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# SCHUR–WEYL DUALITY, VERMA MODULES, AND ROW QUOTIENTS OF ARIKI–KOIKE ALGEBRAS

ABEL LACABANNE AND PEDRO VAZ

ABSTRACT. We prove a Schur–Weyl duality between the quantum enveloping algebra of  $\mathfrak{gl}_m$  and certain quotient algebras of Ariki–Koike algebras, which we describe explicitly. This duality involves several algebraically independent parameters and the module underlying it is a tensor product of a parabolic universal Verma module and a tensor power of the standard representation of  $\mathfrak{gl}_m$ . We also give a new presentation by generators and relations of the generalized blob algebras of Martin and Woodcock as well as an interpretation in terms of Schur–Weyl duality by showing they occur as a special case of our algebras.

## 1. INTRODUCTION

Schur–Weyl duality is a celebrated theorem connecting the finite-dimensional modules over the general linear and the symmetric groups. It states that, over a field  $\mathbb{k}$  that is algebraically closed, the actions of  $GL_m(\mathbb{k})$  and  $\mathfrak{S}_n$  on  $V = (\mathbb{k}^m)^{\otimes n}$  commute and form double centralizers. Several variants of (quantum) Schur–Weyl duality are known, see for example [4, 6, 5, 9, 16, 24] for such variants related to our paper. One particular family of generalizations of interest for us uses a module akin to the one appearing in Schur–Weyl duality, but with an infinite-dimensional module instead of  $V$ . For example, in [15] it is established a Schur–Weyl duality between  $\mathcal{U}_q(\mathfrak{sl}_2)$  and the blob algebra of Martin and Saleur [19] with the underlying module being a tensor product of a projective Verma module with several copies of the standard representation of  $\mathcal{U}_q(\mathfrak{sl}_2)$ . We should warn the reader that in [15] the blob algebra was called the Temperley–Lieb algebra of type  $B$  (see [18] for further explanations).

**1.1. In this paper.** We consider the tensor product of a parabolic universal Verma module with the  $m$ -folded tensor product of the standard representation for  $\mathcal{U}_q(\mathfrak{gl}_m)$  to establish a Schur–Weyl duality with a quotient of Ariki–Koike algebras. Ariki–Koike algebras were first considered by Cherednik in [10] as a cyclotomic quotient of the affine Hecke algebra of type  $A$ . These algebras were later rediscovered and studied by Ariki and Koike [3] from a representation theoretic point of view. Independently, Broué and Malle attached in [7] a Hecke algebra to certain complex reflection groups, and Ariki–Koike algebras turn out to be the Hecke algebras associated to the complex reflection groups  $G(d, 1, n)$ .

Recall that the *Ariki–Koike algebra*  $\mathcal{H}(d, n)$  with parameters  $q \in \mathbb{k}^*$  and  $\underline{u} = (u_1, \dots, u_d) \in \mathbb{k}^d$  is the  $\mathbb{k}$ -algebra with generators  $T_0, T_1, \dots, T_{n-1}$ , where  $T_1, \dots, T_{n-1}$  generate a finite-dimensional Hecke algebra of type  $A$  and  $T_0$  satisfies  $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$ ,  $T_0 T_i = T_i T_0$  for

$i > 1$ , and  $\prod_{i=1}^d (T_0 - u_i) = 0$ . We consider the semisimple case, where the simple modules  $V_\mu$  of  $\mathcal{H}(d, n)$  are indexed by  $d$ -partitions of  $n$ .

Let  $\underline{m} = (m_1, \dots, m_d)$  be a  $d$ -tuple of positive integers and  $\mathcal{P}_{\underline{m}}^n$  be the set of all  $d$ -partitions  $\mu = (\mu^{(1)}, \dots, \mu^{(d)})$  of  $n$  such that  $l(\mu^{(i)}) \leq m_i$  for all  $1 \leq i \leq d$ .

In this paper we introduce the *row-quotient* algebra  $\mathcal{H}_{\underline{m}}(d, n)$ , that depends on  $\underline{m}$  as the quotient of  $\mathcal{H}(d, n)$  by the kernel of the surjection

$$\mathcal{H}(d, n) \twoheadrightarrow \prod_{\mu \in \mathcal{P}_{\underline{m}}^n} \text{End}_{\mathbb{k}}(V_\mu).$$

Let  $M^{\mathfrak{p}}(\Lambda)$  be a parabolic Verma module and  $V$  the standard representation for  $\mathcal{U}_q(\mathfrak{gl}_m)$ . In our conventions,  $\mathfrak{p}$  is standard and has Levi factor  $\mathfrak{l} = \mathfrak{gl}_{m_1} \times \dots \times \mathfrak{gl}_{m_d}$ , with  $m_i \geq 1$  and  $m_1 + m_2 + \dots + m_d = m$  and  $\Lambda$  depends on  $d$  algebraically independent parameters  $\lambda_1, \dots, \lambda_d$  (see Section 3.2 for more details). Thanks to the braided structure on the category of integrable modules over  $\mathcal{U}_q(\mathfrak{gl}_m)$ , we define a left action of  $\mathcal{H}(d, n)$  on  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  in Section 4. Our main result is:

**Theorem A** (Theorem 4.2 and Lemma 4.1).

- The action of  $\mathcal{U}_q(\mathfrak{gl}_m)$  and of  $\mathcal{H}(d, n)$  on  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  commute with each other, which endow  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  with a structure of  $\mathcal{H}(d, n) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$ -module.
- The algebra morphism  $\mathcal{H}(d, n) \rightarrow \text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$  is surjective and factors through an isomorphism

$$\mathcal{H}_{\underline{m}}(d, n) \xrightarrow{\cong} \text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}). \quad (1)$$

- There is an isomorphism of  $\mathcal{H}(d, n) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$ -modules

$$M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n} \simeq \bigoplus_{\mu \in \mathcal{P}_{\underline{m}}^n} V_\mu \otimes M^{\mathfrak{p}}(\Lambda, \mu),$$

where  $M^{\mathfrak{p}}(\Lambda, \mu)$  is a simple module (see 3.2).

The isomorphism in Equation (1) has several particular specializations (Corollaries 4.3–4.7), some of them recovering well-known algebras:

- If  $\mathfrak{p} = \mathfrak{gl}_m$  and  $m \geq n$ , then  $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$  is isomorphic to the Hecke algebra of type  $A$ .
- If  $\mathfrak{p} = \mathfrak{gl}_m$  and  $m = 2$ , then  $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$  is isomorphic to the Temperley–Lieb algebra of type  $A$ .
- For  $\mathfrak{p}$  such that  $m \geq nd$  and  $m_i \geq n$  for all  $1 \leq i \leq d$ , then  $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$  is isomorphic to the Ariki–Koike algebra  $\mathcal{H}(d, n)$ .
- If  $\mathfrak{p}$  is such that  $d = 2$  and  $m_1, m_2 \geq n$ , then  $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$  is isomorphic to the Hecke algebra of type  $B$  with unequal and algebraically independent parameters (see [14, Example 5.2.2, (c)]).
- If the parabolic subalgebra  $\mathfrak{p}$  coincides with the standard Borel subalgebra of  $\mathcal{U}_q(\mathfrak{gl}_m)$  then  $\text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$  is isomorphic to Martin–Woodcock’s [20] generalized blob algebra  $\mathcal{B}(d, n)$ . This generalizes the case of  $\mathcal{U}_q(\mathfrak{sl}_2)$  covered in [15].

In the last case, this gives a new interpretation of the generalized blob algebras  $\mathcal{B}(d, n)$  in terms of Schur–Weyl duality. We also give a new presentation of  $\mathcal{B}(d, n)$  as a quotient of Ariki–Koike algebras:

**Theorem B** (Theorem 2.15). Suppose that  $\mathcal{H}(d, n)$  is semisimple and that for every  $i, j, k$  we have  $(1 + q^{-2})u_k \neq u_i + u_j$ . The generalized blob algebra  $\mathcal{B}(d, n)$  is isomorphic to the quotient of  $\mathcal{H}(d, n)$  by the two-sided ideal generated by the element

$$\tau = \prod_{1 \leq i < j \leq d} \left[ (T_1 - q) \left( T_0 - q \frac{u_i + u_j}{q + q^{-1}} \right) (T_1 - q) \right].$$

**1.2. Connection to other works.** The idea of writing this note originated when we started thinking of possible extensions of our work in [18] to more general Kac–Moody algebras and were not able to find the appropriate generalizations of [15] in the literature. When we were finishing writing this note Peng Shan informed us about [23], whose results are far beyond the ambitions of this article. Nevertheless, we expect our results to be connected to [23, §8] using a braided equivalence of categories between a category of modules for the quantum group  $\mathcal{U}_q(\mathfrak{gl}_m)$  and a category of modules over the affine Lie algebra  $\widehat{\mathfrak{gl}}_m$ , which is due to Kazhdan and Lusztig [17]. However, the explicit description of the endomorphism algebra of  $M^p(\Lambda) \otimes V^{\otimes n}$ , which was our first motivation towards categorification later on, does not seem to appear anywhere in [23] except in the particular case of our Corollary 4.5.

Another motivation for the results presented here resides in the potential applications to low-dimensional topology, as indicated in [22]. We find that it would be also interesting to investigate the use of several Verma modules in a tensor product as suggested in [11].

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## 2. ARIKI–KOIKE ALGEBRAS, ROW QUOTIENTS AND GENERALIZED BLOB ALGEBRAS

We recall the definition of Ariki–Koike algebras and define some quotients which will appear as endomorphism algebras of modules over a quantum group. As a particular case we recover the generalized blob algebras of Martin and Woodcock [20] and we obtain a presentation of these blob algebras that seems to be new.

**2.1. Reminders on Ariki–Koike algebras.** Fix once and for all a field  $\mathbb{k}$  and two positive integers  $d$  and  $n$  and choose elements  $q \in \mathbb{k}^*$  and  $u_1, \dots, u_d \in \mathbb{k}$ . We recall the definition of the Ariki–Koike algebra introduced in [3], which we view as a quotient of the group algebra of the Artin–Tits braid group of type  $B$ .

**Definition 2.1.** The *Ariki–Koike algebra*  $\mathcal{H}(d, n)$  with parameters  $q \in \mathbb{k}^*$  and  $\underline{u} = (u_1, \dots, u_d) \in \mathbb{k}^d$  is the  $\mathbb{k}$ -algebra with generators  $T_0, T_1, \dots, T_{n-1}$ , the relation

$$(T_i - q)(T_i + q^{-1}) = 0,$$

the cyclotomic relation

$$\prod_{i=1}^d (T_0 - u_i) = 0,$$

and the braid relations

$$T_i T_j = T_j T_i \text{ if } |i - j| > 1, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } 1 \leq i \leq n - 2, \\ T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0.$$

**Remark 2.2.** We use different conventions than [3]. In order to recover their definition, one should replace  $q$  by  $q^2$ ,  $T_0$  by  $a_1$ , and  $qT_{i-1}$  by  $a_i$ .

As in the type  $A$  Hecke algebra, for any  $w \in \mathfrak{S}_n$  we can define unambiguously  $T_w$  by choosing any reduced expression of  $w$ .

It is shown in [3] that the algebra  $\mathcal{H}(d, n)$  is of dimension  $d^n n!$  and a basis is given in terms of Jucys–Murphy elements, which are recursively defined by  $X_1 = T_0$  and  $X_{i+1} = T_i X_i T_i$ .

**Theorem 2.3** ([3, Theorem 3.10, Theorem 3.20]). *A basis of  $\mathcal{H}(d, n)$  is given by the set*

$$\{X_1^{r_1} \dots X_d^{r_d} T_w \mid 0 \leq r_i < d, w \in \mathfrak{S}_n\}.$$

*Moreover, the center of  $\mathcal{H}(d, n)$  is generated by the symmetric polynomials in  $X_1, \dots, X_d$ .*

We end this section with a semisimplicity criterion due to Ariki [2], which in our conventions takes the following form.

**Theorem 2.4** ([2, Main Theorem]). *The algebra  $\mathcal{H}(d, n)$  is semisimple if and only if*

$$\left( \prod_{\substack{-n < l < n \\ 1 \leq i < j \leq d}} (q^{2l} u_i - u_j) \right) \left( \prod_{1 \leq i \leq n} (1 + q^2 + q^4 + \dots + q^{2(i-1)}) \right) \neq 0.$$

**2.2. Modules over Ariki–Koike algebras.** In this section, we suppose that the algebra  $\mathcal{H}(d, n)$  is semisimple. In [3], Ariki and Koike gave a construction of the simple  $\mathcal{H}(d, n)$ -modules, using the combinatorics of multipartitions.

**2.2.1.  $d$ -partitions and the Young lattice.** A partition  $\mu$  of  $n$  of length  $l(\mu) = k$  is a non-increasing sequence  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k > 0$  of integers summing to  $|\mu| = n$ . A  $d$ -partition of  $n$  is a  $d$ -tuple of partitions  $\mu = (\mu^{(1)}, \dots, \mu^{(d)})$  such that  $\sum_{i=1}^d |\mu^{(i)}| = n$ . Given a  $d$ -partition  $\mu$  its Young diagram is the set

$$[\mu] = \{(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \{1, \dots, d\} \mid 1 \leq a \leq l(\mu), 1 \leq b \leq \mu_a^{(c)}\},$$

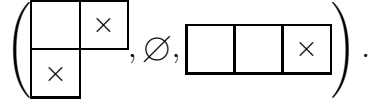
whose elements are called boxes. We usually represents a Young diagram as a  $d$ -tuple of sequences of left-aligned boxes, with  $\mu_a^{(c)}$  boxes in the  $a$ -th row of the  $c$ -th component.

**Example 2.5.** The Young diagram of the 3-partition  $((2, 1), \emptyset, (3))$  of 6 is

$$\left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right).$$

A box  $\gamma$  of  $[\mu]$  is said to be *removable* if  $[\mu] \setminus \{\gamma\}$  is the Young diagram of a  $d$ -partition  $\nu$ , and in this case the box  $\gamma$  is said to be *addable* to  $\nu$ .

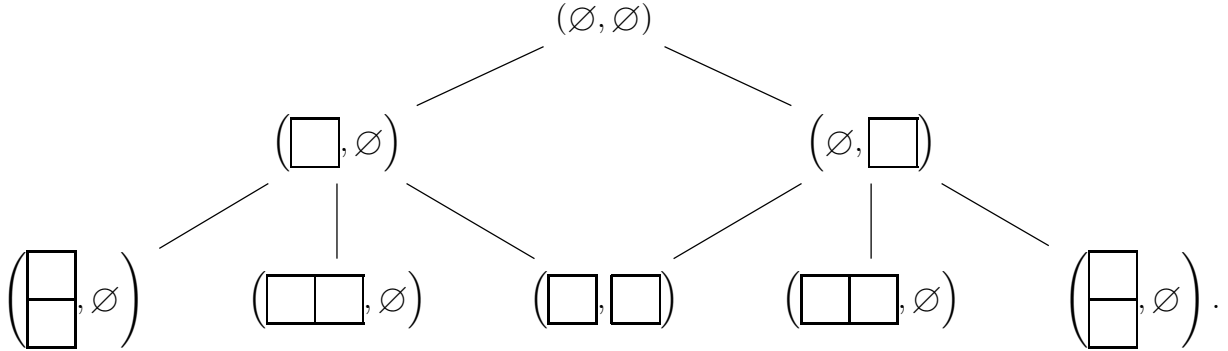
**Example 2.6.** The removable boxes of the 3-partition  $((2, 1), \emptyset, (3))$  below are depicted with a cross



With respect to the above the definitions, we will also use the evident notions of adding a box to a Young diagram or removing a box from a Young diagram.

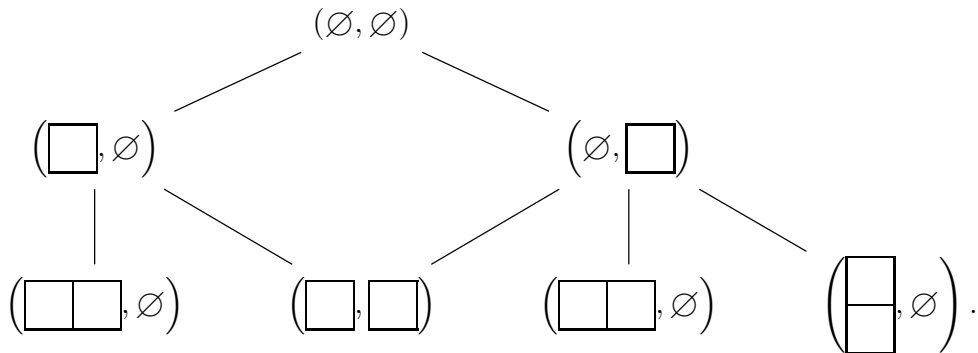
We consider the Young lattice for  $d$ -partitions and some sublattices. It is a graph with vertices consisting of  $d$ -partitions of any integers, and there is an edge between two  $d$ -partitions if and only if one can be obtained from the other by adding or removing a box.

**Example 2.7.** The beginning of the Young lattice for 2-partitions is the following:



If we fix  $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ , we then define  $\mathcal{P}_{\underline{m}}^n$  as the set of  $d$ -partitions  $\mu$  such that  $l(\mu^{(i)}) \leq m_i$ . We will also consider the corresponding sublattice of the Young lattice.

**Example 2.8.** For  $m_1 = 1$  and  $m_2 = 2$ , the beginning of the Young lattice for 2-partitions  $\mu$  with  $l(\mu^{(1)}) \leq 1$  and  $l(\mu^{(2)}) \leq 2$  is the following:



We end this subsection with the notion of a standard tableau of shape  $\mu$  where  $\mu$  is a  $d$ -partition of  $n$ . Such a standard tableau is a bijection  $t: [\mu] \rightarrow \{1, \dots, n\}$  such that for all

boxes  $\gamma = (a, b, c)$  and  $\gamma' = (a', b', c)$  we have  $\mathfrak{t}(\gamma) < \mathfrak{t}(\gamma')$  if  $a = a'$  and  $b < b'$  or  $a < a'$  and  $b = b'$ . Giving a standard tableau of shape  $\mu$  is equivalent to giving a path in the Young lattice from the empty  $d$ -partition to the  $d$ -partition  $\mu$ .

**Example 2.9.** The standard tableau

$$\left( \begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline \end{array}, \emptyset, \begin{array}{|c|c|} \hline 2 & 3 \\ \hline \end{array} \right)$$

of shape  $((1, 1), \emptyset, (2))$  correspond to the path

$$(\emptyset, \emptyset, \emptyset) \longrightarrow (\square, \emptyset, \emptyset) \longrightarrow (\square, \emptyset, \square) \longrightarrow (\square, \emptyset, \square\square) \longrightarrow \left( \begin{array}{|c|} \hline \square \\ \hline \end{array}, \emptyset, \square\square \right).$$

**2.2.2. Constructing the simple modules.** We present the construction of simple modules of the Ariki–Koike algebra following [3, Section 3]. This construction is similar to the classical construction of simple modules of the symmetric group, the Hecke algebra of type  $A$  or of the complex reflection group  $G(d, 1, n)$ . This construction describes explicitly the action of the Ariki–Koike algebra on a vector space. For  $\mu = (\mu^{(1)}, \dots, \mu^{(d)})$  a  $d$ -multipartition of  $n$ , we set

$$V_\mu = \bigoplus_{\mathfrak{t}} \mathbb{k}v_{\mathfrak{t}},$$

where the sum is over all the standard tableaux of shape  $\mu$ . Ariki and Koike gave an explicit action of the generators on the basis of  $V_\mu$  given by the standard tableaux. The action of  $T_0$  is diagonal with respect to this basis:

$$T_0 v_{\mathfrak{t}} = u_c v_{\mathfrak{t}},$$

where  $c$  is such that  $\mathfrak{t}(1, 1, c) = 1$ . The action of  $T_i$  is more involved and depends on the relative positions of the numbers  $i$  and  $i + 1$  in the tableau  $\mathfrak{t}$ :

- (1) if  $i$  and  $i + 1$  are in the same row of the standard tableau  $\mathfrak{t}$ , then  $T_i v_{\mathfrak{t}} = qv_{\mathfrak{t}}$ ,
- (2) if  $i$  and  $i + 1$  are in the same column of the standard tableau  $\mathfrak{t}$ , then  $T_i v_{\mathfrak{t}} = -q^{-1}v_{\mathfrak{t}}$ ,
- (3) if  $i$  and  $i + 1$  neither appear in the same row nor the same column of the standard tableau  $\mathfrak{t}$ , then  $T_i$  will act on the two dimensional subspace generated by  $v_{\mathfrak{t}}$  and  $v_{\mathfrak{s}}$ , where  $\mathfrak{s}$  is the standard tableau obtained from  $\mathfrak{t}$  by permuting the entries  $i$  and  $i + 1$ . The explicit matrix is given in [3] and we will not need it.

**Proposition 2.10** ([3, Theorem 3.7]). *If  $\mu$  is any  $d$ -multipartition of  $n$ , the space  $V_\mu$  is a well-defined  $\mathcal{H}(d, n)$ -module and it is absolutely simple. A set of isomorphism classes of simple  $\mathcal{H}(d, n)$ -modules is moreover given by  $\{V_\mu\}_\mu$ , for  $\mu$  running over the set of  $d$ -partitions of  $n$ .*

The action of the Jucys–Murphy elements is also diagonal in the basis of standard tableaux:

$$X_i v_{\mathfrak{t}} = u_c q^{2(b-a)} v_{\mathfrak{t}}, \quad (2)$$

where  $\mathbf{t}(a, b, c) = i$ . A useful consequence of Proposition 2.10 is the following: if  $V$  is a simple  $\mathcal{H}(d, n)$ -module and  $v \in V$  is a common eigenvector for  $X_1, \dots, X_d$  with eigenvalues as in (2) for some standard tableau  $\mathbf{t}$  of shape  $\mu$ , then  $V$  is isomorphic to  $V_\mu$ .

From the explicit description of the modules  $V_\mu$ , using the standard inclusion  $\mathcal{H}(n, d) \hookrightarrow \mathcal{H}(n+1, d)$ , it is easy to see that for any  $d$ -partition of  $n+1$  we have

$$\operatorname{Res}_{\mathcal{H}(n, d)}^{\mathcal{H}(n+1, d)}(V_\mu) \simeq \bigoplus_{\nu} V_\nu,$$

where the sum is over all  $d$ -partition  $\nu$  of  $n$  whose Young diagram is obtained by deleting one removable box from the Young diagram of  $\mu$ . The branching rule of the inclusions  $\mathcal{H}(1, d) \subset \mathcal{H}(2, d) \subset \dots \subset \mathcal{H}(n, d)$  is therefore governed by the Young lattice of  $d$ -partitions.

**2.3. Row quotients of  $\mathcal{H}(d, n)$  and generalized blob algebras.** We now define the row quotients of  $\mathcal{H}(d, n)$  which will appear later as endomorphism algebras of a tensor product of modules for  $\mathcal{U}_q(\mathfrak{gl}_m)$ .

**Definition 2.11.** Let  $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$  and recall that the algebra  $\mathcal{H}(d, n)$  is assumed to be semisimple, which implies that  $\mathcal{H}(d, n) \simeq \prod_{\mu} \operatorname{End}_{\mathbb{k}}(V_\mu)$ , the product being over all  $d$ -partitions of  $n$ . Recall also that  $\mathcal{P}_{\underline{m}}^n$  is the set of  $d$ -partitions of  $n$  with  $i$ -th component of length at most  $m_i$ .

The  $\underline{m}$ -row quotient of  $\mathcal{H}(d, n)$ , denoted  $\mathcal{H}_{\underline{m}}(d, n)$ , is the quotient of  $\mathcal{H}(d, n)$  by the kernel of the surjection

$$\mathcal{H}(d, n) \twoheadrightarrow \prod_{\mu \in \mathcal{P}_{\underline{m}}^n} \operatorname{End}_{\mathbb{k}}(V_\mu).$$

**Remark 2.12.** If  $m_i \geq n$  for all  $1 \leq i \leq d$  then  $\mathcal{H}_{\underline{m}}(d, n) \simeq \mathcal{H}(d, n)$ .

Similar to the case of  $\mathcal{H}(d, n)$ , we have inclusions  $\mathcal{H}_{\underline{m}}(1, d) \subset \mathcal{H}_{\underline{m}}(2, d) \subset \dots \subset \mathcal{H}_{\underline{m}}(n, d)$  and the branching rule is governed by the corresponding truncation of the Young lattice of  $d$ -partitions.

**2.3.1. Generalized blob algebras.** In the particular case where  $m_i = 1$  for all  $1 \leq i \leq d$ , we recover the definition of the generalized blob algebras [20, Equation (14)], which we denote by  $\mathcal{B}(d, n)$ . Under a mild hypothesis on the parameters, we give a presentation of  $\mathcal{B}(d, n)$ .

We consider the following element of  $\mathcal{H}(d, n)$ :

$$\tau = \prod_{1 \leq i < j \leq d} \left[ (T_1 - q) \left( T_0 - q \frac{u_i + u_j}{q + q^{-1}} \right) (T_1 - q) \right].$$

This element may look cumbersome, but can be better understood thanks to the following lemma:

**Lemma 2.13.** *The two-sided ideal of  $\mathcal{H}(d, n)$  generated by  $\tau$  is equal to the two-sided ideal generated by*

$$(T_1 - q) \prod_{1 \leq i < j \leq d} (X_1 + X_2 - (u_i + u_j)).$$



*Proof.* A simple computation in  $\mathcal{H}(d, n)$  shows that

$$(T_1 - q) \left( T_0 - q \frac{u_i + u_j}{q + q^{-1}} \right) (T_1 - q) = q(X_1 + X_2 - (u_i + u_j)) (T_1 - q).$$

We therefore conclude using the fact that  $(T_1 - q)^2 = -(q + q^{-1})(T_1 - q)$  and that  $T_1$  commutes with  $X_1 + X_2$ .  $\square$

We now investigate which  $\mathcal{H}(d, n)$ -modules  $V_\mu$  factor through the quotient by the two-sided ideal generated by  $\tau$ .

**Proposition 2.14.** *The element  $\tau$  acts by zero on  $V_\mu$  if and only if  $l(\mu^{(k)}) \leq 1$  for every  $k$  such that  $(1 + q^{-2})u_k \neq u_i + u_j$  for all  $i, j$ .*

*Proof.* Suppose that  $\mu$  and  $k$  are such that  $l(\mu^{(k)}) \geq 2$  with  $(1 + q^{-2})u_k \neq u_i + u_j$  for all  $i, j$ . Then there exist a tableau  $\mathfrak{t}$  of shape  $\mu$  such that 1 and 2 are in the first two columns of the  $k$ -th component of the Young diagram of  $\mu$ . By definition of  $V_\mu$ , the generator  $T_1$  acts on  $v_{\mathfrak{t}}$  by multiplication by  $-q^{-1}$ . The Jucys–Murphy element  $X_1$  acts on  $v_{\mathfrak{t}}$  by multiplication by  $u_k$  whereas the Jucys–Murphy element  $X_2$  acts on  $v_{\mathfrak{t}}$  by multiplication by  $q^{-2}u_k$ . Therefore, thanks to Lemma 2.13,  $\tau$  does not act by zero on  $V_\mu$ .

It remains to check that  $\tau$  acts by zero on  $V_\mu$  with  $l(\mu^{(k)}) \leq 1$  whenever  $(1 + q^{-2})u_k \neq u_i + u_j$  for all  $i, j$ . Let  $\mathfrak{t}$  be a standard tableau of shape  $\mu$ . If 1 and 2 are in the same component of the tableau  $\mathfrak{t}$ , then either 1 and 2 are in the same row and  $T_1$  acts on  $v_{\mathfrak{t}}$  by multiplication by  $q$ , either 1 and 2 are in the same column and  $X_1 + X_2$  acts on  $\mathfrak{t}$  by multiplication by  $(1 + q^{-2})u_k$ . The second case is possible only if there exists  $i, j$  such that  $(1 + q^{-2})u_k = u_i + u_j$  and then  $\tau$  acts by zero. If 1 and 2 are in two different Young diagrams and  $X_1 + X_2$  acts on  $\mathfrak{t}$  by  $u_k + u_l$ , where  $k$  (resp.  $l$ ) is such that  $\mathfrak{t}(1, 1, k) = 1$  (resp.  $\mathfrak{t}(1, 1, l) = 2$ ). In both cases,  $\tau$  acts by zero.  $\square$

**Theorem 2.15.** *Suppose that  $\mathcal{H}(d, n)$  is semisimple and that for every  $i, j, k$  we have  $(1 + q^{-2})u_k \neq u_i + u_j$ . The generalized blob algebra  $\mathcal{B}(d, n)$  is isomorphic to the quotient of  $\mathcal{H}(d, n)$  by the two-sided ideal generated by  $\tau$ .*

*Proof.* Recall that we supposed that  $m_1 = \dots = m_d = 1$ . Thanks to Proposition 2.14, the element  $\tau$  is in the kernel of the surjection

$$\mathcal{H}(d, n) \twoheadrightarrow \prod_{\mu \in \mathcal{P}_{\underline{m}}^n} \text{End}_{\mathbb{k}}(V_\mu).$$

Therefore, we have a surjection  $\mathcal{H}(d, n)/\mathcal{H}(d, n)\tau\mathcal{H}(d, n) \twoheadrightarrow \mathcal{B}(d, n)$ . Once again, thanks to Proposition 2.14, the simple modules of  $\mathcal{H}(d, n)/\mathcal{H}(d, n)\tau\mathcal{H}(d, n)$  are exactly the  $V_\mu$  with  $\mu \in \mathcal{P}_{\underline{m}}^n$  which shows that the above surjection is an isomorphism.  $\square$

### 3. QUANTUM $\mathfrak{gl}_m$ , PARABOLIC VERMA MODULES AND TENSOR PRODUCTS

We recall the definition of the quantum enveloping algebra of  $\mathfrak{gl}_m$ , and we also recall some basic properties of its modules, *e.g.* concerning parabolic Verma modules.

**3.1. The quantum enveloping algebra of  $\mathfrak{gl}_m$ .** Let  $q$  be an indeterminate. The following definition of  $\mathcal{U}_q(\mathfrak{gl}_m)$  is over the field  $\mathbb{Q}(q)$ , but, via scalar extension, we will also consider it over a field containing  $\mathbb{Q}(q)$  without further notice.

**Definition 3.1.** The quantum enveloping algebra  $\mathcal{U}_q(\mathfrak{gl}_m)$  is the  $\mathbb{Q}(q)$ -algebra with generators  $L_i^{\pm 1}, E_j$  and  $F_j$ , for  $1 \leq i \leq m$  and  $1 \leq j \leq m-1$  with the following relations:

$$\begin{aligned} L_i^{\pm 1} L_i^{\mp 1} &= 1, & L_i L_j &= L_j L_i, \\ L_i E_j &= q^{\delta_{i,j} - \delta_{i,j+1}} E_j L_i, & L_i F_j &= q^{-\delta_{i,j} + \delta_{i,j+1}} F_j L_i, \end{aligned}$$

$$[E_i, F_j] = \delta_{i,j} \frac{L_i L_{i+1}^{-1} - L_i^{-1} L_{i+1}}{q - q^{-1}},$$

and the quantum Serre relations

$$\begin{aligned} E_i E_j &= E_j E_i \text{ if } |i - j| > 1, & E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 &= 0, \\ F_i F_j &= F_j F_i \text{ if } |i - j| > 1, & F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 &= 0. \end{aligned}$$

We endow it with a structure of a Hopf algebra, with comultiplication  $\Delta$ , counit  $\varepsilon$  and antipode  $S$  given on generators by the following:

$$\begin{aligned} \Delta(L_i) &= L_i \otimes L_i, & \varepsilon(L_i) &= 1, & S(L_i) &= L_i^{-1}, \\ \Delta(E_i) &= E_i \otimes 1 + L_i L_{i+1}^{-1} \otimes E_i, & \varepsilon(E_i) &= 0, & S(E_i) &= -L_i^{-1} L_{i+1} E_i, \\ \Delta(F_i) &= F_i \otimes L_i^{-1} L_{i+1} + 1 \otimes F_i, & \varepsilon(F_i) &= 0, & S(F_i) &= -F_i L_i L_{i+1}^{-1}. \end{aligned}$$

Set  $\mathcal{U}_q(\mathfrak{gl}_m)^0$  as the subalgebra generated by  $(L_i)_{1 \leq i \leq m}$ , and  $\mathcal{U}_q(\mathfrak{gl}_m)^{\geq 0}$  as the subalgebra generated by  $(L_i, E_j)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m-1}}$ .

We denote by  $P = \bigoplus_{i=1}^m \mathbb{Z} \varepsilon_i$  the weight lattice of  $\mathfrak{gl}_m$  with  $\mathbb{Z}$ -basis given by the fundamental weights  $(\varpi_i)_{1 \leq i \leq m}$  where  $\varpi_i = \varepsilon_1 + \dots + \varepsilon_i$ . We denote by  $Q$  the root lattice with  $\mathbb{Z}$ -basis given by the simple roots  $(\alpha_i)_{1 \leq i \leq m-1}$  where  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ . Denote by  $\Phi^+$  the set of positive roots, by  $P^+$  the set of dominant weights for  $\mathfrak{gl}_m$ , that is  $\mu = \sum_{i=1}^m \mu_i \varepsilon_i$  with  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ . We also endow  $P$  with the standard non-degenerate bilinear form:  $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{i,j}$ . The symmetric group  $\mathfrak{S}_m$  acts on  $P$  by permuting the coordinates and leaves the bilinear form  $\langle \cdot, \cdot \rangle$  invariant. Finally, let  $\rho$  be the half-sum of the positive roots.

We will often work with extensions  $\mathbb{Z}[\beta_1, \dots, \beta_k] \otimes_{\mathbb{Z}} P$ , where the  $\beta_i$ 's are indeterminates and we also extend the bilinear form  $\langle \cdot, \cdot \rangle$  to  $\mathbb{Z}[\beta_1, \dots, \beta_k] \otimes_{\mathbb{Z}} P$ .

**3.2. Weights and parabolic Verma modules.** Suppose that our field  $\mathbb{k}$  contains the field  $\mathbb{Q}(q)$  and let  $M$  be an  $\mathcal{U}_q(\mathfrak{gl}_m)$ -module over the ground field  $\mathbb{k}$ . An element  $v \in M$  is said to be a weight vector if  $L_i v = \varphi(\varepsilon_i) v$ , where  $\varphi: P \rightarrow \mathbb{k}$  is the corresponding weight. The module  $M$  is said to be a weight module if the action of the elements  $L_1, \dots, L_m$  is simultaneously diagonalizable. A highest weight module is a weight module  $M$  such that  $M = \mathcal{U}_q(\mathfrak{gl}_m) v$ , where  $v$  is a weight vector such that  $E_i v = 0$  for  $1 \leq i \leq m-1$ .

It is well-known that finite dimensional weight  $\mathcal{U}_q(\mathfrak{gl}_m)$ -modules of type 1 are parameterized by the set  $P^+$  of dominant weights.

In this paper, we will be interested in modules over the field  $\mathbb{Q}(q, \lambda_1, \dots, \lambda_k)$ , where  $\lambda_i = q^{\beta_i}$  is an indeterminate (recall that  $q$  is formal and so  $q^{\beta_i}$  is also formal). Moreover, we only consider type 1 modules, where the weights are of the form

$$\varphi(\nu) = q^{\langle \mu, \nu \rangle},$$

for some  $\mu \in \mathbb{Z}[\beta_1, \dots, \beta_k] \otimes_{\mathbb{Z}} P$  and for all  $\nu \in P$ .

We now turn to parabolic Verma modules. Let  $\mathfrak{p}$  be a standard parabolic subalgebra of  $\mathfrak{gl}_m$  with Levi factor  $\mathfrak{l} = \mathfrak{gl}_{m_1} \times \dots \times \mathfrak{gl}_{m_d}$ , where  $m_i \geq 1$  and  $\sum_{i=1}^d m_i = m$ . Denote by  $I$  the set  $\{\tilde{m}_i \mid 1 \leq i \leq d-1\}$ , where  $\tilde{m}_i = m_1 + \dots + m_i$ , so that  $\mathcal{U}_q(\mathfrak{l})$  is generated by  $L_i, E_j$  and  $F_j$  for  $1 \leq i \leq m$  and  $j \notin I$  and  $\mathcal{U}_q(\mathfrak{p})$  is generated by  $L_i, E_j$  and  $F_k$  for  $1 \leq i \leq m$ ,  $1 \leq j \leq m-1$  and  $k \notin I$ . Denote by  $P_i^+$  the set of dominant weights for  $\mathfrak{gl}_{m_i}$ . We identify the set  $P_1^+ \times \dots \times P_d^+$  with the dominant weights  $P_{\mathfrak{l}}^+$  of  $\mathfrak{l}$  by the following map

$$(\mu^{(1)}, \dots, \mu^{(d)}) \rightarrow \sum_{i=1}^d \left( \sum_{j=1}^{m_i} \mu_j^{(i)} \varepsilon_{\tilde{m}_{i-1}+j} \right).$$

For a dominant weight  $\mu \in P_{\mathfrak{l}}^+$ , we have a simple integrable finite dimensional  $\mathcal{U}_q(\mathfrak{l})$ -module  $V^{\mathfrak{l}}(\Lambda, \mu)$  of highest weight

$$\Lambda_{\mu} = \sum_{i=1}^d \left( \sum_{j=1}^{m_i} (\beta_i + \mu_j^{(i)}) \varepsilon_{\tilde{m}_{i-1}+j} \right).$$

Indeed, one can check that  $\langle \Lambda_{\mu}, \alpha_i \rangle \in \mathbb{N}$  for any  $i \notin I$ . We turn this  $\mathcal{U}_q(\mathfrak{l})$ -module into a  $\mathcal{U}_q(\mathfrak{p})$ -module by setting  $E_i V^{\mathfrak{l}}(\Lambda, \mu) = 0$  for all  $i \in I$ . Then the parabolic Verma module  $M^{\mathfrak{p}}(\Lambda, \mu)$  is

$$M^{\mathfrak{p}}(\Lambda, \mu) = \mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{p})} V^{\mathfrak{l}}(\Lambda, \mu).$$

It is a highest weight module of highest weight  $\Lambda_{\mu}$ . If  $\mu = 0$ , then we will simply denote this module by  $M^{\mathfrak{p}}(\Lambda)$  and its highest weight by  $\Lambda$ .

**Lemma 3.2.** *For any  $\mu \in P_{\mathfrak{l}}^+$ , the parabolic Verma module  $M^{\mathfrak{p}}(\Lambda, \mu)$  is simple.*

*Proof.* Since for any  $i \in I$  the scalar product  $\langle \Lambda_{\mu}, \alpha_i \rangle$  is not an integer, as one easily checks, the claim follows.  $\square$

**Remark 3.3.** If the parabolic subalgebra  $\mathfrak{p}$  is the Borel subalgebra  $\mathfrak{b}$  of upper triangular matrices, we have  $\mathcal{U}_q(\mathfrak{p}) = \mathcal{U}_q(\mathfrak{gl}_m)^{\geq 0}$  and the parabolic Verma module  $M^{\mathfrak{b}}(\Lambda)$  is the universal Verma module. The adjective universal means that any parabolic Verma module can be obtained from  $M^{\mathfrak{b}}(\Lambda)$  by specialization of the parameters.

In the rest of this article, all dominant weights  $\mu \in P_{\mathfrak{l}}^+$  will satisfy  $\mu_{m_i}^{(i)} \geq 0$  for all  $1 \leq i \leq d$ , and it will be convenient to identify such a weight  $\mu$  with the corresponding  $d$ -partition in  $\mathcal{P}_{\underline{m}}^n$ . We will use the same notation  $\mu$  to denote the  $d$ -partition or the corresponding dominant weight.

We also denote by  $V$  the standard representation of  $\mathfrak{gl}_m$  of dimension  $m$ . Explicitly, this is a highest weight module with highest weight  $\varepsilon_1$ , it has as a basis  $v_1, \dots, v_m$  and the action of  $\mathcal{U}_q(\mathfrak{gl}_m)$  is given by

$$L_i \cdot v_j = q^{\delta_{i,j}} v_j, \quad E_i \cdot v_j = \delta_{i+1,j} v_{j-1} \quad \text{and} \quad F_i \cdot v_j = \delta_{i,j} v_{j+1}.$$

**3.3. Tensor products and branching rule.** As  $\mathcal{U}_q(\mathfrak{gl}_m)$  is a Hopf algebra, its category of modules can be endowed with a tensor product. Explicitly, given  $M$  and  $N$  two modules over a ground ring  $R$ , the action of the generators on  $M \otimes_R N$  is given using the comultiplication: for all  $v \in M$  and  $w \in N$ , one have

$$\begin{aligned} L_i \cdot (v \otimes w) &= L_i \cdot v \otimes L_i \cdot w, & E_i \cdot (v \otimes w) &= E_i \cdot v \otimes w + L_i L_{i+1}^{-1} \cdot v \otimes E_i \cdot w \\ & & \text{and} & & F_i \cdot (v \otimes w) &= F_i \cdot v \otimes L_i^{-1} L_{i+1}^{-1} \cdot w + v \otimes F_i \cdot w. \end{aligned}$$

We will write  $\otimes$  instead of  $\otimes_R$  to simplify the notations. Since we will be interested in the endomorphism algebra of  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ , we start by understanding the decomposition of this module.

**Proposition 3.4.** *For any  $\mu \in \mathcal{P}_1^n$ , there is an isomorphism of  $\mathcal{U}_q(\mathfrak{gl}_m)$ -modules*

$$M^{\mathfrak{p}}(\Lambda, \mu) \otimes V \simeq \bigoplus_{\nu \in \mathcal{P}_1^{n+1}} M^{\mathfrak{p}}(\Lambda, \nu),$$

where the sum is over all  $\nu \in \mathcal{P}_1^{n+1}$  whose Young diagram is obtained from the Young diagram of  $\mu$  by adding one addable box.

*Proof.* We start by showing that  $M^{\mathfrak{p}}(\Lambda, \mu) \otimes V$  has a filtration given by the  $M^{\mathfrak{p}}(\Lambda, \nu)$  as in the statement. First, we have the following tensor identity:

$$(\mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{p})} V^{\mathfrak{l}}(\Lambda, \mu)) \otimes V \simeq \mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{p})} (V^{\mathfrak{l}}(\Lambda, \mu) \otimes V).$$

Noticing that  $L \mapsto \mathcal{U}_q(\mathfrak{gl}_m) \otimes_{\mathcal{U}_q(\mathfrak{p})} L$  is an exact functor from the category of finite dimensional  $\mathcal{U}_q(\mathfrak{p})$ -modules to the category of  $\mathcal{U}_q(\mathfrak{gl}_m)$ -modules, it remains to show that

$$V^{\mathfrak{l}}(\Lambda, \mu) \otimes V \simeq \bigoplus_{\nu \in \mathcal{P}_1^{n+1}} V^{\mathfrak{l}}(\Lambda, \nu),$$

where the sum is over all  $\nu \in \mathcal{P}_1^{n+1}$  whose Young diagram is obtained from the Young diagram of  $\mu$  by adding one addable box. This follows from the usual branching rule for  $\mathcal{U}_q(\mathfrak{gl}_m)$ -modules.

To show that the sum is direct, we use arguments from the infinite-dimensional representation theory of Lie algebras. We consider the usual category  $\mathcal{O}$  for  $\mathcal{U}_q(\mathfrak{gl}_m)$  [21, Chapter 4]. We then show that each  $M^{\mathfrak{p}}(\Lambda, \nu)$  lie in a different block of the category  $\mathcal{O}$ , which then implies that the sum is direct.

First, as  $M^{\mathfrak{p}}(\Lambda, \nu)$  is a quotient of the universal Verma module  $M^{\mathfrak{b}}(\Lambda_\nu)$ , these two modules share the same central character. Therefore  $M^{\mathfrak{p}}(\Lambda, \nu)$  and  $M^{\mathfrak{p}}(\Lambda, \nu')$  are in the same block if and only if the central characters afforded by  $M^{\mathfrak{b}}(\Lambda_\nu)$  and  $M^{\mathfrak{b}}(\Lambda_{\nu'})$  are the same. But these central characters are equal if and only if  $\Lambda_\nu$  and  $\Lambda_{\nu'}$  are in the same orbit for

the dot action of the symmetric group, which is the usual action of the symmetric group shifted by the sum of simple roots  $\rho$ .

We obtain that  $M^{\mathbb{P}}(\Lambda, \nu)$  and  $M^{\mathbb{P}}(\Lambda, \nu')$  are in the same block if and only if there exists  $w \in \mathfrak{S}_m$  such that

$$w \cdot \Lambda_\nu = \Lambda_{\nu'}.$$

Now, suppose that  $M^{\mathbb{P}}(\Lambda, \nu)$  and  $M^{\mathbb{P}}(\Lambda, \nu')$  are in the same block. Since the dot action satisfies  $w \cdot (\eta + \gamma) = w \cdot \eta + w(\gamma)$ , we deduce that  $w(\Lambda) = \Lambda$  so that  $w$  lies in  $\mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_d}$ . Then, writing  $w = (w_1, \dots, w_d)$ , we find that  $w_i \cdot \nu^{(i)} = \nu'^{(i)}$  for every  $1 \leq i \leq d$ . Since both  $\nu^{(i)}$  and  $\nu'^{(i)}$  are dominant weights, we deduce that  $\nu^{(i)} = \nu'^{(i)}$  for every  $1 \leq i \leq d$ . Indeed, each orbit for the dot action contains a unique dominant weight.

Hence if  $\nu \neq \nu'$ , the parabolic Verma modules  $M^{\mathbb{P}}(\Lambda, \nu)$  and  $M^{\mathbb{P}}(\Lambda, \nu')$  are in different blocks of the category  $\mathfrak{O}$ .  $\square$

Using the previous proposition and induction, one shows the following corollary.

**Corollary 3.5.** *There is an isomorphism*

$$M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n} \simeq \bigoplus_{\mu \in \mathcal{P}_1^n} M(\Lambda, \mu)^{n_\mu},$$

where  $n_\mu$  is the number of paths from the empty  $d$ -partition to  $\mu$  in the Young lattice of  $d$ -multipartitions.

**3.4. Braiding and an action of the Artin-Tits group of type  $B$ .** The quantized enveloping algebra (or rather a completion of the tensor product with itself) contains an element, called the quasi- $R$ -matrix, which is a crucial tool in defining a braiding on a subcategory of the  $\mathcal{U}_q(\mathfrak{gl}_m)$ -modules. Since there are several possible braidings, we make our choice explicit and refer to [8, 10.1.D] for more details.

In a completion of  $\mathcal{U}_q(\mathfrak{gl}_m) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$ , we define an element  $\Theta$  by

$$\Theta = \prod_{\alpha \in \Phi^+} \left( \sum_{n=0}^{+\infty} q^{\frac{n(n-1)}{2}} \frac{(q - q^{-1})^n}{[n]!} E_\alpha^n \otimes F_\alpha^n \right),$$

where  $[n]! = \prod_{i=1}^n \frac{q^i - q^{-i}}{q - q^{-1}}$  and  $E_\alpha, F_\alpha$  being the root vectors associated to a positive root  $\alpha$ . If  $M$  and  $N$  are two  $\mathcal{U}_q(\mathfrak{gl}_m)$  type 1 weight modules over the ground ring  $\mathbb{Q}(q, \lambda_1, \dots, \lambda_{d-1})$  where  $\mathcal{U}_q(\mathfrak{gl}_m)^{>0}$  act locally nilpotently,  $\Theta$  induces an isomorphism of vector spaces  $\Theta_{M,N}: M \otimes N \rightarrow M \otimes N$ . We then define a morphism of  $\mathcal{U}_q(\mathfrak{gl}_m)$ -modules

$$c_{M,N}: M \otimes N \rightarrow N \otimes M,$$

by

$$c_{M,N} = \tau \circ f \circ \Theta_{M,N},$$

where  $\tau$  is the flip  $v \otimes w \mapsto w \otimes v$  and  $f$  is the map  $v \otimes w \mapsto q^{\langle \mu, \nu \rangle} v \otimes w$  if  $v$  and  $w$  are of respective weights  $\mu$  and  $\nu$ . This endows the category of type 1 weight modules on which  $\mathcal{U}_q(\mathfrak{gl}_m)^{>0}$  acts locally nilpotently with a braiding. In particular, we have the hexagon equation:

$$c_{L \otimes M, N} = (c_{L, N} \otimes \text{Id}_M) \circ (\text{Id}_L \otimes c_{M, N}) \quad \text{and} \quad c_{L, M \otimes N} = (\text{Id}_M \otimes c_{L, N}) \circ (c_{L, M} \otimes \text{Id}_N).$$

Let  $\mathcal{B}_n$  be the Artin-Tits braid group of type  $B_n$ . It has the following presentation in terms of generators and relations:

$$\mathcal{B}_n = \left\langle \tau_0, \tau_1, \dots, \tau_{n-1} \left| \begin{array}{l} \tau_0 \tau_1 \tau_0 \tau_1 = \tau_1 \tau_0 \tau_1 \tau_0, \\ \tau_i \tau_j = \tau_j \tau_i, \quad \text{if } |i - j| > 1, \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \text{for } 1 \leq i \leq n - 2 \end{array} \right. \right\rangle.$$

Using the braiding, we define the following endomorphisms of  $M \otimes N^{\otimes n}$ :

$$\begin{aligned} R_0 &= (c_{N,M} \circ c_{M,N}) \otimes \text{Id}_{N^{\otimes n-1}}, \\ R_i &= \text{Id}_{M \otimes N^{\otimes i-1}} \otimes c_{N,N} \otimes \text{Id}_{N^{\otimes n-i-1}}, \quad \text{for } 1 \leq i \leq n - 1. \end{aligned}$$

Pictorially, one can represent these endomorphisms as

$$R_0 = \begin{array}{c} \text{Diagram of } R_0: \text{A crossing between two strands labeled } M \text{ and } N, \text{ followed by } n-1 \text{ vertical strands labeled } N. \end{array} \quad \text{and} \quad R_i = \begin{array}{c} \text{Diagram of } R_i: \text{ } i \text{ vertical strands labeled } N \text{ with a crossing between the } i \text{th and } (i+1)\text{th strands, followed by } n-i \text{ vertical strands labeled } N. \end{array}$$

**Proposition 3.6.** *The assignment  $\tau_i \mapsto R_i$  defines an action of  $\mathcal{B}_n$  on the module  $M \otimes N^{\otimes n}$  which commutes with the  $\mathcal{U}_q(\mathfrak{gl}_m)$  action.*

*Proof.* The fact that  $R_i$  is a  $\mathcal{U}_q(\mathfrak{gl}_m)$ -morphism follows by definition of  $R_i$ . The fact that the defining relations of  $\mathcal{B}_n$  are satisfied follows from the embedding of the braid group of type  $B_n$  into the braid group of type  $A_{n+1}$  [15, Lemma 2.1].  $\square$

Finally, we end this section with a lemma due to Drinfeld [13, Proposition 5.1 and Remark 4) below] computing the action of the double braiding on highest weight modules, which is related with the action of the ribbon element.

**Lemma 3.7.** *Let  $L, M$  and  $N$  be highest weight modules of respective highest weight  $\lambda, \mu$  and  $\nu$  such that  $L \subset M \otimes N$ . Then the double braiding  $c_{N,M} \circ c_{M,N}$  restricted to  $N$  acts by multiplication by the scalar*

$$q^{\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle}.$$

#### 4. THE ENDOMORPHISM ALGEBRA OF $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$

The aim of this section is to prove the main result of this paper. We first explain why  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  inherits an action of the Ariki-Koike algebra from the action of the braid group of type  $B_n$ . It is a classical result that the eigenvalues of  $R_i$  are  $q$  and  $-q^{-1}$ : the action of the braiding on  $V \otimes V$  is

$$v_i \otimes v_j \mapsto \begin{cases} qv_j \otimes v_i & \text{if } i = j, \\ v_j \otimes v_i & \text{if } i > j, \\ v_j \otimes v_i + (q - q^{-1})v_i \otimes v_j & \text{if } i < j. \end{cases}$$

Moreover, using Lemma 3.7, we easily compute the eigenvalues of the endomorphism  $R_0$  in order to show that the action of  $\mathcal{B}_n$  factors through the Ariki–Koike algebra.

**Lemma 4.1.** *The eigenvalues  $u_1, \dots, u_d$  of  $R_0$  on  $M^{\mathbb{P}}(\Lambda) \otimes V$  are equal to*

$$u_i = (\lambda_i q^{-\tilde{m}_{i-1}})^2.$$

*Proof.* Let  $\Lambda$  be the highest weight of  $M^{\mathbb{P}}(\Lambda)$ . The decomposition of  $M^{\mathbb{P}}(\Lambda) \otimes V$  is given in Proposition 3.4:

$$M^{\mathbb{P}}(\Lambda) \otimes V \simeq \bigoplus_{i=1}^d M^{\mathbb{P}}(\Lambda, \mu_i),$$

where  $\mu_i$  is the  $d$ -partition of 1 whose only non-zero component is the  $i$ -th one and is equal to (1). The highest weight of  $M^{\mathbb{P}}(\Lambda, \mu_i)$  being  $\Lambda + \varepsilon_{\tilde{m}_{i-1}+1}$ , the action of  $R_0$  on  $M^{\mathbb{P}}(\Lambda, \mu_i)$  is given by

$$q^{\langle \Lambda + \varepsilon_{\tilde{m}_{i-1}+1}, \Lambda + \varepsilon_{\tilde{m}_{i-1}+1} + 2\rho \rangle - \langle \Lambda, \Lambda + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle},$$

and we check that

$$\langle \Lambda + \varepsilon_{\tilde{m}_{i-1}+1}, \Lambda + \varepsilon_{\tilde{m}_{i-1}+1} + 2\rho \rangle - \langle \Lambda, \Lambda + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle = 2(\beta_i - \tilde{m}_{i-1}).$$

□

By the definition of the Ariki–Koike algebra, Proposition 3.6 and the previous lemma we thus get an action of the Ariki–Koike algebra for the parameters  $u_i = (\lambda_i q^{-\tilde{m}_{i-1}})^2$  on  $M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n}$ . Therefore, the assignment  $T_i \mapsto R_i$  defines a morphism of algebras

$$\mathcal{H}(d, n) \rightarrow \text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n}).$$

**Theorem 4.2.**

- *The algebra morphism  $\mathcal{H}(d, n) \rightarrow \text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n})$  is surjective and factors through an isomorphism*

$$\mathcal{H}_{\underline{m}}(d, n) \xrightarrow{\simeq} \text{End}_{\mathcal{U}_q(\mathfrak{gl}_m)}(M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n}).$$

- *There is an isomorphism of  $\mathcal{H}(d, n) \otimes \mathcal{U}_q(\mathfrak{gl}_m)$ -module*

$$M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n} \simeq \bigoplus_{\mu \in \mathcal{P}_{\underline{m}}^n} V_{\mu} \otimes M^{\mathbb{P}}(\Lambda, \mu).$$

*Proof.* The first part of the theorem follows immediately from the second part and the definition of the row-quotient  $\mathcal{H}_{\underline{m}}(d, n)$ .

Using Corollary 3.5 and the fact that  $\mathcal{H}(d, n)$  acts on  $M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n}$  by  $\mathcal{U}_q(\mathfrak{gl}_m)$ -linear endomorphisms, we see that

$$M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n} \simeq \bigoplus_{\mu \in \mathcal{P}_1^n} \tilde{V}_{\mu} \otimes M^{\mathbb{P}}(\Lambda, \mu),$$

for some  $\mathcal{H}(d, n)$ -modules  $\tilde{V}_{\mu}$ . Since the multiplicity of  $M^{\mathbb{P}}(\Lambda, \mu)$  in  $M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n}$  is given by the number of paths in the Young lattice from the empty  $d$ -partition to the  $d$ -partition  $\mu$ , we have  $\dim(\tilde{V}_{\mu}) = \dim(V_{\mu})$ . Showing that  $V_{\mu}$  is a submodule of  $\tilde{V}_{\mu}$  will end the proof of the second part of the theorem.

Let  $\mathfrak{t}$  be a standard Young tableau of shape  $\mu$  and denote by  $(a_i, b_i, c_i) = \mathfrak{t}^{-1}(i)$ . Denote by  $\mu[i]$  the  $d$ -partition of  $i$  obtained by adding the boxes labeled by 1 to  $i$  in the chosen standard tableau  $\mathfrak{t}$  to the empty  $d$ -partition. We now choose a highest weight vector  $v \in M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  of weight  $\Lambda_\mu$  such that for all  $1 \leq i \leq n$  we have

$$v \in M^{\mathfrak{p}}(\Lambda, \mu[i]) \otimes V^{\otimes(n-i)} \subset M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}.$$

Using the branching rule, one see that such a vector exists and is unique up to a scalar. Let us show that this vector  $v$  is a common eigenvector of the Jucys–Murphy elements  $X_i$ . It is easy to see that the action of the Jucys–Murphy element  $X_i$  on  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  is given by the double braiding  $(c_{V, M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes(i-1)}} \circ c_{M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes(i-1)}, V}) \otimes \text{Id}_{V^{\otimes(n-i)}}$ . By Lemma 3.7, we obtain that  $X_i$  acts on  $v$  by multiplication by

$$q^{\langle \Lambda_{\mu[i]}, \Lambda_{\mu[i]} + 2\rho \rangle - \langle \Lambda_{\mu[i-1]}, \Lambda_{\mu[i-1]} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle}.$$

Indeed,  $v$  lies in the summand  $M^{\mathfrak{p}}(\Lambda, \mu[i]) \otimes V^{\otimes(n-i)} \subset M^{\mathfrak{p}}(\Lambda, \mu[i-1]) \otimes V \otimes V^{\otimes(n-i)}$  of  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ . But  $\Lambda_{\mu[i]} = \Lambda_{\mu[i-1]} + \varepsilon_{k_i}$  where  $k_i = \tilde{m}_{c_{i-1}} + a_i$  so that

$$\begin{aligned} \langle \Lambda_{\mu[i]}, \Lambda_{\mu[i]} + 2\rho \rangle - \langle \Lambda_{\mu[i-1]}, \Lambda_{\mu[i-1]} + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle &= 2\langle \Lambda_{\mu[i-1]}, \varepsilon_{k_i} \rangle + 2(1 - k_i) \\ &= 2(\beta_{c_i} + b_i - k_i), \end{aligned}$$

since the component of  $\Lambda_{\mu[i-1]}$  on  $\varepsilon_{k_i}$  is  $\beta_{c_i} + (b_i - 1)$ . Therefore,  $X_i$  acts on  $v$  by multiplication by

$$(\lambda_{c_i} q^{b_i - k_i})^2 = u_{c_i} q^{2(b_i - a_i)}.$$

Therefore, the  $\mathcal{H}(d, n)$  submodule spanned by  $v$  is isomorphic to  $V_\mu$  and then  $V_\mu$  is a submodule of  $\tilde{V}_\mu$ .  $\square$

**4.1. Some particular cases.** We finish by giving some special cases of Theorem 4.2 in order to recover various well-known algebras. The two first special cases involve the well-known situation without a parabolic Verma module: it suffices to note that, if  $\mathfrak{p} = \mathfrak{gl}_m$ , then  $M^{\mathfrak{p}}(\Lambda)$  is the trivial module.

**Corollary 4.3.** *If the parabolic subalgebra  $\mathfrak{p}$  is  $\mathfrak{gl}_m$  and  $m \geq n$ , then the endomorphism algebra of  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  is isomorphic to Hecke algebra of type A.*

**Corollary 4.4.** *If the parabolic subalgebra  $\mathfrak{p}$  is  $\mathfrak{gl}_m$  and  $m = 2$ , then the endomorphism algebra of  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  is isomorphic to Temperley–Lieb algebra of type A.*

We now turn to special cases where  $\mathfrak{p}$  is a strict subalgebra of  $\mathfrak{gl}_m$ . The following corollary follows from Remark 2.12.

**Corollary 4.5.** *For  $\mathfrak{p}$  such that  $m \geq nd$  and  $m_i \geq n$  for all  $1 \leq i \leq d$ , the endomorphism algebra of  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  is isomorphic to the Ariki–Koike algebra  $\mathcal{H}(d, n)$ .*

The Hecke algebra of type B with unequal parameters appears when we work with a standard parabolic subalgebra  $\mathfrak{p}$  with Levi factor  $\mathfrak{gl}_{m_1} \times \mathfrak{gl}_{m_2}$ .

**Corollary 4.6.** *If the parabolic subalgebra  $\mathfrak{p}$  is such that  $d = 2$ ,  $m_1 \geq n$  and  $m_2 \geq n$ , then the endomorphism algebra of  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  is isomorphic to the Hecke algebra of type B with unequal and algebraically independent parameters.*



Finally, the last special case is a generalization of the  $\mathfrak{gl}_2$  case of [15], where we recover the generalized blob algebra.

**Corollary 4.7.** *If the parabolic subalgebra  $\mathfrak{p}$  is the standard Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{gl}_m$ , that is  $d = m$  and  $m_i = 1$  for  $1 \leq i \leq d$ , then the endomorphism algebra of  $M(\Lambda) \otimes V^{\otimes n}$  is isomorphic to the generalized blob algebra  $\mathcal{B}(d, n)$ .*

## 5. SOME REMARKS ON THE NON-SEMISIMPLE CASE

This paper deals with the semisimple case, where the decomposition of  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$  as the sum of simple modules is a crucial tool to compute its endomorphism algebra. Non-semisimple situations appear if  $q$  is no longer an indeterminate in the base field  $\mathbb{k}$  but a root of unity. If  $q$  and the parameters  $\lambda_1, \dots, \lambda_d$  appearing in the highest weight of  $M^{\mathfrak{p}}(\Lambda)$  are no longer algebraically independent, a non-semisimple situation may also appear. Indeed, the parabolic Verma module might not be simple anymore as it is readily seen from the case of  $\mathfrak{gl}_2$ . It is then natural to ask whether it is possible to extend the Schur–Weyl duality to the non-semisimple case. Let us remark that if  $q$  is not a root of unity and if  $\lambda_i \lambda_j^{-1} \notin \mathbb{Z}$  for all  $1 \leq i, j \leq d$  then the behavior is similar to the one described in the previous sections.

In order to define the action, we use an “integral version” of the algebras  $\mathcal{U}_q(\mathfrak{gl}_n)$  and  $\mathcal{H}(d, n)$  and of the module  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ , compatible with the specialization at a root of unity.

We start with the Ariki–Koike algebra. The definition given in Section 2.1 is valid for any field  $\mathbb{k}$  and any choice of parameters. Concerning the algebra  $\mathcal{U}_q(\mathfrak{gl}_n)$ , we consider Lusztig’s integral from  $\mathcal{U}_q^{\text{res}}(\mathfrak{gl}_n)$  over  $\mathbb{Z}[q, q^{-1}]$ , see [8, Section 9.3]. It is also known that the quasi- $R$ -matrix  $\Theta$  is an element of (a completion of)  $\mathcal{U}_q^{\text{res}}(\mathfrak{gl}_n) \otimes \mathcal{U}_q^{\text{res}}(\mathfrak{gl}_n)$ . Then for a base field  $\mathbb{k}$  and any  $\xi \in \mathbb{k}^*$ , the quantum group  $\mathcal{U}_\xi(\mathfrak{gl}_n)$  is defined as  $\mathbb{k} \otimes_{\mathbb{Z}[q, q^{-1}]} \mathcal{U}_q^{\text{res}}(\mathfrak{gl}_n)$ , where we see  $\mathbb{k}$  as a  $\mathbb{Z}[q, q^{-1}]$ -module via the morphism sending  $q$  to  $\xi$ .

The parabolic Verma module  $M^{\mathfrak{p}}(\Lambda)$  is a highest weight module and we choose  $v_\Lambda$  a highest weight vector. We then have at our disposal an integral version, which is the submodule generated over  $\mathcal{U}_q^{\text{res}}(\mathfrak{gl}_n)$  by the highest weight  $v_\Lambda$ . Its specialization at  $q = \xi$  will still be denoted  $M^{\mathfrak{p}}(\Lambda)$ . Similarly, we have a version at  $q = \xi$  of the standard module  $V$ , which has a well-known integral form.

Since the quasi- $R$ -matrix  $\Theta$  lies in the Lusztig’s integral form of the quantum group, we can similarly use the braiding to define the endomorphisms  $R_0, R_1, \dots, R_{n-1}$  of the  $\mathcal{U}_\xi(\mathfrak{gl}_n)$ -module  $M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n}$ . As in the semisimple case, we have:

**Proposition 5.1.** *Let  $\mathbb{k}$  be a field,  $q \in \mathbb{k}^*$  and  $\lambda_1, \dots, \lambda_d \in \mathbb{k}$ . Then the assignment  $T_i \mapsto R_i$  is a morphism of algebras from  $\mathcal{H}(d, n)$  to  $\text{End}_{\mathcal{U}_\xi(\mathfrak{gl}_n)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$ . The parameters  $u_i$  of the Ariki–Koike algebra are still given by Lemma 4.1.*

It is more difficult to understand the image of map  $\mathcal{H}(d, n) \rightarrow \text{End}_{\mathcal{U}_\xi(\mathfrak{gl}_n)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$ , or even better to describe the image and the kernel of the map. In [15] Iohara, Lehrer and Zhang studied the particular case of  $\mathfrak{gl}_2$  and  $\mathfrak{p} = \mathfrak{b}$  (this corresponds to  $n = 2$  and  $d = 2$ ) and proved that if  $q$  is an indeterminate in  $\mathbb{k}$  and that  $\lambda_1 \lambda_2^{-1} = q^l$  for  $l \in \mathbb{Z}$ ,  $l \geq -1$ , then the map  $\mathcal{H}(d, n) \rightarrow \text{End}_{\mathcal{U}_q(\mathfrak{gl}_n)}(M^{\mathfrak{p}}(\Lambda) \otimes V^{\otimes n})$  is surjective [15, Proposition 5.11].

In order to extend the Schur–Weyl duality from the semisimple case to a non-semisimple case, a classical strategy [12, 1] is to argue that the dimensions of the various algebras, such as  $\text{End}_{\mathcal{U}_\xi(\mathfrak{gl}_n)}(M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n})$  or  $\mathcal{H}(d, n)$ , are independent of the base field  $\mathbb{k}$ .

Following the arguments of [1], a first step would be to determine whether the parabolic Verma module  $M^{\mathbb{P}}(\Lambda)$  is tilting in an appropriate category  $\mathcal{O}$  of infinite dimensional  $\mathcal{U}_q(\mathfrak{gl}_n)$ -modules. Since  $V$  is tilting and the tensor product of tilting modules is tilting, having  $M^{\mathbb{P}}(\Lambda)$  being tilting would mean that  $M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n}$  is. Since the space of endomorphisms of a tilting module is flat, its dimension does not depend on the base field  $\mathbb{k}$ .

Concerning  $\mathcal{H}(d, n)$ , its definition is valid over the ring  $\mathbb{Z}[q^{\pm 1}, u_1, \dots, u_d]$  and it is known that the basis given in 2.3 is a basis over this ring. This implies that the dimension of the algebra  $\mathcal{H}(d, n)$  is independent of the field  $\mathbb{k}$  and the choice of  $q \in \mathbb{k}^*$  and of  $u_1, \dots, u_d \in \mathbb{k}$ .

Therefore, if  $M^{\mathbb{P}}(\Lambda)$  is tilting in an appropriate category  $\mathcal{O}$  of infinite dimensional  $\mathcal{U}_q(\mathfrak{gl}_n)$ -modules, the map  $\mathcal{H}(d, n) \rightarrow \text{End}_{\mathcal{U}_\xi(\mathfrak{gl}_n)}(M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n})$  would be surjective for any base field  $\mathbb{k}$ .

If we want to consider the row-quotients  $\mathcal{H}_{\underline{m}}(d, n)$  of  $\mathcal{H}(d, n)$ , one must first give a definition which does not rely on the semisimplicity of the algebra  $\mathcal{H}(d, n)$  so that the map  $\mathcal{H}(d, n) \rightarrow \text{End}_{\mathcal{U}_\xi(\mathfrak{gl}_n)}(M^{\mathbb{P}}(\Lambda) \otimes V^{\otimes n})$  factors through  $\mathcal{H}_{\underline{m}}(d, n)$  and then study the existence of an integral basis of  $\mathcal{H}_{\underline{m}}(d, n)$ .

Let us stress that these arguments depend heavily on  $M^{\mathbb{P}}(\Lambda)$  being tilting and on the existence of an integral basis of  $\mathcal{H}_{\underline{m}}(d, n)$ . One may need some extra assumptions on the field  $\mathbb{k}$ , as for example being infinite, or on the parameters of the parabolic Verma module. This non-semisimple behavior deserves further study, which was outside the scope of this paper.

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