u) = \mathrm{L}_{G_1} H|_{x^+} + \frac{u}{2} \left(\mathrm{L}_{G_1}^2 + \mathrm{L}_{G_2} \right) H|_{x^+} + O(u^2), \quad (18)$$

where $O(u^2)$ contains all the remaining terms of higher order in u, and the operator $\mathrm{L}_G H$ denotes the Lie derivative of H along the vector G. Hence, for completeness, the output (15b) is well-defined for $u = 0$ as

$$\lim_{u \to 0} y(x, u) = G_1^\top(x^+) \nabla H(x^+).$$

E. Dirac structure

Given a finite-dimensional linear space $\mathcal{F}$ (the space of flow variables) and its dual $\mathcal{E}$ (the space of effort variables), their product $\mathcal{F} \times \mathcal{E}$ defines the space of variables endowed with the power duality product

$$P = \langle e, f \rangle = e(f), \quad (f, e) \in \mathcal{F} \times \mathcal{E}.$$

The symmetric bilinear form $\langle\langle (f_a, e_a), (f_b, e_b) \rangle\rangle = \langle e_a, f_b \rangle + \langle e_b, f_a \rangle$ is defined on the product space.

Definition 2.5 ([8]): A (constant) Dirac structure on $\mathcal{F}$ is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that[3] $\mathcal{D} = \mathcal{D}^\perp$.

For a finite-dimensional real linear space $\mathcal{F}$ the definition of Dirac structure is equivalent to $\dim \mathcal{D} = \dim \mathcal{F}$ and

$$\langle e, f \rangle = 0, \quad (f, e) \in \mathcal{D}. \quad (19)$$

Any Dirac structure is power-conserving in the sense that the total power entering (or leaving) the Dirac structure is always zero [35]. A subset $\mathcal{R} \subset \mathcal{F}_r \times \mathcal{E}_r = 0$ satisfying $\langle e_r, f_r \rangle = e_r^\top f_r \leq 0$ for all $(f_r, e_r) \in \mathcal{R}$ defines a resistive structure which is expressed by the linear energy dissipation relation

$$e_r = -\overline{R} f_r, \quad (20)$$

for some matrix $\overline{R} = \overline{R}^\top \succeq 0$ capturing the dissipative behaviour, [8, Sec. 2.4]. As usual in the continuous-time domain, port variables $(f_S, e_S) \in \mathcal{F}_S \times \mathcal{E}_S$ of energy storing elements, port variables $(f_r, e_r) \in \mathcal{F}_r \times \mathcal{E}_r$ of resistive elements, and port variables $(f_i, e_i) \in \mathcal{F}_i \times \mathcal{E}_i$ of external elements, define a Dirac structure of the form

$$\mathcal{D} \subset \mathcal{F}_S \times \mathcal{E}_S \times \mathcal{F}_r \times \mathcal{E}_r \times \mathcal{F}_i \times \mathcal{E}_i. \quad (21)$$

A special case of port-Hamiltonian systems arises when no algebraic constraints occur on the state-space variables. It is described by the matrices $J(x) = -J^\top(x)$ and $R(x) = R^\top(x) \succeq 0$ specifying the interconnection and resistive structures respectively. From the dissipative constraint (20), $R(x)$ can be described as $R(x) = g_r(x) \overline{R} g_r^\top(x)$ with $g_r : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ representing the input matrix corresponding to the resistive port. Accordingly, the associated Dirac structure (21) is given by the graph of the skew-symmetric map

$$\begin{bmatrix} f_S \\ f_r \\ f_i \end{bmatrix} = \begin{bmatrix} -J & -g_r & -g \\ g_r^\top & 0 & 0 \\ g^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} e_S \\ e_r \\ e_i \end{bmatrix},$$

in which $g : \mathbb{R}^n \to \mathbb{R}$ represents the input map corresponding to the external ports. We refer to [35, Sec. 2.2] for additional details.

[3]The orthogonal complement with respect to the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ is denoted by $^\perp$.

III. Dirac and port-Hamiltonian structures in discrete time

In this section we introduce a discrete Dirac structure associated with discrete-time dynamics. In particular, inspired by the DDR structure of discrete-time dynamics in (13), we consider the separation of the port variables (f_S, e_S) of the total energy storing elements of the Dirac structure (21) into distinct pairs of ports (f_{S_f}, e_{S_f}) and (f_{S_c}, e_{S_c}) such that

$$\begin{aligned} -\langle e_S, f_S\rangle &= H(x^+(u)) - H(x) \\ &= H(x^+) - H(x) + H(x^+(u)) - H(x^+) \\ &= -\langle e_{S_f}, f_{S_f}\rangle - \langle e_{S_c}, f_{S_c}\rangle. \end{aligned}$$

The separation of the port variable (f_S, e_S) of the total energy storing elements into (f_{S_f}, e_{S_f}) and (f_{S_c}, e_{S_c}) yields an alternative definition of port-Hamiltonian system based on a Dirac structure in the sense of Definition 2.5.

Definition 3.1: Let a state-space $\mathbb{R}^n$ and Hamiltonian function $H : \mathbb{R}^n \to \mathbb{R}$, describing the energy storage. A discrete port-Hamiltonian system is defined by a discrete Dirac structure of the form

$$\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_c, \tag{22}$$

of dimension $\dim\mathcal{D} = \dim\mathcal{D}_f + \dim\mathcal{D}_c$ with subspaces

$$\mathcal{D}_f \subset \mathcal{F}_{S_f} \times \mathcal{E}_{S_f} \times \mathcal{F}_r \times \mathcal{E}_r, \tag{23a}$$

$$\mathcal{D}_c \subset \mathcal{F}_{S_c} \times \mathcal{E}_{S_c} \times \mathcal{F}_i \times \mathcal{E}_i, \tag{23b}$$

port variables $(f_{S_f}, e_{S_f}) \in \mathcal{F}_{S_f} \times \mathcal{E}_{S_f}$ and $(f_{S_c}, e_{S_c}) \in \mathcal{F}_{S_c} \times \mathcal{E}_{S_c}$ of the energy storing elements, port variables $(f_r, e_r) \in \mathcal{F}_r \times \mathcal{E}_r$ of the resistive elements, and port variables $(f_i, e_i) \in \mathcal{F}_i \times \mathcal{E}_i$ of the external elements. The dynamics of the discrete port-Hamiltonian system is implicitly defined by

$$(f_{S_f}, e_{S_f}, f_r, e_r) \in \mathcal{D}_f, \tag{24a}$$

$$(f_{S_c}, e_{S_c}, f_i, e_i) \in \mathcal{D}_c, \tag{24b}$$

$$(f_r, e_r) \in \mathcal{R}, \tag{24c}$$

with and resistive structure $\mathcal{R} \subset \mathcal{F}_r \times \mathcal{E}_r$.

Definition 3.1 introduces two different subspaces, $\mathcal{D}_f$ and $\mathcal{D}_c$, in the construction of a Dirac structure (22). The major departure of the proposed discrete Dirac structure from the standard discrete characterization as in [21] and [17] is the splitting of the storing elements. In particular, the subspace $\mathcal{D}_f$ corresponds to the energy-storing elements in $\mathcal{F}_{S_f} \times \mathcal{E}_{S_f}$, which encodes the internal energy stored not affected by any external phenomenon, and energy-dissipation elements in $\mathcal{F}_r \times \mathcal{E}_r$, which encodes the autonomous dissipation of the system. The subspace $\mathcal{D}_c$ corresponds to the energy-storing elements in $\mathcal{F}_{S_c} \times \mathcal{E}_{S_c}$, which encodes the effect of the control action, and external elements in $\mathcal{F}_i \times \mathcal{E}_i$, which encodes the interaction of the system with the external environment.

The characterization of the Dirac structure in (22) leads to the following result.

Theorem 3.1: Consider the system (15) evolving in $\mathbb{R}^n$ with Hamiltonian function $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. Suppose that $g(x,u) = g(x,0) := g(x)$, and that $R(x) = g_r(x)\overline{R}g_r^\top(x)$ with $g_r : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ and $\overline{R} = \overline{R}^\top \succeq 0$. Then the system (15) is characterized by the discrete Dirac structure (22) with

$$\left(-(x^+ - x), \bar{\nabla}H|_x^{x^+}, f_r, e_r\right) \in \mathcal{D}_f, \tag{25a}$$

$$\left(-(x^+(u) - x^+), \bar{\nabla}H|_{x^+}^{x^+(u)}, y(x,u), u\right) \in \mathcal{D}_c, \tag{25b}$$

$$(f_r, e_r) \in \mathcal{R}, \tag{25c}$$

and dissipative constraint $e_r = -\overline{R}\, f_r$. Moreover, the discrete Dirac structure (22) with elements in (25) satisfies

$$\dim\mathcal{D} = 2n + p + 1. \tag{26}$$

Proof: A constructive proof of the statement is performed by showing that the elements in (25) associated with the system (15) satisfy (19). In particular, with Definition 2.3 in mind, easy algebra shows that

$$\begin{aligned} H(x^+) - H(x) &= \left\langle x^+ - x, \bar{\nabla}H|_x^{x^+}\right\rangle \\ &= -\bar{\nabla}^\top H|_x^{x^+} R(x) \bar{\nabla}H|_x^{x^+}, \end{aligned} \tag{27}$$

and that

$$\begin{aligned} H(x^+(u)) - H(x^+) &= \left\langle x^+(u) - x^+, \bar{\nabla}H|_{x^+}^{x^+(u)}\right\rangle \\ &= u g^\top(x) \bar{\nabla}H|_{x^+}^{x^+(u)}. \end{aligned} \tag{28}$$

Constructing the pair (f_{S_f}, e_{S_f}) such that $f_{S_f} := -(x^+ - x)$ and $e_{S_f} := \bar{\nabla}H|_x^{x^+}$ respectively, we have that

$$(27) \implies -\langle e_{S_f}, f_{S_f}\rangle = -\bar{\nabla}^\top H|_x^{x^+} R(x) \bar{\nabla}H|_x^{x^+} = \langle e_r, f_r\rangle \leq 0,$$

provided $e_r = -\overline{R}\, f_r$ holds due to the structure of $R(x)$, *i.e.* $R(x) = g_r(x)\overline{R}g_r^\top(x)$. Moreover, setting $f_{S_c} := -(x^+(u) - x^+)$ and $e_{S_c} := \bar{\nabla}H|_{x^+}^{x^+(u)}$ we have that

$$(28) \implies -\langle e_{S_c}, f_{S_c}\rangle = u g^\top(x) \bar{\nabla}H|_{x^+}^{x^+(u)} = uy(x,u) = \langle e_i, f_i\rangle$$

with (f_i, e_i) as in (25). Hence, we conclude as (19) that

$$\begin{aligned} \langle e_{S_f}, f_{S_f}\rangle + \langle e_r, f_r\rangle = 0, &\qquad (f_{S_f}, e_{S_f}, f_r, e_r) \in \mathcal{D}_f, \\ \langle e_{S_c}, f_{S_c}\rangle + \langle e_i, f_i\rangle = 0, &\qquad (f_{S_c}, e_{S_c}, f_i, e_i) \in \mathcal{D}_c. \end{aligned}$$

Finally, since $(f_{S_f}, e_{S_f}) \in \mathcal{F}_{S_f} \times \mathcal{E}_{S_f} = \mathbb{R}^n \times \mathbb{R}^n$, $(f_r, e_r) \in \mathcal{F}_r \times \mathcal{E}_r = \mathbb{R}^p \times \mathbb{R}^p$, $(f_{S_c}, e_{S_c}) \in \mathcal{F}_{S_c} \times \mathcal{E}_{S_c} = \mathbb{R}^n \times \mathbb{R}^n$, $(f_i, e_i) \in \mathcal{F}_i \times \mathcal{E}_i = \mathbb{R} \times \mathbb{R}$ we have from (23) that

$$\begin{aligned} \dim\mathcal{D}_f &= \dim\mathcal{F}_{S_f} + \dim\mathcal{F}_{S_r} = n + p \\ \dim\mathcal{D}_c &= \dim\mathcal{F}_{S_c} + \dim\mathcal{F}_{S_i} = n + 1 \end{aligned}$$

hence we conclude (26). ∎

Theorem 3.1 provides the characterization of the underlying Dirac structure of a discrete dynamical system represented by the equations (15). The system expressed in DDR form (15) suggests to decouple the port variables $(f_S, e_S) \in \mathcal{F}_S \times \mathcal{E}_S$ into $(f_{S_f}, e_{S_f}) \in \mathcal{F}_{S_f} \times \mathcal{E}_{S_f}$ and $(f_{S_c}, e_{S_c}) \in \mathcal{F}_{S_c} \times \mathcal{E}_{S_c}$ satisfying $\langle e_S, f_S\rangle = \langle e_{S_f}, f_{S_f}\rangle + \langle e_{S_c}, f_{S_c}\rangle$. The pair (f_{S_f}, e_{S_f}) encodes the effect of the energy variation due to the uncontrolled component and is specified by the discrete gradient from x to x^+. The pair (f_{S_c}, e_{S_c}) encodes the effect of the controlled component and is specified by the discrete gradient from x^+ to $x^+(u)$. Moreover, the obtained Dirac structure is characterized by the graph of the skew-symmetric map

$$\left[\begin{array}{c} f_{S_f} \\ f_r \\ \hdashline f_{S_c} \\ f_i \end{array}\right] = \left[\begin{array}{cc:cc} -J & -g_r & 0 & 0 \\ g_r^\top & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -g \\ 0 & 0 & g^\top & 0 \end{array}\right] \left[\begin{array}{c} e_{S_f} \\ e_r \\ \hdashline e_{S_c} \\ e_i \end{array}\right]. \tag{29}$$

For the sake of completeness, we emphasize that the underlying Dirac structure (25) in case of zero input, *i.e.* for $u = e_i = 0$, recovers the underlying Dirac structure of the uncontrolled dynamics proposed in [21] and [17].

IV. Dirac structures under feedback and interconnection

This section is devoted to the discussion of the transformation of the discrete Dirac structure (22) in two different cases: when a negative output feedback is injected in the system (15); when two systems of the form (15) are interconnected through a power-preserving interconnection.

A. Port-Hamiltonian structures under negative output feedback

To begin with, let the function $\alpha(x)$ be the negative output feedback applied to the port-Hamiltonian system (15) which is obtained as the solution of the implicit equation

$$\alpha(x) = -\kappa y(x, \alpha(x)), \tag{30}$$

with damping factor $\kappa > 0$, output map $y(\cdot,\cdot)$ as in (15b), and input mapping $g(x,u)$ which is assumed as in Theorem 3.1, *i.e.* $g(x,u) = g(x,0) := g(x)$. Consider the input variable as

$$u = \alpha(x) + v, \tag{31}$$

where $v(k) \in \mathbb{R}$ describes the external input variable. Hence, the state-space representation of the port-Hamiltonian system (15) under the control action (31) yields the feedback dynamics

$$x^+ = x + (J(x) - R(x))\bar{\nabla} H|_x^{x^+}, \tag{32a}$$

$$x^+(\alpha + v) = x^+ - \kappa g(x) g^\top(x) \bar{\nabla} H|_{x^+}^{x^+(\alpha)} + g(x) v. \tag{32b}$$

With the above structure in mind, the following result holds denoting for simplicity the feedback $\alpha(x) + v$ by $\alpha + v$.

Proposition 4.1: Consider the feedback dynamics (32) with Hamiltonian function $H : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and output map

$$\tilde{y}(x, v) = g^\top(x) \bar{\nabla} H|_{x^+(\alpha)}^{x^+(\alpha+v)}. \tag{33}$$

Then the dynamics (32) along with (33) is characterized by the discrete Dirac structure (22) with

$$\left(\begin{bmatrix} -(x^+ - x) \\ -(x^+(\alpha) - x^+) \end{bmatrix}, \begin{bmatrix} \bar{\nabla} H|_x^{x^+} \\ \bar{\nabla} H|_{x^+}^{x^+(\alpha)} \end{bmatrix}, f_r, e_r \right) \in \mathcal{D}_f, \tag{34a}$$

$$\left(-(x^+(\alpha+v) - x^+(\alpha)), \bar{\nabla} H|_{x^+(\alpha)}^{x^+(\alpha+v)}, \tilde{y}(x,v), v \right) \in \mathcal{D}_c, \tag{34b}$$

$$(f_r, e_r) \in \mathcal{R}, \tag{34c}$$

and an extended linear energy dissipation relation $e_r = -\overline{R}_e f_r$ with matrix

$$\overline{R}_e = \begin{bmatrix} \overline{R} & 0 \\ 0 & \kappa \end{bmatrix}. \tag{35}$$

Moreover, the discrete Dirac structure (22) with port variables in (34) yields that

$$\dim \mathcal{D} = 3n + p + 2. \tag{36}$$

Proof: In a similar vein of (27) and (28), and with Definition 2.3 in mind, we have that the variation of the Hamiltonian function along the dynamics (32) yields the equation

$$\begin{aligned} &H(x^+(\alpha)) - H(x) \\ &= \left\langle x^+ - x, \bar{\nabla} H|_x^{x^+} \right\rangle + \left\langle x^+(\alpha) - x^+, \bar{\nabla} H|_{x^+}^{x^+(\alpha)} \right\rangle \\ &= -\bar{\nabla}^\top H|_x^{x^+} R(x) \bar{\nabla} H|_x^{x^+} - \kappa g(x) g^\top(x) \bar{\nabla} H|_{x^+}^{x^+(\alpha)}, \end{aligned} \tag{37}$$

and the equation

$$\begin{aligned} &H(x^+(\alpha + v)) - H(x^+(\alpha)) \\ &= \left\langle x^+(\alpha + v) - x^+(\alpha), \bar{\nabla} H|_{x^+(\alpha)}^{x^+(\alpha+v)} \right\rangle \\ &= v g^\top(x) \bar{\nabla} H|_{x^+(\alpha)}^{x^+(\alpha+v)}. \end{aligned} \tag{38}$$

Constructing the elements

$$f_{S_f} := \begin{bmatrix} -(x^+ - x) \\ -(x^+(\alpha) - x^+) \end{bmatrix}, \qquad e_{S_f} := \begin{bmatrix} \bar{\nabla} H|_x^{x^+} \\ \bar{\nabla} H|_{x^+}^{x^+(\alpha)} \end{bmatrix},$$

we have from the structure $R(x) = g_r(x) \overline{R} g_r^\top(x)$ that the equation (37) implies the inequality

$$\begin{aligned} \langle e_{S_f}, f_{S_f} \rangle &= \begin{bmatrix} \bar{\nabla} H|_x^{x^+} \\ \bar{\nabla} H|_{x^+}^{x^+(\alpha)} \end{bmatrix}^\top \begin{bmatrix} g_r \overline{R} g_r^\top & 0 \\ 0 & \kappa g g^\top \end{bmatrix} \begin{bmatrix} \bar{\nabla} H|_x^{x^+} \\ \bar{\nabla} H|_{x^+}^{x^+(\alpha)} \end{bmatrix} \\ &= -\langle e_r, f_r \rangle \geq 0, \end{aligned}$$

which is satisfied providing the dissipative constraint $e_r = -\overline{R}_e f_r$ with extended matrix $\overline{R}_e$ in (35). In addition, setting $f_{S_c} := -(x^+(\alpha+v) - x^+(\alpha))$ and $e_{S_c} := \bar{\nabla} H|_{x^+(\alpha)}^{x^+(\alpha+v)}$ we have that

$$\begin{aligned} (38) \implies -\langle e_{S_c}, f_{S_c} \rangle &= v g^\top(x) \bar{\nabla} H|_{x^+(\alpha)}^{x^+(\alpha+v)} \\ &= v \tilde{y}(x, v) = \langle e_i, f_i \rangle, \end{aligned}$$

with (f_i, e_i) as in (34b). Finally, since the flows and efforts subspaces are by cunstruction such that

$$\begin{aligned} (f_{S_f}, e_{S_f}) &\in \mathcal{F}_{S_f} \times \mathcal{E}_{S_f} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \\ (f_{S_c}, e_{S_c}) &\in \mathcal{F}_{S_c} \times \mathcal{E}_{S_c} = \mathbb{R}^{n} \times \mathbb{R}^{n}, \\ (f_r, e_r) &\in \mathcal{F}_r \times \mathcal{E}_r = \mathbb{R}^{p+1} \times \mathbb{R}^{p+1}, \\ (f_i, e_i) &\in \mathcal{F}_i \times \mathcal{E}_i = \mathbb{R} \times \mathbb{R}, \end{aligned}$$

then this implies that

$$\begin{aligned} \dim \mathcal{D}_f &= \dim \mathcal{F}_{S_f} + \dim \mathcal{F}_{S_r} = 2n + p + 1, \\ \dim \mathcal{D}_c &= \dim \mathcal{F}_{S_c} + \dim \mathcal{F}_{S_i} = n + 1, \end{aligned}$$

hence we conclude (36). ∎

We have shown that the feedback system (32) coupled with the output (33) retains the port-Hamiltonian structure in the sense of Definition 3.1. However, the underlying discrete Dirac structure associated with (34) yields a subspace $\mathcal{D}$ which increases in dimension as provided by the condition (36). The increase in dimension of the subspace $\mathcal{D}$ is due to the closed-loop dissipation matrix

$$\begin{bmatrix} R(x) & 0 \\ 0 & \kappa g(x) g^\top(x) \end{bmatrix} \succeq 0,$$

with $\dim \mathcal{F}_{S_f} = \dim \mathbb{R}^{2n}$, $\dim \mathcal{F}_r = \dim \mathbb{R}^{p+1}$. The Dirac structure is characterized by the graph of the skew-symmetric map

$$\left[\begin{array}{c} f_{S_f} \\ \hdashline f_r \\ \hdashline f_{S_c} \\ f_i \end{array}\right] = \left[\begin{array}{cc:cc:cc} -J & 0 & -g_r & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 & 0 \\ \hdashline g_r^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & g^\top & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & -g \\ 0 & 0 & 0 & 0 & g^\top & 0 \end{array}\right] \left[\begin{array}{c} e_{S_f} \\ \hdashline e_r \\ \hdashline e_{S_c} \\ e_i \end{array}\right]. \tag{39}$$

It worth noting that, from the property in (11), the output $\tilde{y}(x,v)$ satisfies by construction the dissipation inequality

$$H(x^+(\alpha(x) + v)) - H(x) \leq v \tilde{y}(x, v)$$

along the dynamics (32), and thus $\tilde{y}(x,v)$ coincides with the u-average output from $\bar{u}$ in (12) with $\bar{u}=\alpha(x)$. The u-average output from $\alpha(x)$ is computed with respect to the output map (16), that is

$$\tilde{y}(x,v)=\frac{1}{v}\int_0^v Y(x^+(\alpha(x)+s),\alpha(x)+s)\mathrm{d}s. \tag{40}$$

This is reminiscent of the interconnection-based modeling in [24] which shows that the u-average passivity property during the interconnection process, by means of feedback of the form (31), is transformed into u-average passivity from $\alpha(x)$,

B. Closeness under power-preserving interconnection

This section focuses on the composition of Dirac structures. In particular, we show that the feedback interconnection of two discrete-time port-Hamiltonian systems retains the port-Hamiltonian structure in the sense of Definition 3.1. Before proceeding, we recall that one of the key features of the Dirac structures is their compositionality. The composition of Dirac structures is recalled from [8, Sec. 6.1].

Definition 4.1: Given two Dirac structures $(\mathcal{D}_1,\mathcal{D}_2)$ defined respectively on $\mathcal{F}_1\times\mathcal{F}_C$ and $\mathcal{F}_C\times\mathcal{F}_2$ where $\mathcal{F}_C$ is the space of shared flow and effort, their composition is defined on $\mathcal{F}_1\times\mathcal{F}_2$ under the constraints $f_{C_1}=-f_{C_2}$, $e_{C_1}=e_{C_2}$ implying

$$e_{C_1}^\top f_{C_1}+e_{C_2}^\top f_{C_2}=0 \tag{41}$$

on the shared flows and efforts (f_{C_1},e_{C_1}) and (f_{C_2},e_{C_2}).

With the composition of Dirac structures in mind, we consider the power-preserving interconnection which is specified by the equation

$$u_1y_1(x_1,u_1)+u_2y_2(x_2,u_2)=0, \tag{42}$$

or equivalently, in a compact form, is described by

$$\Psi(x,u)=\begin{pmatrix}u_1+y_2(x_2,u_2)\\ u_2-y_1(x_1,u_1)\end{pmatrix}=0.$$

As discussed in [24], by invoking the Implicit Function Theorem the solution of the equation (42) is always well-defined in a neighbourhood of $x_\star=(x_{\star_1},x_{\star_2})$ where $\Psi(x_\star,0)=0$ and the matrix $\frac{\partial\Psi}{\partial u}\big|_{u=0}$ is non-singular at $x_\star$. Hence, denoting by (α_1,α_2) the power-preserving feedback solving the equation (42), *i.e.*

$$\begin{aligned}\alpha_1(x_1,x_2)&=-y_2(x_2,\alpha_2(x_1,x_2)),\\ \alpha_2(x_1,x_2)&=y_1(x_1,\alpha_1(x_1,x_2)),\end{aligned}$$

we define the control laws

$$\begin{aligned}u_1(x_1,x_2,v_1)&=\alpha_1(x_1,x_2)+v_1,\\ u_2(x_1,x_2,v_2)&=\alpha_2(x_1,x_2)+v_2,\end{aligned} \tag{43}$$

with with external input variables $v_1(k)\in\mathbb{R}$ and $v_2(k)\in\mathbb{R}$. Then, the feedback interconnection (43) of two port-Hamiltonian systems of the form (15) with $x_1\in\mathbb{R}^{n_1}$, $x_2\in\mathbb{R}^{n_2}$, $g_{R_1}:\mathbb{R}^{n_1}\to\mathbb{R}^{n_1\times p_1}$, $g_{R_2}:\mathbb{R}^{n_2}\to\mathbb{R}^{n_2\times p_2}$, $\overline{R}_1=\overline{R}_1^\top\succeq 0$, $\overline{R}_2=\overline{R}_2^\top\succeq 0$, $H_1:\mathbb{R}^{n_1}\to\mathbb{R}_{\geq 0}$, and $H_2:\mathbb{R}^{n_2}\to\mathbb{R}_{\geq 0}$ yields the dynamics

$$\begin{aligned}x_1^+&=x_1+(J_1(x_1)-R_1(x_1))\bar{\nabla}H_1|_{x_1}^{x_1^+},\\ x_2^+&=x_2+(J_2(x_2)-R_2(x_2))\bar{\nabla}H_2|_{x_2}^{x_2^+},\\ x_1^+(\alpha_1+v_1)&=x_1^+-g_1(x_1)g_2^\top(x_2)\bar{\nabla}H_2|_{x_2^+}^{x_2^+(\alpha_2)}+g_1(x_1)v_1,\\ x_2^+(\alpha_2+v_2)&=x_2^++g_2(x_2)g_1^\top(x_1)\bar{\nabla}H_1|_{x_1^+}^{x_1^+(\alpha_1)}+g_2(x_2)v_2.\end{aligned} \tag{44}$$

With this in mind, the following result holds.

Theorem 4.1: Consider the interconnected dynamics (44) with Hamiltonian function given by $H(x_1,x_2)=H_1(x_1)+H_2(x_2)$ and output $\tilde{y}(x,v)$ given by

$$\tilde{y}(x_1,x_2,v_1,v_2)=\begin{bmatrix}g_1^\top(x_1)\bar{\nabla}H_1|_{x_1^+}^{x_1^+(\alpha_1+v_1)}\\ g_2^\top(x_2)\bar{\nabla}H_2|_{x_2^+}^{x_2^+(\alpha_2+v_2)}\end{bmatrix}. \tag{45}$$

Then the dynamics (44) with output $\tilde{y}(x_1,x_2,v_1,v_2)$ is characterized by the discrete Dirac structure as in (24) with elements

$$\begin{aligned}f_{S_f}&=\begin{bmatrix}-(x_1^+-x_1)\\ -(x_2^+-x_2)\\ -(x_1^+(\alpha_1)-x_1^+)\\ -(x_2^+(\alpha_2)-x_2^+)\end{bmatrix}, & e_{S_f}&=\begin{bmatrix}\bar{\nabla}H_1|_{x_1}^{x_1^+}\\ \bar{\nabla}H_2|_{x_2}^{x_2^+}\\ \bar{\nabla}H_1|_{x_1^+}^{x_1^+(\alpha_1)}\\ \bar{\nabla}H_2|_{x_2^+}^{x_2^+(\alpha_2)}\end{bmatrix},\\ f_{S_c}&=\begin{bmatrix}-(x_1^+(\alpha_1+v_1)-x_1^+(\alpha_1))\\ -(x_2^+(\alpha_2+v_2)-x_2^+(\alpha_2))\end{bmatrix}, & e_{S_c}&=\begin{bmatrix}\bar{\nabla}H_1|_{x_1^+}^{x_1^+(\alpha_1)}\\ \bar{\nabla}H_2|_{x_2^+}^{x_2^+(\alpha_2)}\end{bmatrix},\\ f_i&=\tilde{y}(x_1,x_2,v_1,v_2), & e_i&=v,\end{aligned} \tag{46}$$

and extended dissipative constraint $e_r=-\overline{R}_e f_r$ with

$$\overline{R}_e=\begin{bmatrix}\overline{R}_1 & 0\\ 0 & \overline{R}_2\end{bmatrix}.$$

Moreover, the discrete Dirac structure (22) yields that

$$\dim\mathcal{D}=3n_1+3n_2+p_1+p_2+2. \tag{47}$$

Proof: To begin with, we note that along (44) the variation of H_1 between $x_1^+(\alpha_1)$ and x_1^+ yields

$$\begin{aligned}&H_1(x_1^+(\alpha_1))-H_1(x_1^+)=\left\langle x_1^+(\alpha_1)-x_1^+,\bar{\nabla}H_1|_{x_1^+}^{x_1^+(\alpha_1)}\right\rangle\\ &=-\left\langle g_1(x_1)g_2^\top(x_2)\bar{\nabla}H_2|_{x_2^+}^{x_2^+(\alpha_2)},\bar{\nabla}H_1|_{x_1^+}^{x_1^+(\alpha_1)}\right\rangle,\end{aligned} \tag{49}$$

and H_2 between $x_2^+(\alpha_2)$ and x_2^+ yields

$$\begin{aligned}&H_2(x_2^+(\alpha_2))-H_2(x_2^+)=\left\langle x_2^+(\alpha_2)-x_2^+,\bar{\nabla}H_2|_{x_2^+}^{x_2^+(\alpha_2)}\right\rangle\\ &=\left\langle g_2(x_2)g_1^\top(x_1)\bar{\nabla}H_1|_{x_1^+}^{x_1^+(\alpha_1)},\bar{\nabla}H_2|_{x_2^+}^{x_2^+(\alpha_2)}\right\rangle,\end{aligned} \tag{50}$$

which implies $\sum_{j=1}^2 H_j(x_j^+(\alpha_j))-H_j(x_j^+)=0$. Then, the flows and efforts given in the equations (46) are by construction such that

$$\begin{aligned}(f_{S_f},e_{S_f})\in\mathcal{F}_{S_f}\times\mathcal{E}_{S_f}&=\mathbb{R}^{2n_1+2n_2}\times\mathbb{R}^{2n_1+2n_2},\\ (f_{S_c},e_{S_c})\in\mathcal{F}_{S_c}\times\mathcal{E}_{S_c}&=\mathbb{R}^{n_1+n_2}\times\mathbb{R}^{n_1+n_2},\\ (f_r,e_r)\in\mathcal{F}_r\times\mathcal{E}_r&=\mathbb{R}^{p_1+p_2}\times\mathbb{R}^{p_1+p_2},\\ (f_i,e_i)\in\mathcal{F}_i\times\mathcal{E}_i&=\mathbb{R}^2\times\mathbb{R}^2,\end{aligned}$$

which implies that

$$\begin{aligned}\dim\mathcal{D}_f&=\dim\mathcal{F}_{S_f}+\dim\mathcal{F}_{S_r}=2n_1+2n_2+p_1+p_2,\\ \dim\mathcal{D}_c&=\dim\mathcal{F}_{S_c}+\dim\mathcal{F}_{S_i}=n_1+n_2+2,\end{aligned}$$

and thus (47). Finally, setting $H(x_1,x_2)=H_1(x_1)+H_2(x_2)$ we have that

$$\begin{aligned}&H(x_1^+(\alpha_1+v_1),x_2^+(\alpha_2+v_2))-H(x_1,x_2)\\ &=-\langle e_{S_f},f_{S_f}\rangle-\langle e_{S_c},f_{S_c}\rangle=\langle e_r,f_r\rangle+\langle e_i,f_i\rangle,\end{aligned}$$

$$\begin{bmatrix} f_{S_f} \\ \hdashline f_r \\ \hdashline f_{S_c} \\ \hdashline f_i \end{bmatrix} = \left[\begin{array}{cccc:cc:cc:cc} -J_1 & 0 & 0 & 0 & -g_{R_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -J_2 & 0 & 0 & 0 & -g_{R_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_1 g_2^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g_2 g_1^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline g_{R_1}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{R_2}^\top & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -g_2 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & g_1^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_2^\top & 0 & 0 \end{array}\right] \begin{bmatrix} e_{S_f} \\ \hdashline e_r \\ \hdashline e_{S_c} \\ \hdashline e_i \end{bmatrix} \tag{48}$$

and thus we conclude that the subspace $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_c$ is a Dirac structure yielding $\langle e_{S_f}, f_{S_f}\rangle + \langle e_{S_c}, f_{S_c}\rangle + \langle e_r, f_r\rangle + \langle e_i, f_i\rangle = 0$. ■

It is worth emphasizing that the dimension of the Dirac structure obtained by interconnecting two port-Hamiltonian systems is determined by the number of compositions of Dirac structures. Increasing the dimension of the structure results in a structure that keeps track of the interconnection order. This is emphasized by the graph of the skew-symmetric map in (48).

V. An illustrative example

Consider the dynamics of a gravity pendulum consisting of a uniform rod of length $\ell > 0$, affected by gravity g, and mass $m > 0$ pivoted at one end. The Hamiltonian function catching respectively the kinetic and the potential energy of the system is given by

$$H(q,p) = \frac{p^2}{2m\ell^2} + mg\ell(1-\cos(q)), \tag{51}$$

where q and p are given by $q = \vartheta$ and $p = m\ell^2\dot{\vartheta}$ respectively, where ϑ is the pendulum angle relative to the vertical axis. Following the discretization procedure in [22], the discrete-time model, which approximates the smooth behaviour of the gravity pendulum over sampling intervals of unitary length, can be expressed by the equations

$$q^+(u) = q + \frac{1}{2m\ell^2}(p^+ + p), \tag{52a}$$

$$p^+(u) = p - \frac{mg\ell}{q^+ - q}(\cos(q) - \cos(q^+)) + u, \tag{52b}$$

$$y(q,p,u) = \frac{1}{2m\ell^2}(p^+(u) + p^+), \tag{52c}$$

where u describes the piecewise constant controlled torque. Since the discrete-time system described by the equations (52) is a discrete port-Hamiltonian system in the sense of Definition 3.1, applying the result of Theorem 3.1 we can construct the port variables $(f_{S_f}, e_{S_f}) \in \mathcal{F}_{S_f} \times \mathcal{E}_{S_f}$, such as

$$f_{S_f} := \begin{cases} f_{q_f} &= -(q^+ - q), \\ f_{p_f} &= -(p^+ - p), \end{cases}$$
$$e_{S_f} := \begin{cases} e_{q_f} &= \frac{mg\ell}{q^+ - q}(\cos(q) - \cos(q^+)), \\ e_{p_f} &= \frac{1}{2m\ell^2}(p^+ + p), \end{cases}$$

the port variables $(f_{S_c}, e_{S_c}) \in \mathcal{F}_{S_c} \times \mathcal{E}_{S_c}$, such as

$$f_{S_c} := \begin{cases} f_{q_u} &= -(q^+(u) - q^+), \\ f_{p_u} &= -(p^+(u) - p^+), \end{cases}$$
$$e_{S_c} := \begin{cases} e_{q_u} &= \frac{mg\ell}{q^+(u) - q^+}(\cos(q^+) - \cos(q^+(u))), \\ e_{p_u} &= \frac{1}{2m\ell^2}(p^+(u) + p^+), \end{cases}$$

and port variables $(f_i, e_i) \in \mathcal{F}_i \times \mathcal{E}_i$ such as $f_i := y$ and $e_i := u$. It is important to note that, in consistency with its continuous-time counterpart, the discrete version of the gravity pendulum (52) is conservative, and thus resistive elements are missing. Nevertheless, the system (52) has an underlying Dirac structure $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_c$ characterized by the graph of the skew-symmetric map

$$\begin{bmatrix} f_{S_f} \\ \hdashline f_{S_c} \\ \hdashline f_i \end{bmatrix} = \left[\begin{array}{cc:cc:c} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \hdashline 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} e_{S_f} \\ \hdashline e_{S_c} \\ \hdashline e_i \end{bmatrix}.$$

In addition, to achieve asymptotic stabilization of the gravity pendulum at the minimum of $H(q,p)$ we consider the controller $u = \alpha(x) + v$ with negative output feedback $\alpha(q,p)$ which is defined as in (30) and specified here as

$$\alpha(q,p) = -\frac{\kappa}{2m\ell^2}(p^+(u) + p^+).$$

Hence following the result in Proposition 4.1, we have that the system (52) under negative output feedback yields an extended Dirac structure $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_c$ which is described by the graph of the extended skew-symmetric map

$$\begin{bmatrix} f_{S_f} \\ \hdashline f_r \\ \hdashline f_{S_c} \\ \hdashline f_i \end{bmatrix} = \left[\begin{array}{cccc:cc:cc} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} e_{S_f} \\ \hdashline e_r \\ \hdashline e_{S_c} \\ \hdashline e_i \end{bmatrix}$$

with additional resistive constraint $e_r = -\kappa f_r$, port variables $(f_{S_f}, e_{S_f}) \in \mathcal{F}_{S_f} \times \mathcal{E}_{S_f}$ with

$$f_{S_f} := \begin{cases} f_{q_f} &= -(q^+ - q), \\ f_{p_f} &= -(p^+ - p), \\ f_{q_\alpha} &= -(q^+(\alpha) - q^+) = 0, \\ f_{p_\alpha} &= -(p^+(\alpha) - p^+), \end{cases}$$
$$e_{S_f} := \begin{cases} e_{q_f} &= \frac{mg\ell}{q^+ - q}(\cos(q) - \cos(q^+)), \\ e_{p_f} &= \frac{1}{2m\ell^2}(p^+ + p), \\ e_{q_\alpha} &= \frac{mg\ell}{q^+(\alpha) - q^+}(\cos(q^+) - \cos(q^+(\alpha)) = 0, \\ e_{p_\alpha} &= \frac{1}{2m\ell^2}(p^+(\alpha) + p^+), \end{cases}$$

port variables $(f_{S_c}, e_{S_c}) \in \mathcal{F}_{S_c} \times \mathcal{E}_{S_c}$ with

$$f_{S_c} := \begin{cases} f_{q_v} &= q^+(\alpha) - q^+(\alpha + v) = 0, \\ f_{p_v} &= p^+(\alpha) - p^+(\alpha + v), \end{cases}$$
$$e_{S_c} := \begin{cases} e_{q_v} &= \frac{mg\ell(\cos(q^+(\alpha)) - \cos(q^+(\alpha+v)))}{q^+(\alpha+v) - q^+(\alpha)} = 0, \\ e_{p_v} &= \frac{(p^+(\alpha+v) + p^+(\alpha))}{2m\ell^2}, \end{cases}$$

and port variables $(f_i, e_i) \in \mathcal{F}_i \times \mathcal{E}_i$ with $f_i := \frac{1}{2m\ell^2}(p^+(\alpha + v) + p^+(\alpha))$, and $e_i := v$.

VI. Conclusion

In this article, we have proposed a new definition of discrete port-Hamiltonian systems, defined by a discrete Dirac structure of the form (22). The key difference from the standard representation is that the spaces of flows and efforts associated with the storing elements of the system are separated into two distinct subspaces. We have demonstrated that the system introduced in [25], which is assumed to be input-affine, can be characterized by an underlying discrete Dirac structure of the form (22) and can be modeled by appropriate flows and efforts. The discrete Dirac structure allows us to show that the closure of the dynamics (32) through negative output feedback and the interconnected system (44), obtained via power-preserving interconnection in [24], preserves the discrete port-Hamiltonian structure in the sense of Definition 3.1. Finally, for discrete-time modeling purposes, we have illustrated the theory by means of a gravity pendulum system. Specifically, we have constructed the flows and efforts for both the conservative dynamics and the dynamics resulting from the injection of negative output feedback. The proposed approach can be extended to the multi-input multi-output case by using a set of partial differential equations that describe the rate of change for each control component as in [28].

These results pave the way for new interconnection-based modeling in discrete time exploiting the DDR structure.

Acknowledgment

The authors are grateful to the Associate Editor and the anonymous Reviewers for their insightful comments and suggestions.

References

[1] G. Kron, *Tensor analysis of networks*. J. Wiley & Sons New York, 1939.

[2] A. J. van der Schaft and B. M. Maschke, "On the Hamiltonian formulation of nonholonomic mechanical systems," *Reports on Mathematical Physics*, vol. 34, no. 2, pp. 225–233, 1994.

[3] B. M. Maschke and A. J. van der Schaft, "Port-controlled Hamiltonian systems: modelling origins and systemtheoretic properties," *IFAC Proceedings Volumes*, vol. 25, no. 13, pp. 359–365, 1992.

[4] A. J. van der Schaft and B. M. Maschke, "The Hamiltonian formulation of energy conserving physical systems with external ports," *AEÜ International Journal of Electronics and Communications*, vol. 49, no. 5/6, pp. 362–371, 1995.

[5] T. J. Courant, "Dirac manifolds," *Transactions of the American Mathematical Society*, vol. 319, no. 2, pp. 631–661, 1990.

[6] B. M. Maschke, A. J. van der Schaft, and P. C. Breedveld, "An intrinsic Hamiltonian formulation of network dynamics: Non-standard poisson structures and gyrators," *Journal of the Franklin institute*, vol. 329, no. 5, pp. 923–966, 1992.

[7] B. M. Maschke and A. J. van der Schaft, "Port-controlled Hamiltonian systems: modelling origins and systemtheoretic properties," in *Nonlinear Control Systems Design 1992*. Elsevier, 1993, pp. 359–365.

[8] A. J. van der Schaft and D. Jeltsema, "Port-Hamiltonian systems theory: An introductory overview," *Foundations and Trends® in Systems and Control*, vol. 1, no. 2-3, pp. 173–378, 2014.

[9] R. Ortega, A. J. van der Schaft, I. Mareels, and B. Maschke, "Putting energy back in control," *IEEE Control Systems Magazine*, vol. 21, no. 2, pp. 18–33, 2001.

[10] A. Macchelli and C. Melchiorri, "Control by interconnection of mixed port Hamiltonian systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 11, pp. 1839–1844, 2005.

[11] R. Ortega, A. van der Schaft, F. Castanos, and A. Astolfi, "Control by interconnection and standard passivity-based control of port-Hamiltonian systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2527–2542, 2008.

[12] S. Stramigioli, C. Secchi, A. J. van der Schaft, and C. Fantuzzi, "Sampled data systems passivity and discrete port-Hamiltonian systems," *IEEE Transactions on Robotics*, vol. 21, no. 4, pp. 574–587, 2005.

[13] A. Moreschini, S. Monaco, and D. Normand-Cyrot, "Gradient and Hamiltonian dynamics under sampling," *IFAC-PapersOnLine*, vol. 52, no. 16, pp. 472 – 477, 2019.

[14] A. Moreschini, M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Stabilization of discrete port-Hamiltonian dynamics via interconnection and damping assignment," *IEEE Control Systems Letters*, vol. 5, no. 1, pp. 103–108, 2021.

[15] P. Kotyczka and T. Thoma, "Symplectic discrete-time energy-based control for nonlinear mechanical systems," *Automatica*, vol. 133, p. 109842, 2021.

[16] A. Macchelli, "Trajectory tracking for discrete-time port-Hamiltonian systems," *IEEE Control Systems Letters*, vol. 6, pp. 3146–3151, 2022.

[17] V. Talasila, J. Clemente-Gallardo, and A. van der Schaft, "Discrete port-Hamiltonian systems," *Systems & Control Letters*, vol. 55, no. 6, pp. 478–486, 2006.

[18] D. S. Laila and A. Astolfi, "Construction of discrete-time models for port-controlled Hamiltonian systems with applications," *Systems & Control Letters*, vol. 55, no. 8, pp. 673–680, 2006.

[19] P. Kotyczka and L. Lefèvre, "Discrete-time port-Hamiltonian systems: A definition based on symplectic integration," *Systems & Control Letters*, vol. 133, p. 104530, 2019.

[20] R. Moulla, L. Lefévre, and B. Maschke, "Pseudo-spectral methods for the spatial symplectic reduction of open systems of conservation laws," *Journal of Computational Physics*, vol. 231, no. 4, pp. 1272–1292, 2012.

[21] S. Aoues, M. Di Loreto, D. Eberard, and W. Marquis-Favre, "Hamiltonian systems discrete-time approximation: Losslessness, passivity and composability," *Systems & Control Letters*, vol. 110, pp. 9–14, 2017.

[22] S. Monaco, D. Normand-Cyrot, M. Mattioni, and A. Moreschini, "Nonlinear Hamiltonian systems under sampling," *IEEE Transactions on Automatic Control*, 2022.

[23] M. Mattioni, A. Moreschini, S. Monaco, and D. Normand-Cyrot, "Discrete-time energy-balance passivity-based control," *Automatica*, vol. 146, p. 110662, 2022.

[24] A. Moreschini, M. Mattioni, S. Monaco, and D. Normand-Cyrot, "Interconnection through u-average passivity in discrete time," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 4234–4239.

[25] ——, "Discrete port-controlled Hamiltonian dynamics and average passivation," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 1430–1435.

[26] S. Monaco and D. Normand-Cyrot, "A unified representation for nonlinear discrete-time and sampled dynamics," *Journal of Mathematical Systems, Estimation and Control*, vol. 7, pp. 50–3, 1995.

[27] O. Gonzalez, "Time integration and discrete Hamiltonian systems," *Journal of Nonlinear Science*, vol. 6, no. 5, p. 449, 1996.

[28] S. Monaco and D. Normand-Cyrot, "Nonlinear average passivity and stabilizing controllers in discrete time," *Systems & Control Letters*, vol. 60, no. 6, pp. 431–439, 2011.

[29] F. Albertini and E. D. Sontag, "Discrete-time transitivity and accessibility: analytic systems," *SIAM Journal on Control and Optimization*, vol. 31, no. 6, pp. 1599–1622, 1993.

[30] B. Jakubczyk and E. D. Sontag, "Controllability of nonlinear discrete-time systems: A Lie-algebraic approach," *SIAM Journal on Control and Optimization*, vol. 28, no. 1, pp. 1–33, 1990.

[31] S. Monaco and D. Normand-Cyrot, "Discrete-time state representations, a new paradigm," in *Perspectives in Control*. Springer, 1998, pp. 191–203.

[32] B. Jakubczyk and D. Normand-Cyrot, "Automatique théorique. orbites de pseudo-groupes de difféomorphismes et commandabilité des systemes non linéaires en temps discret," *Comptes rendus des séances de l'Académie des sciences. Série 1, Mathématique*, vol. 298, no. 11, pp. 257–260, 1984.

[33] R. I. McLachlan, G. Quispel, and N. Robidoux, "Geometric integration using discrete gradients," *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 357, no. 1754, pp. 1021–1045, 1999.

[34] E. Hairer, C. Lubich, and G. Wanner, *Geometric numerical integration: structure-preserving algorithms for ordinary differential equations*. Springer Science & Business Media, 2006, vol. 31.

[35] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx, *Modeling and control of complex physical systems: the port-Hamiltonian approach*. Springer Science & Business Media, 2009.