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ANNULAR WEBS AND LEVI SUBALGEBRAS

ABEL LACABANNE, DANIEL TUBBENHAUER AND PEDRO VAZ

ABSTRACT. For any Levi subalgebra of the form $\mathfrak{l} = \mathfrak{gl}_{l_1} \oplus \dots \oplus \mathfrak{gl}_{l_d} \subseteq \mathfrak{gl}_n$ we construct a quotient of the category of annular quantum \mathfrak{gl}_n webs that is equivalent to the category of finite dimensional representations of quantum \mathfrak{l} generated by exterior powers of the vector representation. This can be interpreted as an annular version of skew Howe duality, gives a description of the representation category of \mathfrak{l} by additive idempotent completion, and a web version of the generalized blob algebra.

CONTENTS

1. Introduction	1
1A. Webs, Schur–Weyl and Howe duality	1
1B. The main result and relation to other works	2
2. Notations and conventions	3
3. Web categories in the plane	4
3A. Preliminaries	4
3B. Exterior \mathfrak{gl}_n -webs	5
4. Annular webs	7
4A. The annular web category	7
4B. Properties of annular webs	8
4C. Quotient by essential circles	10
5. Representation theory of Levi subalgebras	10
5A. The general linear representation category	10
5B. The Levi representation category	12
5C. Levi crossings	14
6. The equivalence	15
6A. A reminder on the \mathfrak{gl}_n story	15
6B. The statement	15
6C. Monoidal behavior of the main functor	16
6D. Ariki–Koike algebras and annular webs	16
6E. Cartan subalgebras and generalized blob algebras	17
6F. Working integrally	18
7. Proof	19
7A. Well-definedness	19
7B. Proof for $q = 1$	21
7C. Proof for $q \neq 1$	23
References	23

1. INTRODUCTION

Throughout fix $n, l_1, \dots, l_d \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^d l_i = n$.

1A. Webs, Schur–Weyl and Howe duality. The so-called *Schur–Weyl duality* has played a key role ever since the early days of representation theory. It relates representations of the symmetric group S_m and the general linear group $GL_n = GL_n(\mathbb{C})$, and has been generalized in many ways. The representation used to related these two groups is $(\mathbb{C}^n)^{\otimes m}$.

Let $L = GL_{l_1} \times \dots \times GL_{l_d} \subseteq GL_n$, and let us for simplicity stay over \mathbb{C} for now. Two generalizations of Schur–Weyl duality are of crucial importance for this paper. Firstly, the Schur–Weyl duality of $(\mathbb{Z}/d\mathbb{Z}) \wr S_m$ (that is, type $G(d, 1, m)$) and L from [ATY95], [SS99] and [Hu01] (see also [MS16] for

a nice and self-contained discussion of this duality). Here the underlying representation is again $(\mathbb{C}^n)^{\otimes m}$. Second, skew (type A) Howe duality, see [How89] and [How95], relating GL_N and GL_n via their action on $\bigwedge^\bullet(\mathbb{C}^N \otimes \mathbb{C}^n)$.

As explained in [CKM14], a diagrammatic interpretation of skew Howe duality is given by (*exterior* GL_n) *webs*. (The same diagrammatics goes under many names, including birdtracks [Cvi08] or spiders [Kup96].) In some sense, in [CKM14] skew Howe duality relating GL_\bullet and GL_n takes the form of an equivalence between the category of webs and the category of GL_n -representations generated by $\{\bigwedge^k \mathbb{C}^n \mid k \in \{1, \dots, n\}\}$, with the web category being obtained by using all GL_N for $N \in \mathbb{Z}_{\geq 0}$. After additive idempotent completion webs even give a diagrammatic interpretation of all finite dimensional GL_n -representations.

In this paper we show that an explicit quotient of the category of *annular* (*exterior* GL_n) *webs* is equivalent to the category of L-representations generated by $\{\bigwedge^k \mathbb{C}^n \mid k \in \{1, \dots, n\}\}$. As before, additive idempotent completion gives a description of all finite dimensional L-representations. This, in some sense, is a form of what could be called *annular skew Howe duality* (we avoid the notion affine as its meaning is context depending) or *skew type $G(d, 1, \bullet)$ Howe duality*.

1B. The main result and relation to other works. We now give a few details and change to the universal enveloping algebras. We consider a *Levi subalgebra* of the form $\mathfrak{l} = \mathfrak{gl}_{l_1} \oplus \dots \oplus \mathfrak{gl}_{l_d} \subseteq \mathfrak{gl}_n$ (in this paper we write \mathfrak{l} instead of the usual notation \mathfrak{l} for readability). Let \mathbb{K}_q denote a field containing an element $q \in \mathbb{K}_q$ that is not a root of unity and additional variables $\mathbb{U} = \{u_1, \dots, u_d\}$, and let further \mathbb{K}_1 denote a field of characteristic zero containing \mathbb{U} . With these ground fields the category of finite dimensional $U_q(\mathfrak{l})$ -representations over \mathbb{K}_q respectively of finite dimensional $U_1(\mathfrak{l})$ -representations over \mathbb{K}_1 are semisimple. (The reader be warned: as explained in the main body of the text there are some nontrivial quantization issues and we carefully need to distinguish the two cases over \mathbb{K}_q and \mathbb{K}_1 .)

In [Section 4](#) we define a \mathbb{K}_q -linear category of annular webs $\mathbf{AWeb}_q \mathfrak{gl}_n$ as well as a quotient $\mathbf{AWeb}_q \mathfrak{l}$ by evaluating essential circles using the variables \mathbb{U} . Similarly over \mathbb{K}_1 , where we write $\mathbf{AWeb}_1 \mathfrak{gl}_n$ and $\mathbf{AWeb}_1 \mathfrak{l}$. Let $\mathbf{Fund}_q \mathfrak{l}$ respectively $\mathbf{Fund}_1 \mathfrak{l}$ denote the categories of $U_q(\mathfrak{l})$ - and $U_1(\mathfrak{l})$ -representations generated by the exterior powers of the vector representation. Our main result is [Theorem 6B.1](#) showing that $\mathbf{AWeb}_q \mathfrak{l}$ is equivalent to $\mathbf{Fund}_q \mathfrak{l}$ and that $\mathbf{AWeb}_1 \mathfrak{l}$ is pivotally equivalent to $\mathbf{Fund}_1 \mathfrak{l}$. The main ingredients in the proof of [Theorem 6B.1](#) are the usual diagrammatic ideas, the Schur–Weyl type dualities from [SS99] as well as the *explosion trick*, which utilizes the semisimplicity.

An almost direct consequence of [Theorem 6B.1](#) is that the endomorphism algebras of annular webs corresponding to tensor products of the vector representation can be described explicitly. As we will see in [Section 6D](#) these are given by certain *row quotients of Ariki–Koike algebras* (Ariki–Koike algebras are Hecke algebras of $(\mathbb{Z}/d\mathbb{Z}) \wr S_m$, see for example [AK94], [BM93], [Che87]) as studied in [LV21]. In the special case of two row quotients, which corresponds to $\mathfrak{l} = \mathfrak{gl}_1 \oplus \dots \oplus \mathfrak{gl}_1$ being the *Cartan subalgebra*, these algebras are Martin–Woodcock’s *generalized blob algebras* [MW03]. We thus obtain a web description of generalized blob algebras, see [Section 6E](#) for details. This web description allows us to answer a conjecture of Cautis–Kamnitzer [CK18, Conjectures 10.2 and 10.3] affirmatively, up to technicalities as detailed in [Remark 6E.7](#).

Let us also mention that our work is inspired by [Que15] (which gives another, very honest, version of annular skew Howe duality), the aforementioned paper [CK18] and [QW18] (which proves [Theorem 6B.1](#) for $\mathbb{K}_1 = \mathbb{C}$ and the special case of the Cartan subalgebra). With respect to [Que15] and [QW18], which are partially motivated by Skein theory, the reader be warned that the monoidal structure on $\mathbf{AWeb}_q \mathfrak{l}$ coming from Skein theory and the one on $\mathbf{Fund}_q \mathfrak{l}$ coming from the Hopf algebra structure of $U_q(\mathfrak{l})$ do not seem to be compatible, see [Section 6C](#) for a discussion.

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2. NOTATIONS AND CONVENTIONS

We start by specifying our notations.

Notation 2.1. Recall that we fixed $n, l_1, \dots, l_d \in \mathbb{Z}_{\geq 0}$ with $\sum_{i=1}^d l_i = n$. These will be used via the general linear Lie algebra \mathfrak{gl}_n and a Levi subalgebra ℓ given by $\mathfrak{h} \subseteq \ell = \mathfrak{gl}_{l_1} \oplus \dots \oplus \mathfrak{gl}_{l_d} \subseteq \mathfrak{gl}_n$. For $l_1 = \dots = l_d = 1$, so that $\ell = \mathfrak{gl}_1 \oplus \dots \oplus \mathfrak{gl}_1$ is the Cartan subalgebra, we will write \mathfrak{h} instead of ℓ .

Notation 2.2. We now specify our underlying ground rings.

- (a) For essential circles, see [Section 4C](#) below, we need extra polynomial elements (these can be ignored otherwise). We denote by $\mathbb{U} = \{u_1, \dots, u_d\}$ variables.
- (b) Let $\mathbb{Z}_v = \mathbb{Z}[v, v^{-1}, \mathbb{U}, \mathbb{U}^{-1}]$ for some indeterminate v .
- (c) We let \mathbb{K}_q denote a field containing \mathbb{U} and \mathbb{U}^{-1} and an element q which is not a root of unity. Let also \mathbb{K}_1 be a field of characteristic zero containing \mathbb{U} and \mathbb{U}^{-1} . (Note that $\text{char}(\mathbb{K}_q)$ is allowed to be a prime, but we assume that $\text{char}(\mathbb{K}_1) = 0$.)
- (d) We also see \mathbb{K}_q as the specialization and scalar extension $- \otimes_{\mathbb{Z}_v} \mathbb{K}_q$ given by $v \mapsto q$, and \mathbb{K}_1 as the specialization and scalar extension $- \otimes_{\mathbb{Z}_v} \mathbb{K}_1$ given by $v \mapsto 1$. We will apply scalar extension to \mathbb{Z}_v -linear categories, and since this will play an important role we will indicate the specialization accordingly. The two important specializations for \mathbb{K}_q and \mathbb{K}_1 are distinguished by using q respectively 1 as a subscript.
- (e) If not specified otherwise, then \otimes denotes the tensor product over the ground ring, which is either \mathbb{Z}_v , \mathbb{K}_q or \mathbb{K}_1 .

Not everything we do in this paper is defined over \mathbb{Z}_v , and statements for \mathbb{K}_q are not strictly related to the ones for \mathbb{K}_1 . If we use \mathbb{Z}_v , then we can specialize without problem. But if we do not work over \mathbb{Z}_v , then, to not double half of the text, we use the following crucial simplification:

Notation 2.3. Throughout (on the diagrammatic and the representation theoretical sides) we always specify our conventions for \mathbb{K}_q and leave the analog conventions and lemmas for \mathbb{K}_1 implicit; the ones for \mathbb{K}_1 are always the $q = 1$ versions of the ones for \mathbb{K}_q . When lemmas etc. are the same for \mathbb{K}_q and \mathbb{K}_1 , then we will use simply q (e.g. we write \mathbb{K}_q and not \mathbb{K}_q and \mathbb{K}_1) to indicate this. We will stress whenever the statements for \mathbb{K}_q and \mathbb{K}_1 are significantly different.

We will use quantum numbers, factorials and binomials viewed as elements of \mathbb{Z}_v . That is, for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$ we let $[0] = 0$, $[0]! = 1 = \begin{bmatrix} a \\ 0 \end{bmatrix}$, $[a] = -[-a]$ for $a < 0$ and otherwise

$$[a] = v^{a-1} + v^{a-3} + \dots + v^{-a+3} + v^{-a+1}, \quad [b]! = [b][b-1]\dots[1], \quad \begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a][a-1]\dots[a-b+1]}{[b]}.$$

The following will be used silently throughout.

Lemma 2.4. *All quantum binomials are invertible in \mathbb{K}_q .*

Proof. Easy and omitted. □

Notation 2.5. We work with strict pivotal (thus, monoidal) categories, where we strictify categories if necessary (by the usual strictification theorems this restriction is for convenience only). We have two directions of composition, vertical \circ_v and horizontal \circ_h , as well as a duality $*$ operation. The monoidal unit is denoted by $\mathbb{1}$, and identity morphisms are denoted by id . We will also distinguish objects and morphisms using different fonts, e.g. K and f .

As we will see, e.g. in [Section 6C](#) below, it will turn out to be important to carefully distinguish the monoidal product on the various categories we consider. This will only play a role on the level of morphisms, and we use various symbols for monoidal products between morphisms if necessary.

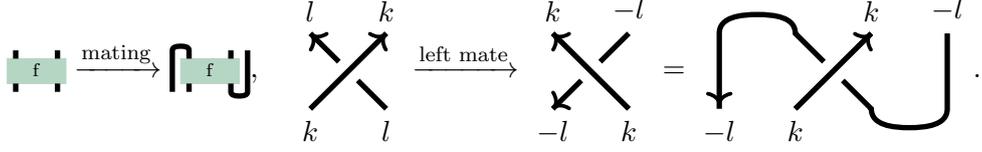
Notation 2.6. We now summarize the diagrammatic conventions that we will use in this paper.

- (a) The following illustration of the interchange law summarizes our reading conventions:

$$(\text{id} \circ_h g) \circ_v (f \circ_h \text{id}) = \circ_v \left(\begin{array}{c} \dots \\ \vdots \\ \text{g} \\ \vdots \\ \text{f} \\ \vdots \\ \dots \end{array} \right) \circ_v = \begin{array}{c} \dots \\ \vdots \\ \text{f} \\ \vdots \\ \text{g} \\ \vdots \\ \dots \end{array} = \circ_v \left(\begin{array}{c} \dots \\ \vdots \\ \text{f} \\ \vdots \\ \text{g} \\ \vdots \\ \dots \end{array} \right) \circ_v = (f \circ_h \text{id}) \circ_v (\text{id} \circ_h g).$$

That is, we read diagrams from bottom to top and left to right.

Definition 3A.5. *Mating* in \mathbf{Dia} is the process of applying (co)evaluation morphisms, for example



Note that mating produces many morphisms from a given set of morphisms.

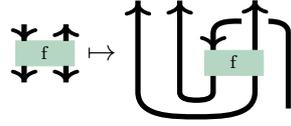
Lemma 3A.6. *Suppose that $\text{End}_{\mathbf{Dia}}(\mathbb{1}) \cong \text{End}_{\mathbf{Dia}}(\uparrow_k) \cong R$. If the left mate of the (k, l) -overcrossing is invertible, then all mates of the (k, l) -overcrossing and its inverse span a pivotal subcategory equivalent to R -linear (labeled) tangles. In general, if the left mate of the (k, l) -overcrossing is invertible, then all mates of the (k, l) -overcrossing and its inverse span a pivotal subcategory equivalent to R -linear (labeled) framed tangles.*

Proof. Note that the assumption $\text{End}_{\mathbf{Dia}}(\uparrow_k) \cong R$ implies the Reidemeister I relation up to scalars, which is enough to copy the argument in [Kas95, Theorem X11.2.2]. The second claim follows similarly. \square

We say $\text{Hom}_{\mathbf{Dia}}(K, L)$ is *determined by* $\text{Hom}_{\mathbf{Dia}}(K', L')$ if there exists an isomorphism of R -modules between them.

Lemma 3A.7. *If the left mates of the (k, l) -overcrossings are invertible, then all hom-spaces in \mathbf{Dia} are determined by upwards hom-spaces.*

Proof. By Lemma 3A.6, the assumptions imply that



is an isomorphism. This easily generalizes. \square

Definition 3A.8. A left invertible morphism $\uparrow_{k+l} \rightarrow \uparrow_k \circ_h \uparrow_l$ is called *explosion*.

Up to rescaling, explosion morphisms and their left inverses can be illustrated by

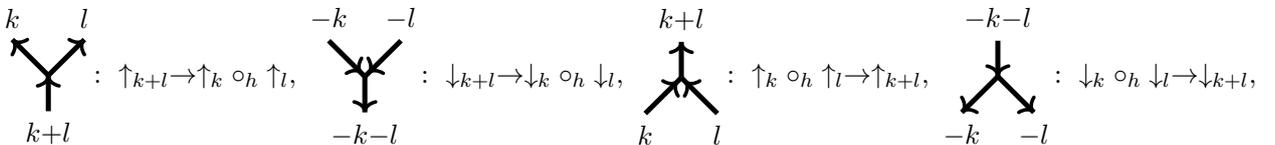


Lemma 3A.9. *If the left mates of the (k, l) -overcrossings are invertible and all explosion morphisms exist, then all hom-spaces in \mathbf{Dia} are determined by end-spaces between objects of the form $1^{\circ_h k} = \uparrow_1 \circ_h \dots \circ_h \uparrow_1$.*

Proof. Turn all strands upwards using Lemma 3A.7, and then explode the strands inductively. \square

3B. Exterior \mathfrak{gl}_n -webs. We now recall the category of exterior \mathfrak{gl}_n webs.

Definition 3B.1. The (exterior \mathfrak{gl}_n) web category $\mathbf{Web}_v \mathfrak{gl}_n$ is the diagram category for $R = \mathbb{Z}_v$ with \circ_h -generating objects of categorical dimension $\begin{bmatrix} n \\ k \end{bmatrix}$ and \circ_v - \circ_h -generating morphisms



such that the left mates of the (k, l) -overcrossing are invertible. The relations imposed on $\mathbf{Web}_v \mathfrak{gl}_n$ are *isotopies* (not displayed zigzag and trivalent-slide relations, see e.g. [LT21, Section 2] for details),

the *exterior relation*, *associativity*, *coassociativity*, *digon removal*, and *dumbbell-crossing relation*. That is, we take the quotient by the $\circ_v\text{-}\circ_h$ -ideal generated by isotopies and

$$\begin{array}{c}
 \begin{array}{c} >n \\ | \\ >n \end{array} = 0, \quad \begin{array}{c} k+l+m \\ \nearrow \\ k \quad l \quad m \\ \searrow \end{array} = \begin{array}{c} k+l+m \\ \nearrow \\ k \quad l \quad m \\ \searrow \end{array}, \quad \begin{array}{c} k \quad l \quad m \\ \nearrow \\ k+l+m \\ \searrow \end{array} = \begin{array}{c} k \quad l \quad m \\ \nearrow \\ k+l+m \\ \searrow \end{array}, \\
 \\
 \begin{array}{c} k+l \\ \nearrow \\ k \quad l \\ \searrow \\ k+l \end{array} = \begin{bmatrix} k+l \\ k \end{bmatrix} \cdot \begin{array}{c} k+l \\ \uparrow \\ k+l \end{array}, \quad \begin{array}{c} k+l \quad r \quad s \\ \nearrow \\ k \quad l \\ \searrow \end{array} = \sum_{k-r=a-b} v^{-(k-a)(l-b)} \begin{array}{c} r \quad s \\ \nearrow \\ a \quad b \\ \searrow \\ k \quad l \end{array} = \sum_{k-r=a-b} v^{(k-a)(l-b)} \begin{array}{c} r \quad s \\ \nearrow \\ a \quad b \\ \searrow \\ k \quad l \end{array},
 \end{array}$$

together with their $(-)^{\uparrow}$ -duals.

We call morphisms in $\mathbf{Web}_v \mathfrak{gl}_n$ (*exterior \mathfrak{gl}_n webs*).

It follows for example from [LT21, Sections 2 and 5] that there is a well-defined functor from the pivotal category of topological webs (where webs are defined as plane labeled oriented trivalent graphs up to planar isotopy) to $\mathbf{Web}_v \mathfrak{gl}_n$ which we will use to draw webs in a topological fashion.

Lemma 3B.2. *In $\mathbf{Web}_v \mathfrak{gl}_n$ we have*

$$(3B.3) \quad \begin{array}{c} l \quad k \\ \nearrow \\ k \quad l \end{array} = (-1)^{kl} \sum_{b-a=k-l} (-v)^{k-b} \begin{array}{c} l \quad k \\ \nearrow \\ a \quad b \\ \searrow \\ k \quad l \end{array}, \quad \begin{array}{c} l \quad k \\ \searrow \\ k \quad l \end{array} = (-1)^{kl} \sum_{b-a=k-l} (-v)^{-k+b} \begin{array}{c} l \quad k \\ \searrow \\ a \quad b \\ \nearrow \\ k \quad l \end{array}.$$

Proof. This is explained in [LT21, Section 5]. □

For completeness, the $k = l = 1$ case of (3B.3) is

$$(3B.4) \quad \begin{array}{c} 1 \quad 1 \\ \nearrow \\ 1 \quad 1 \end{array} = v \cdot \begin{array}{c} 1 \\ \uparrow \\ 1 \end{array} - \begin{array}{c} 1 \quad 1 \\ \nearrow \\ 1 \quad 1 \\ \searrow \end{array}, \quad \begin{array}{c} 1 \quad 1 \\ \searrow \\ 1 \quad 1 \end{array} = v^{-1} \cdot \begin{array}{c} 1 \\ \uparrow \\ 1 \end{array} - \begin{array}{c} 1 \quad 1 \\ \searrow \\ 1 \quad 1 \\ \nearrow \end{array}.$$

Lemma 3B.5. *The following hold in $\mathbf{Web}_v \mathfrak{gl}_n$.*

- (a) *The crossings satisfy the Reidemeister II and III relations, and the Reidemeister I relation holds with scalar, that is,*

$$(3B.6) \quad \begin{array}{c} k \\ \uparrow \\ \text{loop} \\ \downarrow \\ k \end{array} = v^{k(-k+n+1)} \cdot \begin{array}{c} k \\ \uparrow \\ \text{loop} \\ \downarrow \\ k \end{array}, \quad \begin{array}{c} k \quad k \\ \nearrow \\ k \quad k \\ \searrow \end{array} = \begin{array}{c} k \quad k \\ \nearrow \\ k \quad k \\ \searrow \end{array}, \quad \begin{array}{c} k \quad k \\ \searrow \\ k \quad k \\ \nearrow \end{array} = v^{k(k-n-1)} \cdot \begin{array}{c} k \quad k \\ \searrow \\ k \quad k \\ \nearrow \end{array},$$

together with its $(-)^{\uparrow}$ -dual. Various naturality relations hold, see e.g. [LT21, Section 2].

- (b) *Square switches, see e.g. [LT21, Lemma 5.6]. Various other relation that we do not explicitly use, see e.g. [LT21, Section 2] also hold.*

(c) Explosion holds for $\mathbf{Web}_q\mathfrak{gl}_n$ (but not in $\mathbf{Web}_v\mathfrak{gl}_n$), that is

$$\begin{array}{c} k+l \\ \uparrow \\ k+l \end{array} = \begin{bmatrix} k+l \\ k \end{bmatrix}^{-1} \cdot k \begin{array}{c} k+l \\ \uparrow \\ \diamond \\ \downarrow \\ k+l \end{array} l,$$

together with its $(-)^{\uparrow}$ -dual. Moreover, the (k, l) -overcrossings satisfy explosion as well, i.e.

$$(3B.7) \quad \begin{array}{c} l \quad k \\ \diagdown \quad \diagup \\ k \quad l \end{array} = [k]!^{-1} [l]!^{-1} \cdot \begin{array}{c} l \quad k \\ \diagdown \quad \diagup \\ \text{x} \\ \diagup \quad \diagdown \\ k \quad l \end{array}, \quad \text{x} = \begin{array}{c} l \text{ strands} \\ \dots \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \dots \\ k \text{ strands} \end{array},$$

as well as a similar formula for the (k, l) -undercrossings. □

Note that thus Lemma 3A.6 applies in $\mathbf{Web}_q\mathfrak{gl}_n$. Let us also note that following, partially explaining why explosion works well in practice:

Lemma 3B.8. *The additive idempotent completion of $\mathbf{Web}_q\mathfrak{gl}_n$ is semisimple.*

Proof. By Lemma 3A.6 and the existence of certain projectors, see e.g. [RT16] or [TVW17, Section 2.3]. To be precise (and to fix notation for the rest of the paper), the projectors are

$$[k]!^{-1} \cdot \begin{array}{c} 1 \quad 1 \\ \diagdown \quad \diagup \\ k \\ \diagup \quad \diagdown \\ 1 \quad 1 \end{array}, \quad \begin{array}{c} k \\ \diagdown \quad \diagup \\ 1 \quad 1 \end{array} = \begin{array}{c} k \\ \diagdown \quad \diagup \\ 1 \quad 1 \quad 1 \quad 1 \end{array},$$

where the dots indicate an inductive construction as illustrated on the right (the order of how these are constructed is irrelevant due to (co)associativity). □

4. ANNULAR WEBS

This section discusses our main diagram categories of this paper. Similar constructions have appeared in many texts, e.g. [CK18] or [QW18].

4A. The annular web category. The following definition does not use any \circ_h structure.

Definition 4A.1. The (exterior \mathfrak{gl}_n) annular web category $\mathbf{AWeb}_v\mathfrak{gl}_n$ is the category obtained from $\mathbf{Web}_v\mathfrak{gl}_n$ by adding extra \circ_v -generators

$$\rho_K = \begin{array}{c} k_2 \quad k_m \quad k_1 \\ \diagdown \quad \dots \quad \diagup \\ \text{hook} \\ \diagup \quad \dots \quad \diagdown \\ k_1 \quad k_2 \quad k_m \end{array}, \quad \rho^K = \begin{array}{c} k_1 \quad k_2 \quad k_m \\ \text{hook} \\ \diagdown \quad \dots \quad \diagup \\ k_2 \quad k_m \quad k_1 \end{array},$$

for each $K = (k_1, \dots, k_m) \in \mathbb{Z}^m$ to $\mathbf{Web}_v\mathfrak{gl}_n$ modulo the \circ_v -ideal generated by the relations

$$(4A.2) \quad \begin{array}{c} \text{hook} \end{array} = \begin{array}{c} \text{vertical lines} \end{array}, \quad \begin{array}{c} \text{hook} \end{array} = \begin{array}{c} \text{vertical lines} \end{array},$$

Lemma 4B.6. $\mathbf{AWeb}_v\mathfrak{gl}_n$ is equivalent as a diagram category to the affinization $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ (in the sense of e.g. [MS21, Definition 2.1]) of $\mathbf{Web}_v\mathfrak{gl}_n$.

Proof. We will not use this statement explicitly, so we only sketch the proof. First, there is clearly an essentially surjective functor Γ from $\mathbf{AWeb}_v\mathfrak{gl}_n$ to $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ that puts a plane web into the annulus. Next, $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ is defined by adjoining more morphisms and relations to $\mathbf{AWeb}_v\mathfrak{gl}_n$, namely one coil and its inverse for each K and relations [MS21, (2.5)]. But using the coils in Definition 4A.1 one can define these more general coils following [MS21, (2.5), left equation] which satisfy [MS21, (2.5), right equation], showing that Γ is full. Faithfulness of Γ can then be deduced from Theorem 6B.1 below, by showing that the functor therein factors through $\mathbf{Aff}(\mathbf{Web}_v\mathfrak{gl}_n)$ via Γ . \square

4C. Quotient by essential circles. We now define quotients of $\mathbf{AWeb}_v\mathfrak{gl}_n$.

Definition 4C.1. The left and right *essential k -circles* are defined to be

$$c_k^{\leftarrow} = \left[\begin{array}{c} \text{---} \leftarrow \text{---} \\ \text{---} \end{array} \right]_k = \left[\begin{array}{c} \text{---} \leftarrow \text{---} \\ \text{---} \end{array} \right]_k, \quad c_k^{\rightarrow} = \left[\begin{array}{c} \text{---} \rightarrow \text{---} \\ \text{---} \end{array} \right]_k = \left[\begin{array}{c} \text{---} \rightarrow \text{---} \\ \text{---} \end{array} \right]_k.$$

We also say *essential circles* for short.

Note that essential circles are nontrivial endomorphism of $\mathbb{1}$. We want to evaluate them. To this end, let e_k denote the k th elementary symmetric polynomial in n variables, i.e. $e_k = e_k(Z_1, \dots, Z_n)$.

Definition 4C.2. The *Levi evaluation* for ℓ of the essential circles is defined to be

$$(4C.3) \quad \left[\begin{array}{c} \text{---} \leftarrow \text{---} \\ \text{---} \end{array} \right]_k - e_k(v^{-1}u_1, v^{-3}u_1, \dots, v^{-2l_1+1}u_1, \dots, v^{-1}u_d, v^{-3}u_d, \dots, v^{-2l_d+1}u_d) \cdot \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right], \\ \left[\begin{array}{c} \text{---} \rightarrow \text{---} \\ \text{---} \end{array} \right]_k - e_k(vu_1^{-1}, v^3u_1^{-1}, \dots, v^{2l_1-1}u_1^{-1}, \dots, vu_d^{-1}, v^3u_d^{-1}, \dots, v^{2l_d-1}u_d^{-1}) \cdot \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right],$$

which are elements in $\text{End}_{\mathbf{AWeb}_v\mathfrak{gl}_n}(\mathbb{1})$.

The following quotient gives a diagrammatic description of quantum ℓ -representations. We call the kernel of the functor $\mathbf{A}\Gamma_q$ from Section 6B the *Levi ideal* and denote it by \mathcal{I}_ℓ .

Lemma 4C.4. The Levi ideal \mathcal{I}_ℓ is a two-sided \circ_v -ideal in $\mathbf{AWeb}_q\mathfrak{gl}_n$.

Proof. The kernel is a two-sided \circ_v -ideal. \square

Thus, we get a well-defined category by:

Definition 4C.5. Let $\mathbf{AWeb}_q\ell$ denote the quotient of $\mathbf{AWeb}_q\mathfrak{gl}_n$ by the Levi ideal \mathcal{I}_ℓ .

For \mathbb{K}_1 it will turn out that \mathcal{I}_ℓ is the two-sided $\circ_v\text{-}\circ_h^A$ -ideal generated by the Levi evaluations (4C.3). Being careful with the underlying monoidal structure (details are given in Theorem 6B.1), the same holds for \mathbb{K}_q , hence the name.

5. REPRESENTATION THEORY OF LEVI SUBALGEBRAS

This section discusses the representation categories of this paper. The below is (partially) well-known and we will be brief whenever appropriate. A lot of details and also background can be found in text such as [Jan96].

5A. The general linear representation category. We fix our notation regarding the general linear quantum algebra. Let $U_v(\mathfrak{gl}_n)$ be the *divided power quantum enveloping algebra* for \mathfrak{gl}_n , where we use the conventions, excluding the Hopf algebra structure, from [APW91] in the special case of \mathfrak{gl}_n (using $K_i^{\pm 1} = L_i^{\pm 1}L_{i+1}^{\mp 1}$). The algebra $U_v(\mathfrak{gl}_n)$ specializes to either the \mathbb{K}_q -algebra $U_q(\mathfrak{gl}_n)$ or the \mathbb{K}_1 -algebra $U_1(\mathfrak{gl}_n)$ for which we now recall the relevant formulas.

The algebra $U_q(\mathfrak{gl}_n)$ is generated by $L_i^{\pm 1}$ for $i \in \{1, \dots, n\}$ (these are inverses) and E_i, F_i for $i \in \{1, \dots, n-1\}$ and the Hopf algebra structure used in this paper is

$$\begin{aligned} \Delta(E_i) &= E_i \otimes L_i L_{i+1}^{-1} + 1 \otimes E_i, & \epsilon(E_i) &= 0, & S(E_i) &= -E_i L_i^{-1} L_{i+1}, \\ \Delta(F_i) &= F_i \otimes 1 + L_i^{-1} L_{i+1} \otimes F_i, & \epsilon(F_i) &= 0, & S(F_i) &= -L_i L_{i+1}^{-1} F_i, \end{aligned}$$

with $L_i^{\pm 1}$ being group like.

The *vector representation* $\mathbf{V}_q = \mathbf{V}_q(\mathfrak{gl}_n)$ of $U_q(\mathfrak{gl}_n)$ is the \mathbb{K}_q -vector space $\mathbb{K}_q\{v_1, \dots, v_n\}$ with action

$$L_i^{\pm 1} \cdot v_j = q^{\pm \delta_{i,j}} v_j, \quad E_i \cdot v_j = \delta_{i,j} v_{j-1}, \quad F_i \cdot v_j = \delta_{i,j} v_{j+1}.$$

Let TV_q be the tensor algebra. The k th *quantum exterior power* $\Lambda_q^k \mathbf{V}_q$ is defined as the degree k part (in the usual sense) of the *quantum exterior algebra* given by

$$(5A.1) \quad \Lambda_q^\bullet \mathbf{V}_q = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \Lambda_q^k \mathbf{V}_q = TV_q / \langle v_h \otimes v_h, v_j \otimes v_i + q^{-1} v_i \otimes v_j \mid i < j \rangle_{\text{two-sided } \otimes\text{-ideal}}.$$

The exterior powers are $U_q(\mathfrak{gl}_n)$ -representations by using the Hopf algebra structure, and so is $\mathbb{1} = \mathbb{K}_q$ itself and all the duals of the above, denoted by using negative powers: $\Lambda_q^k \mathbf{V}_q = (\Lambda_q^{-k} \mathbf{V}_q)^*$ for $k \in \mathbb{Z}_{< 0}$.

Lemma 5A.2. *For $k \in \mathbb{Z}_{\geq 0}$, the \mathbb{K}_q -vector space $\Lambda_q^k \mathbf{V}_q$ has a basis given by $\{v_S = v_{i_1} \otimes \dots \otimes v_{i_k} \mid S = (i_1 < \dots < i_k) \text{ for } i_j \in \{1, \dots, n\}\}$. If $-k \in \mathbb{Z}_{\geq 0}$, then the \mathbb{K}_q -vector space $\Lambda_q^k \mathbf{V}_q$ has a basis given by $\{v_S^* = v_{i_{-k}} \otimes \dots \otimes v_{i_1} \mid S = (i_1 < \dots < i_{-k}) \text{ for } i_j \in \{1, \dots, n\}\}$.*

Proof. Easy and omitted. \square

Notation 5A.3. We also use the notation v_S from Lemma 5A.2 more generally for any $S = (i_1, \dots, i_k)$ for $i_j \in \{1, \dots, n\}$, and use the usual set operations on them. Recall that such expressions need to potentially be reordered using (5A.1) to match the basis of Lemma 5A.2.

We now consider so-called $U_q(\mathfrak{gl}_n)$ -representations of type 1 which, as usual, is not a serious restriction, see e.g. [Jan96, Section 5.2] for details.

Definition 5A.4. Let $\mathbf{Rep}_q \mathfrak{gl}_n$ denote the category of finite dimensional $U_q(\mathfrak{gl}_n)$ -representations of type 1. We view $\mathbf{Rep}_q \mathfrak{gl}_n$ as pivotal using the above Hopf algebra structure on $U_q(\mathfrak{gl}_n)$. Let further $\mathbf{Fund}_q \mathfrak{gl}_n$ denote the full pivotal subcategory with objects of the form $\Lambda_q^K \mathbf{V}_q = \Lambda_q^{k_1} \mathbf{V}_q \otimes \dots \otimes \Lambda_q^{k_m} \mathbf{V}_q$ for $K = (k_1, \dots, k_m) \in \mathbb{Z}^m$ and $m \in \mathbb{Z}_{\geq 0}$.

We call $\mathbf{Rep}_q \mathfrak{gl}_n$ the *representation category* of $U_q(\mathfrak{gl}_n)$ and $\mathbf{Fund}_q \mathfrak{gl}_n$ its *fundamental category*. (We use the same terminology for ℓ defined below.) The following is crucial, but well-known, and will be used throughout. To state it let $X_{\mathfrak{gl}_n}^+ \subset \mathbb{Z}^n$ denote the set of dominant integral \mathfrak{gl}_n -weight, i.e. tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \dots \geq \lambda_n$.

Lemma 5A.5. *We have the following.*

- (a) *The category $\mathbf{Rep}_q \mathfrak{gl}_n$ is semisimple, its simple objects are of the form $L(\lambda)$ for $\lambda \in X^+$ and their characters are given by Weyl's character formula.*
- (b) *The additive idempotent completion of $\mathbf{Fund}_q \mathfrak{gl}_n$ is pivotally equivalent to $\mathbf{Rep}_q \mathfrak{gl}_n$.*

Proof. Claim (a). See e.g. [Jan96, Theorems 5.15 and 5.17] or [APW91, Section 6].

Claim (b). By (a) classical theory applies, see e.g. [Jan96, Theorems 5.15 and 5.17] or [APW91, Section 6] for details. \square

Let us now list generating $U_q(\mathfrak{gl}_n)$ -equivariant morphisms that will be mapped to the generators of $\mathbf{Web}_q \mathfrak{gl}_n$. The notation is hopefully suggestive.

For tuples S, T as in Notation 5A.3 let $|S < T| = |\{(s, t) \in S \times T \mid s < t\}|$ and $|S, N| = |S < N| - |N < S|$ for $N = (1, 2, \dots, n)$. Using such tuples and this notation we define (here $k, l \in \mathbb{Z}_{\geq 0}$):

$$(5A.6) \quad \begin{aligned} \Lambda_{k,l}^{k+l} &: \Lambda_q^k \mathbf{V}_q \otimes \Lambda_q^l \mathbf{V}_q \rightarrow \Lambda_q^{k+l} \mathbf{V}_q, & v_S \otimes v_T &\mapsto \delta_{S \cap T, \emptyset} (-q)^{-|T < S|} v_{S \cup T}, \\ Y_{k+l}^{k,l} &: \Lambda_q^{k+l} \mathbf{V}_q \rightarrow \Lambda_q^k \mathbf{V}_q \otimes \Lambda_q^l \mathbf{V}_q, & v_U &\mapsto (-1)^{kl} \sum_{S \sqcup T = U, |S|=k} (-q)^{|S < T|} v_S \otimes v_T, \\ \cap_k^{\leftarrow} &: \Lambda_q^{-k} \mathbf{V}_q \otimes \Lambda_q^k \mathbf{V}_q \rightarrow \mathbb{K}_q, & v_S^* \otimes v_T &\mapsto \delta_{S, T}, \\ \cap_k^{\rightarrow} &: \Lambda_q^k \mathbf{V}_q \otimes \Lambda_q^{-k} \mathbf{V}_q \rightarrow \mathbb{K}_q, & v_S \otimes v_T^* &\mapsto q^{|S, N|} \delta_{S, T}, \\ \cup_k^{\leftarrow} &: \mathbb{K}_q \rightarrow \Lambda_q^k \mathbf{V}_q \otimes \Lambda_q^{-k} \mathbf{V}_q, & 1 &\mapsto \sum_{|S|=k} v_S \otimes v_S^*, \\ \cup_k^{\rightarrow} &: \mathbb{K}_q \rightarrow \Lambda_q^{-k} \mathbf{V}_q \otimes \Lambda_q^k \mathbf{V}_q, & 1 &\mapsto \sum_{|S|=k} q^{-|S, N|} v_S^* \otimes v_S. \end{aligned}$$

Lemma 5A.7. *The morphisms in (5A.6) are \circ_v - \otimes -generators of $\mathbf{Fund}_q \mathfrak{gl}_n$.*

Proof. A careful check of the relations shows that these maps are $U_q(\mathfrak{gl}_n)$ -equivariant. That they generate follows from Lemma 5A.5 and classical theory. \square

We have an *algebraic version of explosion*:

Lemma 5A.8. *Explosion holds for $\mathbf{Fund}_q \mathfrak{gl}_n$, that is*

$$\mathrm{id}_{k+l} = \begin{bmatrix} k+l \\ k \end{bmatrix}^{-1} \Lambda_{k,l}^{k+l} Y_{k+l}^{k,l}.$$

Proof. A direct computation. \square

We denote by $Y_k^{1,\dots,1}$ and $\Lambda_{1,\dots,1}^k$ the successive explosion of k strands.

We also have the $(1,1)$ -overcrossings and $(1,1)$ -undercrossings

$$\begin{aligned} \mathbf{R}_{1,1} &= q \mathrm{id}_{1,1} - Y_2^{1,1} \Lambda_{1,1}^2 : \mathbf{V}_q \otimes \mathbf{V}_q \rightarrow \mathbf{V}_q \otimes \mathbf{V}_q, & v_i \otimes v_j &\mapsto \begin{cases} qv_i \otimes v_i & \text{if } i = j, \\ v_j \otimes v_i & \text{if } i < j, \\ v_j \otimes v_i + (q - q^{-1})v_i \otimes v_j & \text{if } i > j, \end{cases} \\ \mathbf{R}_{1,1}^{-1} &= q^{-1} \mathrm{id}_{1,1} - Y_2^{1,1} \Lambda_{1,1}^2 : \mathbf{V}_q \otimes \mathbf{V}_q \rightarrow \mathbf{V}_q \otimes \mathbf{V}_q, & v_i \otimes v_j &\mapsto \begin{cases} q^{-1}v_i \otimes v_i & \text{if } i = j, \\ v_j \otimes v_i & \text{if } i > j, \\ v_j \otimes v_i + (q^{-1} - q)v_i \otimes v_j & \text{if } i < j. \end{cases} \end{aligned}$$

We also get crossings $\mathbf{R}_{k,l}^{\pm 1} : \Lambda_q^k \mathbf{V}_q \otimes \Lambda_q^l \mathbf{V}_q \rightarrow \Lambda_q^l \mathbf{V}_q \otimes \Lambda_q^k \mathbf{V}_q$ for all $k, l \in \mathbb{Z}$ by mimicking (3B.3) and mating.

Lemma 5A.9. *The crossings satisfy the Reidemeister II and III relations and various naturality relations, and can be alternatively defined by explosion.*

Proof. Well-known and omitted (for the explosion statement see Lemma 3B.5 imported via Theorem 6A.1). \square

5B. The Levi representation category. Recall that we have fixed $\ell = \mathfrak{gl}_{l_1} \oplus \dots \oplus \mathfrak{gl}_{l_d}$ of which we think as being

$$\ell = \left(\begin{array}{ccc} \mathfrak{gl}_{l_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathfrak{gl}_{l_d} \end{array} \right) \subset \mathfrak{gl}_n, \quad \text{generators in } \mathfrak{gl}_{l_i} : \begin{cases} L_{i,1} & E_{i,1} & & & \\ F_{i,1} & \ddots & \ddots & & \\ & \ddots & \ddots & & E_{i,l_i-1} \\ & & F_{i,l_i-1} & & L_{i,l_i} \end{cases}$$

where we reindex the elements L_i^\pm , E_i and F_i as indicated. (Note that all L_i^\pm appear in this reindexing, but not all E_i and F_i .)

Definition 5B.1. Let $U_v(\ell)$ be the \mathbb{Z}_v -subalgebra of $U_v(\mathfrak{gl}_n)$ generated by $L_{i,k}^\pm$, $E_{i,k}$ and $F_{i,k}$. We endow $U_v(\ell)$ with the structure of a Hopf algebra by restricting the one for $U_v(\mathfrak{gl}_n)$.

The following gives us a *block decomposition* and will be used without further reference:

Lemma 5B.2. *We have $U_v(\ell) \cong U_v(\mathfrak{gl}_{l_1}) \otimes \dots \otimes U_v(\mathfrak{gl}_{l_d})$ as \mathbb{Z}_v -algebras.*

Proof. By definition. \square

The representation theory of $U_q(\ell)$ is easy (knowing the representation theory for $U_q(\mathfrak{gl}_m)$), but nevertheless we state a few lemmas that we will use. For starters, note that all $U_q(\mathfrak{gl}_n)$ -representations restrict to $U_q(\ell)$ -representations. Note also that the vector representation $\mathbf{V}_q(\mathfrak{gl}_{l_i})$ of $U_q(\mathfrak{gl}_{l_i})$ is a $U_q(\ell)$ -module with action inflated to $U_q(\ell)$. The same holds for the exterior powers.

Lemma 5B.3. *As a $U_q(\ell)$ -module we have $\Lambda_q^k \mathbf{V}_q \cong \Lambda_q^k \bigoplus_{i=1}^d \mathbf{V}_q(\mathfrak{gl}_{l_i}) \cong \bigoplus_{k_1+\dots+k_d=k} \Lambda_q^{k_1} \mathbf{V}_q(\mathfrak{gl}_{l_1}) \otimes \dots \otimes \Lambda_q^{k_d} \mathbf{V}_q(\mathfrak{gl}_{l_d})$. We also have $\mathbf{V}_q^{\otimes k} \cong \bigoplus_{k_1+\dots+k_d=k} \mathbf{V}_q(\mathfrak{gl}_{l_1})^{\otimes k_1} \otimes \dots \otimes \mathbf{V}_q(\mathfrak{gl}_{l_d})^{\otimes k_d}$.*

Note that Lemma 5B.3 implies that \mathbf{V}_q is not simple as a $U_q(\ell)$ -representation.

Proof. The case $k = 1$ is clear by e.g. using the usual diagrammatic description of V_q (the actions of the $L_i^{\pm 1}$ are omitted in the following illustration):

$$\begin{array}{cccccccc} v_1 & \xrightarrow{F_{1,1}} & v_2 & \xrightarrow{F_{1,2}} & v_3 & \xrightarrow{F_3} & v_4 & \xrightarrow{F_{2,1}} & v_5 & \xrightarrow{F_{2,2}} & v_6 & \xrightarrow{F_6} & v_7 & \xrightarrow{F_{3,1}} & v_8 \\ & \xleftarrow{E_{1,1}} & & \xleftarrow{E_{1,2}} & & \xleftarrow{E_3} & & \xleftarrow{E_{2,1}} & & \xleftarrow{E_{2,2}} & & \xleftarrow{E_6} & & \xleftarrow{E_{3,1}} & \end{array} .$$

This illustrates the case $l_1 = 3$, $l_2 = 3$, $l_3 = 2$ and $n = 8$. The case of general k and the second isomorphism are similar and omitted. \square

We now reindex the basis of V_q to $\{v_{1,1}, \dots, v_{1,l_1}, \dots, v_{d,1}, \dots, v_{d,l_d}\}$, which then induces a reindexing of the basis of $\bigwedge_q^k V_q$ that we will use below.

Lemma 5B.4. *As a $U_q(\ell)$ -module we have $V_q^{\otimes k} \cong \bigwedge_q^k V_q \oplus W$ and no simple constituent of $\bigwedge_q^k V_q$ appears in W .*

Proof. By looking at highest weight vectors and classical theory, this follows directly from the first and the second decomposition in Lemma 5B.3. \square

Definition 5B.5. Let $\mathbf{Rep}_q \ell$ denote the category of finite dimensional $U_q(\ell)$ -representations of type 1. We view $\mathbf{Rep}_q \ell$ as pivotal using the above Hopf algebra structure on $U_q(\mathfrak{gl}_n)$. Let further $\mathbf{Fund}_q \ell$ denote the full pivotal subcategory with objects of the form $\bigwedge_q^K V_q$ for $K = (k_1, \dots, k_m) \in \mathbb{Z}^m$ and $m \in \mathbb{Z}_{\geq 0}$.

We write \circ_h^ℓ for the monoidal structure on $\mathbf{Rep}_q \ell$ and $\mathbf{Fund}_q \ell$ on the morphism level, and \circ_h on the object level. With contrast to (4B.1), a picture for the monoidal structure on $\mathbf{Fund}_q \ell$ is

$$(5B.6) \quad \begin{array}{c} \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\ \hline \text{f} \\ \hline \begin{array}{|c|} \hline \dots \\ \hline \end{array} \end{array} \circ_h^\ell \begin{array}{c} \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\ \hline \text{g} \\ \hline \begin{array}{|c|} \hline \dots \\ \hline \end{array} \end{array} = \begin{array}{c} \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\ \hline \text{f} \\ \hline \begin{array}{|c|} \hline \dots \\ \hline \end{array} \end{array} \begin{array}{c} \begin{array}{|c|} \hline \dots \\ \hline \end{array} \\ \hline \text{g} \\ \hline \begin{array}{|c|} \hline \dots \\ \hline \end{array} \end{array} .$$

Remark 5B.7. (5B.6) is a standard construction in the theory of Hopf algebras, see e.g. [TV17].

Lemma 5B.8. *We have the following.*

- The category $\mathbf{Rep}_q \ell$ is semisimple, and its simple objects are of the form $L(\lambda_1) \otimes \dots \otimes L(\lambda_d)$ with the factors being simple objects of $\mathbf{Rep}_q \mathfrak{gl}_{l_i}$.
- The additive idempotent completion of $\mathbf{Fund}_q \ell$ is pivotally equivalent to $\mathbf{Rep}_q \ell$.

Proof. Lemma 5A.5 applies componentwise. \square

We define the *Levi (1,1)-overcrossings* and *Levi (1,1)-undercrossing* to be

$$\begin{aligned} R_{1,1}^\ell : V_q \otimes V_q &\rightarrow V_q \otimes V_q, & v_{i,j} \otimes v_{k,l} &\mapsto \begin{cases} R_{1,1}(v_{i,j} \otimes v_{i,l}) & \text{if } i = k, \\ v_{k,l} \otimes v_{i,j} & \text{else,} \end{cases} \\ (R_{1,1}^\ell)^{-1} : V_q \otimes V_q &\rightarrow V_q \otimes V_q, & v_{i,j} \otimes v_{k,l} &\mapsto \begin{cases} (R_{1,1})^{-1}(v_{i,j} \otimes v_{i,l}) & \text{if } i = k, \\ v_{k,l} \otimes v_{i,j} & \text{else.} \end{cases} \end{aligned}$$

In words, $R_{1,1}^\ell$ is the respective \mathfrak{gl}_{l_i} braiding within one block, and the swap map otherwise, and similarly for its inverse.

Definition 5B.9. For $k, l \in \mathbb{Z}_{\geq 0}$ define $R_{k,l}^\ell$ as the composition

$$\bigwedge_q^k V_q \otimes \bigwedge_q^l V_q \xrightarrow{i} V_q^{\otimes(k+l)} \xrightarrow{x} V_q^{\otimes(k+l)} \xrightarrow{p} \bigwedge_q^l V_q \otimes \bigwedge_q^k V_q,$$

where i and p are inclusion and projection, respectively, and x is defined as in (3B.7) but with Levi crossings. Define $(R_{k,l}^\ell)^{-1}$ similarly.

We also get various mates of which we think as rotated versions of the ones in Definition 5B.9.

Lemma 5B.10. *The Levi crossings are $U_q(\ell)$ -equivariant, satisfy the Reidemeister II and III relations and various naturality relations.*

Proof. This follows from a calculation, [Lemma 5B.2](#) and [Lemma 5A.9](#). \square

Define now the following *coil maps*:

$$(5B.11) \quad \begin{aligned} r_{1,K}: \mathbb{V}_q \otimes \bigwedge_q^K \mathbb{V}_q &\rightarrow \bigwedge_q^K \mathbb{V}_q \otimes \mathbb{V}_q, & v_{i,j} \otimes w &\mapsto u_i R_{1,K}^\ell(v_{i,j} \otimes w), \\ r^{1,K}: \bigwedge_q^K \mathbb{V}_q \otimes \mathbb{V}_q &\rightarrow \mathbb{V}_q \otimes \bigwedge_q^K \mathbb{V}_q, & v \otimes w &\mapsto u_i^{-1} (R_{1,K}^\ell)^{-1}(w \otimes v_{i,j}). \end{aligned}$$

Note that $r_{1,K}$ and $r^{1,K}$ are inverses.

Lemma 5B.12. *The morphisms in (5A.6) and (5B.11) are $\circ_v\text{-}\circ_h^\ell$ -generators of \mathbf{Fund}_q^ℓ .*

Proof. The morphisms in (5A.6) are $U_q(\mathfrak{gl}_n)$ -equivariant, so they are also $U_q(\ell)$ -equivariant, and one easily checks that the morphisms in (5B.11) are $U_q(\ell)$ -equivariant. That these generate follows from [Lemma 5A.7](#) and [Lemma 5B.3](#). \square

Definition 5B.13. For $i \in \{1, \dots, d\}$ define

$$\text{pr}_i^\ell = \prod_{i \neq j} \frac{r_1 - u_i}{u_j - u_i} \in \text{End}_{\mathbf{Fund}_q^\ell}(\mathbb{V}_q),$$

which we call *Levi block projectors*.

Lemma 5B.14. *We have $\text{pr}_i^\ell \text{pr}_j^\ell = \delta_{i,j} \text{pr}_i^\ell$ and $\text{id}_{\mathbb{V}_q} = \sum_{i=1}^d \text{pr}_i^\ell$. These projectors realize the decomposition $\mathbb{V}_q \cong \bigoplus_{i=1}^d \mathbb{V}_q(\mathfrak{gl}_{l_i})$.*

Proof. Note that r_1 is given by multiplication by u_i on $v_{i,j}$. Thus, the formula for pr_i^ℓ is the usual Lagrange-type interpolation and the claims follow. \square

5C. Levi crossings. The Levi crossings are not $U_q(\mathfrak{gl}_n)$ -equivariant in general, and there is no planar web picture for it. However, it will be helpful to have the following diagrammatic notation. For the Levi (k, l) -overcrossings and the Levi (k, l) -undercrossings we use:

$$\text{over} : R_{k,l}^\ell \rightsquigarrow \begin{array}{c} l \quad k \\ \diagup \quad \diagdown \\ \text{red dot} \\ \diagdown \quad \diagup \\ k \quad l \end{array}, \quad \text{under} : (R_{k,l}^\ell)^{-1} \rightsquigarrow \begin{array}{c} l \quad k \\ \diagdown \quad \diagup \\ \text{red dot} \\ \diagup \quad \diagdown \\ k \quad l \end{array}.$$

We also use rotated pictures for their mates.

By [Lemma 5B.10](#), we have the *Reidemeister II and III relations*, e.g.

We can use this to define *Levi braids* associated to any braid word. Of particular importance will be the (positive) *Levi full twist* on k strands (denote by a box notation). By definition, this map is the square of the positive lift, using Levi overcrossings, of the longest word in the symmetric group on $\{1, \dots, k\}$. For example, for $k = 4$ this full twist is

$$\text{ft} = \left(\begin{array}{c} \text{diagram of 4 strands with 6 crossings} \\ \text{red dots on crossings} \end{array} \right)^2.$$

We also have various naturality relations such as

including the various $(-)^{\downarrow}$ and $(-)^{\leftrightarrow}$ -duals.

However, we need to be careful with the *Reidemeister I relation* as $\bigwedge_q^k \mathbf{V}_q$ needs not to be simple, see [Lemma 5B.3](#). Nevertheless, we still have:

Lemma 5C.1. *The Levi crossings are diagonal matrices in the basis given the decomposition of $\bigwedge_q^k \mathbf{V}_q$ from [Lemma 5B.3](#).*

Proof. Note that the decomposition of $\bigwedge_q^k \mathbf{V}_q$ from [Lemma 5B.3](#) is multiplicity free, and Schur's lemma applies. (Note that Schur's lemma in this setting does not need the underlying field to be algebraically closed, see e.g. [\[APW91, Corollary 7.4\]](#) or [\[AT17, Remark 2.29\]](#).) \square

One can check that the diagonal entries mentioned in [Lemma 5C.1](#) are given by products of the Reidemeister I scalars in [\(3B.6\)](#).

6. THE EQUIVALENCE

We now state and prove our main result.

6A. A reminder on the \mathfrak{gl}_n story. We first recall the relationship between \mathfrak{gl}_n webs and the representation theory of $U_q(\mathfrak{gl}_n)$. Define a functor

$$\Gamma_q = \Gamma_q^{ext}(\mathfrak{gl}_n): \mathbf{Web}_q \mathfrak{gl}_n \rightarrow \mathbf{Fund}_q \mathfrak{gl}_n$$

sending the object K to $\bigwedge_q^K \mathbf{V}_q$ and the generating morphisms of $\mathbf{Web}_q \mathfrak{gl}_n$ as follows:

$$\begin{array}{c} \begin{array}{ccc} k & & l \\ & \swarrow & \searrow \\ & k+l & \\ & \swarrow & \searrow \\ k & & l \end{array} \mapsto Y_{k+l}^{k,l}, \quad \begin{array}{ccc} & k+l & \\ & \uparrow & \\ k & & l \end{array} \mapsto \Lambda_{k,l}^{k+l}, \quad \begin{array}{ccc} & \curvearrowright & \\ k & & -k \end{array} \mapsto \cap_k^{\rightarrow}, \quad \begin{array}{ccc} & \curvearrowleft & \\ -k & & k \end{array} \mapsto \cap_k^{\leftarrow}, \quad \begin{array}{ccc} -k & & k \\ & \curvearrowright & \\ & k & \end{array} \mapsto \cup_k^{\rightarrow}, \quad \begin{array}{ccc} k & & -k \\ & \curvearrowleft & \\ & -k & \end{array} \mapsto \cup_k^{\leftarrow}. \end{array}$$

Theorem 6A.1. *The functor Γ_q is an equivalence of pivotal categories, and it induces an equivalence of pivotal categories between the additive idempotent completion of $\mathbf{Web}_q \mathfrak{gl}_n$ and $\mathbf{Rep}_q \mathfrak{gl}_n$.*

Proof. The fact that Γ_q is fully faithful is [\[CKM14, Theorem 3.3.1\]](#) and the fact that the relevant hom-spaces stay of the same dimension when restricting from \mathfrak{gl}_n to \mathfrak{sl}_n . The second claim follows from [Lemma 5A.5\(b\)](#), or [\[CKM14, Theorem 3.3.1\]](#) and flatness of restriction from \mathfrak{gl}_n to \mathfrak{sl}_n . \square

6B. The statement. We now extend the functor Γ_q into a functor

$$\mathbf{A}\Gamma_q = \mathbf{A}\Gamma_q^{ext}(\ell): \mathbf{AWeb}_q \mathfrak{gl}_n \rightarrow \mathbf{Fund}_q \ell.$$

On objects and the generators of $\mathbf{Web}_q \mathfrak{gl}_n$ the functor $\mathbf{A}\Gamma_q$ is defined to be Γ_q . We define $\mathbf{A}\Gamma_q$ on the two coils ρ_K and ρ^K by

$$\begin{array}{ccc} \begin{array}{ccc} k_2 & k_m & 1 \\ & \swarrow & \searrow \\ & \dots & \\ & \swarrow & \searrow \\ 1 & k_2 & k_m \end{array} \mapsto \Gamma_{1,K}, & \begin{array}{ccc} 1 & k_2 & k_m \\ & \swarrow & \searrow \\ & \dots & \\ & \swarrow & \searrow \\ k_2 & k_m & 1 \end{array} \mapsto \Gamma^{1,K}. \end{array}$$

Theorem 6B.1. *Let $\mathbf{AdId}(-)$ denote additive idempotent completion.*

(a) *We have the commuting diagram of categories*

$$\begin{array}{ccccc} \mathbf{AWeb}_q \mathfrak{gl}_n & \xrightarrow{(4C.3)} & \mathbf{AWeb}_q \ell & \xrightarrow{\mathbf{AdId}(-)} & \mathbf{AdId}(\mathbf{AWeb}_q \ell) \\ & \searrow \mathbf{A}\Gamma_q & \mathbf{A}\Gamma_q^\ell \cong & & \cong \downarrow \mathbf{AdId}(\mathbf{A}\Gamma_q^\ell) \\ & & \mathbf{Fund}_q \ell & \xrightarrow{\mathbf{AdId}(-)} & \mathbf{Rep}_q \ell. \end{array}$$

The Levi ideal \mathcal{I}_ℓ is the two-sided $\circ_v\text{-}\circ_h^\ell$ -ideal generated by the Levi evaluations. Here \circ_h^ℓ is the pullback of the monoidal structure from $\mathbf{Fund}_q \ell$ to $\mathbf{AWeb}_q \mathfrak{gl}_n$.

(b) We have the commuting diagram of pivotal categories

$$\begin{array}{ccccc}
\mathbf{AWeb}_1 \mathfrak{gl}_n & \xrightarrow{(4C.3)} & \mathbf{AWeb}_1 \ell & \xrightarrow{\text{AdId}(-)} & \text{AdId}(\mathbf{AWeb}_1 \ell) \\
& \searrow \mathbf{A}\Gamma_1 & \mathbf{A}\Gamma_1^\ell \cong & & \cong \downarrow \text{AdId}(\mathbf{A}\Gamma_1^\ell) \\
& & \mathbf{Fund}_1 \ell & \xrightarrow{\text{AdId}(-)} & \mathbf{Rep}_1 \ell.
\end{array}$$

The Levi ideal \mathcal{I}_ℓ is the two-sided $\circ_v\text{-}\circ_h^A$ -ideal generated by the Levi evaluations.

The proof of [Theorem 6B.1](#) is postponed to [Section 7](#), since we want to focus on applications of this theorem first. For the rest of the section we assume that [Theorem 6B.1](#) holds (with the exception of the next subsection where we only assume that $\mathbf{A}\Gamma_q$ is well-defined).

6C. Monoidal behavior of the main functor. The categories $\mathbf{AWeb}_q \mathfrak{gl}_n$ and $\mathbf{Rep}_q \ell$ are endowed with monoidal structures that are natural from two different perspectives, as explained in [Remark 4B.2](#). However, as we will elaborate now, these need not to be the same under the equivalence [Theorem 6B.1](#).

Lemma 6C.1. *Assume that the functor $\mathbf{A}\Gamma_q$ is well-defined.*

- (a) *The functor $\mathbf{A}\Gamma_1$ is pivotal.*
- (b) *The functor $\mathbf{A}\Gamma_q$ is not monoidal (and thus not pivotal).*

Proof. (a). Let us consider $q = 1$. In this case the braiding on $\mathbf{Rep}_1 \ell$ is given by permutation. Comparing [\(4B.1\)](#) and [\(5B.6\)](#), and observing that coils in $\mathbf{Rep}_1 \ell$ are permutations, up to diagonal entries of the form u_i on blocks, shows that the functor $\mathbf{A}\Gamma_1$ is monoidal. Pivotality is clear, and the final claim follows since $\mathbf{A}\Gamma_1$ is monoidal.

(b). For $q \neq 1$, one can check that the images under $\mathbf{A}\Gamma_q$ of [\(4B.1\)](#) is not the same as [\(5B.6\)](#). Explicitly, taking $K = L = (1)$, $f = \rho_K$ and $g = \text{id}_L$ verifies that $\mathbf{A}\Gamma_q$ is not monoidal. \square

Remark 6C.2. We note that [Lemma 6C.1](#) shows that the choice which side goes over or under in [\(4B.1\)](#) matters and gives different results on the representation theoretical side. This indicates that one might need to use the notion of module categories rather than monoidal categories to describe the representation theory associated to $\mathbf{AWeb}_q \mathfrak{gl}_n$. This is similar as in, for example, [\[ST19\]](#) or [\[ES18\]](#) (via [\[MS16, Remark 12\]](#)), so coideal subalgebras might play a role.

Remark 6C.3. One could use the equivalence in [Theorem 6B.1](#) to pullback the monoidal structure of $\mathbf{Rep}_q \ell$, resulting in a monoidal structure on $\mathbf{AWeb}_q \ell$ that is distinct from the one we give in [Lemma 4B.3](#) above. This pullback monoidal structure would not satisfy the conditions in [\[MS21, \(2.11\)\]](#), which are necessary for it to be unique, see [\[MS21, Proposition 2.5\]](#). Conversely, one could push the monoidal structure from $\mathbf{AWeb}_q \ell$ to $\mathbf{Rep}_q \ell$, resulting in a monoidal structure on $\mathbf{Rep}_q \ell$ satisfying [\[MS21, \(2.11\)\]](#).

6D. Ariki–Koike algebras and annular webs. The *Ariki–Koike algebra* $\mathcal{H}_q^{m,d}$ from [\[AK94\]](#), [\[BM93\]](#), [\[Che87\]](#), using different conventions, is:

Definition 6D.1. Fix $m \in \mathbb{Z}_{\geq 0}$, the number of strands, and let $\mathcal{H}_v^{m,d}$ denote the \mathbb{Z}_v -algebra with algebra generators T_0, T_1, \dots, T_{m-1} modulo the two-sided ideal generated by

$$\prod_{k=1}^d (T_0 - u_k), \quad T_0 T_1 T_0 T_1 - T_1 T_0 T_1 T_0,$$

$$(T_i - q)(T_i + q^{-1}) \text{ if } |i - j| = 0, \quad T_i T_j T_i - T_j T_i T_j \text{ if } |i - j| = 1, \quad T_i T_j - T_j T_i \text{ if } |i - j| > 1,$$

where $i, j \in \{1, \dots, m-1\}$.

The Ariki–Koike algebra acts on $1^{\circ_h m}$.

Proposition 6D.2. *We have a surjective \mathbb{K}_q -algebra homomorphism*

$$\pi_{m,d}: \mathcal{H}_q^{m,d} \rightarrow \text{End}_{\mathbf{AWeb}_q \mathfrak{gl}_n}(1^{\circ_h m}), \quad T_0 \mapsto \begin{array}{c} 1 \quad 1^{\circ_h(m-1)} \\ \uparrow \quad \uparrow \\ \text{---} \\ \downarrow \quad \downarrow \\ 1 \quad 1^{\circ_h(m-1)} \end{array}, \quad T_i \mapsto \begin{array}{c} 1 \quad 1 \\ \swarrow \quad \searrow \\ 1 \quad 1 \end{array}.$$

Proof. By [Theorem 6B.1](#), this follows from [[SS99](#), Theorem 4.2] after adjustment of conventions. \square

We define the usual *Jucys–Murphy elements*:

Definition 6D.3. Define elements of $\mathcal{H}_v^{m,d}$ by $X_1 = T_0$ and for $i \in \mathbb{Z}_{\geq 1}$ recursively $X_i = T_i X_{i-1} T_i$.

Let $P(m, d)$ denote the set of d -partitions of m (which we identify with d -tuples of Young diagrams in the English convention), and for $\lambda \in P(m, d)$ let $\text{Std}(\lambda)$ denote the set of all *standard d -tableaux of shape λ* .

Lemma 6D.4. *For all $\lambda \in P(m, d)$ there exists a simple $\mathcal{H}_q^{m,d}$ -module V_λ , and these form a complete and nonredundant set of simple $\mathcal{H}_q^{m,d}$ -representations. Moreover, we have an \mathbb{K}_q -algebra isomorphism $\mathcal{H}_q^{m,d} \cong \bigoplus_{\lambda \in P(m,d)} \text{End}_{\mathbb{K}_q}(V_\lambda)$. Finally, V_λ can be given a \mathbb{K}_q -basis $\{v_T \mid T \in \text{Std}(\lambda)\}$ such that X_i acts by*

$$(6D.5) \quad X_i \cdot v_T = u_a q^{2b-2c} v_T,$$

where a is the component of the entry i in T , b is the column number where i appears and c is the row number of i .

Proof. Our assumptions on the involved parameters imply that $\mathcal{H}_q^{m,d}$ is semisimple and the lemma follows from [[AK94](#), Theorem 3.7] and results in the same section, e.g. [[AK94](#), Proposition 3.16]. \square

Let $\mathcal{I}_{>l_1, \dots, >l_d} \subset \mathcal{H}_q^{m,d}$ denote the two-sided ideal generated by the idempotents, realizing the Artin–Wedderburn decomposition in [Lemma 6D.4](#), for d -partitions of m with strictly more than l_i rows in the i th entry. The following definition appears in [[LV21](#), Section 2C]:

Definition 6D.6. Let \mathcal{B}_q^ℓ be the algebra quotient of $\mathcal{H}_q^{m,d}$ by $\mathcal{I}_{>l_1, \dots, >l_d}$.

Proposition 6D.7. *The map $\pi_{m,d}$ from [Proposition 6D.2](#) induces a \mathbb{K}_q -algebra isomorphism*

$$\bar{\pi}_\ell: \mathcal{B}_q^\ell \xrightarrow{\cong} \text{End}_{\mathbf{AWeb}_q \mathfrak{gl}_\ell}(1^{\circ_h m}).$$

Thus, the kernel of $\pi_{m,d}$ is $\ker(\pi_{m,d}) = \mathcal{I}_{>l_1, \dots, >l_d}$.

Proof. As in the proof of [Proposition 6D.2](#). \square

6E. Cartan subalgebras and generalized blob algebras. We now consider the case of the Cartan subalgebra in detail.

Proposition 6E.1. *The functor $\mathbf{A}\Gamma_q$ descends to an equivalence of categories $\mathbf{A}\Gamma_q^{\mathfrak{h}}: \mathbf{AWeb}_q \mathfrak{h} \rightarrow \mathbf{Fund}_q \mathfrak{h}$, and it induces an equivalence of categories between the additive idempotent completion of $\mathbf{AWeb}_q \mathfrak{h}$ and $\mathbf{Rep}_q \mathfrak{h}$.*

Proof. Directly from [Theorem 6B.1](#) and [Lemma 6C.1](#). \square

Remark 6E.2. [Proposition 6E.1](#) should be compared with [[QW18](#), Corollary 43], and can be seen as a quantum version of that corollary.

Now recall the so-called *generalized blob algebra* $\mathcal{B}_q^{m,d}$ in the sense of [[MW03](#)] (which is a special case of [Definition 6D.6](#)):

Definition 6E.3. Let $\mathcal{B}_q^{m,d}$ be the algebra quotient of $\mathcal{H}_q^{m,d}$ by $\mathcal{I}_{>1, \dots, >1}$.

Proposition 6E.4. *The map $\pi_{m,d}$ from [Proposition 6D.2](#) induces a \mathbb{K}_q -algebra isomorphism*

$$\bar{\pi}_{m,d}: \mathcal{B}_q^{m,d} \xrightarrow{\cong} \text{End}_{\mathbf{AWeb}_q \mathfrak{h}}(1^{\circ_h m}).$$

Thus, the kernel of $\pi_{m,d}$ is $\ker(\pi_{m,d}) = \mathcal{I}_{>1, \dots, >1}$.

Proof. By [Proposition 6E.1](#), this follows from [\[ATY95, Theorem 3.1\]](#). \square

Recall from [Section 4C](#) that e_k denotes the k th elementary symmetric polynomial in d variables. In the special case of the Cartan subalgebra we have $d = n$, and we let $e_k = e_k(u_1, \dots, u_d)$.

Definition 6E.5. Define $e_k^{(1)} = e_k$ and for $i \in \mathbb{Z}_{\geq 1}$ recursively $e_k^{(i)} = e_k^{(i-1)} + (q^2 - 1)(X_{i-1}e_{k-1}^{(i-1)} - X_{i-1}^2e_{k-2}^{(i-1)} + \dots \pm X_{i-1}^k)$, and let $\mathcal{J}_{>1} \subset \mathcal{H}_q^{m,d}$ denote the two-sided ideal generated by $R_i = X_i^d - e_1^{(i)}X_i^{d-1} + \dots \pm e_d^{(i)}$. Let $\mathcal{J}_2 \subset \mathcal{H}_q^{m,d}$ denote the two-sided ideal generated by R_2 .

Proposition 6E.6. *The kernel of $\pi_{m,d}$ is alternatively given by $\ker(\pi_{m,d}) = \mathcal{J}_2 = \mathcal{J}_{>1}$. The same holds for $q = 1$.*

Proof. Claim 1. R_2 acts on V_λ as zero if and only if the d -partition λ has at most one row per component.

Proof of the Claim 1. The case where λ has one node is easy, so assume that λ has at least two nodes. We use that V_λ has a \mathbb{K}_q -basis given by v_T for T a standard d -tableaux of shape λ on which X_i acts by [\(6D.5\)](#). Using [\(6D.5\)](#), we can calculate the action of R_2 on the \mathbb{K}_q -basis given by the v_T . There are three cases depending on the positions of 1 and 2 in T one needs to check:

$$(i): (\dots, \boxed{1}, \dots, \boxed{2}, \dots) \text{ or vice versa, } (ii): (\dots, \boxed{1 \ 2}, \dots), (iii): (\dots, \boxed{\frac{1}{2}}, \dots).$$

All of these are annoying but straightforward calculations, and details are omitted.

Claim 2. $\mathcal{J}_2 = \mathcal{J}_{>1}$.

Proof of the Claim 2. To see this, we note that on $v_T \in V_\lambda$ for λ a one row d -partition and T a standard d -tableau of shape λ we have

$$e_k^{(i)} \cdot v_T = e_k(q^{2\alpha_1}u_1, \dots, q^{2\alpha_d}u_d)v_T,$$

where $\alpha_r = |\{s < i \mid s \text{ is in the } r\text{th component of } T\}|$. Using this formula and [\(6D.5\)](#) one can recursively check that the above claim holds for R_i for $i \geq 2$, which implies that $\mathcal{J}_{>1} \subset \mathcal{J}_2$, and the proof of the claim completes.

The first claim implies that $\mathcal{J}_2 = \mathcal{I}_2$, and this together with the second claim and [Proposition 6E.4](#) proves the lemma. \square

Remark 6E.7. By [Lemma 4B.6](#), [Proposition 6E.6](#) answers [\[CK18, Conjectures 10.2 and 10.3\]](#) affirmatively in the following sense. The category we consider in [Proposition 6E.6](#) is defined using \mathfrak{gl}_n webs, a quotient of $\mathbf{AWeb}_q\mathfrak{gl}_n$ by essential circles and has a field containing \mathbb{U} as its ground field. This setting is mildly different from [\[CK18\]](#) as they consider \mathfrak{sl}_n webs, take no quotient (these are both negligible differences, see e.g. [\[TVW17, Remark 1.1\]](#), discussing the first difference, and [\[CK18, Section 4.7\]](#) for the second difference) and work over a different ground ring.

6F. Working integrally. Note that our main result [Theorem 6B.1](#) is not stated or proven over \mathbb{Z}_v , and we work over \mathbb{K}_q in which case $\mathbf{AWeb}_q\mathfrak{gl}$ and $\mathbf{Rep}_q\mathfrak{gl}$ are semisimple. Working *integrally*, that is, over \mathbb{Z}_v or even $\mathbb{Z}[v, v^{-1}, \mathbb{U}]$ needs some nontrivial extra steps:

- (i) [Theorem 6A.1](#) works over \mathbb{Z}_v , see [\[Eli15, Theorem 2.58\]](#), which uses the light ladder strategy from that paper and [\[AST18, Theorem 3.1\]](#). Passing to an appropriate field (e.g. $\overline{\mathbb{F}_p}$ for a prime p) one gets an equivalence of pivotal categories between the additive idempotent completion of $\mathbf{Web}_q\mathfrak{gl}_n$ and $\mathbf{Tilt}_v\mathfrak{gl}_n$, the latter being the category of tilting modules.
- (ii) The relation from diagram categories to tilting modules is a folk observation in the field, see e.g. [\[Eli15, Theorem 2.58\]](#), [\[AST18, Section 5A\]](#), [\[TW21, Proposition 2.28\]](#) or [\[Bod20, Theorem 1.1\]](#) for some examples.
- (iii) Thus, it is tempting to conjecture that integral versions of [Theorem 6B.1](#) and its consequences involve $\mathbf{Tilt}_v\mathfrak{gl}$, e.g. under appropriate assumptions on the underlying field the additive idempotent completion of $\mathbf{AWeb}_v\mathfrak{gl}$ should be equivalent to $\mathbf{Tilt}_v\mathfrak{gl}$. However, there is a nontrivial catch: the quantization behaves not very well, see e.g. [Section 6C](#) or [\[CK18, Section 10\]](#). As sketched in [Remark 6C.2](#), this might indicate that quantum groups are not the correct objects to use in this setting.

- (iv) Note that the blob algebra is not defined integrally, and it is also not clear from the definition how to work integrally. Let us however point out that the description of $\mathcal{B}_q^{m,d}$ from [LV21, Theorem 2.15] works integrally and might play a role in the integral story.

We decided not to pursue this further in this work.

7. PROOF

7A. Well-definedness. Recall that the images of the coils are defined by explosion. We define $u: \mathbb{V}_q^{\otimes k} \rightarrow \mathbb{V}_q^{\otimes k}$ as the map sending $v_{i_1, j_1} \otimes \cdots \otimes v_{i_k, j_k}$ to $u_{i_1} \cdots u_{i_k} v_{i_1, j_1} \otimes \cdots \otimes v_{i_k, j_k}$.

Lemma 7A.1. *We have*

$$\mathbf{A}\Gamma_q(\rho_{(k,K)}) = [k]!^{-1} \cdot \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{ft} \\ \text{u} \\ \diagdown \quad \diagup \\ \text{---} \\ k \quad K \end{array}.$$

Proof. Using the Levi crossings introduced in Section 5C, we easily check that $\mathbf{A}\Gamma_q$ sends the coil $\rho_{(k,K)}$ to the claimed picture. \square

Lemma 7A.2. *In $\text{Rep}_q \ell$ we have*

$$[k]!^{-1} [l]!^{-1} \cdot \begin{array}{c} k \quad l \\ \diagup \quad \diagdown \\ \text{ft} \\ \diagdown \quad \diagup \\ k+l \end{array} = [k+l]!^{-1} \cdot \begin{array}{c} k \quad l \\ \diagup \quad \diagdown \\ \text{ft} \\ \diagdown \quad \diagup \\ k+l \end{array}.$$

Proof. We first suppose that the strands are oriented upward. The image of $Y_k^{1, \dots, 1} k+l$ is an $U_q(\ell)$ -submodule of $\mathbb{V}_q^{\otimes k+l}$. The full twist ft is $U_q(\ell)$ -invariant, so this submodule remains invariant. But this submodule is isomorphic to $\bigwedge_q^{k+l} \mathbb{V}_q$ and since the weight spaces of $\bigwedge_q^{k+l} \mathbb{V}_q$ are of dimension one, we deduce that the image of a vector v_S by $\text{ft} \circ Y_k^{1, \dots, 1} k+l$ is a multiple of $Y_k^{1, \dots, 1} k+l(v_S)$, say $\lambda_S Y_k^{1, \dots, 1} k+l(v_S)$. Hence, $Y_{k+l}^{k,l} \circ \Lambda_{1, \dots, 1}^k k+l \circ \text{ft} \circ Y_k^{1, \dots, 1} k+l$ sends v_S to $\lambda_S Y_{k+l}^{k,l} \circ \Lambda_{1, \dots, 1}^k k+l \circ Y_k^{1, \dots, 1} k+l(v_S) = \lambda_S [k+l]! Y_{k+l}^{k,l}(v_S)$ and $(\Lambda_{1, \dots, 1}^k \otimes \Lambda_{1, \dots, 1}^l) \circ \text{ft} \circ Y_k^{1, \dots, 1} k+l$ sends v_S to $\lambda_S (\Lambda_{1, \dots, 1}^k \otimes \Lambda_{1, \dots, 1}^l) \circ Y_k^{1, \dots, 1} k+l(v_S) = \lambda_S [k]! [l]! Y_{k+l}^{k,l}(v_S)$.

The same arguments shows that the equality also holds with strands oriented downward. \square

Lemma 7A.3. *The relation (4A.2) is satisfied after applying the functor $\mathbf{A}\Gamma_q$.*

Proof. Clear. \square

Lemma 7A.4. *The relation (4A.3) is satisfied after applying the functor $\mathbf{A}\Gamma_q$.*

Proof. Using the associativity of splits and merges and the fact that Levi crossings satisfy the Reidemeister III relation, it remains to prove

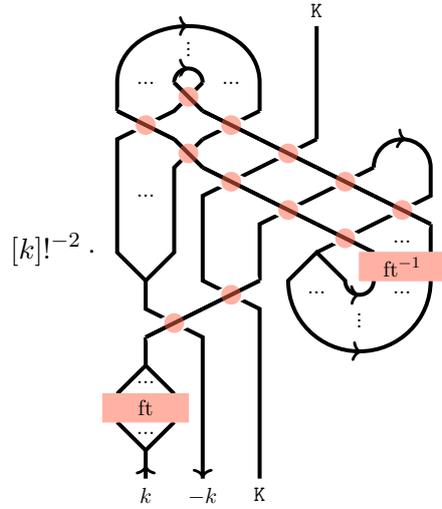
$$[k+l]!^{-1} \cdot \begin{array}{c} K \quad k \quad l \\ \diagup \quad \diagdown \\ \text{ft} \\ \diagdown \quad \diagup \\ k+l \quad K \end{array} = [k]!^{-1} [l]!^{-1} \cdot \begin{array}{c} K \quad k \quad l \\ \diagup \quad \diagdown \\ \text{ft} \\ \diagdown \quad \diagup \\ k+l \quad K \end{array},$$

which follows immediately from Lemma 7A.2. \square

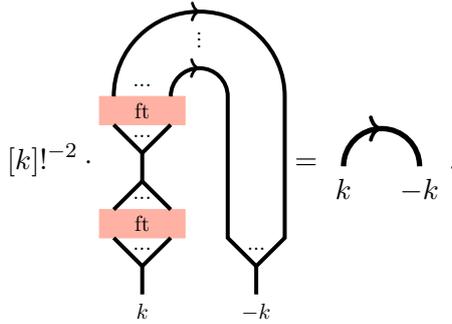
We can show similarly that merges slide through coils.

Lemma 7A.5. *The relation (4A.4) is satisfied after applying the functor $\mathbf{A}\Gamma_q$.*

Proof. The image of the right hand side of (4A.4) is given, up to the multiplication by elements of \mathbb{U} that cancel out, by



Using Reidemeister II relations and the fact that mates of merges are splits and vice versa, it remains to prove the following equality:

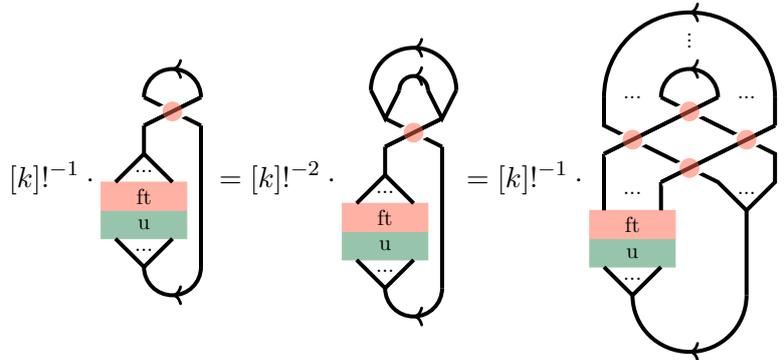


This equality follows from a repeated application of Lemma 7A.2 and from the digon removal relation. The argument for the leftward oriented cap is similar and omitted. \square

We can show similarly that cups slide through coils.

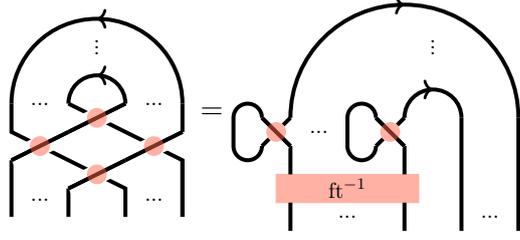
Lemma 7A.6. *The relations in (4C.3) are in the kernel of $\mathbf{A}\Gamma_q$.*

Proof. The image of the leftward oriented essential circle with a strand of thickness k through $\mathbf{A}\Gamma_q$ is given by

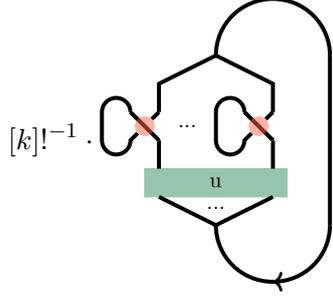


In this calculation the first equality is obtained from explosion of strands, and the last equality is obtained from Lemma 7A.2.

Now, since



it remains to compute the following scalar:



We easily check that the twist sends $v_{i,j}$ to $q^{-n+2(l_1+\dots+l_d)}v_{i,j}$ and therefore the previous scalar is equal to

$$\begin{aligned} & \sum_{k_1+\dots+k_d=k} \sum_{\substack{1 \leq j_{1,1} < \dots < j_{1,k_1} \leq l_1 \\ \dots \\ 1 \leq j_{d,1} < \dots < j_{d,k_d} \leq l_d}} u_1^{k_1} \dots u_d^{k_d} \prod_{i=1}^d q^{-n+2(l_1+\dots+l_d)k_i + \sum_{r=1}^{k_i} (n+1-2(l_1+\dots+l_{i-1}+j_{i,r}))} \\ &= \sum_{k_1+\dots+k_d=k} \sum_{\substack{1 \leq j_{1,1} < \dots < j_{1,k_1} \leq l_1 \\ \dots \\ 1 \leq j_{d,1} < \dots < j_{d,k_d} \leq l_d}} u_1^{k_1} \dots u_d^{k_d} \prod_{i=1}^d q^{k_i - \sum_{r=1}^{k_i} 2j_{i,r}} \\ &= e_k(q^{-1}u_1, q^{-3}u_1, \dots, q^{-2l_1+1}u_1, \dots, q^{-1}u_d, q^{-3}u_d, \dots, q^{-2l_d+1}u_d), \end{aligned}$$

which is what we needed to show. \square

Lemma 7A.7. *The functor $\mathbf{A}\Gamma_q$ is well-defined and descends to the functor $\mathbf{A}\Gamma_q^\ell$. Moreover, for $q \neq 1$ the Levi ideal \mathcal{I}_ℓ contains the two-sided \circ_v -ideal and right \circ_h^A -ideal generated by the Levi evaluations, while for $q = 1$ the Levi ideal \mathcal{I}_ℓ contains the two-sided \circ_v - \circ_h^A -ideal generated by the Levi evaluations.*

Proof. We need to check that the relations in [Definition 4A.1](#) are satisfied and that the Levi evaluations from [\(4C.3\)](#) are in the kernel of $\mathbf{A}\Gamma_q$. This follows as a combination of [Lemma 7A.5](#) and [Lemma 7A.6](#). By [Lemma 6C.1](#), the statement about the Levi ideal for $q = 1$ follows from that. To verify the claim for $q \neq 1$, we recall from [Remark 4A.5](#) that the coils pass in front of the annulus. Now we observe that e.g.

The diagram shows an equality between two web configurations. On the left, a crossing of two strands is shown with a red box labeled \circ_h^A placed over it. On the right, the same crossing is shown but with the strands separated. The strands are black with arrows.

More generally, multiplying essential circles from the right (but not from the left) by any morphism f produces a picture with f not interfering with the essential circles. \square

For the remainder we assume [Lemma 7A.7](#).

7B. Proof for $q = 1$. Recall the web block projectors from [Definition 4A.7](#) respectively the Levi block projectors from [Definition 5B.13](#). We use these to define a certain basis in the following definition, where S_k is the symmetric group on $\{1, \dots, k\}$:

Definition 7B.1. Fix $k \in \mathbb{Z}_{\geq 0}$. Let $s = (s_1, \dots, s_k)$, $t \in \{1, \dots, d\}^k$ and σ a colored permutation from s to t such that the longest word of $S_{l_i+1} \subset S_k$ does not appear in the permutation of color i . (This condition is vacuous for $l_i \geq k$ or if the color i does not appear strictly more than l_i times.) Define

$$\nu_{s,t,\sigma}^1 = \sigma \circ_v (\text{pr}_{s_1}^w \circ_h^A \dots \circ_h^A \text{pr}_{s_d}^w), \quad n_{s,t,\sigma}^1 = \sigma \circ_v (\text{pr}_{s_1}^\ell \circ_h^\ell \dots \circ_h^\ell \text{pr}_{s_d}^\ell),$$

where we view σ as an element of $\mathbf{AWeb}_1 \mathfrak{gl}_n$ and $\mathbf{Fund}_1 \ell$, respectively. The respective sets (collecting these elements for all $k \in \mathbb{Z}_{\geq 0}$) of these are denoted by $B(\mathbf{AWeb}_1 \ell)$ and $B(\mathbf{Fund}_1 \ell)$.

Lemma 7B.2. *We have $\mathbf{A}\Gamma_1(B(\mathbf{AWeb}_1 \ell)) = B(\mathbf{Fund}_1 \ell)$ and this set is a \mathbb{K}_1 -linear independent set in $\coprod_k \text{End}_{\mathbf{Fund}_1 \ell}(\mathbf{V}_1^{\otimes k})$.*

Proof. We get $\mathbf{A}\Gamma_1(B(\mathbf{AWeb}_1 \ell)) = B(\mathbf{Fund}_1 \ell)$ directly from the definition and Lemma 6C.1.

To prove faithfulness, let $\text{pr}_s^\ell = \text{pr}_{s_1}^w \circ_h^A \dots \circ_h^A \text{pr}_{s_d}^w$. By construction and Lemma 5B.14 we have $n_{s,t,\sigma}^1 \circ_v \text{pr}_s^\ell = \text{pr}_t^\ell \circ_v n_{s,t,\sigma}^1 = n_{s,t,\sigma}^1$ and $n_{s,t,\sigma}^1 \circ_v \text{pr}_q^\ell = \text{pr}_r^\ell \circ_v n_{s,t,\sigma}^1 = 0$ for $q \neq s$ and $r \neq t$. Thus, it suffices to show that $(n_{s,t,\sigma}^1)_\sigma$ is \mathbb{K}_1 -linear independent (that is, we can fix s and t). After sorting the colors of s and t , it remains to verify faithfulness within a block, i.e. for \mathfrak{gl}_{l_i} . Thus, classical theory applies: for \mathfrak{gl}_{l_i} and S_k it is known that Schur–Weyl duality gives the generalized Temperley–Lieb algebra as the endomorphism algebra. This algebra admits a description in terms of a quotient of S_k by the longest word of S_{l_i+1} , see e.g. [Här99, Section 3], and the elements of $B(\mathbf{Fund}_1 \ell)$ describe the associated standard-type basis within each block. \square

Lemma 7B.3. *The set $B(\mathbf{AWeb}_1 \ell)$ is a \mathbb{K}_1 -linear spanning set of $\coprod_k \text{End}_{\mathbf{AWeb}_1 \ell}(1^{\circ_h k})$. Moreover, the set $B(\mathbf{Fund}_1 \ell)$ is a \mathbb{K}_1 -linear spanning set of $\coprod_k \text{End}_{\mathbf{Fund}_1 \ell}(\mathbf{V}_1^{\otimes k})$.*

Proof. We first recall that the full antisymmetrizer is zero in $\mathbf{Fund}_1 \ell$, see e.g. [MS16, Lemma 11]. Thus, the same holds in $\mathbf{AWeb}_1 \ell$ by Lemma 7B.2. That is, in formulas we have

$$\left(\sum_{\sigma \in S_{l_i+1}} (-1)^{l(\sigma)} \sigma(\text{pr}_i^\ell)^{\circ_h^{(l_i+1)}} = 0 \right) \Rightarrow \left(\sum_{\sigma \in S_{l_i+1}} (-1)^{l(\sigma)} \sigma(\text{pr}_i^w)^{\circ_h^{(l_i+1)}} = 0 \right),$$

and both hold.

To address the first statement of the lemma, observe that $\coprod_k \text{End}_{\mathbf{AWeb}_1 \ell}(1^{\circ_h k})$ is generated as a \mathbb{K}_1 -algebra by crossings and coils since we can remove essential circles by Lemma 7A.7. Moreover, the sliding relations (4A.3) and (4A.4) imply that we have the usual \mathbb{K}_1 -linear spanning set given by first coils and then crossings. Observe next that the web block projectors \mathbb{K}_1 -linear span the subalgebra generated by coils, so it remains to see that the symmetric group part is \mathbb{K}_1 -linear spanned by σ such that the longest word of S_{l_i+1} does not appear in the permutation of color i . This however is a consequence of the vanishing of the antisymmetrizer.

For the second statement of the lemma we use Lemma 5B.3, Lemma 5B.14 and Schur’s lemma. (As explained in the proof of Lemma 5C.1, Schur’s lemma still holds in this setting although \mathbb{K}_1 is not necessary algebraically closed.) \square

Lemma 7B.4. *The set $B(\mathbf{AWeb}_1 \ell)$ is a \mathbb{K}_1 -basis of $\coprod_k \text{End}_{\mathbf{AWeb}_1 \ell}(1^{\circ_h k})$. Furthermore, the set $B(\mathbf{Fund}_1 \ell)$ is a \mathbb{K}_1 -basis of $\coprod_k \text{End}_{\mathbf{Fund}_1 \ell}(\mathbf{V}_1^{\otimes k})$.*

Proof. Combine Lemma 7B.2 and Lemma 7B.3. (Note that Lemma 7B.2 also proves that $B(\mathbf{AWeb}_1 \ell)$ is a \mathbb{K}_1 -linear independent set.) \square

Proposition 7B.5. *The functor $\mathbf{A}\Gamma_1^\ell$ is fully faithful and the Levi ideal \mathcal{I}_ℓ is the two-sided $\circ_v \circ_h^A$ -ideal generated by the Levi evaluations.*

Proof. By Lemma 3A.9 we only need to show that $\coprod_k \text{End}_{\mathbf{AWeb}_1 \ell}(1^{\circ_h k})$ and $\coprod_k \text{End}_{\mathbf{Fund}_1 \ell}(\mathbf{V}_1^{\otimes k})$ are isomorphic \mathbb{K}_1 -vector spaces with $\mathbf{A}\Gamma_1^\ell$ inducing an isomorphism, and this follows from Lemma 7B.2 and Lemma 7B.4. The proof completes. \square

Proof of Theorem 6B.1.(b). Lemma 7A.7 shows that the functors $\mathbf{A}\Gamma_1$ and $\mathbf{A}\Gamma_1^\ell$ are well-defined and Lemma 6C.1 shows the statements involving the pivotal structure. Fully faithfulness follows from Proposition 7B.5, and Lemma 5B.8 ensures that we have that $\mathbf{AdId}(\mathbf{Fund}_1 \ell)$ is equivalent to $\mathbf{Rep}_1 \ell$. These statements taken together complete the proof using the usual properties of the additive idempotent completion. \square

7C. **Proof for $q \neq 1$.** We start by:

Lemma 7C.1. *The functor $\mathbf{A}\Gamma_q$ is full.*

Proof. It is clear that the image of the crossings and the coils span. (Note that there is no issue with essential circles in $\mathbf{Fund}_q\ell$.) \square

For $q \neq 1$, we note that we can mimic [Definition 7B.1](#) on the Levi side (the only difference is that we use a positive lift, in Levi crossings, of σ instead of σ itself) to define $n_{s,t,\sigma}^q$ as well as $B(\mathbf{Fund}_q\ell)$. Now we use that and [Lemma 7C.1](#) to define $\nu_{s,t,\sigma}^q$ as well as $B(\mathbf{AWeb}_q\ell)$ by pulling back the elements from $n_{s,t,\sigma}^q$ by choosing a preimage.

Lemma 7C.2. *The set $B(\mathbf{AWeb}_q\ell)$ is a \mathbb{K}_q -linear independent set of $\coprod_k \text{End}_{\mathbf{AWeb}_q\ell}(1^{\circ h^k})$. Moreover, the set $B(\mathbf{Fund}_q\ell)$ is a \mathbb{K}_q -linear independent set of $\coprod_k \text{End}_{\mathbf{Fund}_q\ell}(\mathbf{V}_q^{\otimes k})$.*

Proof. The claim on the Levi side can be proven verbatim as in [Lemma 7B.2](#), so our focus is on the web side. However, by construction, the set $B(\mathbf{AWeb}_q\ell)$ is then sent to $B(\mathbf{Fund}_q\ell)$, so $B(\mathbf{AWeb}_q\ell)$ is \mathbb{K}_q -linear independent because $B(\mathbf{Fund}_q\ell)$ is. \square

Lemma 7C.3. *The set $B(\mathbf{AWeb}_q\ell)$ is a \mathbb{K}_q -linear spanning set of $\coprod_k \text{End}_{\mathbf{AWeb}_q\ell}(1^{\circ h^k})$. Moreover, the set $B(\mathbf{Fund}_q\ell)$ is a \mathbb{K}_q -linear spanning set of $\coprod_k \text{End}_{\mathbf{Fund}_q\ell}(\mathbf{V}_q^{\otimes k})$.*

Proof. We can remove essential circles labeled by 1 by using

$$\begin{array}{ccc} \begin{array}{c} 1 \quad 1 \\ \swarrow \quad \nearrow \\ 1 \quad 1 \end{array} & - & \begin{array}{c} 1 \quad 1 \\ \nearrow \quad \swarrow \\ 1 \quad 1 \end{array} = (q - q^{-1}) \cdot \begin{array}{c} 1 \quad 1 \\ \uparrow \quad \uparrow \\ 1 \quad 1 \end{array}, \end{array}$$

which is the classical Skein relation (that holds in our setting by using [\(3B.4\)](#)), and [Lemma 7A.7](#) which removes essential circles labeled by 1 that are in the front of the cylinder. Using explosion, the rest of the argument is the same as in the proof of [Lemma 7B.3](#) by using that the $q \neq 1$ basis agrees with $B(\mathbf{AWeb}_1\ell)$ on the associated graded by filtration by number of crossings (using the Skein relations). \square

Lemma 7C.4. *The set $B(\mathbf{AWeb}_q\ell)$ is a \mathbb{K}_q -basis of $\coprod_k \text{End}_{\mathbf{AWeb}_q\ell}(1^{\circ h^k})$. Furthermore, the set $B(\mathbf{Fund}_q\ell)$ is a \mathbb{K}_q -basis of $\coprod_k \text{End}_{\mathbf{Fund}_q\ell}(\mathbf{V}_q^{\otimes k})$.*

Proof. By [Lemma 7C.2](#) and [Lemma 7C.3](#). \square

Remark 7C.5. We do not have or need any explicitly description of the elements of $B(\mathbf{AWeb}_q\ell)$ in terms of webs.

Proposition 7C.6. *The functor $\mathbf{A}\Gamma_q^\ell$ is fully faithful and the Levi ideal \mathcal{I}_ℓ is the two-sided $\circ_v - \circ_h^\ell$ -ideal generated by the Levi evaluations. Here \circ_h^ℓ is the pullback of the monoidal structure from $\mathbf{Fund}_q\ell$ to $\mathbf{AWeb}_q\mathfrak{gl}_n$.*

Proof. As in the proof of [Proposition 7B.5](#). \square

Proof of [Theorem 6B.1\(a\)](#). Using the above statements, this can be proven verbatim as [Theorem 6B.1\(b\)](#). \square

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