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# Necessary and sufficient Tauberian condition for both Cesàro and Abel summability

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## Abstract

We prove that the Weakly-Vanishing Mean Oscillation (W-VMO) property of a sequence or a series is a necessary and sufficient condition under which either the convergence  $(\mathbf{C}_0)$  or the Cesàro summability  $(\mathbf{C}_1)$  follows from the Abel summability  $(\mathbf{A}_0)$  to the same limit. Hence, this result shows the Tauberian converse, with the largest possible space of sequences, of both the Abel (1826) theorem, *i.e.*  $(\mathbf{C}_0) \Rightarrow (\mathbf{A}_0)$ , and the Frobenius (1880) theorem, *i.e.*  $(\mathbf{C}_1) \Rightarrow (\mathbf{A}_0)$ . The inversion of the Cesàro summability  $(\mathbf{C}_1) \Rightarrow (\mathbf{C}_0)$  is also addressed within the same unified setting and solved with the necessary and sufficient W-VMO condition.

**Keywords:** Abel summability, Cesàro summability, Power series, Tauberian theory, Weakly-vanishing mean oscillation

**2020 MSC Classification:** 40A05 , 40E05 (primary) , 42A16 , 42A20 , 42A24 (secondary)

## 1 Introduction and historical review

Founding our knowledge in the monographs by Hardy [5] and Korevaar [8], there exists in the literature no necessary and sufficient condition to ensure the Tauberian converse of Abel summability  $(A_0)$ <sup>1</sup> of a sequence  $(u_n)_{n \in \mathbb{N}}$ , except the following one. Derived from Tauber (1897) 'second theorem' [18], an Abel summable sequence  $(u_n)$

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<sup>1</sup>The definitions of Cesàro and Abel summabilities are precised in Theorems 3.1 and 4.1, respectively.

is convergent  $(C_0)$  if and only if, see [8][Theorem 5.3]:

$$\lim_{n \rightarrow +\infty} (u_n - \sigma_n) = 0, \quad \text{where} \quad \sigma_n := \frac{1}{n+1} \sum_{k=0}^n u_k. \quad (1)$$

This shows that the Abel summability  $(A_0)$  and the Cesàro summability  $(C_1)$  are intimately connected. However, there are many sufficient conditions, either two-sided boundedness in the case of complex coefficients or one-sided for real coefficients that are possibly relaxed. In this paper, we present a unified framework to tackle the inversion of both Cesàro and Abel summabilities of a sequence or a series, indifferently composed of real or complex values. In both cases of summability, our main results in Theorems 3.1 and 4.1 show that the Tauberian converses hold true if and only if the sequence  $(u_n)$  is of Weakly-Vanishing Mean Oscillation (W-VMO), see (12) in Definition 2.

Without being exhaustive, we remind below the main Tauberian conditions found in the literature and observe how they were relaxed with time. Among the existing sufficient conditions to ensure that  $(A_0) \Rightarrow (C_0)$  holds, Landau (1913) [10] proposed the two-sided condition of *slow oscillation* of  $(u_n)$ :

$$\lim_{n \rightarrow +\infty} |u_m - u_n| = 0, \quad \text{for} \quad 1 < \frac{m}{n} \xrightarrow{n \rightarrow +\infty} 1. \quad (2)$$

This can be reformulated as in (14). For real coefficients, a corresponding unilateral condition of so-called *slow decrease*<sup>2</sup>:

$$\liminf_{n \rightarrow +\infty} (u_m - u_n) \geq 0, \quad \text{for} \quad 1 < \frac{m}{n} \xrightarrow{n \rightarrow +\infty} 1, \quad (3)$$

was also shown to be sufficient by Schmidt (1925) [16]. Let us notice that (2) implies (3) for real values but the converse is clearly not true. The emblematic '*big  $\mathcal{O}$ '-condition* of Hardy and Littlewood  $n(u_n - u_{n-1}) = \mathcal{O}(1)$ , see Littlewood (1911) [11, 14], implies (2), e.g. see Proposition 2.1. This is of course the case for the stronger condition  $n(u_n - u_{n-1}) = o(1)$  of Tauber (1897) '*first theorem*' [18]. In the real case, the '*big  $\mathcal{O}$ '-condition* can be replaced by the equivalent one-sided condition, see Hardy and Littlewood (1914) [4]:

$$n(u_n - u_{n-1}) \geq -C \quad (C > 0), \quad \forall n \geq 1, \quad (4)$$

that implies (3). Several alternative Tauberian conditions have been introduced by Szász (1951) [17] or earlier, the best possible being below:

$$\lim_{n \rightarrow +\infty} \sum_{k=n+1}^{n+m} (|u_k - u_{k-1}| - (u_k - u_{k-1})) = 0, \quad \text{as} \quad \frac{m}{n} \xrightarrow{n \rightarrow +\infty} 0. \quad (5)$$

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<sup>2</sup>The terminology is misleading since only the decrease of such sequences is restricted, not their increase and any increasing sequence is of '*slow decrease*'.

For the 'Abel to Cesàro summability', the well-known one-sided condition  $(u_n) \geq -C$  for any constant  $C \geq 0$  has been proved by Hardy and Littlewood (1914) [4] to be sufficient to ensure that  $(A_0) \Rightarrow (C_1)$  holds. Later, Bingham (1985) [1] has shown that it can be replaced by a weaker one which involves boundedness from below in some average sense:

$$\lim_{\lambda \rightarrow 1^+} \left( \liminf_{n \rightarrow +\infty} \min_{n < m \leq \lambda n} \frac{1}{n} \sum_{k=n+1}^m u_k \right) \geq 0. \quad (6)$$

Concerning the 'Cesàro summability to convergence', Hardy (1910) [3] early proved that the 'big  $\mathcal{O}$ '-condition  $n(u_n - u_{n-1}) = \mathcal{O}(1)$  is sufficient to ensure that  $(C_1) \Rightarrow (C_0)$  holds. At the same time, Landau (1910) [9] has shown that the unilateral boundedness (4) also works. Moreover, the weaker condition of slow oscillation (2), or slow decrease (3) for the real case, is proved to be sufficient too in [5][Theorem 68]. However, Maric and Tomić (1984) [12] have still relaxed the latter condition for real coefficients with the two one-sided conditions below:

$$\begin{cases} \liminf_{p \rightarrow +\infty} \frac{1}{p} \sum_{k=1}^p (u_{n+k} - u_n) \geq 0, \\ \liminf_{p \rightarrow +\infty} \frac{1}{p} \sum_{k=1}^{p-1} (u_n - u_{n-k}) \geq 0, \end{cases} \quad \text{with } \frac{p}{n} \xrightarrow{n \rightarrow +\infty} 0. \quad (7)$$

More recently, Móricz (1994) [13] has proved that  $(C_1) \Rightarrow (C_0)$  holds for a real sequence  $(u_n)$  if and only if the two following one-sided conditions are satisfied:

$$\begin{cases} \limsup_{\lambda \rightarrow 1^+} \left( \liminf_{n \rightarrow +\infty} \frac{1}{[n\lambda] - n} \sum_{k=n+1}^{[n\lambda]} (u_k - u_n) \right) \geq 0, \\ \limsup_{\lambda \rightarrow 1^-} \left( \liminf_{n \rightarrow +\infty} \frac{1}{n - [n\lambda]} \sum_{k=[n\lambda]+1}^n (u_n - u_k) \right) \geq 0, \end{cases} \quad (8)$$

where  $[n\lambda]$  denotes the integer part of the product  $n\lambda$ . For the general case of complex coefficients, Móricz (1994) [13] also asserts that the condition (9) below

$$\liminf_{\lambda \rightarrow 1^+} \limsup_{n \rightarrow +\infty} \left| \frac{1}{[n\lambda] - n} \sum_{k=n+1}^{[n\lambda]} (u_k - u_n) \right| = 0, \quad (9)$$

or symmetrically :

$$\liminf_{\lambda \rightarrow 1^-} \limsup_{n \rightarrow +\infty} \left| \frac{1}{n - [n\lambda]} \sum_{k=[n\lambda]+1}^n (u_n - u_k) \right| = 0,$$

is necessary and sufficient but left the proof to the reader. However, it seems that the conditions (8) or (9) have not yet been investigated as Tauberian conditions for the Abel summability. Hereafter, we shall prove the result in a different way in Theorem 3.1 with the necessary and sufficient condition (12) that is slightly weaker than (9), by introducing the so-called generalized Cesàro means and representation (11) of any sequence  $(u_n)$ . Further, we shall see that (11) will also play a crucial role to prove Theorem 4.1 and its Corollary 4.2 for the Tauberian converses of Abel summability  $(A_0)$  with the necessary and sufficient condition (12) for a (W-VMO) sequence.

The paper is hereafter organized as follows. Some preliminary results are shown in the next Section 2. Then, the Tauberian inversion of Cesàro summability, *i.e.*  $(C_1) \Rightarrow (C_0)$  is addressed in Section 3. Finally, the Tauberian inversion of Abel summability, *i.e.*  $(A_0) \Rightarrow (C_0)$  or  $(A_0) \Rightarrow (C_1)$  is analyzed in Section 4.

## 2 Preliminary results

We first define the Weakly-Vanishing Mean Oscillation (W-VMO) property of any sequence of complex numbers, which mean roughly speaking that the mean oscillation of the sequence tends to zero at infinite. A stronger counterpart has been introduced by Sarason (1975) [15] for the space of Vanishing Mean Oscillation (VMO) functions as a subspace of the set of Bounded Mean Oscillation (BMO) functions defined in [6] by John and Nirenberg (1961). Then, we give some technical results that are interesting by themselves and will be crucial further.

**Notation 1.** For any real number  $\lambda \in \mathbb{R}$ ,  $[\lambda]$  denotes its (lower) integer part, *i.e.*  $[\lambda] \in \mathbb{Z}$  such that  $[\lambda] \leq \lambda < [\lambda] + 1$ .

**Definition 1 (Mean Oscillation).**

For any sequence  $(u_n)_{n \in \mathbb{N}}$  of complex or real numbers, the sequence of mean oscillation of  $(u_n)$  is defined by:

$$\omega_n^+(\lambda) := \frac{1}{[n\lambda] - n} \sum_{k=n+1}^{[n\lambda]} (u_n - u_k), \quad \forall n \geq \frac{1}{\lambda - 1}, \quad \forall \lambda > 1,$$

such that  $[n\lambda] - n > 0$ , or symmetrically :

$$\omega_n^-(\lambda) := \frac{1}{n - [n\lambda]} \sum_{k=[n\lambda]+1}^n (u_n - u_k), \quad \forall n \geq 1, \quad \forall \lambda \in ]0, 1[.$$

Therefore, any sequence  $u := (u_n)_{n \in \mathbb{N}}$  admits the following decomposition :

$$\left\{ \begin{array}{l} u_n = \sigma_n^+(\lambda) + \omega_n^+(\lambda), \quad \text{or} \quad u_n = \sigma_n^-(\lambda) + \omega_n^-(\lambda), \quad \forall n \geq 1, \\ \text{where: } \sigma_n^+(\lambda) := \frac{1}{[n\lambda] - n} \sum_{k=n+1}^{[n\lambda]} u_k, \quad \forall n \geq \frac{1}{\lambda - 1}, \quad \forall \lambda > 1, \\ \sigma_n^-(\lambda) := \frac{1}{n - [n\lambda]} \sum_{k=[n\lambda]+1}^n u_k, \quad \forall n \geq 1, \quad \forall \lambda \in ]0, 1[. \end{array} \right. \quad (11)$$

The sequences  $(\sigma_n^+(\lambda))_{n \geq 1}$  and  $(\sigma_n^-(\lambda))_{n \geq 1}$  can be interpreted as generalized Cesàro means of  $(u_n)$ , e.g. we have:

$$\lim_{\lambda \rightarrow 0^+} \sigma_n^-(\lambda) = \frac{1}{n} \sum_{k=1}^n u_k, \quad \forall n \geq 1.$$

**Definition 2 (Weakly-Vanishing Mean Oscillation (W-VMO)).**

Any sequence  $(u_n)_{n \in \mathbb{N}}$  of complex or real numbers is said to be of Weakly-Vanishing Mean Oscillation, or shortly W-VMO, if it satisfies the property:

$$\inf_{\lambda > 1} \left( \limsup_{n \rightarrow +\infty} |\omega_n^+(\lambda)| \right) = 0,$$

or symmetrically :

$$\inf_{0 < \lambda < 1} \left( \limsup_{n \rightarrow +\infty} |\omega_n^-(\lambda)| \right) = 0. \tag{12}$$

The next proposition shows that the main properties proved to be sufficient in the literature [5, 8] for the issue of inversion of the Cesàro or Abel summability are in fact sufficient conditions so that  $(u_n)$  is a W-VMO sequence.

**Proposition 2.1 (Sufficient conditions for a W-VMO sequence).**

Let  $\sum_{n \in \mathbb{N}} a_n$  be a series of complex coefficients and  $u_n := \sum_{k=0}^n a_k$  for all  $n \in \mathbb{N}$ , be the partial sums. Let us consider the following properties.

- a) The sequence  $(u_n)$  satisfies the 'big  $\mathcal{O}$ '-condition  $n a_n = \mathcal{O}(1)$  or equivalently  $n(u_n - u_{n-1}) = \mathcal{O}(1)$  when  $n \rightarrow +\infty$ , i.e.

$$C := \sup_{n \geq 1} (n|a_n|) = \sup_{n \geq 1} (n|u_n - u_{n-1}|) < +\infty. \tag{13}$$

- b) By reformulating (2), the sequence  $(u_n)$  is said to be slowly oscillating if it satisfies:

$$\lim_{\lambda \rightarrow 1^-} \left( \limsup_{n \rightarrow +\infty} \max_{[n\lambda] < k \leq n} |u_n - u_k| \right) = 0,$$

or symmetrically

$$\lim_{\lambda \rightarrow 1^+} \left( \limsup_{n \rightarrow +\infty} \max_{n < k \leq [n\lambda]} |u_k - u_n| \right) = 0. \tag{14}$$

Then, we have: (13)  $\Rightarrow$  (14)  $\Rightarrow$  (12) and thus, these properties are sufficient conditions for  $(u_n)$  to be a W-VMO sequence in the sense of Definition 2.

*Proof.*

- a) Let us show that (13)  $\Rightarrow$  (14). By using the assumption  $n|a_n| \leq C$  for some  $C > 0$  and the comparison between the harmonic series and integral of the function  $t \mapsto 1/t$

for  $t > 0$ , we find for  $\lfloor n\lambda \rfloor + 1 \leq k \leq n$  with any  $\lambda \in ]0, 1[$  and  $n$  sufficiently large:

$$\begin{aligned} |u_n - u_k| &= \left| \sum_{j=k+1}^n a_j \right| \leq \sum_{j=k+1}^n |a_j| \leq \sum_{j=k+1}^n \frac{C}{j} = \sum_{j=k+1}^n \int_{j-1}^j \frac{C}{j} dt \\ &\leq \sum_{j=k+1}^n \int_{j-1}^j \frac{C}{t} dt = C \int_k^n \frac{1}{t} dt = C \ln \left( \frac{n}{k} \right). \end{aligned}$$

Then, we get the upper bound below independent of  $n$ :

$$\max_{\lfloor n\lambda \rfloor < k \leq n} |u_n - u_k| \leq C \ln \left( \frac{n}{n\lambda} \right) = C |\ln \lambda|, \quad \forall \lambda \in ]0, 1[.$$

Hence, it follows immediately:

$$\lim_{\lambda \rightarrow 1^-} \left( \limsup_{n \rightarrow +\infty} \max_{\lfloor n\lambda \rfloor < k \leq n} |u_n - u_k| \right) = 0,$$

which means that  $(u_n)$  verifies (14).

b) We have with Definition 1:

$$|\omega_n^-(\lambda)| \leq \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor + 1}^n |u_n - u_k| \leq \max_{\lfloor n\lambda \rfloor < k \leq n} |u_n - u_k|, \quad \forall n \geq 1, \quad \forall \lambda \in ]0, 1[.$$

Thus, we get with the property (14):

$$\lim_{\lambda \rightarrow 1^-} \left( \limsup_{n \rightarrow +\infty} |\omega_n^-(\lambda)| \right) = 0,$$

which means that a slowly oscillating sequence  $(u_n)$  satisfies (12) and is necessarily a W-VMO sequence. ■

The statements in the lemma below can be considered as generalizations of the so-called "staircase" lemma of Cauchy (1821) or Cesàro's means (1870).

**Lemma 2.2 (Generalizations of Cauchy-Cesàro's lemma).**

Let  $(u_n)_{n \in \mathbb{N}}$  be a sequence of numbers in  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  and let us define the sequence of means for all integer  $n \geq 1$  by:

$$\sigma_n^-(\lambda) := \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor + 1}^n u_k, \quad \forall n \geq 1 \quad \forall \lambda \in ]0, 1[.$$

Then we have the following assertions:

- a)  $(u_n)$  bounded by  $U := \sup_{n \in \mathbb{N}} |u_n| < +\infty \Rightarrow \sup_{0 < \lambda < 1} \left( \sup_{n \geq 1} |\sigma_n^-(\lambda)| \right) \leq U < +\infty$ .
- b)  $(u_n)$  converges to  $\ell \in \mathbb{K} \Rightarrow (\sigma_n^-(\lambda))$  converges to  $\ell$  for all  $\lambda \in ]0, 1[$ .
- c)  $|u_n| \rightarrow +\infty$  when  $n \rightarrow +\infty \Rightarrow \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n |u_k| \xrightarrow{n \rightarrow +\infty} +\infty$ , for all  $\lambda \in ]0, 1[$ .
- d)  $u_n \rightarrow +\infty$  (for  $u_n \in \mathbb{R}$ ) when  $n \rightarrow +\infty \Rightarrow \sigma_n^-(\lambda) \xrightarrow{n \rightarrow +\infty} +\infty$ , for all  $\lambda \in ]0, 1[$ .

Similar properties hold by replacing the mean terms  $\sigma_n^-(\lambda)$  by  $\sigma_n^+(\lambda)$ , now for any  $\lambda > 1$ .

*Proof.*

- a) If  $U := \sup_{n \in \mathbb{N}} |u_n| < +\infty$ , then we have for all integer  $n \geq 1$  and any  $\lambda \in ]0, 1[$ :

$$|\sigma_n^-(\lambda)| \leq \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n |u_k| \leq U \frac{n - \lfloor n\lambda \rfloor}{n - \lfloor n\lambda \rfloor} = U,$$

and thus:

$$\sup_{0 < \lambda < 1} \left( \sup_{n \geq 1} |\sigma_n^-(\lambda)| \right) \leq U < +\infty.$$

- b) If  $u_n \rightarrow \ell \in \mathbb{K}$  when  $n \rightarrow +\infty$ , by noticing that:  $\ell = \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n \ell$ , then we have for all  $n \geq 1$  and  $\lambda \in ]0, 1[$ :

$$|\sigma_n^-(\lambda) - \ell| \leq \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n |u_k - \ell| \leq \sup_{k \geq \lfloor n\lambda \rfloor} |u_k - \ell| \xrightarrow{n \rightarrow +\infty} 0, \quad \forall \lambda \in ]0, 1[,$$

since  $\lfloor n\lambda \rfloor \rightarrow +\infty$  when  $n \rightarrow +\infty$  with  $\lfloor n\lambda \rfloor \leq n\lambda < \lfloor n\lambda \rfloor + 1$ , and:

$$\lim_{n \rightarrow +\infty} (\sup_{k \geq n} |u_k - \ell|) := \limsup_{n \rightarrow +\infty} |u_n - \ell| = \lim_{n \rightarrow +\infty} |u_n - \ell| = 0.$$

Thus we get:

$$\lim_{n \rightarrow +\infty} |\sigma_n^-(\lambda) - \ell| = 0, \quad \forall \lambda \in ]0, 1[.$$

- c) Let us assume that  $|u_n| \rightarrow +\infty$  when  $n \rightarrow +\infty$ , *i.e.* for any  $A > 0$ , there exists  $N_A \in \mathbb{N}$  such that for all integer  $n > N_A$ , we have  $|u_n| \geq A$ . We define the sequence  $(v_n)_{n \in \mathbb{N}}$  by  $v_n := \min(|u_n|, A)$ ,  $\forall n \in \mathbb{N}$ . Then by construction, we have:  $0 \leq v_n \leq |u_n|$  and  $v_n \rightarrow A$  when  $n \rightarrow +\infty$ . Thus, we get for all  $n \geq 1$  and  $\lambda \in ]0, 1[$ :

$$\frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n v_k \leq \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n |u_k|.$$

Now taking the lower limit by using that  $\frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n v_k \xrightarrow{n \rightarrow +\infty} A$  with  $b$ ) since  $v_n \rightarrow A$ , it gives:

$$A = \liminf_{n \rightarrow +\infty} \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n v_k \leq \liminf_{n \rightarrow +\infty} \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n |u_k|, \quad \forall \lambda \in ]0, 1[.$$

Since  $A > 0$  can be chosen arbitrarily large, it yields by taking  $A \rightarrow +\infty$ :

$$+\infty = \liminf_{n \rightarrow +\infty} \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n |u_k| \leq \limsup_{n \rightarrow +\infty} \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n |u_k|, \quad \forall \lambda \in ]0, 1[,$$

and the result is obtained.

d) It is a direct consequence of c). ■

**Lemma 2.3 (Generalization of Abel theorem for the W-VMO property).**

Let us consider a W-VMO sequence  $(u_n)_{n \in \mathbb{N}}$  of real or complex coefficients and for any  $\lambda \in ]0, 1[$ , let us define the functions  $g_\lambda, G_\lambda : [0, 1[ \rightarrow \mathbb{C}$  by:

$$\begin{aligned} g_\lambda(x) &:= (1-x) \sum_{n=1}^{+\infty} \omega_n^-(\lambda) x^n, \\ G_\lambda(x) &:= (1-x) \sum_{n=1}^{+\infty} |\omega_n^-(\lambda)| x^n, \quad \forall x \in [0, 1[, \end{aligned}$$

where the mean oscillation sequence  $(\omega_n^-(\lambda))_{n \geq 1}$  is defined by (10). Then, we have with (12):

$$\inf_{0 < \lambda < 1} \left( \limsup_{x \rightarrow 1^-} |g_\lambda(x)| \right) = 0 = \inf_{0 < \lambda < 1} \left( \limsup_{x \rightarrow 1^-} G_\lambda(x) \right) \quad (15)$$

A similar property holds by replacing in  $g_\lambda$  the mean oscillation terms  $\omega_n^-(\lambda)$  by  $\omega_n^+(\lambda)$ , now for  $\lambda > 1$ .

*Proof.* It is clear that for all  $x \in [0, 1[$  and any  $\lambda \in ]0, 1[$ , we have:

$$0 \leq |g_\lambda(x)| \leq G_\lambda(x) \quad \text{and thus} \quad 0 \leq \limsup_{x \rightarrow 1^-} |g_\lambda(x)| \leq \limsup_{x \rightarrow 1^-} G_\lambda(x), \quad \forall \lambda \in ]0, 1[.$$

Moreover, by separating the sum into two parts for any integer  $n \geq 1$ , we get for all  $x \in [0, 1[$ :

$$\begin{aligned}
G_\lambda(x) &= (1-x) \sum_{k=1}^n |\omega_k^-(\lambda)| x^k + (1-x) \sum_{k=n+1}^{+\infty} |\omega_k^-(\lambda)| x^k \\
&\leq (1-x) \sum_{k=1}^n |\omega_k^-(\lambda)| + (1-x) \sup_{k \geq n} |\omega_k^-(\lambda)| \sum_{k=n+1}^{+\infty} x^k \\
&\leq (1-x) \sum_{k=1}^n |\omega_k^-(\lambda)| + (1-x) \sup_{k \geq n} |\omega_k^-(\lambda)| \sum_{k=0}^{+\infty} x^k \\
&\leq (1-x) \sum_{k=1}^n |\omega_k^-(\lambda)| + \sup_{k \geq n} |\omega_k^-(\lambda)|, \quad \forall \lambda \in ]0, 1[.
\end{aligned}$$

Now, by passing to the upper limit when  $x \rightarrow 1^-$  for a fixed  $n \geq 1$ , it gives:

$$\limsup_{x \rightarrow 1^-} G_\lambda(x) \leq \sup_{k \geq n} |\omega_k^-(\lambda)|,$$

which then yields by taking now the limit when  $n \rightarrow +\infty$ :

$$0 \leq \limsup_{x \rightarrow 1^-} G_\lambda(x) \leq \limsup_{n \rightarrow +\infty} \sup_{k \geq n} |\omega_k^-(\lambda)|, \quad \forall \lambda \in ]0, 1[.$$

Finally using the infimum property with (12), we get:

$$0 \leq \inf_{0 < \lambda < 1} \left( \limsup_{x \rightarrow 1^-} G_\lambda(x) \right) \leq \inf_{0 < \lambda < 1} \left( \limsup_{n \rightarrow +\infty} \sup_{k \geq n} |\omega_k^-(\lambda)| \right) = 0,$$

that is (15) is verified. ■

The next result can be viewed as weak versions of Frobenius (1880) theorem, e.g. [5, 8], and Hardy-Littlewood (1914) positive Tauberian theorem [4].

**Lemma 2.4 (Weak Frobenius Tauberian converse).**

Let  $\sum_{n \in \mathbb{N}} u_n x^n$  be a power series with real or complex coefficients such that the convergence radius is  $R = 1$ . We define for all  $n \in \mathbb{N}$ :

$$S_n := \sum_{k=0}^n u_k, \quad \text{the Cesàro means of } (u_n): \quad \sigma_n := \frac{1}{n+1} \sum_{k=0}^n u_k = \frac{S_n}{n+1},$$

and the function  $f : [0, 1[ \rightarrow \mathbb{C}$  by:

$$f(x) := (1-x) \sum_{n=0}^{+\infty} u_n x^n, \quad \forall x \in [0, 1[.$$

Then, we have the following properties:

a) If the sequence  $(\sigma_n)$  is bounded by  $K := \sup_{n \in \mathbb{N}} |\sigma_n| < +\infty$ , then we have:

$$|f(x)| \leq K, \quad \forall x \in [0, 1[, \quad \text{and} \quad \limsup_{x \rightarrow 1^-} |f(x)| \leq K < +\infty.$$

- b) A Tauberian converse of a): if  $(u_n) \geq 0$  (for real coefficients), then  $f \geq 0$  in  $[0, 1[$  and the converse assertion of a) holds true, i.e.

$$\limsup_{x \rightarrow 1^-} f(x) < +\infty \quad \Rightarrow \quad \exists C > 0; \quad 0 \leq \limsup_{n \rightarrow +\infty} \sigma_n \leq C \limsup_{x \rightarrow 1^-} f(x) < +\infty.$$

Hence, we have:

$$\lim_{x \rightarrow 1^-} f(x) = 0 \quad \Rightarrow \quad \lim_{n \rightarrow +\infty} \sigma_n = 0.$$

*Proof.*

- a) By considering the Cauchy product of the power series  $\sum_{n \in \mathbb{N}} u_n x^n$  and  $\sum_{n \in \mathbb{N}} x^n$ , we first observe that  $f$  also reads as:

$$f(x) := (1-x) \sum_{n=0}^{+\infty} u_n x^n = (1-x)^2 \sum_{n=0}^{+\infty} S_n x^n, \quad \forall x \in [0, 1[. \quad (16)$$

Since the sequence  $(\sigma_n)$  is bounded by  $K := \sup_{n \in \mathbb{N}} |\sigma_n| < +\infty$ , we easily get:

$$\begin{aligned} |f(x)| &\leq (1-x)^2 \sum_{n=0}^{+\infty} |S_n| x^n = (1-x)^2 \sum_{n=0}^{+\infty} (n+1) |\sigma_n| x^n \\ &\leq K (1-x)^2 \sum_{n=0}^{+\infty} (n+1) x^n, \quad \forall x \in [0, 1[. \end{aligned}$$

By calculating the derivative of the sum  $x/(1-x)$  of the power series  $\sum_{n \in \mathbb{N}} x^{n+1} = x \sum_{n \in \mathbb{N}} x^n$ , it yields:

$$|f(x)| \leq K, \quad \forall x \in [0, 1[, \quad \text{and thus} \quad \limsup_{x \rightarrow 1^-} |f(x)| \leq K < +\infty.$$

- b) Let  $(u_n) \geq 0$  be a real positive sequence (thus  $f \geq 0$  in  $[0, 1[$ ) such that  $\limsup_{x \rightarrow 1^-} f(x) < +\infty$ , i.e.  $f$  is bounded in  $[0, 1[$ . We consider the cut off function  $k$ , positive and bounded, defined in  $[0, 1[$  by:

$$k(x) := \begin{cases} 0 & \text{for } 0 \leq x < 1/e, \\ 1/x & \text{for } 1/e \leq x \leq 1, \end{cases} \quad \text{such that: } 0 \leq k(x) \leq e, \quad \forall x \in [0, 1[.$$

Then, we have with the bounds of  $f$  and  $k$ :

$$\begin{aligned} T(x) &:= (1-x) \sum_{n=0}^{+\infty} u_n x^n k(x^n) \leq e (1-x) \sum_{n=0}^{+\infty} u_n x^n = e f(x) \\ &\leq e \sup_{x \in [0, 1[} f(x) < +\infty, \quad \forall x \in [0, 1[. \end{aligned} \quad (17)$$

Now choosing  $x_n := e^{-1/n}$  for all integer  $n \geq 1$  such that  $x_n \rightarrow 1^-$  with

$$(1 - x_n) = (1 - e^{-1/n}) = \frac{1}{n} + o\left(\frac{1}{n}\right) \sim \frac{1}{n+1} \quad \text{when } n \rightarrow +\infty,$$

we have:  $x_n^k k(x_n^k) = 1$  for  $0 \leq k \leq n$  and  $x_n^k k(x_n^k) = 0$  for all  $k > n$ . Thus, we get the truncature:

$$\sum_{k=0}^{+\infty} u_k x_n^k k(x_n^k) = \sum_{k=0}^n u_k = S_n = (n+1)\sigma_n, \quad \forall n \geq 1. \quad (18)$$

Hence, with the previous bound (17), we finally obtain from (18):

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \sigma_n &= \limsup_{n \rightarrow +\infty} T(x_n) \leq e \limsup_{n \rightarrow +\infty} f(x_n) \\ &\leq e \limsup_{x \rightarrow 1^-} f(x) < +\infty, \end{aligned} \quad (19)$$

which means that the positive sequence  $(\sigma_n)$  is bounded.

Then, if  $\lim_{x \rightarrow 1^-} f(x) = 0$ , a direct consequence of the inequality (19) is  $\lim_{n \rightarrow +\infty} \sigma_n = 0$ . ■

### 3 Tauberian results for Cesàro summability

The following equivalence result for Cesàro summability, here formulated for a series  $\sum_{n \in \mathbb{N}} a_n$  of real or complex numbers, provides the Tauberian converse of the Cauchy (1821) "staircase" lemma equivalent to Cesàro (1890) lemma.

**Theorem 3.1 (Necessary and sufficient condition for Cesàro converse).**

Let  $\sum_{n \in \mathbb{N}} a_n$  be a series of real or complex coefficients. We define the partial sums  $(u_n)$  and their Cesàro means  $(\sigma_n)$  by:

$$u_n := \sum_{k=0}^n a_k, \quad \sigma_n := \frac{1}{n+1} \sum_{k=0}^n u_k, \quad \forall n \in \mathbb{N},$$

and the properties below for any finite value  $\ell \in \mathbb{C}$ :

$$(C_0) \quad (u_n) \text{ (or } \sum_{n \in \mathbb{N}} a_n \text{) converges, i.e. } \sum_{n=0}^{+\infty} a_n := \lim_{n \rightarrow +\infty} u_n = \ell,$$

$$(C_1) \quad (u_n) \text{ (or } \sum_{n \in \mathbb{N}} a_n \text{) is Cesàro summable, i.e. } \lim_{n \rightarrow +\infty} \sigma_n = \ell.$$

Then,  $(C_1)$  is equivalent to  $(C_0)$  if and only if the sequence  $(u_n)$  is *W-VMO*.

*Proof.* Let us first recall that  $(C_0) \Rightarrow (C_1)$  is ensured with no condition by the classical Cauchy-Cesàro lemma. It is also well-known that the converse is not generally true, e.g.

$a_n := (-1)^n$  gives Grandi's series  $\sum_{n \in \mathbb{N}} (-1)^n$  that is diverging with no limit since it oscillates between the two values 1 and 0. But, it is easy to verify that the Cesàro mean is converging with  $\sigma_n \rightarrow 1/2$  when  $n \rightarrow +\infty$ .

1. *W-VMO necessary condition for  $(C_1) \Rightarrow (C_0)$ .*

If we assume that the conclusion  $(C_0)$  is satisfied, *i.e.*  $u_n \rightarrow \ell \in \mathbb{C}$  when  $n \rightarrow +\infty$ , then Lemma 2.2[b)] shows that  $\sigma_n^-(\lambda) \rightarrow \ell$  for all  $\lambda \in ]0, 1[$ , and thus with Eq. (11):

$$|\omega_n^-(\lambda)| = |u_n - \sigma_n^-(\lambda)| \xrightarrow{n \rightarrow +\infty} 0, \quad \forall \lambda \in ]0, 1[.$$

Hence, we have:

$$\lim_{n \rightarrow +\infty} |\omega_n^-(\lambda)| = 0 = \lim_{n \rightarrow +\infty} \omega_n^-(\lambda), \quad \forall \lambda \in ]0, 1[. \quad (20)$$

Therefore,  $(u_n)$  satisfies (12) and is necessarily a W-VMO sequence.

2. *W-VMO sufficient condition for  $(C_1) \Rightarrow (C_0)$ .*

Let us now assume that  $(u_n)$  is a W-VMO sequence such that  $(C_1)$  is satisfied, *i.e.*  $\sigma_n \rightarrow \ell \in \mathbb{C}$  when  $n \rightarrow +\infty$ . By definition, we have for all integer  $n \geq 1$  and for all  $\lambda \in ]0, 1[$ :

$$\begin{aligned} \sigma_n^-(\lambda) &:= \frac{1}{n - \lfloor n\lambda \rfloor} \sum_{k=\lfloor n\lambda \rfloor+1}^n u_k = \frac{1}{n - \lfloor n\lambda \rfloor} \left( \sum_{k=0}^n u_k - \sum_{k=0}^{\lfloor n\lambda \rfloor} u_k \right) \\ &= \frac{1}{n - \lfloor n\lambda \rfloor} \left( (n+1)\sigma_n - (\lfloor n\lambda \rfloor + 1)\sigma_{\lfloor n\lambda \rfloor} \right) \\ &= \sigma_n + \frac{\lfloor n\lambda \rfloor + 1}{n - \lfloor n\lambda \rfloor} (\sigma_n - \sigma_{\lfloor n\lambda \rfloor}). \end{aligned} \quad (21)$$

Moreover, with  $\lfloor n\lambda \rfloor \leq n\lambda < \lfloor n\lambda \rfloor + 1$ , we have also  $n - \lfloor n\lambda \rfloor \geq (n - n\lambda) > 0$  and we get the bound :

$$\begin{aligned} 0 < \frac{\lfloor n\lambda \rfloor + 1}{n - \lfloor n\lambda \rfloor} &\leq \frac{n\lambda + 1}{n - n\lambda} = \frac{\lambda + \frac{1}{n}}{1 - \lambda} \\ &\leq \frac{1 + \lambda}{1 - \lambda}, \quad \forall n \geq 1, \quad \forall \lambda \in ]0, 1[. \end{aligned} \quad (22)$$

Hence combining (21) with (22), we get the inequality below:

$$|\sigma_n^-(\lambda) - \ell| \leq |\sigma_n - \ell| + \frac{1 + \lambda}{1 - \lambda} |\sigma_n - \sigma_{\lfloor n\lambda \rfloor}|, \quad \forall n \geq 1, \quad \forall \lambda \in ]0, 1[. \quad (23)$$

Then, if  $\sigma_n \rightarrow \ell \in \mathbb{C}$  when  $n \rightarrow +\infty$  with  $(C_1)$ , we have also  $\sigma_{\lfloor n\lambda \rfloor} \rightarrow \ell$  for any  $\lambda \in ]0, 1[$  and thus it yields with (23):

$$\lim_{n \rightarrow +\infty} \sigma_n^-(\lambda) = \ell, \quad \forall \lambda \in ]0, 1[. \quad (24)$$

Now, it follows with Eq. (11):

$$|u_n - \ell| = |(\sigma_n^-(\lambda) - \ell) + \omega_n^-(\lambda)| \leq |\sigma_n^-(\lambda) - \ell| + |\omega_n^-(\lambda)|,$$

which gives with (24) by first taking the upper limit when  $n \rightarrow +\infty$ :

$$0 \leq \limsup_{n \rightarrow +\infty} |u_n - \ell| \leq \limsup_{n \rightarrow +\infty} |\omega_n^-(\lambda)|, \quad \forall \lambda \in ]0, 1[.$$

Now using the infimum property for  $\lambda \in ]0, 1[$  and the W-VMO hypothesis (12) yields:

$$\limsup_{n \rightarrow +\infty} |u_n - \ell| = 0,$$

which means that  $u_n \rightarrow \ell$  when  $n \rightarrow +\infty$ .

Finally, the W-VMO property of  $(u_n)$  is actually a necessary and sufficient condition to ensure that the statement  $(C_1) \Rightarrow (C_0)$  holds true.  $\blacksquare$

Among others, let us mention one direct consequence in Fourier analysis.

**Corollary 3.2 (Application to Fourier series).**

Let  $f$  be any continuous complex-valued function of period  $2\pi$  and let us denote by  $S_n(f)$  its Fourier partial sums defined by:

$$S_n(f)(x) := \sum_{k=-n}^n \widehat{f}(k) e^{ikx}, \quad \forall x \in \mathbb{R}, \quad \forall n \in \mathbb{N},$$

where  $(\widehat{f}(k))_{k \in \mathbb{Z}}$  are the Fourier coefficients of  $f$ .

Then for any  $x \in \mathbb{R}$ ,  $S_n(f)(x)$  converges to  $f(x)$  if and only if the Fourier sequence  $(S_n(f)(x))_{n \in \mathbb{N}}$  is W-VMO.

*Proof.* The Fejér sums  $\sigma_N(f)$  of  $f$  are classically defined by the Cesàro mean of its Fourier sums, *i.e.*

$$\sigma_N(f) := \frac{1}{N+1} \sum_{n=0}^N S_n(f), \quad \forall N \in \mathbb{N}.$$

The Fejér (1904) theorem [2] states that the Fourier series is uniformly Cesàro summable to  $f$ , *i.e.*

$$\lim_{N \rightarrow +\infty} \|\sigma_N(f) - f\|_\infty = 0,$$

and thus  $\lim_{N \rightarrow +\infty} \sigma_N(f)(x) = f(x)$ , for all  $x \in \mathbb{R}$ . Then, the desired result is a direct consequence of Theorem 3.1.  $\blacksquare$

## 4 Tauberian results for Abel summability

We can now prove our main result for Abel summability, here formulated for a series  $\sum_{n \in \mathbb{N}} a_n$  of real or complex numbers, which provides the Tauberian converse of Abel theorem (1826) of radial limit for a power series.

**Theorem 4.1 (Necessary and sufficient condition for Abel converse).**

Let  $\sum_{n \in \mathbb{N}} a_n$  be a series of real or complex coefficients such that the convergence radius of the power series  $\sum_{n \in \mathbb{N}} a_n x^n$  is  $R = 1$ . We define the partial sums  $(u_n)$  by:

$$u_n := \sum_{k=0}^n a_k, \quad \forall n \in \mathbb{N},$$

and the properties below for any finite value  $\ell \in \mathbb{C}$ :

$$(C_0) \quad (u_n) \text{ (or } \sum_{n \in \mathbb{N}} a_n \text{) converges to } \ell, \text{ i.e. } \sum_{n=0}^{+\infty} a_n := \lim_{n \rightarrow +\infty} u_n = \ell,$$

$$(A_0) \quad (u_n) \text{ (or } \sum_{n \in \mathbb{N}} a_n \text{) is Abel summable to } \ell, \text{ i.e.}$$

$$f(x) := \sum_{n=0}^{+\infty} a_n x^n = (1-x) \sum_{n=0}^{+\infty} u_n x^n \xrightarrow{x \rightarrow 1^-} \ell.$$

Then,  $(A_0)$  is equivalent to  $(C_0)$  if and only if the sequence  $(u_n)$  is W-VMO.

*Proof.* Let us first recall that  $(C_0) \Rightarrow (A_0)$  is ensured with no condition by the Abel theorem (1826) of radial limit for a power series, e.g. [5, 8]. It is also well-known that the converse is not generally true, e.g.  $a_n := (-1)^n$  gives:

$$\sum_{n=0}^{+\infty} (-1)^n x^n = \sum_{n=0}^{+\infty} (-x)^n = \frac{1}{1+x} \xrightarrow{x \rightarrow 1^-} \frac{1}{2},$$

but the series  $\sum_{n \in \mathbb{N}} (-1)^n$  is diverging with no limit.

1. *W-VMO necessary condition for  $(A_0) \Rightarrow (C_0)$ .*

If we assume that the conclusion  $(C_0)$  is satisfied, i.e.  $u_n \rightarrow \ell \in \mathbb{C}$  when  $n \rightarrow +\infty$ , then Lemma 2.2[b)] shows that  $\sigma_n^-(\lambda) \rightarrow \ell$  for all  $\lambda \in ]0, 1[$ , and thus with Eq. (11):

$$|\omega_n^-(\lambda)| = |u_n - \sigma_n^-(\lambda)| \xrightarrow{n \rightarrow +\infty} 0, \quad \forall \lambda \in ]0, 1[.$$

Hence, we have:

$$\lim_{n \rightarrow +\infty} |\omega_n^-(\lambda)| = 0 = \lim_{n \rightarrow +\infty} \omega_n^-(\lambda), \quad \forall \lambda \in ]0, 1[.$$

Therefore,  $(u_n)$  satisfies (12) and is necessarily a W-VMO sequence.

2. *W-VMO sufficient condition for  $(A_0) \Rightarrow (C_0)$ .*

Let us now assume that  $(u_n)$  is a W-VMO sequence, i.e. (12) is satisfied, and let us denote by  $f$  the sum of the power series  $\sum_{n \in \mathbb{N}} a_n x^n$ . It is classical, either with Abel's summation by parts or by calculating the Cauchy product of the power series  $\sum_{n \in \mathbb{N}} a_n x^n$  and  $\sum_{n \in \mathbb{N}} x^n$ , that  $f(x)$  also reads as:

$$f(x) := \sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} u_n x^n, \quad \forall x \in ]-1, 1[. \quad (25)$$

Then, the assumption of radial limit  $(A_0)$  reads  $f(1^-) := \lim_{x \rightarrow 1^-} f(x) = \ell \in \mathbb{C}$ . For the sake of convenience, we notice that it suffices to study the case with  $\ell = 0$  because it amounts to replace  $f(x)$  by  $f(x) - \ell$ , that is to replace the coefficient  $a_0$  by  $a_0 - \ell$ . Thus from now on, we assume that:

$$\lim_{x \rightarrow 1^-} f(x) = 0 = \lim_{x \rightarrow 1^-} |f(x)|. \quad (26)$$

Moreover, with no loss of generality, it suffices to prove the result for  $u_0 = a_0 = 0$  because the first term  $(1-x)u_0$  in the sum  $f(x)$  tends to 0 when  $x \rightarrow 1^-$ . Then, we follow two steps.

*Step 1.* By using (11) and linearity, we have the following decomposition of  $f(x)$  and its consequence by the triangle inequality:

$$\begin{aligned} f(x) &= g_\lambda(x) + h_\lambda(x), \quad \forall x \in [0, 1[, \quad \forall \lambda \in ]0, 1[, \\ \text{with } 0 \leq |f(x)| &\leq F(x) \leq G_\lambda(x) + H_\lambda(x), \end{aligned} \quad (27)$$

where:

$$\left\{ \begin{array}{l} f(x) := (1-x) \sum_{n=1}^{+\infty} u_n x^n, \quad F(x) := (1-x) \sum_{n=1}^{+\infty} |u_n| x^n, \\ g_\lambda(x) := (1-x) \sum_{n=1}^{+\infty} \omega_n^-(\lambda) x^n, \quad G_\lambda(x) := (1-x) \sum_{n=1}^{+\infty} |\omega_n^-(\lambda)| x^n, \\ h_\lambda(x) := (1-x) \sum_{n=1}^{+\infty} \sigma_n^-(\lambda) x^n, \quad H_\lambda(x) := (1-x) \sum_{n=1}^{+\infty} |\sigma_n^-(\lambda)| x^n. \end{array} \right. \quad (28)$$

Moreover, since the sequence  $(u_n)$  is W-VMO, Eq. (15) in Lemma 2.3 ensures that:

$$\inf_{0 < \lambda < 1} \left( \limsup_{x \rightarrow 1^-} |g_\lambda(x)| \right) = 0 = \inf_{0 < \lambda < 1} \left( \limsup_{x \rightarrow 1^-} G_\lambda(x) \right). \quad (29)$$

Since we have with (27) and (28):

$$|f(x)| \leq F(x) \leq G_\lambda(x) + H_\lambda(x),$$

we also get:

$$||f(x)| - H_\lambda(x)| \leq |G_\lambda(x)| = G_\lambda(x), \quad \forall \lambda \in ]0, 1[.$$

Now, taking the upper limit when  $x \rightarrow 1^-$  yields with  $|f(x)| \rightarrow 0$  by (26):

$$\limsup_{x \rightarrow 1^-} |H_\lambda(x)| \leq \limsup_{x \rightarrow 1^-} G_\lambda(x), \quad \forall \lambda \in ]0, 1[.$$

and thus with the inequality  $0 \leq |h_\lambda(x)| \leq H_\lambda(x)$  and Eq. (29), we get:

$$\inf_{0 < \lambda < 1} \left( \limsup_{x \rightarrow 1^-} |h_\lambda(x)| \right) = 0 = \inf_{0 < \lambda < 1} \left( \limsup_{x \rightarrow 1^-} H_\lambda(x) \right). \quad (30)$$

Then, combining (27) with (29) and (30) yields:

$$0 \leq \limsup_{x \rightarrow 1^-} F(x) \leq 0,$$

and thus

$$\lim_{x \rightarrow 1^-} F(x) = 0 \quad (= \lim_{x \rightarrow 1^-} f(x)). \quad (31)$$

*Step 2.* Let us introduce the Cesàro mean  $(\sigma_n)$  of the sequence  $(u_n)$ :

$$\sigma_n := \frac{1}{n+1} \sum_{k=0}^n u_k, \quad \forall n \in \mathbb{N}. \quad (32)$$

Then, since the sequence  $(|u_n|)$  is nonnegative satisfying (31), Lemma 2.4[b)] ensures that the Cesàro mean of the sequence  $(|u_n|)$  converges to 0, *i.e.*

$$\lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n |u_k| = 0. \quad (33)$$

Thus, it follows:

$$|\sigma_n| \leq \frac{1}{n+1} \sum_{k=0}^n |u_k| \xrightarrow{n \rightarrow +\infty} 0.$$

Hence, we get:

$$\lim_{n \rightarrow +\infty} |\sigma_n| = 0 = \lim_{n \rightarrow +\infty} \sigma_n. \quad (34)$$

Therefore, the Tauberian converse  $(A_0) \Rightarrow (C_1)$  of Frobenius theorem (1880) is already shown under the necessary and sufficient W-VMO condition.

Furthermore, by applying Theorem 3.1 since  $(u_n)$  is a W-VMO sequence, we get that  $(C_1) \Rightarrow (C_0)$ , that's to say:

$$\lim_{n \rightarrow +\infty} |u_n| = 0 = \lim_{n \rightarrow +\infty} u_n, \quad (35)$$

which is the desired result.

Finally, the W-VMO property of  $(u_n)$  actually proves to be a necessary and sufficient condition to ensure that the statement  $(A_0) \Rightarrow (C_0)$  holds true. ■

**Remark 1 (Alternative argument for Step 2 in the proof of Theorem 4.1).**

*At this step, since the sequence  $(|u_n|)$  is nonnegative satisfying (31), we can apply the Hardy-Littlewood (1914) theorem [4] which provides a strong Tauberian converse of Frobenius theorem (1880); a simpler short proof being given by Karamata (1930) [7]. Hence, we deduce that the Cesàro mean of the sequence  $(|u_n|)$  converges to 0, *i.e.* (33).*

However, it is more suitable to remain self contained here and to provide a simple proof of Step 2 by applying Lemma 2.4 without using the previous argument. Moreover, Lemma 2.4 provides an alternative simple proof of Hardy-Littlewood (1914) positive Tauberian theorem.

Let us observe that the necessary and sufficient condition (1) pointed out in the introduction is clearly satisfied by the present result. By the way, Theorem 4.1 also provides the Tauberian converse of Frobenius theorem (1880) of radial limit for a power series.

**Corollary 4.2 (Necessary and sufficient condition for Frobenius converse).**

Let  $\sum_{n \in \mathbb{N}} a_n$  be a series of real or complex coefficients such that the convergence radius of the power series  $\sum_{n \in \mathbb{N}} a_n x^n$  is  $R = 1$ . We define the partial sums  $(u_n)$  by and the Cesàro means  $(\sigma_n)$  by:

$$u_n := \sum_{k=0}^n a_k, \quad \sigma_n := \frac{1}{n+1} \sum_{k=0}^n u_k, \quad \forall n \in \mathbb{N},$$

and the properties below for any finite value  $\ell \in \mathbb{C}$ :

$$(C_1) \quad (u_n) \text{ (or } \sum_{n \in \mathbb{N}} a_n) \text{ is Cesàro summable to } \ell, \text{ i.e. } \lim_{n \rightarrow +\infty} \sigma_n = \ell,$$

$$(A_0) \quad (u_n) \text{ (or } \sum_{n \in \mathbb{N}} a_n) \text{ is Abel summable to } \ell, \text{ i.e.}$$

$$f(x) := \sum_{n=0}^{+\infty} a_n x^n = (1-x) \sum_{n=0}^{+\infty} u_n x^n \xrightarrow{x \rightarrow 1^-} \ell.$$

Then,  $(A_0)$  is equivalent to  $(C_1)$  if and only if the sequence  $(u_n)$  is W-VMO.

*Proof.* From one side, the Frobenius theorem (1880) ensures that  $(C_1) \Rightarrow (A_0)$ . From the other side, since the Cauchy-Cesàro lemma ensures that the convergence of any sequence  $(C_0)$  implies the convergence of the Cesàro mean  $(C_1)$  towards the same limit, Theorem 4.1 shows that  $(A_0) \Rightarrow (C_1)$  if and only if the sequence  $(u_n)$  is W-VMO. ■

**Remark 2 (On the W-BMO necessary condition).**

In the proof of the necessity of the W-VMO condition in Theorems 3.1 and 4.1, it appears that if  $u_n \rightarrow \ell \in \mathbb{C}$  when  $n \rightarrow +\infty$ , then the sequence  $(u_n)$  is bounded by some  $U := \sup_{n \in \mathbb{N}} |u_n| < +\infty$  and Lemma 2.2[a)] shows that  $(\sigma_n^-(\lambda))$  is also bounded by  $U$  for all  $\lambda \in ]0, 1[$ . Thus, it follows with Eq. (11):

$$\sup_{n \geq 1} |\omega_n^-(\lambda)| \leq 2U < +\infty, \quad \forall \lambda \in ]0, 1[.$$

This means that the condition

$$\sup_{0 < \lambda < 1} \left( \sup_{n \geq 1} |\omega_n^-(\lambda)| \right) < +\infty,$$

or symmetrically :

$$\sup_{\lambda > 1} \left( \sup_{n \geq 1} |\omega_n^+(\lambda)| \right) < +\infty, \tag{36}$$

is a necessary condition for the inversion of both the Cesàro or Abel summability. Therefore, the property (36) called Weakly-Bounded Mean Oscillation or shortly W-BMO can be added to the definition 2 of the W-VMO sequence  $(u_n)$ . In such a way, the space of W-VMO sequences is actually a subspace of the W-BMO space.

The present results appeal several comments.

**Remark 3 (Final comments).**

1. Using Proposition 2.1, Theorem 3.1 gives another proof of Hardy (1910) theorem [3] for the inversion of Cesàro summability with the 'big  $\mathcal{O}$ -condition'  $n a_n = \mathcal{O}(1)$ . Moreover, unilateral bounds can be considered as well in the case of real coefficients, e.g. the condition  $n |a_n| \leq K$  can be equivalently replaced by the one-sided condition of Landau (1910) [9]:  $n a_n \leq K$  or  $n a_n \geq -K$  for some  $K > 0$ .
2. Still with Proposition 2.1, Theorem 4.1 provides an alternative proof of Littlewood (1911) Tauberian theorem [11] for the inversion of Abel summability with the 'big  $\mathcal{O}$ -condition', that has strongly extended the two theorems of Tauber (1897) [18]. Besides, Theorem 4.1 also provides an alternative proof of the so-called generalized Littlewood Tauberian theorem for a slowly oscillating sequence  $(u_n)$  that was proved by Landau (1913) [10].
3. Furthermore, considering the result of Hardy-Littlewood-Karamata strong Tauberian theorem, the necessary W-VMO condition in Theorem 4.1 actually shows that a positive sequence  $(u_n) \geq 0$  such that  $f(x) \rightarrow \ell < +\infty$  when  $x \rightarrow 1^-$  is necessarily a W-VMO sequence, which is not a trivial result. This is also the case for other sufficient conditions found in the literature to get either  $(A_0) \Rightarrow (C_1)$  or  $(A_0) \Rightarrow (C_0)$ , e.g. those in Szász (1951) [17]. Therefore, Theorems 4.1 and 3.1 allow us to assert that all of them yield a W-VMO sequence  $(u_n)$  with the assumption  $(A_0)$ .
4. The theorems 3.1 and 4.1 involve interesting consequences when applied to Fourier analysis, that are not precised here for the sake of brevity. Moreover, some results from the extension of the present setting to Dirichlet series and Laplace integrals are expected.

## 5 Conclusion

As a result, we have provided in this work a tauberian condition that is sufficiently weak to remain necessary and sufficient for the inversion of both Cesàro and Abel summabilities.

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