

# On Ventcel-type transmission conditions for a Helmholtz problem with a non-uniform thin layer

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## Abstract

We study the asymptotic behaviour of the electric field in the transverse magnetic (TM) mode, propagating in a material composed of a two-dimensional object surrounded by a thin layer of non-constant thickness and embedded in an ambient medium. Using an asymptotic expansion of the solution  $u^\varepsilon$  to the Helmholtz problem, we derive Ventcel-type transmission conditions on the limit interface  $\Gamma$  modelling the effect of the thin layer with accuracy up to  $O(\varepsilon^2)$ .

**Keywords:** Helmholtz equation, asymptotic analysis, Ventcel-type transmission conditions, thin layer.

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## 1 Introduction

In physics, a thin layer is a structure formed by applying a very thin coating, which is significantly smaller in thickness compared to the size of the domain being considered. Thin layers find extensive applications in optics, electronics, chemistry, materials science, and many other fields. They are used in the production of flat screens, solar cells, sensors, anti-reflective coatings, hard disks, integrated circuits, and catalysts. They can be deposited on various substrates such as glass, metals, or semiconductors using techniques like sputtering, vacuum evaporation, or epitaxial growth. They are used to modify the optical, electronic, or mechanical properties of the substrate, protect materials against oxidation, or create interfaces between different materials.

The numerical resolution of problems defined in domains with thin layers, like finite element method or finite difference method, sometimes leads to numerical instabilities and significant computation time: a sufficiently fine discretization at the scale of the thin layer is necessary, which leads to systems with a conditioning number that increases considerably as the thickness of the thin layer tends to 0.

To overcome these difficulties, asymptotic methods are developed to replace the effect of the thin layer with approximate transmission conditions called impedance transmission conditions or Ventcel-type transmission conditions involving tangential differential operators of order greater than or equal to that of the interior differential operator. These last years, a significant amount of literature has been devoted to the modelling of thin layers. We refer the reader to [1, 2, 4, 12, 14, 15, 17, 18, 23], and references therein.

In this paper, we study the asymptotic behavior of the solution to the Helmholtz problem in a two-dimensional domain separated by a thin layer with a non-constant thickness intended to tend to 0. We present approximate transmission conditions that take into account the effect of the thin layer. These conditions can be useful for computing an approximation of the solution to the Helmholtz problem using finite elements because they avoid meshing the thin layer, which can be a very expensive task.

Let us give precise notations. Let  $\Omega_-^\varepsilon$  an open bounded smooth domain of  $\mathbb{R}^2$  with connected boundary  $\Gamma_-^\varepsilon$  surrounded by a thin layer  $\Omega_{m,2}^\varepsilon$  of non-constant thickness with boundary  $\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon$  and which the curve  $\Gamma_-^\varepsilon$  is parametrized by its curvilinear coordinate:

$$\Gamma_-^\varepsilon = \left\{ \gamma_-^\varepsilon(t), t \in (0, l_{\Gamma_-^\varepsilon}) \right\},$$

where  $l_{\Gamma_-^\varepsilon}$  is the length of  $\Gamma_-^\varepsilon$  and  $t$  is the arc length of  $\gamma_-^\varepsilon$ . The curve  $\Gamma_+^\varepsilon$  is defined by

$$\Gamma_+^\varepsilon = \left\{ \gamma_-^\varepsilon(t) + \varepsilon f(t) \mathbf{n}_-, t \in (0, l_{\Gamma_-^\varepsilon}) \right\},$$

where  $\mathbf{n}_-$  is the unit normal vector  $\Gamma_-^\varepsilon$ ,  $\varepsilon$  is a small parameter destined to tend to 0 and  $f$  is a smooth, positive function such that  $\exists m, M > 0; 0 < m \leq f(t) < M, \forall t \in (0, l_{\Gamma_-^\varepsilon})$ . We also denote by  $\Omega_+^\varepsilon$  the exterior domain of  $\mathbb{R}^2$  defined by  $\Omega_+^\varepsilon = \mathbb{R}^2 \setminus (\Omega_-^\varepsilon \cup \overline{\Omega_{m,2}^\varepsilon})$  (cf. Fig. 1).

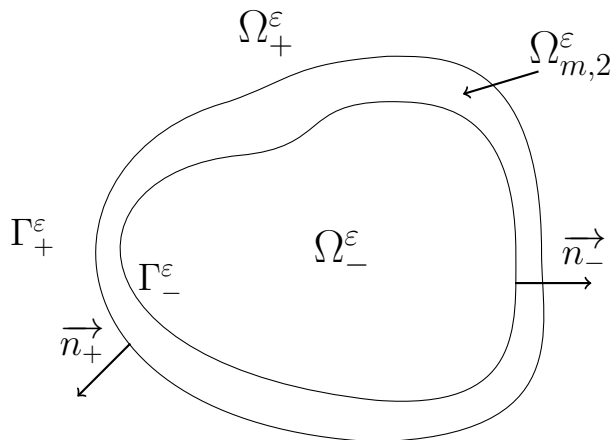


Figure 1: Geometry of the problem

Let  $\alpha_\varepsilon$  et  $k_\varepsilon$  be two piecewise strictly positive constant functions defined by

$$\alpha_\varepsilon(x) = \begin{cases} \alpha_+ & \text{if } x \in \Omega_+^\varepsilon, \\ 1 & \text{if } x \in \Omega_{m,2}^\varepsilon, \\ \alpha_- & \text{if } x \in \Omega_-^\varepsilon, \end{cases} \quad \text{and} \quad k_\varepsilon(x) = \begin{cases} k_+ & \text{if } x \in \Omega_+^\varepsilon, \\ k_m & \text{if } x \in \Omega_{m,2}^\varepsilon, \\ k_- & \text{if } x \in \Omega_-^\varepsilon. \end{cases} \quad (1)$$

In (1),  $\alpha_\varepsilon$  and  $k_\varepsilon$  describe respectively the contrast and the refractive properties of the media  $\Omega_-^\varepsilon$  and  $\Omega_{m,2}^\varepsilon$  relative to the exterior domain  $\Omega_+^\varepsilon$ . We assume that all the constants  $\alpha_\pm$ ,  $k_\pm$  and  $k_m$  are independent of  $\varepsilon$  and  $\alpha_\pm < 1 < \alpha_\mp$ . Our aim is to study the asymptotic behaviour as  $\varepsilon \rightarrow 0$  of the solution  $u^\varepsilon$  to the following problem

$$\begin{cases} \Delta u^\varepsilon + k_\varepsilon^2 u^\varepsilon = 0, & \text{in } \Omega_-^\varepsilon \cup \Omega_{m,2}^\varepsilon \cup \Omega_+^\varepsilon, & (2a) \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) (u_+^\varepsilon - u_{inc}) = 0, & & (2b) \end{cases}$$

with transmission conditions on the interfaces  $\Gamma_+^\varepsilon$  and  $\Gamma_-^\varepsilon$

$$\begin{cases} u_{m,2}^\varepsilon = u_\pm^\varepsilon, & \text{on } \Gamma_\pm^\varepsilon, & (2c) \\ \partial_{\mathbf{n}_\pm} u_{m,2}^\varepsilon = \alpha_\pm \partial_{\mathbf{n}_\pm} u_\pm^\varepsilon, & \text{on } \Gamma_\pm^\varepsilon, & (2d) \end{cases}$$

where  $\partial_{\mathbf{n}_+}$  and  $\partial_{\mathbf{n}_-}$  denote the derivatives in the direction of the unit normal vectors  $\mathbf{n}_+$  and  $\mathbf{n}_-$  to  $\Gamma_+^\varepsilon$  and  $\Gamma_-^\varepsilon$  respectively (Fig. 1);  $u_+^\varepsilon$ ,  $u_{m,2}^\varepsilon$  and  $u_-^\varepsilon$  are the restrictions of  $u^\varepsilon$  respectively to the domains  $\Omega_+^\varepsilon$ ,  $\Omega_{m,2}^\varepsilon$ ,  $\Omega_-^\varepsilon$  and  $u_{inc}$  is the incident wave defined by  $u_{inc} = e^{ik_+(x \cdot d)}$ , with  $d$  being a unit vector of  $\mathbb{R}^2$  giving the direction of the plane wave  $u_{inc}$ .

This work is a continuation of a series of two papers [9, 10] in which the authors studied a similar problem defined in a domain with a uniform thin layer.

Especially, in [10], the authors considered the case  $\alpha_\pm \in ]1, +\infty[$  or  $\alpha_\pm \in ]0, 1[$  and determined a third-order model to take into account the effect of the thin layer by Ventcel-type transmission conditions defined on an interface  $\Gamma$  parallel to  $\Gamma_+^\varepsilon$  and  $\Gamma_-^\varepsilon$  located inside the thin layer  $\Omega_{m,2}^\varepsilon$ . However, the well-posedness of the approximate problem has only been proven in the case  $\alpha_\pm \in ]1, +\infty[$  or  $\alpha_\pm \in ]0, 1[$ .

Our goal is twofold. We consider in this work the case of a non-uniform thin layer and treat the case where  $\alpha_\pm < 1 < \alpha_\mp$ . We use a multiscale expansion of the solution in powers of  $\varepsilon$  to derive, in the case of the non-uniform layer, an equivalent model of order 2 with Ventcel-type transmission conditions involving tangential differential operators of order 2 defined on a parallel interface  $\Gamma$  to  $\Gamma_+^\varepsilon$  and  $\Gamma_-^\varepsilon$  located outside the thin layer  $\Omega_{m,2}^\varepsilon$ . We also derive an equivalent model of order 3 in the particular case where  $f \equiv 1$ , i.e., a uniform thin layer.

The outline of the paper is the following. In the next section, we give precisely the statement of the considered model problem. In section 3, we recall suitable notations and definitions from differential geometry on curves. Section 4 contains asymptotic analysis of the solution of our problem. We calculate the first three terms of a formal asymptotic expansion of the solution and give a convergence theorem justifying the ansatz. The main result of this paper is presented in section 5. We begin by establishing an approximate model of order 2 with Ventcel-type transmission conditions for the case of a non-uniform thin layer. We prove a uniform estimate (Theorem 5) in order to establish the uniqueness of the solution of the approximate problem. Next, we present an equivalent system of pseudo-differential equations to prove the existence result (Theorem 8). Finally, we provide an estimate of the modeling error (Theorem 9). In the

second part of section 5, we address the case of a uniform thin layer and provide an approximate model of order 3 with Ventcel-type transmission conditions. In this case, the uniqueness of the solution is shown (Theorem 12) using the Rellich lemma ([19]). We leave the proof of the existence of the solution and the error estimate to the reader, as they are practically the same as those in Theorem 8 and Theorem 9.

## 2 Problem setting

We would like to use the same technique as that used in [10], where the authors derived Ventcel-type transmission conditions on an interface  $\Gamma$  separating the thin layer into two subdomains. However, it is not suitable for the problem studied in this paper. Therefore, we adopt similar approach to that used in [8]; we consider a parallel interface  $\Gamma$  to  $\Gamma_-^\varepsilon$  dividing  $\Omega_-^\varepsilon$  into two subdomains: a bounded domain  $\Omega_-$  and a thin layer  $\Omega_{m,1}^\varepsilon$  of thickness  $d\varepsilon$  where  $d$  is a positive real number independent of  $\varepsilon$  (see Fig. 2).

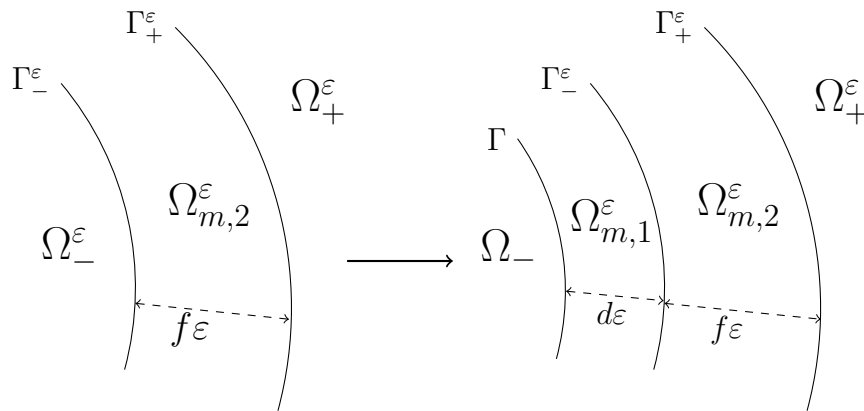


Figure 2: A zoom on the thin layer, with and without the interface  $\Gamma$

Then problem (2) is equivalent to the following problem

$$\begin{cases} \Delta u_+^\varepsilon + k_+^2 u_+^\varepsilon = 0, & \text{in } \Omega_+^\varepsilon, & (3a) \\ \Delta u_{m,2}^\varepsilon + k_{m,2}^2 u_{m,2}^\varepsilon = 0, & \text{in } \Omega_{m,2}^\varepsilon, & (3b) \\ \Delta u_{m,1}^\varepsilon + k_-^2 u_{m,1}^\varepsilon = 0, & \text{in } \Omega_{m,1}^\varepsilon, & (3c) \\ \Delta u_-^\varepsilon + k_-^2 u_-^\varepsilon = 0, & \text{in } \Omega_-, & (3d) \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) (u_+^\varepsilon - u_{inc}) = 0, & & (3e) \end{cases}$$

with transmission conditions

$$\begin{cases} u_{m,2}^\varepsilon = u_+^\varepsilon, & \text{on } \Gamma_+^\varepsilon, & (3f) \\ u_{m,1}^\varepsilon = u_{m,2}^\varepsilon, & \text{on } \Gamma_-^\varepsilon, & (3g) \\ u_-^\varepsilon = u_{m,1}^\varepsilon, & \text{on } \Gamma, & (3h) \\ \partial_{\mathbf{n}_+} u_{m,2}^\varepsilon = \alpha_+ \partial_{\mathbf{n}_+} u_+^\varepsilon, & \text{on } \Gamma_+^\varepsilon, & (3i) \\ \alpha_- \partial_{\mathbf{n}} u_{m,1}^\varepsilon = \partial_{\mathbf{n}_-} u_{m,2}^\varepsilon, & \text{on } \Gamma_-^\varepsilon, & (3j) \\ \partial_{\mathbf{n}} u_-^\varepsilon = \partial_{\mathbf{n}} u_{m,1}^\varepsilon, & \text{on } \Gamma, & (3k) \end{cases}$$

where  $\partial_{\mathbf{n}}$  denotes the derivative in the direction of the unit normal vectors  $\mathbf{n}$  to  $\Gamma$  and  $u_{m,1}^\varepsilon$  is the restriction of  $u^\varepsilon$  to the domains  $\Omega_{m,1}^\varepsilon$ .

We derive an asymptotic expansion of the solution  $u^\varepsilon$  with respect to the thickness  $\varepsilon$  up to any order. We then provide Ventcel-type approximate transmission conditions on the interface  $\Gamma$  modelling the effect of the thin layer and we discuss according to the parameter  $d$  the existence and the uniqueness of the approximate solution.

We start by the following theorem in which we state the existence and the uniqueness result and give a uniform estimate of the solution  $u^\varepsilon$  with respect to  $\varepsilon$ , for which a proof can be found in [9, 7].

**Theorem 1** *If  $u_{inc}$  is a plane wave, then Problem (3) admits a unique solution  $u^\varepsilon$  in  $H_{loc}^1(\mathbb{R}^2)$ . Furthermore, there exists a constant  $c$  independent of  $\varepsilon$  such that*

$$\|u^\varepsilon\|_{H^1(\Omega)} \leq c,$$

where  $\Omega$  is a bounded smooth domain with boundary  $\Gamma_\infty$  enclosing the domain  $\Omega_-^\varepsilon$  as well as the thin layer  $\Omega_{m,2}^\varepsilon$  (cf. Fig. 3).

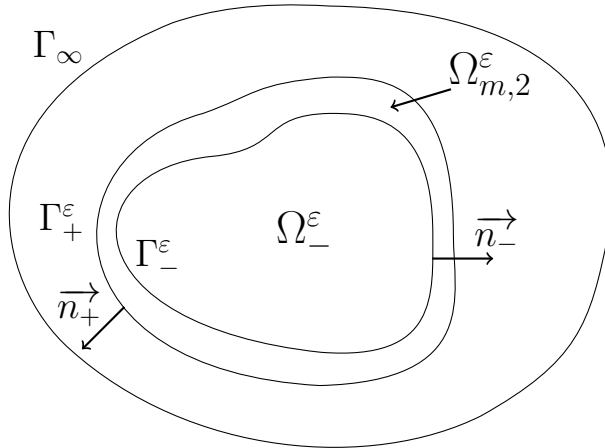


Figure 3: The domain  $\Omega$

### 3 Geometrical tools - Notations

The goal of this section is to define and to collect the main features of differential geometry to formulate our problem in a fixed domain (independent of  $\varepsilon$ ). This technique is a key tool to determine the asymptotic expansion of the solution  $u^\varepsilon$ .

#### 3.1 Parameterization of $\Gamma$

Let  $\Gamma$  be a regular parameterized closed curve through the  $\mathcal{C}^\infty$  map  $\gamma$  defined by

$$\begin{aligned} \gamma : (0, l_\Gamma) &\longrightarrow \Gamma \subset \mathbb{R}^2 \\ t &\longrightarrow \gamma(t) = (\gamma_1(t), \gamma_2(t)), \end{aligned}$$

where  $l_\Gamma$  is the length of  $\Gamma$  and  $t$  is the arc length of  $\gamma$ . The tangent and normal unit vectors  $\tau(t)$  and  $\mathbf{n}(t)$  to  $\Gamma$  at  $\gamma(t)$  are given by

$$\tau(t) := \frac{d\gamma(t)}{dt} = (n_2(t), -n_1(t)), \quad \mathbf{n}(t) := (n_1(t), n_2(t)).$$

We recall Frénet's formulas defining the curvature  $c(t)$  of  $\Gamma$  at point  $\gamma(t)$  (cf. [13])

$$\frac{d\tau(t)}{dt} = -c(t)\mathbf{n}, \quad \frac{d\mathbf{n}(t)}{dt} = c(t)\tau.$$

#### 3.2 Parameterization of $\Omega_m^\varepsilon$

Let  $I_1 = (0, d)$  and  $I_2 = (0, 1)$ . As well-known [13], if  $\varepsilon$  is small enough then the thin layer  $\Omega_{m,\beta}^\varepsilon$ ,  $\beta \in \{1, 2\}$ , can be parameterized by the manifold  $(0, l_\Gamma) \times I_\beta$  with coordinates  $(t, s)$ :

$$\begin{aligned} \Omega_{m,1}^\varepsilon &= \{x = \gamma(t) + \varepsilon s \mathbf{n}(t) \in \Omega_{m,1}^\varepsilon, (t, s) \in \Omega_{m,1} := (0, l_\Gamma) \times I_1 \}, \\ \Omega_{m,2}^\varepsilon &= \{x = \gamma(t) + \varepsilon(d + sf(t))\mathbf{n}(t) \in \Omega_{m,2}^\varepsilon, (t, s) \in \Omega_{m,2} := (0, l_\Gamma) \times I_2 \}. \end{aligned}$$

To each function  $v_\beta$  defined on  $\Omega_{m,\beta}^\varepsilon$ , we associate the function  $V_\beta$  defined on  $\Omega_{m,\beta}$  by

$$V_\beta(t, s) := v_\beta(x), \quad (t, s) \in (0, l_\Gamma) \times I_\beta.$$

Hence the expression of the Laplacian in coordinates  $(t, s)$  is

$$\Delta v_1 = \varepsilon^{-2} \left( \partial_s^2 - \sum_{j=1}^N \varepsilon^j A_{1,j} + \varepsilon^{N+1} T_{N,1} \right) V_1, \quad (4)$$

$$\Delta v_2 = \varepsilon^{-2} \left[ \frac{1}{f^2} \partial_s^2 - \sum_{j=1}^N \varepsilon^j A_{2,j} + \varepsilon^{N+1} T_{N,2} \right] V_2, \quad (5)$$

where  $T_{N,\beta}$  is a bounded operator with respect to  $\varepsilon$ . In particular

$$A_{1,1} = -c(t)\partial_s, \quad (6)$$

$$A_{1,2} = sc^2(t)\partial_s - \partial_t^2, \quad (7)$$

$$A_{1,3} = -s^2 c^3(t) \partial_s + 2sc(t) \partial_t^2 + sc'(t) \partial_t, \quad (8)$$

$$A_{2,1} = -\frac{c(t)}{f(t)} \partial_s, \quad (9)$$

$$A_{2,2} = 2sf'^{-1}(t) \partial_t \partial_s - s^2 f'^2(t) f^{-2}(t) \partial_s^2 - \partial_t^2 - [2sf'^2(t) f^{-2}(t) - sf''(t) f^{-1}(t) - sc^2(t) - dc^2 f^{-1}] \partial_s, \quad (10)$$

$$\begin{aligned} A_{2,3} = & \left( \frac{2s^3 c(t)}{f(t)} + \frac{2s^2 dc(t)}{f^2(t)} \right) f'^2(t) \partial_s^2 + \left( 4s^2 c(t) + \frac{4sdc(t)}{f(t)} \right) f'(t) \partial_t \partial_s \\ & + (2sc(t) f(t) + 2dc(t)) \partial_t^2 - (d + sf(t)) c'(t) \partial_t + \left( -s^2 c^3(t) f(t) - 2sdc^3(t) - \frac{dc^3(t)}{f(t)} \right. \\ & - 2s^2 c(t) f''(t) + \frac{4s^2 c(t) f'^2(t)}{f(t)} - \frac{2sdc(t) f''(t)}{f(t)} + \frac{4sdc(t) f'^2(t)}{f^2(t)} - s^2 c'(t) f'(t) \\ & \left. - \frac{sd c'(t) f'(t)}{f(t)} \right) \partial_s. \end{aligned} \quad (11)$$

The normal derivative on  $\Gamma \cup \Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon$  expands in power of  $\varepsilon$  as

$$\partial_{\mathbf{n}} v_1|_\Gamma = \varepsilon^{-1} \partial_s V_1(t, 0), \quad (12)$$

$$\partial_{\mathbf{n}_-} v_1|_{\Gamma_-^\varepsilon} = \varepsilon^{-1} \partial_s V_1(t, d), \quad (13)$$

$$\partial_{\mathbf{n}_-} v_2|_{\Gamma_-^\varepsilon} = \varepsilon^{-1} f^{-1} \partial_s V_2(t, 0), \quad (14)$$

$$\partial_{\mathbf{n}_+} v_2|_{\Gamma_+^\varepsilon} = \left[ \varepsilon^{-1} f^{-1} \partial_s + \varepsilon \left( \frac{1}{2} f'^2 f^{-1} \partial_s - f' \partial_t \right) + \dots \right] V_2(t, 1). \quad (15)$$

Otherwise, if  $v$  is a function defined on  $\Omega_m^\varepsilon$ , which is parameterized with coordinates  $(t, \eta)$ :

$$\Omega_m^\varepsilon = \{x = \gamma(t) + \eta(d + f(t)) \mathbf{n}(t) \in \Omega_m^\varepsilon, (t, \eta) \in \Omega_m := (0, l_\Gamma) \times (0, \varepsilon)\}, \quad (16)$$

then, using a Taylor expansion in the normal variable  $\eta$ , the normal derivative on  $\Gamma_+^\varepsilon$  has the expansion

$$\partial_{\mathbf{n}_+} v|_{\Gamma_+^\varepsilon} = \partial_{\mathbf{n}} v|_\Gamma + \varepsilon [(f + d) \partial_{\mathbf{n}}^2 v|_\Gamma - f' \partial_t v|_\Gamma] + \dots \quad (17)$$

## 4 The asymptotic analysis

In this section, we derive a multiscale expansion for the solution  $u^\varepsilon$  of the model problem (2) in power series of the small parameter  $\varepsilon$ . We present a hierarchy of equations defined in a fixed domain (independent of  $\varepsilon$ ). Then we give the first three terms of the asymptotic expansions and the convergence theorem justifying our ansatz.

### 4.1 Hierarchy of equations

Due to boundary layer phenomena, the solution  $u^\varepsilon$  admits two asymptotic expansions: exterior asymptotic expansions corresponding to the expansion of the solution  $u^\varepsilon$  restricted to  $\Omega_+^\varepsilon$  and to  $\Omega_-$ , written in cartesian coordinates  $x = (x_1, x_2)$  (macroscopic scale) and given by the ansatz

$$u_+^\varepsilon = \sum_{n \geq 0} \varepsilon^n u_+^n \text{ in } \Omega_+^\varepsilon, \quad (18)$$

$$u_-^\varepsilon = \sum_{n \geq 0} \varepsilon^n u_-^n \text{ in } \Omega_-, \quad (19)$$

where the terms  $u_-^n$  and  $u_+^n$  ( $n \in \mathbb{N}$ ) are independent of  $\varepsilon$  and respectively defined on  $\Omega_-$  and on  $\Omega_+ := \Omega_+^\varepsilon \cup \Gamma_+^\varepsilon \cup \Omega_{m,2}^\varepsilon \cup \Gamma_-^\varepsilon \cup \Omega_{m,1}^\varepsilon$ . They satisfy

$$\begin{cases} \Delta u_+^n + k_+^2 u_+^n = 0 & \text{in } \Omega_+, \\ \Delta u_-^n + k_-^2 u_-^n = 0 & \text{in } \Omega_-, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) (u_+^n - \delta_{0,n} u_{inc}) = 0, \end{cases} \quad (20)$$

in which  $\delta_{0,n}$  indicates the Kronecker symbol. And an interior expansion corresponding to the asymptotic expansion of  $u^\varepsilon$  restricted to  $\Omega_m^\varepsilon$ , written in local coordinates  $(t, s)$  (microscopic scale) and defined by the ansatz

$$u_{m,\beta}^\varepsilon(x_1, x_2) = U_{m,\beta}^\varepsilon(t, s) = \sum_{n \geq 0} \varepsilon^n U_{m,\beta}^n(t, s) \text{ in } \Omega_{m,\beta}, \beta = 1, 2, \quad (21)$$

where  $U_{m,\beta}^n$  are independent of  $\varepsilon$ .

Using a Taylor expansion in the normal variable and formula (16), transmission conditions (3f)-(3h) become

$$\begin{aligned} U_{m,2}^0(t, 1) + \varepsilon U_{m,2}^1(t, 1) + \varepsilon^2 U_{m,2}^2(t, 1) + \dots \\ = u_+^0|_\Gamma + \varepsilon (u_+^1|_\Gamma + (d + f(t)) \partial_n u_+^0|_\Gamma) \\ + \varepsilon^2 \left( u_+^2|_\Gamma + (d + f(t)) \partial_n u_+^1|_\Gamma + \frac{(d + f(t))^2}{2} \partial_n^2 u_+^0|_\Gamma \right) + \dots, \end{aligned} \quad (22)$$

$$U_{m,1}^0(t, d) + \varepsilon U_{m,1}^1(t, d) + \varepsilon^2 U_{m,1}^2(t, d) + \dots = U_{m,2}^0(t, 0) + \varepsilon U_{m,2}^1(t, 0) + \varepsilon^2 U_{m,2}^2(t, 0) + \dots, \quad (23)$$

$$U_{m,1}^0(t, 0) + \varepsilon U_{m,1}^1(t, 0) + \varepsilon^2 U_{m,1}^2(t, 0) + \dots = u_-^0|_\Gamma + \varepsilon u_-^1|_\Gamma + \varepsilon^2 u_-^2|_\Gamma + \dots, \quad (24)$$

Upon using (12)-(15) and condition (17), we get

$$\begin{aligned} \frac{\varepsilon^{-1}}{f(t)} \partial_s U_{m,2}^0(t, 1) + \frac{1}{f(t)} \partial_s U_{m,2}^1(t, 1) + \varepsilon \left( \frac{1}{f(t)} \partial_s U_{m,2}^2(t, 1) + \frac{f'^2(t)}{2f(t)} \partial_s U_{m,2}^0(t, 1) \right. \\ \left. - f'(t) \partial_t U_{m,2}^0(t, 1) \right) + \varepsilon^2 \left( \frac{1}{f(t)} \partial_s U_{m,2}^3(t, 1) + \frac{f'^2(t)}{2f(t)} \partial_s U_{m,2}^1(t, 1) \right. \\ \left. - f'(t) \partial_t U_{m,2}^1(t, 1) - c f'^2(t) \left( \frac{d}{f(t)} + 1 \right) \partial_s U_{m,2}^0(t, 1) \right. \\ \left. + 2c(t) f'(t) (d + f(t)) \partial_t U_{m,2}^0(t, 1) \right) + \dots \\ = \alpha_+ \partial_n u_+^0|_\Gamma + \varepsilon (\alpha_+ \partial_n u_+^1|_\Gamma + (d + f(t)) \alpha_+ \partial_n^2 u_+^0|_\Gamma - f'(t) \alpha_+ \partial_t u_+^0|_\Gamma) \\ + \varepsilon^2 \left( \alpha_+ \partial_n u_+^2|_\Gamma + (d + f(t)) \alpha_+ \partial_n^2 u_+^1|_\Gamma + \frac{(d + f(t))^2}{2} \alpha_+ \partial_n^3 u_+^0|_\Gamma \right. \\ \left. - f'(t) \partial_t \alpha_+ u_+^1|_\Gamma + \frac{f'^2(t)}{2} \alpha_+ \partial_n u_+^0|_\Gamma + 2c(t) f'(t) (d + f(t)) \alpha_+ \partial_t u_+^0|_\Gamma \right. \\ \left. - f'(t) (d + f(t)) \alpha_+ \partial_t \partial_n u_+^0|_\Gamma \right) + \dots, \end{aligned} \quad (25)$$



$$\begin{aligned}
& \varepsilon^{-1} \alpha_- \partial_s U_{m,1}^0(t, d) + \alpha_- \partial_s U_{m,1}^1(t, d) + \varepsilon \alpha_- \partial_s U_{m,1}^2(t, d) + \varepsilon^2 \alpha_- \partial_s U_{m,1}^3(t, d) + \dots \\
& = \frac{1}{f(t)} \left( \varepsilon^{-1} \partial_s U_{m,2}^0(t, 0) + \partial_s U_{m,2}^1(t, 0) + \varepsilon \partial_s U_{m,2}^2(t, 0) + \varepsilon^2 \partial_s U_{m,2}^3(t, 0) \right) + \dots,
\end{aligned} \tag{26}$$

$$\begin{aligned}
& \varepsilon^{-1} \partial_s U_{m,1}^0(t, 0) + \partial_s U_{m,1}^1(t, 0) + \varepsilon \partial_s U_{m,1}^2(t, 0) + \varepsilon^2 \partial_s U_{m,1}^3(t, 0) + \dots \\
& = \partial_{\mathbf{n}} u_-^0|_{\Gamma} + \varepsilon \partial_{\mathbf{n}} u_-^1|_{\Gamma} + \varepsilon^2 \partial_{\mathbf{n}} u_-^2|_{\Gamma} + \dots.
\end{aligned} \tag{27}$$

Now, inserting serie (21) in (3b) and (3c), using (4), (5) and matching the same powers of  $\varepsilon$ , we get, for all  $(t, s) \in (0, l_{\Gamma}) \times I_{\beta}$ , the following families of equations

$$\partial_s^2 U_{m,\beta}^0 = 0, \tag{28}$$

$$\partial_s^2 U_{m,\beta}^1 = A_{\beta,1} U_{m,\beta}^0, \tag{29}$$

$$\partial_s^2 U_{m,1}^2 = A_{1,1} U_{m,1}^1 + A_{1,2} U_{m,1}^0 - k_-^2 U_{m,1}^0, \tag{30}$$

$$\partial_s^2 U_{m,2}^2 = A_{2,1} U_{m,2}^1 + A_{2,2} U_{m,2}^0 - k_m^2 U_{m,2}^0, \tag{31}$$

$$\partial_s^2 U_{m,1}^3 = A_{1,1} U_{m,1}^2 + A_{1,2} U_{m,1}^1 + A_{1,3} U_{m,1}^0 - k_-^2 U_{m,1}^1, \tag{32}$$

$$\partial_s^2 U_{m,2}^3 = A_{2,1} U_{m,2}^2 + A_{2,2} U_{m,2}^1 + A_{2,3} U_{m,2}^0 - k_m^2 U_{m,2}^1. \tag{33}$$

## 4.2 Calculation of the first terms

In this paragraph, we exhibit the first three terms of the asymptotic expansion of the solution  $u^\varepsilon$  to problem (2). All the details and formal calculi are presented in Appendix A.

By resolving equations (28)-(33), with the help of transmission conditions (22)-(26), the terms  $(u_-^n, u_+^n)$  are solution of the following boundary-value problems

$$\begin{cases}
\Delta u_+^n + k_+^2 u_+^n = 0 & \text{in } \Omega_+, \\
\Delta u_-^n + k_-^2 u_-^n = 0 & \text{in } \Omega_-, \\
\lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) (u_+^n - \delta_{0,n} u_{inc}) = 0,
\end{cases}$$

with transmission conditions on  $\Gamma$ :

**At order 0.**

$$u_+^0 - u_-^0 = 0, \tag{34}$$

$$\alpha_+ \partial_{\mathbf{n}} u_+^0 - \alpha_- \partial_{\mathbf{n}} u_-^0 = 0. \tag{35}$$

**At order 1.**

$$u_+^1 - u_-^1 = \frac{f(t) \alpha_+ \alpha_- + d \alpha_+ - (f(t) + d) \alpha_-}{2 \alpha_+ \alpha_-} (\alpha_+ \partial_{\mathbf{n}} u_+^0 + \alpha_- \partial_{\mathbf{n}} u_-^0), \tag{36}$$

$$\alpha_+ \partial_{\mathbf{n}} u_+^1 - \alpha_- \partial_{\mathbf{n}} u_-^1 = \vartheta_1 (\alpha_+ \partial_{\mathbf{n}} u_+^0 + \alpha_- \partial_{\mathbf{n}} u_-^0) + \vartheta_2 (u_+^0 + u_-^0)$$

$$+\vartheta_3 (\partial_t u_+^0 + \partial_t u_-^0) + \vartheta_4 (\partial_t^2 u_+^0 + \partial_t^2 u_-^0). \quad (37)$$

where

$$\begin{aligned} \vartheta_1 &= \frac{1}{2} \left( f(t) - \frac{1}{f(t)} \right) c(t), \\ \vartheta_2 &= \frac{1}{2} \left( (d + f(t)) \alpha_+ k_+^2 - \frac{1}{f(t)} k_m^2 - d \alpha_- k_-^2 \right), \\ \vartheta_3 &= \frac{1}{2} (\alpha_+ - 1) f'(t), \\ \vartheta_4 &= \frac{1}{2} \left( (d + f) \alpha_+ - \frac{1}{f(t)} - d \alpha_- \right). \end{aligned}$$

**At order 2.**

$$\begin{aligned} u_+^2 - u_-^2 &= \frac{f(t) \alpha_+ \alpha_- + d \alpha_+ - (d + f(t)) \alpha_-}{2 \alpha_+ \alpha_-} (\alpha_+ \partial_{\mathbf{n}} u_+^1 + \alpha_- \partial_{\mathbf{n}} u_-^1) \\ &+ \left( \frac{(-2f^2(t) f'(t) - 2df^2(t) - f^3(t)) \alpha_+ \alpha_- + (d - d^2 f(t) - df^2(t)) \alpha_+}{4f(t) \alpha_+ \alpha_-} \right. \\ &\quad \left. - \frac{(d + f(t) + d^2 f(t) + df^2(t)) \alpha_-}{4f(t) \alpha_+ \alpha_-} \right) c(t) (\alpha_+ \partial_{\mathbf{n}} u_+^0 + \alpha_- \partial_{\mathbf{n}} u_-^0) \\ &+ \left( \frac{-(d + f(t)) f^2(t) \alpha_+^2 \alpha_- - df^2(t) \alpha_+ \alpha_-^2 - df(t) (d + f(t)) (\alpha_+^2 - \alpha_-^2)}{4f(t) \alpha_+ \alpha_-} \right. \\ &\quad \left. + \frac{d \alpha_+ + (d + f(t)) \alpha_-}{4f(t) \alpha_+ \alpha_-} \right) (\partial_t^2 u_+^0 + \partial_t^2 u_-^0) \\ &+ \left( \frac{-(d + f(t)) f^2(t) k_+^2 \alpha_+^2 \alpha_- - df^2(t) k_-^2 \alpha_+ \alpha_-^2 - d(d + f(t)) f(t) (k_+^2 \alpha_+^2 - k_-^2 \alpha_-^2)}{4f(t) \alpha_+ \alpha_-} \right. \\ &\quad \left. + \frac{dk_m^2 \alpha_+ + (d + f(t)) k_m^2 \alpha_-}{4f(t) \alpha_+ \alpha_-} \right) (u_+^0 + u_-^0) \\ &+ \frac{f(t) \alpha_+^2 \alpha_- - d \alpha_+^2 - d \alpha_+ \alpha_- + d \alpha_+ + (d + f(t)) \alpha_-}{4 \alpha_+ \alpha_-} f'(t) (\partial_t u_+^0 + \partial_t u_-^0), \quad (38) \end{aligned}$$

$$\begin{aligned} \alpha_+ \partial_{\mathbf{n}} u_+^2 - \alpha_- \partial_{\mathbf{n}} u_-^2 &= \vartheta_1 (\alpha_+ \partial_{\mathbf{n}} u_+^1 + \alpha_- \partial_{\mathbf{n}} u_-^1) + \vartheta_2 (u_+^1 + u_-^1) \\ &+ \vartheta_3 (\partial_t u_+^1 + \partial_t u_-^1) + \vartheta_4 (\partial_t^2 u_+^1 + \partial_t^2 u_-^1) \\ &+ \vartheta_5 (\partial_t^2 u_+^0 + \partial_t^2 u_-^0) + \vartheta_6 (\partial_t u_+^0 + \partial_t u_-^0) \\ &+ \vartheta_7 (u_+^0 + u_-^0) + \vartheta_8 (\alpha_+ \partial_t^2 \partial_{\mathbf{n}} u_+^0 + \partial_t^2 \alpha_- \partial_{\mathbf{n}} u_-^0) \\ &+ \vartheta_9 (\alpha_+ \partial_t \partial_{\mathbf{n}} u_+^0 + \partial_t \alpha_- \partial_{\mathbf{n}} u_-^0) \\ &+ \vartheta_{10} (\alpha_+ \partial_{\mathbf{n}} u_+^0 + \alpha_- \partial_{\mathbf{n}} u_-^0), \end{aligned}$$

where

$$\vartheta_5 = \frac{3d^2}{4} \alpha_- c(t) - \frac{c(t)}{4f^2(t)} + \frac{c(t)}{2f(t)} (d + f(t)) \alpha_+$$

$$\begin{aligned}
& +\frac{1}{2}c(t) + \frac{dc(t)}{f(t)} - \frac{3(d+f(t))^2}{4}\alpha_+c(t) \\
& +\frac{1}{2}\left(f(t) - \frac{1}{f(t)}\right)c(t)\left[\frac{(d+f(t))f(t)\alpha_+ - df(t)\alpha_- - 1}{2f(t)}\right] \\
& +dc(t)\frac{(d+f(t))f(t)\alpha_+ - df(t)\alpha_- - 1}{2f(t)}, \\
\vartheta_6 & = -\frac{1}{4}c'(t) + \frac{d^2}{4}\alpha_-c'(t) - \frac{c(t)}{2f(t)}f'(t) + \alpha_+\frac{c(t)}{2f(t)}f'(t) - c(t)f'(t)(d+f(t))(\alpha_+ - 1) \\
& -\frac{(d+f(t))^2}{4}\alpha_+c'(t) - \frac{dc'(t)}{2f(t)} + dc(t)\frac{f'(t)(\alpha_+ - 1)}{2} \\
& +\frac{1}{2}\left(f(t) - \frac{1}{f(t)}\right)c(t)\left[\frac{f'(t)(\alpha_+ - 1)}{2}\right], \\
\vartheta_7 & = \frac{d^2}{4}\alpha_-c(t)k_-^2 - \frac{c(t)}{4f^2(t)}k_m^2 + \frac{c(t)}{2f(t)}(d+f(t))\alpha_+k_+^2 - \frac{(d+f(t))^2}{4}\alpha_+c(t)k_+^2 \\
& +\frac{1}{2}\left(f(t) - \frac{1}{f(t)}\right)c(t)\left[\frac{(d+f(t))f(t)k_+^2\alpha_+ - df(t)k_-^2\alpha_- - k_m^2}{2f(t)}\right] \\
& +dc(t)\frac{(d+f(t))f(t)k_+^2\alpha_+ - df(t)k_-^2\alpha_- - k_m^2}{2f(t)}, \\
\vartheta_8 & = \frac{(d+f(t))^2 - d^2 - 1}{4} - \frac{1}{2}\frac{1}{f(t)}\left(\frac{d+f(t) - f(t)\alpha_+}{\alpha_+}\right) \\
& +\frac{1}{2}\left((d+f(t))\alpha_+ + d\alpha_- - \frac{1}{f(t)}\right)\left[\frac{f(t)\alpha_+\alpha_- - (d+f(t))\alpha_- + d\alpha_+}{2\alpha_+\alpha_-}\right], \\
\vartheta_9 & = \frac{f'(t)(\alpha_+ - 1)}{f(t)\alpha_+} + \frac{f'(t)(d+f(t))(\alpha_+ - 1)}{2\alpha_+} \\
& +\frac{1}{2}(\alpha_+ - 1)f'(t)\frac{f(t)\alpha_+\alpha_- - (d+f(t))\alpha_- + d\alpha_+}{2\alpha_+\alpha_-} \\
& +\left((d+f(t))\alpha_+ + d\alpha_- - \frac{1}{f(t)}\right)\left(\frac{f'(t)\alpha_+\alpha_- - (f'(t))\alpha_-}{2\alpha_+\alpha_-}\right), \\
\vartheta_{10} & = -\frac{d^2}{4}k_-^2 + \frac{1}{2}d^2c^2(t) + \frac{1}{4}c^2(t) - \frac{(f'(t))^2}{2f^2(t)} - \frac{1}{2}\frac{1}{f(t)}\left[\frac{(f''(t) - f''(t)\alpha_+)}{\alpha_+}\right] \\
& +\frac{1}{2}dc^2(t)f^{-1}(t) - \frac{(f'(t))^2}{2\alpha_+} + \frac{1}{2f(t)}c^2(t)(d+f(t)) - \frac{(d+f(t))^2}{4}(2c^2(t) - k_+^2) \\
& -k_m^2\left(\frac{d+f(t) - f(t)\alpha_+}{2\alpha_+f(t)}\right) - \frac{c^2(t)}{4f^2(t)} + \frac{(f'(t))^2}{2f^2(t)} + d\frac{c^2(t)(f^2(t) - 1)}{2f(t)} \\
& -\left(\frac{1}{f(t)} - f(t)\right)c^2(t)\frac{(f^2(t) - 1)}{4f(t)} + \frac{1}{4\alpha_+}(1 - \alpha_+)^2(f'(t))^2 - \frac{1}{4}k_m^2 \\
& -\frac{1}{2}\left(\frac{1}{f(t)}k_m^2 - (d+f)\alpha_+k_+^2 - d\alpha_-k_-^2\right)\left(\frac{f(t)\alpha_+\alpha_- - (d+f(t))\alpha_- + d\alpha_+}{2\alpha_+\alpha_-}\right) \\
& -\frac{1}{2}\left(\frac{1}{f(t)} - (d+f)\alpha_+ - d\alpha_-\right)\left(\frac{f''(t)\alpha_+\alpha_- - f''(t)\alpha_-}{2\alpha_+\alpha_-}\right).
\end{aligned}$$

And  $(U_{m,\beta}^n)_{0 \leq n \leq 1}$  are determined by  $\forall (t, s) \in (0, l_\Gamma) \times I_\beta$

$$U_{m,\beta}^0(t, s) = u_-^0|_\Gamma = u_+^0|_\Gamma, \quad (39)$$

$$U_{m,1}^1(t, s) = u_-^1|_\Gamma + s \partial_{\mathbf{n}} u_-^0|_\Gamma, \quad (40)$$

$$U_{m,2}^1(t, s) = u_+^1|_\Gamma + (f + d - f(t)\alpha_+ + sf(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma, \quad (41)$$

$$U_{m,1}^2(t, s) = [-c(t)\partial_{\mathbf{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma - k_-^2 u_-^0|_\Gamma] \frac{s^2}{2} + \partial_{\mathbf{n}} u_-^1|_\Gamma s + u_-^2|_\Gamma, \quad (42)$$

$$\begin{aligned} U_{m,2}^2(t, s) &= [-c(t)\alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma - \partial_t^2 u_+^0|_\Gamma - k_m^2 u_+^0|_\Gamma] \frac{s^2 - 2s + 1}{2} + [f(t)\alpha_+ \partial_{\mathbf{n}} u_+^1|_\Gamma \\ &\quad - c(t)f(t)f'(t)\alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma + f(t)f'(t)\partial_t u_+^0|_\Gamma + (d + f(t))f(t)\alpha_+ \partial_{\mathbf{n}}^2 u_+^0|_\Gamma] (s - 1) \\ &\quad + (d + f(t)) \partial_{\mathbf{n}} u_+^1|_\Gamma + \frac{(d + f(t))^2}{2} \partial_{\mathbf{n}}^2 u_+^0|_\Gamma + u_+^2|_\Gamma. \end{aligned} \quad (43)$$

**Remark 2** We note that by setting, in [10],  $d_1 = -d$  and  $d_2 = 1 + d$ , we formally obtain the asymptotic expansions and the equivalent models presented here in the case of a uniform thin layer where  $f = 1$ .

### 4.3 Convergence theorem

The goal of this paragraph is to validate the ansatz (18), (19) and (21) by providing an error estimate made by approximating the exact solution of (2) by a truncated asymptotic expansion at order  $N \in \mathbb{N}$ . Let  $\tilde{\Omega}_+^\varepsilon$  be a domain of  $\mathbb{R}^2$  defined by  $\tilde{\Omega}_+^\varepsilon := \Omega_+^\varepsilon \cap \Omega$ . We set

$$u_-^{\varepsilon,(N)} := \sum_{n=0}^N \varepsilon^n u_-^n, \quad u_+^{\varepsilon,(N)} := \sum_{n=0}^N \varepsilon^n u_+^n \quad \text{and} \quad u_{m,\beta}^{\varepsilon,(N)} := \sum_{n=0}^N \varepsilon^n u_{m,\beta}^n,$$

where  $u_{m,\beta}^n(x) := U_{m,\beta}^n(t, s)$ ;  $\forall x = \Phi_\beta(t, s) \in \Omega_{m,\beta}^\varepsilon$ . Then we have the following theorem.

**Theorem 3 (Convergence Theorem)** *For all integers  $N$ , if  $u_{inc}$  is a plane wave, then there exists a constant  $c$  independent of  $\varepsilon$  such as*

$$\begin{aligned} \left\| u_-^\varepsilon - u_-^{\varepsilon,(N)} \right\|_{H^1(\Omega_-)} + \varepsilon^{1/2} \left\| u_{m,\beta}^\varepsilon - u_{m,1}^{\varepsilon,(N)} \right\|_{H^1(\Omega_{m,1}^\varepsilon)} + \varepsilon^{1/2} \left\| u_{m,2}^\varepsilon - u_{m,2}^{\varepsilon,(N)} \right\|_{H^1(\Omega_{m,2}^\varepsilon)} \\ + \left\| u_+^\varepsilon - u_+^{\varepsilon,(N)} \right\|_{H^1(\tilde{\Omega}_+^\varepsilon)} \leq c\varepsilon^{N+1}. \end{aligned}$$

**Proof.** The existence and uniqueness of the series  $(u_-^n)_n$ ,  $(u_+^n)_n$  and  $(U_{m,\beta}^n)_n$  as well as the error estimate follow the same lines as Theorem 13 in [9] and Theorem 5.1 in [7]. ■

## 5 Approximate transmission conditions

In this section, we derive Ventcel-type transmission conditions and approximate models to exact problem (2). We start with the case of a thin layer with a variable thickness. We show the existence and the uniqueness of the solution and prove an error estimate to validate the convergence of the equivalent conditions with accuracy up to  $O(\varepsilon^2)$ . In the next paragraph, we deal with the case of a thin layer with a constant thickness. We determine approximate model with Ventcel-type transmission conditions up to order 3.

## 5.1 The case of a thin layer with a variable thickness

To obtain equivalent transmission conditions of order 2, we truncate the series (18) and (19) up to a residual term in  $O(\varepsilon^2)$ . An approximation of the exact solution  $u^\varepsilon$  far from the thin layer is solution  $u_\infty^{\varepsilon,ap} := (u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$  of the following problem

$$\begin{cases} \Delta u_+^{\varepsilon,ap} + k_+^2 u_+^{\varepsilon,ap} = 0 & \text{in } \Omega_+, \\ \Delta u_-^{\varepsilon,ap} + k_-^2 u_-^{\varepsilon,ap} = 0 & \text{in } \Omega_-, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) (u_+^{\varepsilon,ap} - u_{inc}) = 0, \end{cases} \quad (44a)$$

with transmission conditions on  $\Gamma$

$$\begin{pmatrix} [u_\infty^{\varepsilon,ap}] \\ [\alpha \partial_{\mathbf{n}} u_\infty^{\varepsilon,ap}] \end{pmatrix} = \varepsilon \begin{pmatrix} 2\mathcal{C}_1 Id & 0 \\ c(t) \left( f(t) - \frac{1}{f(t)} \right) Id & 2(\mathcal{D}_1 \partial_t^2 + \mathcal{D}_4 \partial_t + \mathcal{D}_2 Id) \end{pmatrix} \begin{pmatrix} \langle \alpha \partial_{\mathbf{n}} u_\infty^{\varepsilon,ap} \rangle \\ \langle u_\infty^{\varepsilon,ap} \rangle \end{pmatrix}, \quad (44b)$$

in which

$$\alpha(x) = \begin{cases} \alpha_+ & \text{if } x \in \Omega_+, \\ \alpha_- & \text{if } x \in \Omega_-, \end{cases}$$

$$\begin{aligned} [v] &= v_{+|\Gamma} - v_{-|\Gamma}, & \langle v \rangle &= \frac{1}{2} (v_{+|\Gamma} + v_{-|\Gamma}), \\ [\alpha \partial_{\mathbf{n}} v] &= \alpha_+ \partial_{\mathbf{n}} v_{+|\Gamma} - \alpha_- \partial_{\mathbf{n}} v_{-|\Gamma}, & \langle \alpha \partial_{\mathbf{n}} v \rangle &= \frac{1}{2} (\alpha_+ \partial_{\mathbf{n}} v_{+|\Gamma} + \alpha_- \partial_{\mathbf{n}} v_{-|\Gamma}), \end{aligned}$$

and

$$\mathcal{C}_1 = \frac{f(t)\alpha_+\alpha_- - (d + f(t))\alpha_- + d\alpha_+}{2\alpha_+\alpha_-}, \quad (45)$$

$$\mathcal{D}_1 = \frac{(d + f(t))f(t)\alpha_+ - df(t)\alpha_- - 1}{2f(t)}, \quad (46)$$

$$\mathcal{D}_2 = \frac{(d + f(t))f(t)k_+^2\alpha_+ - df(t)k_-^2\alpha_- - k_m^2}{2f(t)}, \quad (47)$$

$$\mathcal{D}_3 = c(t) \frac{(f^2(t) - 1)\alpha_+\alpha_-}{f(t)(f(t)\alpha_+\alpha_- - (d + f(t))\alpha_- + d\alpha_+)}, \quad (48)$$

$$\mathcal{D}_4 = \frac{f'(t)(\alpha_+ - 1)}{2}. \quad (49)$$

Note that  $\mathcal{C}_1 \neq 0$  since we have assumed that  $\alpha_\pm < 1 < \alpha_\mp$ .

### 5.1.1 Uniform stability

We now prove  $\varepsilon$ -uniform a priori estimate for the solution of problem (44). This latter is actually going to be useful for proving the unicity of the solution.

To do, we rewrite problem (44) in a truncated domain (see [5] for a similar reduction) in order to give a suitable variational formulation of problem (44).

We recall that  $\Omega$  is a bounded smooth domain with boundary  $\Gamma_\infty$  enclosing the domain  $\Omega_-^\varepsilon$  as well as the thin layer  $\Omega_{m,2}^\varepsilon$  (see Fig. 3) and let us denote by  $\tilde{\Omega}_\infty$  the exterior domain of  $\mathbb{R}^2$  with boundary  $\Gamma_\infty$  defined by  $\tilde{\Omega}_\infty = \mathbb{R}^2 \setminus \bar{\Omega}$ .

We introduce the DtN operator  $T$  (Dirichlet-to-Neumann) defined from  $H^{1/2}(\Gamma_\infty)$  onto  $H^{-1/2}(\Gamma_\infty)$  by  $T\varphi := -\partial_{\mathbf{n}_{\tilde{\Omega}_\infty}} \omega$ , where  $\mathbf{n}_{\tilde{\Omega}_\infty}$  indicates the unit normal to  $\Gamma_\infty$  outwardly directed to  $\Omega$  and  $\omega$  is the unique solution to the following problem

$$\begin{cases} \text{Find } \omega \in H_{loc}^1(\tilde{\Omega}_\infty) \\ \Delta\omega + k_+^2\omega = 0 & \text{in } \tilde{\Omega}_\infty, \\ \omega = \varphi & \text{on } \Gamma_\infty, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) \omega = 0. \end{cases} \quad (50)$$

The DtN operator  $T$  is a pseudodifferential operator of order one [5] and is linear continuous from  $H^{1/2}(\Gamma_\infty)$  to  $H^{-1/2}(\Gamma_\infty)$ . The next lemma, whose proof uses standard elliptic regularity (cf., e.g., [5]), gives some useful decomposition of the DtN operator.

**Lemma 4** *Let  $\phi \in H^{1/2}(\Gamma_\infty)$  and  $\varphi_0 \in H^1(\tilde{\Omega}_\infty)$  be the unique solution to the following coercive scattering problem*

$$\begin{cases} \Delta\varphi_0 - \varphi_0 = 0 & \text{in } \tilde{\Omega}_\infty \\ \varphi_0 = \phi & \text{on } \Gamma_\infty. \end{cases}$$

Now let us consider  $T_0\phi = -\partial_{\mathbf{n}_{\tilde{\Omega}_\infty}} \varphi_0$ . Then  $T_0$  is bounded and coercive from  $H^{1/2}(\Gamma_\infty)$  into  $H^{-1/2}(\Gamma_\infty)$ . In addition, there exists a compact operator  $K$  acting from  $H^{1/2}(\Gamma_\infty)$  into  $H^{3/2}(\Gamma_\infty)$  such that

$$T = T_0 + K. \quad (51)$$

We introduce the Hilbert space  $\mathbb{H}^1(\Omega, \Gamma)$  defined by

$$\mathbb{H}^1(\Omega, \Gamma) = \left\{ v = (v_-, v_+) \in H^1(\Omega_-) \times H^1(\tilde{\Omega}_+); u_{\pm|\Gamma} \in H^1(\Gamma) \right\},$$

where  $\tilde{\Omega}_+ := \Omega_+ \cap \Omega$ . Then a variational formulation of (44) is given by

$$\begin{cases} \text{Find } u_\infty^{\varepsilon, ap} = (u_-^{\varepsilon, ap}, u_+^{\varepsilon, ap}) \in \mathbb{H}^1(\Omega, \Gamma), \forall v \in \mathbb{H}^1(\Omega, \Gamma) \\ a_\varepsilon(u_\infty^{\varepsilon, ap}, v) := \alpha_- \int_{\Omega_-} \nabla u_-^{\varepsilon, ap} \cdot \nabla v_- \, d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} u_-^{\varepsilon, ap} v_- \, d\Omega_- \\ + \alpha_+ \int_{\tilde{\Omega}_+} \nabla u_+^{\varepsilon, ap} \cdot \nabla v_+ \, d\tilde{\Omega}_+ - \alpha_+ k_+^2 \int_{\tilde{\Omega}_+} u_+^{\varepsilon, ap} v_+ \, d\tilde{\Omega}_+ + \int_\Gamma \frac{1}{2\varepsilon c_1} [u_\infty^{\varepsilon, ap}] [v] \, d\Gamma \\ - 2\varepsilon \int_\Gamma \mathcal{D}_1 \langle \partial_t u_\infty^{\varepsilon, ap} \rangle \langle \partial_t v \rangle \, d\Gamma + 2\varepsilon \int_\Gamma \mathcal{D}_2 \langle u_\infty^{\varepsilon, ap} \rangle \langle v \rangle \, d\Gamma \\ + \int_\Gamma \mathcal{D}_3 [u_\infty^{\varepsilon, ap}] \langle v \rangle \, d\Gamma + 2\varepsilon \int_\Gamma (\mathcal{D}_4 - \mathcal{D}'_1) \langle \partial_t u_\infty^{\varepsilon, ap} \rangle \langle v \rangle \, d\Gamma \\ + \alpha_+ \langle T u_+^{\varepsilon, ap}, v_+ \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)} = l(v), \end{cases} \quad (52)$$

where  $\langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)}$  denotes the duality pairing between  $H^{-1/2}(\Gamma_\infty)$  and  $H^{1/2}(\Gamma_\infty)$  and  $l$  is defined by

$$l(v) := \alpha_+ \int_{\Gamma_\infty} \left( \partial_{\mathbf{n}_{\tilde{\Omega}_\infty}} + T \right) u_{inc} \bar{v} \, d\sigma.$$

We have the following theorem about the dependance of  $u_\infty^{\varepsilon, ap}$  as  $\varepsilon$  goes to zero.

**Theorem 5** For all  $l$  in  $(\mathbb{H}^1(\Omega, \Gamma))'$ , there exists a positive constant  $c$ , independent of  $\varepsilon$ , such that the solution  $u_\infty^{\varepsilon, ap}$  of problem (52) satisfy

$$\|u_\infty^{\varepsilon, ap}\|_{\mathbb{H}^1(\Omega, \Gamma)} \leq c\varepsilon^{-\frac{1}{2}} \|l\|_{(\mathbb{H}^1(\Omega, \Gamma))'}.$$

**Proof.** We need to prove that the next estimate holds true

$$\|u_\infty^{\varepsilon, ap}\|_{\mathbb{H}^1(\Omega, \Gamma)} \leq c\varepsilon^{-\frac{1}{2}} \sup_{v \in \mathbb{H}^1(\Omega, \Gamma)} \frac{|a_\varepsilon(u_\infty^{\varepsilon, ap}, v)|}{\|v\|_{\mathbb{H}^1(\Omega, \Gamma)}}.$$

To do so, we proceed by contradiction hence assuming there exists sequences  $(\varepsilon_n)_{n \in \mathbb{N}}$  and  $(u_\infty^{\varepsilon_n, ap})_{n \in \mathbb{N}}$  (denoted by  $(u_\infty^{n, ap})_{n \in \mathbb{N}}$ ) such that

$$\lim_{n \rightarrow +\infty} \varepsilon_n = 0, \quad \|\sqrt{\varepsilon_n} u_\infty^{n, ap}\|_{\mathbb{H}^1(\Omega, \Gamma)} = 1, \quad \forall n \in \mathbb{N} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \sup_{\|\varphi\|_{\mathbb{H}^1(\Omega, \Gamma)}=1} |a_{\varepsilon_n}(u_\infty^{n, ap}, \varphi)| = 0.$$

From Rellich theorem, we can extract a subsequence (still denoted by  $(u_\infty^{n, ap})_{n \in \mathbb{N}}$ ) such that

$$\begin{cases} \sqrt{\varepsilon_n} u_\infty^{n, ap} \rightarrow u_\infty^{0, ap} & \text{in } \mathbb{L}^2(\Omega), \\ \sqrt{\varepsilon_n} u_\infty^{n, ap} \rightharpoonup u_\infty^{0, ap} & \text{in } \mathbb{H}^1(\Omega, \Gamma). \end{cases}$$

Otherwise, for all  $(v_-, v_+) \in \mathcal{C}^\infty(\overline{\Omega_-}) \times \mathcal{C}^\infty(\overline{\widetilde{\Omega}_+})$ , we have

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \frac{1}{2\varepsilon_n} \int_\Gamma \frac{\sqrt{\varepsilon_n}}{\mathcal{C}_1} [u_\infty^{n, ap}] [v] \, d\Gamma = \lim_{n \rightarrow +\infty} \alpha_- \int_{\Omega_-} \sqrt{\varepsilon_n} \nabla u_-^{n, ap} \cdot \nabla v_- \, d\Omega_- \\ & - \alpha_- k_-^2 \int_{\Omega_-} \sqrt{\varepsilon_n} u_-^{n, ap} v_- \, d\Omega_- + \alpha_+ \int_{\widetilde{\Omega}_+} \sqrt{\varepsilon_n} \nabla u_+^{n, ap} \cdot \nabla v_+ \, d\widetilde{\Omega}_+ - \alpha_+ k_+^2 \int_{\widetilde{\Omega}_+} \sqrt{\varepsilon_n} u_+^{n, ap} v_+ \, d\widetilde{\Omega}_+ \\ & - 2\varepsilon_n \int_\Gamma \mathcal{D}_1 \sqrt{\varepsilon_n} \langle \partial_t u_\infty^{n, ap} \rangle \langle \partial_t v \rangle \, d\Gamma + 2\varepsilon_n \int_\Gamma \mathcal{D}_2 \sqrt{\varepsilon_n} \langle u_\infty^{n, ap} \rangle \langle v \rangle \, d\Gamma \\ & + \int_\Gamma \mathcal{D}_3 \sqrt{\varepsilon_n} [u_\infty^{n, ap}] \langle v \rangle \, d\Gamma + 2\varepsilon_n \int_\Gamma (\mathcal{D}_4 - \mathcal{D}'_1) \sqrt{\varepsilon_n} \langle \partial_t u_\infty^{n, ap} \rangle \langle v \rangle \, d\Gamma \\ & - \sqrt{\varepsilon_n} a_\varepsilon(u_+^{n, ap}, v) + \alpha_+ \langle \sqrt{\varepsilon_n} T u_+^{n, ap}, v_+ \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)} \\ & = \alpha_- \int_{\Omega_-} \nabla u_-^{0, ap} \cdot \nabla v_- \, d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} u_-^{0, ap} v_- \, d\Omega_- + \alpha_+ \int_{\widetilde{\Omega}_+} \nabla u_+^{0, ap} \cdot \nabla v_+ \, d\widetilde{\Omega}_+ \\ & - \alpha_+ k_+^2 \int_{\widetilde{\Omega}_+} u_+^{0, ap} v_+ \, d\widetilde{\Omega}_+ + \int_\Gamma \mathcal{D}_3 [u_\infty^{0, ap}] \langle v \rangle \, d\Gamma \\ & + \alpha_+ \langle T u_+^{0, ap}, v_+ \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)}. \end{aligned} \tag{53}$$

The right-side term being independent of  $\varepsilon_n$ , it follows that,  $\forall (v_-, v_+) \in \mathcal{C}^\infty(\overline{\Omega_-}) \times \mathcal{C}^\infty(\overline{\widetilde{\Omega}_+})$ ,

$$\frac{1}{2\varepsilon_n} \int_\Gamma \frac{\sqrt{\varepsilon_n}}{\mathcal{C}_1} [u_\infty^{n, ap}] [v] \, d\Gamma = O(1).$$

Using the fact that  $\mathcal{C}^\infty(\overline{\Omega_-}) \times \mathcal{C}^\infty(\overline{\widetilde{\Omega}_+})$  is dense in  $\mathbb{H}^1(\Omega, \Gamma)$  and  $\mathcal{C}_1$  and  $\frac{1}{\mathcal{C}_1}$  are bounded in  $\Gamma$ , we obtain for  $v = \overline{u_\infty^{n, ap}}$

$$\| [u_\infty^{n, ap}] \|_{L^2(\Gamma)} \leq c\varepsilon_n^{1/4}. \tag{54}$$

As a consequence  $u_+^{0,ap} = u_-^{0,ap}$  on  $\Gamma$  and for all  $v$  in  $\mathbb{H}^1(\Omega, \Gamma)$ , we have

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sqrt{\varepsilon_n} a_{\varepsilon_n} (u_\infty^{n,ap}, v) &= \alpha_- \int_{\Omega_-} \nabla u_-^{0,ap} \cdot \nabla v_- \, d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} u_-^{0,ap} v_- \, d\Omega_- \\ &+ \alpha_+ \int_{\tilde{\Omega}_+} \nabla u_+^{0,ap} \cdot \nabla v_+ \, d\tilde{\Omega}_+ - \alpha_+ k_+^2 \int_{\tilde{\Omega}_+} u_+^{0,ap} v_+ \, d\tilde{\Omega}_+ \\ &+ \alpha_+ \langle T u_+^{0,ap}, v_+ \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)} = 0. \end{aligned} \quad (55)$$

As a result, well-know properties of uniqueness of the solution of problem : Find  $u_\infty^{0,ap}$  in  $H^1(\Omega_- \cup \Gamma \cup \tilde{\Omega}_+)$  satisfying (55), based on Rellich ([19]) lemma and the operator  $T$  imply that  $u_\infty^{0,ap} = 0$ .

To obtain the contradiction, we show that  $\lim_{n \rightarrow +\infty} \|\sqrt{\varepsilon_n} u_\infty^{n,ap}\|_{\mathbb{H}^1(\Omega, \Gamma)} = 0$ . Note that, since  $u_\infty^{0,ap}$  is uniquely determined, it is all the sequence  $(\sqrt{\varepsilon_n} u_\infty^{n,ap})_{n \in \mathbb{N}}$  that converges to  $u_\infty^{0,ap}$ . It only remains to show that  $\lim_{n \rightarrow +\infty} \|\sqrt{\varepsilon_n} \nabla u_\infty^{n,ap}\|_{\mathbb{L}^2(\Omega)} = 0$ . We have

$$\begin{aligned} \|\sqrt{\varepsilon_n} \nabla u_\infty^{n,ap}\|_{\mathbb{L}^2(\Omega)}^2 &\leq c \varepsilon_n \left( \alpha_- \int_{\Omega_-} \sqrt{\varepsilon_n} |\nabla u_-^{n,ap}|^2 \, d\Omega_- + \alpha_+ \int_{\tilde{\Omega}_+} \sqrt{\varepsilon_n} |\nabla u_+^{n,ap}|^2 \, d\tilde{\Omega}_+ \right) \\ &= c \mathcal{R} \left( \varepsilon_n a_{\varepsilon_n} (u_\infty^{n,ap}, \overline{u_\infty^{n,ap}}) + \alpha_- k_-^2 \int_{\Omega_-} \varepsilon_n |u_-^{n,ap}|^2 \, d\Omega_- + \alpha_+ k_+^2 \int_{\tilde{\Omega}_+} \varepsilon_n |u_+^{n,ap}|^2 \, d\tilde{\Omega}_+ \right. \\ &\quad - \int_{\Gamma} \frac{1}{2\mathcal{C}_1} | [u_\infty^{n,ap}] |^2 \, d\Gamma + 2\varepsilon_n^2 \int_{\Gamma} \mathcal{D}_1 | \langle \partial_t u_\infty^{n,ap} \rangle |^2 \, d\Gamma \\ &\quad - 2\varepsilon_n^2 \int_{\Gamma} \mathcal{D}_2 | \langle u_\infty^{n,ap} \rangle |^2 \, d\Gamma - \varepsilon_n \int_{\Gamma} \mathcal{D}_3 [u_\infty^{n,ap}] \overline{\langle u_\infty^{n,ap} \rangle} \, d\Gamma \\ &\quad - 2\varepsilon_n^2 \int_{\Gamma} (\mathcal{D}_4 - \mathcal{D}'_1) \langle \partial_t u_\infty^{n,ap} \rangle \overline{\langle u_\infty^{n,ap} \rangle} \, d\Gamma \\ &\quad \left. - \varepsilon_n \alpha_+ \langle T u_+^{n,ap}, \overline{u_+^{n,ap}} \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)} \right). \end{aligned}$$

Using Lemma 4, we deduce

$$\begin{aligned} \|\sqrt{\varepsilon_n} \nabla u_\infty^{n,ap}\|_{\mathbb{L}^2(\Omega)}^2 &\leq \mathcal{R} \left( \varepsilon_n a_{\varepsilon_n} (u_\infty^{n,ap}, \overline{u_\infty^{n,ap}}) + \alpha_- k_-^2 \int_{\Omega_-} \varepsilon_n |u_-^{n,ap}|^2 \, d\Omega_- + \alpha_+ k_+^2 \int_{\tilde{\Omega}_+} \varepsilon_n |u_+^{n,ap}|^2 \, d\tilde{\Omega}_+ \right. \\ &\quad - \int_{\Gamma} \frac{1}{2\mathcal{C}_1} | [u_\infty^{n,ap}] |^2 \, d\Gamma + 2\varepsilon_n^2 \int_{\Gamma} \mathcal{D}_1 | \langle \partial_t u_\infty^{n,ap} \rangle |^2 \, d\Gamma \\ &\quad - 2\varepsilon_n^2 \int_{\Gamma} \mathcal{D}_2 | \langle u_\infty^{n,ap} \rangle |^2 \, d\Gamma - \varepsilon_n \int_{\Gamma} \mathcal{D}_3 ([u_\infty^{n,ap}]) \overline{\langle u_\infty^{n,ap} \rangle} \, d\Gamma \\ &\quad - 2\varepsilon_n^2 \int_{\Gamma} (\mathcal{D}_4 - \mathcal{D}'_1) \langle \partial_t u_\infty^{n,ap} \rangle \overline{\langle u_\infty^{n,ap} \rangle} \, d\Gamma \\ &\quad \left. - \varepsilon_n \alpha_+ \langle K u_+^{n,ap}, \overline{u_+^{n,ap}} \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)} \right). \end{aligned}$$

Since  $K$  is compact and  $\sqrt{\varepsilon_n} u_\infty^{n,ap} \rightarrow 0$  in  $\mathbb{H}^1(\Omega, \Gamma)$ , we obtain

$$\langle \sqrt{\varepsilon_n} K u_+^{n,ap}, \sqrt{\varepsilon_n} u_+^{n,ap} \rangle_{\mathbb{H}^{-1/2}(\Gamma_\infty) \times \mathbb{H}^{1/2}(\Gamma_\infty)} \rightarrow 0.$$

Finally, the hypothesis  $\lim_{n \rightarrow +\infty} \mathcal{R} \left[ a_{\varepsilon_n} (u_\infty^{n,ap}, \sqrt{\varepsilon_n} \overline{u_\infty^{n,ap}}) \right] = 0$  and (54) yield  $\|\sqrt{\varepsilon_n} \nabla u_\infty^{n,ap}\|_{\mathbb{L}^2(\Omega)} = 0$ , contradicting  $\|\sqrt{\varepsilon_n} u_\infty^{n,ap}\|_{\mathbb{H}^1(\Omega, \Gamma)} = 1$ . ■



### 5.1.2 Existence and uniqueness

To show the existence of the solution  $u_\infty^{\varepsilon,ap} = (u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$ , we introduce the Steklov-Poincaré operators (cf., e.g., [5]) (called also Dirichlet-to-Neumann operators)  $T_-$  and  $T_+$  defined from  $H^{1/2}(\Gamma)$  onto  $H^{-1/2}(\Gamma)$  by  $T_- \varphi := \alpha_- \partial_{\mathbf{n}} u_-|_\Gamma$ , where  $u_-$  is the solution to the boundary-value problem

$$\begin{cases} \Delta u_- + k_-^2 u_- = 0 & \text{in } \Omega_-, \\ u_- = \varphi & \text{on } \Gamma, \end{cases}$$

and by  $T_+ \psi := \alpha_+ \partial_{-\mathbf{n}} u_+|_\Gamma$ , where  $u_+$  is the solution to the boundary-value problem

$$\begin{cases} \Delta u_+ + k_+^2 u_+ = 0 & \text{in } \Omega_+, \\ u_+ = \psi & \text{on } \Gamma, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) u_+ = 0, \end{cases}$$

The Dirichlet-to-Neumann operators  $T_-$  and  $T_+$  are elliptic pseudodifferential operators (cf., e.g., [20, 21, 22, 11]) of real symbol of order 1.

**Remark 6** *The function  $u_-$  is defined only in the case where the constant  $k_-^2$  does not belong to the spectrum of the closed operator  $(-\Delta, H_0^1(\Omega_-))$ . We will therefore assume that this condition holds.*

We also assume that

$$d > \frac{1 - m^2 \alpha_+}{m(\alpha_+ - \alpha_-)} \text{ or } d < \frac{1 - M^2 \alpha_+}{M(\alpha_+ - \alpha_-)}, \quad (56)$$

if  $\alpha_+ > \alpha_-$  and

$$d > \frac{M^2 \alpha_+ - 1}{m(\alpha_- - \alpha_+)} \text{ or } d < \frac{m^2 \alpha_+ - 1}{M(\alpha_- - \alpha_+)}, \quad (57)$$

if  $\alpha_- > \alpha_+$ , where  $m = \min_{t \in \Gamma} f(t)$  and  $M = \max_{t \in \Gamma} f(t)$ .

**Remark 7** *Note that if  $m^2 \alpha_+ > 1$  when  $\alpha_+ > \alpha_-$  or  $M^2 \alpha_+ < 1$  when  $\alpha_- > \alpha_+$  then the hypotheses (56)-(57) are superfluous.*

Using the definition of The Steklov-Poincaré operators, we rewrite Problem (44) into an equivalent system of boundary equations: Find  $(\omega, \varkappa) \in H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ , such that:

$$\begin{cases} \Lambda_{11} \omega + \Lambda_{12} \varkappa = \varepsilon \mathcal{C}_1 h, \\ \Lambda_{21} \omega + \Lambda_{22} \varkappa = -h, \end{cases} \quad (58)$$

where  $\omega$  and  $\varkappa$  are the traces of  $u_+^{\varepsilon,ap}$  and  $u_-^{\varepsilon,ap}$  on  $\Gamma$  respectively,  $h = T_+(u_{inc}|_\Gamma) + \alpha_+ \partial_{\mathbf{n}} u_{inc}|_\Gamma$ , and  $(\Lambda_{ij})_{1 \leq i, j \leq 2}$  are pseudodifferential operators defined by

$$\begin{aligned} \Lambda_{11} &= I + \varepsilon \mathcal{C}_1 T_+, \\ \Lambda_{12} &= -I - \varepsilon \mathcal{C}_1 T_-, \\ \Lambda_{21} &= (-\mathcal{D}_3 - \varepsilon \mathcal{D}_1 - \varepsilon \mathcal{D}_2) I - T_+ + \varepsilon \mathcal{D}_1 (I - \partial_t^2) - \varepsilon \mathcal{D}_4 \partial_t, \\ \Lambda_{22} &= (\mathcal{D}_3 - \varepsilon \mathcal{D}_1 - \varepsilon \mathcal{D}_2) I - T_- + \varepsilon \mathcal{D}_1 (I - \partial_t^2) - \varepsilon \mathcal{D}_4 \partial_t. \end{aligned}$$

We are now in position to state the existence theorem. We have:

**Theorem 8** For any integer  $k \geq 1$ , if  $u_{inc}$  is a plane wave,  $h \in H^{k-\frac{3}{2}}(\Gamma)$  and

$$-\frac{1}{\varepsilon} \notin \sigma(\mathcal{C}_1 T_+), \quad (59)$$

then Problem (44) admits a unique solution  $(u_-^{\varepsilon, ap}, u_+^{\varepsilon, ap}) \in H^k(\Omega_-) \times H_{loc}^k(\overline{\Omega}_+)$ .

**Proof.** Let  $k$  be a positive integer. In view of (59),  $\Lambda_{11}^{-1}$  is a well-defined pseudodifferential operator of order  $-1$ . Thus, from the first equation of System (58) we get

$$\omega = -\Lambda_{11}^{-1} \Lambda_{12} \varkappa + \varepsilon \Lambda_{11}^{-1} \mathcal{C}_1 h,$$

so

$$[\Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12} - \Lambda_{22}] \varkappa = [I + \varepsilon \Lambda_{21} \Lambda_{11}^{-1} \mathcal{C}_1] h.$$

The operator  $I - \partial_t^2$  is an elliptic self-adjoint semibounded from below pseudodifferential operator of order 2 (cf. [6]), it is Fredholm with zero index and maps  $H^s(\Gamma)$  to  $H^{s-2}(\Gamma)$ , for any  $s \in \mathbb{R}$ . In view of (56)-(57),  $\mathcal{D}_1 \neq 0$ , then the operator  $\mathcal{D}_1(I - \partial_t^2) : H^s(\Gamma) \rightarrow H^{s-2}(\Gamma)$  is Fredholm with zero index. Since  $\partial_t$ ,  $T_+$ , and  $T_-$  are pseudodifferential operators of order 1, they map  $H^s(\Gamma)$  to  $H^{s-1}(\Gamma)$ , and since the injection  $H^{s-1}(\Gamma) \hookrightarrow H^{s-2}(\Gamma)$  is compact,

$$(\mp \mathcal{D}_3 - \varepsilon \mathcal{D}_1 - \varepsilon \mathcal{D}_2)I - T_{\pm} - \varepsilon \mathcal{D}_4 \partial_t, \Lambda_{11} \text{ and } \Lambda_{12} : H^s(\Gamma) \rightarrow H^{s-2}(\Gamma)$$

are compact operators. Then the operator  $\Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12} - \Lambda_{22}$  is a compact perturbation of  $\varepsilon \mathcal{D}_1(I - \partial_t^2)(\Lambda_{11}^{-1} \Lambda_{12} - I)$ . It follows that to show that the operator  $\Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12} - \Lambda_{22}$  is Fredholm with zero index, it remains to prove that the operator  $\Lambda_{11} - \Lambda_{12} = \varepsilon \mathcal{C}_1(T_+ + T_-) + 2I$  defined from  $H^{k-1/2}(\Gamma)$  to  $H^{k-3/2}(\Gamma)$  is invertible. Therefore, we consider the equation

$$\left(T_+ + T_- + \frac{2}{\varepsilon \mathcal{C}_1} I\right) \varphi = \psi, \quad \psi \in H^{k-3/2}(\Gamma), \quad k \geq 1. \quad (60)$$

The definition of the operators  $T_+$  and  $T_-$  shows that equation (60) is equivalent to the following boundary-value problem

$$\begin{cases} \Delta u_+ + k_+^2 u_+ = 0 & \text{in } \Omega_+, \\ \Delta u_- + k_-^2 u_- = 0 & \text{in } \Omega_-, \\ u_- = u_+ & \text{on } \Gamma, \\ \alpha_- \partial_{\mathbf{n}} u_- - \alpha_+ \partial_{\mathbf{n}} u_+ + \frac{2}{\varepsilon \mathcal{C}_1} u_+ = \psi & \text{on } \Gamma, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) u_+ = 0, \end{cases} \quad (61)$$

where  $\varphi = u_-|_{\Gamma} = u_+|_{\Gamma}$ . Standard arguments based on Rellich's lemma and the Fredholm alternative show that, for all  $k$  in  $\mathbb{N}^*$ , if  $\psi \in H^{k-3/2}(\Gamma)$ , then problem (61) admits a unique solution  $(u_-, u_+)$  in  $H^k(\Omega_-) \times H_{loc}^k(\overline{\Omega}_+)$ , and hence there exists a unique Dirichlet trace  $\varphi \in H^{k-1/2}(\Gamma)$ .

As a consequence, the operator  $T_+ + T_- + \frac{2}{\varepsilon \mathcal{C}_1} I$ , defined from  $H^{k-1/2}(\Gamma)$  to  $H^{k-3/2}(\Gamma)$ , is invertible, so  $\Lambda_{21} \Lambda_{11}^{-1} \Lambda_{12} - \Lambda_{22}$  is Fredholm with index 0. The equivalence of System (58) to Problem (44) and Theorem 5 shows that the uniqueness of  $(u_-^{\varepsilon, ap}, u_+^{\varepsilon, ap})$  implies that for any integer  $k \geq 1$ , if  $(\varepsilon \mathcal{C}_1 h, -h) \in H^{k-\frac{3}{2}}(\Gamma) \times H^{k-\frac{3}{2}}(\Gamma)$ , there exists a unique solution  $(\omega, \varkappa) \in H^{k-\frac{1}{2}}(\Gamma) \times H^{k-\frac{1}{2}}(\Gamma)$  of (58) which leads to the existence of a unique solution  $(u_-^{\varepsilon, ap}, u_+^{\varepsilon, ap}) \in H^k(\Omega_-) \times H_{loc}^k(\overline{\Omega}_+)$ , as we wished. ■

### 5.1.3 Error estimate

In this paragraph, we define an approximate solution  $u^{\varepsilon,ap}$  of the exact solution  $u^\varepsilon$  and prove  $\varepsilon$ -uniform a priori estimate of the convergence of  $u^{\varepsilon,ap}$  to  $u^\varepsilon$ . Using  $u_\infty^{\varepsilon,ap}$ , the solution of (44), we define  $u^{\varepsilon,ap}$  by

$$u^{\varepsilon,ap} = \begin{cases} u_+^{\varepsilon,ap} & \text{in } \Omega_+^\varepsilon, \\ u_{m,\beta}^{\varepsilon,ap} & \text{in } \Omega_{m,\beta}^\varepsilon, \\ u_-^{\varepsilon,ap} & \text{in } \Omega_-, \end{cases}$$

where  $u_{m,\beta}^{\varepsilon,ap}$  is obtained from (39)-(41) by adding and subtracting a term of order 1 in  $\varepsilon$  and neglecting all terms in  $O(\varepsilon^2)$ . It is defined, for all  $(t, s)$  in  $\Omega_{m,\beta}^\varepsilon$ , by

$$u_{m,1}^{\varepsilon,ap}(x) := U_{m,1}^{\varepsilon,ap}(t, s) := u_{-\Gamma}^{\varepsilon,ap} + s\varepsilon\partial_{\mathbf{n}}u_{-\Gamma}^{\varepsilon,ap}, \quad (62)$$

$$u_{m,2}^{\varepsilon,ap}(x) := U_{m,2}^{\varepsilon,ap}(t, s) := u_{+\Gamma}^{\varepsilon,ap} + \varepsilon(f(t) + d - f(t)\alpha_+ + sf(t)\alpha_+) \partial_{\mathbf{n}}u_{+\Gamma}^{\varepsilon,ap}. \quad (63)$$

We have the following error estimate :

**Theorem 9** *If  $u_{inc}$  is a plane wave, then there exists a constant  $c$  independent of  $\varepsilon$  such that, for  $\varepsilon$  small enough, we have*

$$\|u_-^\varepsilon - u_-^{\varepsilon,ap}\|_{H^1(\Omega_-)} + \varepsilon^{1/2} \|u_{m,1}^\varepsilon - u_{m,1}^{\varepsilon,ap}\|_{H^1(\Omega_{m,1}^\varepsilon)} + \varepsilon^{1/2} \|u_{m,2}^\varepsilon - u_{m,2}^{\varepsilon,ap}\|_{H^1(\Omega_{m,2}^\varepsilon)} + \|u_+^\varepsilon - u_+^{\varepsilon,ap}\|_{H^1(\tilde{\Omega}_+^\varepsilon)} \leq c\varepsilon^2.$$

.

Before starting the proof of Theorem 9, we need a stability result. Let  $\mathbb{H}^1(\Omega)$  be the Hilbert space defined by

$$\mathbb{H}^1(\Omega) := \left\{ v = (v_-, v_+) \in H^1(\Omega_-) \times H^1(\tilde{\Omega}_+) \right\},$$

and  $b_\varepsilon(\cdot, \cdot)$  be the bilinear form on  $\mathbb{H}^1(\Omega)$  defined by

$$\begin{aligned} b_\varepsilon(u, v) &:= \alpha_- \int_{\Omega_-} \nabla u_- \cdot \nabla v_- \, d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} u_- v_- \, d\Omega_- \\ &+ \alpha_+ \int_{\tilde{\Omega}_+} \nabla u_+ \cdot \nabla v_+ \, d\tilde{\Omega}_+ - \alpha_+ k_+^2 \int_{\tilde{\Omega}_+} u_+ v_+ \, d\tilde{\Omega}_+ \\ &- \frac{1}{2} \lambda_\varepsilon \int_{\Gamma} [u] [v] \, d\Gamma \\ &+ \alpha_+ \langle Tu_+, v_+ \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)}, \end{aligned}$$

where  $\lambda_\varepsilon = O(\varepsilon^{-1})$ . We have the following lemma, which proof is very close to that of Theorem 5 and can be founded in [7].

**Lemma 10** *For all  $h_\varepsilon \in (\mathbb{H}^1(\Omega))'$ , there exists a positive constant  $c$  independent of  $\varepsilon$  such that the solution to the variational problem*

$$\begin{cases} \text{Find } u \in \mathbb{H}^1(\Omega), \forall v \in \mathbb{H}^1(\Omega), \\ b_\varepsilon(u, v) = h_\varepsilon(v), \end{cases}$$

satisfies

$$\|u\|_{\mathbb{H}^1(\Omega)} \leq c\varepsilon^{-1/2} \|h_\varepsilon\|_{(\mathbb{H}^1(\Omega))'}.$$

**Proof of Theorem 9.** According to

$$\left\| U_{m,1}^{\varepsilon,ap} - U_{m,1}^{\varepsilon,(1)} \right\|_{H^1(\Omega_{m,1})} \leq c \left( \|u_-^{\varepsilon,ap} - u_-^{\varepsilon,(1)}\|_{H^1(\Omega_-)} + \varepsilon^2 \|u_-^1\|_{H^1(\Omega_-)} \right),$$

and

$$\left\| U_{m,2}^{\varepsilon,ap} - U_{m,2}^{\varepsilon,(1)} \right\|_{H^1(\Omega_{m,2})} \leq c \left( \|u_+^{\varepsilon,ap} - u_+^{\varepsilon,(1)}\|_{H^1(\Omega_+)} + \varepsilon^2 \|u_+^1\|_{H^1(\Omega_+)} \right),$$

where  $c$  is a constant independent of  $\varepsilon$ , and convergence Theorem (Theorem 3), it suffices to estimate  $\|u_-^{\varepsilon,ap} - u_-^{\varepsilon,(1)}\|_{H^1(\Omega_-)}$  and  $\|u_+^{\varepsilon,ap} - u_+^{\varepsilon,(1)}\|_{H^1(\Omega_+)}$ . As in [9], we derive asymptotic expansions for  $u_-^{\varepsilon,ap}$  and  $u_+^{\varepsilon,ap}$  through the ansatz

$$u_-^{\varepsilon,ap} = \sum_{n \geq 0} \varepsilon^n w_-^n \quad \text{and} \quad u_+^{\varepsilon,ap} = \sum_{n \geq 0} \varepsilon^n w_+^n, \quad (64)$$

where the terms  $w_-^n$  and  $w_+^n$  are independent of  $\varepsilon$ . Inserting (64) in Problem (44) and identifying the same powers of  $\varepsilon$ , we get the following hierarchy of boundary-value problems

$$\begin{cases} \Delta w_+^n + k_+^2 w_+^n = 0 & \text{in } \Omega_+, \\ \Delta w_-^n + k_-^2 w_-^n = 0 & \text{in } \Omega_-, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) (w_+^n - \delta_{0,n} u_{inc}) = 0, \end{cases}$$

with transmission conditions on  $\Gamma$

$$\begin{pmatrix} [w^n] \\ [\alpha \partial_{\mathbf{n}} w^n] \end{pmatrix} = \begin{pmatrix} 2\mathcal{C}_1 Id & 0 \\ c(t) \left( f(t) - \frac{1}{f(t)} \right) Id & 2(\mathcal{D}_1 \partial_t^2 + \mathcal{D}_4 \partial_t + \mathcal{D}_2 Id) \end{pmatrix} \begin{pmatrix} \langle \alpha \partial_{\mathbf{n}} w^{n-1} \rangle \\ \langle w^{n-1} \rangle \end{pmatrix},$$

in which  $(\mathcal{C}_i)$  and  $(\mathcal{D}_i)_{1 \leq i \leq 4}$  are defined by formulas (45)-(49) with the convention that  $w_-^{-1} = w_+^{-1} = 0$ . A simple calculation shows that the terms  $(w_-^n, w_+^n)$  coincide with  $(u_-^n, u_+^n)$ , for  $n \in \{0, 1\}$ . Furthermore, each term in (64) is bounded in  $\mathbb{H}^1(\Omega)$  (cf., e.g., [9, Theorem 4.1]). Let  $R_w$  be the remainder obtained by truncating the asymptotic expansions (64) at order 3

$$\begin{aligned} R_w|_{\Omega_-} &= R_{w_-} = u_-^{\varepsilon,ap} - w_-^0 - \varepsilon w_-^1 - \varepsilon^2 w_-^2 - \varepsilon^3 w_-^3, \\ R_w|_{\Omega_+} &= R_{w_+} = u_+^{\varepsilon,ap} - w_+^0 - \varepsilon w_+^1 - \varepsilon^2 w_+^2 - \varepsilon^3 w_+^3. \end{aligned}$$

Hence  $R_w$  is the solution of the following problem

$$\begin{cases} \Delta R_{w_+} + k_+^2 R_{w_+} = 0 & \text{in } \Omega_+, \\ \Delta R_{w_-} + k_-^2 R_{w_-} = 0 & \text{in } \Omega_-, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) R_{w_+} = 0, \end{cases}$$

with transmission conditions on  $\Gamma$

$$\begin{aligned} R_{w_+} - R_{w_-} &= \varepsilon \mathcal{C}_1 (\alpha_+ \partial_{\mathbf{n}} R_{w_+} + \alpha_- \partial_{\mathbf{n}} R_{w_-}) + \varepsilon^4 \mathcal{C}_1 (\alpha_+ \partial_{\mathbf{n}} w_+^3 + \alpha_- \partial_{\mathbf{n}} w_-^3), \\ \alpha_+ \partial_{\mathbf{n}} R_{w_+} - \alpha_- \partial_{\mathbf{n}} R_{w_-} &= \varepsilon \mathcal{D}_1 (\partial_t^2 R_{w_+} + \partial_t^2 R_{w_-}) + \varepsilon \mathcal{D}_2 (R_{w_+} + R_{w_-}) \end{aligned}$$

$$\begin{aligned}
& + \mathcal{D}_3 (R_{w_+} - R_{w_-}) + \varepsilon \mathcal{D}_4 (\partial_t R_{w_+} + \partial_t R_{w_-}) \\
& + \mathcal{D}_1 \varepsilon^4 (\partial_t^2 w_+^3 + \partial_t^2 w_-^3) + \mathcal{D}_2 \varepsilon^4 (w_+^3 + w_-^3) \\
& + \mathcal{D}_4 \varepsilon^4 (\partial_t w_+^3 + \partial_t w_-^3).
\end{aligned}$$

So for all  $v = (v_-, v_+) \in \mathbb{H}^1(\Omega)$ , we get

$$\begin{aligned}
& \alpha_- \int_{\Omega_-} \nabla R_{w_-} \cdot \nabla v_- \, d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} R_{w_-} v_- \, d\Omega_- + \alpha_+ \int_{\tilde{\Omega}_+} \nabla R_{w_+} \cdot \nabla v_+ \, d\tilde{\Omega}_+ \\
& - \alpha_+ k_+^2 \int_{\tilde{\Omega}_+} R_{w_+} v_+ \, d\tilde{\Omega}_+ + \int_{\Gamma} \frac{1}{2\mathcal{C}_1 \varepsilon} [R_w] [v] \, d\Gamma \\
& + \alpha_+ \langle T R_{w_+}, v_+ \rangle_{H^{-1/2}(\Gamma_\infty) \times H^{1/2}(\Gamma_\infty)} = h_\varepsilon(v),
\end{aligned}$$

where

$$\begin{aligned}
h_\varepsilon(v) & = -2\varepsilon \int_{\Gamma} \mathcal{D}_1 \langle \partial_t^2 R_w \rangle \langle v \rangle \, d\Gamma - 2\varepsilon \int_{\Gamma} \mathcal{D}_2 \langle R_w \rangle \langle v \rangle \, d\Gamma \\
& - 2\varepsilon \int_{\Gamma} \mathcal{D}_4 \langle \partial_t R_w \rangle \langle v \rangle \, d\Gamma - 2\varepsilon \int_{\Gamma} \mathcal{C}_1 \mathcal{D}_3 \langle \alpha \partial_{\mathbf{n}} R_w \rangle \langle v \rangle \, d\Gamma \\
& - 2\varepsilon^4 \int_{\Gamma} \mathcal{D}_1 \langle \partial_t^2 w^3 \rangle \langle v \rangle \, d\Gamma - 2\varepsilon^4 \int_{\Gamma} \mathcal{D}_2 \langle w^3 \rangle \langle v \rangle \, d\Gamma \\
& - 2\varepsilon \int_{\Gamma} \mathcal{C}_1 \mathcal{D}_3 \langle \alpha \partial_{\mathbf{n}} w^3 \rangle \langle v \rangle \, d\Gamma - 2\varepsilon^4 \int_{\Gamma} \mathcal{D}_4 \langle \partial_t w^3 \rangle \langle v \rangle \, d\Gamma \\
& + \varepsilon^3 \int_{\Gamma} \langle \alpha \partial_{\mathbf{n}} w^3 \rangle [v] \, d\Gamma.
\end{aligned}$$

By using Lemma 10, we obtain

$$\|R_w\|_{\mathbb{H}^1(\Omega)} \leq c\varepsilon^{-1/2} \left[ \varepsilon \|R_w\|_{\mathbb{H}^1(\Omega)} + \varepsilon^3 \|w_3\|_{\mathbb{H}^1(\Omega)} \right],$$

where  $c$  is a positive constant independent of  $\varepsilon$ . Therefore

$$\|R_w\|_{\mathbb{H}^1(\Omega)} \leq \frac{c\varepsilon^{5/2}}{1 - c\varepsilon^{1/2}} \|w_3\|_{\mathbb{H}^1(\Omega)}.$$

Since  $\varepsilon$  is very small, it follows that

$$\|R_w\|_{\mathbb{H}^1(\Omega)} \leq c\varepsilon^2 \|w_3\|_{\mathbb{H}^1(\Omega)},$$

as we wished. ■

**Remark 11** *In Section 4.2, we explicitly calculated the first three terms of the asymptotic expansions (18), (19) and (21). This should have led us to determine a third-order model for a non-uniform thin layer. However, this was not achieved because we do not have a result on the existence and uniqueness of the solution for the third-order model, but this could be the subject of future work.*

*The calculation of the first three terms in Section 4.2 will be used in the next section where a third-order model will be determined for a uniform thin layer.*

## 5.2 The case of a thin layer with a constant thickness

In this paragraph, we consider the case where the thickness of the thin layer is constant, i.e.,  $f \equiv 1$ . We will derive approximate transmission conditions of order 3 and prove  $\varepsilon$ -uniform a priori estimate of the convergence of  $u^{\varepsilon,ap}$  to  $u^\varepsilon$ .

Once again, as in the previous paragraph, we define the approximate solution  $u^{\varepsilon,ap}$  by

$$u^{\varepsilon,ap} = \begin{cases} u_+^{\varepsilon,ap} & \text{in } \Omega_+^\varepsilon, \\ u_{m,\beta}^{\varepsilon,ap} & \text{in } \Omega_{m,\beta}^\varepsilon, \\ u_-^{\varepsilon,ap} & \text{in } \Omega_-, \end{cases}$$

where  $u_{m,\beta}^{\varepsilon,ap}$  is obtained from (39)-(43) by adding and subtracting a term of order 2 in  $\varepsilon$  and neglecting all terms in  $O(\varepsilon^3)$  and defined, for all  $x = \Phi_\beta(t, s) \in \Omega_{m,\beta}^\varepsilon$ , by

$$\begin{aligned} u_{m,1}^{\varepsilon,ap}(x) := & U_{m,1}^{\varepsilon,ap}(t, s) := u_-^{\varepsilon,ap}|_\Gamma + s\varepsilon\partial_{\mathbf{n}}u_-^{\varepsilon,ap}|_\Gamma - \frac{s^2k_-^2}{2}\varepsilon^2u_-^{\varepsilon,ap}|_\Gamma - \frac{s^2}{2}\varepsilon^2\partial_t^2u_-^{\varepsilon,ap}|_\Gamma \\ & - \frac{s^2}{2}\varepsilon^2c(t)\partial_{\mathbf{n}}u_-^{\varepsilon,ap}|_\Gamma, \end{aligned} \quad (65)$$

$$\begin{aligned} u_{m,2}^{\varepsilon,ap}(x) := & U_{m,2}^{\varepsilon,ap}(t, s) := u_+^{\varepsilon,ap}|_\Gamma + \varepsilon((1+d)(1-\alpha_+) + s\alpha_+)\partial_{\mathbf{n}}u_+^{\varepsilon,ap}|_\Gamma \\ & - \frac{2(s-1-d)(1+d)k_+^2\alpha_+ + (1+d)^2k_+^2 + (s-1-d)^2k_m^2}{2}\varepsilon^2u_+^{\varepsilon,ap}|_\Gamma \\ & - \frac{2(s-1-d)(1+d)\alpha_+ + (1+d)^2 + (s-1-d)^2}{2}\varepsilon^2\partial_t^2u_+^{\varepsilon,ap}|_\Gamma \\ & - \frac{2(s-1-d)(1+d)\alpha_+ + (1+d)^2 + (s-1-d)^2\alpha_+}{2}\varepsilon^2c(t)\partial_{\mathbf{n}}u_+^{\varepsilon,ap}|_\Gamma, \end{aligned} \quad (66)$$

and  $u_\infty^{\varepsilon,ap} := (u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$  is the solution to the following problem

$$\begin{cases} \Delta u_+^{\varepsilon,ap} + k_+^2 u_+^{\varepsilon,ap} = 0 & \text{in } \Omega_+, \\ \Delta u_-^{\varepsilon,ap} + k_-^2 u_-^{\varepsilon,ap} = 0 & \text{in } \Omega_-, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) (u_+^{\varepsilon,ap} - u_{inc}) = 0, \end{cases} \quad (67a)$$

with transmission conditions on  $\Gamma$

$$\begin{aligned} u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap} &= \varepsilon\mathcal{A}_1(\alpha_+\partial_{\mathbf{n}}u_+^{\varepsilon,ap} + \alpha_-\partial_{\mathbf{n}}u_-^{\varepsilon,ap}) + \varepsilon^2\mathcal{A}_2(\partial_t^2u_+^{\varepsilon,ap} + \partial_t^2u_-^{\varepsilon,ap}) \\ &+ \varepsilon\mathcal{A}_3(u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}) + \varepsilon^2\mathcal{A}_4(u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}), \end{aligned} \quad (67b)$$

$$\begin{aligned} \alpha_+\partial_{\mathbf{n}}u_+^{\varepsilon,ap} - \alpha_-\partial_{\mathbf{n}}u_-^{\varepsilon,ap} &= (\varepsilon\mathcal{B}_1 + \varepsilon^2\mathcal{B}_2)(\partial_t^2u_+^{\varepsilon,ap} + \partial_t^2u_-^{\varepsilon,ap}) \\ &+ (\varepsilon\mathcal{B}_3 + \varepsilon^2\mathcal{B}_4)(u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}) + \varepsilon\mathcal{B}_5(u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}) \\ &+ \varepsilon^2\mathcal{B}_6(\partial_tu_+^{\varepsilon,ap} + \partial_tu_-^{\varepsilon,ap}) + \varepsilon\mathcal{B}_7(\partial_t^2u_+^{\varepsilon,ap} - \partial_t^2u_-^{\varepsilon,ap}), \end{aligned} \quad (67c)$$

in which

$$\mathcal{A}_1 = \frac{\alpha_+\alpha_- - (1+d)\alpha_- + d\alpha_+}{2\alpha_+\alpha_-}, \quad (68)$$

$$\mathcal{A}_2 = \frac{-(1+d)\alpha_+^2\alpha_- - d\alpha_+\alpha_-^2 - d(1+d)\alpha_+^2 + d(1+d)\alpha_-^2 + d\alpha_+ + (1+d)\alpha_-}{4\alpha_+\alpha_-}, \quad (69)$$

$$\mathcal{A}_3 = c(t) \frac{(-1-2d)\alpha_+\alpha_- + (1+d)^2\alpha_- - d^2\alpha_+}{2(\alpha_+\alpha_- - (1+d)\alpha_- + d\alpha_+)}, \quad (70)$$

$$\begin{aligned} \mathcal{A}_4 &= \left( \frac{-(1+d)k_+^2\alpha_+^2\alpha_- - dk_-^2\alpha_+\alpha_-^2 - d(1+d)k_+^2\alpha_+^2 + d(1+d)k_-^2\alpha_-^2 + dk_m^2\alpha_+}{4\alpha_+\alpha_+} \right. \\ &\quad \left. + \frac{(1+d)k_m^2\alpha_-}{4\alpha_+\alpha_+} \right), \end{aligned} \quad (71)$$

$$\mathcal{B}_1 = \frac{(1+d)\alpha_+ - d\alpha_- - 1}{2}, \quad (72)$$

$$\mathcal{B}_2 = c(t) \frac{d^2\alpha_- - (1+d)^2\alpha_+ - 2d - 1}{4}, \quad (73)$$

$$\mathcal{B}_3 = \frac{(1+d)k_+^2\alpha_+ - dk_-^2\alpha_- - k_m^2}{2}, \quad (74)$$

$$\mathcal{B}_4 = c(t) \frac{(1+d)^2k_+^2\alpha_+ - d^2k_-^2\alpha_- - (1+2d)k_m^2}{4}, \quad (75)$$

$$\begin{aligned} \mathcal{B}_5 &= \left( \frac{(1+d)k_+^2\alpha_+^2\alpha_- + dk_-^2\alpha_+\alpha_-^2 + d(1+d)k_+^2\alpha_+^2 - d(1+d)k_-^2\alpha_-^2 - dk_m^2\alpha_+}{2(\alpha_+\alpha_- - (1+d)\alpha_- + d\alpha_+)} \right. \\ &\quad \left. - \frac{(1+d)k_m^2\alpha_-}{2(\alpha_+\alpha_- - (1+d)\alpha_- + d\alpha_+)} \right) = -\frac{\mathcal{A}_4}{\mathcal{A}_1}, \end{aligned} \quad (76)$$

$$\mathcal{B}_6 = c'(t) \frac{d^2\alpha_- - (1+d)^2\alpha_+ - 2d - 1}{4} = \partial_t \mathcal{B}_2, \quad (77)$$

$$\mathcal{B}_7 = \frac{(1+d)\alpha_+^2\alpha_- + d\alpha_+\alpha_-^2 + d(1+d)\alpha_+^2 - d(1+d)\alpha_-^2 - d\alpha_+ - (1+d)\alpha_-}{2(\alpha_+\alpha_- - (1+d)\alpha_- + d\alpha_+)} = -\frac{\mathcal{A}_2}{\mathcal{A}_1} \quad (78)$$

Note that  $\mathcal{A}_1 \neq 0$  since we have assumed that  $\alpha_{\pm} < 1 < \alpha_{\mp}$ .

### 5.2.1 Existence, Uniqueness and error estimate

The uniqueness of the solution of problem (67) is obtained in a different way to that of problem (44). It is based on standard arguments of the solution of this type of problems (cf., eg., [16]) and Rellich's lemma. However, the existence of the solution and error estimate are obtained in a similar way. We focus then on the proof of the uniqueness of the solution, Theorem 12 below, and we leave the existence theorem and uniform estimate for the reader.

**Theorem 12** *If  $u_{inc}$  is a plane wave, then Problem (67) admits at most one solution.*

**Proof.** Let us consider the homogeneous problem associated with Problem (67):

$$\begin{cases} \Delta u_+^{\varepsilon, ap} + k_+^2 u_+^{\varepsilon, ap} = 0 & \text{in } \Omega_+, \\ \Delta u_-^{\varepsilon, ap} + k_-^2 u_-^{\varepsilon, ap} = 0 & \text{in } \Omega_-, \end{cases} \quad (79)$$

$$\begin{cases} \Delta u_+^{\varepsilon, ap} + k_+^2 u_+^{\varepsilon, ap} = 0 & \text{in } \Omega_+, \\ \Delta u_-^{\varepsilon, ap} + k_-^2 u_-^{\varepsilon, ap} = 0 & \text{in } \Omega_-, \end{cases} \quad (80)$$

$$\begin{cases} \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_+) u_+^{\varepsilon, ap} = 0, \\ \lim_{|x| \rightarrow +\infty} \sqrt{|x|} (\partial_{|x|} - ik_-) u_-^{\varepsilon, ap} = 0, \end{cases} \quad (81)$$

with transmission conditions on  $\Gamma$

$$\begin{aligned} u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap} &= \varepsilon \mathcal{A}_1 (\alpha_+ \partial_{\mathbf{n}} u_+^{\varepsilon,ap} + \alpha_- \partial_{\mathbf{n}} u_-^{\varepsilon,ap}) + \varepsilon^2 \mathcal{A}_2 (\partial_t^2 u_+^{\varepsilon,ap} + \partial_t^2 u_-^{\varepsilon,ap}) \\ &+ \varepsilon \mathcal{A}_3 (u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}) + \varepsilon^2 \mathcal{A}_4 (u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}), \end{aligned} \quad (82)$$

$$\begin{aligned} \alpha_+ \partial_{\mathbf{n}} u_+^{\varepsilon,ap} - \alpha_- \partial_{\mathbf{n}} u_-^{\varepsilon,ap} &= (\varepsilon \mathcal{B}_1 + \varepsilon^2 \mathcal{B}_2) (\partial_t^2 u_+^{\varepsilon,ap} + \partial_t^2 u_-^{\varepsilon,ap}) \\ &+ (\varepsilon \mathcal{B}_3 + \varepsilon^2 \mathcal{B}_4) (u_+^{\varepsilon,ap} + u_-^{\varepsilon,ap}) + \varepsilon \mathcal{B}_5 (u_+^{\varepsilon,ap} - u_-^{\varepsilon,ap}) \\ &+ \varepsilon^2 \mathcal{B}_6 (\partial_t u_+^{\varepsilon,ap} + \partial_t u_-^{\varepsilon,ap}) + \varepsilon \mathcal{B}_7 (\partial_t^2 u_+^{\varepsilon,ap} - \partial_t^2 u_-^{\varepsilon,ap}). \end{aligned} \quad (83)$$

Standard regularity results for elliptic problems (cf., e.g., [3]) ensure that  $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap}) \in \mathcal{C}^\infty(\overline{\Omega_-}) \times \mathcal{C}^\infty(\overline{\Omega_+})$ . Let  $B_R$  be the ball with centre  $O$  and radius  $R$  sufficiently large to enclose  $\Omega_-$  and let  $\Omega_R$  be the domain of  $\mathbb{R}^2$  defined by  $\Omega_R := B_R \cap \Omega_+$ . Multiplying Eqs. (79) and (80) by  $\overline{u_+^{\varepsilon,ap}}$  and  $\overline{u_-^{\varepsilon,ap}}$  respectively and using Green's formula, we get

$$\begin{aligned} &\alpha_- \int_{\Omega_-} |\nabla u_-^{\varepsilon,ap}|^2 d\Omega_- - \alpha_- k_-^2 \int_{\Omega_-} |u_-^{\varepsilon,ap}|^2 d\Omega_- + \alpha_+ \int_{\Omega_R} |\nabla u_+^{\varepsilon,ap}|^2 d\Omega_R \\ &- \alpha_+ k_+^2 \int_{\Omega_R} |u_+^{\varepsilon,ap}|^2 d\Omega_R - \int_{\Gamma} \frac{\varepsilon \mathcal{A}_3 - 1}{2\varepsilon \mathcal{A}_1} |[u^{\varepsilon,ap}]|^2 d\Gamma \\ &+ 2 \int_{\Gamma} (\varepsilon \mathcal{B}_3 + \varepsilon^2 \mathcal{B}_4) |\langle u^{\varepsilon,ap} \rangle|^2 d\Gamma - 2 \int_{\Gamma} (\varepsilon \mathcal{B}_1 + \varepsilon^2 \mathcal{B}_2) |\langle \partial_t u^{\varepsilon,ap} \rangle|^2 d\Gamma \\ &+ \varepsilon \mathcal{B}_5 \int_{\Gamma} (|u_+^{\varepsilon,ap}|^2 - |u_-^{\varepsilon,ap}|^2) d\Gamma - \varepsilon \mathcal{B}_7 \int_{\Gamma} (|\partial_t u_+^{\varepsilon,ap}|^2 - |\partial_t u_-^{\varepsilon,ap}|^2) d\Gamma \\ &= \alpha_+ \int_{S_R} \partial_{\mathbf{R}} u_+^{\varepsilon,ap} \overline{u_+^{\varepsilon,ap}} dS_R, \end{aligned} \quad (84)$$

where  $S_R$  denotes the circle with centre  $O$  and radius  $R$ . Taking the imaginary part of (84), we obtain

$$\Im \left( \int_{S_R} \partial_{\mathbf{R}} u_+^{\varepsilon,ap} \overline{u_+^{\varepsilon,ap}} dS_R \right) = 0.$$

It follows from radiation condition (81) and Rellich's lemma [19] that  $u_+^{\varepsilon,ap} = 0$  in  $\Omega_+$ . Problem (79)-(83) is thus reduced to

$$\Delta u_-^{\varepsilon,ap} + k_-^2 u_-^{\varepsilon,ap} = 0 \quad \text{in } \Omega_-,$$

with the following transmission conditions on  $\Gamma$

$$\begin{aligned} u_-^{\varepsilon,ap} &= -\varepsilon \mathcal{A}_1 (\alpha_- \partial_{\mathbf{n}} u_-^{\varepsilon,ap}) - \varepsilon^2 \mathcal{A}_2 (\partial_t^2 u_-^{\varepsilon,ap}) + \varepsilon \mathcal{A}_3 (u_-^{\varepsilon,ap}) - \varepsilon^2 \mathcal{A}_4 (u_-^{\varepsilon,ap}), \\ \alpha_- \partial_{\mathbf{n}} u_-^{\varepsilon,ap} &= - (\varepsilon \mathcal{B}_1 + \varepsilon^2 \mathcal{B}_2) (\partial_t^2 u_-^{\varepsilon,ap}) - (\varepsilon \mathcal{B}_3 + \varepsilon^2 \mathcal{B}_4) (u_-^{\varepsilon,ap}) + \varepsilon \mathcal{B}_5 (u_-^{\varepsilon,ap}) \\ &- \varepsilon^2 \mathcal{B}_6 \partial_t u_-^{\varepsilon,ap} + \varepsilon \mathcal{B}_7 (\partial_t^2 u_-^{\varepsilon,ap}). \end{aligned}$$

This implies that

$$\int_{\Gamma} \left( \frac{1}{\mathcal{A}_1} - \varepsilon \frac{\mathcal{A}_3}{\mathcal{A}_1} - \varepsilon^2 \mathcal{B}_3 - \varepsilon^3 \mathcal{B}_4 \right) |u_-^{\varepsilon,ap}|^2 d\Gamma + \int_{\Gamma} (\varepsilon^2 \mathcal{B}_1 + \varepsilon^3 \mathcal{B}_2) |\partial_t u_-^{\varepsilon,ap}|^2 d\Gamma = 0. \quad (85)$$



As  $\alpha_{\pm} < 1 < \alpha_{\mp}$ , then  $\mathcal{A}_1\mathcal{B}_1 > 0$ , and since  $\varepsilon$  is small enough, there exists a positive constant  $C$  such as

$$\begin{aligned} C \left( \|u_-^{\varepsilon,ap}\|_{L^2(\Gamma)}^2 + \|\partial_t u_-^{\varepsilon,ap}\|_{L^2(\Gamma)}^2 \right) &\leq \int \left| \frac{1}{\mathcal{A}_1} - \varepsilon \frac{\mathcal{A}_3}{\mathcal{A}_1} - \varepsilon^2 \mathcal{B}_3 - \varepsilon^3 \mathcal{B}_4 \right| |u_-^{\varepsilon,ap}|^2 d\Gamma \\ &+ \varepsilon^2 \int_{\Gamma} |\mathcal{B}_1 + \varepsilon \mathcal{B}_2| |\partial_t u_-^{\varepsilon,ap}|^2 d\Gamma \\ &= 0. \end{aligned}$$

This leads to  $u_-^{\varepsilon,ap}|_{\Gamma} = 0$ . Then, we obtain

$$\begin{cases} \Delta u_-^{\varepsilon,ap} + k_-^2 u_-^{\varepsilon,ap} = 0 & \text{in } \Omega_-, \\ u_-^{\varepsilon,ap} = 0 & \text{on } \Gamma, \\ \partial_{\mathbf{n}} u_-^{\varepsilon,ap} = 0 & \text{on } \Gamma. \end{cases}$$

Well-know arguments of uniqueness of the solution of this type of problems (cf. [16]) leads to  $u_-^{\varepsilon,ap} = 0$  in  $\Omega_-$ , which proves the uniqueness of the solution  $(u_-^{\varepsilon,ap}, u_+^{\varepsilon,ap})$ . ■

Finally, we have  $\varepsilon$ -uniform estimate of the convergence of the approximate solution  $u^{\varepsilon,ap}$  to  $u^{\varepsilon}$  which proof follows the same lines as the one of Theorem 9.

**Theorem 13** *If  $u_{inc}$  is a plane wave, then there exists a constant  $c$  independent of  $\varepsilon$  such that*

$$\|u_-^{\varepsilon} - u_-^{\varepsilon,ap}\|_{H^1(\Omega_-)} + \varepsilon^{1/2} \|u_{m,1}^{\varepsilon} - u_{m,1}^{\varepsilon,ap}\|_{H^1(\Omega_{m,1}^{\varepsilon})} + \varepsilon^{1/2} \|u_{m,2}^{\varepsilon} - u_{m,2}^{\varepsilon,ap}\|_{H^1(\Omega_{m,2}^{\varepsilon})} + \|u_+^{\varepsilon} - u_+^{\varepsilon,ap}\|_{H^1(\tilde{\Omega}_+^{\varepsilon})} \leq c\varepsilon^3.$$

## 6 Conclusion and perspectives

In this work, we have derived and justified an asymptotic expansion of the exact solution of problem (3) defined in a domain of  $\mathbb{R}^2$  with a thin layer having a variable thickness. We have also modeled the effect of the thin layer with Ventcel-type transmission conditions in two different situations: first, the case with a thin layer with a variable thickness with accuracy up to  $O(\varepsilon^2)$  and the second, with a constant thickness with accuracy up to  $O(\varepsilon^3)$ .

Interesting aspects left open in this work, might be considered in forthcoming publications. First, there is the question of confirming our results obtained here by numerical simulations. Then, investigating Ventcel-type transmission conditions for others problems like the Maxwell or elasticity equations and elasto-acoustic couplings.

## A Calculation of the first three terms of the asymptotic expansion

### A.1 Term of order 0

Eq. (28) and conditions (25) and (27) give

$$\partial_s U_{m,\beta}^0 = 0, \quad \beta \in \{1, 2\}.$$

Using (22), (23) and (24), we get  $\forall(t, s) \in \Omega_{m,\beta}$

$$U_{m,\beta}^0(t, s) = u_-^0|_\Gamma = u_+^0|_\Gamma. \quad (86)$$

Eq.(29) implies

$$\partial_s^2 U_{m,\beta}^1 = A_{\beta,1} U_{m,\beta}^0 = 0.$$

Using (25), (26) and (27), we obtain  $\forall(t, s) \in \Omega_{m,\beta}$

$$\partial_s U_{m,2}^1(t, s) = f(t)\alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma, \quad \partial_s U_{m,1}^1(t, s) = \partial_{\mathbf{n}} u_-^0|_\Gamma \quad \text{and} \quad \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma = \alpha_- \partial_{\mathbf{n}} u_-^0|_\Gamma. \quad (87)$$

## A.2 Term of order 1

Relation (87) with conditions (22) and (24) yield  $\forall(t, s) \in \Omega_{m,\beta}$

$$\begin{aligned} U_{m,1}^1(t, s) &= u_-^1|_\Gamma + (\partial_{\mathbf{n}} u_-^0|_\Gamma) s, \\ U_{m,2}^1(t, s) &= u_+^1|_\Gamma + (d + f(t) - f(t)\alpha_+ + sf(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma. \end{aligned}$$

So (23) gives

$$u_+^1|_\Gamma - u_-^1|_\Gamma = \frac{f(t)\alpha_+\alpha_- - (d + f(t))\alpha_- + d\alpha_+}{2\alpha_+\alpha_-} (\alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma + \alpha_- \partial_{\mathbf{n}} u_-^0|_\Gamma). \quad (88)$$

From (30) and (31), we have

$$\begin{aligned} \partial_s^2 U_{m,1}^2 &= A_{1,1} U_{m,1}^1 + A_{1,2} U_{m,1}^0 - k_-^2 U_{m,1}^0 = -c(t) \partial_{\mathbf{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma - k_-^2 u_-^0|_\Gamma, \\ \partial_s^2 U_{m,2}^2 &= A_1 U_{m,2}^1 + A_2 U_{m,2}^0 - k_m^2 U_{m,2}^0 = -c(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma - \partial_t^2 u_+^0|_\Gamma - k_m^2 u_+^0|_\Gamma. \end{aligned}$$

Using (25) and (27), we obtain  $\forall(t, s) \in \Omega_{m,\beta}$

$$\partial_s U_{m,1}^2(t, s) = [-c(t) \partial_{\mathbf{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma - k_-^2 u_-^0|_\Gamma] s + \partial_{\mathbf{n}} u_-^1|_\Gamma. \quad (89)$$

$$\begin{aligned} \partial_s U_{m,2}^2(t, s) &= [-c(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma - \partial_t^2 u_+^0|_\Gamma - k_m^2 u_+^0|_\Gamma] (s - 1) + f(t) \alpha_+ \partial_{\mathbf{n}} u_+^1|_\Gamma \\ &\quad - c(t) f(t) f'(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma + f(t) f'(t) \partial_t u_+^0|_\Gamma + (d + f(t)) f(t) \alpha_+ \partial_{\mathbf{n}}^2 u_+^0|_\Gamma. \end{aligned} \quad (90)$$

Now, from Condition (26) at order 1, we get

$$\begin{aligned} \alpha_+ \partial_{\mathbf{n}} u_+^1|_\Gamma - \alpha_- \partial_{\mathbf{n}} u_-^1|_\Gamma &= \frac{c(t)(f'(t) - df(t) - 1)}{f(t)} \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma - \frac{df(t)\alpha_- + 1}{f(t)} \partial_t^2 u_+^0|_\Gamma \\ &\quad - \frac{dk_-^2 \alpha_- + k_m^2}{f(t)} u_+^0|_\Gamma - f'(t) \partial_t u_+^0|_\Gamma - (d + f(t)) \alpha_+ \partial_{\mathbf{n}}^2 u_+^0|_\Gamma. \end{aligned} \quad (91)$$

As

$$\begin{aligned} \Delta u_+ &= \left[ \frac{(1 + \eta c(t)(d + f(t))^2 + \eta^2 f'^2(t))}{(d + f(t))^2 (1 + \eta c(t)(d + f(t))^2)} \partial_\eta^2 - \frac{2\eta f'(t)}{(d + f(t))(1 + \eta c(t)(d + f(t))^2)} \partial_t \partial_\eta \right. \\ &\quad \left. + \frac{1}{(1 + \eta c(t)(d + f(t))^2} \partial_t^2 - \frac{\varepsilon dc'(t) + \eta c'(t) f(t)}{(1 + \eta c(t)(d + f(t))^3} \partial_t + \left( \frac{c(t)}{(d + f(t))(1 + \eta c(t)(d + f(t))} \right) \right. \end{aligned}$$

$$+ \left. \frac{\eta(-f(t)f''(t) + 2f'(t)^2)}{(d+f(t))^2(1+\eta c(t)(d+f(t))^2)} + \frac{\eta \varepsilon d c'(t) f(t) f'(t) + \eta^2 c'(t) f^2(t) f'(t)}{(d+f(t)) f^2(1+\eta c(t)(d+f(t))^3)} \right] \partial_\eta \tilde{u}_+.$$

and

$$\partial_\eta u_+(t, \eta)|_{\eta=0} = (d+f(t)) \partial_n u_+|_\Gamma,$$

it follows by taking the limit  $\eta \rightarrow 0$ , we obtain

$$\partial_t^2 u_+|_{\eta=0} + \frac{c(t)}{d+f(t)} \partial_\eta u_+^n|_{\eta=0} + \frac{1}{(d+f(t))^2} \partial_\eta^2 u_+^n|_{\eta=0} + k_+^2 u_+|_{\eta=0} = 0,$$

i.e.

$$\partial_{\mathbf{n}}^2 u_+|_\Gamma = -\partial_t^2 u_+|_\Gamma - c(t) \partial_n u_+|_\Gamma - k_+^2 u_+|_\Gamma. \quad (92)$$

So (91) becomes

$$\begin{aligned} \alpha_+ \partial_{\mathbf{n}} u_+^1 - \alpha_- \partial_{\mathbf{n}} u_-^1 &= \frac{(d+f(t))f(t)\alpha_+ - df(t)\alpha_- - 1}{2f(t)} (\partial_t^2 u_+^0 + \partial_t^2 u_-^0) \\ &+ \frac{(d+f(t))f(t)k_+^2\alpha_+ - df(t)k_-^2\alpha_- - k_m^2}{2f(t)} (u_+^0 + u_-^0) \\ &+ \frac{c(t)(f^2(t) - 1)}{2f(t)} (\alpha_+ \partial_n u_+^0 + \alpha_- \partial_n u_-^0) \\ &+ \frac{f'(t)(\alpha_+ - 1)}{2} (\partial_t u_+^0 + \partial_t u_-^0). \end{aligned} \quad (93)$$

### A.3 Term of order 2

Relations (89) and (90) with conditions (22) and (24) yield  $\forall(t, s) \in \Omega_{m,\beta}$

$$U_{m,1}^2(t, s) = [-c(t) \partial_{\mathbf{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma - k_-^2 u_-^0|_\Gamma] \frac{s^2}{2} + \partial_{\mathbf{n}} u_-^1|_\Gamma s + u_-^2|_\Gamma. \quad (94)$$

$$\begin{aligned} U_{m,2}^2(t, s) &= [-c(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma - \partial_t^2 u_+^0|_\Gamma - k_m^2 u_+^0|_\Gamma] \frac{s^2 - 2s + 1}{2} + [f(t) \alpha_+ \partial_n u_+^1|_\Gamma \\ &\quad - c(t) f(t) f'(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma + f(t) f'(t) \partial_t u_+^0|_\Gamma + (d+f(t)) f(t) \alpha_+ \partial_{\mathbf{n}}^2 u_+^0|_\Gamma] (s-1) \\ &\quad + (d+f(t)) \partial_{\mathbf{n}} u_+^1|_\Gamma + \frac{(d+f(t))^2}{2} \partial_{\mathbf{n}}^2 u_+^0|_\Gamma + u_+^2|_\Gamma. \end{aligned} \quad (95)$$

So (23) with (92) and (93) we get

$$\begin{aligned} u_+^2|_\Gamma - u_-^2|_\Gamma &= -\frac{1}{2} [-c(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma - \partial_t^2 u_+^0|_\Gamma - k_m^2 u_+^0|_\Gamma] + [f(t) \alpha_+ \partial_n u_+^1|_\Gamma \\ &\quad - c(t) f(t) f'(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma + f(t) f'(t) \partial_t u_+^0|_\Gamma + (d+f(t)) f(t) \alpha_+ \partial_{\mathbf{n}}^2 u_+^0|_\Gamma] \\ &\quad - (d+f(t)) \partial_{\mathbf{n}} u_+^1|_\Gamma - \frac{(d+f(t))^2}{2} \partial_{\mathbf{n}}^2 u_+^0|_\Gamma + [-c(t) \partial_{\mathbf{n}} u_-^0|_\Gamma - \partial_t^2 u_-^0|_\Gamma \\ &\quad - k_-^2 u_-^0|_\Gamma] \frac{d^2}{2} + d \partial_n u_-^1|_\Gamma. \end{aligned} \quad (96)$$

$$\begin{aligned}
u_+^2|_\Gamma - u_-^2|_\Gamma &= \frac{f(t)\alpha_+\alpha_- + d\alpha_+ - (d+f(t))\alpha_-}{\alpha_+\alpha_-} \alpha_+ \partial_{\mathbf{n}} u_+^1|_\Gamma \\
&+ \left( \frac{(f(t) - 2f^2(t)f'(t) - 2df^2(t) - 2f^3(t))\alpha_+\alpha_- - (d^2f(t) + 2df^2(t) - 2d)\alpha_+}{2f(t)\alpha_+\alpha_-} \right. \\
&+ \left. \frac{(d+f(t))^2f(t)\alpha_-}{2f(t)\alpha_+\alpha_-} \right) c(t)\alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma + \left( \frac{-2(d+f(t))f^2(t)\alpha_+^2\alpha_-}{2f(t)\alpha_+\alpha_-} \right. \\
&+ \left. \frac{-2d(d+f(t))f(t)\alpha_+^2 + (f(t) + 2d^2f(t) + f^3(t) + 2df^2(t))\alpha_+\alpha_- + 2d\alpha_+}{2f(t)\alpha_+\alpha_-} \right) \\
&\times \partial_t^2 u_+^0|_\Gamma + \left( \frac{-2(d+f(t))f^2(t)k_+^2\alpha_+^2\alpha_- - 2d(d+f(t))f(t)k_+^2\alpha_+^2}{2f(t)\alpha_+\alpha_-} \right. \\
&+ \left. \frac{+[k_m^2 + d^2k_-^2 + (d+f(t))^2k_+^2]f(t)\alpha_+\alpha_- + 2dk_m^2\alpha_+}{2f(t)\alpha_+\alpha_-} \right) u_+^0|_\Gamma \\
&+ \frac{f(t)\alpha_+\alpha_- - d\alpha_+^2 + d\alpha_+}{\alpha_+\alpha_-} f'(t)\partial_t u_+^0|_\Gamma. \tag{97}
\end{aligned}$$

Using (93) we get

$$\begin{aligned}
u_+^2|_\Gamma - u_-^2|_\Gamma &= \frac{f(t)\alpha_+\alpha_- + d\alpha_+ - (d+f(t))\alpha_-}{\alpha_+\alpha_-} \alpha_- \partial_{\mathbf{n}} u_-^1|_\Gamma \\
&+ \left( \frac{(-f(t) - 2f^2(t)f'(t) - 2df^2(t))\alpha_+\alpha_- + (-f^3(t) + d^2f(t) + 2f(t) + 2d)\alpha_-}{2f(t)\alpha_+\alpha_-} \right. \\
&- \left. \frac{d^2f(t)\alpha_+}{2f(t)\alpha_+\alpha_-} \right) c(t)\alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma + \left( \frac{-2df^2(t)\alpha_+\alpha_-^2 + 2df(t)(d+f(t))\alpha_-^2}{2f(t)\alpha_+\alpha_-} \right. \\
&+ \left. \frac{-(f(t) + 2d^2f(t) + f^3(t) + 2df^2(t))\alpha_+\alpha_- + 2(d+f(t))\alpha_-}{2f(t)\alpha_+\alpha_-} \right) \partial_t^2 u_+^0|_\Gamma \\
&+ \left( \frac{-2df^2(t)k_-^2\alpha_+\alpha_-^2 + 2d(d+f(t))f(t)k_-^2\alpha_-^2}{2f(t)\alpha_+\alpha_-} \right. \\
&+ \left. \frac{-[k_m^2 + d^2k_-^2 + (d+f(t))^2k_+^2]f(t)\alpha_+\alpha_- + 2(d+f(t))k_m^2\alpha_-}{2f(t)\alpha_+\alpha_-} \right) u_+^0|_\Gamma \\
&+ \frac{f(t)\alpha_+^2\alpha_- - (d+f(t))\alpha_+\alpha_- + (d+f(t))\alpha_-}{\alpha_+\alpha_-} f'(t)\partial_t u_+^0|_\Gamma. \tag{98}
\end{aligned}$$

Then

$$\begin{aligned}
u_+^2|_\Gamma - u_-^2|_\Gamma &= \frac{f(t)\alpha_+\alpha_- + d\alpha_+ - (d+f(t))\alpha_-}{2\alpha_+\alpha_-} (\alpha_+ \partial_{\mathbf{n}} u_+^1|_\Gamma + \alpha_- \partial_{\mathbf{n}} u_-^1|_\Gamma) \\
&+ \left( \frac{(-2f^2(t)f'(t) - 2df^2(t) - f^3(t))\alpha_+\alpha_- + (d - d^2f(t) - df^2(t))\alpha_+}{4f(t)\alpha_+\alpha_-} \right. \\
&- \left. \frac{(d+f(t) + d^2f(t) + df^2(t))\alpha_-}{4f(t)\alpha_+\alpha_-} \right) c(t) (\alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma + \alpha_- \partial_{\mathbf{n}} u_-^0|_\Gamma) \\
&+ \left( \frac{-(d+f(t))f^2(t)\alpha_+^2\alpha_- - df^2(t)\alpha_+\alpha_-^2 - df(t)(d+f(t))(\alpha_+^2 - \alpha_-^2)}{4f(t)\alpha_+\alpha_-} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{d\alpha_+ + (d + f(t))\alpha_-}{4f(t)\alpha_+\alpha_-} \left( \partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma \right) \\
& + \left( \frac{-(d + f(t))f^2(t)k_+^2\alpha_+^2\alpha_- - df^2(t)k_-^2\alpha_+\alpha_-^2 - d(d + f(t))f(t)(k_+^2\alpha_+^2 - k_-^2\alpha_-^2)}{4f(t)\alpha_+\alpha_-} \right. \\
& \left. + \frac{dk_m^2\alpha_+ + (d + f(t))k_m^2\alpha_-}{4f(t)\alpha_+\alpha_-} \right) (u_+^0|_\Gamma + u_-^0|_\Gamma) \\
& + \frac{f(t)\alpha_+^2\alpha_- - d\alpha_+^2 - d\alpha_+\alpha_- + d\alpha_+ + (d + f(t))\alpha_-}{4\alpha_+\alpha_-} f'(t) (\partial_t u_+^0|_\Gamma + \partial_t u_-^0|_\Gamma). \quad (99)
\end{aligned}$$

Equations (32) et (33) implique that,  $\forall(t, s) \in \Omega_{m,\beta}$ ,

$$\begin{aligned}
\partial_s U_{m,1}^3(t, s) &= [c^2 \partial_{\mathbf{n}} u_-^0|_\Gamma + c \partial_t^2 u_-^0|_\Gamma + ck_-^2 u_-^0|_\Gamma - k_-^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma) - \partial_t^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma)] \frac{s^2}{2} \\
&+ [c^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma) + 2c(t) \partial_t^2 u_-^0|_\Gamma + c'(t) \partial_t u_-^0|_\Gamma] \frac{s^2}{2} \\
&+ (-\partial_t^2 u_-^1|_\Gamma - k_-^2 u_-^1|_\Gamma - c \partial_{\mathbf{n}} u_-^1|_\Gamma) s + \partial_s U_{m,1}^3(t, 0),
\end{aligned}$$

$$\begin{aligned}
\partial_s U_{m,2}^3(t, s) &= -\frac{c(t)}{f(t)} [-c(t)\alpha_+ \partial_{\mathbf{n}} u_+^0|_\Gamma - \partial_t^2 u_+^0|_\Gamma - k_m^2 u_+^0|_\Gamma] \left( \frac{1}{2} s^2 - s + \frac{1}{2} \right) \\
&+ [f'(t)f^{-1}(t) \partial_t ((f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma) 2 - \partial_t^2 [(f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma] \\
&- f'^2(t)f^{-2}(t) (f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma 2 + f''(t)f^{-1}(t) (f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma + c^2(t) (f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma \\
&+ c(t)f(t) \partial_t^2 u_+^0|_\Gamma 2 - f(t)c'(t) \partial_t u_+^0|_\Gamma - k_m^2 (f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma] \left( \frac{1}{2} s^2 - \frac{1}{2} \right) \\
&+ [dc^2 f^{-1} (f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma - \partial_t^2 u_+^1|_\Gamma - \partial_t^2 [(d + f(t) - f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma] \\
&- c(t)\alpha_+ \partial_{\mathbf{n}} u_+^1|_\Gamma - c(t)(d + f(t))\alpha_+ \partial_{\mathbf{n}}^2 u_+^0|_\Gamma + c(t)f'(t)\alpha_+ \partial_t u_+^0|_\Gamma - c(t)f'(t) \partial_t u_+^0|_\Gamma \\
&+ 2dc(t) \partial_t^2 u_+^0|_\Gamma - k_m^2 u_+^1|_\Gamma - k_m^2 (d + f(t) - f(t)\alpha_+) \partial_{\mathbf{n}} u_+^0|_\Gamma - dc'(t) \partial_t u_+^0|_\Gamma] (s - 1) \\
&+ \partial_s U_{m,2}^3(t, 1)
\end{aligned}$$

From Condition (27) at order 2, we get

$$\begin{aligned}
\partial_s U_{m,1}^3(t, s) &= [c^2 \partial_{\mathbf{n}} u_-^0|_\Gamma + c \partial_t^2 u_-^0|_\Gamma + ck_-^2 u_-^0|_\Gamma - k_-^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma) - \partial_t^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma)] \frac{s^2}{2} \\
&+ [c^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma) + 2c(t) \partial_t^2 u_-^0|_\Gamma + c'(t) \partial_t u_-^0|_\Gamma] \frac{s^2}{2} \\
&+ (-\partial_t^2 u_-^1|_\Gamma - k_-^2 u_-^1|_\Gamma - c \partial_{\mathbf{n}} u_-^1|_\Gamma) s + \partial_{\mathbf{n}} u_-^2|_\Gamma,
\end{aligned}$$

It follows from Transmission conditions (26) at order 2

$$\begin{aligned}
& \frac{d^2}{2} \alpha_- c^2 \partial_{\mathbf{n}} u_-^0|_\Gamma + \frac{d^2}{2} \alpha_- c \partial_t^2 u_-^0|_\Gamma + \frac{d^2}{2} \alpha_- ck_-^2 u_-^0|_\Gamma - \frac{d^2}{2} \alpha_- k_-^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma) - \frac{d^2}{2} \alpha_- \partial_t^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma) \\
& + \frac{d^2}{2} \alpha_- c^2 (\partial_{\mathbf{n}} u_-^0|_\Gamma) + d^2 \alpha_- c(t) \partial_t^2 u_-^0|_\Gamma + \frac{d^2}{2} \alpha_- c'(t) \partial_t u_-^0|_\Gamma \\
& - d\alpha_- \partial_t^2 u_-^1|_\Gamma - d\alpha_- k_-^2 u_-^1|_\Gamma - d\alpha_- c \partial_{\mathbf{n}} u_-^1|_\Gamma + \alpha_- \partial_{\mathbf{n}} u_-^2|_\Gamma
\end{aligned}$$

$$\begin{aligned}
&= \frac{c}{2f^2}c(t)\alpha_+\partial_{\mathbf{n}}u_+^0|_{\Gamma} + \frac{c}{2f^2}\partial_t^2u_+^0|_{\Gamma} + \frac{c}{2f^2}k_m^2u_-^0|_{\Gamma} \\
&\quad - \frac{1}{f}f'(t)f^{-1}(t)\partial_t((f(t)\alpha_+)\partial_{\mathbf{n}}u_+^0|_{\Gamma}) + \frac{1}{2}\frac{1}{f}\partial_t^2[(f(t)\alpha_+)\partial_{\mathbf{n}}u_+^0|_{\Gamma}] \\
&\quad + \frac{1}{f}f'^2(t)f^{-2}(t)(f(t)\alpha_+)\partial_{\mathbf{n}}u_+^0|_{\Gamma} - \frac{1}{2}f''(t)f^{-1}(t)\alpha_+\partial_{\mathbf{n}}u_+^0|_{\Gamma} - \frac{1}{2}c^2(t)\alpha_+\partial_{\mathbf{n}}u_+^0|_{\Gamma} \\
&\quad - c(t)\partial_t^2u_+^0|_{\Gamma} + \frac{1}{2}c'(t)\partial_tu_+^0|_{\Gamma} + \frac{1}{2}k_m^2\alpha_+\partial_{\mathbf{n}}u_+^0|_{\Gamma} \\
&\quad - dc^2f^{-1}\alpha_+\partial_{\mathbf{n}}u_+^0|_{\Gamma} + \frac{1}{f}\partial_t^2u_+^1|_{\Gamma} + \frac{1}{f}\partial_t^2[(d+f(t)-f(t)\alpha_+)\partial_{\mathbf{n}}u_+^0|_{\Gamma}] \\
&\quad + \frac{1}{f}c(t)\alpha_+\partial_{\mathbf{n}}u_+^1|_{\Gamma} + \frac{1}{f}c(t)(d+f(t))\alpha_+\partial_{\mathbf{n}}^2u_+^0|_{\Gamma} - \frac{1}{f}c(t)f'(t)\alpha_+\partial_tu_+^0|_{\Gamma} + \frac{1}{f}c(t)f'(t)\partial_tu_+^0|_{\Gamma} \\
&\quad - \frac{1}{f}2dc(t)\partial_t^2u_+^0|_{\Gamma} + \frac{1}{f}k_m^2u_+^1|_{\Gamma} + k_m^2(d+f(t)-f(t)\alpha_+)\frac{1}{f}\partial_{\mathbf{n}}u_+^0|_{\Gamma} + \frac{1}{f}dc'(t)\partial_tu_+^0|_{\Gamma} \\
&\quad + \frac{1}{f}\partial_sU_{m,2}^3(t,1). \tag{100}
\end{aligned}$$

Now calculate the term  $\partial_sU_{m,2}^3(t,1)$  in (100). From (25) at order 2, we have

$$\begin{aligned}
&\frac{1}{f(t)}\partial_sU_{m,2}^3(t,1) \\
&= (f'(t))^2\partial_{\mathbf{n}}u_+^0|_{\Gamma} + 2c(t)f'(t)(d+f(t))(\alpha_+-1)\partial_tu_+^0|_{\Gamma} \\
&\quad + \alpha_+\partial_{\mathbf{n}}u_+^2|_{\Gamma} + (1-\alpha_+)f'(t)\partial_tu_+^1|_{\Gamma} + f'(t)(d+f(t))(1-\alpha_+)\partial_t\partial_{\mathbf{n}}u_+^0|_{\Gamma} \\
&\quad + \frac{(d+f(t))^2}{2}\alpha_+\partial_{\mathbf{n}}^3u_+^0|_{\Gamma} + (d+f(t))\alpha_+\partial_{\mathbf{n}}^2u_+^1|_{\Gamma}. \tag{101}
\end{aligned}$$

Recall that

$$\Delta u_+^n + k_+^2u_+^n = \frac{-\eta c'(t)}{[1+\eta c(t)]^3}\partial_tu_+^n + \frac{1}{[1+\eta c(t)]^2}\partial_t^2u_+^n + \frac{c(t)}{1+\eta c(t)}\partial_{\eta}u_+^n + \partial_{\eta}^2u_+^n + k_+^2u_+^n = 0.$$

The partial derivative with respect to  $\eta$  leads to

$$\begin{aligned}
&\frac{-c'(t)[1+\eta c(t)]^3 + 3c(t)[1+\eta c(t)]^2\eta c'}{[1+\eta c(t)]^6}\partial_tu_+^n + \frac{\eta c'(t)}{[1+\eta c(t)]^3}\partial_{\eta}\partial_tu_+^n \\
&- \frac{2c}{[1+\eta c(t)]^3}\partial_t^2u_+^n + \frac{1}{[1+\eta c(t)]^2}\partial_{\eta}\partial_t^2u_+^n - \frac{c^2(t)}{[1+\eta c(t)]^2}\partial_{\eta}u_+^n + \frac{c(t)}{1+\eta c(t)}\partial_{\eta}^2u_+^n + \partial_{\eta}^3u_+^n + k_+^2\partial_{\eta}u_+^n \\
&= 0.
\end{aligned}$$

Taking the limit  $\eta \rightarrow 0$ , we obtain

$$-c'(t)\partial_tu_+^n|_{\Gamma} - 2c(t)\partial_t^2u_+^n|_{\Gamma} + \partial_{\mathbf{n}}\partial_t^2u_+^n|_{\Gamma} - c^2(t)\partial_{\mathbf{n}}u_+^n|_{\Gamma} + c(t)\partial_{\mathbf{n}}^2u_+^n|_{\Gamma} + \partial_{\mathbf{n}}^3u_+^n|_{\Gamma} + k_+^2\partial_{\mathbf{n}}u_+^n|_{\Gamma} = 0,$$

hence

$$\partial_{\mathbf{n}}^3u_+^n|_{\Gamma} = c'(t)\partial_tu_+^n|_{\Gamma} + 2c\partial_t^2u_+^n|_{\Gamma} - \partial_{\mathbf{n}}\partial_t^2u_+^n|_{\Gamma} + c^2(t)\partial_{\mathbf{n}}u_+^n|_{\Gamma} - c(t)\partial_{\mathbf{n}}^2u_+^n|_{\Gamma} - k_+^2\partial_{\mathbf{n}}u_+^n|_{\Gamma}. \tag{102}$$

Using Identity (92), Relation (102) becomes

$$\begin{aligned}\partial_{\mathbf{n}}^3 u_+^n|_{\Gamma} &= 3c(t)\partial_t^2 u_+^n|_{\Gamma} + (2c^2(t) - k_+^2) \partial_{\mathbf{n}} u_+^n|_{\Gamma} + c'(t)\partial_t u_+^n|_{\Gamma} \\ &- \partial_{\mathbf{n}}\partial_t^2 u_+^n|_{\Gamma} + c(t)k_+^2 u_+^n|_{\Gamma}, \quad \forall n \geq 0.\end{aligned}\quad (103)$$

Therefore, using (103) for  $n = 0$  and (92) for  $n \in \{0, 1\}$ , Relation (101) leads to

$$\begin{aligned}\frac{1}{f(t)}\partial_s U_{m,2}^3(t, 1) &= (f'(t))^2 \partial_{\mathbf{n}} u_+^0|_{\Gamma} + 2c(t)f'(t)(d + f(t))(\alpha_+ - 1) \partial_t u_+^0|_{\Gamma} + \frac{(d + f(t))^2}{2} \alpha_+ c'(t) \partial_t u_+^0|_{\Gamma} \\ &+ \alpha_+ \partial_{\mathbf{n}} u_+^2|_{\Gamma} + (1 - \alpha_+) f'(t) \partial_t u_+^1|_{\Gamma} + f'(t)(d + f(t))(1 - \alpha_+) \partial_t \partial_{\mathbf{n}} u_+^0|_{\Gamma} \\ &- (d + f(t)) \alpha_+ \partial_t^2 u_+^1|_{\Gamma} - (d + f(t)) \alpha_+ c(t) \partial_{\mathbf{n}} u_+^1|_{\Gamma} - (d + f(t)) \alpha_+ k_+^2 u_+^1|_{\Gamma} \\ &+ \frac{(d + f(t))^2}{2} \alpha_+ 3c(t) \partial_t^2 u_+^0|_{\Gamma} + \frac{(d + f(t))^2}{2} \alpha_+ (2c^2(t) - k_+^2) \partial_{\mathbf{n}} u_+^0|_{\Gamma} \\ &- \frac{(d + f(t))^2}{2} \alpha_+ \partial_{\mathbf{n}} \partial_t^2 u_+^0|_{\Gamma} + \frac{(d + f(t))^2}{2} \alpha_+ c(t) k_+^2 u_+^0|_{\Gamma}\end{aligned}\quad (104)$$

So (100) becomes

$$\begin{aligned}&\alpha_+ \partial_{\mathbf{n}} u_+^2|_{\Gamma} \\ &= \alpha_- \partial_{\mathbf{n}} u_-^2|_{\Gamma} - \frac{1}{f(t)} c(t) \alpha_+ \partial_{\mathbf{n}} u_+^1|_{\Gamma} + (d + f(t)) \alpha_+ c(t) \partial_{\mathbf{n}} u_+^1|_{\Gamma} - d \alpha_- c(t) \partial_{\mathbf{n}} u_-^1|_{\Gamma} - (1 - \alpha_+) f'(t) \partial_t u_+^1|_{\Gamma} \\ &- \frac{1}{f(t)} k_m^2 u_+^1|_{\Gamma} + (d + f(t)) \alpha_+ k_+^2 u_+^1|_{\Gamma} - d \alpha_- k_-^2 u_-^1|_{\Gamma} - \frac{1}{f(t)} \partial_t^2 u_+^1|_{\Gamma} + (d + f(t)) \alpha_+ \partial_t^2 u_+^1|_{\Gamma} \\ &- d \alpha_- \partial_t^2 u_-^1|_{\Gamma} + \frac{d^2}{2} \alpha_- c^2(t) \partial_{\mathbf{n}} u_-^0|_{\Gamma} + \frac{d^2}{2} \alpha_- c(t) \partial_t^2 u_-^0|_{\Gamma} + \frac{d^2}{2} \alpha_- c(t) k_-^2 u_-^0|_{\Gamma} \\ &- \frac{d^2}{2} \alpha_- k_-^2 (\partial_{\mathbf{n}} u_-^0|_{\Gamma}) - \frac{d^2}{2} \alpha_- \partial_t^2 (\partial_{\mathbf{n}} u_-^0|_{\Gamma}) + \frac{d^2}{2} \alpha_- c^2(t) \partial_{\mathbf{n}} u_-^0|_{\Gamma} + d^2 \alpha_- c(t) \partial_t^2 u_-^0|_{\Gamma} \\ &+ \frac{d^2}{2} \alpha_- c'(t) \partial_t u_-^0|_{\Gamma} - \frac{1}{2f^2(t)} c^2(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} - \frac{c(t)}{2f^2(t)} \partial_t^2 u_+^0|_{\Gamma} - \frac{c(t)}{2f^2(t)} k_m^2 u_-^0|_{\Gamma} \\ &+ \frac{1}{f^2(t)} f'(t) \partial_t ((f(t) \alpha_+) \partial_{\mathbf{n}} u_+^0|_{\Gamma}) - \frac{1}{2} \frac{1}{f(t)} \partial_t^2 [(f(t) \alpha_+) \partial_{\mathbf{n}} u_+^0|_{\Gamma}] - \frac{f'^2(t)}{f^2(t)} \alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} \\ &+ \frac{1}{2} \frac{f''(t)}{f(t)} \alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \frac{1}{2} c^2(t) \alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} + c(t) \partial_t^2 u_+^0|_{\Gamma} - \frac{1}{2} c'(t) \partial_t u_+^0|_{\Gamma} - \frac{1}{2} k_m^2 \alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} \\ &+ dc^2(t) \frac{1}{f(t)} \alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} - \frac{1}{f(t)} \partial_t^2 [(d + f(t) - f(t) \alpha_+) \partial_{\mathbf{n}} u_+^0|_{\Gamma}] + \frac{1}{f(t)} c(t) (d + f(t)) \alpha_+ \partial_t^2 u_+^0|_{\Gamma} \\ &+ \frac{1}{f(t)} c^2(t) (d + f(t)) \alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \frac{1}{f(t)} c(t) (d + f(t)) \alpha_+ k_+^2 u_+^0|_{\Gamma} + \frac{1}{f(t)} c(t) f'(t) \alpha_+ \partial_t u_+^0|_{\Gamma} \\ &- \frac{1}{f(t)} c(t) f'(t) \partial_t u_+^0|_{\Gamma} + \frac{1}{f(t)} 2dc(t) \partial_t^2 u_+^0|_{\Gamma} - k_m^2 (d + f(t) - f(t) \alpha_+) \frac{1}{f(t)} \partial_{\mathbf{n}} u_+^0|_{\Gamma} \\ &- \frac{1}{f(t)} dc'(t) \partial_t u_+^0|_{\Gamma} - (f'(t))^2 \partial_{\mathbf{n}} u_+^0|_{\Gamma} - 2c(t) f'(t) (d + f(t)) (\alpha_+ - 1) \partial_t u_+^0|_{\Gamma} \\ &- \frac{(d + f(t))^2}{2} \alpha_+ c'(t) \partial_t u_+^0|_{\Gamma} - f'(t) (d + f(t)) (1 - \alpha_+) \partial_t \partial_{\mathbf{n}} u_+^0|_{\Gamma} - \frac{(d + f(t))^2}{2} \alpha_+ 3c(t) \partial_t^2 u_+^0|_{\Gamma} \\ &- \frac{(d + f(t))^2}{2} \alpha_+ (2c^2(t) - k_+^2) \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \frac{(d + f(t))^2}{2} \alpha_+ \partial_{\mathbf{n}} \partial_t^2 u_+^0|_{\Gamma} - \frac{(d + f(t))^2}{2} \alpha_+ c(t) k_+^2 u_+^0|_{\Gamma}\end{aligned}\quad (105)$$

Using again Transmission conditions (86), (87), (88) and (91), we obtain

$$\begin{aligned}
\alpha_+ \partial_n u_+^2|_\Gamma &= \alpha_- \partial_n u_-^2|_\Gamma - \mathcal{M} - \frac{1}{2} \left[ \frac{d^2}{2} k_-^2 - d^2 c^2(t) + \frac{1}{f^2(t)} f'^2(t) - \frac{1}{2} c^2(t) + \right. \\
&+ \frac{1}{2} k_m^2 + \frac{1}{f(t)} \left[ \frac{(f''(t) - f''(t)\alpha_+)}{\alpha_+} \right] - d c^2(t) \frac{1}{f(t)} + \frac{(f'(t))^2}{\alpha_+} + \frac{(d+f(t))^2}{2} (2c^2(t) - k_+^2) \\
&+ k_m^2 \left( \frac{d+f(t) - f(t)\alpha_+}{\alpha_+} \right) \frac{1}{f(t)} - \frac{1}{f(t)} c^2(t) (d+f(t)) + \frac{c^2(t)}{2f^2(t)} - \frac{(f'(t))^2}{f^2(t)} \left. \right] (\alpha_+ \partial_n u_+^0|_\Gamma + \alpha_- \partial_n u_-^0|_\Gamma) \\
&+ \frac{1}{2} \left[ \frac{d^2}{2} \alpha_- c k_-^2 - \frac{c}{2f^2} k_m^2 + \frac{1}{f} c(t) (d+f(t)) \alpha_+ k_+^2 - \frac{(d+f(t))^2}{2} \alpha_+ c(t) k_+^2 \right] (u_+^0|_\Gamma + u_-^0|_\Gamma) \\
&+ \left[ \frac{d^2}{4} \alpha_- c'(t) - \frac{1}{4} c'(t) - \frac{1}{2f(t)} c(t) f'(t) - c(t) f'(t) (d+f(t)) (\alpha_+ - 1) \right. \\
&+ \left. \frac{1}{2f(t)} c(t) f'(t) \alpha_+ - \frac{(d+f(t))^2}{4} \alpha_+ c'(t) - \frac{1}{2} \frac{1}{f(t)} d c'(t) \right] (\partial_t u_+^0|_\Gamma + \partial_t u_-^0|_\Gamma) \\
&+ \frac{1}{2} \left[ \frac{c(t)}{2f^2(t)} - \frac{1}{f(t)} c(t) (d+f(t)) \alpha_+ - c(t) - \frac{1}{f(t)} 2dc(t) \right. \\
&+ \left. \frac{(d+f(t))^2}{2} \alpha_+ 3c(t) - \frac{3d^2}{2} \alpha_- c(t) \right] (\partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma) \\
&+ \frac{1}{2} \left[ \frac{(d+f(t))^2}{2} - \frac{d^2}{2} - \frac{1}{2} - \frac{1}{f(t)} \left( \frac{d+f(t) - f(t)\alpha_+}{\alpha_+} \right) \right] (\alpha_+ \partial_t^2 \partial_n u_+^0|_\Gamma + \partial_t^2 \alpha_- \partial_n u_-^0|_\Gamma) \\
&+ \frac{1}{2} \left[ -2 \frac{1}{f(t)} \left( \frac{(f'(t) - f'(t)\alpha_+)}{\alpha_+} \right) - \frac{f'(t) (d+f(t)) (1 - \alpha_+)}{\alpha_+} \right] (\alpha_+ \partial_t \partial_n u_+^0|_\Gamma + \partial_t \alpha_- \partial_n u_-^0|_\Gamma),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M} &= \frac{1}{2} \left( \frac{1}{f(t)} - f(t) \right) c(t) (\alpha_+ \partial_n u_+^1|_\Gamma + \alpha_- \partial_n u_-^1|_\Gamma) \\
&+ \frac{1}{2} \left( \frac{1}{f(t)} k_m^2 - (d+f(t)) \alpha_+ k_+^2 + d \alpha_- k_-^2 \right) (u_+^1|_\Gamma + u_-^1|_\Gamma) \\
&+ \frac{1}{2} (1 - \alpha_+) f'(t) (\partial_t u_+^1|_\Gamma + \partial_t u_-^1|_\Gamma) + \frac{1}{2} \left( \frac{1}{f(t)} - (d+f(t)) \alpha_+ + d \alpha_- \right) (\partial_t^2 u_+^1|_\Gamma + \partial_t^2 u_-^1|_\Gamma) \\
&+ \frac{1}{2} \left( \frac{1}{f(t)} - f(t) \right) c(t) \left[ \frac{(d+f(t)) f(t) \alpha_+ - d f(t) \alpha_- - 1}{2f(t)} (\partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma) \right. \\
&+ \left. \frac{(d+f(t)) f(t) k_+^2 \alpha_+ - d f(t) k_-^2 \alpha_- - k_m^2}{2f(t)} (u_+^0|_\Gamma + u_-^0|_\Gamma) \right. \\
&+ \left. \frac{c(t)(f^2(t) - 1)}{2f(t)} (\alpha_+ \partial_n u_+^0|_\Gamma + \alpha_- \partial_n u_-^0|_\Gamma) + \frac{f'(t)(\alpha_+ - 1)}{2} (\partial_t u_+^0|_\Gamma + \partial_t u_-^0|_\Gamma) \right] \\
&- d c(t) \frac{(d+f(t)) f(t) \alpha_+ - d f(t) \alpha_- - 1}{2f(t)} (\partial_t^2 u_+^0|_\Gamma + \partial_t^2 u_-^0|_\Gamma) \\
&- d c(t) \frac{(d+f(t)) f(t) k_+^2 \alpha_+ - d f(t) k_-^2 \alpha_- - k_m^2}{2f(t)} (u_+^0|_\Gamma + u_-^0|_\Gamma)
\end{aligned}$$



$$\begin{aligned}
& -dc^2(t) \frac{(f^2(t) - 1)}{2f(t)} (\alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \alpha_- \partial_{\mathbf{n}} u_-^0|_{\Gamma}) - dc(t) \frac{f'(t)(\alpha_+ - 1)}{2} (\partial_t u_+^0|_{\Gamma} + \partial_t u_-^0|_{\Gamma}) \\
& + \left( \frac{1}{f(t)} k_m^2 - (d + f(t))\alpha_+ k_+^2 - d\alpha_- k_-^2 \right) \left( \frac{f(t)\alpha_+ \alpha_- - (d + f(t))\alpha_- + d\alpha_+}{4\alpha_+ \alpha_-} (\alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \alpha_- \partial_{\mathbf{n}} u_-^0|_{\Gamma}) \right) \\
& + (1 - \alpha_+) f'(t) \left( \frac{f'(t)\alpha_+ \alpha_- - f'(t)\alpha_-}{4\alpha_+ \alpha_-} \right) (\alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \alpha_- \partial_{\mathbf{n}} u_-^0|_{\Gamma}) \\
& + (1 - \alpha_+) f'(t) \frac{f(t)\alpha_+ \alpha_- - (d + f(t))\alpha_- + d\alpha_+}{4\alpha_+ \alpha_-} (\alpha_+ \partial_t \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \alpha_- \partial_t \partial_{\mathbf{n}} u_-^0|_{\Gamma}) \\
& + \left( \frac{1}{f(t)} - (d + f(t))\alpha_+ - d\alpha_- \right) \left( \frac{f''(t)\alpha_+ \alpha_- - (f''(t))\alpha_-}{4\alpha_+ \alpha_-} \right) (\alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \alpha_- \partial_{\mathbf{n}} u_-^0|_{\Gamma}) \\
& + \left( \frac{1}{f(t)} - (d + f(t))\alpha_+ - d\alpha_- \right) \left( \frac{f'(t)\alpha_+ \alpha_- - f'(t)\alpha_-}{2\alpha_+ \alpha_-} \right) (\alpha_+ \partial_t \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \alpha_- \partial_t \partial_{\mathbf{n}} u_-^0|_{\Gamma}) \\
& + \left( \frac{1}{f(t)} - (d + f(t))\alpha_+ - d\alpha_- \right) \frac{f(t)\alpha_+ \alpha_- - (d + f(t))\alpha_- + d\alpha_+}{4\alpha_+ \alpha_-} (\alpha_+ \partial_t^2 \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \alpha_- \partial_t^2 \partial_{\mathbf{n}} u_-^0|_{\Gamma}).
\end{aligned}$$

Then

$$\begin{aligned}
\alpha_+ \partial_{\mathbf{n}} u_+^2|_{\Gamma} - \alpha_- \partial_{\mathbf{n}} u_-^2|_{\Gamma} &= \vartheta_1 (\alpha_+ \partial_{\mathbf{n}} u_+^1|_{\Gamma} + \alpha_- \partial_{\mathbf{n}} u_-^1|_{\Gamma}) + \vartheta_2 (u_+^1|_{\Gamma} + u_-^1|_{\Gamma}) \\
&+ \vartheta_3 (\partial_t u_+^1|_{\Gamma} + \partial_t u_-^1|_{\Gamma}) + \vartheta_4 (\partial_t^2 u_+^1|_{\Gamma} + \partial_t^2 u_-^1|_{\Gamma}) \\
&+ \vartheta_5 (\partial_t^2 u_+^0|_{\Gamma} + \partial_t^2 u_-^0|_{\Gamma}) + \vartheta_6 (\partial_t u_+^0|_{\Gamma} + \partial_t u_-^0|_{\Gamma}) \\
&+ \vartheta_7 (u_+^0|_{\Gamma} + u_-^0|_{\Gamma}) + \vartheta_8 (\alpha_+ \partial_t^2 \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \partial_t^2 \alpha_- \partial_{\mathbf{n}} u_-^0|_{\Gamma}) \\
&+ \vartheta_9 (\alpha_+ \partial_t \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \partial_t \alpha_- \partial_{\mathbf{n}} u_-^0|_{\Gamma}) \\
&+ \vartheta_{10} (\alpha_+ \partial_{\mathbf{n}} u_+^0|_{\Gamma} + \alpha_- \partial_{\mathbf{n}} u_-^0|_{\Gamma}),
\end{aligned}$$

where  $(\vartheta_i)_{1 \leq i \leq 10}$  are defined in Section 4.2.

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