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ASYMPTOTICS IN FINITE MONOIDAL CATEGORIES

ABEL LACABANNE, DANIEL TUBBENHAUER AND PEDRO VAZ

ABSTRACT. We give explicit formulas for the asymptotic growth rate of the number of summands in tensor powers in certain monoidal categories with finitely many indecomposable objects, and related structures.

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1. INTRODUCTION

Let $R = (R, C)$ be a *finite based* $\mathbb{R}_{\geq 0}$ -algebra with basis $C = \{1 = c_0, \dots, c_{r-1}\}$ (recalled in [Section 3](#) together with some other notions used in this introduction). Recall that we thus have

$$(1A.1) \quad c_i c_j = \sum_k m_{i,j}^k \cdot c_k \quad \text{with} \quad m_{i,j}^k \in \mathbb{R}_{\geq 0}.$$

Iterating this gives us coefficients $m_{i,j,\dots,l}^k \in \mathbb{R}_{\geq 0}$, and summing gives us coefficients for all $c \in R$.

Fix $c \in R$. We write $m_n^*(c)$ for these coefficients as they appear in c^n where $*$ $\in \{0, \dots, r-1\}$. Define

$$b_n^{R,c} := \# \text{total sum of coefficients } m_n^*(c).$$

Moreover, we define the function

$$b^{R,c}(n): \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}, n \mapsto b_n^{R,c}.$$

We are interested in the *asymptotic behavior* of the function $b^{R,c}(n)$. We main question we address is:

Question 1A.2. Find an explicit formula $a(n)$ such that

$$b^{R,c}(n) \sim a(n),$$

where we write \sim for asymptotically equal. ◇

We answer [Question 1A.2](#) as follows.

For $a_i \in \mathbb{R}_{\geq 0}$, the (transposed) *action matrix* of $c = a_0 \cdot c_0 + \dots + a_{r-1} \cdot c_{r-1} \in R$ is the matrix $(\sum_i a_i m_{i,j}^k)_{k,j}$. Abusing language, we will call the submatrix of it corresponding to the connected component of 1 also the action matrix and use this below.

Assume that the *Perron–Frobenius theorem* holds, that is the action matrix of $c \in R$ has a leading eigenvalue $\lambda_0 = \text{PFdim } c$ of multiplicity one that we call the *Perron–Frobenius dimension* of c . Moreover, the action matrix has some period $h \in \mathbb{Z}_{\geq 0}$ such that there are precisely $h-1$ other eigenvalues $\lambda_i = \zeta^i \text{PFdim } c$, and all of these are of multiplicity one, where $\zeta = \exp(2\pi i/h)$. We will drop this assumption in [Section 3](#) below.

Let us denote the right (the one with for $Mv = \lambda_i \cdot v_i$) and left (the one with $w_i^T M = \lambda_i \cdot w_i^T$) eigenvectors by v_i and w_i , normalized such that $w_i^T v_i = 1$.

Let $v_i w_i^T[1]$ denote taking the sum of the first column of the matrix $v_i w_i^T$. Define

$$(1A.3) \quad a(n) = (v_0 w_0^T[1] \cdot 1 + v_1 w_1^T[1] \cdot \zeta^n + v_2 w_2^T[1] \cdot (\zeta^2)^n + \dots + v_{h-1} w_{h-1}^T[1] \cdot (\zeta^{h-1})^n) \cdot (\text{PFdim } c)^n \in \overline{\mathbb{Z}}.$$

Let λ^{sec} be the second largest eigenvalue of the action matrix of c . We will prove (see [Section 3](#) below):

Theorem 1A.4. *We have*

$$b^{R,c}(n) \sim a(n),$$

and the convergence is geometric with ratio $|\lambda^{sec}/\text{PFdim } c|$. In particular,

$$\beta^{R,c} := \lim_{n \rightarrow \infty} \sqrt[n]{b_n^{R,c}} = \text{PFdim } c.$$

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The reason why [Theorem 1A.4](#) is interesting from the categorical point of view is the following. For us a *finite monoidal category* is a category such that:

- (i) It is monoidal.
- (ii) It is additive Krull–Schmidt.
- (iii) It has finitely many (isomorphism classes of) indecomposable objects.

Example 1A.5. Here are a few examples:

- (a) Let G be a finite group and consider $\mathbf{Rep}(G) = \mathbf{Rep}(G, \mathbb{C})$ the category of finite dimensional complex representations of G . This is a prototypical example of a finite monoidal category.
- (b) More generally, all fusion categories are finite monoidal.
- (c) For a finite group G , and arbitrary field, we can consider projective $\mathbf{Proj}(G)$ or injective $\mathbf{Inj}(G)$ representations. These are finite monoidal categories. More generally, one can take any finite dimensional Hopf algebra instead of a finite group.
- (d) If we assume that a Hopf algebra H is of finite type, then we can even consider $\mathbf{Rep}(H)$. An explicit example is the Taft algebra by [\[CVOZ14, Theorem 2.5\]](#).
- (e) In any additive Krull–Schmidt monoidal category one can take $\{\mathbf{X}^{\otimes d} | d \in \mathbb{Z}_{\geq 0}\}^{\oplus, \mathbb{C}^{\oplus}}$, the additive idempotent completion of the full subcategory generated by an object \mathbf{X} , as long as this has finitely many indecomposable objects. Explicitly, for a finite group G one can take any two dimensional G -representation for \mathbf{X} , which follows from [\[Alp79\]](#). There are many more examples, see e.g. [\[Cra13\]](#).
- (f) Consider Soergel bimodules $\mathbf{SBim}(W)$ as in [\[Soe92\]](#). These are finite monoidal categories if $W = (W, S)$ is of finite Coxeter type.

There are of course many more examples. ◇

The following is very easy and omitted:

Lemma 1A.6. *The additive Grothendieck ring of a finite monoidal category is a finite based $\mathbb{R}_{\geq 0}$ -algebra with basis given by the classes of indecomposable objects.* □

Fix a finite monoidal category \mathbf{C} and an object $\mathbf{X} \in \mathbf{C}$. Following [\[COT23\]](#), we define

$$b_n^{\mathbf{C}, \mathbf{X}} := \#\text{indecomposable summands in } \mathbf{X}^{\otimes n} \text{ counted with multiplicities.}$$

Note that $a(n)$ has an analog in this context, denoted by the same symbol, obtained for the (transposed) action matrix for left tensoring. Similarly as before we also have λ^{sec} . We then get:

Theorem 1A.7. *Under the same assumption as in [Theorem 1A.4](#), we have*

$$b^{\mathbf{C}, \mathbf{X}}(n) \sim a(n),$$

and the convergence is geometric with ratio $|\lambda^{\text{sec}}/\text{PFdim } \mathbf{X}|$. In particular,

$$\beta^{\mathbf{C}, \mathbf{X}} := \lim_{n \rightarrow \infty} \sqrt[n]{b_n^{\mathbf{C}, \mathbf{X}}} = \text{PFdim } \mathbf{X}.$$

Proof. From [Theorem 1A.4](#) and [Lemma 1A.6](#). □

In the next section we will discuss examples of [Theorem 1A.7](#), and then we will prove [Theorem 1A.4](#). We also generalize these two theorems in [Section 3](#) by getting rid of the assumption on the action matrix. Before that, let us finish the introduction with some (historical) remarks.

Remark 1A.8.

- (a) To study asymptotic properties of tensor powers is a rather new subject and most things are still quite mysterious. Let us mention a few facts that are known. An early reference we know is [\[Bia93\]](#), which studies questions similar to the one in this note but for Lie algebras, and this was carried on in several works such as [\[PR20\]](#). As another example, the paper [\[BS20\]](#) studies the growth rate of the dimensions of the non-projective part of tensor powers of a representation of a finite group. More generally, the paper [\[CEO23b\]](#) studies, working in certain tensor categories, the growth rates of summands of categorical dimension prime to the underlying characteristic. The paper [\[COT23\]](#) studies the growth rate of all summands, while [\[KST22\]](#) studies the Schur–Weyl dual question.
- (b) [Theorem 1A.4](#) and [Theorem 1A.7](#) generalize [\[CEO23a, Proposition 2.1\]](#). And for us one of the main features of that proposition is its simplicity, having a simple statement and proof. As we will see, the same is true for [Theorem 1A.4](#) and [Theorem 1A.7](#) as well: Clearly, the statements themselves are (surprisingly) simple yet general. Moreover, the proof of [Theorem 1A.4](#), and therefore the proof of [Theorem 1A.7](#) as well, is rather straightforward as soon as the key ideas are in place.

- (c) The second statements in [Theorem 1A.4](#) and [Theorem 1A.7](#) were already observed in [\[COT23\]](#), but the (finer) asymptotic behavior appears to be new.

Finally, let us mention that similar questions have been studied much earlier, see for example [\[AE81\]](#) for a related notion involving length of projective resolutions, or [\[LS77\]](#) for counting and Young diagrams. \diamond

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2. EXAMPLES

Let us call [Theorem 1A.4](#) and [Theorem 1A.7](#) our *main theorems* or *MT* for short. To underpin the explicit nature of these theorems, we now list examples of *MT* and also add that all the below can be double checked using the code on [\[LTV23\]](#). That page also contains a (potentially empty) Erratum.

2A. Finite groups. Let G be a finite group. Given a finite dimensional complex G -representation V , we denote its character by χ_V . Denote by $Z_V(G) \subset G$ the subgroup consisting of elements of g that acts as a scalar on V and by $\omega_V(g) \in \mathbb{C}$ the corresponding scalar. If V is simple, then ω_V is known as the *central character* of V .

Suppose that V is a faithful G -representation. Since V is faithful we get that $Z_V(G)$ is a subgroup of $Z(G)$ and also that the action graph of tensoring with V is connected (in the oriented sense). Then *MT* implies:

$$(2A.1) \quad a(n) = \left(\frac{1}{\#G} \sum_{g \in Z_V(G)} \left(\sum_{L \in S(G)} \omega_L(g) \dim_{\mathbb{C}} L \right) \cdot \omega_V(g)^n \right) \cdot (\dim_{\mathbb{C}} V)^n,$$

where $S(G) = \{\text{simple } G\text{-representations}\} / \cong$. This follows directly from *MT* after recalling the connection from Perron–Frobenius theory to character theory as explained in e.g. [\[EGNO15, Chapter 3\]](#).

Remark 2A.2. If V is not faithful, then the action graph of tensoring with V needs not to be connected, but that is not an issue in our main theorems. Thus, the assumption that V is faithful can be easily relaxed. \diamond

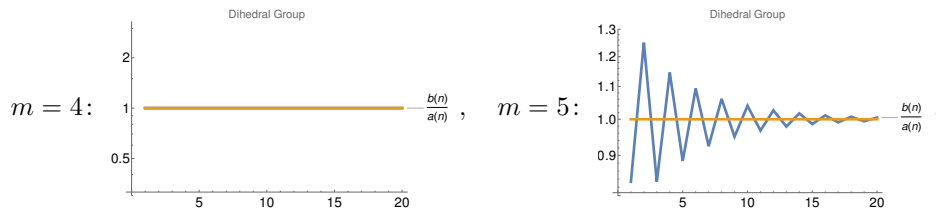
Remark 2A.3. Alternatively one can prove [\(2A.1\)](#) using character theory, similarly to [\[CEO23a, Proposition 2.1\]](#). [\(2A.1\)](#) still generalizes [\[CEO23a, Proposition 2.1\]](#). \diamond

Let us give a few explicit examples.

Example 2A.4 (Dihedral groups). Let $m \in \mathbb{Z}_{\geq 3}$ and let G be the *dihedral group* of order $2m$. Let $m' = m/2$, if m is even, and $m' = (m-1)/2$, if m is odd. Choose V any faithful representation of dimension 2 of G . Then [\(2A.1\)](#) gives the formulas

$$a(n) = \begin{cases} \frac{m+1}{2m} \cdot 2^n & \text{if } m \text{ is odd,} \\ \frac{m+2}{2m} \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is odd,} \\ \left(\frac{m+2}{2m} \cdot 1 + \frac{1}{m} \cdot (-1)^n \right) \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is even.} \end{cases}$$

Two explicit examples are $m \in \{4, 5\}$ and V is the G -representation corresponding to rotation by $2\pi/m$. Then:



Here and throughout, we display the graphs of $b(n)/a(n)$ in the usual way but log plotted. Moreover, for $m=4$ we have $b(n)=a(n)$ and we will omit plots in case that happens. \diamond

The next example can be seen as a $m=4$ and $p>2$ version of [Example 2A.4](#).

Example 2A.5 (Extraspecial groups). Let p be a prime and $m \in \mathbb{Z}_{\geq 1}$. Recall that a p -group of order p^{1+2m} is called *extraspecial* if its center $Z(G)$ is of order p and the quotient $G/Z(G)$ is a p -elementary abelian group. For each p and m , there exists two isomorphism classes of extraspecial groups of order p^{1+2m} , and they have the same character table. Thus, by [\(2A.1\)](#) we can take any of these two without difference. In the special case $p=2$ and $m=1$, we recover the dihedral group and the quaternion group of order 8.

Fix now an extraspecial group G of order p^{1+2m} . The simple G -representations are given as follows:

- (i) There are p^{2m} nonisomorphic one dimensional representations that arise from the representations of $G/Z(G)$.
- (ii) There are $p-1$ nonisomorphic irreducible representations of dimension p^m which are characterized by their central character.

Choose V any of the simple G -representation of dimension p^n . Then $Z_V(G) = Z(G)$ and (2A.1) gives

$$a(n) = \begin{cases} (p^m)^n & \text{if } p \mid n, \\ (p^m)^{n-1} & \text{otherwise.} \end{cases}$$

It turns out that this formula is not only asymptotic: we have $b(n) = a(n)$. This is due to the fact that the character of V vanishes outside of $Z(G)$. \diamond

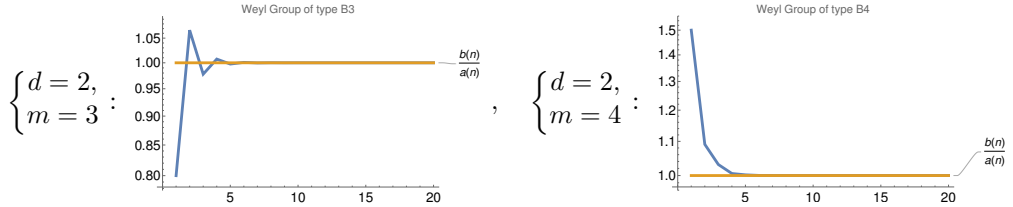
Example 2A.6 (Imprimitive complex reflection groups). Let d and m be integers in $\mathbb{Z}_{\geq 1}$ and consider the *imprimitive complex reflection group* $G = G(d, 1, m)$. This group can be seen as the group of m -by- m monomial matrices with entries being d th roots of unity. For $d = 1$ we recover the symmetric group, covered by [CEO23a, Example 2.3], and if $d = 2$ we recover the Weyl group of type B_m .

Choose V the standard representation given by the matrix description of G . Then (2A.1) gives a formula akin to [CEO23a, Example 2.3] which we decided not to write down as its a bit tedious.

In any case, for the special cases $d \in \{1, 2\}$ and $m = 2$, or $d = 2$ and $m = 4$ we get

$$\begin{cases} d=1, \\ m=3 \end{cases} : a(n) = \frac{2}{3} \cdot 3^n, \quad \begin{cases} d=2, \\ m=3 \end{cases} : a(n) = \frac{5}{12} \cdot 3^n, \quad \begin{cases} d=2, \\ m=4 \end{cases} : a(n) = \left(\frac{19}{96} \cdot 1 + \frac{1}{32} \cdot (-1)^n \right) \cdot 4^n.$$

We get the plots



Moreover, the formula $a(n) = \frac{2}{3} \cdot 3^n$ is exact for $d = 1$ and $m = 3$. \diamond

2B. Fusion categories. This section discusses Fusion categories over \mathbb{C} different from $\mathbf{Rep}(G)$.

Example 2B.1 (Fibonacci category). Let \mathcal{F} be the *Fibonacci category*, see for example [EGNO15, Exercise 8.18.7] where \mathcal{F} is denoted \mathcal{YL}_+ (or \mathcal{YL}_- , depending on conventions). All we need to know is that \mathcal{F} is \otimes -generated by one object \mathbf{X} with action matrix $M(\mathbf{X}) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

We want to estimate $b^{\mathbf{F}, \mathbf{X}}(n)$. To this end, the eigenvalues of $M(\mathbf{X})$ are the two roots of $x^2 = x + 1$, in particular, $\text{PFdim } \mathbf{X} = \phi$, the golden ratio. Its Perron–Frobenius eigenvectors are $v = w = \left(\frac{\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}}, \sqrt{\frac{1}{10}(\sqrt{5}+5)} \right)^T$ and we therefore get

$$a(n) = \frac{1}{10}(\sqrt{5}+5) \cdot \phi^n = \frac{1}{\sqrt{5}}\phi^{n-1},$$

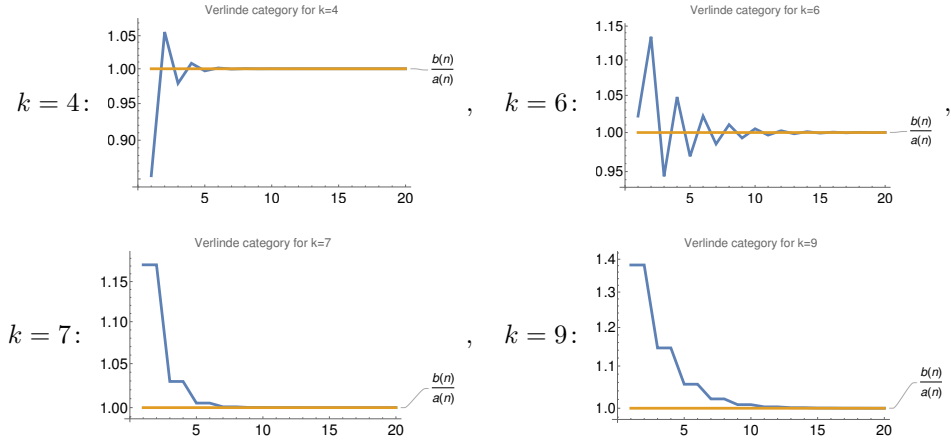
from MT . Note that the classical asymptotic for the Fibonacci numbers is $\frac{1}{\sqrt{5}}\phi^n$ and not $\frac{1}{\sqrt{5}}\phi^{n-1}$, but $b^{\mathbf{F}, \mathbf{X}}(n)$ is equal to the $(n+1)$ th Fibonacci number and hence the off-by-one-error in the exponent. \diamond

Example 2B.2 (Verlinde category). We now consider the *Verlinde category* $\mathbf{Ver}_k(\text{SL}_2)$ for $k \in \mathbb{Z}_{\geq 2}$, see for example [EGNO15, Section 8.18.2] (denoted differently therein). This fusion category has k simple objects, and we take the generating object \mathbf{X} of categorical dimension $2 \cos(\pi/(k+1))$. The case $k = 2$ compares to super vector spaces.

The action matrix for \mathbf{X} has the type A Dynkin diagram as its associated graph, and the eigenvalues and eigenvectors of this graph are well-known, see for example [Smi70]. In particular, $\text{PFdim } \mathbf{X} = 2 \cos(\pi/(k+1))$. Let $q = \exp(\pi i/(k+1))$. Then MT gives us

$$a(n) = \begin{cases} \left(\frac{[1]_q + \dots + [k]_q}{[1]_q^2 + \dots + [k]_q^2} \cdot (2 \cos(\pi/(k+1))) \right)^n & \text{if } k \text{ is even,} \\ \left(\frac{[1]_q + \dots + [k]_q}{[1]_q^2 + \dots + [k]_q^2} \cdot 1 + \frac{[1]_q - [2]_q + \dots - [k-1]_q + [k]_q}{[1]_q^2 + \dots + [k]_q^2} \cdot (-1)^n \right) \cdot (2 \cos(\pi/(k+1)))^n & \text{if } k \text{ is odd.} \end{cases}$$

Here $[a]_q$ denotes the a th quantum number evaluated at q . We get, for example:

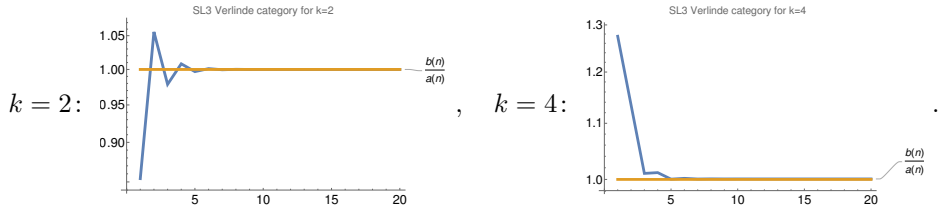


Moreover, for $k \in \{3, 5\}$ the formula $a(n)$ is spot on. ◇

Example 2B.3 (Higher rank Verlinde categories). Verlinde categories can be defined for all simple Lie algebras as quotients of representations of quantum groups at a root of unity as explained in [AP95]. Let us focus in this example on $\mathbf{Ver}_k(\mathrm{SL}_3)$, the one for the special linear group of rank three (with k determined as e in [MMMT20, Section 2]).

For $\mathbf{Ver}_k(\mathrm{SL}_3)$ we take the generating object \mathbf{X} corresponding to the vector representation of $\mathrm{SL}_3(\mathbb{C})$. Its action matrix is the oriented version of the graph displayed in [MMMT20, Fig. A1] with the orientation as in [MMMT20, (3-1)]. Using this, and omitting $k = 1$ since this is trivial, MT gives

$$k=2: a(n) = \frac{1}{10}(\sqrt{5} + 5) \cdot \phi^k, \quad k=3: a(n) = 1/2 \cdot 2^n, \quad k=4: a(n) = \frac{1}{7} \left(2 + 2 \cos\left(\frac{3\pi}{7}\right) \right) \cdot \left(1 + 2 \cos\left(\frac{2\pi}{7}\right) \right)^n,$$



For $k = 3$ the displayed formulas are exact. Moreover, one can find $a(n)$ explicitly in general using the formulas in [Zub98] or [MMMT20, Section 2]. ◇

2C. Nonsemisimple examples. We now discuss to two nonsemisimple examples.

Example 2C.1 ($\mathrm{SL}_2(\mathbb{F}_p)$ in defining characteristic). Let our ground field be \mathbb{F}_p for some prime $p > 2$. We consider the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ and its representations over \mathbb{F}_p . Take $V = \mathbb{F}_p^2$ to be the vector representation of $\mathrm{SL}_2(\mathbb{F}_p)$. In this case the action matrices are exemplified by

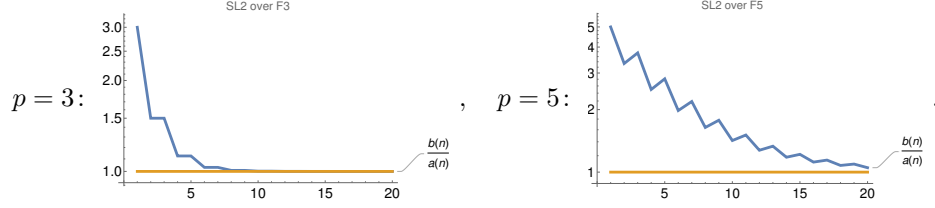
$$p=3: \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad p=5: \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad p=7: \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

These can be described as follows. The matrix is the one obtained as a $(2p-1)$ -by- $(2p-1)$ cut-off of the matrix for the infinite group over $\overline{\mathbb{F}_p}$ that can be obtained from [STWZ23, Proposition 4.4], together with an extra entry 1 in position $(2p-2, p)$.

Then MT gives

$$a(n) = \left(\frac{1}{2p-2} \cdot 1 + \frac{1}{2p^2-2p} \cdot (-1)^n \right) \cdot 2^n.$$

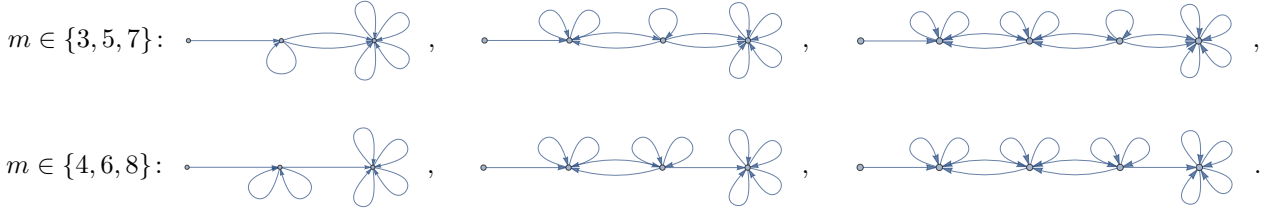
Explicitly, for $p \in \{3, 5\}$ we get



The convergence is rather slow (but still geometric). \diamond

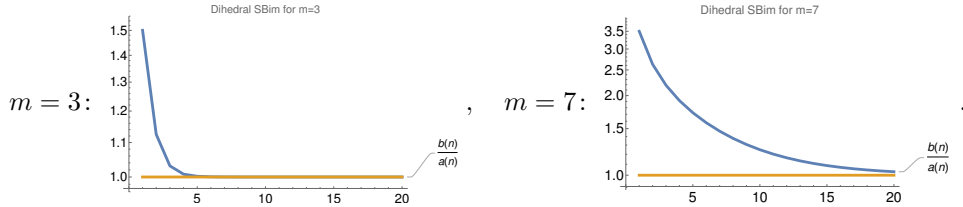
Example 2C.2 (Dihedral Soergel bimodules). We now look at the category of dihedral Soergel bimodules as studied in details in, for example, [Eli16], [MT19] or [Tub22]. In particular, [Tub22, Section 3C] lists all the formulas relevant for MT . To get a finite based $\mathbb{R}_{\geq 0}$ -algebra we collapse the grading, meaning we specialize [Tub22, Section 3C] at $q = 1$.

Fix $\langle s, t | s^2 = t^2 = (st)^m \rangle$ as the presentation for the dihedral group of order $2m$ where $m \in \mathbb{Z}_{\geq 3}$. Let us take X to be the Bott–Samelson generator for st . By the explicit formulas in [Tub22, Section 3C], the action graph of X is almost the same as the action graph of tensoring with \mathbb{C}^3 as a $SO_3(\mathbb{C})$ -representation. The first ones are (read from left to right):



The pattern generalizes. It is then easy to show that the leading eigenvalues is always 4 and the absolute values of all other eigenvalues are strictly smaller. Moreover, MT gives:

$$a(n) = \frac{1}{2m} \cdot 4^n,$$



The rate of convergence is rather slow for $m \gg 0$. \diamond

3. GENERALIZATIONS AND PROOFS

We will prove several versions of [Theorem 1A.4](#).

3A. Perron–Frobenius theory. We start with the main player, the *Perron–Frobenius theorem*. To this end, recall that one can associate an oriented and weighted graph, its *adjacency graph*, to an m -by- m matrix $M = (m_{ij})_{1 \leq i, j \leq m} \in \text{Mat}_m(\mathbb{R}_{\geq 0})$ as follows:

- (i) The vertices are $\{1, \dots, m\}$.
- (ii) There is an edge with weight m_{ij} from i to j .

We call a nonzero matrix $M \in \text{Mat}_m(\mathbb{R}_{\geq 0})$ *irreducible* if its associated graph is connected in the oriented sense (this is called *strongly connected*). Recall that in this note a right eigenvector satisfies $Mv = \lambda \cdot v$, and a left eigenvector satisfies $wM = \lambda \cdot w$.

Theorem 3A.1 (Perron–Frobenius theorem part I). *Let $M \in \text{Mat}_m(\mathbb{R}_{\geq 0})$ be irreducible.*

- (a) *M has a Perron–Frobenius eigenvalue, that is, $\lambda \in \mathbb{R}_{>0}$ such that $\lambda \geq |\mu|$ for all other eigenvalues μ . This eigenvalue appears with multiplicity one, and all other eigenvalues with $\lambda = |\mu|$ also appear with multiplicity one.*
- (b) *There exists $h \in \mathbb{Z}_{\geq 1}$, the period, such that all eigenvalues μ with $\lambda = |\mu|$ are $\exp(k2\pi i/h)\lambda$ for $k \in \{0, \dots, h-1\}$. We call these pseudo-dominant eigenvalues.*
- (c) *The eigenvectors, left and right, for the Perron–Frobenius eigenvalues can be normalized to have values in $\mathbb{R}_{\geq 0}$.*

Proof. Well-known. See for example, Frobenius' paper 92 in Band 3 of [Fro68]. (This is the paper "Über Matrizen aus nicht negativen Elementen".) \square

Fix a function $f: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$. We say that $f(n)$ converges *geometrically* to $a \in \mathbb{R}$ with ratio $\beta \in [0, 1)$ if for all $\gamma \in (\beta, 1)$ we have that $\{(f(n) - a)/\gamma^n\}_{n \in \mathbb{Z}_{\geq 0}}$ is bounded. The following accompanies Theorem 3A.1:

Theorem 3A.2 (Perron–Frobenius theorem part II). *Let $M \in \text{Mat}_m(\mathbb{R}_{\geq 0})$ be irreducible, λ be its Perron–Frobenius eigenvalue and h be its period. Let $\zeta = \exp(2\pi i/h)$. For each $k \in \{0, \dots, h-1\}$, choose a left eigenvector v_i and a right eigenvector w_i with eigenvalue $\zeta^k \lambda$, normalized such that $w_i^T v_i = 1$.*

Then we have:

$$M^n \sim v_0 w_0^T \cdot \lambda^n + v_1 w_1^T \cdot (\zeta \lambda)^n + v_2 w_2^T \cdot (\zeta^2 \lambda)^n + \dots + v_{h-1} w_{h-1}^T \cdot (\zeta^{h-1} \lambda)^n.$$

Moreover, the convergence is geometric with ratio $|\lambda^{\text{sec}}/\lambda|$, where λ^{sec} is any second largest (in the sense of absolute value) eigenvalue.

Proof. This is known, but proofs are a bit tricky to find in the literature, so we give one. The proof also shows where the vectors v_i and w_j come from.

For any $\mu \in \mathbb{C}$, let V_μ be the generalized eigenspace associated to the eigenvalue μ . Then we have

$$\mathbb{C}^m \cong \bigoplus_{k=0}^{h-1} V_{\zeta^k \lambda} \oplus \bigoplus_{\mu, |\mu| < \lambda} V_\mu.$$

By Theorem 3A.1, the space $V_{\zeta^k \lambda}$ is the eigenspace associated to the eigenvalue $\zeta^k \lambda$ and $v_k w_k^T$ is the projection onto that subspace.

This implies that we have

$$M^n = v_0 w_0^T \cdot \lambda^n + v_1 w_1^T \cdot (\zeta \lambda)^n + v_2 w_2^T \cdot (\zeta^2 \lambda)^n + \dots + v_{h-1} w_{h-1}^T \cdot (\zeta^{h-1} \lambda)^n + R(n),$$

where $R(n)$ is the multiplication action of M^n onto the rest. Since the eigenvalues μ of M on the rest satisfies $|\mu| < \lambda$, we have $R(n)/\lambda^n \rightarrow_{n \rightarrow \infty} 0$ geometrically with ratio $|\lambda^{\text{sec}}/\lambda|$. \square

For a general matrix $M \in \text{Mat}_m(\mathbb{R}_{\geq 0})$ things change, but not too much:

Theorem 3A.3 (Perron–Frobenius theorem part III). *Let $M \in \text{Mat}_m(\mathbb{R}_{\geq 0})$.*

- (a) *M has a Perron–Frobenius eigenvalue, that is, $\lambda \in \mathbb{R}_{\geq 0}$ such that $\lambda \geq |\mu|$ for all other eigenvalues μ .*
- (b) *Let s be the multiplicity of the Perron–Frobenius eigenvalue. There exists $(h_1, \dots, h_s) \in \mathbb{Z}_{\geq 1}^s$, the periods, such that all eigenvalues μ with $\lambda = |\mu|$ are $\exp(k2\pi i/h_i)\lambda$ for $k \in \{0, \dots, h_i-1\}$, for some period. We call these pseudo-dominant eigenvalues.*
- (c) *The eigenvectors, left and right, for the Perron–Frobenius eigenvalues can be normalized to have values in $\mathbb{R}_{\geq 0}$.*
- (d) *Let $h = \text{lcm}(h_1, \dots, h_s)$, and let ν the maximal dimension of the Jordan blocks of M containing λ . There exist matrices $S^i(n)$ with polynomial entries of degree $\leq (\nu-1)$ for $i \in \{0, \dots, h-1\}$ such that*

$$\lim_{n \rightarrow \infty} |(M/\lambda)^{hn+i} - S^i(n)| \rightarrow 0 \quad \forall i \in \{0, \dots, h-1\},$$

and the convergence is geometric with ratio $|\lambda^{\text{sec}}/\lambda|^h$. There are also explicit formulas for the matrices $S^i(n)$, see [Rot81, Section 5].

Proof. This can be found in [Rot81]. See also [Hog07, Section I.10] (in the second version) for a useful list of properties of nonnegative matrices. \square

3B. Three versions of the MT. We recall based algebras. These algebras originate in work of Lusztig on so-called *special* representations of Weyl groups [Lus79]. We follow [KM16, Section 2] with our definition.

Let $\mathbb{K} \subset \mathbb{C}$ be a unital subring. A \mathbb{K} -algebra R with a finite \mathbb{K} -basis $C = \{1 = c_0, \dots, c_{r-1}\}$ is called a *finite based $\mathbb{R}_{\geq 0}$ -algebra* if all structure constants are in $\mathbb{R}_{\geq 0}$ with respect to the basis C . That is, (1A.1) holds.

Example 3B.1. Examples include:

- (a) The Grothendieck rings of all the examples in Example 1A.5.
- (b) Group or more general semigroup algebras for finite groups or semigroups.
- (c) There are many interesting infinite examples coming from skein theory, see e.g. [Thu14].

Decategorifications are our main examples where $\mathbb{R}_{\geq 0}$ can be replaced by $\mathbb{Z}_{\geq 0}$. \diamond

A finite based $\mathbb{R}_{\geq 0}$ -algebra is actually a pair (R, C) , but we will write R for short. Next, fix such an R and $c \in \mathbb{R}_{\geq 0}C$. In this setting we can define the (pre) action matrix $M'(c)_{k,j} = \sum_i a_i m_{i,j}^k \in \mathbb{R}_{\geq 0}$. The action matrix $M(c)$ is then the adjacency matrix for the connected component, in the nonoriented sense, of the identity $1 \in C$ in the adjacency graph of $M'(c)$. Note that $M(c) \in \text{Mat}_m(\mathbb{R}_{\geq 0})$ is a submatrix of $M'(c) \in \text{Mat}_r(\mathbb{R}_{\geq 0})$ for some $1 \leq m \leq r$.

We give three versions of *MT*, stated in terms of finite based $\mathbb{R}_{\geq 0}$ -algebras. The categorical version then follows immediately from [Lemma 1A.6](#).

Theorem 3B.2 (Version 1). *Fix a finite based $\mathbb{R}_{\geq 0}$ -algebra R , and c a $\mathbb{R}_{\geq 0}$ -linear combination of elements from C . Assume that the action matrix $M(c)$ is **irreducible**. Then [Theorem 1A.4](#) holds with $a(n)$ as in [\(1A.3\)](#).*

Proof. Consider the following matrix equation:

$$M(c)c(n-1) = c(n),$$

where $c(k) = (c_0(k), \dots, c_{r-1}(k)) \in \mathbb{R}_{\geq 0}^r$ are vectors such that their i th entry is the multiplicity of c_i in c^k , and $a(0) = (1, 0, \dots, 0)^T$ with the one is the slot of $c_0 = 1$. This equation holds by the definition of the action matrix. Iterating this process, we get

$$M(c)^n c(0) = c(n).$$

Note that $M(c)^n c(0)$ is the same as taking the first column of $M(c)^n$. Hence,

$$b^{R,c}(n) = M(c)^n [1]$$

in the notation of the introduction. Thus, [Theorem 3A.2](#) implies the result. \square

Remark 3B.3. [Theorem 3B.2](#) is sufficient for many example. Explicitly, [Theorem 3B.2](#) works for all *transitive* finite based $\mathbb{R}_{\geq 0}$ -algebras. Examples include all finite monoidal categories that are rigid by [\[EGNO15, Proposition 4.5.4\]](#). \diamond

We say that $M \in \text{Mat}_m(\mathbb{R}_{\geq 0})$ has the *Perron–Frobenius property* if its Perron–Frobenius eigenvalue has multiplicity one.

Theorem 3B.4 (Version 2). *Fix a finite based $\mathbb{R}_{\geq 0}$ -algebra R , and c an $\mathbb{R}_{\geq 0}$ -linear combination of elements from C . Assume that the action matrix $M(c)$ has the **Perron–Frobenius property**. Then [Theorem 1A.4](#) holds with $a(n)$ as in [\(1A.3\)](#).*

Proof. The iteration works as in the proof of [Theorem 3B.2](#), so let us focus on the growth rate. We will use [Theorem 3A.3](#) for $s = 1$. This implies that $\nu = 1$, by its definition. In particular, we only have $S^i(n)$ with entries of degree zero, so these are matrices that do not depend on n , so we can simply write S^i . We will argue that they are essentially the matrices $v_i w_i^T$.

Precisely, as follows from [\[Rot81, Section 5\]](#), we have

$$\begin{aligned} S^i &= v_0 w_0^T + v_1 w_1^T \cdot \zeta^i + v_2 w_2^T \cdot \zeta^{2i} + \dots + v_{h-1} w_{h-1}^T \cdot \zeta^{(h-1)i} \\ &= v_0 w_0^T + v_1 w_1^T \cdot \zeta^{nh+i} + v_2 w_2^T \cdot \zeta^{2(nh+i)} + \dots + v_{h-1} w_{h-1}^T \cdot \zeta^{(h-1)(nh+i)i}. \end{aligned}$$

Now we apply [Theorem 3A.3\(c\)](#). \square

Remark 3B.5. [Theorem 3B.4](#) is the version we used in [Section 2](#). \diamond

Recall that the polynomials $S^i(n)$ are explicitly given in [\[Rot81, Section 5\]](#) and define:

$$(3B.6) \quad a(n) = \frac{1}{h} \sum_{i=0}^{h-1} \sum_{j=0}^{h-1} S^j(\lfloor n/h \rfloor) \cdot \zeta^{i(n-j)}.$$

Theorem 3B.7 (Version 3). *Fix a finite based $\mathbb{R}_{\geq 0}$ -algebra R , and c an $\mathbb{R}_{\geq 0}$ -linear combination of elements from C . Then [Theorem 1A.4](#) holds with $a(n)$ as in [\(3B.6\)](#).*

Proof. Observing that $1 + \zeta^i + \dots + \zeta^{(h-1)i} = 0$ if $i \not\equiv 0 \pmod h$, this follows as for the previous theorems. \square

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