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# The intertwined derivative Schrödinger system of Calogero–Moser–Sutherland type

Ruoci Sun\*

**Abstract** This paper is dedicated to extending the focusing/defocusing Calogero–Moser–Sutherland cubic derivative Schrödinger equation (CMSdNLS), introduced in Gérard–Lenzmann [22] and Badreddine [2, 3], to a system of two matrix-valued variables, leading to the following intertwined system,

$$\begin{cases} i\partial_t U + \partial_x^2 U = -\frac{1}{2}U(D + |D|)(V^*U) - \frac{1}{2}V(D + |D|)(U^*U), \\ i\partial_t V + \partial_x^2 V = -\frac{1}{2}V(D + |D|)(U^*V) - \frac{1}{2}U(D + |D|)(V^*V), \end{cases} \quad D = -i\partial_x.$$

This system still enjoys a Lax pair structure, which allows to establish an explicit formula for general solutions on the torus and on the line. As a consequence, this system can be considered as an integrable perturbation of both the linear Schrödinger equation and the CMSdNLS equations.

**Keywords** derivative Schrödinger equation, Lax pair, explicit formula, Toeplitz operators.

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# 1 Introduction

Given any two positive integers  $M, N \in \mathbb{N}_+$ , the cubic  $M \times N$  matrix-valued intertwined derivative Schrödinger system of Calogero–Moser–Sutherland (CMS) type reads as

$$\begin{cases} i\partial_t U + \partial_x^2 U = -\frac{1}{2}U(D + |D|)(V^*U) - \frac{1}{2}V(D + |D|)(U^*U), \\ i\partial_t V + \partial_x^2 V = -\frac{1}{2}V(D + |D|)(U^*V) - \frac{1}{2}U(D + |D|)(V^*V), \end{cases} \quad D = -i\partial_x. \quad (1.1)$$

where  $t \in \mathfrak{J}$  for some interval  $\mathfrak{J} \subset \mathbb{R}$ ,  $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$  or  $x \in \mathbb{R}$  and  $U = U(t, x), V = V(t, x) \in \mathbb{C}^{M \times N}$ . If  $V = \mu U$ , for some  $\mu \in \mathbb{C}$ , the system (1.1) is reduced to the following equation,

$$i\partial_t U + \partial_x^2 U = -(\operatorname{Re}\mu)U(D + |D|)(U^*U), \quad t \in \mathfrak{J}, \quad x \in \mathbb{M}, \quad (1.2)$$

where  $\mathbb{M} = \mathbb{T}$  or  $\mathbb{R}$ . When  $\operatorname{Re}\mu > 0$ , equation (1.2) is called the focusing CMS derivative cubic Schrödinger equation up to scaling,

$$i\partial_t U + \partial_x^2 U = -U(D + |D|)(U^*U), \quad t \in \mathfrak{J}, \quad x \in \mathbb{M}. \quad (1.3)$$

When  $\operatorname{Re}\mu < 0$ , equation (1.2) is called the defocusing CMS derivative cubic Schrödinger equation up to scaling,

$$i\partial_t U + \partial_x^2 U = U(D + |D|)(U^*U), \quad t \in \mathfrak{J}, \quad x \in \mathbb{M}. \quad (1.4)$$

When  $\operatorname{Re}\mu = 0$ , equation (1.2) becomes the linear Schrödinger equation,

$$i\partial_t U + \partial_x^2 U = 0_{M \times N}, \quad t \in \mathbb{R}, \quad x \in \mathbb{M}. \quad (1.5)$$

In the scalar case  $M = N = 1$ , we refer to Abanov–Bettelheim–Wiegmann [1] and the references therein to see the physical backgrounds of both defocusing and focusing CMSdNLS equations (1.3), (1.4) on  $\mathbb{M}$ .

1. If  $\mathbb{M} = \mathbb{T}$ , both (1.3) and (1.4) on the torus can be interpreted as the thermodynamic limit of the following Calogero–Sutherland model (CSM)

$$\mathcal{H}_{CSM} = \frac{1}{2} \sum_{j=1}^N p_j^2 + \frac{\pi^2}{2L^2} \sum_{1 \leq j \neq k \leq N} \frac{g^2}{\sin^2\left(\frac{\pi}{L}(x_j - x_k)\right)}, \quad N = cL \rightarrow +\infty, \quad (1.6)$$

for some constant  $c > 0$ , which is introduced in Calogero [7] and Sutherland [45].

2. If  $\mathbb{M} = \mathbb{R}$ , both (1.3) and (1.4) on the line can be interpreted as the continuum limit of the following classical Calogero–Moser model (CM)

$$\mathcal{H}_{CM} = \frac{1}{2} \sum_{j=1}^N p_j^2 + \sum_{1 \leq j \neq k \leq N} \frac{1}{2(x_j - x_k)^2}, \quad N \rightarrow +\infty, \quad (1.7)$$

which is introduced in Moser [30] and Olshanetsky–Perelomov [32].

Equations (1.3) and (1.4) involve the  $L^2$ -spaces with only nonnegative Fourier modes,

$$L_+^2(\mathbb{T}; \mathbb{C}) := \{f \in L^2(\mathbb{T}; \mathbb{C}) : \hat{f}(n) = 0, \forall n < 0\}, \quad L_+^2(\mathbb{R}; \mathbb{C}) := \{f \in L^2(\mathbb{R}; \mathbb{C}) : \operatorname{supp}(\hat{f}) \subset [0, +\infty)\}. \quad (1.8)$$

The filtered Sobolev spaces are denoted by  $H_+^s(\mathbb{M}; \mathbb{C}) = H^s \cap L_+^2(\mathbb{M}; \mathbb{C})$ ,  $\mathbb{M} = \mathbb{T}$  or  $\mathbb{R}$ ,  $\forall s \geq 0$ . Thanks to P. Gérard and E. Lenzmann's work [22] and R. Badreddine's work [2], both (1.3) and (1.4) enjoy the Lax pair structure, which allows to deduce an explicit expression of the Poisson integral of general solutions. The defocusing equation (1.4) is globally wellposed in all filtered Sobolev spaces  $H_+^s(\mathbb{M}; \mathbb{C})$ ,  $s \geq 0$ , for arbitrary initial data; the focusing equation (1.3) is globally wellposed in  $H_+^s(\mathbb{M}; \mathbb{C})$  for small  $L_+^2(\mathbb{M}; \mathbb{C})$ -data,  $s \geq 0$ . Moreover, P. Gérard and E. Lenzmann have completely classified all traveling waves of (1.3) and obtained the turbulent behavior of multi-solitons in [22]. The traveling waves of both (1.3) and (1.4) are characterized in R. Badreddine's work [3]. The  $1 \times 1$  CMS dNLS equations (1.3) and (1.4) can be considered as  $L^2$ -critical versions of the Benjamin–Ono equation,

$$\partial_t v = \partial_x (|D|v - v^2), \quad (t, x) \in \mathbb{R} \times \mathbb{M}. \quad (1.9)$$

Inspired from the cubic Schrödinger system of two variables in Zakharov–Shabat [48] and Grébert–Kappeler [25], and the spin generalization of the Benjamin–Ono equation introduced in Berntson–Langmann–Lenells [4], Gérard [9], we generalize both (1.3) and (1.4) to the matrix intertwined system (1.1) in order to study the perturbation of the original equations (1.3), (1.4) and understand their deep structure of integrable system. Since the intertwined system (1.1) has the same linear part as both (1.3) and (1.4), Kato's classical iterative scheme can be used to prove the  $H_+^s(\mathbb{M})$  local wellposedness of (1.1), thanks to the ideas in subsection 2.1 of Gérard–Lenzmann [22].

**Proposition 1.1.** *Given any  $s > \frac{3}{2}$ ,  $M, N \in \mathbb{N}_+$  and  $R > 0$ ,  $\mathbb{M} \in \{\mathbb{T}, \mathbb{R}\}$ , there exists  $T(R) > 0$  such that, if  $U_0, V_0 \in H_+^s(\mathbb{M}; \mathbb{C}^{M \times N})$  with  $\|U_0\|_{H_+^s(\mathbb{M}; \mathbb{C}^{M \times N})} + \|V_0\|_{H_+^s(\mathbb{M}; \mathbb{C}^{M \times N})} \leq R$ , there exists a unique pair of functions  $U, V \in C^0([-T(R), T(R)]; H_+^s(\mathbb{M}; \mathbb{C}^{M \times N}))$  solving the intertwined system (1.1) with initial datum  $(U(0), V(0)) = (U_0, V_0)$ .*

The goal of this paper is to provide an explicit expression for all general solutions  $(U(t), V(t))$  to the intertwined system (1.1) on  $\mathbb{T}$  and on  $\mathbb{R}$  in terms of the initial datum  $(U(0), V(0)) = (U_0, V_0)$  and the time variable  $t$ . In order to establish this explicit formula, we need to generalize Gérard–Lenzmann's Lax pair structure of (1.3) and (1.4) to the intertwined system (1.1).

## 1.1 The Lax pair

Let  $\Pi_{\geq 0} : L^2(\mathbb{M}; \mathbb{C}^{M \times N}) \rightarrow L^2(\mathbb{M}; \mathbb{C}^{M \times N})$  denotes Szegő operator which cancels all negative Fourier modes and preserves the nonnegative Fourier modes, i.e. for any  $U = \sum_{n \in \mathbb{Z}} \hat{U}(n)e^{inx} \in L^2(\mathbb{T}; \mathbb{C}^{M \times N})$ , for any  $V \in L^2(\mathbb{R}; \mathbb{C}^{M \times N})$ , we have

$$\begin{aligned} (\Pi_{\geq 0}U)(x) &= \sum_{n \geq 0} \hat{U}(n)e^{inx}, \quad \hat{U}(n) = \frac{1}{2\pi} \int_0^{2\pi} U(x)e^{-inx} dx \in \mathbb{C}^{M \times N}, \quad \forall n \in \mathbb{Z}, \\ (\widehat{\Pi_{\geq 0}V})(\xi_1) &= \hat{V}(\xi) \in \mathbb{C}^{M \times N}, \quad (\widehat{\Pi_{\geq 0}V})(\xi_2) = 0_{M \times N}, \quad \forall \xi_1 > 0 > \xi_2. \end{aligned} \quad (1.10)$$

Since  $(D + |D|) = 2D\Pi_{\geq 0}$ , the intertwined system (1.1) also reads as

$$\begin{cases} \partial_t U = i\partial_x^2 U + U\partial_x \Pi_{\geq 0}(V^*U) + V\partial_x \Pi_{\geq 0}(U^*U), \\ \partial_t V = i\partial_x^2 V + V\partial_x \Pi_{\geq 0}(U^*V) + U\partial_x \Pi_{\geq 0}(V^*V), \end{cases} \quad t \in \mathfrak{J}, \quad x \in \mathbb{M}. \quad (1.11)$$

The matrix-valued Hardy spaces are denoted by  $L_+^2(\mathbb{M}; \mathbb{C}^{M \times N}) := \Pi_{\geq 0}(L^2(\mathbb{M}; \mathbb{C}^{M \times N}))$ ,  $\mathbb{M} \in \{\mathbb{T}, \mathbb{R}\}$ . The filtered Sobolev spaces are given by  $H_+^s(\mathbb{M}; \mathbb{C}^{M \times N}) := H^s \cap L_+^2(\mathbb{M}; \mathbb{C}^{M \times N}) = \Pi_{\geq 0}(H^s(\mathbb{M}; \mathbb{C}^{M \times N}))$ ,  $\forall s \geq 0$ . For any  $d \in \mathbb{N}_+$ , the (right) Toeplitz operator of symbol  $V \in L^2(\mathbb{M}; \mathbb{C}^{M \times N})$  is defined by

$$\mathbf{T}_V = \mathbf{T}_V^{(r)} : G \in H_+^1(\mathbb{M}; \mathbb{C}^{N \times d}) \mapsto \mathbf{T}_V(G) = \Pi_{\geq 0}(VG) \in L_+^2(\mathbb{M}; \mathbb{C}^{M \times d}). \quad (1.12)$$

If  $\Psi = (U, V) \in H_+^1(\mathbb{M}; (\mathbb{C}^{M \times N})^2)$ , the operators  $\mathbf{L}_\Psi, \mathbf{B}_\Psi : H_+^1(\mathbb{M}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{M}; \mathbb{C}^{M \times d})$  are given by

$$\begin{aligned} \mathbf{L}_\Psi &= \mathbf{D} - \frac{1}{2}(\mathbf{T}_U \mathbf{T}_{V^*} + \mathbf{T}_V \mathbf{T}_{U^*}), \quad \mathbf{D} = -i\partial_x; \\ \mathbf{B}_\Psi &= \frac{1}{2}(\mathbf{T}_U \mathbf{T}_{\partial_x V^*} + \mathbf{T}_V \mathbf{T}_{\partial_x U^*} - \mathbf{T}_{\partial_x V} \mathbf{T}_{U^*} - \mathbf{T}_{\partial_x U} \mathbf{T}_{V^*}) + \frac{i}{4}(\mathbf{T}_U \mathbf{T}_{V^*} + \mathbf{T}_V \mathbf{T}_{U^*})^2. \end{aligned} \quad (1.13)$$

The operator  $\mathbf{L}_\Psi$  is a densely defined self-adjoint operator on  $L_+^2(\mathbb{M}; \mathbb{C}^{M \times d})$ , which is bounded from below. If  $\Psi = (U, V) \in H_+^2(\mathbb{M}; (\mathbb{C}^{M \times N})^2)$  in addition, the operator  $i\mathbf{B}_\Psi$  is bounded self-adjoint on  $L_+^2(\mathbb{M}; \mathbb{C}^{M \times d})$ ,  $\mathbb{M} \in \{\mathbb{T}, \mathbb{R}\}$ . The first result of this paper is stated as follows.

**Theorem 1.2.** *Given  $M, N, d \in \mathbb{N}_+$  and  $\mathbb{M} \in \{\mathbb{T}, \mathbb{R}\}$ , if  $\Psi = (U, V) \in C^0(T^-, T^+; H_+^2(\mathbb{M}; (\mathbb{C}^{M \times N})^2))$  solves the intertwined system (1.11), then following Heisenberg–Lax equation holds*

$$\frac{d}{dt} \mathbf{L}_{\Psi(t)} = [\mathbf{B}_{\Psi(t)}, \mathbf{L}_{\Psi(t)}] : H_+^1(\mathbb{M}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{M}; \mathbb{C}^{M \times d}), \quad \forall t \in (T^-, T^+). \quad (1.14)$$

Then  $(\mathbf{L}_\Psi, \mathbf{B}_\Psi)$  defined in (1.13) is a Lax pair of the intertwined derivative Schrödinger system (1.1).

**Remark 1.3.** *In the case  $V = \mu U$ , the Lax pair structure for (1.3) and (1.4), which is constructed in Gérard–Lenzmann [22] and Badreddine [2], can be revisited.*

1. When  $\mu = 1$ , the intertwined system (1.1) becomes the focusing equation (1.3) and the Lax pair  $(\mathbf{L}_\Psi, \mathbf{B}_\Psi)$  is reduced to the following operators,

$$\mathfrak{L}_U^- = \mathbf{D} - \mathbf{T}_U \mathbf{T}_{U^*}, \quad \mathfrak{B}_U^- = \mathbf{T}_U \mathbf{T}_{\partial_x U^*} - \mathbf{T}_{\partial_x U} \mathbf{T}_{U^*} + i(\mathbf{T}_U \mathbf{T}_{U^*})^2. \quad (1.15)$$

2. When  $\mu = -1$ , the intertwined system (1.1) becomes the defocusing equation (1.4) and the Lax pair  $(\mathbf{L}_\Psi, \mathbf{B}_\Psi)$  is reduced to the following operators,

$$\mathfrak{L}_U^+ = \mathbf{D} + \mathbf{T}_U \mathbf{T}_{U^*}, \quad \mathfrak{B}_U^+ = \mathbf{T}_{\partial_x U} \mathbf{T}_{U^*} - \mathbf{T}_U \mathbf{T}_{\partial_x U^*} + i(\mathbf{T}_U \mathbf{T}_{U^*})^2. \quad (1.16)$$

3. When  $\operatorname{Re} \mu = 0$ , the system (1.1) is reduced to the linear Schrödinger equation (1.5) and the Lax pair degenerates, i.e.  $(\mathbf{L}_\Psi, \mathbf{B}_\Psi) = (\mathbf{D}, 0)$ .

As a consequence, the intertwined system (1.1) is an integrable perturbation of equations (1.3), (1.4) and (1.5) simultaneously.

Inspired from the works Gérard–Grellier [15], Gérard [10], Gérard–Lenzmann [22], Badreddine [2] and Sun [44], we establish an explicit expression for general solutions to the intertwined derivative Schrödinger system (1.1) both on the torus and on the line, thanks to its Lax pair structure (1.14) and the conjugation acting method.

## 1.2 The explicit formula on the torus

In the space-periodic case, every function  $U \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$  has Fourier series expression, i.e.

$$U = \sum_{n \geq 0} \hat{U}(n) \mathbf{e}_n, \quad \hat{U}(n) = \frac{1}{2\pi} \int_0^{2\pi} U(x) e^{-inx} dx = \mathbf{I}(U \mathbf{e}_{-n}) \in \mathbb{C}^{M \times N}, \quad \forall n \in \mathbb{Z}, \quad (1.17)$$

where  $\mathbf{e}_n : x \in \mathbb{T} \mapsto e^{inx} \in \mathbb{C}$  and  $\mathbf{I}$  denotes the integral operator, i.e.

$$\mathbf{I} : F \in L^1(\mathbb{T}; \mathbb{C}^{M \times N}) \mapsto \mathbf{I}(F) = \hat{F}(0) = \frac{1}{2\pi} \int_0^{2\pi} F(x) dx \in \mathbb{C}^{M \times N}. \quad (1.18)$$

The Hardy space  $L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$  is equipped with the shift operator  $\mathbf{S}$ , which is defined by

$$\mathbf{S} : F \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N}) \mapsto \mathbf{e}_1 F \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N}). \quad (1.19)$$

Its  $L_+^2$ -adjoint is denoted by  $\mathbf{S}^* \in \mathcal{B}(L_+^2(\mathbb{T}; \mathbb{C}^{M \times N}))$ . Then every nonnegative Fourier mode of an arbitrary solution to the intertwined system (1.1) can be expressed explicitly in terms of its initial datum and the time variable, leading to the explicit formula of the solution.

**Theorem 1.4.** *If  $U = \sum_{n \geq 0} \hat{U}(t, n) \mathbf{e}_n, V = \sum_{n \geq 0} \hat{V}(t, n) \mathbf{e}_n \in C^0(T^-, T^+; H_+^3(\mathbb{T}; \mathbb{C}^{M \times N}))$  solve the cubic intertwined derivative Schrödinger system (1.1) for some integers  $M, N \in \mathbb{N}_+$  with initial data  $\Psi_0 = (U_0, V_0) = (U(0), V(0)) \in H_+^3(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$ , the following explicit formula holds:*

$$\hat{U}(t, n) = e^{-int} \mathbf{I}((e^{-2it\mathbf{L}_{\Psi_0}} \mathbf{S}^*)^n(U_0)), \quad \hat{V}(t, n) = e^{-int} \mathbf{I}((e^{-2it\mathbf{L}_{\Psi_0}} \mathbf{S}^*)^n(V_0)), \quad (1.20)$$

where  $\mathbf{I}$  is given by (1.18),  $\forall n \in \mathbb{N}, \forall t \in (T_-, T_+)$ .

Inspired from subsection 2.2 of Badreddine [2], we have the following global wellposedness result of (1.1) on the torus.

**Proposition 1.5.** *Given any  $s > \frac{3}{2}$  and  $M, N \in \mathbb{N}_+$ , if  $\Psi_0 = (U_0, V_0) \in H_+^s(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$  satisfies that*

$$\|U_0\|_{H_+^s(\mathbb{T}; \mathbb{C}^{M \times N})} \|V_0\|_{H_+^s(\mathbb{T}; \mathbb{C}^{M \times N})} < 1, \quad (1.21)$$

then there exists a unique global solution  $\Psi = (U, V) \in C^0(\mathbb{R}; H_+^s(\mathbb{T}; (\mathbb{C}^{M \times N})^2))$  to the intertwined system (1.1) with initial datum  $(U(0), V(0)) = (U_0, V_0)$ . Moreover, the global solution is uniformly bounded in time, i.e.  $\sup_{t \in \mathbb{R}} \|\Psi(t)\|_{H_+^s(\mathbb{T}; (\mathbb{C}^{M \times N})^2)} < +\infty$ .

### 1.3 The explicit formula on the line

Every function  $U \in L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$  can be extended to a holomorphic function on the upper half plane  $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im}z > 0\}$  via its Poisson integral,

$$\underline{U}(z) = \mathcal{P}[U](z) := \int_{\mathbb{R}} \mathfrak{P}_y(x-t)U(t)dt = \frac{1}{2\pi} \int_0^{+\infty} e^{iz\xi} \hat{U}(\xi) d\xi, \quad \forall z = x + yi \in \mathbb{C}_+, \quad (1.22)$$

where  $\mathfrak{P}_y(x) = \frac{y}{\pi(x^2+y^2)}$ , denotes the Poisson kernel on  $\mathbb{C}_+$ . For any fixed  $y > 0$ , when restricting to  $\mathbb{R}$ , the function  $\tau_{-yi}\underline{U}|_{\mathbb{R}} : x \in \mathbb{R} \mapsto \underline{U}(x + yi) \in \mathbb{C}^{M \times N} \in L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$  satisfies that

$$\sup_{y>0} \|\tau_{-yi}\underline{U}|_{\mathbb{R}}\|_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} \leq \|U\|_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})}, \quad \lim_{y \rightarrow 0^+} \|\tau_{-yi}\underline{U}|_{\mathbb{R}} - U\|_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} = 0. \quad (1.23)$$

The mapping  $U \mapsto \underline{U}$  provides a  $\mathbb{C}$ -Hilbert isomorphism from  $L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$  to the Hardy space

$$\mathbb{H}^2(\mathbb{C}_+; \mathbb{C}^{M \times N}) := \{U \in \text{Hol}(\mathbb{C}_+; \mathbb{C}^{M \times N}) : \sup_{y>0} \int_{\mathbb{R}} \text{tr}(\underline{U}(x+yi)(\underline{U}(x+yi))^*) dx < +\infty\}. \quad (1.24)$$

Instead of one single shift operator, we consider the Lax–Beurling shift semigroup  $(\mathbf{S}(\eta))_{\eta \geq 0}$  of isometries on  $L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$  and its adjoint semigroup  $(\mathbf{S}(\eta)^*)_{\eta \geq 0}$ , which are defined as

$$\mathbf{S}(\eta)U = \mathbf{e}_\eta U, \quad \mathbf{S}(\eta)^*U = \Pi_{\geq 0}(\mathbf{e}_{-\eta}U), \quad \forall U \in L_+^2(\mathbb{R}; \mathbb{C}^{M \times N}), \quad \forall \eta \geq 0, \quad (1.25)$$

where  $\mathbf{e}_\eta(x) = e^{inx}$ ,  $\forall x \in \mathbb{R}$ . Both  $(\mathbf{S}(\eta))_{\eta \geq 0}$  and  $(\mathbf{S}(\eta)^*)_{\eta \geq 0}$  are contraction semi-groups on  $L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$ . Let  $-i\mathbf{G}$  denote the infinitesimal generator of  $(\mathbf{S}(\eta)^*)_{\eta \geq 0}$ , i.e.  $\mathbf{G}(F) := i \frac{d}{d\eta} \Big|_{\eta=0^+} \mathbf{S}(\eta)^*(F) \in L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$ ,  $\forall F \in \text{Dom}(\mathbf{G})^{M \times N}$ , where the domain of definition of  $\mathbf{G}$  is given by

$$\text{Dom}(\mathbf{G})^{M \times N} := \{F \in L_+^2(\mathbb{R}; \mathbb{C}^{M \times N}) : \hat{F}|_{\mathbb{R}_+^*} \in H^1((0, +\infty); \mathbb{C}^{M \times N})\}, \quad (1.26)$$

with  $\mathbb{R}_+^* = (0, +\infty)$ . For any  $\delta > 0$ , the Sobolev extension theorem yields that  $\varphi(0^+) := \lim_{\epsilon \rightarrow 0^+} \varphi(\epsilon)$  exists for any  $\varphi \in H^1((0, \delta); \mathbb{C})$ . For any  $F \in L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$  such that  $\hat{F} \in H^1((0, \delta); \mathbb{C}^{M \times N})$  for some  $\delta > 0$ , we define that

$$\mathcal{J}(F) := \hat{F}(0^+) \in \mathbb{C}^{M \times N}. \quad (1.27)$$

Precisely, for any  $F \in \text{Dom}(\mathbf{G})^{M \times N}$ , we have

$$\mathbf{G}(F)(x) = xF(x) - \frac{i}{2\pi} \hat{F}(0^+), \quad \forall x \in \mathbb{R}. \quad (1.28)$$

In order to establish an explicit formula for general solutions to the intertwined system (1.1) on  $\mathbb{R}$ , it is enough to express their Poisson integrals explicitly in terms of the initial datum and the time variable, thanks to the  $L_+^2$ -convergence formula (1.23).

**Theorem 1.6.** *Given  $M, N \in \mathbb{N}_+$ , if  $U, V \in C^0(T^-, T^+; H_+^3(\mathbb{R}; \mathbb{C}^{M \times N}))$  solve the intertwined system (1.1) with initial datum  $\Psi_0 = (U_0, V_0) = (U(0), V(0)) \in H_+^3(\mathbb{R}; (\mathbb{C}^{M \times N})^2)$ , then we have the following explicit formula of the Poisson integrals of  $U(t)$  and  $V(t)$ ,*

$$\underline{U}(t, z) = \frac{1}{2\pi i} \mathcal{J} \left( (\mathbf{G} + 2t\mathbf{L}_{\Psi_0} - z)^{-1} (U_0) \right), \quad \underline{V}(t, z) = \frac{1}{2\pi i} \mathcal{J} \left( (\mathbf{G} + 2t\mathbf{L}_{\Psi_0} - z)^{-1} (V_0) \right), \quad (1.29)$$

for any  $z \in \mathbb{C}_+$ , where  $\mathcal{J}$  is given by (1.27),  $\forall t \in (T^-, T_+)$ .

Then  $U(t)$  and  $V(t)$  can be described as the  $L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$ -limit of their Poisson integrals  $\underline{U}(t)$  and  $\underline{V}(t)$  when  $\text{Im}z \rightarrow 0^+$ , due to formula (1.23). Inspired from subsection 2.3 of Gérard–Lenzmann [22], we have the following global wellposedness result of (1.1) on the line.

**Proposition 1.7.** *Given any  $s > \frac{3}{2}$  and  $M, N \in \mathbb{N}_+$ , if  $\Psi_0 = (U_0, V_0) \in H_+^s(\mathbb{R}; (\mathbb{C}^{M \times N})^2)$  satisfies that*

$$\|U_0\|_{H_+^s(\mathbb{R}; \mathbb{C}^{M \times N})} \|V_0\|_{H_+^s(\mathbb{R}; \mathbb{C}^{M \times N})} < 2\pi, \quad (1.30)$$

then there exists a unique global solution  $\Psi = (U, V) \in C^0(\mathbb{R}; H_+^s(\mathbb{R}; (\mathbb{C}^{M \times N})^2))$  to the intertwined system (1.1) on  $\mathbb{R}$  with initial datum  $(U(0), V(0)) = (U_0, V_0)$ . Moreover, the global solution is uniformly bounded in time, i.e.  $\sup_{t \in \mathbb{R}} \|\Psi(t)\|_{H_+^s(\mathbb{R}; (\mathbb{C}^{M \times N})^2)} < +\infty$ .

## 1.4 Organization of the paper

This paper is organized as follows. Theorem 1.2 about the Lax pair of (1.1) is proved in section 2. Section 3 and 4 are devoted to establishing the explicit formulas of solutions of the intertwined system (1.1) on the torus and on the line respectively.

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## 2 Proof of the Lax pair structure

This section is devoted to establishing the Lax pair structure (1.14) of the intertwined system (1.1). In order to prove theorem 1.2, we need some preliminaries about matrix-valued functions and Toeplitz operators.

In the space-periodic case, if  $U = \sum_{n \in \mathbb{Z}} \hat{U}(n) \mathbf{e}_n \in L^2(\mathbb{T}; \mathbb{C}^{M \times N})$ , then

$$\Pi_{<0} U = (\Pi_{\geq 0}(U^*))^* - \hat{U}(0) \in L^2_-(\mathbb{T}; \mathbb{C}^{M \times N}), \quad (\Pi_{\geq 0} U)^* = \Pi_{<0}(U^*) + (\hat{U}(0))^* \in L^2(\mathbb{T}; \mathbb{C}^{N \times M}) \quad (2.1)$$

**Lemma 2.1.** *Given  $M, N, d \in \mathbb{N}_+$ , if  $A \in L^2_+(\mathbb{T}; \mathbb{C}^{M \times N})$  and  $B \in L^2_+(\mathbb{T}; \mathbb{C}^{N \times d})$  and one of  $A, B$  is essentially bounded, then  $AB \in L^2_+(\mathbb{T}; \mathbb{C}^{M \times d})$ .*

*Proof.* See lemma 2.4 of Sun [44]. □

Thanks to Lemma 2.1, the intertwined derivative Schrödinger system (1.11) reads as

$$\begin{cases} \partial_t U = i \partial_x^2 U + \mathbf{T}_U \partial_x \mathbf{T}_{V^*}(U) + \mathbf{T}_V \partial_x \mathbf{T}_{U^*}(U), \\ \partial_t V = i \partial_x^2 V + \mathbf{T}_U \partial_x \mathbf{T}_{V^*}(V) + \mathbf{T}_V \partial_x \mathbf{T}_{U^*}(V). \end{cases} \quad (2.2)$$

**Lemma 2.2.** *For any  $M, N, P \in \mathbb{N}_+$ , we choose  $A, W \in L^2(\mathbb{T}; \mathbb{C}^{M \times N})$ ,  $B, V \in L^2(\mathbb{T}; \mathbb{C}^{N \times P})$ , such that  $B \in L^2_+(\mathbb{T}; \mathbb{C}^{N \times P})$  and  $W - \widehat{W}(0) \in L^2_-(\mathbb{T}; \mathbb{C}^{M \times N})$ .*

- (a). *If one of  $A, B$  belongs to  $H^1(\mathbb{T})$ , then  $\mathbf{T}_A \mathbf{T}_B = \mathbf{T}_{AB} : H^1_+(\mathbb{T}; \mathbb{C}^{P \times d}) \rightarrow L^2_+(\mathbb{T}; \mathbb{C}^{M \times d})$ .*
- (b). *If one of  $W, V$  belongs to  $H^1(\mathbb{T})$ , then  $\mathbf{T}_W \mathbf{T}_V = \mathbf{T}_{WV} : H^1_+(\mathbb{T}; \mathbb{C}^{P \times d}) \rightarrow L^2_+(\mathbb{T}; \mathbb{C}^{M \times d})$ .*

*Proof.* If  $F \in H^1_+(\mathbb{T}; \mathbb{C}^{P \times d})$ , we have  $\mathbf{T}_A \mathbf{T}_B(F) = ABF = \mathbf{T}_{AB}(F) \in L^2_+(\mathbb{T}; \mathbb{C}^{M \times d})$  and  $W \Pi_{<0}(VF) = \sum_{l \leq -1} (\sum_{n=l+1}^0 \widehat{W}(n) \widehat{VF}(l-n)) \mathbf{e}_l \in L^2_-(\mathbb{T}; \mathbb{C}^{M \times d})$ . Then  $\mathbf{T}_W \mathbf{T}_V(F) = \Pi_{\geq 0}(W(VF - \Pi_{<0}(VF))) = \Pi_{\geq 0}(WV F) = \mathbf{T}_{WV}(F) \in L^2_+(\mathbb{T}; \mathbb{C}^{M \times d})$ . □

In the space non periodic case, if  $F \in L^2(\mathbb{R}; \mathbb{C}^{M \times N})$ ,  $\Pi_{<0} F = (\Pi_{\geq 0}(F^*))^* \in L^2_-(\mathbb{R}; \mathbb{C}^{M \times N})$  and

$$F = \Pi_{\geq 0} F + \Pi_{<0} F = \Pi_{\geq 0} F + (\Pi_{\geq 0}(F^*))^*. \quad (2.3)$$

**Lemma 2.3.** *Given  $A \in L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})$  and  $B \in L^2_+(\mathbb{R}; \mathbb{C}^{N \times d})$  for some  $M, N, d \in \mathbb{N}_+$ , if one of  $A, B$  is essentially bounded, then  $AB \in L^2_+(\mathbb{R}; \mathbb{C}^{M \times d})$ .*

*Proof.* If  $f \in L^2_+(\mathbb{R}; \mathbb{C})$  and  $g \in L^\infty \cap L^2_+(\mathbb{R}; \mathbb{C})$ , then  $\widehat{fg}(\xi) = \int_{\mathbb{R}} \hat{f}(\xi - \eta) \hat{g}(\eta) \frac{d\eta}{2\pi} = 0, \forall \xi < 0$ . □

**Corollary 2.4.** *Given  $A \in L^2_-(\mathbb{R}; \mathbb{C}^{M \times N})$  and  $B \in L^2_-(\mathbb{R}; \mathbb{C}^{N \times d})$  for some  $M, N, d \in \mathbb{N}_+$ , if one of  $A, B$  is essentially bounded, then  $AB \in L^2_-(\mathbb{R}; \mathbb{C}^{M \times d})$ .*

**Lemma 2.5.** *For any  $M, N, P \in \mathbb{N}_+$ , we choose  $A, W \in L^2(\mathbb{R}; \mathbb{C}^{M \times N})$ ,  $B, V \in L^2(\mathbb{R}; \mathbb{C}^{N \times P})$ , such that  $B \in L^2_+(\mathbb{R}; \mathbb{C}^{N \times P})$  and  $W \in L^2_-(\mathbb{R}; \mathbb{C}^{M \times N})$ .*

- (a). *If one of  $A, B$  belongs to  $H^1(\mathbb{R})$ , then  $\mathbf{T}_A \mathbf{T}_B = \mathbf{T}_{AB} : H^1_+(\mathbb{R}; \mathbb{C}^{P \times d}) \rightarrow L^2_+(\mathbb{R}; \mathbb{C}^{M \times d})$ .*
- (b). *If one of  $W, V$  belongs to  $H^1(\mathbb{R})$ , then  $\mathbf{T}_W \mathbf{T}_V = \mathbf{T}_{WV} : H^1_+(\mathbb{R}; \mathbb{C}^{P \times d}) \rightarrow L^2_+(\mathbb{R}; \mathbb{C}^{M \times d})$ .*

*Proof.* This is a direct consequence of lemma 2.3 and corollary 2.4 by following the proof of lemma 2.2. □

We are ready to prove the Heisenberg–Lax equation (1.14).

*Proof of theorem 1.2.* We prove at first (1.14) for the space-periodic case, i.e. assume that  $\mathbb{M} = \mathbb{T}$ . If  $U, V \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times N})$ , we have  $[D, \mathbf{T}_U] = \mathbf{T}_{DU} : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$  and

$$[\mathbf{T}_U \mathbf{T}_{V^*}, D] = i(\mathbf{T}_U \mathbf{T}_{\partial_x V^*} + \mathbf{T}_{\partial_x U} \mathbf{T}_{V^*}) : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}). \quad (2.4)$$

We define  $\mathcal{A} := \mathcal{A}(U, V) = \mathbf{T}_U \mathbf{T}_{V^*} + \mathbf{T}_V \mathbf{T}_{U^*} \in B(H_+^s(\mathbb{T}; \mathbb{C}^{M \times d}))$  for some  $s = 0$  or  $\frac{1}{2} < s \leq 1$ , and  $\Omega := \Omega(U, V) = \mathbf{T}_U \mathbf{T}_{\partial_x V^*} + \mathbf{T}_V \mathbf{T}_{\partial_x U^*} - \mathbf{T}_{\partial_x V} \mathbf{T}_{U^*} - \mathbf{T}_{\partial_x U} \mathbf{T}_{V^*} \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$ . Then

$$\mathbf{L}_\Psi = D - \frac{1}{2}\mathcal{A}, \quad \mathbf{B}_\Psi = \frac{1}{2}\Omega + \frac{i}{4}\mathcal{A}^2, \quad \text{where } \Psi = (U, V). \quad (2.5)$$

Formula (2.4) implies  $-i[\mathcal{A}, D] = \mathbf{T}_U \mathbf{T}_{\partial_x V^*} + \mathbf{T}_{\partial_x V} \mathbf{T}_{U^*} + \mathbf{T}_{\partial_x U} \mathbf{T}_{V^*} + \mathbf{T}_V \mathbf{T}_{\partial_x U^*}$ , so

$$\begin{aligned} i[\mathcal{A}^2, D] &= i\mathcal{A}[\mathcal{A}, D] + i[\mathcal{A}, D]\mathcal{A} : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \\ &= -\mathcal{A} \mathbf{T}_U \mathbf{T}_{\partial_x V^*} - \mathcal{A} \mathbf{T}_{\partial_x V} \mathbf{T}_{U^*} - \mathcal{A} \mathbf{T}_{\partial_x U} \mathbf{T}_{V^*} - \mathcal{A} \mathbf{T}_V \mathbf{T}_{\partial_x U^*} \\ &\quad - \mathbf{T}_U \mathbf{T}_{\partial_x V^*} \mathcal{A} - \mathbf{T}_{\partial_x V} \mathbf{T}_{U^*} \mathcal{A} - \mathbf{T}_{\partial_x U} \mathbf{T}_{V^*} \mathcal{A} - \mathbf{T}_V \mathbf{T}_{\partial_x U^*} \mathcal{A}. \end{aligned} \quad (2.6)$$

The operator  $[\Omega, \mathcal{A}] : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$  is given by

$$\begin{aligned} [\Omega, \mathcal{A}] &= -\mathcal{A} \mathbf{T}_U \mathbf{T}_{\partial_x V^*} + \mathcal{A} \mathbf{T}_{\partial_x V} \mathbf{T}_{U^*} + \mathcal{A} \mathbf{T}_{\partial_x U} \mathbf{T}_{V^*} - \mathcal{A} \mathbf{T}_V \mathbf{T}_{\partial_x U^*} \\ &\quad + \mathbf{T}_U \mathbf{T}_{\partial_x V^*} \mathcal{A} - \mathbf{T}_{\partial_x V} \mathbf{T}_{U^*} \mathcal{A} - \mathbf{T}_{\partial_x U} \mathbf{T}_{V^*} \mathcal{A} + \mathbf{T}_V \mathbf{T}_{\partial_x U^*} \mathcal{A}. \end{aligned} \quad (2.7)$$

Plugging  $\mathcal{A} = \mathbf{T}_U \mathbf{T}_{V^*} + \mathbf{T}_V \mathbf{T}_{U^*}$  into (2.6) and (2.7), we obtain that

$$\begin{aligned} i[\mathcal{A}^2, D] - [\Omega, \mathcal{A}] &: H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \\ &= -2\mathbf{T}_U (\mathbf{T}_{V^*} \mathbf{T}_{\partial_x V} + \mathbf{T}_{\partial_x V^*} \mathbf{T}_V) \mathbf{T}_{U^*} - 2\mathbf{T}_V (\mathbf{T}_{U^*} \mathbf{T}_{\partial_x U} + \mathbf{T}_{\partial_x U^*} \mathbf{T}_U) \mathbf{T}_{V^*} \\ &\quad - 2\mathbf{T}_V (\mathbf{T}_{U^*} \mathbf{T}_{\partial_x U} + \mathbf{T}_{\partial_x U^*} \mathbf{T}_U) \mathbf{T}_{V^*} - 2\mathbf{T}_U (\mathbf{T}_{V^*} \mathbf{T}_{\partial_x U} + \mathbf{T}_{\partial_x V^*} \mathbf{T}_U) \mathbf{T}_{V^*}. \end{aligned} \quad (2.8)$$

Since  $(\partial_x(U^*V))^\wedge(0) = 0_{N \times N}$ , formula (2.1) implies that  $\partial_x(U^*V) = \partial_x \Pi_{\geq 0}(U^*V) + (\partial_x \Pi_{\geq 0}(V^*U))^*$ . For any  $W, Y \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times N})$ , lemma 2.2 yields that

$$\begin{aligned} \mathbf{T}_W (\mathbf{T}_{U^*} \mathbf{T}_{\partial_x V} + \mathbf{T}_{\partial_x U^*} \mathbf{T}_V) \mathbf{T}_{Y^*} &= \mathbf{T}_W \mathbf{T}_{\partial_x(U^*V)} \mathbf{T}_{Y^*} : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \\ &= \mathbf{T}_W \mathbf{T}_{\partial_x \Pi_{\geq 0}(U^*V)} \mathbf{T}_{Y^*} + \mathbf{T}_W \mathbf{T}_{(\partial_x \Pi_{\geq 0}(V^*U))^*} \mathbf{T}_{Y^*} = \mathbf{T}_W \mathbf{T}_{\partial_x \Pi_{\geq 0}(U^*V)} \mathbf{T}_{Y^*} + \mathbf{T}_W \mathbf{T}_{(Y \partial_x \Pi_{\geq 0}(V^*U))^*}. \end{aligned} \quad (2.9)$$

Plugging (2.9) into (2.8), we obtain that  $\forall U, V \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times N})$ , then

$$\begin{aligned} i[\mathcal{A}^2, D] - [\Omega, \mathcal{A}] &: H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \\ &= -2\mathbf{T}_U \mathbf{T}_{\partial_x \Pi_{\geq 0}(V^*V) + V \partial_x \Pi_{\geq 0}(U^*V)} \mathbf{T}_{U^*} - 2\mathbf{T}_U \mathbf{T}_{(U \partial_x \Pi_{\geq 0}(V^*V) + V \partial_x \Pi_{\geq 0}(U^*V))^*} \\ &\quad - 2\mathbf{T}_U \mathbf{T}_{\partial_x \Pi_{\geq 0}(V^*U) + V \partial_x \Pi_{\geq 0}(U^*U)} \mathbf{T}_{V^*} - 2\mathbf{T}_V \mathbf{T}_{(U \partial_x \Pi_{\geq 0}(V^*U) + V \partial_x \Pi_{\geq 0}(U^*U))^*}. \end{aligned} \quad (2.10)$$

Now assume that  $U, V \in H_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$ , then  $\partial_x U, \partial_x V \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times N})$  and (2.4) yields that

$$[\Omega, D] = -\mathbf{T}_U \mathbf{T}_{(i\partial_x^2 V)^*} - \mathbf{T}_V \mathbf{T}_{(i\partial_x^2 U)^*} - \mathbf{T}_{i\partial_x^2 V} \mathbf{T}_{U^*} - \mathbf{T}_{i\partial_x^2 U} \mathbf{T}_{V^*} : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2. \quad (2.11)$$

As a consequence, formulas (2.5), (2.10) and (2.11) imply that

$$\begin{aligned} 2[\mathbf{B}_\Psi, \mathbf{L}_\Psi] &= [\Omega, D] + \frac{i}{2}[\mathcal{A}^2, D] - \frac{1}{2}[\Omega, \mathcal{A}] : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \\ &= -\mathbf{T}_U \mathbf{T}_{(i\partial_x^2 V + U \partial_x \Pi_{\geq 0}(V^*V) + V \partial_x \Pi_{\geq 0}(U^*V))^*} - \mathbf{T}_{i\partial_x^2 V + U \partial_x \Pi_{\geq 0}(V^*V) + V \partial_x \Pi_{\geq 0}(U^*V)} \mathbf{T}_{U^*} \\ &\quad - \mathbf{T}_V \mathbf{T}_{(i\partial_x^2 U + U \partial_x \Pi_{\geq 0}(V^*U) + V \partial_x \Pi_{\geq 0}(U^*U))^*} - \mathbf{T}_{i\partial_x^2 U + U \partial_x \Pi_{\geq 0}(V^*U) + V \partial_x \Pi_{\geq 0}(U^*U)} \mathbf{T}_{V^*}, \end{aligned} \quad (2.12)$$

$\forall \Psi = (U, V) \in H_+^2(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$ . So  $[\mathbf{B}_\Psi, \mathbf{L}_\Psi] = d\mathbf{L}_\Psi(X_{\mathbf{E}}(\Psi))$ , where  $X_{\mathbf{E}}(\Psi) = (X_{\mathbf{E}}^{(1)}(\Psi), X_{\mathbf{E}}^{(2)}(\Psi))$  denotes the Hamiltonian vector field of the energy function of (1.11), i.e.

$$\begin{cases} X_{\mathbf{E}}^{(1)}(\Psi) = i\partial_x^2 U + U\partial_x \Pi_{\geq 0}(V^*U) + V\partial_x \Pi_{\geq 0}(U^*U), \\ X_{\mathbf{E}}^{(2)}(\Psi) = i\partial_x^2 V + V\partial_x \Pi_{\geq 0}(U^*V) + U\partial_x \Pi_{\geq 0}(V^*V). \end{cases}$$

In the space non periodic case, i.e. assume that  $\mathbb{M} = \mathbb{R}$ , formula (2.8) can be obtained by the same calculus as above. However,  $L^2(\mathbb{R})$ -functions are almost everywhere defined, then the zero mode no longer plays an important role in (2.3). Thanks to (2.3), formula (2.9) still holds for all  $W, Y \in H_+^1(\mathbb{R}; \mathbb{C}^{M \times N})$  by using lemma 2.5. The rest of proof is the same as the space-periodic case.  $\square$

### 3 Proof of the explicit formula on $\mathbb{T}$

The explicit formula (1.20) for general solutions of the intertwined system (1.1) is proved in this section. Let  $\mathfrak{D} := \{z \in \mathbb{C} : |z| < 1\}$  denote the open unit disc. Given any  $M, N \in \mathbb{N}_+$ , the Fourier series  $U = \sum_{n \geq 0} \hat{U}(n)\mathbf{e}_n$  of an arbitrary function  $U \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$  is identified to the power series of its Poisson integral,

$$\underline{U}(z) = \mathcal{P}[U](r, \theta) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{p}_r(\theta - x)U(x)dx = \sum_{n \geq 0} z^n \hat{U}(n) \in \mathbb{C}, \quad \forall z = re^{i\theta} \in \mathfrak{D}, \quad (3.1)$$

where  $\mathbf{p}_r(x) = \frac{1-r^2}{1-2r\cos(x)+r^2} = \sum_{n \in \mathbb{Z}} r^{|n|}e^{inx}$ , denotes the Poisson kernel on  $\mathfrak{D}$ . Thanks to Theorem 11.16 of Rudin [41], the function  $\mathcal{P}[U](r) = \sum_{n \geq 0} r^n \hat{U}(n)\mathbf{e}_n \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$  satisfies that

$$\sup_{0 \leq r < 1} \|\mathcal{P}[U](r)\|_{L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})} \leq \|U\|_{L_+^2}, \quad \lim_{r \rightarrow 1^-} \|\mathcal{P}[U](r) - U\|_{L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})} = 0. \quad (3.2)$$

In addition, if  $U \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$  is continuous, then theorem 11.8 and 11.16 of Rudin [41] yield that

$$\sup_{0 \leq r < 1} \|\mathcal{P}[U](r)\|_{L^\infty(\mathbb{T}; \mathbb{C}^{M \times N})} \leq \|U\|_{L^\infty}, \quad \lim_{r \rightarrow 1^-} \|\mathcal{P}[U](r) - U\|_{L^\infty(\mathbb{T}; \mathbb{C}^{M \times N})} = 0. \quad (3.3)$$

The mapping  $U \mapsto \underline{U}$  provides a  $\mathbb{C}$ -Hilbert isomorphism from  $L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$  to the Hardy space

$$\mathbb{H}^2(\mathfrak{D}; \mathbb{C}^{M \times N}) := \{U \in \text{Hol}(\mathfrak{D}; \mathbb{C}^{M \times N}) : \sup_{0 \leq r < 1} \int_0^{2\pi} \text{tr}(\underline{U}(re^{i\theta})(\underline{U}(re^{i\theta}))^*) dx < +\infty\}. \quad (3.4)$$

If  $F = \sum_{n \geq 0} \hat{F}(n)\mathbf{e}_n \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$ ,

$$\mathbf{S}^*(F) = \Pi_{\geq 0}(\mathbf{e}_{-1}F) = \sum_{n \geq 0} \hat{F}(n+1)\mathbf{e}_n \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N}). \quad (3.5)$$

If  $F = (F_{kj})_{1 \leq k \leq M, 1 \leq j \leq N} \in L^1(\mathbb{T}; \mathbb{C}^{M \times N})$ , we have  $\langle F, \mathbb{E}_{kj} \rangle_{L^2(\mathbb{T}; \mathbb{C}^{M \times N})} = \hat{F}_{kj}(0)$  and

$$\mathbf{I}(F) = \sum_{k=1}^M \sum_{j=1}^N \langle F, \mathbb{E}_{kj} \rangle_{L^2(\mathbb{T}; \mathbb{C}^{M \times N})} \mathbb{E}_{kj} \in \mathbb{C}^{M \times N}, \quad (3.6)$$

where  $\mathbb{E}_{kj} \in \mathbb{C}^{M \times N}$  denotes the  $M \times N$  matrix whose  $kj$ -entry is 1 and the other entries are all 0.

**Lemma 3.1.** *If  $U \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})$  for some  $M, N \in \mathbb{N}_+$ , then*

$$\underline{U}(z) = \mathbf{I} \left( (\text{id}_{L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})} - z\mathbf{S}^*)^{-1}U \right), \quad \forall z \in \mathfrak{D}. \quad (3.7)$$

*Proof.* See Lemma 2.8 of Sun [44]. □

Recall the following commutator formulas of the Toeplitz operator and the shift operator.

**Lemma 3.2.** *If  $B \in L^\infty(\mathbb{T}; \mathbb{C}^{M \times N})$ ,  $F \in L_+^2(\mathbb{T}; \mathbb{C}^{N \times d})$ , for some  $d, M, N \in \mathbb{N}_+$ ,*

$$[\mathbf{S}^*, \mathbf{T}_B](F) = \mathbf{S}^* (\Pi_{\geq 0}B) \hat{F}(0) \in L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}). \quad (3.8)$$

*Proof.* See lemma 3.8 of Sun [44]. □

If  $\Psi = (U, V) \in H_+^2(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$ , then (2.5) yields that  $\mathbf{L}_\Psi^2 = \mathbf{D}^2 - \frac{1}{2}\mathbf{D}\mathcal{A} - \frac{1}{2}\mathcal{A}\mathbf{D} + \frac{i}{4}\mathcal{A}^2$  is an unbounded self-adjoint operator on  $L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$ , whose domain of definition is  $H_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$ .

**Lemma 3.3.** *Given  $d, M, N \in \mathbb{N}_+$ , if  $\Psi = (U, V) \in H_+^1(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$ , we have*

$$\tilde{\mathbf{B}}_\Psi := \mathbf{B}_\Psi - i\mathbf{L}_\Psi^2 = i\partial_x^2 + \mathbf{T}_U\partial_x\mathbf{T}_{V^*} + \mathbf{T}_V\partial_x\mathbf{T}_{U^*} : H_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}). \quad (3.9)$$

*Proof.* If  $\Psi = (U, V) \in H_+^1(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$ , then (2.5) yields that  $\mathbf{L}_\Psi^2 = \mathbf{D}^2 - \frac{1}{2}\mathbf{D}\mathcal{A} - \frac{1}{2}\mathcal{A}\mathbf{D} + \frac{i}{4}\mathcal{A}^2$ . So we conclude by formulas (1.13), (2.5) and the following identity:

$$\mathbf{T}_U\partial_x\mathbf{T}_{V^*} = \partial_x\mathbf{T}_U\mathbf{T}_{V^*} - \mathbf{T}_{\partial_x U}\mathbf{T}_{V^*} = \mathbf{T}_U\mathbf{T}_{V^*}\partial_x + \mathbf{T}_U\mathbf{T}_{\partial_x V^*} : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}). \quad (3.10)$$

□

Then formulas (2.2) and (3.9) yield the following corollary.

**Corollary 3.4.** *The function  $\Psi = (U, V) \in C^0((T^-, T^+), H_+^2(\mathbb{T}; (\mathbb{C}^{M \times N})^2))$  solves the intertwined derivative Schrödinger system (1.11) if and only if*

$$\partial_t U(t) = \tilde{\mathbf{B}}_{\Psi(t)}(U(t)), \quad \partial_t V(t) = \tilde{\mathbf{B}}_{\Psi(t)}(V(t)), \quad \forall t \in (T_-, T_+). \quad (3.11)$$

**Corollary 3.5.** *If  $\Psi \in H_+^1(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$  and  $A \in \mathbb{C}^{M \times d}$ , then  $\mathbf{B}_\Psi(A) = i\mathbf{L}_\Psi^2(A) \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times d})$ .*

*Proof.* If  $V = \sum_{n \geq 0} \hat{V}(n)\mathbf{e}_n \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times N})$ ,  $\mathbf{T}_U\partial_x\mathbf{T}_{V^*}(A) = \mathbf{T}_U\partial_x(\hat{V}(0)^*A) = \partial_x^2 A = 0_{M \times d}$ . □

In order to prove (1.20) by conjugation acting method, we need the following commutator formula.

**Lemma 3.6.** *Given  $M, N, d \in \mathbb{N}_+$ , if  $\Psi = (U, V) \in H_+^1(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$ , then*

$$[\mathbf{S}^*, \mathbf{B}_\Psi] = i\mathbf{S}^*\mathbf{L}_\Psi^2 - (\mathbf{L}_\Psi + \text{id})^2\mathbf{S}^* : H_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}). \quad (3.12)$$

*Proof.* Since  $\mathbf{D}\mathbf{e}_n = n\mathbf{e}_n$ ,  $\forall n \in \mathbb{Z}$ , we have  $[\mathbf{S}^*, \mathbf{D}] = \mathbf{S}^*$  and  $[\mathbf{S}^*, \mathbf{D}^2] = \mathbf{S}^*\mathbf{D} + \mathbf{D}\mathbf{S}^* = \mathbf{S}^* + 2\mathbf{D}\mathbf{S}^*$ . If  $V = \sum_{n \geq 0} \hat{V}(n)\mathbf{e}_n \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times N})$ , then  $\mathbf{S}^*\Pi_{\geq 0}(V^*) = \Pi_{\geq 0}(\hat{V}(0)^*\mathbf{e}_{-1}) = 0_{N \times M}$ . If  $G \in H_+^1(\mathbb{T}; \mathbb{C}^{N \times d})$ , then  $\widehat{\mathbf{D}G}(0) = 0\hat{G}(0) = 0_{N \times d}$ . Thanks to formula (3.8),  $\forall U \in H_+^1(\mathbb{T}; \mathbb{C}^{M \times N})$ , we have

$$[\mathbf{S}^*, \mathbf{T}_{V^*}] = 0 : L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow 0_{N \times d} \quad \text{and} \quad [\mathbf{S}^*, \mathbf{T}_U]\mathbf{D}\mathbf{T}_{V^*} = 0 : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow 0_{M \times d}. \quad (3.13)$$

As a consequence, formula (3.13) yields that

$$[\mathbf{S}^*, \mathbf{T}_U D \mathbf{T}_{V^*}] = [\mathbf{S}^*, \mathbf{T}_U D] \mathbf{T}_{V^*} = \mathbf{T}_U [\mathbf{S}^*, D] \mathbf{T}_{V^*} = \mathbf{T}_U \mathbf{T}_{V^*} \mathbf{S}^* : H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}). \quad (3.14)$$

Then we use formulas (1.13), (3.9) and (3.14) to deduce that

$$i[\mathbf{S}^*, \tilde{\mathbf{B}}_\Psi] = \mathbf{S}^* + 2D\mathbf{S}^* - \mathbf{T}_U \mathbf{T}_{V^*} \mathbf{S}^* - \mathbf{T}_V \mathbf{T}_{U^*} \mathbf{S}^* = \mathbf{S}^* + 2\mathbf{L}_\Psi \mathbf{S}^* : H_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow H_+^1(\mathbb{T}; \mathbb{C}^{M \times d}). \quad (3.15)$$

So formula (3.12) is obtained by plugging (3.9) into (3.15).  $\square$

Recall the following wellposedness theorem for linear equations.

**Proposition 3.7** (Cauchy). *Let  $\mathcal{E}$  be a Banach space,  $\mathcal{I}$  is an open interval of  $\mathbb{R}$  and  $A \in C^0(\mathcal{I}; \mathcal{B}(\mathcal{E}))$ , if  $(t_0, x_0) \in \mathcal{I} \times \mathcal{E}$ , there exists a unique function  $x \in C^1(\mathcal{I}; \mathcal{B}(\mathcal{E}))$  such that  $x(t_0) = x_0$  and*

$$\frac{d}{dt}x(t) = A(t)(x(t)), \quad \forall t \in \mathcal{I}. \quad (3.16)$$

*Proof.* See Theorem 1.1.1 of Chemin [8].  $\square$

Then we prove theorem 1.4 by comparing two families of unitary operators which act on the shift operator  $\mathbf{S}^*$  by conjugation simultaneously.

*Proof of theorem 1.4.* If  $\Psi \in H_+^3(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$ , then  $i\mathbf{B}_\Psi$  is a bounded self-adjoint operator on the Hardy space  $L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$ ,  $\forall d \in \mathbb{N}_+$ . In addition,  $\mathbf{B}_\Psi \in \mathbb{X}_s := \mathcal{B}(H_+^s(\mathbb{T}; \mathbb{C}^{M \times d}))$ , if  $\frac{1}{2} < s \leq 2$ . The operators  $\mathbf{L}_\Psi$  and  $\mathbf{L}_\Psi^2$  are unbounded self-adjoint operators on  $L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$  by Kato–Rellich theorem. For any  $F \in \text{Dom}(\mathbf{L}_\Psi^2) = H_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$ , the operator theory implies that  $e^{it\mathbf{L}_\Psi^2}(F) \in H_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$ ,  $\forall t \in \mathbb{R}$ , see formula (X.97) in Page 236 of Reed–Simon [38].

Now assume that  $\Psi = (U, V) \in C^0((T^-, T^+), H_+^3(\mathbb{T}; (\mathbb{C}^{M \times N})^2))$  solves the intertwined derivative Schrödinger system (1.11) with initial data  $\Psi_0 = (U_0, V_0) = (U(0), V(0)) \in H_+^3(\mathbb{T}; (\mathbb{C}^{M \times N})^2)$ , the mapping  $\mathcal{A}_j : t \in (T_-, T_+) \mapsto \mathcal{A}_j(t) \in \mathcal{B}(\mathbb{X}_s)$  is continuous with respect to the norm topology,  $j = 1$  or  $2$ , if  $s = 0$  or  $\frac{1}{2} < s \leq 2$ , where  $\mathcal{A}_1(t)(W) = \mathbf{B}_{\Psi(t)}W$  and  $\mathcal{A}_2(t)(W) = -W\mathbf{B}_{\Psi(t)}$ . Thanks to Proposition 3.7, there exists a unique function  $\mathbf{W} \in C^1(T_-, T_+; \mathcal{B}(L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})))$  such that

$$\partial_t \mathbf{W}(t) = \mathbf{B}_{\Psi(t)} \mathbf{W}(t), \quad \partial_t \mathbf{W}(t)^* = -\mathbf{W}(t)^* \mathbf{B}_{\Psi(t)}, \quad \mathbf{W}(0) = \text{id}_{L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})}. \quad (3.17)$$

Here  $\mathbf{W}(t)^* = \mathbf{W}(t)^{-1}$  is the  $L_+^2$ -adjoint of  $\mathbf{W}(t)$ . But  $\mathbf{W}, \mathbf{W}^* \in C^1(T_-, T_+; \mathcal{B}(H_+^s(\mathbb{T}; \mathbb{C}^{M \times d})))$ ,  $s = 0$  or  $\frac{1}{2} < s \leq 2$ . The Lax pair structure (1.14) yields the following unitary equivalence:  $\forall \mu \in \mathbb{C}$ ,

$$(\mathbf{L}_{\Psi(t)} + \mu)^2 = \mathbf{W}(t) (\mathbf{L}_{\Psi(0)} + \mu)^2 \mathbf{W}(t)^* : H_+^2(\mathbb{T}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{T}; \mathbb{C}^{M \times d}). \quad (3.18)$$

Set  $\mathbf{Y}(t) = \mathbf{W}(t)^* \mathbf{S}^* \mathbf{W}(t)$ . Then  $Y \in C^1(T_-, T_+; \mathcal{B}(H_+^s(\mathbb{T}; \mathbb{C}^{M \times d})))$ ,  $s = 0$  or  $\frac{1}{2} < s \leq 2$ . So

$$\begin{aligned} \partial_t \mathbf{Y}(t) &= \mathbf{W}(t)^* [\mathbf{S}^*, \mathbf{B}_{\Psi(t)}] \mathbf{W}(t) = i\mathbf{W}(t)^* \mathbf{S}^* \mathbf{L}_{\Psi(t)}^2 \mathbf{W}(t) - i\mathbf{W}(t)^* \mathbf{S}^* (\mathbf{L}_{\Psi(t)} + 1)^2 \mathbf{W}(t) \\ &= i\mathbf{Y}(t) \mathbf{L}_{\Psi_0}^2 - i(\mathbf{L}_{\Psi_0} + 1)^2 \mathbf{Y}(t) \in \mathcal{B}(H_+^s(\mathbb{T}; \mathbb{C}^{M \times d})). \end{aligned} \quad (3.19)$$

Set  $\mathbf{Z}(t) = e^{it(\mathbf{L}_{\Psi_0} + 1)^2} \mathbf{Y}(t) e^{-it\mathbf{L}_{\Psi_0}^2}$ . If  $F \in H_+^2(\mathbb{T}; \mathbb{C}^{M \times d})$ , then formula (3.19) yields that  $\partial_t \mathbf{Z}(t)(F) = e^{it(\mathbf{L}_{\Psi_0} + 1)^2} (i(\mathbf{L}_{\Psi_0} + 1)^2 \mathbf{Y}(t) + \partial_t \mathbf{Y}(t) - i\mathbf{Y}(t) \mathbf{L}_{\Psi_0}^2) e^{-it\mathbf{L}_{\Psi_0}^2}(F) = 0_{M \times d}$ . So we have

$$\mathbf{W}(t)^* \mathbf{S}^* \mathbf{W}(t) = \mathbf{Y}(t) = e^{-it(\mathbf{L}_{\Psi_0} + 1)^2} \mathbf{S}^* e^{it\mathbf{L}_{\Psi_0}^2} \in \mathcal{B}(L_+^2(\mathbb{T}; \mathbb{C}^{M \times d})), \quad \forall t \in (T_-, T_+). \quad (3.20)$$

Fix  $d = N$ . So  $\partial_t(\mathbf{W}(t)^*U(t)) = \mathbf{W}(t)^*(\partial_t U(t) - \mathbf{B}_{\Psi(t)}U(t)) = -i\mathbf{W}(t)^*\mathbf{L}_{\Psi(t)}^2 U(t) = -i\mathbf{L}_{\Psi_0}^2 \mathbf{W}(t)^*U(t)$ , thanks to formulas (3.9), (3.11), (3.17) and (3.18). Recall that  $\mathbb{E}_{kj} \in \mathbb{C}^{M \times N}$  denotes  $M \times N$  matrix whose  $kj$ -entry is 1 and the other entries are all 0. Similarly,  $\partial_t \mathbf{W}(t)^*\mathbb{E}_{kj} = -i\mathbf{L}_{\Psi_0}^2 \mathbf{W}(t)^*\mathbb{E}_{kj}$  is obtained by Corollary 3.5,  $1 \leq k \leq M$ ,  $1 \leq j \leq N$ . Consequently, we have

$$\mathbf{W}(t)^*U(t) = e^{-it\mathbf{L}_{\Psi_0}^2}(U_0), \quad \mathbf{W}(t)^*\mathbb{E}_{kj} = e^{-it\mathbf{L}_{\Psi_0}^2}(\mathbb{E}_{kj}) \in H_+^2(\mathbb{T}; \mathbb{C}^{M \times N}). \quad (3.21)$$

For any  $z \in \mathbb{C}$  such that  $|z| < 1$ , we use formulas (3.20) and (3.21) to obtain that

$$\begin{aligned} & \langle (\text{id} - z\mathbf{S}^*)^{-1}U(t), \mathbb{E}_{kj} \rangle_{L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})} = \langle (\text{id} - z\mathbf{W}(t)^*\mathbf{S}^*\mathbf{W}(t))^{-1}\mathbf{W}(t)^*U(t), \mathbf{W}(t)^*\mathbb{E}_{kj} \rangle_{L_+^2} \\ & = \langle (\text{id} - ze^{-it(\mathbf{L}_{\Psi_0} + 1)^2} \mathbf{S}^* e^{it\mathbf{L}_{\Psi_0}^2})^{-1} e^{-it\mathbf{L}_{\Psi_0}^2}(U_0), e^{-it\mathbf{L}_{\Psi_0}^2}(\mathbb{E}_{kj}) \rangle_{L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})} \\ & = \langle (\text{id} - ze^{-it} e^{-2it\mathbf{L}_{\Psi_0}} \mathbf{S}^*)^{-1}(U_0), \mathbb{E}_{kj} \rangle_{L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})}. \end{aligned} \quad (3.22)$$

Thanks to (3.6) and (3.7), the Poisson integral of  $U(t) = \sum_{n \geq 0} \hat{U}(t, n) \mathbf{e}_n \in H_+^3(\mathbb{T}; \mathbb{C}^{M \times N})$  is given by

$$\underline{U}(t, z) = \sum_{n \geq 0} z^n \hat{U}(t, n) = \sum_{k=1}^M \sum_{j=1}^N \langle (\text{id} - z\mathbf{S}^*)^{-1}U(t), \mathbb{E}_{kj} \rangle_{L_+^2(\mathbb{T}; \mathbb{C}^{M \times N})} \mathbb{E}_{kj} \in \mathbb{C}^{M \times N}. \quad (3.23)$$

Plugging formula (3.22) into (3.23), we use (3.6) again to deduce that

$$\underline{U}(t, z) = \mathbf{I}((\text{id} - ze^{-it} e^{-2it\mathbf{L}_{\Psi_0}} \mathbf{S}^*)^{-1}U_0) = \sum_{n \geq 0} z^n e^{-int} \mathbf{I}((e^{-2it\mathbf{L}_{\Psi_0}} \mathbf{S}^*)^n U_0). \quad (3.24)$$

Since  $\Psi = (U, V)$  solves (1.11), so does  $\Phi := (V, U)$  with initial data  $\Phi_0 = (V_0, U_0)$ . We observe that  $\mathbf{L}_{\Psi_0} = \mathbf{L}_{\Phi_0} = \mathbf{D} - \frac{1}{2}(\mathbf{T}_{U_0} \mathbf{T}_{V_0}^* + \mathbf{T}_{V_0} \mathbf{T}_{U_0}^*)$ . Formula (3.24) yields that

$$\underline{V}(t, z) = \mathbf{I}((\text{id} - ze^{-it} e^{-2it\mathbf{L}_{\Phi_0}} \mathbf{S}^*)^{-1}V_0) = \mathbf{I}((\text{id} - ze^{-it} e^{-2it\mathbf{L}_{\Psi_0}} \mathbf{S}^*)^{-1}V_0),$$

and hence the second formula of (1.20) holds.  $\square$

## 4 Proof of the explicit formula on $\mathbb{R}$

The Lax–Beurling semigroup  $(\mathbf{S}(\eta)^*)_{\eta \geq 0}$  on  $L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$  defined by (1.25) has been used to establish the explicit formula of solutions of various integrable equations on the Hardy space. We refer to Pocovnicu [37] and Gérard–Pushnitski [24] to see its application to the cubic Szegő equation on the line. We refer to Sun [43], Gérard [10] and Killip–Laurens–Viřan [27] to see its application to the Benjamin–Ono equation and its hierarchies on the line. Its infinitesimal generator  $\mathbf{G} : \text{Dom}(\mathbf{G})^{M \times N} \rightarrow L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$  is a closed and densely defined operator, which can be interpreted as the adjoint of the operator of multiplication by the monomial  $x$  on  $L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$ . For any  $F \in \text{Dom}(\mathbf{G})^{M \times N}$ , the definition formula (1.28) is equivalent to the following identity,

$$\widehat{\mathbf{G}(F)}(\xi) = i\partial_\xi \widehat{F}(\xi), \quad \forall \xi > 0 \quad (4.1)$$

According to formula (X.97) in P. 236 of Reed–Simon [38],  $(\mathbf{S}(\eta)^*)_{\eta \geq 0}$  can be expressed as

$$\mathbf{S}(\eta)^* = e^{-i\eta \mathbf{G}} \in \mathcal{B}(L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})), \quad \forall \eta \geq 0. \quad (4.2)$$

The commutator formula between a Toeplitz operator and  $\mathbf{G}$  is given as follows.

**Lemma 4.1.** *If  $B \in L^2 \cap L^\infty(\mathbb{R}; \mathbb{C}^{M \times N})$  for some  $M, N \in \mathbb{N}_+$ , then  $\mathbf{T}_B(\text{Dom}(\mathbf{G})^{N \times d}) \subset \text{Dom}(\mathbf{G})^{M \times d}$  for any  $d \in \mathbb{N}_+$  and the following identity holds,*

$$[\mathbf{G}, \mathbf{T}_B](F) = \frac{i}{2\pi} (\Pi_{\geq 0} B) \hat{F}(0^+) \in L^2_+(\mathbb{R}; \mathbb{C}^{M \times d}), \quad \forall F \in \text{Dom}(\mathbf{G})^{N \times d}. \quad (4.3)$$

*Proof.* The map  $\frac{1}{\eta}[\mathbf{S}(\eta)^*, \mathbf{T}_B](F) = \sum_{k=1}^M \sum_{j=1}^d \sum_{n=1}^N \frac{1}{\eta}[\mathbf{S}(\eta)^*, \mathbf{T}_{B_{kn}}](F_{nj})\mathbb{E}_{kj}$  converges to  $\frac{\Pi_{\geq 0} B}{2\pi} \hat{F}(0^+)$  in  $L^2_+(\mathbb{R}; \mathbb{C}^{M \times d})$  when  $\eta \rightarrow 0^+$ , thanks to lemma 3.1 of Sun [43].  $\square$

Inspired from [37], [10] and [24], we introduce an auxiliary approximate identity

$$\chi_\epsilon(x) = (1 - i\epsilon x)^{-1}, \quad \forall 0 < \epsilon < 1. \quad (4.4)$$

Then  $\mathfrak{P}_y = \frac{1}{y\pi} \text{Re} \chi_{y^{-1}}$ ,  $\forall y > 0$ , where  $\mathfrak{P}_y$  is the Poisson kernel defined in (1.22). Then we have

$$\lim_{\epsilon \rightarrow 0^+} \|\chi_\epsilon f - f\|_{L^2(\mathbb{R}; \mathbb{C})} = 0, \quad \lim_{\epsilon \rightarrow 0^+} \langle g, \chi_\epsilon \rangle_{L^2(\mathbb{R}; \mathbb{C})} = \hat{g}(0^+) = \mathcal{S}(g), \quad (4.5)$$

for any  $f \in L^2(\mathbb{R}; \mathbb{C})$  and  $\forall g \in \text{Dom}(\mathbf{G})^{1 \times 1}$ , thanks to Lebesgue's dominated convergence theorem. Since  $\chi_\epsilon \in H^{+\infty}(\mathbb{R}; \mathbb{C}_+) := \bigcap_{s \geq 0} H^s_+(\mathbb{R}; \mathbb{C}_+)$ , for any integer  $p \geq 1$  and  $f \in H^p(\mathbb{R}; \mathbb{C})$ , we have

$$\lim_{\epsilon \rightarrow 0^+} \|\partial_x^p \chi_\epsilon\|_{L^2_+(\mathbb{R}; \mathbb{C})} = \lim_{\epsilon \rightarrow 0^+} \|f \chi_\epsilon - f\|_{H^p(\mathbb{R}; \mathbb{C})} = 0. \quad (4.6)$$

For any  $F \in \text{Dom}(\mathbf{G})^{M \times N}$ , if  $F_{kj} \in \text{Dom}(\mathbf{G})^{1 \times 1}$  denotes the  $kj$ -entry of  $F$ , then

$$\langle F, \chi_\epsilon \mathbb{E}_{kj} \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})} = \langle F_{kj}, \chi_\epsilon \rangle_{L^2_+(\mathbb{R}; \mathbb{C})} \rightarrow \hat{F}_{kj}(0^+), \quad (4.7)$$

as  $\epsilon \rightarrow 0^+$ . Recall that  $\mathcal{S}$  is given by (1.27). Then the following identity holds,

$$\mathcal{S}(F) = \hat{F}(0^+) = \sum_{k=1}^M \sum_{j=1}^N \lim_{\epsilon \rightarrow 0^+} \langle F, \chi_\epsilon \mathbb{E}_{kj} \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})} \mathbb{E}_{kj} \in \mathbb{C}^{M \times N}. \quad (4.8)$$

The next lemma gives an inversion formula, which expresses the Poisson integral  $\underline{U}$  in terms of the resolvent of  $\mathbf{G}$  and the original function  $U \in L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})$ .

**Lemma 4.2.** *If  $U \in L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})$  for some  $M, N \in \mathbb{N}_+$ , then we have*

$$\underline{U}(z) = \frac{1}{2\pi i} \mathcal{S}((\mathbf{G} - z)^{-1}(U)) = \frac{1}{2\pi i} \sum_{k=1}^M \sum_{j=1}^N \lim_{\epsilon \rightarrow 0^+} \langle (\mathbf{G} - z)^{-1}(U), \chi_\epsilon \mathbb{E}_{kj} \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})} \mathbb{E}_{kj}, \quad (4.9)$$

for any  $z \in \mathbb{C}_+$ , where  $\underline{U}$  denotes the Poisson integral given by (1.22).

*Proof.* Set  $\mathbf{e}_z : x \in \mathbb{R} \mapsto e^{izx} \in \mathbb{C}$ ,  $\forall z \in \mathbb{C}$ . If  $U = \sum_{k=1}^M \sum_{j=1}^N U_{kj} \mathbb{E}_{kj} \in L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})$ , then

$$\widehat{U_{kj} \chi_\epsilon}(\xi) = \langle \mathbf{e}_{-\xi} U_{kj}, \chi_\epsilon \rangle_{L^2(\mathbb{R}; \mathbb{C})} = \langle \mathbf{S}(\xi)^*(U), \chi_\epsilon \mathbb{E}_{kj} \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})}, \quad \forall \xi \geq 0, \quad \forall \epsilon \in (0, 1). \quad (4.10)$$

Since  $\mathbf{e}_z \in L^2(\mathbb{R}^*_+; \mathbb{C})$  with  $\mathbb{R}^*_+ = (0, +\infty)$ ,  $\forall z \in \mathbb{C}_+$ , formulas (1.22), (4.2), (4.5) and (4.10) yield that

$$\begin{aligned} \underline{U_{kj}}(z) &= \frac{1}{2\pi} \langle \hat{U}_{kj}, \overline{\mathbf{e}_z} \rangle_{L^2(\mathbb{R}^*_+; \mathbb{C})} = \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0^+} \langle \widehat{U_{kj} \chi_\epsilon}, \overline{\mathbf{e}_z} \rangle_{L^2(\mathbb{R}^*_+; \mathbb{C})} \\ &= \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0^+} \int_0^{+\infty} \langle e^{i\xi(z - \mathbf{G})}(U), \chi_\epsilon \mathbb{E}_{kj} \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})} d\xi, \quad \forall z \in \mathbb{C}_+. \end{aligned} \quad (4.11)$$

Since  $\text{Im}z > 0$ , the vector-valued integration  $\int_a^b e^{i\xi(z-\mathbf{G})}(U)d\xi$  with  $a, b \in \mathbb{R}$ , defined in theorems 3.27 and 3.29 of Rudin [42] can be extended to the Riemann sum over  $[0, +\infty)$ . Formula (X.98) of Reed–Simon [38] and definition 3.26 of Rudin [42] yield that

$$\underline{U}_{kj}(z) = \frac{1}{2\pi i} \lim_{\epsilon \rightarrow 0^+} \langle (\mathbf{G} - z)^{-1}(U), \chi_\epsilon \mathbb{E}_{kj} \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})}, \quad \forall z \in \mathbb{C}_+. \quad (4.12)$$

The proof is completed by using  $\underline{U}(z) = \sum_{k=1}^M \sum_{j=1}^N \underline{U}_{kj}(z) \mathbb{E}_{kj}$  and formula (4.8).  $\square$

Formula (3.9) still holds for the space non periodic case, i.e. if  $\Psi = (U, V) \in H^1_+(\mathbb{R}; (\mathbb{C}^{M \times N})^2)$ , then

$$\tilde{\mathbf{B}}_\Psi := \mathbf{B}_\Psi - i\mathbf{L}_\Psi^2 = i\partial_x^2 + \mathbf{T}_U \partial_x \mathbf{T}_{V^*} + \mathbf{T}_V \partial_x \mathbf{T}_{U^*} : H^2_+(\mathbb{R}; \mathbb{C}^{M \times d}) \rightarrow L^2_+(\mathbb{R}; \mathbb{C}^{M \times d}). \quad (4.13)$$

Thus the intertwined system (1.1) on  $\mathbb{R}$  also reads as

$$\partial_t U(t) = \tilde{\mathbf{B}}_{\Psi(t)}(U(t)), \quad \partial_t V(t) = \tilde{\mathbf{B}}_{\Psi(t)}(V(t)). \quad (4.14)$$

The next lemma shows that  $[\mathbf{G}, \mathbf{B}_\Psi]$  can be expressed in terms of the Lax operator  $\mathbf{L}_\Psi$ .

**Lemma 4.3.** *Given any  $d, M, N \in \mathbb{N}_+$ , if  $\Psi = (U, V) \in H^2_+(\mathbb{R}; (\mathbb{C}^{M \times N})^2)$ , then*

$$[\mathbf{G}, \mathbf{B}_\Psi] = i[\mathbf{G}, \mathbf{L}_\Psi^2] + 2\mathbf{L}_\Psi. \quad (4.15)$$

*Proof.*  $[\mathbf{G}, \partial_x] = -\text{id}$ ,  $[\mathbf{G}, \partial_x^2] = -2\partial_x$ . For any  $F \in \text{Dom}(\mathbf{G})^{M \times d}$ , (4.3) yields that

$$[\mathbf{G}, \mathbf{T}_{V^*}](F) = \frac{i}{2\pi} (\Pi_{\geq 0}(V^*)) \hat{F}(0^+) = 0_{N \times d}. \quad (4.16)$$

Then  $[\mathbf{G}, \partial_x \mathbf{T}_{V^*}](F) = [\mathbf{G}, \partial_x] \mathbf{T}_{V^*}(F) + \partial_x [\mathbf{G}, \mathbf{T}_{V^*}](F) = -\mathbf{T}_{V^*}(F)$ . Thanks to lemma 4.1, for any  $F \in \text{Dom}(\mathbf{G})^{M \times d} \cap H^1_+(\mathbb{R}; \mathbb{C}^{M \times d})$ , we have  $\mathbf{T}_{V^*}(F) \in \text{Dom}(\mathbf{G})^{N \times d} \cap H^1_+(\mathbb{R}; \mathbb{C}^{N \times d})$ , then the Fourier transform  $(\partial_x \mathbf{T}_{V^*}(F))^\wedge : \xi \mapsto i\xi \widehat{\mathbf{T}_{V^*}(F)} \in H^1((0, 1); \mathbb{C}^{N \times d})$ , then  $\mathcal{S}(\partial_x \mathbf{T}_{V^*}(F)) = 0_{N \times d}$  by Sobolev extension theorem. As a consequence,  $[\mathbf{G}, \mathbf{T}_U] \partial_x \mathbf{T}_{V^*}(F) = \frac{iU}{2\pi} (\partial_x \mathbf{T}_{V^*}(F))^\wedge(0^+) = 0_{M \times d}$ . Thus

$$[\mathbf{G}, \mathbf{T}_U \partial_x \mathbf{T}_{V^*}](F) = [\mathbf{G}, \mathbf{T}_U] \partial_x \mathbf{T}_{V^*}(F) + \mathbf{T}_U [\mathbf{G}, \partial_x \mathbf{T}_{V^*}](F) = -\mathbf{T}_U \mathbf{T}_{V^*}(F). \quad (4.17)$$

Similarly,  $[\mathbf{G}, \mathbf{T}_V \partial_x \mathbf{T}_{U^*}] = -\mathbf{T}_V \mathbf{T}_{U^*}$ . Finally we have

$$[\mathbf{G}, \tilde{\mathbf{B}}_\Psi] = -2i\partial_x - \mathbf{T}_U \mathbf{T}_{V^*} - \mathbf{T}_V \mathbf{T}_{U^*} = 2\mathbf{L}_\Psi. \quad (4.18)$$

The proof is completed by comparing formulas (4.13) and (4.18).  $\square$

The convergence formula (4.6) yields the following lemma.

**Lemma 4.4.** *Given any  $d, M, N \in \mathbb{N}_+$ , if  $\Psi = (U, V) \in H^2_+(\mathbb{R}; (\mathbb{C}^{M \times N})^2)$  and  $P \in \mathbb{C}^{M \times d}$ , we have*

$$\lim_{\epsilon \rightarrow 0^+} \left( \|\mathbf{L}_\Psi^2(\chi_\epsilon P)\|_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times d})} + \|\mathbf{B}_\Psi(\chi_\epsilon P)\|_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times d})} \right) = 0. \quad (4.19)$$

Moreover, there exists a constant  $\mathcal{C}(\|\Psi\|_{H^2_+(\mathbb{R}; (\mathbb{C}^{M \times N})^2)}) > 0$  such that

$$\sup_{0 < \epsilon < 1} \|\mathbf{B}_\Psi(\chi_\epsilon P)\|_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times d})} \leq \mathcal{C}(\|\Psi\|_{H^2_+(\mathbb{R}; (\mathbb{C}^{M \times N})^2)}) \|P\|_{\mathbb{C}^{M \times d}}. \quad (4.20)$$

The explicit formula (1.29) of the intertwined system (1.1) on the line is also proved by using the conjugation acting method. However, the constant function does not belong to  $L^2$ -spaces. So we have to use the auxiliary approximation identity  $\chi_\epsilon$  given by (4.4) and study the limit as  $\epsilon \rightarrow 0^+$ .

*Proof of theorem 1.6.* For any  $d \in \mathbb{N}_+$ , there exists a unique function  $\mathscr{W} \in C^1(T_-, T_+; \mathcal{B}(L_+^2(\mathbb{R}; \mathbb{C}^{M \times d})))$  such that

$$\partial_t \mathscr{W}(t) = \mathbf{B}_{\Psi(t)} \mathscr{W}(t), \quad \partial_t \mathscr{W}(t)^* = -\mathscr{W}(t)^* \mathbf{B}_{\Psi(t)}, \quad \mathscr{W}(0) = \text{id}_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times d})}, \quad (4.21)$$

thanks to proposition 3.7. Here  $\mathscr{W}(t)^* = \mathscr{W}(t)^{-1}$  denotes the  $L_+^2(\mathbb{R})$ -adjoint of  $\mathscr{W}(t)$ . But  $\mathscr{W}, \mathscr{W}^* \in C^1(T_-, T_+; \mathcal{B}(H_+^s(\mathbb{R}; \mathbb{C}^{M \times d})))$ ,  $s = 0$  or  $\frac{1}{2} < s \leq 2$ . The Lax pair structure (1.14) yields the following unitary equivalence:  $\forall \mu \in \mathbb{C}$ ,  $\mathbf{L}_{\Psi(t)} + \mu = \mathscr{W}(t) (\mathbf{L}_{\Psi(0)} + \mu) \mathscr{W}(t)^*$  and

$$(\mathbf{L}_{\Psi(t)} + \mu)^2 = \mathscr{W}(t) (\mathbf{L}_{\Psi(0)} + \mu)^2 \mathscr{W}(t)^* : H_+^2(\mathbb{R}; \mathbb{C}^{M \times d}) \rightarrow L_+^2(\mathbb{R}; \mathbb{C}^{M \times d}). \quad (4.22)$$

Set  $\mathscr{Y}(t) := \mathscr{W}(t)^* \mathbf{G} \mathscr{W}(t)$ . Then  $\mathscr{Y}$  satisfies the following linear differential equation,

$$\begin{aligned} \partial_t \mathscr{Y}(t) &= \mathscr{W}(t)^* [\mathbf{G}, \mathbf{B}_{\Psi(t)}] \mathscr{W}(t) = i \mathscr{W}(t)^* \mathbf{G} \mathbf{L}_{\Psi(t)}^2 \mathscr{W}(t) - i \mathscr{W}(t)^* \mathbf{L}_{\Psi(t)}^2 \mathbf{G} \mathscr{W}(t) + 2\mathbf{L}_{\Psi(0)} \\ &= i[\mathscr{Y}(t), \mathbf{L}_{\Psi_0}^2] + 2\mathbf{L}_{\Psi_0}, \quad T_- < t < T_+. \end{aligned} \quad (4.23)$$

Then we integrate (4.23) in order to obtain the following identity,

$$\mathscr{W}(t)^* \mathbf{G} \mathscr{W}(t) = 2t\mathbf{L}_{\Psi_0} + e^{-it\mathbf{L}_{\Psi_0}^2} \mathbf{G} e^{it\mathbf{L}_{\Psi_0}^2}, \quad T_- < t < T_+. \quad (4.24)$$

Fix  $d = N$ . Using the same idea of proving (3.21), we have

$$\mathscr{W}(t)^* U(t) = e^{-it\mathbf{L}_{\Psi_0}^2} (U(0)), \quad T_- < t < T_+. \quad (4.25)$$

Similarly, we integrate  $\partial_t (\mathscr{W}(t)^* (\chi_\epsilon \mathbb{E}_{kj})) = -\mathscr{W}(t)^* \mathbf{B}_{\Psi(t)} (\chi_\epsilon \mathbb{E}_{kj})$  to obtain

$$\mathscr{W}(t)^* (\chi_\epsilon \mathbb{E}_{kj}) - \chi_\epsilon \mathbb{E}_{kj} = - \int_0^t \mathscr{W}(\tau)^* \mathbf{B}_{\Psi(\tau)} (\chi_\epsilon \mathbb{E}_{kj}) d\tau. \quad (4.26)$$

Recall that  $\mathscr{W}(\tau)$  is a unitary operator on  $L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})$ . Thanks to the convergence formula (4.19) and the estimate (4.20), we use dominated convergence theorem to deduce that

$$\|\mathscr{W}(t)^* (\chi_\epsilon \mathbb{E}_{kj}) - \chi_\epsilon \mathbb{E}_{kj}\|_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} \leq \left| \int_0^t \|\mathbf{B}_{\Psi(\tau)} (\chi_\epsilon \mathbb{E}_{kj})\|_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} d\tau \right| \rightarrow 0, \quad (4.27)$$

as  $\epsilon \rightarrow 0^+$ . Similarly, (4.19) yields that

$$\limsup_{\epsilon \rightarrow 0^+} \|e^{-it\mathbf{L}_{\Psi_0}^2} (\chi_\epsilon \mathbb{E}_{kj}) - \chi_\epsilon \mathbb{E}_{kj}\|_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} \leq \lim_{\epsilon \rightarrow 0^+} \|\mathbf{L}_{\Psi_0}^2 (\chi_\epsilon \mathbb{E}_{kj})\|_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} |t| = 0. \quad (4.28)$$

Thanks to the previous estimates (4.27) and (4.28), we have the following linearization when  $\epsilon \rightarrow 0^+$ ,

$$\lim_{\epsilon \rightarrow 0^+} \|\mathscr{W}(t)^* (\chi_\epsilon \mathbb{E}_{kj}) - e^{-it\mathbf{L}_{\Psi_0}^2} (\chi_\epsilon \mathbb{E}_{kj})\|_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} = 0. \quad (4.29)$$

Then the conjugation acting formula (4.24) yields that for any  $1 \leq k \leq M$ ,  $1 \leq j \leq N$ ,

$$\begin{aligned} &\langle (\mathbf{G} - z)^{-1} (U(t)), \chi_\epsilon \mathbb{E}_{kj} \rangle_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} = \langle (\mathscr{W}(t)^* \mathbf{G} \mathscr{W}(t) - z)^{-1} (\mathscr{W}(t)^* U(t)), \mathscr{W}(t)^* (\chi_\epsilon \mathbb{E}_{kj}) \rangle_{L_+^2} \\ &= \langle \left( e^{-it\mathbf{L}_{\Psi_0}^2} \mathbf{G} e^{it\mathbf{L}_{\Psi_0}^2} - z + 2t\mathbf{L}_{\Psi_0} \right)^{-1} (e^{-it\mathbf{L}_{\Psi_0}^2} U(0)), e^{-it\mathbf{L}_{\Psi_0}^2} (\chi_\epsilon \mathbb{E}_{kj}) \rangle_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} + \mathbf{r}_\epsilon \\ &= \langle (\mathbf{G} - z + 2t\mathbf{L}_{\Psi_0})^{-1} (U(0)), \chi_\epsilon \mathbb{E}_{kj} \rangle_{L_+^2(\mathbb{R}; \mathbb{C}^{M \times N})} + \mathbf{r}_\epsilon, \end{aligned} \quad (4.30)$$

where  $\mathbf{r}_\epsilon$  is given by

$$\mathbf{r}_\epsilon := \langle \mathscr{W}(t)^* (\mathbf{G} - z)^{-1} (U(t)), \mathscr{W}(t)^* (\chi_\epsilon \mathbb{E}_{kj}) - e^{-it\mathbf{L}_{\Psi_0}} (\chi_\epsilon \mathbb{E}_{kj}) \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})} \rightarrow 0, \quad (4.31)$$

as  $\epsilon \rightarrow 0^+$ , by using (4.29). Plugging formulas (4.30) and (4.31) into the inversion formula (4.9) for Poisson integral, we deduce that

$$\begin{aligned} 2\pi i \underline{U}(t, z) &= \sum_{k=1}^M \sum_{j=1}^N \lim_{\epsilon \rightarrow 0^+} \langle (\mathbf{G} - z)^{-1} (U(t)), \chi_\epsilon \mathbb{E}_{kj} \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})} \mathbb{E}_{kj} \\ &= \sum_{k=1}^M \sum_{j=1}^N \lim_{\epsilon \rightarrow 0^+} \langle (\mathbf{G} - z + 2t\mathbf{L}_{\Psi_0})^{-1} (U(0)), \chi_\epsilon \mathbb{E}_{kj} \rangle_{L^2_+(\mathbb{R}; \mathbb{C}^{M \times N})} \mathbb{E}_{kj} \\ &= \mathscr{I} \left( (\mathbf{G} + 2t\mathbf{L}_{\Psi_0} - z)^{-1} (U_0) \right). \end{aligned} \quad (4.32)$$

The explicit formula  $\underline{V}(t, z)$  can be obtained by interchanging  $U, V$  and using the fact that  $\mathbf{L}_{(U,V)} = \mathbf{L}_{(V,U)}$  as in the proof of theorem 1.4.  $\square$

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