Nonlocal time delays reaction-diffusion problems of gradient flow type: existence, stochastic homogenization

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NONLOCAL TIME DELAYS REACTION-DIFFUSION PROBLEMS OF GRADIENT FLOW TYPE: EXISTENCE, STOCHASTIC HOMOGENIZATION

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Abstract. In this paper we continue our study of nonlocal reaction-diffusion problems of gradient flow type that we developed in a previous paper [AHMM23] by using the notion of CP-structured reaction functionals as introduced in [AHMM19]. We consider here problems with times delays and DCP-structured reaction functionals as introduced in [AHMM20]. We present an existence result, investigate stochastic homogenization and provide an application to spatial population dynamics.

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1. Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, let $T > 0$ and let $O \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary. In this paper we continue our study of nonlocal reaction-diffusion problems of gradient flow type that we developed in a previous paper [AHMM23] by considering the stochastic homogenization of nonlocal time delays reaction-diffusion problems of the form:

$$(\mathscr{P}_{\varepsilon,\omega}) \begin{cases} \dfrac{du_\varepsilon^\omega}{dt}(t) + \nabla \mathscr{E}_\varepsilon(\omega, u_\varepsilon^\omega(t)) = F(\omega, t, u_\varepsilon^\omega(t), \mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u_\varepsilon^\omega(t) = \psi_\varepsilon^\omega(t) \text{ for all } t \in]-\infty, 0], \end{cases}$$

where, for each $\varepsilon > 0$, $\psi_\varepsilon^\omega \in C_{\mathrm{c}}(]-\infty, O]; L^2(O))$ is the history function and $\mathscr{R}_\varepsilon^\omega : C_{\mathrm{c}}(]-\infty, T]; L^2(O)) \to L^2([0,T]; L^2(O))$ is the delays operator, see Definition 2.3. As in [AHMM23], the diffusion term is the gradient of a random nonlocal functional $\mathscr{E}_\varepsilon : \Omega \times L^2(O) \to [0, \infty[$ of type:

$$\mathscr{E}_\varepsilon(\omega, u) = \frac{1}{4\varepsilon^d} \int_O \int_O J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) \left(\frac{u(x) - u(y)}{\varepsilon}\right)^2 dxdy + \mathscr{D}_\varepsilon(\omega, u) \tag{1.1}$$

with $J : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty[$ and $\mathscr{D}_\varepsilon : \Omega \times L^2(O) \to \mathbb{R}$ a nonlocal functional characterizing the fact that $(\mathscr{P}_{\varepsilon,\omega})$ is of Neumann-Cauchy nonhomogenous or Dirichlet-Cauchy type. As introduced in [AHMM20], the reaction term is a random DCP-structured reaction functional $F_\varepsilon : \Omega \times [0,T] \times L^2(O) \times L^2(O) \to L^2(O)$, see Definition 2.16. Our framework allows us to consider both single-delays and multi-delays cases, as well as delays distributed by a diffuse delays kernel in nonlocal reaction-diffusion problems, see §2.1.

The problems of the form $(\mathscr{P}_{\varepsilon,\omega})$ model various situations within the context of the biological invasion of a single-species population. The reaction functional represents the regulation of population growth, involving complex maturation or resource regeneration periods. The nonlocal diffusion accounts for the interaction between individuals at short or long ranges in a heterogeneous but statistically homogeneous space environment. The parameter ε accounts for the size of heterogeneities. To our knowledge, in this general form, the existence and convergence of such problems were first studied in [AHMM20], where the diffusion operator is local and the reaction term admits a more restrictive structure.

In the present paper we extend the results of [AHMM20] to the nonlocal case. Specifically, in a nonlocal framework, we prove that any problem of the form $(\mathscr{P}_{\varepsilon,\omega})$ has a unique solution (see Theorem 2.19) and establish the almost sure homogenization of $(\mathscr{P}_{\varepsilon,\omega})$ as $\varepsilon \to 0$ (see Theorem 3.8 and Corollary 3.16) to the problem

$$(\mathscr{P}_{\mathrm{hom},\omega}) \begin{cases} \dfrac{du^\omega}{dt}(t) + \nabla \mathscr{E}_{\mathrm{hom}}(\omega, u^\omega(t)) = G^\omega(t, u^\omega(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u^\omega(t) = \psi^\omega(t) \text{ for all } t \in]-\infty, 0] \text{ and } \psi^\omega(0) \in \mathrm{dom}(\mathscr{E}_{\mathrm{hom}}(\omega, \cdot)), \end{cases} \tag{1.2}$$

where $\psi_\varepsilon^\omega \to \psi^\omega$ in $C(]-\infty,0];L^2(O))$ and

$$G(t,u^\omega(t)) = \begin{cases} F(\omega,u^\omega)(t) & \text{if } (F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega) \xrightarrow{\text{mix}} F(\omega,\cdot) \\ \\ F(\omega,t,u^\omega(t),\mathscr{R}^\omega u^\omega(t)) & \text{if } (F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega) \xrightarrow{\text{stable}} (F(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega). \end{cases}$$

The limit functional $F(\omega,\cdot): C_{\rm c}(]-\infty,T];L^2(O)) \to L^2([0,T];L^2(O))$, $F(\omega,\cdot,\cdot,\cdot):[0,T]\times L^2(O)\times L^2(O)\to L^2(O)$ and the limit operator $\mathscr{R}^\omega: C_{\rm c}(]-\infty,T];L^2(O)) \to L^2([0,T];L^2(O))$, are obtained via the notions mixing and stable convergence, denoted by " $\xrightarrow{\text{mix}}$ " and " $\xrightarrow{\text{stable}}$ " respectively. These notions are introduced in §3.3 (see Definitions 3.10 and 3.12). As established in [AHMM23, Theorem 4.8], the functional $\mathscr{E}_{\rm hom}:\Omega\times L^2(O)\to[0,\infty]$ is the almost sure Mosco-limit of $\mathscr{E}_\varepsilon$ and is given in its domain by

$$\mathscr{E}_{\rm hom}(\omega,u)=\int_O f_{\rm hom}(\omega,\nabla u(x))dx$$

with $f_{\rm hom}:\Omega\times\mathbb{R}^d\to[0,\infty[$ a quadratic function defined as the limit of a suitable subadditive process (see [AHMM23, Propositions 3.14 and 3.17]).

Nonlocal problems of type $(\mathscr{P}_{\varepsilon,\omega})$ are well adapted for spatial population dynamics where the density J in (1.1) accounts for the numbers of individuals at time t in O which jump from y to x. The nonlocal diffusion term can be explained by the fact that spatial movement is possible without constraints in the sense that individuals can disperse and interact with other individuals over relatively long distances. In Section 5 we consider such a population dynamics model with a reaction term including age structure, of the form:

$$F_\varepsilon(\omega,t,u_\varepsilon^\omega(t),\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega(t)) = b\left(\omega,t,\frac{x}{\varepsilon},\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega(t)\right) - d\left(\omega,t,\frac{x}{\varepsilon},u_\varepsilon^\omega(t)\right) + a\left(\omega,t,\frac{x}{\varepsilon}\right),$$

where $b:\Omega\times[0,\infty[\times\mathbb{R}^d\times\mathbb{R}\to]0,\infty[$ and $d:\Omega\times[0,\infty[\times\mathbb{R}^d\times\mathbb{R}\to]0,\infty[$ represent "birth" and "death" respectively, and $a:\Omega\times[0,\infty[\times\mathbb{R}^d\to\mathbb{R}$ is the immigration or harvesting rate. To incorporate the age structure, it is necessary to introduce time delays $\mathscr{R}_\varepsilon^\omega: C_{\rm c}(]-\infty,T];L^2(O))\to L^2([0,T];L^2(O))$, specifically in the "birth" component, given by

$$\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega(t) = \frac{1}{k}u_\varepsilon^\omega(t)\left(t-\sigma\left(\omega,\frac{\cdot}{\varepsilon}\right)\right)$$

with $\sigma:\Omega\times\mathbb{R}^d\to[0,\infty[$, $\sigma(\omega,\mathbb{R}^d)=\{\sigma_1^\omega,\cdots,\sigma_k^\omega\}$ where $\sigma_1^\omega,\cdots,\sigma_k^\omega\in[0,\infty[$. By applying our homogenization result, we show (see Corollary 5.8) that as $\varepsilon\to0$, the nonlocal reaction-diffusion problems $(\mathscr{P}_{\varepsilon,\omega})$ almost surely converge to a local reaction-diffusion problem of type (1.2) with $G^\omega(t,u^\omega(t))=F(\omega,u^\omega)(t)$ where

$$\begin{aligned} F(\omega,u^\omega)(t) := {} & \mathbb{E}^{\mathscr{J}}\left[\int_Y b\left(\cdot,t,y,u^\omega\left(\frac{1}{k}(t-\sigma(\cdot,y))\right)\right)dy\right](\omega) - \mathbb{E}^{\mathscr{J}}\left[\int_Y d(\cdot,t,y,u^\omega(t))dy\right](\omega) \\ & + \mathbb{E}^{\mathscr{J}}\left[\int_Y a(\cdot,t,y)dy\right](\omega). \end{aligned}$$

The symbol $\mathbb{E}^{\mathscr{J}}$ denotes the conditional mathematical expectation with respect to σ-algebra $\mathscr{J}$ of invariant sets with respect to the dynamical system $(\Omega,\mathscr{F},\mathbb{P},\{T_z\}_{z\in\mathbb{Z}^d})$ (see §3.1 for more details).

Plan of the paper. Section 2 is devoted to existence and uniqueness of bounded solutions for nonlocal reaction time delays diffusion problems of gradient flow type when the reaction term is a DCP-structured reaction functional (see Definition 2.16 and Theorem 2.19). For this, in §2.1, we begin by recalling the notion of time delays operator and then, in §2.2, we develop the nonlocal framework for dealing with Neumann-Cauchy homogeneous (see §2.2.1), nonhomogeneous (see §2.2.2) and Dirichlet-Cauchy (see 2.2.3) nonlocal time delays reaction-diffusion problems. The proof of Theorem 2.19 is given in §2.3.

Section 3 is devoted to the stochastic homogenization for nonlocal time delays problems with DCP-structured reaction functionals. In §3.1 we begin by specifying the probability setting and recall some tools from ergodic theory (see Definitions 3.1–3.2). Next, by applying Theorem 2.19, we obtain existence and uniqueness of bounded solutions for random Neumann-Cauchy homogeneous and Dirichlet-Cauchy nonlocal time delays reaction-diffusion problems (see Corollary 3.5). We state the stochastic homogenization theorem in §3.2 (see Theorem 3.8). To identify the homogenized diffusion term we need a suitable subadditive theorem that we have already proved in [AHMM23, Propositions 3.14 and 3.17]). To provide information about the specific form of the limit reaction term in Theorem 3.8, in §3.3 we introduce two notions of convergence for both the reaction functionals and the time delays operators (see Definitions 3.10) and 3.12). These notions allows us to refine Theorem 3.8 (see Corollary 3.16). Note that we do not address the convergence of Neumann-Cauchy nonhomogeneous nonlocal time delays reaction-diffusion problems. Indeed, the mathematical analysis seems technically more tricky but we hope to cover this case in the future.

Section 4 is devoted to the proof of Theorem 3.8. The proof, presented in §4.2, relies on two theorems. The first one (see Theorem 4.1) is an abstract convergence result for passing from nonlocal to local. This theorem is stated and proved in §4.1. The second theorem establishes the almost sure Mosco-convergence of the energies associated with the diffusion term. This theorem has already been proved in [AHMM23, Theorem 4.8].

Section 5 is devoted to the application of the results to a nonlocal spatial population model with age structure. In §5.1 we begin by providing a heuristic derivation of the model. Then, in §5.2, we specify the structure of the reaction term together with the time delays operator. Finally, in §5.3, we precise the mathematical description of the model, showing that it can studied within the general framework developed in Sections 2–3. By applying Corollary 3.16 we obtain the homogenized model (see Corollary 5.8).

For convenience of the reader, in the appendix we recall some classical definitions and results that we use in the paper.

Notation. Throughout the paper we will use the following notation.

- The cardinal of a set is denoted by $|\cdot|$.
- Given $x_0 \in \mathbb{R}^d$ we denote the open (resp. closed) ball of radius $r > 0$ centered at x_0 by $B_r(x_0)$ (resp. $\overline{B}_r(x_0)$).
- The closure (resp. interior) of a set $A \subset \mathbb{R}^d$ is denoted by $\overline{A}$ (resp. $\mathrm{int}(A)$).
- The Lebesgue measure on $\mathbb{R}^d$ with $d \in \mathbb{N}^*$ is denoted by $\mathscr{L}^d$ and for each Borel set $A \subset \mathbb{R}^d$, the measure of A with respect to $\mathscr{L}^d$ is denoted by $\mathscr{L}^d(A)$.

- The class of bounded Borel subsets of $\mathbb{R}^d$ is denoted by $\mathscr{B}_{\rm b}(\mathbb{R}^d)$.
- The space of continuous piecewise affine functions from O to $\mathbb{R}$ is denoted by $\mathrm{Aff}(O)$.
- Given $(a,b)\in\mathbb{R}^2$ with $a\leqslant b$, the space of $u\in L^2(O)$ (resp. $u\in L^\infty(O)$) such that $a\leqslant u\leqslant b$ is denoted by $L^2(O;[a,b])$ (resp. $L^\infty(O;[a,b])$).
- Given a Banach space X and a closed interval $I\subset\mathbb{R}$, the space of continuous functions from I to X is denoted by $C(I;X)$, and the space of absolutely continuous functions from I to X by $AC(I;X)$.
- Given $\{u_n\}_n\subset C([0,T];L^2(O))$, by $u_n\to u$ in $C([0,T];L^2(O))$ we mean that $\lim\limits_{n\to\infty}\sup_{t\in[0,T]}\|u_n(t)-u(t)\|_{L^2(O)}=0$. By $\frac{du_n}{dt}\rightharpoonup\frac{du}{dt}$ in $L^2([0,T];L^2(O))$ we mean that for every $v\in L^2([0,T];L^2(O))$, $\int_0^T\langle\frac{du_n}{dt}(t),v(t)\rangle dt\to\int_0^T\langle\frac{du}{dt}(t),v(t)\rangle dt$ as $n\to\infty$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(O)$.
- The space of step functions from $\mathbb{R}$ to $L^2(O)$ is denoted by $S^{\rm tp}(\mathbb{R};L^2(O))$.
- The space of functions from $\mathbb{R}$ to $L^2(O)$ which are uniform limit of step functions is denoted by $S^{\rm tp}_\infty(\mathbb{R};L^2(O))$.
- The space of continuous functions with compact support from $]-\infty,T]$ to $L^2(O)$ is denoted by $C_{\rm c}(]-\infty,T];L^2(O))$.
- The class of positive vector measures $\boldsymbol{m}:\mathscr{B}(\mathbb{R})\to L^\infty(O)$ such that $\|\boldsymbol{m}\|(\mathbb{R})\leqslant 1$ is denoted by $\mathscr{M}_1^+(\mathbb{R};L^\infty(O))$.

2. Existence and uniqueness of bounded solutions for nonlocal time delays problems with DCP-structured reaction functionals

From now on, $O\subset\mathbb{R}^d$ is a bounded open domain with Lipschitz boundary.

2.1. **Time delays operators associated with vector measures.**

Throughout the paper we use the theory of integration with respect to vector measures. This theory is briefly recalled in Appendix A (for more details we refer to [Din67]). We begin with the following definition.

Definition 2.1. Let $\boldsymbol{m}:\mathscr{B}(\mathbb{R})\to L^\infty(O)$ be a vector measure. We say that $\boldsymbol{m}$ is positive if for every $u\in S^{\rm tp}_\infty(\mathbb{R};L^2(O))$, $u\geqslant 0$ implies $\int_{\mathbb{R}}u(t)d\boldsymbol{m}(t)\geqslant 0$.

From now on, we denote the class of positive vector measures $\boldsymbol{m}:\mathscr{B}(\mathbb{R})\to L^\infty(O)$ such that $\|\boldsymbol{m}\|(\mathbb{R})\leqslant 1$ by $\mathscr{M}_1^+(\mathbb{R};L^\infty(O))$, where $\|\boldsymbol{m}\|$, the variation of $\boldsymbol{m}$, is defined in Appendix A, and the integral with respect to $\boldsymbol{m}$ is associated with the continuous bilinear map $B:L^\infty(O)\times L^2(O)\to L^2(O)$ defined by $B(u,v)=uv$.

Remark 2.2. For every $\boldsymbol{m}\in\mathscr{M}_1^+(\mathbb{R};L^\infty(O))$, $\boldsymbol{m}(A)\in L^\infty(O;[0,1])$.

Let $\{\boldsymbol{m}_t\}_{t\geqslant 0}\subset\mathscr{M}_1^+(\mathbb{R};L^\infty(O))$ be such that for every $u\in S^{\rm tp}_\infty(\mathbb{R};L^2(O))$, the map $t\mapsto\int_{-\infty}^t u(\tau)d\boldsymbol{m}_t(\tau)$ is $(\mathscr{B}(\mathbb{R}),\mathscr{B}(L^2(O))$-measurable.

Definition 2.3. Let $T>0$. By time delays operator associated with $\{\boldsymbol{m}_t\}_{t\geqslant 0}$ we mean the linear continuous map $\mathscr{R}:C_{\rm c}(]-\infty,T];L^2(O))\to L^2([0,T];L^2(O))$ defined by

$$\mathscr{R}u(t):=\int_{-\infty}^t u(\tau)d\boldsymbol{m}_t(\tau).$$

Remark 2.4. By using Proposition A.5, we see that for every $u \in C_c(]-\infty,T];L^2(O))$,

$$\|\mathscr{R}u(t)\|_{L^2(O)} \leqslant 2\|u\|_{C_c(]-\infty,T];L^2(O))} \text{ for all } t \in [0,T]$$

and so

$$\|\mathscr{R}u\|_{L^2([0,T];L^2(O))} \leqslant 2\sqrt{T}\|u\|_{C_c(]-\infty,T];L^2(O))}.$$

Remark 2.5. Given $\alpha \geqslant 0$, if $u \in C_c(]-\infty,T];L^2(O))$ is such that $0 \leqslant u \leqslant \alpha$ then $0 \leqslant \mathscr{R}u \leqslant \alpha$.

In what follows we give three basic examples of time delays operators associated with vector measures.

Example 2.6 (multi-delays case). Let $\{\boldsymbol{m}_t\}_{t\geqslant 0} \subset \mathscr{M}_1^+([0,T];L^\infty(O))$ be given by

$$\boldsymbol{m}_t := \sum_{k\in\mathbb{N}} \boldsymbol{d}_k \delta_{t-t_k} \text{ with } \begin{cases} \{t_k\}_{k\in\mathbb{N}} \subset]0,\infty[\\ \{\boldsymbol{d}_k\}_{k\in\mathbb{N}} \subset L^\infty(O;[0,\infty[) \\ \displaystyle\sum_{k\in\mathbb{N}} \|\boldsymbol{d}_k\|_{L^\infty(O)} = 1. \end{cases}$$

Then, $\mathscr{R} : C_c(]-\infty,T];L^2(O)) \to L^2([0,T];L^2(O))$ defined by

$$\mathscr{R}u(t) := \int_{-\infty}^t u(\tau)d\boldsymbol{m}_t(\tau) = \sum_{k\in\mathbb{N}} \boldsymbol{d}_k u(t-t_k)$$

is the time delays operator associated with $\{\boldsymbol{m}_t\}_{t\geqslant 0}$.

Example 2.7 (single delays depending on the space variable). Let $\sigma : \mathbb{R}^d \to [0,\infty[$ be such that $|\sigma(\mathbb{R}^d)| < \infty$, i.e. $\sigma(\mathbb{R}^d) = \{\sigma_1,\cdots,\sigma_k\}$ with $k \in \mathbb{N}^*$, and let $\{\boldsymbol{m}_t\}_{t\geqslant 0} \subset \mathscr{M}_1^+(\mathbb{R};L^\infty(O))$ be defined by

$$\boldsymbol{m}_t := \frac{1}{k}\delta_{t-\sigma(\cdot)}.$$

Then, $\mathscr{R} : C_c(]-\infty,T];L^2(O)) \to L^2([0,T];L^2(O))$ defined by

$$\mathscr{R}u(t) := \int_{-\infty}^t u(\tau)d\boldsymbol{m}_t(\tau) = \frac{1}{k}u(t-\sigma(\cdot))$$

is the time delays operator associated with $\{\boldsymbol{m}_t\}_{t\geqslant 0}$.

Example 2.8 (delays distributed by a diffuse delays kernel). For each $t \geqslant 0$, let $\mathcal{K}_t : \mathbb{R} \to L^\infty(O)$ be defined by

$$\mathcal{K}_t(\tau) = \mathcal{K}(t-\tau),$$

where $\mathcal{K} : \mathbb{R} \to L^\infty(O)$ is such that:

- $\int_{-\infty}^{\infty} \|\mathcal{K}(t)\|_{L^\infty(O)}\, dt = 1$;
- for every $t \in \mathbb{R}$ and $\mathscr{L}^p$-a.a. $x \in O$, $\mathcal{K}(t)(x) \geqslant 0$;
- for every $t \in]-\infty,0]$, $\mathcal{K}(t) = 0$,

and let $\{\boldsymbol{m}_t\}_{t\geqslant 0} \subset \mathscr{M}_1^+(\mathbb{R};L^\infty(O))$ be given by

$$\boldsymbol{m}_t := \mathcal{K}_t\mathscr{L}^1.$$

Then, $\mathscr{R} : C_c(]-\infty,T];L^2(O)) \to L^2([0,T];L^2(O))$ defined by

$$\mathscr{R}u(t) := \int_{-\infty}^{t} u(\tau)d\boldsymbol{m}_t(\tau) = \int_{-\infty}^{t} u(\tau)\mathcal{K}_t(t-\tau)d\tau = \int_0^{\infty} u(t-\tau)\mathcal{K}_t(\tau)d\tau = (u * \mathcal{K}_t)(t).$$

is the time delays operator associated with $\{\boldsymbol{m}_t\}_{t\geqslant 0}$.

Remark 2.9. In the three examples above, we have $\mathscr{R}u \in AC([0,T];L^2(O))$ for all $u \in C_c(]-\infty,T];L^2(O))$. Indeed, by an easy calculation we see that

(R) there exists a probability measure μ on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ with $\mu(]-\infty,0[) = 0$ such that for every $t_1,t_2 \in [0,T]$ with $t_1 < t_2$,

$$\|\mathscr{R}u(t_1) - \mathscr{R}u(t_2)\|_{L^2(O)} \leqslant \int_{t_1}^{t_2} \left(\left\|\frac{du}{dt}\right\|_{L^2(O)} * \mu\right)(t)dt$$

with $\left(\left\|\frac{du}{dt}\right\|_{L^2(O)} * \mu\right)(t) := \int_{\mathbb{R}} \left\|\frac{du}{dt}(t-\tau)\right\|_{L^2(O)} d\mu(\tau)$. For the previous examples we have:

$$\mu = \begin{cases} \displaystyle\sum_{k\in\mathbb{N}} \|\boldsymbol{d}_k\|_{L^\infty(O)}\delta_{t_k} & \text{in Example 2.6} \\ \\ \displaystyle\frac{1}{k}\sum_{i=1}^{k}\delta_{\sigma_i} & \text{in Example 2.7} \\ \\ \|K(\cdot-\tau)\|_{L^\infty(O)}d\tau & \text{in Example 2.8.} \end{cases}$$

2.2. **Nonlocal time delays reaction-diffusion problems of gradient flow type.** Given $T > 0$ and $\mathscr{R} : C_c(]-\infty,T];L^2(O)) \to L^2([0,T];L^2(O))$ a time delays operator associated with $\{\boldsymbol{m}_t\}_{t\geqslant 0} \subset \mathscr{M}_1^+(\mathbb{R};L^\infty(O))$ and $\psi \in C_c(]-\infty,0];L^2(O))$ we consider time delays reaction-diffusion problems of type:

$$(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R}}) \begin{cases} \displaystyle\frac{du}{dt}(t) + \nabla\mathscr{E}(u(t)) = F(t,u(t),\mathscr{R}u(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u(t) = \psi(t) \text{ for all } t \in]-\infty,0], \end{cases}$$

where the diffusion term is the gradient of a nonlocal, convex and Fréchet-differentiable functional $\mathscr{E} : L^2(O) \to [0,\infty[$ (see §2.2.1–2.2.3 for more details) and the reaction term $F : [0,T]\times L^2(O)\times L^2(O) \to L^2(O)$ is a DCP-structured reaction functional (see Definition 2.14).

2.2.1. *Neumann-Cauchy homogeneous nonlocal time delays problems.* Let $J : \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty[$ be a Borel measurable function satisfying the following conditions:

(NL_1) J is symmetric, i.e. for every $(x,y) \in \mathbb{R}^d\times\mathbb{R}^d$,

$$J(x,y,x-y) = J(y,x,y-x);$$

(NL_2) there exists a $(\mathscr{B}(\mathbb{R}^d),\mathscr{B}(\mathbb{R}))$-measurable function $\overline{J} : \mathbb{R}^d \to [0,\infty[$ with $\mathrm{supp}(\overline{J}) = \overline{B}_{R_J}(0)$ for some $R_J > 0$ and $\int_{\mathbb{R}^d}\overline{J}(\xi)d\xi = 1$ such that for every $(x,y,\xi) \in \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$,

$$0 \leqslant J(x,y,\xi) \leqslant \overline{J}(\xi).$$

Remark 2.10. The function $\overline{J}$ is assumed to be compactly supported for simplifying certain calculations. Without major difficulties, by using a truncation argument, we could take $\overline{J}$ growing as $\frac{1}{1+|\xi|^{d+2+\kappa}}$ with $\kappa > 0$.

Let $O \subset \mathbb{R}^d$ be a bounded open set and let $\mathscr{J} : L^2(O) \to [0, \infty[$ be defined by

$$\mathscr{J}(u) := \frac{1}{4} \int_O \int_O J(x, y, x-y)(u(x) - u(y))^2 dxdy. \tag{2.1}$$

It is easy to see that $\mathscr{J}$ is convex and Fréchet-differentiable, and by the Riesz representation theorem, for each $u \in L^2(O)$, the gradient of $\mathscr{J}$ at u, denoted by $\nabla\mathscr{J}(u)$, is such that

$$\mathscr{J}'(u)(v) = \langle \nabla\mathscr{J}(u), v\rangle = \int_O \nabla\mathscr{J}(u)(x)v(x)dx \text{ for all } v \in L^2(O),$$

where $\nabla\mathscr{J}(u) \in L^2(O)$ and is given by

$$\nabla\mathscr{J}(u)(x) = -\int_O J(x, y, x-y)(u(y) - u(x))dy.$$

The problem $(\mathscr{P}^{\psi,F,T}_{\mathscr{J},\mathscr{R}})$, which corresponds to $(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R}})$ with $\mathscr{E} = \mathscr{J}$, is a nonlocal time delays reaction-diffusion problem of gradient flow type that is called "*Neumann-Cauchy homogeneous nonlocal time delays reaction-diffusion problem*". Note that $(\mathscr{P}^{\psi,F,T}_{\mathscr{J},\mathscr{R}})$ can be rewritten as follows:

$$(\mathscr{P}^{\psi,F,T}_{\mathscr{J},\mathscr{R}}) \begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \displaystyle\int_O J(x,y,x-y)(u(t,y) - u(t,x))dy \\ = F\left(t, u(t,x), \displaystyle\int_{-\infty}^t u(\tau,\cdot)d\boldsymbol{m}_t(\tau)\right) \text{ in } [0,T]\times O \\ \\ u(t,\cdot) = \psi(t,\cdot) \text{ for all } t \in]-\infty, 0]. \end{cases}$$

Remark 2.11. The term "*Neumann-Cauchy homogeneous nonlocal time delays problem*" refers to homogeneous Neumann-Cauchy boundary conditions for local reaction-diffusion problems. Indeed, by suitably rescaling J and K, it can be established that the solutions of the rescaled corresponding problems converges to the solution of a "standard" local reaction-diffusion problem with the homogeneous Neumann boundary condition (see [AVMRTM10, Chapter 3, §3.1, pp. 41] for $J = \overline{J}$ and $F = 0$).

2.2.2. *Neumann-Cauchy nonhomogeneous nonlocal time delays problems.* Let $h \in L^1(\mathbb{R}^d\backslash O)$, let $K \in L^\infty(O \times \mathbb{R}^d)$ and let $\mathscr{N}_{h,K} : L^2(O) \to \mathbb{R}$ be defined by

$$\mathscr{N}_{h,K}(u) := \int_O \left(\int_{\mathbb{R}^d\backslash O} K(x, x-y)h(y)dy\right) u(x)dx.$$

Clearly, $\mathscr{N}_{h,K}$ is a continuous linear form and for every $u \in L^2(O)$, $\nabla\mathscr{N}_{h,K}(u) \in L^2(O)$, and is given by

$$\nabla\mathscr{N}_{h,K}(u)(x) = \int_{\mathbb{R}^d\backslash O} K(x, x-y)h(y)dy.$$

The problem $(\mathscr{P}^{\psi,F,T}_{\mathscr{J}-\mathscr{N}_{h,K},\mathscr{R}})$, which corresponds to $(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R}})$ with $\mathscr{E} = \mathscr{J} - \mathscr{N}_{h,K}$, is a nonlocal time delays reaction-diffusion problem of gradient flow type that is called "*Neumann-Cauchy*

nonhomogeneous nonlocal time delays reaction-diffusion problem". Note that $(\mathscr{P}^{\psi,F,T}_{\mathscr{J}-\mathscr{N}_{h,K},\mathscr{R}})$ can be rewritten as follows:

$$(\mathscr{P}^{\psi,F,T}_{\mathscr{J}-\mathscr{N}_{h,K},\mathscr{R}})\begin{cases}\dfrac{\partial u}{\partial t}(t,x)-\displaystyle\int_O J(x,y,x-y)(u(t,y)-u(t,x))dy\\ -\displaystyle\int_{\mathbb{R}^d\backslash O}K(x,x-y)h(y)dy=F\left(t,u(t,x),\displaystyle\int_{-\infty}^t u(\tau,\cdot)d\boldsymbol{m}_t(\tau)\right)\text{ in }[0,T]\times O\\ \\ u(t,\cdot)=\psi(t,\cdot)\text{ for all }t\in]-\infty,0].\end{cases}$$

Remark 2.12. The term "*Neumann-Cauchy nonhomogeneous nonlocal time delays problem*" refers to nonhomogeneous Neumann-Cauchy boundary conditions for local reaction-diffusion problems. Indeed, by suitably rescaling J and K, it can be established that the solutions of the rescaled corresponding problems converges to the solution of a "standard" local reaction-diffusion problem with the nonhomogeneous Neumann boundary condition $\frac{du}{d\mathbf{n}}=h$ where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$ (see [AVMRTM10, Chapter 3, §3.2, pp. 45] for $J=\overline{J}$ and $F=0$).

2.2.3. *Dirichlet-Cauchy nonlocal time delays problems.* Set $O^J:=O+\mathrm{supp}(\overline{J})=O+\overline{B}_{R_J}(0)$, let $g\in L^2(O^J\backslash\overline{O})$ and let $\mathscr{D}_g:L^2(O)\to\mathbb{R}$ be defined by

$$\mathscr{D}_g(u):=\frac{1}{2}\int_O\int_{O^J\backslash\overline{O}}J(x,y,x-y)(g(y)-u(x))^2dxdy. \tag{2.2}$$

It is easy to see that $\mathscr{D}_g$ is convex and Fréchet-differentiable, and for every $u\in L^2(O)$, $\nabla\mathscr{D}_g(u)\in L^2(O)$ and is given by

$$\nabla\mathscr{D}_g(u)(x)=-\int_{O^J\backslash\overline{O}}J(x,y,x-y)(g(y)-u(x))dy.$$

The problem $(\mathscr{P}^{\psi,F,T}_{\mathscr{J}+\mathscr{D}_g,\mathscr{R}})$, which corresponds to $(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R}})$ with $\mathscr{E}=\mathscr{J}+\mathscr{D}_g$, is a nonlocal times delays reaction-diffusion problem of gradient flow type that is called "*Dirichlet-Cauchy nonlocal time delays reaction-diffusion problem*". Note that $(\mathscr{P}^{\psi,F,T}_{\mathscr{J}+\mathscr{D}_g,\mathscr{R}})$ can be rewritten as follows:

$$(\mathscr{P}^{\psi,F,T}_{\mathscr{J}+\mathscr{D}_g,\mathscr{R}})\begin{cases}\dfrac{\partial u}{\partial t}(t,x)-\displaystyle\int_O J(x,y,x-y)(u(t,y)-u(t,x))dy\\ -\displaystyle\int_{O^J\backslash\overline{O}}J(x,y,x-y)(g(y)-u(t,x))dy\\ =F\left(t,u(t,x),\displaystyle\int_{-\infty}^t u(\tau,\cdot)d\boldsymbol{m}_t(\tau)\right)\text{ in }[0,T]\times O\\ \\ u(t,\cdot)=\psi(t,\cdot)\text{ for all }t\in]-\infty,0].\end{cases}$$

Remark 2.13. In the spirit of Remarks 2.11–2.12, the term "*Dirichlet-Cauchy nonlocal time delays problem*" refers to Dirichlet-Cauchy boundary conditions for local reaction-diffusion problems (see [AVMRTM10, Chapter 2, §2.1, pp. 31] for $J=\overline{J}$ and $F=0$).

2.3. **Existence and uniqueness for nonlocal time delays problems.** We begin by recalling the existence and uniqueness result for nonlocal problem (without time delays) that we proved in [AHMM23] (see Theorem 2.15). For this, we need the class of CP-structured reaction functionals that we denote by $\mathscr{F}_{\mathrm{CP}}^{T}$ with $T > 0$. (This class was introduced in [AHMM19], see also [AHMM22, §2.2.2, pp. 27].)

Definition 2.14. Let $T > 0$. A map $\Phi : [0,T] \times L^2(O) \to L^2(O)$ is called a CP-structured reaction functional if

$$\Phi(t,u)(x) = \phi(t,x,u(x))$$

for all $(t,u,x) \in [0,T] \times L^2(O) \times O$, where $\phi : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function satisfying the following three properties:

- (CP_1^T) $\phi(t,x,\zeta)$ is locally Lipschitz continuous in ζ uniformly with respect to $(t,x) \in [0,T] \times \mathbb{R}^d$;
- (CP_2^T) $\phi(\cdot,\cdot,0) \in L^2([0,T]; L^2(O))$;
- (CP_3^T) there exist $\underline{f}, \overline{f} : [0,T] \times \mathbb{R} \to \mathbb{R}$ with $\underline{f} \leqslant 0 \leqslant \overline{f}$ and $(\underline{\rho}, \overline{\rho}) \in \mathbb{R}^2$ with $\underline{\rho} \leqslant \overline{\rho}$ such that each of the two following ordinary differential equations

$$(\underline{\mathrm{ODE}}) \begin{cases} y'(t) = \underline{f}(t, y(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ y(0) = \underline{\rho} \end{cases}$$

$$(\overline{\mathrm{ODE}}) \begin{cases} y'(t) = \overline{f}(t, y(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ y(0) = \overline{\rho} \end{cases}$$

admits at least a solution, $\underline{y}$ for $(\underline{\mathrm{ODE}})$ and $\overline{y}$ for $(\overline{\mathrm{ODE}})$, satisfying

$$\begin{cases} \underline{f}(t, \underline{y}(t)) \leqslant \phi(t,x,\underline{y}(t)) \\ \overline{f}(t, \overline{y}(t)) \geqslant \phi(t,x,\overline{y}(t)) \end{cases}$$

for $\mathscr{L}^1 \otimes \mathscr{L}^1$-a.a. $(t,x) \in [0,T] \times \mathbb{R}$.

For each $T > 0$ and each $(a,b) \in \mathbb{R}^2$ with $a \leqslant b$, we consider the following problem:

$$(\mathscr{P}_{\mathscr{E},[a,b]}^{u_0,\Phi,T}) \begin{cases} \dfrac{du}{dt}(t) + \nabla \mathscr{E}(u(t)) = \Phi(t,u(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u(0) = u_0 \in L^2(O;[a,b]). \end{cases}$$

In [AHMM23, Corollary 2.11] we established the existence and uniqueness of bounded solutions for nonlocal reaction-diffusion equations with CP-structured reaction functionals.

Theorem 2.15. *Let $T > 0$, let $\Phi \in \mathscr{F}_{\mathrm{CP}}^T$ with $(\underline{f}, \overline{f})$, $(\underline{\rho}, \overline{\rho})$ and $(\underline{y}, \overline{y})$ given by (CP_3^T), let $u_0 \in L^2(O; [\underline{\rho}, \overline{\rho}])$ and let $g \in L^2(O^J \backslash \overline{O})$ be such that:*

$$I_g := \operatorname*{ess\,inf}_{x \in O^J \backslash \overline{O}} \frac{\int_{O^J \backslash \overline{O}} J(x, y, x-y) g(y) dy}{\int_{O^J \backslash \overline{O}} J(x, y, x-y) dy} > -\infty; \tag{2.3}$$

$$S_g := \operatorname*{ess\,sup}_{x \in O^J \backslash \overline{O}} \frac{\int_{O^J \backslash \overline{O}} J(x, y, x-y) g(y) dy}{\int_{O^J \backslash \overline{O}} J(x, y, x-y) dy} < \infty; \tag{2.4}$$

$$\underline{\rho} \leqslant I_g \text{ and } \overline{\rho} \geqslant S_g.$$

Then $(\mathscr{P}^{u_0,\Phi,T}_{\mathscr{J},[\underline{\rho},\overline{\rho}]})$ (resp. $(\mathscr{P}^{u_0,\Phi,T}_{\mathscr{J}+\mathscr{D}_g,[\underline{\rho},\overline{\rho}]})$) admits a unique solution $u \in AC([0,T]; L^2(O))$ such that

$$\underline{y}(T) \leqslant \underline{y}(t) \leqslant u(t) \leqslant \overline{y}(t) \leqslant \overline{y}(T) \text{ for all } t \in [0,T].$$

Moreover, if $\Phi(\cdot, u) \in AC([0,T]; L^2(O))$ then u admits a right derivative $\frac{d^+u}{dt}(t)$ at every $t \in [0,T[$ which satisfies $\frac{d^+u}{dt}(t) + \nabla \mathscr{E}(u(t)) = \Phi(t, u(t))$ with $\mathscr{E} = \mathscr{J}$ (resp. $\mathscr{E} = \mathscr{J} + \mathscr{D}_g$).

In order to establish existence and uniqueness of bounded solutions for nonlocal time delays reaction-diffusion problems, we need to consider the following class of reaction functionals, called the class of DCP-structured reaction functionals and denoted by $\mathscr{F}_{\mathrm{DCP}}$. (This class was introduced in [AHMM20], see also [AHMM22, §3.2.1, pp. 83].)

Definition 2.16. A map $F : [0, \infty[\times L^2(O) \times L^2(O) \to L^2(O)$ is called a DCP-structured reaction functional if

$$F(t, u, v)(x) = f(t, x, u(x), v(x))$$

for all $(t, u, v, x) \in [0, \infty[\times L^2(O) \times L^2(O) \times O$, where $f : [0, \infty[\times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function satisfying the following three properties:

- (DCP_1) for every $T > 0$, $f(t, x, \xi, \zeta)$ is locally Lipschitz continuous in ξ (resp. ζ) uniformly with respect to $(t, x, \zeta) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$ (resp. $(t, x, \xi) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$);
- (DCP_2) for every $T > 0$, $f(\cdot, \cdot, 0, 0) \in L^2([0,T]; L^2(O))$ for all $\zeta \in \mathbb{R}$;
- (DCP_3) there exist $\overline{f} : [0, \infty[\times \mathbb{R} \to [0, \infty[$ and $\overline{\rho} \in]0, \infty[$ such that the ordinary differential equation $(\overline{\mathrm{ODE}})$ admits at least a solution $\overline{y}$ such that for every $T > 0$,

$$\begin{cases} f(t, x, 0, \zeta) \geqslant 0 \\ \\ \overline{f}(t, \overline{y}(t)) \geqslant f(t, x, \overline{y}(t), \zeta) \end{cases}$$

  for $\mathscr{L}^1 \otimes \mathscr{L}^1$-a.a. $(t, x) \in [0,T] \times \mathbb{R}$ and all $\zeta \in [0, \overline{y}(T)]$.

Remark 2.17. From (DCP_3) we see that $\overline{y}$ is increasing so that $0 \leqslant \overline{\rho} \leqslant \overline{y}(0) \leqslant \overline{y}(t) \leqslant \overline{y}(T)$ for all $t \in [0,T]$ and all $T > 0$.

Remark 2.18. Let $F \in \mathscr{F}_{\mathrm{DCP}}$ with $(\overline{f}, \overline{\rho}, \overline{y})$ given by (DCP_3) and let $T > 0$. Then for every $v \in L^2([0,T]; L^2(O))$ with $v(t) \in L^2(O; [0, \overline{y}(T)])$ for all $t \in [0,T]$, we have $\Phi_v \in \mathscr{F}_{\mathrm{CP}}^T$ with $\Phi_v : [0,T] \times L^2(O) \to L^2(O)$ defined by $\Phi_v(t, u) := F(t, u, v(t))$. The function $\phi_v : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is given by $\phi_v(t, x, \xi) := f(t, x, \xi, v(t, x))$ and satisfies (CP_1^T)–(CP_2^T) and (CP_3^T) with $(\underline{f}, \underline{\rho}, \underline{y}) = (0, 0, 0)$.

For each $T>0$ and each $a\in]0,\infty[$, we consider the following problem:

$$(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R},a})\begin{cases}\dfrac{du}{dt}(t)+\nabla\mathscr{E}(u(t))=F(t,u(t),\mathscr{R}u(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t\in[0,T]\\[2ex] u(t)=\psi(t)\in L^2(O;[0,a]) \text{ for all } t\in]-\infty,0].\end{cases}$$

By using Theorem 2.15 and the Banach fixed point theorem, we can establish the following existence and uniqueness of bounded solutions for nonlocal time delays reaction-diffusion equations with DCP-structured reaction functionals.

Theorem 2.19. *Let $F\in\mathscr{F}_{\rm DCP}$ with $(\overline{f},\overline{\rho},\overline{y})$ given by* (DCP$_3$)*, let $\psi\in C_{\rm c}(]-\infty,0];L^2(O))$ be such that $\psi(t)\in L^2(O;[0,\overline{\rho}])$ for all $t\in]-\infty,0]$ and let $g\in L^2(O^J\backslash O)$ be satisfying* (2.3)–(2.4) *and such that*

$$0\leqslant I_g \text{ and } \overline{\rho}\geqslant S_g.$$

Let $T>0$ and let $\mathscr{R}:C_{\rm c}(]-\infty,T];L^2(O))\to L^2([0,T];L^2(O))$ be a time delays operator associated with $\{\boldsymbol{m}_t\}_{t\geqslant 0}\subset\mathscr{M}_1^+(\mathbb{R};L^\infty(O))$. Then $(\mathscr{P}^{\psi,F,T}_{\mathscr{J},\mathscr{R},\overline{\rho}})$ (resp. $(\mathscr{P}^{\psi,F,T}_{\mathscr{J}+\mathscr{D}_g,\mathscr{R},\overline{\rho}})$) admits a unique solution $u\in C(]-\infty,T];L^2(O))$ satisfying $u\in AC([0,T];L^2(O))$ and

$$0\leqslant u(t)\leqslant\overline{y}(t)\leqslant\overline{y}(T) \text{ for all } t\in[0,T].$$

Moreover, if $F(\cdot,u,\mathscr{R}u)\in AC([0,T];L^2(O))$ then u admits a right derivative $\frac{d^+u}{dt}(t)$ at every $t\in[0,T[$ which satisfies $\frac{d^+u}{dt}(t)+\nabla\mathscr{E}(u(t))=F(t,u(t),\mathscr{R}u(t))$ with $\mathscr{E}=\mathscr{J}$ (resp. $\mathscr{E}=\mathscr{J}+\mathscr{D}_g$).

Proof of Theorem 2.19. Let $\mathscr{E}\in\{\mathscr{J},\mathscr{J}+\mathscr{D}_g\}$. The proof is divided into three steps.

Step 1: local existence. Let $M>0$ be such that $\mathrm{supp}(\psi)\subset[-M,0]$ and, for each $T\in]0,T]$, set

$$X_T:=\Big\{u\in C(]-\infty,T];L^2(O)):0\leqslant u\leqslant\overline{y}(T) \text{ and } u|_{]-\infty,0]}=\psi\Big\}.$$

It is clear X_T is a closed subset of $C([-M,T];L^2(O))$ endowed with the uniform norm. Therefore, X_T is a complete metric space with respect to the uniform distance.

Fix any $u\in X_T$. Then, by Remark 2.5 we see that $\mathscr{R}u(t)\in L^2(O;[0,\overline{y}(T)])$ for all $t\in[0,T]$. Hence, by Remark 2.18, $\Phi_u\in\mathscr{F}^T_{\rm CP}$ with $\Phi_u:[0,T]\times L^2(O)\to L^2(O)$ defined by

$$\Phi_u(t,w):=F(t,w,\mathscr{R}u(t)) \tag{2.5}$$

and $(\underline{f},\underline{\rho},\underline{y})=(0,0,0)$. From Theorem 2.15 we deduce that there exists $\Lambda u\in C([0,T];L^2(O))$ such that

$$\begin{cases}\Lambda u\in AC([0,T];L^2(O))\\[2ex] \Lambda u \text{ is the unique solution of } (\mathscr{P}^{\psi,\Phi_u,T}_{\mathscr{E},[0,\overline{\rho}]})\\[2ex] 0\leqslant\Lambda u(t)\leqslant\overline{y}(t)\leqslant\overline{y}(T) \text{ for all } t\in[0,T]\end{cases} \tag{2.6}$$

and, moreover,

$$\Phi_u(\cdot,\Lambda u)\in AC([0,T];L^2(O))\Rightarrow\begin{cases}\Lambda u \text{ has a right derivative } \frac{d^+\Lambda u}{dt}(t) \text{ at all } t\in[0,T[\\ \\ \frac{d^+\Lambda u}{dt}(t)+\nabla\mathscr{E}(\Lambda u(t))=\Phi_u(t,\Lambda u(t)) \text{ for all } t\in[0,T[.\end{cases}\tag{2.7}$$

Extending Λu by ψ on $]-\infty,0]$ we see that $\Lambda u\in X_T$ and is a solution of the following problem:

$$(\mathscr{P}_u)\begin{cases}\dfrac{dw}{dt}(t)+\nabla\mathscr{E}(w(t))=\Phi_u(t,w(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t\in[0,T]\\ \\ w(t)=\psi(t)\in L^2(O;[0,a]) \text{ for all } t\in]-\infty,0].\end{cases}$$

Let us prove that $\Lambda: X_T\to X_T$ is a contraction for T small enough. Fix any $(u,v)\in X_T\times X_T$. Then Λu (resp. Λv) is a solution of $(\mathscr{P}_u)$ (resp. $(\mathscr{P}_v)$). Hence, for $\mathscr{L}^1$-a.e. $t\in[0,T]$,

$$\begin{aligned}&\left\langle\frac{d}{dt}\left(\Lambda u-\Lambda v\right)(t),\left(\Lambda u-\Lambda v\right)(t)\right\rangle\\ &+\left\langle\nabla\mathscr{E}(\Lambda u(t))-\nabla\mathscr{E}(\Lambda v(t)),\Lambda u(t)-\Lambda v(t)\right\rangle=\left\langle\Phi_u(t,\Lambda u(t))-\Phi_v(t,\Lambda v(t)),\Lambda u(t)-\Lambda v(t)\right\rangle.\end{aligned}$$

Since $\nabla\mathscr{E}$ is a monotone operator, it follows that for $\mathscr{L}^1$-a.e. $t\in[0,T]$,

$$\frac{1}{2}\frac{d}{dt}\|\left(\Lambda u-\Lambda v\right)(t)\|^2_{L^2(O)}\leqslant\left\langle\Phi_u(t,\Lambda u(t))-\Phi_v(t,\Lambda v(t)),\Lambda u(t)-\Lambda v(t)\right\rangle,$$

and consequently

$$\begin{aligned}\frac{d}{dt}\|\left(\Lambda u-\Lambda v\right)(t)\|^2_{L^2(O)}&\leqslant 2\|\Phi_u(t,\Lambda u(t))-\Phi_v(t,\Lambda v(t))\|_{L^2(O)}\|\Lambda u(t)-\Lambda v(t)\|_{L^2(O)}\\ &\leqslant\|\Phi_u(t,\Lambda u(t))-\Phi_v(t,\Lambda v(t))\|^2_{L^2(O)}+\|\Lambda u(t)-\Lambda v(t)\|^2_{L^2(O)}.\end{aligned}\tag{2.8}$$

On the other, taking (2.5) and Remark 2.4 together with the fact that $u|_{]-\infty,0]}=v|_{]-\infty,0]}$ into account, from (DCP_1) we can assert that for $\mathscr{L}^1$-a.e. $t\in[0,T]$,

$$\|\Phi_u(t,\Lambda u(t))-\Phi_v(t,\Lambda v(t))\|^2_{L^2(O)}\leqslant L(T)\left(4T\|u(t)-v(t)\|^2_{L^2(O)}+\|\Lambda u(t)-\Lambda v(t)\|^2_{L^2(O)}\right)\tag{2.9}$$

with $L(T)\in[0,\infty[$ given by $L(T):=2\max(L_1(T),L_2(T))$ where

$$\begin{cases}L_1(T):=\sup\limits_{t\in[0,T]}\sup\limits_{x\in\mathbb{R}^d}\sup\limits_{\zeta\in\mathbb{R}}\sup\limits_{\xi\neq\xi'}\dfrac{|f(t,x,\xi,\zeta)-f(t,x,\xi',\zeta)|}{|\xi-\xi'|}\\ \\ L_2(T):=\sup\limits_{t\in[0,T]}\sup\limits_{x\in\mathbb{R}^d}\sup\limits_{\xi\in\mathbb{R}}\sup\limits_{\zeta\neq\zeta'}\dfrac{|f(t,x,\xi,\zeta)-f(t,x,\xi,\zeta')|}{|\zeta-\zeta'|}.\end{cases}\tag{2.10}$$

Combining (2.8) with (2.9) we deduce that for $\mathscr{L}^1$-a.e. $t\in[0,T]$,

$$\frac{d}{dt}\|\left(\Lambda u-\Lambda v\right)(t)\|^2_{L^2(O)}\leqslant 4TL(T)\|u(t)-v(t)\|^2_{L^2(O)}+(1+L(T))\|\Lambda u(t)-\Lambda v(t)\|^2_{L^2(O)}.\tag{2.11}$$

Noticing that $\Lambda u(0) = \Lambda v(0) = \psi(0)$, by integrating over $[0, s]$ with $s \in [0, T]$ it follows that

$$\begin{aligned}\|\Lambda u(s) - \Lambda v(s)\|^2_{L^2(O)} &\leqslant 4T^2 L(T)\|u - v\|^2_{C(]-\infty,T];L^2(O))} \\ &\quad + \int_0^s (1 + L(T))\|\Lambda u(t) - \Lambda v(t)\|^2_{L^2(O)} dt \text{ for all } s \in [0, T].\end{aligned}$$

From Grönwall's lemma (see Lemma D.1 that we apply with $\phi(s) = \|\Lambda u(s) - \Lambda v(s)\|^2_{L^2(O)}$, $a = 4T^2 L(T)\|u - v\|^2_{C(]-\infty,T];L^2(O))}$ and $\mathrm{m}(t) = 1 + L(T)$) and the fact that $\Lambda u(s) = \Lambda v(s) = \psi(s)$ for all $s \in]-\infty, 0]$, we see that

$$\|\Lambda u - \Lambda v\|^2_{C(]-\infty,T];L^2(O))} \leqslant 4T^2 L(T)\mathrm{e}^{T(1+L(T))}\|u - v\|^2_{C(]-\infty,T];L^2(O))}.$$

Thus, for every $u, v \in X_T$,

$$\|\Lambda u - \Lambda v\|_{C(]-\infty,T];L^2(O))} \leqslant K(T)\|u - v\|_{C(]-\infty,T];L^2(O))}$$

with $K(T) := 2T\sqrt{L(T)}\mathrm{e}^{\frac{T}{2}(1+L(T))}$. From (2.10) it easily seen that $T \mapsto L(T)$ is decreasing, hence $\lim_{T\to 0} K(T) = 0$ and so $K(T) < 1$ for $T > 0$ small enough. Then, for a such T, $\Lambda : X_T \to X_T$ is a contraction.

By applying Banach's fixed point theorem, we conclude that Λ admits a unique fixed point $u \in X_T$. From (2.6) it is clear that u is a solution of $(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R},\overline{\rho}})$, $u \in AC([0, T]; L^2(O))$ and $0 \leqslant u(t) \leqslant \overline{y}(t) \leqslant \overline{y}(T)$ for all $t \in [0, T]$. By (2.7), if moreover $F(\cdot, u, \mathscr{R}u) \in AC([0, T]; L^2(O))$ then u admits a right derivative $\frac{d^+u}{dt}(t)$ at every $t \in [0, T[$ which satisfies $\frac{d^+u}{dt}(t) + \nabla\mathscr{E}(u(t)) = F(t, u(t), \mathscr{R}u(t))$.

Step 2: global existence. First of all, by [ABM14, Theorem 17.2.5, pp. 701] we can assert that

(I) for every $T > 0$, if $u \in C(]-\infty, T]; L^2(O))$ is a solution of $(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R},\overline{\rho}})$ then for every $\delta > 0$, $\int_\delta^T \|\frac{du}{dt}(t)\|^2_{L^2(O)} dt \leqslant \frac{C_T}{\delta}$ with $C_T := \int_0^T \|\sqrt{t}\frac{du}{dt}(t)\|_{L^2(O)} dt < \infty$.

Fix $\delta \in]0, T^*[$ where $T^* > 0$ is given by the step 1 and define $E \subset \mathbb{R}$ by

$$E := \Big\{T > \delta : \exists u \in X_T \text{ such that } u \text{ is a solution of } (\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R},\overline{\rho}})\Big\}.$$

Then $E \neq \varnothing$. (Moreover, it is clear that E is an interval of $\mathbb{R}$.) Set $T_{\max} := \sup E \in [\delta, \infty]$. We are going to prove that $T_{\max} = \infty$. For this we argue by contradiction by assuming that $T_{\max} < \infty$.

Step 2-1: proving that $T_{\max} \in E$. Let $\{T_n\}_{n\geqslant 1} \subset E$ be such that $T_n \to T_{\max}$ and let $u \in C(]-\infty, T_{\max}[; L^2(O))$ be such that for every $n \geqslant 1$, $u|_{]-\infty,T_n]} \in X_{T_n}$ and is a solution of $(\mathscr{P}^{\psi,F,T_n}_{\mathscr{E},\mathscr{R},\overline{\rho}})$. Fix any $n \geqslant 1$. Then

$$\left\langle \frac{du}{dt}(t), \frac{du}{dt}(t) \right\rangle + \left\langle \nabla\mathscr{E}(u(t)), \frac{du}{dt}(t) \right\rangle = \left\langle F(t, u(t), \mathscr{R}u(t)), \frac{du}{dt}(t) \right\rangle \ \mathscr{L}^1\text{-a-e. } t \in [0, T_n].$$

But $\frac{d}{dt}\mathscr{E}(u(t)) = \langle \nabla\mathscr{E}(u(t)), \frac{du}{dt}(t)\rangle$, hence

$$\int_\delta^{T_n} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt + \mathscr{E}(u(T_n)) - \mathscr{E}(u(\delta))$$
$$\leqslant \left(\int_0^{T_n} \|F(t,u(t),\mathscr{R}u(t))\|^2_{L^2(O)} dt\right)^{\frac{1}{2}} \left(\int_\delta^{T_n} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt\right)^{\frac{1}{2}}. \tag{2.12}$$

On the other hand, by using (DCP_1) we have

$$\int_0^{T_n} \|F(t,u(t),\mathscr{R}u(t))\|^2_{L^2(O)} dt \leqslant 2L_1^2(T_{\max}) \int_0^{T_n} \|u(t)\|^2_{L^2(O)} dt + 4L_2^2(T_{\max}) \int_0^{T_n} \|\mathscr{R}u(t)\|^2_{L^2(O)} dt$$
$$+4\|f(\cdot,\cdot,0,0)\|^2_{L^2([0,T_{\max}];L^2(O))},$$

where $L_1(T_{\max}) \in [0,\infty[$ and $L_2(T_{\max}) \in [0,\infty[$ are given by (2.10) with $T = T_{\max}$ and $\|f(\cdot,\cdot,0,0)\|^2_{L^2([0,T_{\max}];L^2(O))} < \infty$ by (DCP_2). As $u|_{]-\infty,T_n]} \in X_{T_n}$ we have $0 \leqslant u \leqslant \overline{y}(T_n)$ and $u|_{]-\infty,T_n]} \in C_{\mathrm{c}}(]-\infty,T_n];L^2(O))$, hence $0 \leqslant \mathscr{R}u(t) \leqslant \overline{y}(T_n)$ (see Remark 2.5), and consequently, since $\overline{y}(T_n) \leqslant \overline{y}(T_{\max})$ (see Remark 2.17),

$$\int_0^{T_n} \|F(t,u(t),\mathscr{R}u(t))\|^2_{L^2(O)} dt \leqslant C_1 \tag{2.13}$$

with $C_1 \in]0,\infty[$ given by

$$C_1 := 2T_{\max}\mathscr{L}^d(O)(\overline{y}(T_{\max}))^2(L_1^2(T_{\max}) + 2L_2^2(T_{\max})) + 4\|f(\cdot,\cdot,0,0)\|^2_{L^2([0,T_{\max}];L^2(O))}.$$

Set $C_2 := \mathscr{E}(u(\delta)) - \inf\mathscr{E}$. (Note that $C_2 \in [0,\infty[$ because $\inf\mathscr{E} > -\infty$.) From (2.12) and (2.14) we deduce that

$$\int_\delta^{T_n} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt \leqslant C_2 + C_1 \left(\int_\delta^{T_n} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt\right)^{\frac{1}{2}}.$$

Moreover, as $u|_{]-\infty,T_n]}$ is a solution of $(\mathscr{P}^{\psi,F,T_n}_{\mathscr{E},\mathscr{R},\overline{\rho}})$ we have $\int_\delta^{T_n} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt < \infty$ by (I). Hence

$$\left(\int_\delta^{T_n} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt\right)^{\frac{1}{2}} \left[\left(\int_\delta^{T_n} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt\right)^{\frac{1}{2}} - C_1\right] \leqslant C_2 \text{ for all } n \geqslant 1. \tag{2.14}$$

Noticing that $(\int_\delta^{T_{\max}} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt)^{\frac{1}{2}} = \lim_{n\to\infty}(\int_\delta^{T_n} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt)^{\frac{1}{2}}$ by the monotone convergence theorem, from (2.14) it follows that

$$L := \left(\int_\delta^{T_{\max}} \left\|\frac{du}{dt}(t)\right\|^2_{L^2(O)} dt\right)^{\frac{1}{2}} < \infty.$$

Then, for every $t_1, t_2 \in [\delta, T_{\max}[$,

$$\|u(t_1) - u(t_2)\|_{L^2(O)} \leqslant \int_{t_1}^{t_2} \left\|\frac{du}{dt}(t)\right\|_{L^2(O)} dt \leqslant L|t_1 - t_2|^{\frac{1}{2}},$$

which means that u is $\frac{1}{2}$-Hölder continuous on $[\delta, T_{\max}[$. According to the continuous extension principle, u has a unique continuous extension $\widetilde{u}$ to $]-\infty, T_{\max}]$ with $\widetilde{u}(T_{\max}) = \lim_{t\to T_{\max}} u(t) \in L^2(O)$. Thus, $\widetilde{u} \in X_{T_{\max}}$ and is a solution of $(\mathscr{P}^{\psi,F,T_{\max}}_{\mathscr{E},\mathscr{R},\overline{\rho}})$, which shows that $T_{\max} \in E$.

Step 2-2: contradiction. Let $u_{\max} \in C(]-\infty, T_{\max}]; L^2(O))$ be a solution of $(\mathscr{P}^{\psi,F,T_{\max}}_{\mathscr{E},\mathscr{R},\overline{\rho}})$ and let $\psi_{\max} :]-\infty, 0] \to L^2(O)$ be defined by

$$\psi_{\max}(t) := u_{\max}(t + T_{\max}).$$

As $u_{\max}(t) = \psi(t)$ for all $t \in]-\infty, 0]$ and $\psi \in C_c(]-\infty, 0]; L^2(O)) \cap AC(]-\infty, 0]; L^2(O))$ we have $\psi_{\max} \in C_c(]-\infty, 0]; L^2(O)) \cap AC(]-\infty, 0]; L^2(O))$. Hence, by using the step 1, there exists $\overline{T} > 0$ such that $(\mathscr{P}^{\psi_{\max},F,\overline{T}}_{\mathscr{E},\mathscr{R},\overline{\rho}})$ admits a solution $\overline{u} \in C(]-\infty, \overline{T}]; L^2(O))$. Setting

$$\overline{u}_{\max}(t) := \begin{cases} u_{\max}(t) & \text{if } t \in]-\infty, T_{\max}] \\ \\ \overline{u}(t - T_{\max}) & \text{if } t \in [T_{\max}, T_{\max} + \overline{T}] \end{cases}$$

and noticing that $\overline{u}(0) = \psi_{\max}(0) = u_{\max}(T_{\max})$ we see that $\overline{u}_{\max} \in C(]-\infty, T_{\max} + \overline{T}]; L^2(O))$ and is a solution of $(\mathscr{P}^{\psi,F,T_{\max}+\overline{T}}_{\mathscr{E},\mathscr{R},\overline{\rho}})$. This contradicts the maximality of $T_{\max}$. It follows that $T_{\max} = \infty$.

Step 3: uniqueness. Let $u, v \in C(]-\infty, T]; L^2(O))$ be two solutions of $(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R},\overline{\rho}})$. Then, by letting $\Lambda u = u$ and $\Lambda v = v$ in (2.11) we see that for $\mathscr{L}^1$-a.e. $t \in [0, T]$,

$$\frac{d}{dt}\|u(t) - v(t)\|^2_{L^2(O)} \leqslant C(T)\|u(t) - v(t)\|^2_{L^2(O)}$$

with $C(T) := 1 + L(T)(4T + 1)$. Noticing that $u(0) = v(0) = \psi(0)$, by integrating over $[0, s]$ with $s \in [0, T]$ it follows that

$$\|u(s) - v(s)\|^2_{L^2(O)} \leqslant C(T) \int_0^s \|u(t) - v(t)\|^2_{L^2(O)} dt \text{ for all } s \in [0, T].$$

From Grönwall's lemma (see Lemma D.1 that we apply with $\phi(s) = \|u(s) - v(s)\|^2_{L^2(O)}$, $a = 0$ and $\mathrm{m}(t) = C(T)$) and the fact that $u(s) = v(s) = \psi(s)$ for all $s \in]-\infty, 0]$, we deduce that $\|u - v\|_{C(]-\infty,T];L^2(O))} = 0$, i.e. $u = v$.

This completes the proof of Theorem 2.19. ■

To be satisfied, the regularity hypothesis on $F(\cdot, u, \mathscr{R}u)$ in Theorem 2.19, i.e. $F(\cdot, u, \mathscr{R}u) \in AC([0, T]; L^2(O))$, requires some conditions on f, $\mathscr{R}$ and ψ, as specified in the following proposition (for a proof we refer to [AHMM20, Theorem 3.1], see also [AHMM22, Theorem 3.1, pp. 88]).

Proposition 2.20. *Let $F \in \mathscr{F}_{\mathrm{DCP}}$ be such that*

$$f(t, x, \xi, \zeta) = \langle r(t, x) \odot h(\zeta), g(\xi)\rangle + q(t, x),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^m$ and $r : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^m$, $g, h : \mathbb{R} \to \mathbb{R}^m$ and $q : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, with $m \in \mathbb{N}^*$, satisfy the following conditions:

- *g and h are locally Lipschitz continuous*;
- *for every $T > 0$, $r \in L^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}^m) \cap AC([0, T]; L^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m))$;*

- *for every* $T > 0$, $q \in AC([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^d))$.

Let $T > 0$, *let* $\mathscr{E} \in \{\mathscr{J}, \mathscr{J} + \mathscr{D}_g\}$ *and let* u *be the solution of* $(\mathscr{P}^{\psi,F,T}_{\mathscr{E},\mathscr{R},\overline{\rho}})$. *If* (R) *holds and if* $\psi \in AC(]-\infty,0]; L^2(O))$ *then* $F(\cdot, u, \mathscr{R}u) \in AC([0,T]; L^2(O))$.

3. Stochastic homogenization for nonlocal time delays problems with DCP-structured reaction functionals

3.1. **Random nonlocal time delays reaction-diffusion problems of gradient flow type.** From now on we consider a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a family $\{T_z\}_{z\in\mathbb{Z}^d}$ satisfying the following three properties:

- (mesurability) $T_z : \Omega \to \Omega$ is $\mathscr{F}$-measurable for all $z \in \mathbb{Z}^d$;
- (group property) $T_z \circ T_{z'} = T_{z+z'}$ and $T_{-z} = T_z^{-1}$ for all $z, z' \in \mathbb{Z}^d$;
- (mass invariance) $\mathbb{P}(T_z A) = \mathbb{P}(A)$ for all $A \in \mathscr{F}$ and all $z \in \mathbb{Z}^d$.

Definition 3.1. The family $\{T_z\}_{z\in\mathbb{Z}^d}$ is said to be a (discrete) group of $\mathbb{P}$-preserving transformation on $(\Omega, \mathscr{F}, \mathbb{P})$ and the quadruplet $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is called a (discrete) dynamical system.

Let $\mathscr{I} := \{A \in \mathscr{F} : \mathbb{P}(T_z A \Delta A) = 0 \text{ for all } z \in \mathbb{Z}^d\}$ be the σ-algebra of invariant sets with respect to $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$.

Definition 3.2. When $\mathbb{P}(A) \in \{0,1\}$ for all $A \in \mathscr{I}$, the measurable dynamical system $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is said to be ergodic.

Remark 3.3. A sufficient condition to ensure the ergodicity of $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is the so-called mixing condition, i.e. for every $(E, F) \in \mathscr{F} \times \mathscr{F}$,

$$\lim_{|z|\to\infty} \mathbb{P}(T_z E \cap F) = \mathbb{P}(E)\mathbb{P}(F).$$

For each $X \in L^1_{\mathbb{P}}(\Omega)$, $\mathbb{E}^{\mathscr{I}}(X)$ denotes the conditional mathematical expectation of X with respect to $\mathscr{I}$, i.e. the unique $(\mathscr{I}, \mathscr{B}(\overline{\mathbb{R}}))$-measurable function in $L^1_{\mathbb{P}}(\Omega)$ such that for every $E \in \mathscr{I}$,

$$\int_E \mathbb{E}^{\mathscr{I}}(X)(\omega) d\mathbb{P}(\omega) = \int_E X(\omega) d\mathbb{P}(\omega).$$

Remark 3.4. If $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is ergodic then $\mathbb{E}^{\mathscr{I}}(X)$ is constant and equal to the mathematical expectation $\mathbb{E}(X)$ of X, i.e. $\mathbb{E}^{\mathscr{I}}(X) = \mathbb{E}(X) := \int_\Omega X(\omega) d\mathbb{P}(\omega)$.

Let $J : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty[$ be a $(\mathscr{F} \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$-measurable satisfying the following conditions:

(PNL_1) J is symmetric, i.e. for every (ω, x, y, ξ),

$$J(\omega, x, y, \xi) = J(\omega, y, x, \xi),$$

and J is bi-stationary with respect to $(T_z)_{z\in\mathbb{Z}^d}$, i.e. for every $z \in \mathbb{Z}^d$ and every $(\omega, x, y, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$J(\omega, x + z, y + z, \xi) = J(T_z\omega, x, y, \xi);$$

(PNL_2) there exist $\underline{J}, \overline{J} : \mathbb{R}^d \to [0, \infty[$ with

$$\begin{cases} \underline{J} \not\equiv 0 \\ \text{for every } (\xi, \zeta) \in \mathbb{R}^d \times \mathbb{R}^d, \text{ if } |\xi| \leqslant |\zeta| \text{ then } \underline{J}(\xi) \geqslant \underline{J}(\zeta) \\ \mathrm{supp}(\overline{J}) = \overline{B}_{R_J}(0) \text{ is compact with } R_J > 0, \end{cases} \tag{3.1}$$

such that for every $(\omega, x, y, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\underline{J}(\xi) \leqslant J(\omega, x, y, \xi) \leqslant \overline{J}(\xi).$$

Fix any $\varepsilon > 0$. Let $O \subset \mathbb{R}^d$ be an open set and define $\mathscr{J}_\varepsilon : \Omega \times L^2(O) \to [0, \infty[$ by

$$\mathscr{J}_\varepsilon(\omega, u) := \frac{1}{4\varepsilon^d} \int_O \int_O J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) \left(\frac{u(x) - u(y)}{\varepsilon}\right)^2 dxdy. \tag{3.2}$$

Let $F_\varepsilon : \Omega \times [0, \infty[\times L^2(O) \times L^2(O) \to L^2(O)$ be such that $F_\varepsilon(\omega, \cdot, \cdot, \cdot) \in \mathscr{F}_{\mathrm{DCP}}$ for all $\omega \in \Omega$. For each $\omega \in \Omega$, let $(\overline{\rho}_\varepsilon^\omega, \overline{y}_\varepsilon^\omega)$ be given by (DCP_3) with $F = F_\varepsilon(\omega, \cdot, \cdot, \cdot)$ and let $\psi_\varepsilon^\omega \in C_{\mathrm{c}}(]-\infty, 0]; L^2(O))$. Let $T > 0$ and $\mathscr{R}_\varepsilon^\omega : C_{\mathrm{c}}(]-\infty, T]; L^2(O)) \to L^2([0, T]; L^2(O))$ be a time delays operator associated with $\{\boldsymbol{m}_{\varepsilon,t}^\omega\}_{t \geqslant 0} \subset \mathscr{M}_1^+(\mathbb{R}; L^\infty(O))$[1]. Taking Remark 2.17 into account, we assume that

$$0 \leqslant \sup_{\varepsilon > 0} \overline{y}_\varepsilon^\omega(T) < \infty, \tag{3.3}$$

and consider the Neumann-Cauchy homogeneous nonlocal problem $(\mathscr{P}_{\varepsilon,\omega}^{\mathrm{NH}}) := \left(\mathscr{P}_{\mathscr{J}_\varepsilon(\omega,\cdot),\mathscr{R}_\varepsilon^\omega,\overline{\rho}_\varepsilon^\omega}^{\psi_\varepsilon^\omega, F_\varepsilon(\omega,\cdot,\cdot,\cdot), T}\right)$, i.e.

$$(\mathscr{P}_{\varepsilon,\omega}^{\mathrm{NH}}) \begin{cases} \dfrac{du_\varepsilon^\omega}{dt}(t) + \nabla \mathscr{J}_\varepsilon(\omega, u_\varepsilon^\omega(t)) = F_\varepsilon(\omega, t, u_\varepsilon^\omega(t), \mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ \\ u_\varepsilon^\omega(t) = \psi_\varepsilon^\omega(t) \in L^2(O; [0, \overline{\rho}_\varepsilon^\omega]) \text{ for all } t \in]-\infty, 0]. \end{cases}$$

Let $g \in H^1(O^J \backslash \overline{O})$ with $O^J := O + \mathrm{supp}(\overline{J}) = O + \overline{B}_{R_J}(0)$ be such that:

$$\begin{aligned} I_g^{\varepsilon,\omega} &:= \operatorname*{ess\,inf}_{x \in O^J \backslash \overline{O}} \frac{\int_{O^J \backslash \overline{O}} J(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}) g(y) dy}{\int_{O^J \backslash \overline{O}} J(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}) dy} > -\infty; \\ S_g^{\varepsilon,\omega} &:= \operatorname*{ess\,sup}_{x \in O^J \backslash \overline{O}} \frac{\int_{O^J \backslash \overline{O}} J(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}) g(y) dy}{\int_{O^J \backslash \overline{O}} J(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}) dy} < \infty; \\ 0 &\leqslant I_g^{\varepsilon,\omega} \text{ and } \overline{\rho}_\varepsilon^\omega \geqslant S_g^{\varepsilon,\omega}. \end{aligned} \tag{3.4}$$

for all $x \in O$ and and all $\omega \in \Omega$. Define $\mathscr{D}_g^\varepsilon : \Omega \times L^2(O) \to [0, \infty[$ by

$$\mathscr{D}_g^\varepsilon(\omega, u) := \frac{1}{2\varepsilon^d} \int_O \int_{O^J \backslash \overline{O}} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) \left(\frac{g(y) - u(x)}{\varepsilon}\right)^2 dxdy \tag{3.5}$$

[1]For each ε and t, we assume that $\boldsymbol{m}_{\varepsilon,t}^\omega$ is measurable with respect to ω in the following sense: for all $B \in \mathscr{B}(\mathbb{R})$, the map $\omega \mapsto \mathbf{m}_{\varepsilon,t}^\omega(B)$ is $(\mathscr{F}, \mathscr{B}(L^\infty(O)))$-measurable.

and consider the Dirichlet-Cauchy nonlocal problem $(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega}) := \left(\mathscr{P}^{\psi^\omega_\varepsilon, F_\varepsilon(\omega,\cdot,\cdot,\cdot),T}_{\mathscr{J}_\varepsilon(\omega,\cdot)+\mathscr{D}^\varepsilon_g(\omega,\cdot),\mathscr{R}^\omega_\varepsilon,\overline{\rho}^\omega_\varepsilon}\right)$, i.e.

$$(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega}) \begin{cases} \dfrac{du^\omega_\varepsilon}{dt}(t) + \nabla \mathscr{J}^g_\varepsilon(\omega, u^\omega_\varepsilon(t)) = F_\varepsilon(\omega, t, u^\omega_\varepsilon(t), \mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u^\omega_\varepsilon(t) = \psi^\omega_\varepsilon(t) \in L^2(O;[0,\overline{\rho}^\omega_\varepsilon]) \text{ for all } t \in]-\infty, 0] \end{cases}$$

with $\mathscr{J}^g_\varepsilon := \mathscr{J}_\varepsilon + \mathscr{D}^\varepsilon_g$. The following result is a straightforward consequence of Theorem 2.19.

Corollary 3.5. *For each $\omega \in \Omega$ and each $\varepsilon > 0$, $(\mathscr{P}^{\mathrm{NH}}_{\varepsilon,\omega})$ (resp. $(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega})$) admits a unique solution $u^\omega_\varepsilon \in AC([0,T]; L^2(O))$ satisfying*

$$0 \leqslant u^\omega_\varepsilon(t) \leqslant \overline{y}^\omega_\varepsilon(t) \leqslant \overline{y}^\omega_\varepsilon(T)$$

for all $t \in [0,T]$. Moreover, if $F_\varepsilon(\omega,\cdot,u^\omega_\varepsilon, \mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon) \in AC([0,T]; L^2(O))^2$ then u^ω_ε admits a right derivative $\frac{d^+u^\omega_\varepsilon}{dt}(t)$ at every $t \in [0,T[$ which satisfies $\frac{d^+u^\omega_\varepsilon}{dt}(t) + \nabla \mathscr{J}_\varepsilon(\omega, u^\omega_\varepsilon(t)) = F_\varepsilon(\omega, t, u^\omega_\varepsilon(t), \mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon(t))$ (resp. $\frac{d^+u^\omega_\varepsilon}{dt}(t) + \nabla \mathscr{J}^g_\varepsilon(\omega, u^\omega_\varepsilon(t)) = F_\varepsilon(\omega, t, u^\omega_\varepsilon(t), \mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon(t))$).

Our purpose is to look for the almost sure limit of $(\mathscr{P}^{\mathrm{NH}}_{\varepsilon,\omega})$ and $(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega})$ as $\varepsilon \to 0$. This is the object of the next section.

3.2. **Stochastic homogenization theorem.** For each $\theta \in \mathbb{R}^d$, each $R > 0$ and each $A \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$, set

$$L^2_{\mathrm{loc},\theta,R,A}(\mathbb{R}^d) := \Big\{ u \in L^2_{\mathrm{loc}}(\mathbb{R}^d) : u = \ell_\theta \text{ in } \partial_R(A) \Big\}, \tag{3.6}$$

where $\ell_\theta : \mathbb{R}^d \to \mathbb{R}$ is the linear map defined by $\ell_\theta(x) = \theta x$ and $\partial_R(A)$ denotes the R-neighborhood of the boundary ∂A of A, i.e.

$$\partial_R(A) := \Big\{ x \in \mathbb{R}^d : \mathrm{dist}(x, \partial A) < R \Big\}. \tag{3.7}$$

Let $\mathscr{S} : \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d) \times \Omega \times \mathbb{R}^d \to [0,\infty[$ be defined by

$$\mathscr{S}_A(\omega,\theta) := \inf \left\{ \mathscr{J}(\omega, u, \mathbb{R}^d, A) : u \in L^2_{\mathrm{loc},\theta,R_J,A}(\mathbb{R}^d) \right\},$$

where $R_J > 0$ is given by (PNL_2) and $\mathscr{J} : \Omega \times L^2_{\mathrm{loc}}(\mathbb{R}^d) \times \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d) \times \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d) \to [0,\infty[$ is defined by

$$\mathscr{J}(\omega, u, A, B) := \frac{1}{4} \int_A \int_B J(\omega, x, y, x-y)(u(x) - u(y))^2 dxdy.$$

Let $f_{\mathrm{hom}} : \Omega \times \mathbb{R}^d \to [0,\infty[$ be defined by

$$f_{\mathrm{hom}}(\omega,\theta) := \inf_{k \in \mathbb{N}^*} \mathbb{E}^{\mathscr{J}} \left(\frac{\mathscr{S}_{[0,k[^d}(\cdot,\theta)}{k^d} \right)(\omega).$$

Remark 3.6. It is easy to see that $f_{\mathrm{hom}}(\omega,\cdot)$ is quadratic, i.e. there exists a symmetric $d \times d$ matrix A^ω_{hom} such that for every $\theta \in \mathbb{R}^d$,

$$f_{\mathrm{hom}}(\omega,\theta) = \frac{1}{2} \langle A^\omega_{\mathrm{hom}} \theta, \theta \rangle, \tag{3.8}$$

where $\langle \cdot,\cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$ (see [AHMM23, Propositions 3.14 and 3.17] for more details on the definition of f_{hom}).

[2]See Proposition 2.20.

Let $\mathscr{J}_{\rm hom}, \mathscr{J}^g_{\rm hom} : \Omega \times L^2(O) \to [0,\infty]$ be defined by:

$$\mathscr{J}_{\rm hom}(\omega,u) := \begin{cases} \displaystyle\int_O f_{\rm hom}(\omega,\nabla u(x))dx & \text{if } u \in H^1(O) \\ \infty & \text{if } u \in L^2(O)\backslash H^1(O); \end{cases} \tag{3.9}$$

$$\mathscr{J}^g_{\rm hom}(\omega,u) := \begin{cases} \displaystyle\int_O f_{\rm hom}(\omega,\nabla u(x))dx & \text{if } u \in H^1_g(O) \\ \infty & \text{if } L^2(O)\backslash H^1_g(O) \end{cases} \tag{3.10}$$

with $H^1_g(O) := \{u \in H^1(O) : \gamma(u) = \gamma_J(g)\}$, where γ (resp. γ_J) is the trace operator $\gamma : H^1(O) \to L^2(\partial O)$ (resp. $\gamma_J : H^1(O^J\backslash\overline{O}) \to L^2(\partial O)$).

Remark 3.7. By Remark 3.6, $\mathscr{J}_{\rm hom}(\omega,\cdot)$ (resp. $\mathscr{J}^g_{\rm hom}(\omega,\cdot)$) is proper, convex and lower semi-continuous, and Fréchet-differentiable on $\mathrm{dom}(\partial\mathscr{J}_{\rm hom}(\omega,\cdot))$ (resp. $\mathrm{dom}(\partial\mathscr{J}^g_{\rm hom}(\omega,\cdot))$).

For $\mathbb{P}$-a.e. $\omega \in \Omega$, let $G^\omega : [0,T] \times L^2(O) \to L^2(O)$ be such that $G^\omega(\cdot,v) \in L^2([0,T];L^2(O))$ for all $v \in L^2(O)$, let $\psi^\omega \in C_{\rm c}(]-\infty,0];L^2(O))$ and consider the following Neumann-Cauchy homogeneous local problem:

$$(\mathscr{P}^{\rm NH}_{{\rm hom},\omega}) \begin{cases} \dfrac{du^\omega}{dt}(t) + \nabla\mathscr{J}_{\rm hom}(\omega,u^\omega(t)) = G^\omega(t,u^\omega(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u^\omega(t) = \psi^\omega(t) \text{ for all } t \in]-\infty,0] \text{ and } \psi^\omega(0) \in \mathrm{dom}(\mathscr{J}_{\rm hom}(\omega,\cdot)) \end{cases}$$

and the following Dirichlet-Cauchy local problem:

$$(\mathscr{P}^{\rm D}_{{\rm hom},\omega}) \begin{cases} \dfrac{du^\omega}{dt}(t) + \nabla\mathscr{J}^g_{\rm hom}(\omega,u^\omega(t)) = G^\omega(t,u^\omega(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u^\omega(t) = \psi^\omega(t) \text{ for all } t \in]-\infty,0] \text{ and } \psi^\omega(0) \in \mathrm{dom}(\mathscr{J}^g_{\rm hom}(\omega,\cdot)). \end{cases}$$

Here is the main result of the paper.

Theorem 3.8. *For $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $\varepsilon > 0$, let u^ω_ε be the unique solution of $(\mathscr{P}^{\rm NH}_{\varepsilon,\omega})$ (resp. $(\mathscr{P}^{\rm D}_{\varepsilon,\omega})$), see Corollary* 3.5*, and assume that*:

(H^ω_1) $\sup_{\varepsilon>0} \mathscr{J}_\varepsilon(\omega,\psi^\omega_\varepsilon(0)) < \infty$ (*resp.* $\sup_{\varepsilon>0} \mathscr{J}^g_\varepsilon(\omega,\psi^\omega_\varepsilon(0)) < \infty$);

(H^ω_2) $\psi^\omega_\varepsilon \to \psi^\omega$ *in* $C(]-\infty,0];L^2(O))$;

(H^ω_3) $\sup_{\varepsilon>0} \|F_\varepsilon(\omega,\cdot,u^\omega_\varepsilon,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)\|_{L^2([0,T];L^2(O))} < \infty$.

Then, there exists $\widehat{\Omega} \in \mathscr{F}$ with $\mathbb{P}(\widehat{\Omega}) = 1$ such that for every $\omega \in \widehat{\Omega}$ there exists $u^\omega \in C([0,T];L^2(O))$ such that up to a subsequence:

$$u^\omega_\varepsilon \to u^\omega \text{ in } C([0,T];L^2(O)); \tag{3.11}$$

$$\frac{du^\omega_\varepsilon}{dt} \rightharpoonup \frac{du^\omega}{dt} \text{ in } L^2([0,T];L^2(O)). \tag{3.12}$$

Moreover, we have

$$0 \leqslant u^\omega(t) \leqslant \sup_{\varepsilon>0} \overline{y}^\omega_\varepsilon(T) \text{ for all } t \in [0,T].$$

Assume furthermore that:

(H_4^ω) *for every* $v \in C([0,T];L^2(O))$,

$$u_\varepsilon^\omega \to v \text{ in } C([0,T];L^2(O)) \Longrightarrow F_\varepsilon(\omega,\cdot,u_\varepsilon^\omega,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega) \rightharpoonup G^\omega(\cdot,v) \text{ in } L^2([0,T];L^2(O));$$

(H_5^ω) *the problem* $(\mathscr{P}^{\mathrm{NH}}_{\mathrm{hom},\omega})$ *(resp.* $(\mathscr{P}^{\mathrm{D}}_{\mathrm{hom},\omega})$*) admits at most one solution.*

Then, (3.11)–(3.12) *hold for the whole sequence* ε *and*

u^ω *extended by* ψ^ω *in* $]-\infty,0]$ *is the unique solution of* $(\mathscr{P}^{\mathrm{NH}}_{\mathrm{hom},\omega})$ *(resp.* $(\mathscr{P}^{\mathrm{D}}_{\mathrm{hom},\omega})$*).*

Moreover, $\psi^\omega(t) \in [0,\overline{\rho}^\omega]$ *for all* $t \in]-\infty,0]$ *where* $\overline{\rho}^\omega := \sup_{\varepsilon>0}\overline{\rho}_\varepsilon^\omega$.

Remark 3.9. In most applications, (H_3^ω) is automatically checked (see Section 5).

3.3. **Notions of convergence for the reaction functionals and stochastic homogenization.** Here, we introduce two notions of convergence for both the reaction functionals and the time delays operators. The first one is called mixing convergence (see Definition 3.10) and the second one is stable convergence (see Definition 3.12). These two notions allow us to establish the assumption (H_4^ω)–(H_5^ω) of Theorem 3.8 and provide information on the specific form of G^ω (see Lemma 3.15) with $\omega \in \Omega$.

Definition 3.10. Let $T > 0$ and let $F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega} : C_{\mathrm{c}}(]-\infty,T];L^2(O;[0,\sup_{\varepsilon>0}\overline{y}_\varepsilon^\omega(T)])) \to L^2([0,T];L^2(O))$ be defined by

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(v)(t) := F_\varepsilon(\omega,t,v(t),\mathscr{R}_\varepsilon^\omega v(t)). \tag{3.13}$$

We say that $\{(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega)\}_{\varepsilon>0}$ mixing converges to $F(\omega,\cdot) : C_{\mathrm{c}}(]-\infty,T];L^2(O)) \to L^2([0,T];L^2(O))$, and we write

$$(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega) \xrightarrow{\text{mix}} F(\omega,\cdot),$$

if $\{F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}\}_{\varepsilon>0}$ pointwise weakly converges to $F(\omega,\cdot)$, i.e.

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(v) \rightharpoonup F(\omega,v) \text{ in } L^2([0,T];L^2(O))$$

for all $v \in C_{\mathrm{c}}(]-\infty,T];L^2(O;[0,\sup_{\varepsilon>0}\overline{y}_\varepsilon^\omega(T)]))$.

For the notion of stable convergence, we need to introduce the following subclass of DCP-structured reaction functionals.

Definition 3.11. By the class of special DCP-structured reaction functionals, that we denote by $\mathscr{F}^{\mathrm{s}}_{\mathrm{DCP}}$, we mean the subclass of $\mathscr{F}_{\mathrm{DCP}}$ for which $f : [0,\infty[\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by Definition 2.16 is of the form

$$f(t,x,\xi,\zeta) = \langle r(t,x), g(\xi,\zeta)\rangle$$

where $r \in L^\infty([0,T]\times\mathbb{R}^d;\mathbb{R}^m)$ for all $T > 0$ and $g : \mathbb{R}\times\mathbb{R}\to\mathbb{R}^m$ is such that $g(\xi,\zeta)$ is locally Lipschitz continuous in ξ (resp. ζ) uniformly with respect to ζ (resp. ξ), with $\langle\cdot,\cdot\rangle$ denoting the scalar product in $\mathbb{R}^m$ ($m \in \mathbb{N}^*$).

Definition 3.12. Let $T > 0$, let $\mathscr{R}^\omega : C_{\mathrm{c}}(]-\infty,T];L^2(O)) \to L^2([0,T];L^2(O))$ be a time delays operator associated with $\{\boldsymbol{m}_t^\omega\}_{t\geqslant 0} \subset \mathscr{M}_1^+(\mathbb{R};L^\infty(O))$, let $F(\omega,\cdot,\cdot,\cdot) : [0,T]\times L^2(O)\times L^2(O) \to L^2(O)$ be such that $F(\omega,t,u,v)(x) = \langle r(\omega,t,x), g(\omega,u(x),v(x))\rangle$ for all $u,v \in$

$L^2(O;[0,\sup_{\varepsilon>0}\overline{y}^\omega_\varepsilon(T)])$, and assume that $\{F_\varepsilon(\omega,\cdot,\cdot,\cdot)\}_{\varepsilon>0}\subset\mathscr{F}^{\rm s}_{\rm DCP}$, i.e. $F_\varepsilon(\omega,t,u,v)(x)=\langle r_\varepsilon(\omega,t,x),g_\varepsilon(\omega,u(x),v(x))\rangle$, with in addition that every $g_\varepsilon(\omega,\cdot,\cdot)$ is locally Lipschitz continuous in ξ (resp. ζ) uniformly with respect ε and to ζ (resp. ξ).
We say that $\{(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega_\varepsilon)\}_{\varepsilon>0}$ stable converges to $(F(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega)$, and we write

$$(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega_\varepsilon)\overset{\rm stable}{\longrightarrow}(F(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega),$$

if the following three convergences hold:

- $r_\varepsilon(\omega,\cdot,\cdot)\rightharpoonup r(\omega,\cdot,\cdot)$ in $L^2([0,T];L^2(O;\mathbb{R}^m))$;
- for every $(\xi,\zeta)\in\mathbb{R}\times\mathbb{R}$, $g_\varepsilon(\omega,\xi,\zeta)\to g(\omega,\xi,\zeta)$ in $\mathbb{R}^m$;
- for every $t\in[0,T]$, $\boldsymbol{m}^\omega_{\varepsilon,t}\lfloor_{]-\infty,t]}\rightharpoonup\boldsymbol{m}^\omega_t\lfloor_{]-\infty,t]}$ for the weak convergence of measures in $L^2(O)$ strong, i.e.

$$\mathscr{R}^\omega_\varepsilon\varphi\lfloor_{]-\infty,t]}(t)=\int_{-\infty}^t\varphi d\boldsymbol{m}^\omega_{\varepsilon,t}\to\mathscr{R}^\omega\varphi\lfloor_{]-\infty,t]}(t)\int_{-\infty}^t\varphi d\boldsymbol{m}^\omega_t\text{ in }L^2(O)$$

for all $\varphi\in C_{\rm c}(\mathbb{R};L^2(O))$.

Remark 3.13. The stable convergence of $\{(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega_\varepsilon)\}_{\varepsilon>0}$ to $(F(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega)$ does not imply that $F(\omega,\cdot,\cdot,\cdot)$ satisfies (DCP$_3$).

To make precise the expression of the limits problems according to each convergence introduced in Definitions 3.10 and 3.12, we adopt the following notation: given $T>0$, for every $a\in]0,\infty[$ and $\mathbb{P}$-a.e. $\omega\in\Omega$, we consider the following problems:

$$(\mathscr{P}^{\omega,F}_{\mathscr{E}})\begin{cases}\dfrac{du^\omega}{dt}(t)+\nabla\mathscr{E}(\omega,u^\omega(t))=F(\omega,u^\omega)(t)\quad\text{for }\mathscr{L}^1\text{-a.a. }t\in[0,T]\\[2ex] u^\omega(t)=\psi^\omega(t)\in L^2(O;[0,\overline{\rho}^\omega])\text{ for all }t\in]-\infty,0]\text{ and }\psi^\omega(0)\in\mathrm{dom}(\mathscr{E}(\omega,\cdot))\end{cases}$$

$$(\mathscr{P}^{\omega,F}_{\mathscr{E},\mathscr{R}^\omega})\begin{cases}\dfrac{du^\omega}{dt}(t)+\nabla\mathscr{E}(\omega,u^\omega(t))=F(\omega,t,u^\omega(t),\mathscr{R}^\omega u^\omega(t))\quad\text{for }\mathscr{L}^1\text{-a.a. }t\in[0,T]\\[2ex] u^\omega(t)=\psi^\omega(t)\in L^2(O;[0,\overline{\rho}^\omega])\text{ for all }t\in]-\infty,0]\text{ and }\psi^\omega(0)\in\mathrm{dom}(\mathscr{E}(\omega,\cdot)),\end{cases}$$

where $\overline{\rho}^\omega:=\sup_{\varepsilon>0}\overline{\rho}^\omega_\varepsilon$. The interest of Definitions 3.10 and 3.12 comes from Lemma 3.15 below. In what follows, for each $\varepsilon>0$, u^ω_ε is the unique solution of $(\mathscr{P}^{\rm NH}_{\varepsilon,\omega})$ (resp. $(\mathscr{P}^{\rm D}_{\varepsilon,\omega})$), see Corollary 3.5.

Remark 3.14. Since $0\leqslant u^\omega_\varepsilon\leqslant\overline{y}^\omega_\varepsilon$ for all $\varepsilon>0$, if $u^\omega_\varepsilon\to v$ in $C([0,T];L^2(O))$ then $0\leqslant v\leqslant\sup_{\varepsilon>0}\overline{y}^\omega_\varepsilon(T):=\overline{y}_T$ so that, by Remark 2.5, $0\leqslant\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon\leqslant\overline{y}_T$ and $0\leqslant\mathscr{R}^\omega_\varepsilon v\leqslant\overline{y}_T$ for all $\varepsilon>0$. Consequently, if $u^\omega_\varepsilon\to v$ in $C([0,T];L^2(O))$ then $u^\omega_\varepsilon(t),v(t),\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon(t),\mathscr{R}^\omega_\varepsilon v(t)\in[0,\overline{y}_T]$ for all $\varepsilon>0$ and all $t\in[0,T]$.

Lemma 3.15. (a) *If* $(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega_\varepsilon)\overset{\rm mix}{\longrightarrow}F(\omega,\cdot)$ *then for every* $v\in C([0,T];L^2(O))$,

$$u^\omega_\varepsilon\to v\text{ in }C([0,T];L^2(O))\Longrightarrow F_\varepsilon(\omega,\cdot,u^\omega_\varepsilon,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)\rightharpoonup F(\omega,v)\text{ in }L^2([0,T];L^2(O)),$$

and the problem $(\mathscr{P}^{\omega,F}_{\mathscr{J}_{\rm hom}})$ *(resp.* $(\mathscr{P}^{\omega,F}_{\mathscr{J}^g_{\rm hom}})$*) admits at most one solution.*

(b) *If* $(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega) \overset{\text{stable}}{\longrightarrow} (F(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega)$ *then for every* $v \in C_{\rm c}(]-\infty,T];L^2(O))$, $u_\varepsilon^\omega \to v$ *in* $C_{\rm c}(]-\infty,T];L^2(O)) \Longrightarrow F_\varepsilon(\omega,\cdot,u_\varepsilon^\omega,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega) \rightharpoonup F(\omega,\cdot,v,\mathscr{R}^\omega v)$ *in* $L^2([0,T];L^2(O))$, *and the problem* $(\mathscr{P}^{\omega,F}_{\mathscr{J}_{\rm hom},\mathscr{R}^\omega})$ (*resp.* $(\mathscr{P}^{\omega,F}_{\mathscr{J}^g_{\rm hom},\mathscr{R}^\omega})$) *admits at most one solution.*

Proof of Lemma 3.15. (a) Fix any $\varepsilon>0$. It is clear that

$$\begin{aligned} F_\varepsilon(\omega,\cdot,u_\varepsilon^\omega,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega) &= F_\varepsilon(\omega,\cdot,v,\mathscr{R}_\varepsilon^\omega v)+F_\varepsilon(\omega,\cdot,v,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega)-F_\varepsilon(\omega,\cdot,v,\mathscr{R}_\varepsilon^\omega v)\\ &\quad +F_\varepsilon(\omega,\cdot,u_\varepsilon^\omega,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega)-F_\varepsilon(\omega,\cdot,v,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega), \end{aligned} \tag{3.14}$$

and, by (DCP_1) and Remarks 3.14 and 2.4, we can assert that there exist $L_1,L_2\in]0,\infty[$ (which do not depend on ε) such that:

$$\begin{cases} \|F_\varepsilon(\omega,\cdot,v,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega)-F_\varepsilon(\omega,\cdot,v,\mathscr{R}_\varepsilon^\omega v)\|_{L^2([0,T];L^2(O))}\leqslant 2\sqrt{T}L_1\|u_\varepsilon^\omega-v\|_{L^2([0,T];L^2(O))}\\ \|F_\varepsilon(\omega,\cdot,u_\varepsilon^\omega,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega)-F_\varepsilon(\omega,\cdot,v,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega)\|_{L^2([0,T];L^2(O))}\leqslant L_2\|u_\varepsilon^\omega-v\|_{L^2([0,T];L^2(O))}. \end{cases} \tag{3.15}$$

By using Cauchy-Schwarz's inequality, from (3.14) and (3.15) it follows that for every $\phi\in L^2([0,T];L^2(O))$,

$$\begin{aligned} \left|\int_0^T\langle F_\varepsilon(\omega,t,u_\varepsilon^\omega(t),\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega(t))-F(\omega,v(t)),\phi(t)\rangle dt\right| &\leqslant \left|\int_0^T\langle F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(v)(t)-F(\omega,v(t)),\phi(t)\rangle dt\right|\\ &\quad +\alpha(T)\|u_\varepsilon^\omega-v\|_{L^2([0,T];L^2(O))}\|\phi\|_{L^2([0,T];L^2(O))} \end{aligned}$$

with $\alpha(T):=2\sqrt{T}L_1+L_2$ and $\langle\cdot,\cdot\rangle$ denoting the scalar product in $L^2(O)$, which shows that $F_\varepsilon(\omega,\cdot,u_\varepsilon^\omega,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega)\rightharpoonup F(\omega,v)$ in $L^2([0,T];L^2(O))$ because $(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega)\overset{\text{mix}}{\longrightarrow}F(\omega,\cdot)$ and $u_\varepsilon^\omega\to v$ in $C([0,T];L^2(O))$.

It remains to establish uniqueness: we only give the proof for the problem $(\mathscr{P}^{\omega,F}_{\mathscr{J}_{\rm hom}})$, while the proof for the problem $(\mathscr{P}^{\omega,F}_{\mathscr{J}^g_{\rm hom}})$ is an easy adaptation. Let u_1 and u_2 be two solutions of $(\mathscr{P}^{\omega,F}_{\mathscr{J}_{\rm hom}})$. First of all, by repeating the proof of (2.9) (in replacing Λu and Λv by u_1 and u_2 respectively) we see that there is $C(T)\in[0,\infty[$ such that for $\mathscr{L}^1$-a.a. $t\in[0,T]$,

$$\|F_\varepsilon(\omega,t,u_1(t),\mathscr{R}_\varepsilon^\omega u_1(t))-F_\varepsilon(\omega,t,u_2(t),\mathscr{R}_\varepsilon^\omega u_2(t))\|^2_{L^2(O)}\leqslant C(T)\|u_1(t)-u_2(t)\|^2_{L^2(O)}.$$

Hence, for any $s\in[0,T]$,

$$\|F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(u_1)-F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(u_2)\|^2_{L^2([0,s];L^2(O))}\leqslant C(T)\int_0^s\|u_1(t)-u_2(t)\|^2_{L^2(O)}dt. \tag{3.16}$$

As $(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega)\overset{\text{mix}}{\longrightarrow}F(\omega,\cdot)$ we have

$$\begin{cases} F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(u_1)\rightharpoonup F(\omega,u_1) \text{ in } L^2([0,s];L^2(O))\\ F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(u_2)\rightharpoonup F(\omega,u_2) \text{ in } L^2([0,s];L^2(O)), \end{cases}$$

so that, by passing to the limit in (3.16) and by using the weak lower semicontinuity of the norm of $L^2([0,s];L^2(O))$,

$$\|F(\omega,u_1)-F(\omega,u_2)\|_{L^2([0,s];L^2(O))}\leqslant\sqrt{C(T)}\|u_1-u_2\|_{L^2([0,s];L^2(O))} \text{ for all } s\in[0,T]. \tag{3.17}$$

On the other hand, for $\mathscr{L}^1$-a.e. $t \in [0,T]$, by taking $u_1(t)-u_2(t)$ as a test function in $(\mathscr{P}^{\omega,F}_{\mathscr{J}_{\rm hom}})$ for the solution u_1 and u_2 respectively, by subtracting and by using the monotonicity of $\nabla \mathscr{J}_{\rm hom}(\omega,\cdot)$, we easily obtain

$$\frac{d}{dt}\|u_1(t)-u_2(t)\|^2_{L^2(O)} \leqslant 2\|F(\omega,u_1)(t)-F(\omega,u_2)(t)\|_{L^2(O))}\|u_1(t)-u_2(t)\|_{L^2(O)}.$$

Noticing that $u_1(0)=u_2(0)=\psi(0)$, by integrating over $[0,s]$ with $s\in[0,T]$ and by using Cauchy-Schwarz, it follows that

$$\|u_1(s)-u_2(s)\|^2_{L^2(O)} \leqslant 2\|F(\omega,u_1)-F(\omega,u_2)\|_{L^2([0,s];L^2(O))}\|u_1-u_2\|_{L^2([0,s];L^2(O))},$$

which combined with (3.17) gives

$$\begin{aligned}\|u_1(s)-u_2(s)\|^2_{L^2(O)} &\leqslant 2\sqrt{C(T)}\|u_1-u_2\|^2_{L^2([0,S];L^2(O))}\\ &= 2\sqrt{C(T)}\int_0^s \|u_1(t)-u_2(t)\|^2_{L^2(O)}dt \text{ for all } s\in[0,T].\end{aligned}$$

From Grönwall's lemma (see Lemma D.1 that we apply with $\phi(s)=\|u_1(s)-u_2(s)\|^2_{L^2(O)}$, $a=0$ and $\mathrm{m}(t)=2\sqrt{C(T)}$) and the fact that $u_1(s)=u_2(s)=\psi(s)$ for all $s\in]-\infty,0]$, we deduce that $\|u_1-u_2\|_{C(]-\infty,T];L^2(O))}=0$, i.e. $u_1=u_2$.

(b) Since $r_\varepsilon(\omega,\cdot,\cdot)\rightharpoonup r(\omega,\cdot,\cdot)$ in $L^2([0,T];L^2(O;\mathbb{R}^m))$, we are reduced to show that

$$g_\varepsilon(\omega,u^\omega_\varepsilon,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)\to g(\omega,v,\mathscr{R}^\omega v) \text{ in } L^2([0,T];L^2(O)). \tag{3.18}$$

First of all, it is clear that for every $\varepsilon>0$,

$$\begin{aligned}\|g_\varepsilon(\omega,u^\omega_\varepsilon,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)-g(\omega,v,\mathscr{R}v)\|_{L^2([0,T];L^2(O))} \leqslant{}& \|g_\varepsilon(\omega,u^\omega_\varepsilon,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)-g_\varepsilon(\omega,v,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)\|_{L^2([0,T];L^2(O))}\\ &+\|g_\varepsilon(\omega,v,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)-g_\varepsilon(\omega,v,\mathscr{R}^\omega v)\|_{L^2([0,T];L^2(O))}\\ &+\|g_\varepsilon(\omega,v,\mathscr{R}^\omega v)-g(\omega,v,\mathscr{R}^\omega v)\|_{L^2([0,T];L^2(O))}\end{aligned} \tag{3.19}$$

On the other hand, taking Remark 3.14 into account, as every g_ε is locally Lipschitz continuous in ξ (resp. ζ) uniformly with respect ε and to ζ (resp. ξ), we can assert that there exists $L(T)\in[0,\infty[$ such that:

$$\|g_\varepsilon(\omega,u^\omega_\varepsilon,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)-g_\varepsilon(\omega,v,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)\|_{L^2([0,T];L^2(O))} \leqslant L(T)\|u^\omega_\varepsilon - v\|_{L^2([0,T];L^2(O))}; \tag{3.20}$$

$$\|g_\varepsilon(\omega,v,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)-g_\varepsilon(\omega,v,\mathscr{R}^\omega v)\|_{L^2([0,T];L^2(O))} \leqslant L(T)\|\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon - \mathscr{R}^\omega v\|_{L^2([0,T];L^2(O))} \tag{3.21}$$

for all $\varepsilon>0$. Since $u^\omega_\varepsilon\to v$ in $C([0,T];L^2(O))$, from (3.20) we have

$$\|g_\varepsilon(\omega,u^\omega_\varepsilon,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)-g_\varepsilon(\omega,v,\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon)\|_{L^2([0,T];L^2(O))}\to 0. \tag{3.22}$$

According to Remark 2.4 we have

$$\|\mathscr{R}^\omega_\varepsilon u^\omega_\varepsilon - \mathscr{R}^\omega v\|_{L^2([0,T];L^2(O))} \leqslant 2\sqrt{T}\|u^\omega_\varepsilon - v\|_{C_{\rm c}(]-\infty,T];L^2(O))} + \|\mathscr{R}^\omega_\varepsilon v - \mathscr{R}^\omega v\|_{L^2([0,T];L^2(O))} \tag{3.23}$$

For each $t\in[0,T]$, taking $\varphi\lfloor_{]-\infty,t]}= v$ as a test function in the definition of the stable convergence (see Definition 3.12) we see that $\mathscr{R}^\omega_\varepsilon v(t)\to\mathscr{R}^\omega v(t)$ in $L^2(O)$ for all $t\in[0,T]$, and according to Remark 3.14, from Lebesgue's dominated convergence theorem we deduce that

$$\|\mathscr{R}^\omega_\varepsilon v - \mathscr{R}^\omega v\|_{L^2([0,T];L^2(O))}\to 0. \tag{3.24}$$

As $u_\varepsilon^\omega \to v$ in $C(]-\infty,T];L^2(O))$, from (3.23) and (3.24) it follows that

$$\|\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega - \mathscr{R}v\|_{L^2([0,T];L^2(O))} \to 0,$$

and consequently, by using (3.21),

$$\|g_\varepsilon(\omega,v,\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega) - g_\varepsilon(\omega,v,\mathscr{R}^\omega v)\|_{L^2([0,T];L^2(O))} \to 0. \tag{3.25}$$

Taking Remark 3.14 into account and noticing that $\{g_\varepsilon(\omega,\cdot,\cdot)\}_{\varepsilon>0}$ pointwise converges to $g(\omega,\cdot,\cdot)$ (see Definition 3.12), from Lebesgue's dominated convergence theorem we see that

$$\|g_\varepsilon(\omega,v,\mathscr{R}^\omega v) - g(\omega,v,\mathscr{R}^\omega v)\|_{L^2([0,T];L^2(O))} \to 0, \tag{3.26}$$

and (3.18) folows by combining (3.19) with (3.22), (3.25) and (3.26).

Finally, the uniqueness can be established by using similar arguments as in the step 3 of the proof of Theorem 2.19. ■

The following result is a direct consequence of Theorem 3.8 and Lemma 3.15.

Corollary 3.16. *Assume that for $\mathbb{P}$-a.e. $\omega\in\Omega$, the assumptions (H_1^ω)–(H_3^ω) of Theorem 3.8 hold and suppose furthermore that either*

$$(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega) \xrightarrow{\text{mix}} F(\omega,\cdot) \tag{3.27}$$

or

$$(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega) \xrightarrow{\text{stable}} (F(\omega,\cdot,\cdot,\cdot),\mathscr{R}^\omega) \tag{3.28}$$

holds. Then:

- $u_\varepsilon^\omega \to u^\omega$ *in* $C([0,T];L^2(O))$;
- $\dfrac{du_\varepsilon^\omega}{dt} \rightharpoonup \dfrac{du^\omega}{dt}$ *in* $L^2([0,T];L^2(O))$;
- $0 \leqslant u^\omega(t) \leqslant \sup\limits_{\varepsilon>0} \overline{y}_\varepsilon^\omega(T)$ *for all* $t\in[0,T]$;
- u^ω *extended by* ψ^ω *in* $]-\infty,0]$ *is the unique solution of*

$$\begin{cases} (\mathscr{P}^{\omega,F}_{\mathscr{J}_{\mathrm{hom}}}) \text{ (resp. } (\mathscr{P}^{\omega,F}_{\mathscr{J}^g_{\mathrm{hom}}}) \text{ if (3.27) holds} \\ \\ (\mathscr{P}^{\omega,F}_{\mathscr{J}_{\mathrm{hom}},\mathscr{R}^\omega}) \text{ (resp. } (\mathscr{P}^{\omega,F}_{\mathscr{J}^g_{\mathrm{hom}},\mathscr{R}^\omega})) \text{ if (3.28) holds.} \end{cases}$$

4. Proof of the stochastic homogenization theorem

In this section we prove Theorem 3.8.

4.1. Convergence of time delays reaction-diffusion problems of gradient flow type.

Let $T>0$, let $\{\overline{a}_\varepsilon\}_{\varepsilon>0}\subset[0,\infty[$ and

$$\sup_{\varepsilon>0}\overline{a}_\varepsilon<\infty,$$

let $\{\overline{z}_\varepsilon\}_{\varepsilon>0}\subset C([0,T];\mathbb{R})$ be such that $0\leqslant\overline{z}_\varepsilon\leqslant\overline{z}_\varepsilon(T)$ for all $\varepsilon>0$ and

$$\sup_{\varepsilon>0}\overline{z}_\varepsilon(T)<\infty. \tag{4.1}$$

For each $\varepsilon>0$, let $\mathscr{E}_\varepsilon : L^2(O)\to[0,\infty[$ be a convex and Fréchet-differentiable functional, let $F_\varepsilon : [0,\infty[\times L^2(O)\times L^2(O)\to L^2(O)$, let $\mathscr{R}_\varepsilon : C_{\mathrm{c}}(]-\infty,T];L^2(O))\to L^2([0,T];L^2(O))$ be a

time delays operator associated with $\{\boldsymbol{m}_t^\varepsilon\}_{t\geqslant 0} \subset \mathscr{M}_1^+(\mathbb{R}; L^\infty(O))$, let $\psi_\varepsilon \in C_{\rm c}(]-\infty, 0]; L^2(O))$ and consider the following time delays reaction-diffusion problem of gradient flow type:

$$(\mathscr{P}_\varepsilon)\begin{cases} \dfrac{du_\varepsilon}{dt}(t) + \nabla\mathscr{E}_\varepsilon(u_\varepsilon(t)) = F_\varepsilon(t, u_\varepsilon(t), \mathscr{R}_\varepsilon u_\varepsilon(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u_\varepsilon(t) = \psi_\varepsilon(t) \in L^2(O; [0, \overline{a}_\varepsilon]) \text{ for all } t \in]-\infty, 0]. \end{cases}$$

Let $\mathscr{E}_0 : L^2(O) \to [0,\infty]$ be a proper, convex and lower semicontinuous functional, let $G : [0,T] \times L^2(O) \to L^2(O)$ be such that $G(\cdot, v) \in L^2([0,T]; L^2(O))$ for all $v \in L^2(O)$, let $\psi \in C_{\rm c}(]-\infty, 0]; L^2(O))$ and consider the following problem of gradient flow type:

$$(\mathscr{P}_0)\begin{cases} \dfrac{du}{dt}(t) + \partial\mathscr{E}_0(u(t)) \ni G(t, u(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u(t) = \psi(t) \text{ for all } t \in]-\infty, 0] \text{ and } \psi(0) \in \mathrm{dom}(\mathscr{E}_0). \end{cases}$$

To establish the following result, which gives sufficient conditions for the convergence of $(\mathscr{P}_\varepsilon)$ to $(\mathscr{P}_0)$ as $\varepsilon \to 0$, we do not need the existence but only the uniqueness of the solution of $(\mathscr{P}_0)$.

Theorem 4.1. *Assume that*:

(C_1) $\sup_{\varepsilon>0} \mathscr{E}_\varepsilon(\psi_\varepsilon(0)) < \infty$;

(C_2) $\psi_\varepsilon \to \psi$ *in* $C(]-\infty, 0]; L^2(O))$;

(C_3) *for each* $\varepsilon > 0$, $(\mathscr{P}_\varepsilon)$ *admits a solution* $u_\varepsilon \in AC([0,T]; L^2(O))$ *with* $0 \leqslant u_\varepsilon \leqslant \overline{z}_\varepsilon \leqslant \overline{z}_\varepsilon(T)$ *and* $\sup_{\varepsilon>0} \|F_\varepsilon(\cdot, u_\varepsilon, \mathscr{R}_\varepsilon u_\varepsilon)\|_{L^2([0,T];L^2(O))} < \infty$.

(C_4) *for every* $\{v_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$, *if* $\sup_{\varepsilon>0} \mathscr{E}_\varepsilon(v_\varepsilon) < \infty$ *then* $\{v_\varepsilon\}_{\varepsilon>0}$ *is relatively compact in* $L^2(O)$.

Then, there exists $u \in C([0,T]; L^2(O))$ *such that up to a subsequence*:

$$u_\varepsilon \to u \text{ in } C([0,T]; L^2(O)); \tag{4.2}$$

$$\frac{du_\varepsilon}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2([0,T]; L^2(O)). \tag{4.3}$$

Moreover we have

$$0 \leqslant u(t) \leqslant \sup_{\varepsilon>0} \overline{z}_\varepsilon(T) \text{ for all } t \in [0,T].$$

Assume furthermore that:

(C_5) *for every* $v \in C([0,T]; L^2(O))$,

$$u_\varepsilon \to v \text{ in } C([0,T]; L^2(O)) \Longrightarrow F_\varepsilon(\cdot, u_\varepsilon, \mathscr{R}_\varepsilon u_\varepsilon) \rightharpoonup G(\cdot, v) \text{ in } L^2([0,T]; L^2(O));$$

(C_6) $\mathscr{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathscr{E}_0$[3];

(C_7) *the problem* $(\mathscr{P}_0)$ *admits at most one solution.*

[3]By $\mathscr{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathscr{E}_0$ we mean that $\{\mathscr{E}_\varepsilon\}_{\varepsilon>0}$ Mosco-converges to $\mathscr{E}_0$, see Appendix C for more details.

Then, (4.2)–(4.3) *hold for the whole sequence* ε *and*

$$u \text{ extended by } \psi \text{ in }]-\infty,0] \text{ is the unique solution of } (\mathscr{P}_0).$$

Moreover, $\psi(t) \in [0,\overline{a}]$ *for all* $t \in]-\infty,0]$ *where* $\overline{a} := \sup_{\varepsilon>0} \overline{a}_\varepsilon$.

Proof of Theorem 4.1. In what follows the scalar product in $L^2(O)$ is denoted by $\langle\cdot,\cdot\rangle$. The proof is divided into three steps.

Step 1: bounds. First of all, from (C_3) and (4.1) we see that

$$0 \leqslant u_\varepsilon \leqslant \sup_{\varepsilon>0} \overline{z}_\varepsilon(T) < \infty. \tag{4.4}$$

Hence

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{C([0,T];L^2(O))} < \infty. \tag{4.5}$$

Fix any $\varepsilon > 0$. From (C_3) and $(\mathscr{P}_\varepsilon)$ we deduce that for $\mathscr{L}^1$-a.e. $t \in [0,T]$,

$$\left\|\frac{du_\varepsilon}{dt}(t)\right\|^2_{L^2(O)} + \left\langle \nabla\mathscr{E}_\varepsilon(u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t)\right\rangle = \left\langle F_\varepsilon(t,u_\varepsilon(t),\mathscr{R}_\varepsilon u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t)\right\rangle,$$

and so, by integrating over $[0,T]$,

$$\int_0^T \left\|\frac{du_\varepsilon}{dt}(t)\right\|^2_{L^2(O)} dt + \int_0^T \left\langle \nabla\mathscr{E}_\varepsilon(u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t)\right\rangle dt = \int_0^T \left\langle F_\varepsilon(t,u_\varepsilon(t),\mathscr{R}_\varepsilon u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t)\right\rangle dt.$$

But $\frac{d}{dt}\mathscr{E}_\varepsilon(u_\varepsilon(t)) = \left\langle \nabla\mathscr{E}_\varepsilon(u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t)\right\rangle$ for $\mathscr{L}^1$-a.a. $t \in [0,T]$ and $u_\varepsilon(0) = \psi_\varepsilon(0)$ by $(\mathscr{P}_\varepsilon)$, hence

$$\int_0^T \left\langle \nabla\mathscr{E}_\varepsilon(u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t)\right\rangle dt = \int_0^T \frac{d}{dt}\mathscr{E}_\varepsilon(u_\varepsilon(t))dt = \mathscr{E}_\varepsilon(u_\varepsilon(T)) - \mathscr{E}_\varepsilon(\psi_\varepsilon(0)),$$

and consequently

$$\begin{aligned}
\left\|\frac{du_\varepsilon}{dt}\right\|^2_{L^2([0,T];L^2(O))} &= \int_0^T \left\langle F_\varepsilon(t,u_\varepsilon(t),\mathscr{R}u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t)\right\rangle dt + \mathscr{E}_\varepsilon(\psi_\varepsilon(0)) - \mathscr{E}_\varepsilon(u_\varepsilon(T)) \\
&\leqslant \|F_\varepsilon(\cdot,u_\varepsilon,\mathscr{R}_\varepsilon u_\varepsilon)\|_{L^2([0,T];L^2(O))} \left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))} + \mathscr{E}_\varepsilon(u_{0,\varepsilon}) - \mathscr{E}_\varepsilon(u_\varepsilon(T)) \qquad (4.6)\\
&\leqslant \|F_\varepsilon(\cdot,u_\varepsilon)\|_{L^2([0,T];L^2(O))} \left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))} + \mathscr{E}_\varepsilon(\psi_\varepsilon(0)).
\end{aligned}$$

Noticing that by (C_1) and (C_3) we have:

$$\begin{aligned}
c_1 &:= \sup_{\varepsilon>0} \mathscr{E}_\varepsilon(\psi_\varepsilon(0)) < \infty; \\
c_2 &:= \sup_{\varepsilon>0} \|F_\varepsilon(\cdot,u_\varepsilon,\mathscr{R}_\varepsilon u_\varepsilon)\|_{L^2([0,T];L^2(O))} < \infty,
\end{aligned} \tag{4.7}$$

it follows that for every $\varepsilon > 0$,

$$\left\|\frac{du_\varepsilon}{dt}\right\|^2_{L^2([0,T];L^2(O))} \leqslant C\left(\left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))} + 1\right)$$

with $C := \max(c_1, c_2)$, which implies that

$$\sup_{\varepsilon>0} \left\| \frac{du_\varepsilon}{dt} \right\|_{L^2([0,T];L^2(O))} < \infty. \tag{4.8}$$

Step 2: compactness. By (4.5), $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $C([0,T];L^2(O))$. Moreover, For every $(s_1, s_2) \in [0,T] \times [0,T]$ with $s_1 < s_2$,

$$\begin{aligned} \|u_\varepsilon(s_1) - u_\varepsilon(s_2)\|_{L^2(O)} &\leqslant \int_{s_1}^{s_2} \left\| \frac{du_\varepsilon}{dt}(t) \right\|_{L^2(O)} dt \\ &\leqslant (s_2 - s_1)^{\frac{1}{2}} \sup_{\varepsilon>0} \left\| \frac{du_\varepsilon}{dt} \right\|_{L^2([0,T];L^2(O))}, \end{aligned}$$

which, by (4.8), implies the equi-continuity of $\{u_\varepsilon\}_{\varepsilon>0}$. On the other hand, from (C_1) and (C_4) it is clear $\{u_\varepsilon(0)\}_{\varepsilon>0} = \{\psi_\varepsilon(0)\}_{\varepsilon>0}$ is relatively compact in $L^2(O)$. Morever, if $s \in]0,T]$ then, by replacing T by s in (4.6), we have

$$\mathscr{E}_\varepsilon(u_\varepsilon(s)) \leqslant \left\| \frac{du_\varepsilon}{dt} \right\|_{L^2([0,T];L^2(O))} \left(\|F_\varepsilon(\cdot, u_\varepsilon, \mathscr{R}_\varepsilon u_\varepsilon)\|_{L^2([0,T];L^2(O))} - \left\| \frac{du_\varepsilon}{dt} \right\|_{L^2([0,T];L^2(O))} \right) + \mathscr{E}_\varepsilon(\psi_\varepsilon(0)).$$

From (C_1), (4.8) and (4.7), it follows that $\sup_{\varepsilon>0} \mathscr{E}_\varepsilon(u_\varepsilon(s)) < \infty$. Hence, by ($C_4$), $\{u_\varepsilon(s)\}_{\varepsilon>0}$ is relatively compact in $L^2(O)$. Consequently, by Arzelà-Ascoli's compactness theorem there exists $u \in C([0,T];L^2(O))$ such that, up to a subsequence,

$$u_\varepsilon \to u \text{ in } C([0,T];L^2(O)). \tag{4.9}$$

From (4.8) we deduce that

$$\frac{du_\varepsilon}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2([0,T];L^2(O)) \tag{4.10}$$

and from (C_3) and (4.9) it follows that $0 \leqslant u(t) \leqslant \sup_{\varepsilon>0} \overline{z}_\varepsilon(T)$ for all $t \in [0,T]$.

Step 3: convergence to the solution of ($\mathscr{P}_0$). We are going to prove that u is a solution of ($\mathscr{P}_0$).

Step 3-1: Legendre-Fenchel transform of ($\mathscr{P}_\varepsilon$). Fix any $\varepsilon > 0$ and denote the Legendre-Fenchel conjugates of $\mathscr{E}_\varepsilon$ and $\mathscr{E}_0$ by $\mathscr{E}_\varepsilon^*$ and $\mathscr{E}_0^*$ respectively. From Fenchel's extremality relation (see Proposition B.4(b)) we see that ($\mathscr{P}_\varepsilon$) is equivalent to

$$\begin{cases} \mathscr{E}_\varepsilon(u_\varepsilon(t)) + \mathscr{E}_\varepsilon^*(G_\varepsilon(t) - \dfrac{du_\varepsilon}{dt}(t)) + \left\langle \dfrac{du_\varepsilon}{dt}(t) - G_\varepsilon(t), u_\varepsilon(t) \right\rangle = 0 \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u_\varepsilon(t) = \psi_\varepsilon(t) \in L^2(O;[0,\overline{a}_\varepsilon]) \text{ for all } t \in]-\infty, 0]. \end{cases}$$

with $G_\varepsilon := F_\varepsilon(\cdot, u_\varepsilon, \mathscr{R}_\varepsilon u_\varepsilon)$. Using Legendre-Fenchel's inequality (see Theorem B.2(b)) it follows that

$$(\mathscr{P}_\varepsilon) \Longleftrightarrow \begin{cases} \displaystyle\int_0^T \left[\mathscr{E}_\varepsilon(u_\varepsilon(t)) + \mathscr{E}_\varepsilon^*(G_\varepsilon(t) - \frac{du_\varepsilon}{dt}(t)) + \left\langle \frac{du_\varepsilon}{dt}(t) - G_\varepsilon(t), u_\varepsilon(t) \right\rangle \right] dt = 0 \\ u_\varepsilon(t) = \psi_\varepsilon(t) \in L^2(O;[0,\overline{a}_\varepsilon]) \text{ for all } t \in]-\infty, 0]. \end{cases}$$

On the other hand, we have

$$\begin{aligned}\int_0^T \left\langle \frac{du_\varepsilon}{dt}(t) - G_\varepsilon(t), u_\varepsilon(t) \right\rangle dt &= \int_0^T \left[\frac{d}{dt}(\frac{1}{2}\|u_\varepsilon\|^2)(t) - \langle G_\varepsilon(t), u_\varepsilon(t)\rangle \right] dt \\ &= \frac{1}{2}(\|u_\varepsilon(T)\|^2 - \|\psi_\varepsilon(0)\|^2) - \int_0^T \langle G_\varepsilon(t), u_\varepsilon(t)\rangle\, dt.\end{aligned}$$

Hence, for every $\varepsilon > 0$,

$$(\mathscr{P}_\varepsilon) \Longleftrightarrow \begin{cases} \displaystyle\int_0^T \left[\mathscr{E}_\varepsilon(u_\varepsilon(t)) + \mathscr{E}_\varepsilon^*(G_\varepsilon(t) - \frac{du_\varepsilon}{dt}(t)) \right] dt + \frac{1}{2}\left(\|u_\varepsilon(T)\|^2 - \|\psi_\varepsilon(0)\|^2 \right) \\ \quad \displaystyle - \int_0^T \langle G_\varepsilon(t), u_\varepsilon(t)\rangle\, dt = 0 \\ \\ u_\varepsilon(t) = \psi_\varepsilon(t) \in L^2(O; [0, \overline{a}_\varepsilon]) \text{ for all } t \in]-\infty, 0]. \end{cases} \tag{4.11}$$

Step 3-2: passing to the limit. First of all, from (C_2) we have

$$\psi_\varepsilon(t) \to \psi(t) \text{ in } L^2(O) \text{ for all } t \in]-\infty, 0]. \tag{4.12}$$

But, by $(\mathscr{P}_\varepsilon)$, $\psi_\varepsilon(t) \in [0, \overline{a}_\varepsilon]$ for all $t \in]-\infty, 0]$, hence

$$\psi(t) \in [0, \overline{a}] \text{ for all } t \in]-\infty, 0]$$

with $\overline{a} := \sup_{\varepsilon>0} \overline{a}_\varepsilon$. By (4.9) we have

$$u_\varepsilon(0) \to u(0) \text{ in } L^2(O),$$

and from $(\mathscr{P}_\varepsilon)$ and (4.12) we see that

$$\psi_\varepsilon(0) = u_\varepsilon(0) \to \psi(0) \text{ in } L^2(O). \tag{4.13}$$

Hence:

$$u(0) = \psi(0); \tag{4.14}$$

$$\lim_{\varepsilon\to 0} \|\psi_\varepsilon(0)\|^2_{L^2(O)} = \|\psi(0)\|^2_{L^2(O)}. \tag{4.15}$$

Moreover, from (C_1), (4.13) and (C_6) we have

$$\mathscr{E}_0(\psi(0)) \leqslant \underline{\lim}_{\varepsilon\to 0} \mathscr{E}_\varepsilon(\psi_\varepsilon(0)) \leqslant \sup_{\varepsilon>0} \mathscr{E}_\varepsilon(\psi_\varepsilon(0)) < \infty,$$

and consequently

$$\psi(0) \in \mathrm{dom}(\mathscr{E}_0). \tag{4.16}$$

Since $u_\varepsilon(T) = u_\varepsilon(0) + \int_0^T \frac{du_\varepsilon}{dt}(t)dt$ and $u(T) = u(0) + \int_0^T \frac{du}{dt}(t)dt$, from (4.10), (4.13) and (4.14) we deduce that

$$\underline{\lim}_{\varepsilon\to 0} \|u_\varepsilon(T)\|^2_{L^2(O)} \geqslant \|u(T)\|^2_{L^2(O)}. \tag{4.17}$$

Let $E_0, E_0^* : L^2([0,T];L^2(O)) \to [0,\infty]$ be defined by

$$\begin{cases} E_0(u) := \displaystyle\int_0^T \mathscr{E}_0(u(t))dt \\ \\ E_0^*(u) := \displaystyle\int_0^T \mathscr{E}_0^*(u(t))dt \end{cases}$$

and, for each $\varepsilon > 0$, let Let $E_\varepsilon : L^2([0,T];L^2(O)) \to [0,\infty]$ be defined by

$$\begin{cases} E_\varepsilon(u) := \displaystyle\int_0^T \mathscr{E}_\varepsilon(u(t))dt \\ \\ E_\varepsilon^*(u) := \displaystyle\int_0^T \mathscr{E}_\varepsilon^*(u(t))dt. \end{cases}$$

From (C_6) and Theorem C.4 we have $\mathscr{E}_\varepsilon^* \xrightarrow{M} \mathscr{E}_0^*$. Hence $E_\varepsilon \xrightarrow{M} E_0$ and $E_\varepsilon^* \xrightarrow{M} E_0^*$ by Theorem C.5. From (4.9), (C_5) and (4.10) it follows that:

$$\begin{aligned} &\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geqslant E_0(u), \text{ i.e.} \\ &\underline{\lim}_{\varepsilon\to 0} \int_0^T \mathscr{E}_\varepsilon(u_\varepsilon(t))dt \geqslant \int_0^T \mathscr{E}_0(u(t))dt; \qquad (4.18) \\ &\underline{\lim}_{\varepsilon\to 0} E_\varepsilon^*(G_\varepsilon - \frac{du_\varepsilon}{dt}) \geqslant E_0^*(G_0 - \frac{du}{dt}), \text{ i.e.} \\ &\underline{\lim}_{\varepsilon\to 0} \int_0^T \mathscr{E}_\varepsilon^*(G_\varepsilon(t) - \frac{du_\varepsilon}{dt}(t))dt \geqslant \int_0^T \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t))dt \qquad (4.19) \end{aligned}$$

with $G_0 := G(\cdot, u)$. Extending u by ψ in $]-\infty, 0]$ and taking (4.13), (4.14), (4.15), (4.16), (4.17), (4.18) and (4.19) into account, by letting $\varepsilon \to 0$ in (4.11) we obtain

$$\begin{cases} \displaystyle\int_0^T \left[\mathscr{E}_0(u(t)) + \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t))\right] dt + \frac{1}{2}(\|u(T)\|^2 - \|u(0)\|^2) \\ \quad \displaystyle -\int_0^T \langle G_0(t), u(t)\rangle\, dt \leqslant 0 \\ \\ u(t) = \psi(t) \text{ for all } t \in]-\infty, 0] \text{ and } \psi(0) \in \mathrm{dom}(\mathscr{E}_0), \end{cases}$$

i.e.

$$\begin{cases} \displaystyle\int_0^T \left[\mathscr{E}_0(u(t)) + \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t)) + \left\langle \frac{du}{dt}(t) - G_0(t), u(t)\right\rangle\right] dt \leqslant 0 \\ \\ u(t) = \psi(t) \text{ for all } t \in]-\infty, 0] \text{ and } \psi(0) \in \mathrm{dom}(\mathscr{E}_0). \end{cases}$$

But, by using again Legendre-Fenchel's inequality (see Theorem B.2(b)), we have

$$\mathscr{E}_0(u(t)) + \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t)) + \left\langle \frac{du}{dt}(t) - G_0(t), u(t)\right\rangle \geqslant 0 \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T],$$

hence

$$\begin{cases} \displaystyle\int_0^T \left[\mathscr{E}_0(u(t)) + \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t)) + \left\langle \frac{du}{dt}(t) - G_0(t), u(t) \right\rangle\right] dt = 0 \\ \\ u(t) = \psi(t) \text{ for all } t \in]-\infty, 0] \text{ and } \psi(0) \in \mathrm{dom}(\mathscr{E}_0). \end{cases} \tag{4.20}$$

Using again Fenchel's extremality relation (see Proposition B.4(b)) we see that (4.20) is equivalent to

$$\begin{cases} \displaystyle\frac{du}{dt}(t) + \partial\mathscr{E}_0(u(t)) \ni G_0(t) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u(t) = \psi(t) \text{ for all } t \in]-\infty, 0] \text{ and } \psi(0) \in \mathrm{dom}(\mathscr{E}_0), \end{cases}$$

which shows that u is a solution of $(\mathscr{P}_0)$, and the proof is complete because of the uniqueness of the solution of $(\mathscr{P}_0)$ by (C_7). ■

4.2. **Proof of Theorem 3.8.** By [AHMM23, Theorem 4.8] there exists $\Omega' \in \mathscr{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$, $\{\mathscr{J}_\varepsilon(\omega,\cdot)\}_{\varepsilon>0}$ (resp. $\{\mathscr{J}_\varepsilon^g(\omega,\cdot)\}_{\varepsilon>0}$) Mosco-convergence to $\mathscr{J}_{\mathrm{hom}}(\omega,\cdot)$ (resp. $\mathscr{J}^g_{\mathrm{hom}}(\omega,\cdot)$). Let $\Omega'' \in \mathscr{F}$ be such that $\mathbb{P}(\Omega'') = 1$ and (H_1^ω)–(H_5^ω) (in Theorem 3.8) hold. Set $\widehat{\Omega} = \Omega' \cap \Omega''$. Then $\widehat{\Omega} \in \mathscr{F}$ and $\mathbb{P}(\widehat{\Omega}) = 1$. We are going to apply Theorem 4.1.

Firstly, from (H_1^ω)–(H_4^ω) it is easy to see that (C_1)–(C_3) and (C_5) hold with $\psi_\varepsilon = \psi_\varepsilon^\omega$, $\psi = \psi^\omega$, $u_\varepsilon = u_\varepsilon^\omega$, $\overline{z}_\varepsilon = \overline{y}_\varepsilon^\omega$, $\overline{a}_\varepsilon = \overline{\rho}_\varepsilon^\omega$, $\overline{a} = \overline{\rho}^\omega$, $F_\varepsilon = F_\varepsilon(\omega,\cdot,\cdot,\cdot)$, $G = G^\omega$ and $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon(\omega,\cdot)$ (resp. $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon^g(\omega,\cdot)$). Note that (C_4) is verified with $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon(\omega,\cdot)$ (resp. $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon^g(\omega,\cdot)$) by using [AHMM23, Lemma 4.2] (resp. [AHMM23, Lemma 4.3]). Secondly, (C_6) is satisfied with $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon(\omega,\cdot)$ and $\mathscr{E}_0 = \mathscr{J}_{\mathrm{hom}}(\omega,\cdot)$ (resp. $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon^g(\omega,\cdot)$ and $\mathscr{E}_0 = \mathscr{J}^g_{\mathrm{hom}}(\omega,\cdot)$) because $\{\mathscr{J}_\varepsilon(\omega,\cdot)\}_{\varepsilon>0}$ (resp. $\{\mathscr{J}_\varepsilon^g(\omega,\cdot)\}_{\varepsilon>0}$) Mosco-convergence to $\mathscr{J}_{\mathrm{hom}}(\omega,\cdot)$ (resp. $\mathscr{J}^g_{\mathrm{hom}}(\omega,\cdot)$). Finally, by (H_5^ω) we sse that (C_7) is verified with $G = G^\omega$, and the conclusion of Theorem 3.8 follows by applying Theorem 4.1 and noticing that $\partial\mathscr{J}_{\mathrm{hom}}(\omega,\cdot) = \{\nabla\mathscr{J}_{\mathrm{hom}}(\omega,\cdot)\}$ (resp. $\partial\mathscr{J}^g_{\mathrm{hom}}(\omega,\cdot) = \{\nabla\mathscr{J}^g_{\mathrm{hom}}(\omega,\cdot)\}$). ■

5. Application to spatial population dynamics

Here we apply Corollary 3.16 to nonlocal spatial population model with age structure.

5.1. **Nonlocal spatial population model.** In general terms, without considering spatial interactions between individuals and potential delays caused by age differences among individuals, the mathematical description of a population dynamics model can be described by a differential equation of the form:

$$\frac{du}{dt}(t) = \text{birth - death + (immigration or harvesting)}$$

where $u(t)$ represents the population size at time t. Here we assume that spatial movement is possible without constraints in the sense that individuals can disperse and interact with

other individuals over relatively long distances. This means that the model is nonlocal and can be described by a partial differential equation of the type:

$$\frac{\partial u}{dt}(t,x) + \text{nonlocal diffusion term} = \text{birth - death} + (\text{immigration or harvesting}),$$

where, in order to use our results from Section 3, the nonlocal diffusion term will be a nonlocal integral functional of type $\mathscr{J} + \mathscr{D}_g$ with $\mathscr{J}$ and $\mathscr{D}_g$ given by (2.1) and (2.2) respectively. The term "birth - death + (immigration or harvesting)" represents the reaction process. In order to incorporate the age structure, it will be necessary to introduce time delays specifically in the "birth" component of the reaction term (see §5.3). In this way, in §5.2 we present a class of reaction functionals within a stochastic framework. From now on, as in Section 3, $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is a dynamical system and $O \subset \mathbb{R}^d$ (with $d = 1, 2$ or 3) is a bounded domain with Lipschitz boundary.

5.2. **Random reaction functionals.** Let $f : \Omega \times [0,\infty[\times\mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(\omega, t, x, \xi, \zeta) := b(\omega, t, x, \zeta) - d(\omega, t, x, \xi) + a(\omega, t, x), \tag{5.1}$$

where $b : \Omega\times[0,\infty[\times\mathbb{R}^d\times\mathbb{R} \to]0,\infty[$ and $d : \Omega\times[0,\infty[\times\mathbb{R}^d\times\mathbb{R} \to]0,\infty[$ are two $(\mathscr{F}\otimes\mathscr{B}(\mathbb{R})\otimes \mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(\mathbb{R}), \mathscr{B}(\mathbb{R}))$-measurable functions representing "birth" and "death" respectively, and $a : \Omega\times[0,\infty[\times\mathbb{R}^d \to \mathbb{R}$ is a $(\mathscr{F}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$-measurable function representing the immigration or harvesting rate.

We further consider the following assumptions:

- (A_1) for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $T \in [0,\infty[$, $b(\omega,\cdot,\cdot,\cdot) \in L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R})$;
- (A_2) for every $T \in [0,\infty[$, $\overline{b}(\cdot) := \operatorname{ess\,sup}\{b(\cdot,t,x,\zeta) : (t,x,\zeta) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}\} \in L^1_{\mathbb{P}}(\Omega)$;
- (A_3) for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $T \in [0,\infty[$, $b(\omega,t,x,\zeta)$ is locally Lipschitz continuous in ζ uniformly with respect to $(t,x) \in [0,T]\times\mathbb{R}^d$;
- (A_4) for $\mathbb{P}$-a.e. $\omega \in \Omega$, $b(\omega,\cdot,\cdot,0) = 0$;
- (A_5) for $\mathbb{P}$-a.e. $\omega \in \Omega$, every $x \in \mathbb{R}^d$ and every $z \in \mathbb{Z}^d$, $b(T_z\omega,\cdot,x,\cdot) = b(\omega,\cdot,x+z,\cdot)$;
- (A_6) for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $T \in [0,\infty[$, $d(\omega,\cdot,\cdot,\cdot) \in L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R})$;
- (A_7) for every $T \in [0,\infty[$, $\overline{d}(\cdot) := \operatorname{ess\,sup}\{d(\cdot,t,x,\xi) : (t,x,\xi) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}\} \in L^1_{\mathbb{P}}(\Omega)$;
- (A_8) for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $T \in [0,\infty[$, $\underline{d}(\omega,\cdot) := \operatorname{ess\,inf}\{d(\omega,t,x,\cdot) : (t,x) \in [0,T]\times\mathbb{R}^d\}$ is continuous and increasing on $[0,\infty[$;
- (A_9) for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $T \in [0,\infty[$, $d(\omega,t,x,\xi)$ is locally Lipschitz continuous in ξ uniformly with respect to $(t,x) \in [0,T]\times\mathbb{R}^d$;
- (A_{10}) for $\mathbb{P}$-a.e. $\omega \in \Omega$, $d(\omega,\cdot,\cdot,0) = 0$;
- (A_{11}) for $\mathbb{P}$-a.e. $\omega \in \Omega$, every $x \in \mathbb{R}^d$ and every $z \in \mathbb{Z}^d$, $d(T_z\omega,\cdot,x,\cdot) = d(\omega,\cdot,x+z,\cdot)$;
- (A_{12}) for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for every $T \in [0,\infty[$, $a(\omega,\cdot,\cdot) \in L^\infty([0,T]\times\mathbb{R}^d)$;
- (A_{13}) for every $T \in [0,\infty[$, $\overline{a}(\cdot) := \operatorname{ess\,sup}\{a(\cdot,t,x) : (t,x) \in [0,T]\times\mathbb{R}^d\} \in L^1_{\mathbb{P}}(\Omega)$;
- (A_{14}) for $\mathbb{P}$-a.e. $\omega \in \Omega$, every $x \in \mathbb{R}^d$ and every $z \in \mathbb{Z}^d$, $a(T_z\omega,\cdot,x) = a(\omega,\cdot,x+z)$;
- (A_{15}) for $\mathbb{P}$-a.e. $\omega \in \Omega$, $b(\omega,\cdot,\cdot,\cdot) + a(\omega,\cdot,\cdot) \geqslant 0$.

Remark 5.1. The assumptions (A_4) and (A_{10}) are not restrictive. In fact, if we have a triple (b, d, a) that satisfies (A_1)–(A_3), (A_5)–(A_9), and (A_{11})–(A_{15}), we can obtain a new triple that satisfies (A_1)–(A_{15}) without modifying the function f defined by (5.1). We achieve this by considering the triple $(b - b(\cdot,\cdot,\cdot,0), d - d(\cdot,\cdot,\cdot,0), a + b(\cdot,\cdot,\cdot,0) - d(\cdot,\cdot,\cdot,0))$ instead of the original triple (b, d, a).

For each $\varepsilon > 0$, let $f_\varepsilon : \Omega \times [0,\infty[\times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f_\varepsilon(\omega, t, x, \xi, \zeta) := f\left(\omega, t, \frac{x}{\varepsilon}, \xi, \zeta\right)$$

with $f : \Omega \times [0,\infty[\times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by (5.1) and let $F_\varepsilon : \Omega \times [0,\infty[\times L^2(O) \times L^2(O) \to L^2(O)$ be defined by

$$\begin{aligned} F_\varepsilon(\omega, t, u, v)(x) &:= f_\varepsilon\left(\omega, t, x, u(x), v(x)\right) \\ &= f\left(\omega, t, \frac{x}{\varepsilon}, u(x), v(x)\right) \\ &= b\left(\omega, t, \frac{x}{\varepsilon}, v(x)\right) - d\left(\omega, t, \frac{x}{\varepsilon}, u(x)\right) + a\left(\omega, t, \frac{x}{\varepsilon}\right). \end{aligned} \tag{5.2}$$

The following lemma shows that under the previous assumptions the family $\{F_\varepsilon(\omega,\cdot,\cdot,\cdot)\}_{\varepsilon>0}$ is a family of DCP-structured reaction functionals (see Definition 2.16).

Lemma 5.2. *Assume that* (A_1)–(A_{15}) *hold. Then, for every* $\varepsilon > 0$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$, $F_\varepsilon(\omega,\cdot,\cdot,\cdot) \in \mathscr{F}_{\mathrm{DCP}}$ *with* $(\overline{f}_\varepsilon(\omega,\cdot,\cdot), \overline{\rho}^\omega_\varepsilon, \overline{y}^\omega_\varepsilon(\cdot)) = (0, \overline{\rho}^\omega, \overline{\rho}^\omega)$ *where* $\overline{\rho}^\omega \geqslant \underline{d}^{-1}(\omega, \overline{b}(\omega) + \overline{a}(\omega))$ *with* $\underline{d}^{-1}(\omega,\cdot)$ *denoting the inverse function of* $\underline{d}(\omega,\cdot)$[4].

Proof of Lemma 5.2. Fix $\varepsilon > 0$ and a suitable $\omega \in \Omega$. We only need to establish the condition (DCP_3). First of all, by (A_{15}) we have $f_\varepsilon(\omega, t, x, 0, \zeta) = b(\omega, t, \frac{x}{\varepsilon}, \zeta) + a(\omega, t, \frac{x}{\varepsilon}) \geqslant 0$ for all $(t, x, \zeta) \in [0,\infty[\times \mathbb{R} \times \mathbb{R}$. On the other hand, let $\overline{\rho}^\omega \in]0,\infty[$ be such that $\overline{\rho}^\omega \geqslant \underline{d}^{-1}(\omega, \overline{b}(\omega) + \overline{a}(\omega))$ and set $\overline{y}^\omega_\varepsilon(\cdot) := \overline{\rho}^\omega$. Taking (A_8) into account we see that for every $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\begin{aligned} d\left(\omega, t, \frac{x}{\varepsilon}, \overline{y}^\omega_\varepsilon(t)\right) = d\left(\omega, t, \frac{x}{\varepsilon}, \overline{\rho}^\omega\right) &\geqslant \underline{d}(\omega, \overline{\rho}^\omega) \\ &\geqslant \underline{d}(\omega, \underline{d}^{-1}(\omega, \overline{b}(\omega) + \overline{a}(\omega))) \\ &= \overline{b}(\omega) + \overline{a}(\omega). \end{aligned}$$

According to (A_2) and (A_{13}) it follows that

$$d\left(\omega, t, \frac{x}{\varepsilon}, \overline{y}^\omega_\varepsilon(t)\right) \geqslant b\left(\omega, t, \frac{x}{\varepsilon}, \zeta\right) + a\left(\omega, t, \frac{x}{\varepsilon}\right),$$

i.e. $f_\varepsilon(\omega, t, x, \overline{y}^\omega_\varepsilon(t)) \leqslant 0$ for all $(t, x, \zeta) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$, which establishes (DCP_3) with $\overline{f}_\varepsilon(\omega,\cdot,\cdot) = 0$. ■

To illustrate our purpose, here are a few examples of well-known models that are compatible with our framework.

[4]Such an inverse function exists because of the condition (A_8).

Example 5.3 (Nicklson's blowflies model). For this model we have:

$$\begin{aligned} b(\omega,t,x,\zeta) &= \beta(\omega,t,x)\zeta \mathrm{e}^{-\gamma\zeta};\\ d(\omega,t,x,\xi) &= \delta(\omega,t,x)\xi \end{aligned} \tag{5.3}$$

with $\gamma > 0$ and $\beta, \delta : \Omega \times [0,\infty[\times\mathbb{R}^d \to]0,\infty[$ such that

$$\text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega, \ \begin{cases} \overline{\beta}(\omega) := \operatorname*{ess\,sup}_{(t,x)\in[0,\infty[\times\mathbb{R}^d} \beta(\omega,t,x) < \infty \\ \underline{\delta}(\omega) := \operatorname*{ess\,inf}_{(t,x)\in[0,\infty[\times\mathbb{R}^d} \delta(\omega,t,x) > 0. \end{cases} \tag{5.4}$$

Here, $\overline{\rho}^\omega$ has to verify the following condition:

$$\overline{\rho}^\omega \geqslant \frac{\overline{\beta}(\omega)\mathrm{e}^{-1}}{\underline{\delta}(\omega)\gamma}.$$

Example 5.4 (Wazewska-Czyziewska and Lasota model). For this model we have

$$b(\omega,t,x,\zeta) = \beta(\omega,t,x)\mathrm{e}^{-\gamma\zeta}$$

and $d : \Omega\times[0,\infty[\times\mathbb{R}^d\times\mathbb{R} \to]0,\infty[$ given by (5.3) with $\gamma > 0$ and $\beta, \delta : \Omega\times[0,\infty[\times\mathbb{R}^d \to]0,\infty[$ satisfying (5.4). Here, $\overline{\rho}^\omega$ has to verify the following condition:

$$\overline{\rho}^\omega \geqslant \frac{\overline{\beta}(\omega)}{\underline{\delta}(\omega)}.$$

Example 5.5 (Mackey-Glass model). For this model we have

$$b(\omega,t,x,\zeta) = \beta(\omega,t,x)g(\zeta),$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined by

$$g(\zeta) := \begin{cases} \frac{\zeta}{1+\zeta^m} & \text{if } \zeta > 0 \\ 0 & \text{if } \zeta \leqslant 0, \end{cases}$$

and $d : \Omega\times[0,\infty[\times\mathbb{R}^d\times\mathbb{R} \to]0,\infty[$ given by (5.3) with $\gamma > 0$ and $\beta, \delta : \Omega\times[0,\infty[\times\mathbb{R}^d \to]0,\infty[$ satisfying (5.4). Here, $\overline{\rho}^\omega$ has to verify the following condition:

$$\overline{\rho}^\omega \geqslant \frac{\overline{\beta}(\omega)}{\underline{\delta}(\omega)} g\left(\left(\frac{1}{m-1}\right)^{\frac{1}{m}}\right).$$

Remark 5.6. Several reactions functions are incompatible with our framework, including the logistic function with emigration. For further details about these reaction functions, we refer to [AHMM22].

5.3. **Nonlocal reaction-diffusion problems with age structure.** Let $\sigma : \Omega \times \mathbb{R}^d \to [0,\infty[$ be a $(\mathscr{F}\otimes\mathscr{B}(\mathbb{R}^d),\mathscr{B}(\mathbb{R}))$-measurable function satisfying the following properties:

- (A_{16}) there exists $k \in \mathbb{N}^*$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $|\sigma(\omega,\mathbb{R}^d)| = k$, i.e. $\sigma(\omega,\mathbb{R}^d) = \{\sigma_1^\omega,\cdots,\sigma_k^\omega\}$ with $\sigma_1^\omega,\cdots,\sigma_k^\omega \in [0,\infty[$;
- (A_{17}) for every $T \in [0,\infty[$ there exists $M > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $(t,x) \in [0,T]\times\mathbb{R}^d$, $t - \sigma(\omega,x) \in [-M,T]$;
- (A_{18}) for $\mathbb{P}$-a.e. $\omega \in \Omega$, every $x \in \mathbb{R}^d$ and every $z \in \mathbb{Z}^d$, $\sigma(T_z\omega,x) = \sigma(\omega,x+z)$.

Let $T>0$. For each $\omega\in\Omega$ and each $\varepsilon>0$, let $\mathscr{R}_\varepsilon^\omega : C_{\rm c}(]-\infty,T];L^2(O))\to L^2([0,T];L^2(O))$ be a time delays operator associated with $\{\boldsymbol{m}_{\varepsilon,t}^\omega\}_{t\geqslant 0}\subset\mathscr{M}_1^+(\mathbb{R};L^\infty(O))$ defined by

$$\boldsymbol{m}_{\varepsilon,t}^\omega := \frac{1}{k}\delta_{t-\sigma\left(\omega,\frac{\cdot}{\varepsilon}\right)}.$$

Then, $\mathscr{R}_\varepsilon^\omega : C_{\rm c}(]-\infty,T];L^2(O))\to L^2([0,T];L^2(O))$ is given by

$$\mathscr{R}_\varepsilon^\omega u(t)=\frac{1}{k}u\left(t-\sigma\left(\omega,\frac{\cdot}{\varepsilon}\right)\right).$$

For each $\varepsilon>0$, let $F_\varepsilon:\Omega\times[0,\infty[\times L^2(O)\times L^2(O)\to L^2(O)$ be given by (5.2) and let $\mathscr{J}_\varepsilon^g:\Omega\times L^2(O)\to[0,\infty]$ be given by (3.10) and, for each $\omega\in\Omega$, let $\overline{\rho}^\omega\in]0,\infty[$ be given by Lemma 5.2, let $\psi_\varepsilon^\omega\in C_{\rm c}(]-\infty,0];L^2(O))$ and consider the following Dirichlet-Cauchy nonlocal reaction-diffusion problem with age structure of gradient flow type:

$$(\mathscr{P}_{\varepsilon,\omega}^{\rm AS})\begin{cases}\dfrac{du_\varepsilon^\omega}{dt}(t)+\nabla\mathscr{J}_\varepsilon^g(\omega,u_\varepsilon^\omega(t))=F_\varepsilon(\omega,t,u_\varepsilon^\omega(t),\mathscr{R}_\varepsilon^\omega u_\varepsilon^\omega(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,T]\\[2ex] u_\varepsilon^\omega(t)=\psi_\varepsilon^\omega(t)\in L^2(O;[0,\overline{\rho}^\omega]) \text{ for all } t\in]-\infty,0].\end{cases}$$

Note that $(\mathscr{P}_{\varepsilon,\omega}^{\rm AS})$ can be rewritten as follows:

$$(\mathscr{P}_{\varepsilon,\omega}^{\rm AS})\begin{cases}\dfrac{\partial u_\varepsilon^\omega}{\partial t}(t,x)-\dfrac{1}{\varepsilon^{d+1}}\displaystyle\int_O J\left(\omega,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{x-y}{\varepsilon}\right)(u_\varepsilon^\omega(t,y)-u_\varepsilon^\omega(t,x))\,dy\\ -\dfrac{1}{\varepsilon^{d+1}}\displaystyle\int_{O^J\setminus\overline{O}} J\left(\omega,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{x-y}{\varepsilon}\right)(g(y)-u_\varepsilon^\omega(t,x))\,dy\\ =b\Big(\omega,t,\dfrac{x}{\varepsilon},\dfrac{1}{k}u_\varepsilon^\omega\left(t-\sigma\left(\omega,\dfrac{x}{\varepsilon}\right),x\right)\Big)-d\Big(\omega,t,\dfrac{x}{\varepsilon},u_\varepsilon^\omega(t,x)\Big)+a\Big(\omega,t,\dfrac{x}{\varepsilon}\Big) \text{ in } [0,T]\times O\\[2ex] u_\varepsilon^\omega(t,x)=\psi_\varepsilon^\omega(t,x)\in[0,\overline{\rho}^\omega] \text{ in }]-\infty,0]\times O,\end{cases}$$

where $J:\Omega\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\to[0,\infty[$ satisfies ($\rm PNL_1$)–($\rm PNL_2$) and $g\in L^2(O^J\setminus O)$ verifies (3.4) with $O^J:=O+\operatorname{supp}(\overline{J})=O+\overline{B}_{R_J}(0)$ (see §3.1 for more details).

In order to apply Corollary 3.16 we need to establish the mixing convergence (see Definition 3.10) of $\{(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega)\}_{\varepsilon>0}$ which is the object of the following lemma.

Lemma 5.7. *Assume that* ($\rm A_1$)–($\rm A_{18}$) *hold. Given* $T>0$ *we further assume that*:

($\rm A_{19}$) *for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$*, the functions* $a(\omega,\cdot,x)$*,* $b(\omega,\cdot,x,\xi)$ *and* $d(\omega,\cdot,x,\zeta)$ *are Lipschitz continuous uniformly with respect to* x*,* (x,ξ) *and* (x,ζ) *respectively*;

($\rm A_{20}$) *for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$ *and for each* $\varepsilon>0$*,* $\operatorname{supp}(\psi_\varepsilon^\omega)\subset[-M,0]$ *where* $M>0$ *is given by* ($\rm A_{17}$).

Then, for $\mathbb{P}$*-a.e.* $\omega\in\Omega$*,*

$$(F_\varepsilon(\omega,\cdot,\cdot,\cdot),\mathscr{R}_\varepsilon^\omega)\xrightarrow{\rm mix}F(\omega,\cdot),$$

i.e.

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(v)\rightharpoonup F(\omega,v) \text{ in } L^2([0,T];L^2(O)) \text{ for all } v\in C_{\rm c}([-M,T];L^2(O;[0,\overline{\rho}^\omega])),$$

where $F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega} : C_{\rm c}([-M,T];L^2(O;[0,\overline{\rho}^\omega])) \to L^2([0,T];L^2(O))$ *is given by* (3.13) *and* $F(\omega,\cdot) : C_{\rm c}([-M,T];L^2(O;[0,\overline{\rho}^\omega])) \to L^2([0,T];L^2(O))$ *is defined by*

$$F(\omega,v)(t) := \mathbb{E}^{\mathscr{I}}\left[\int_Y b\left(\cdot,t,y,v\left(\frac{1}{k}(t-\sigma(\cdot,y))\right)\right)dy\right](\omega) - \mathbb{E}^{\mathscr{I}}\left[\int_Y d(\cdot,t,y,v(t))dy\right](\omega) + \mathbb{E}^{\mathscr{I}}\left[\int_Y a(\cdot,t,y)dy\right](\omega). \tag{5.5}$$

Proof of Lemma 5.7. Let $S_{\rm C}^{\rm tp}([-M,T];L^2(O))$ denote the subset of $C([-M,T];L^2(O))$ made up of step functions. As $L^2(O)$ is separable it is easy to show that there is a countable set $\mathscr{D} \subset S_{\rm C}^{\rm tp}([-M,T];L^2(O))$ which is dense in $C([-M,T];L^2(O))$. Fix any $w \in \mathscr{D}$, i.e. $w = \sum_{i\in I} \mathbb{1}_{B_i} w_i$ with $\mathrm{card}(I) < \infty$, $\{w_i\}_{i\in I} \subset L^2(O)$ and $\{B_i\}_{i\in I}$ is disjoint covering of intervals in $[-M,T]$. Taking (A_4) and (A_{10}) into account, it easily seen that for $\mathbb{P}$-a.e. $\omega \in \Omega$, every $t \in [0,T]$ and every $\varepsilon > 0$,

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(w)(t) = \sum_{i\in I} \mathbb{1}_{B_i}\left(\frac{1}{k}\left(t-\sigma\left(\omega,\frac{\cdot}{\varepsilon}\right)\right)\right) b\left(\omega,t,\frac{\cdot}{\varepsilon},w_i\right) - \sum_{i\in I}\mathbb{1}_{B_i}(t) d\left(\omega,t,\frac{\cdot}{\varepsilon},w_i\right) + a\left(\omega,t,\frac{\cdot}{\varepsilon}\right).$$

From $((\mathrm{A}_2))$, (A_5), (A_7),(A_{11}), (A_{13})–(A_{14}) and (A_{18}), by using Chabi-Michaille's theorem (see [CM94, Theorem 4.2]) we can assert that for any $t \in [0,T]$ there exists $\Omega_{w,t} \in \mathscr{F}$ with $\mathbb{P}(\Omega_{w,t}) = 1$ such that $F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(w)(t) \rightharpoonup F(\omega,w)(t)$ in $L^2(O)$ for all $\omega \in \Omega_{w,t}$. Setting $\Omega' := \cap_{(w,t)\in\mathscr{D}\times\mathbb{Q}\cap[0,T]}\Omega_{w,t}$ we have $\Omega' \in \mathscr{F}$ with $\mathbb{P}(\Omega') = 1$ and

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(w)(t) \rightharpoonup F(\omega,w)(t) \text{ in } L^2(O) \text{ for all } \omega \in \Omega', \text{ all } w \in \mathscr{D} \text{ and all } t \in \mathbb{Q}\cap[0,T]. \tag{5.6}$$

By (A_{16}) and (A_{19}) it can be in fact established that (5.6) holds for $\mathscr{L}^1$-a.e. $t \in [0,T]$. Taking (A_2), (A_7) and (A_{13}) into account, from Lebesgue's dominated convergence theorem it follows that for every $\omega \in \Omega'$,

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(w) \rightharpoonup F(\omega,w) \text{ in } L^2([0,T];L^2(O)) \text{ for all } w \in \mathscr{D}. \tag{5.7}$$

Fix $\omega \in \Omega'$ and consider $v \in C_{\rm c}([-M,T];L^2(O;[0,\overline{\rho}^\omega]))$. First of all, by density, there exists $\{w_n\}_{n\geqslant 1} \subset \mathscr{D}$ with $w_n \in L^2(O;[0,\overline{\rho}^\omega])$ such that

$$w_n \to v \text{ in } C([-M,T];L^2(O)), \tag{5.8}$$

and it is clear that

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(v) = F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(w_n) + \left(F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(v) - F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(w_n)\right) \text{ for all } \varepsilon > 0 \text{ and all } n \geqslant 1. \tag{5.9}$$

On the other hand, by using (5.8), (A_3) and (A_9) we see that

$$\overline{\lim_{n\to\infty}}\,\overline{\lim_{\varepsilon\to 0}} \left\|F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(v) - F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(w_n)\right\|_{L^2([0,T];L^2(O))} = 0, \tag{5.10}$$

and by (5.7) we deduce that

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(w_n) \rightharpoonup F(\omega,w_n) \text{ in } L^2([0,T];L^2(O)) \text{ for all } n \geqslant 1. \tag{5.11}$$

Moreover, by using (3.17) with $u_1 = v$ and $u_2 = w_n$, (5.8) and the conditional Lebesgue's dominated convergence theorem it can be established that

$$\lim_{n\to\infty} \|F(\omega,v) - F(\omega,w_n)\|_{L^2([0,T];L^2(O))} = 0. \tag{5.12}$$

From (5.10), (5.11) and (5.12), letting $\varepsilon \to 0$ and $n \to \infty$ in (5.9) we conclude that

$$F_\varepsilon^{\omega,\mathscr{R}_\varepsilon^\omega}(v) \rightharpoonup F(\omega, v) \text{ in } L^2([0,T]; L^2(O)),$$

and the proof is complete. ■

According to Lemmas 5.2 and 5.7, as a direct consequence of Corollary 3.16 we obtain the following homogenization result for $(\mathscr{P}_{\varepsilon,\omega}^{\mathrm{AS}})$ as $\varepsilon \to 0$.

Corollary 5.8. *For $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $\varepsilon > 0$, let u_ε^ω be the unique solution of $(\mathscr{P}_{\varepsilon,\omega}^{\mathrm{AS}})$, see Lemma* 5.2 *and Corollary* 3.5*, and assume that the assumptions* (H_1^ω)–(H_2^ω) *of Theorem* 3.8 *hold. Suppose furthermore that* (A_1)–(A_{20}) *are satisfied. Then*:

- $u_\varepsilon^\omega \to u^\omega$ *in* $C([0,T]; L^2(O))$;
- $\dfrac{du_\varepsilon^\omega}{dt} \rightharpoonup \dfrac{du^\omega}{dt}$ *in* $L^2([0,T]; L^2(O))$;
- $0 \leqslant u^\omega(t) \leqslant \sup\limits_{\varepsilon>0} \overline{y}_\varepsilon^\omega(T)$ *for all* $t \in [0,T]$;
- u^ω *extended by* ψ^ω *in* $]-\infty, 0]$ *is the unique solution of*

$$\begin{cases} \dfrac{du^\omega}{dt}(t) + \nabla \mathscr{J}_{\mathrm{hom}}^g(\omega, u^\omega(t)) = F(\omega, u^\omega)(t) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \\ u^\omega(t) = \psi^\omega(t) \in L^2(O; [0, \overline{\rho}^\omega]) \text{ for all } t \in]-\infty, 0] \text{ and } \psi^\omega(0) \in \mathrm{dom}(\mathscr{J}_{\mathrm{hom}}^g(\omega, \cdot)) \end{cases}$$

with $F : \Omega \times C_{\mathrm{c}}([-M,T]; L^2(O; [0, \overline{\rho}^\omega])) \to L^2([0,T]; L^2(O))$ *given by* (5.5).

Appendix A. Vector measures

Let E be a finite dimensional normed space, let $\mathscr{B}(E)$ be the σ-algebra of Borel subsets of E and let Y be a Banach space. We begin with the following two definitions.

Definition A.1. We say that $\boldsymbol{m} : \mathscr{B}(E) \to Y$ is a Y-valued vector measure on E if $\boldsymbol{m}$ is σ-additive.

Let X be another Banach space.

Definition A.2. By a step function from E to X we mean $s : E \to X$ given by

$$s = \sum_{i \in I} \mathbb{1}_{A_i} a_i,$$

where I is a finite set, $\{A_i\}_{i\in I} \subset \mathscr{B}(E)$ and $\{a_i\}_{i\in I} \subset X$. We denote the class of step functions from M to X by $S^{\mathrm{tp}}(E; X)$.

Assume further that there exists a continuous bilinear map $B : Y \times X \to X$. Then, we define the integral of $s \in S^{\mathrm{tp}}(E; X)$ with respect to $\boldsymbol{m}$ by

$$\int_E s(t) d\boldsymbol{m}(t) := \sum_{i \in I} B(\boldsymbol{m}(A_i), a_i) \in X. \tag{A.1}$$

The definition of $\int_E s(t) d\boldsymbol{m}(t)$ only depends on s (and is independent $(\{A_i\}_{i\in I}, \{a_i\}_{i\in I})$, see [Din67, Chapter II, §7, Proposition 1, pp. 107]). Given a X-valued vector measure $\boldsymbol{m}$ on E,

let $\|\boldsymbol{m}\| : \mathscr{B}(E) \to [0,\infty]$ be defined by

$$\|\boldsymbol{m}\|(A) := \sup\left\{\sum_{i\in I} \|\boldsymbol{m}(A_i)\| : |I| < \infty, \{A_i\}_{i\in I} \subset \mathscr{B}(E) \text{ is pairwise disjoint and } A = \bigcup_{i\in I} A_i\right\}.$$

The set function $\|\boldsymbol{m}\|$ is called the variation of $\boldsymbol{m}$.

Definition A.3. We say that $\boldsymbol{m}$ has finite variation if $\|\boldsymbol{m}\|(E) < \infty$.

The following proposition makes clear the interest of Definition A.3.

Proposition A.4. *If $\boldsymbol{m}$ has finite variation then $\|\boldsymbol{m}\|$ is a positive Borel measure and, for every $s \in S^{\mathrm{tp}}(E;X)$,*

$$\left\|\int_E s(t)d\boldsymbol{m}(t)\right\|_X \leqslant C_B \sup_{t\in E} \|s(t)\|_X \|\boldsymbol{m}\|(E),$$

where C_B denotes the norm of the continuous bilinear map B.

Let $S^{\mathrm{tp}}_\infty(E;X)$ be the space of functions from E to X which are uniform limit of step functions. Then, we can extend the integral with respect to $\boldsymbol{m}$ defined in (A.1) for step functions $s \in S^{\mathrm{tp}}(E;X)$ to functions $u \in S^{\mathrm{tp}}_\infty(E;X)$ by defining $\int_E u(t)d\boldsymbol{m}(t)$ as the limit in X of the Cauchy sequence $\left\{\int_E s_n(t)d\boldsymbol{m}(t)\right\}_{n\geqslant 1}$ where $\{s_n\}_{n\geqslant 1} \subset S^{\mathrm{tp}}(E;X)$ is such that $\lim\limits_{n\to\infty} \sup_{t\in E} \|s_n(t) - u(t)\|_X = 0$, i.e.

$$\int_E u(t)d\boldsymbol{m}(t) := \lim_{n\to\infty} \int_E s_n(t)d\boldsymbol{m}(t).$$

This extension clearly does not depend on $\{s_n\}_{n\geqslant 1}$. Moreover, we have

Proposition A.5. *If $\boldsymbol{m}$ has finite variation then, for every $u \in S^{\mathrm{tp}}_\infty(E;X)$,*

$$\left\|\int_E u(t)d\boldsymbol{m}(t)\right\|_X \leqslant \int_E \|u(t)\|_X d\|\boldsymbol{m}\|(t).$$

Proposition A.5 is easy to establish for $u \in S^{\mathrm{tp}}(E;X)$, see [Din67, Chapter III, Section 7, §2, pp. 110]. Then, we argue by density. We refer to [Din67] for a comprehensive review on vector measures.

Appendix B. Elements of Legendre-Fenchel calculus

Let X be a normed space and let X^* be its topological dual. In what follows, for any $u \in X$ and any $u^* \in X^*$, we write $u^*(u) = \langle u^*, u\rangle$. We begin with the following definition.

Definition B.1. Let $\Phi : X \to]-\infty,\infty]$ be a proper[5] function. The Legendre-Fenchel conjugate (or the conjugate) of Φ is the function $\Phi^* : X^* \to]-\infty,\infty]$ defined by

$$\Phi^*(u^*) := \sup\left\{\langle u^*, u\rangle - \Phi(u) : u \in X\right\}.$$

(As Φ is proper and $\Phi > -\infty$ we have $\Phi^* > -\infty$.) The Legendre-Fenchel biconjugate (or the biconjugate) of Φ is the function $\Phi^{**} : X \to [-\infty,\infty]$ defined by

$$\Phi^{**}(u) := \sup\left\{\langle u^*, u\rangle - \Phi^*(u^*) : u^* \in X^*\right\}.$$

[5]We say that $\Phi : X \to]-\infty,\infty]$ is proper if (its effective domain) $\mathrm{dom}(\Phi) := \{u \in X : \Phi(u) < \infty\} \neq \varnothing$.

(Since $\Phi^* > -\infty$, $u^* \in \mathrm{dom}(\Phi^*)$ if and only if there exists $\alpha \in \mathbb{R}$ such that $\Phi^*(u^*) \leqslant \alpha$, i.e. $\Phi(u) \geqslant \langle u^*, u\rangle - \alpha$ for all $u \in X$. Hence, if Φ admits a continuous affine minorant function[6] then Φ^* is proper and $\Phi^{**} > -\infty$.) The following theorem gives the main properties of the Legendre-Fenchel conjugate and biconjugate (see [ABM14, §9.3, pp. 343] for more details).

Theorem B.2. *Let $\Phi : X \to]-\infty, \infty]$ be a proper function.*

(a) *If Φ is convex and lower semicontinuous then Φ^* is proper, convex and lower semicontinuous.*

(b) *(Legendre-Fenchel's inequality.) For every $u \in X$ and every $u^* \in X^*$,*

$$\Phi(u) + \Phi^*(u^*) - \langle u^*, u\rangle \geqslant 0.$$

(c) *(Fenchel-Moreau-Rockafellar's theorem.) If Φ is convex and lower semicontinuous then*

$$\Phi^{**} = \Phi.$$

(d) *If Φ is convex and admits a continuous affine minorant function then*

$$\Phi^{**} = \overline{\Phi},$$

where $\overline{\Phi}$ denotes the lower semicontinuous envelope of Φ.

Here is the definition of the subdifferential of a function.

Definition B.3. Let $\Phi : X \to]-\infty, \infty]$ be a proper function. The subdifferential of Φ is the multivalued operator $\partial\Phi : X \rightrightarrows X^*$ defined by

$$\partial\Phi(u) := \big\{u^* \in X^* : \Phi(v) \geqslant \Phi(u) + \langle u^*, v - u\rangle \text{ for all } v \in X\big\}.$$

(Note that $\mathrm{dom}(\Phi) \supset \mathrm{dom}(\partial\Phi) := \big\{u \in X : \partial\Phi(u) \neq \varnothing\big\}$.)

For the subdifferentials of convex functions we have the following result (see [ABM14, §9.5, pp. 355 and Lemma 17.4.1, pp. 737] for more details).

Proposition B.4. *Let $\Phi : X \to]-\infty, \infty]$ be a proper and convex function.*

(a) *If Φ is Fréchet-differentiable at $u \in X$ then*

$$\partial\Phi(u) = \big\{\nabla\Phi(u)\big\}.$$

(b) *(Fenchel's extremality relation.) If Φ is lower semicontinuous then*

$$u^* \in \partial\Phi(u) \iff \Phi(u) + \Phi^*(u^*) - \langle u^*, u\rangle = 0.$$

(c) *(Brønsted-Rockafellar's lemma) If Φ is lower semicontinuous then*

$$\overline{\mathrm{dom}(\partial\Phi)} = \overline{\mathrm{dom}(\Phi)}.$$

[6] This is true if $\Phi : X \to]-\infty, \infty]$ is a proper, convex and lower semicontinuous function, because Φ is then equal to the supremum of all its continuous affine minorant functions.

Appendix C. Mosco-convergence

Let X be a Banach space and let X^* be its topological dual. In what follows, "$\to$" (resp. "$\rightharpoonup$") denotes the strong (resp. the weak) convergence. We begin with the definition of De Giorgi Γ-convergence (see [DM93, BD98, Bra06] for more details).

Definition C.1. Let $\Phi : X \to]-\infty, \infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty, \infty]$. We say that $\{\Phi_\varepsilon\}_{\varepsilon>0}$ strongly Γ-converges (resp. weakly Γ-converges) to Φ, and we write

$$\Phi = \Gamma_{\mathrm{s}}\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\Gamma_{\mathrm{s}}} \Phi \quad (\text{resp. } \Phi = \Gamma_{\mathrm{w}}\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\Gamma_{\mathrm{w}}} \Phi),$$

if the following two assertions hold:

- for every $u \in X$, $\Gamma_{\mathrm{s}}\text{-}\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) \geqslant \Phi(u)$(resp. $\Gamma_{\mathrm{w}}\text{-}\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) \geqslant \Phi(u)$) with

$$\Gamma_{\mathrm{s}}\text{-}\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) := \inf\left\{\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\right\}$$

$$(\text{resp. } \Gamma_{\mathrm{w}}\text{-}\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) := \inf\left\{\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \rightharpoonup u\right\})$$

  or equivalently, for every $u \in X$ and every $\{u_\varepsilon\}_{\varepsilon>0} \subset X$, if $u_\varepsilon \to u$ (resp. $u_\varepsilon \rightharpoonup u$) then

$$\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) \geqslant \Phi(u);$$

- for every $u \in X$, $\Gamma_{\mathrm{s}}\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) \leqslant \Phi(u)$(resp. $\Gamma_{\mathrm{w}}\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) \leqslant \Phi(u)$) with

$$\Gamma_{\mathrm{s}}\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) := \inf\left\{\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\right\}$$

$$(\text{resp. } \Gamma_{\mathrm{w}}\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) := \inf\left\{\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \rightharpoonup u\right\})$$

  or equivalently, for every $u \in X$ there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset X$ such that $u_\varepsilon \to u$ (resp. $u_\varepsilon \rightharpoonup u$) and

$$\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) \leqslant \Phi(u).$$

From Γ-convergence we can define Mosco-convergence (which was introduced by Mosco, see [Mos71]).

Definition C.2. Let $\Phi : X \to]-\infty, \infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty, \infty]$. We say that $\{\Phi_\varepsilon\}_{\varepsilon>0}$ Mosco-converges to Φ, and we write

$$\Phi = \mathrm{M}\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\mathrm{M}} \Phi,$$

if $\Phi = \Gamma_{\mathrm{s}}\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon = \Gamma_{\mathrm{w}}\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon$ or equivalently $\Gamma_{\mathrm{s}}\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon \leqslant \Phi \leqslant \Gamma_{\mathrm{w}}\text{-}\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon$.

From Definition C.2, it is easy to see that under a suitable compactness condition strong Γ-convergence is equivalent to Mosco-convergence.

Proposition C.3. *Let $\Phi : X \to]-\infty, \infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty, \infty]$. Assume that the following compactness condition hold*:

- *for every* $\{u_\varepsilon\}_{\varepsilon>0} \subset X$, *if* $\sup_{\varepsilon>0} \Phi_\varepsilon(u_\varepsilon) < \infty$ *then* $\{u_\varepsilon\}_{\varepsilon>0}$ *is strongly relatively compact in* X.

Then, $\Phi_\varepsilon \xrightarrow{\Gamma_s} \Phi$ *if and only if* $\Phi_\varepsilon \xrightarrow{M} \Phi$.

As stated in the following theorem due to Mosco (see [Mos71, Theorem 1]), in the reflexive case and for lower semicontinuous, convex and proper functions, the Legendre-Fenchel transform is continuous with respect to Mosco-convergence.

Theorem C.4. *Let* $\Phi : X \to]-\infty, \infty]$ *be a proper, convex and lower semicontinuous function and, for each* $\varepsilon > 0$, *let* $\Phi_\varepsilon : X \to]-\infty, \infty]$ *be a proper, convex and lower semicontinuous function. If* X *is reflexive then* $\Phi_\varepsilon \xrightarrow{M} \Phi$ *if and only if* $\Phi_\varepsilon^* \xrightarrow{M} \Phi^*$.

The following result allows to pass from Mosco-convergence in X to Mosco-convergence in $L^2([0,T];X)$ (see [AHMM22, Lemma 2.6, pp. 50] for a proof).

Theorem C.5. *Fix* $T > 0$ *and assume that* X *is a Hilbert space. Let* $\Phi : X \to [0,\infty]$ *be a proper, convex and lower semicontinuous function, let* $\Theta : L^2([0,T];X) \to [0,\infty]$ *be defined by*

$$\Theta(u) := \int_0^T \Phi(u(t))dt$$

and, for each $\varepsilon > 0$, *let* $\Phi_\varepsilon : X \to [0,\infty]$ *be a lower semicontinuous, proper and convex function and let* $\Theta_\varepsilon : L^2([0,T];X) \to [0,\infty]$ *be defined by*

$$\Theta_\varepsilon(u) := \int_0^T \Phi_\varepsilon(u(t))dt.$$

If $\Phi_\varepsilon \xrightarrow{M} \Phi$ *then* $\Theta_\varepsilon \xrightarrow{M} \Theta$.

Appendix D. Grönwall's lemma

In the paper we use the following version of the so-called Grönwall's lemma (for a proof we refer to [AHMM22, Lemma A.1, pp. 277]).

Lemma D.1. *Let* $T > 0$, *let* $a \in [0,\infty[$, *let* $\mathrm{m} \in L^1([0,T])$ *be such that* $\mathrm{m}(s) \geqslant 0$ *for* $\mathscr{L}^1$*-a.a.* $s \in [0,T]$ *and let* $\phi \in C([0,T];\mathbb{R})$ *be such that* $\phi(s) \leqslant a + \int_0^s \phi(t)\mathrm{m}(t)dt$ *for all* $s \in [0,T]$. *Then* $\phi(s) \leqslant a e^{\int_0^s \mathrm{m}(t)dt}$ *for all* $s \in [0,T]$.

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