Coherent Taylor expansion as a bimonad

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Abstract

We extend the recently introduced setting of coherent differentiation for taking into account not only differentiation, but also Taylor expansion in categories which are not necessarily (left)additive. The main idea consists in extending summability into an infinitary functor which intuitively maps any object to the object of its countable summable families. This functor is endowed with a canonical structure of bimonad. In a linear logical categorical setting, Taylor expansion is then axiomatized as a distributive law between this summability functor and the resource comonad (aka. exponential), allowing to extend the summability functor into a bimonad on the Kleisli category of the resource comonad: this extended functor computes the Taylor expansion of the (nonlinear) morphisms of the Kleisli category. We also show how this categorical axiomatizations of Taylor expansion can be generalized to arbitrary cartesian categories, leading to a general theory of Taylor expansion formally similar to that of differential cartesian categories, although it does not require the underlying cartesian category to be left additive. We provide several examples of concrete categories which arise in denotational semantics and feature such analytic structures.

Differentiation, Lambda-Calculus and Linear Logic. Linear Logic (LL) is a field that arose from semantics ([Gir87]), following the observation that some interesting models of the λ-calculus were actually the Kleisli category $\mathcal{L}_!$ of some category $\mathcal{L}$ of linear morphisms. That is, a morphism from object X to Y can be seen as a morphism in $\mathcal{L}(!X, Y)$, where $\mathcal{L}$ is a category of linear morphisms and ! is a comonad on $\mathcal{L}$. The syntactical counterpart of a morphism in $\mathcal{L}$ is a proof/program that uses its input (data or hypothesis) exactly once, and the syntactical version of !_ features a construction (called promotion) that makes a resource a duplicable and discardable.

It turns out that in multiple models of LL, the morphisms $f \in \mathcal{L}(!X, Y)$ are differentiable in some sense, strongly suggesting that differentiation of proofs and programs should be considered as a natural LL operation. Remember that for any Banach spaces E and F, a function $f : E \to F$ is differentiable at $x \in E$ if there is a neighborhood U of 0 in E and a linear and continuous function $\varphi : E \to F$ such that, for all $u \in U$

$$f(x+u) = f(x) + \varphi(u) + o(\|u\|)\,. \tag{1}$$

When φ exists, it is unique and is denoted as $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$. When $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$ exists for all $x \in E$, the function $\frac{\mathrm{d}f}{\mathrm{d}x} : E \to \mathcal{L}(E, F)$, where $\mathcal{L}(E, F)$ is the Banach space of linear and continuous functions $E \to F$, is called the *differential* of f. This function can itself admit a differential

and so on. When all these iterated differentials exist one says that f is *smooth* and the nth derivative of f is a function $\frac{\mathsf{d}^n f}{\mathsf{d}x^n} : E \to \mathcal{L}_n(E,F)$ where $\mathcal{L}_n(E,F)$ is the space of continuous n-linear symmetric functions $E^n \to F$. It can even happen that f is locally (or even globally) expressed using its iterated derivatives by means of the *Taylor Formula*

$$f(x+u) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathsf{d}^n f}{\mathsf{d}^n x}(x)(u,\dots,u) \tag{2}$$

When this holds locally at any point x, f is said to be *analytic*.

Based on categorical models of LL where morphisms $f \in \mathcal{L}(!X, Y)$ are analytic functions, the differential λ-calculus ([ER03]) and differential LL provide a logical and syntactical account of differentiation. If $\Gamma, x : A \vdash P : B$ and $\Gamma \vdash Q : A$, then one can define in these systems, by purely syntactical means, a program $\Gamma, x : A \vdash \frac{\partial P}{\partial x} \cdot Q : B$ whose denotational semantics in such models is the derivative of the interpretation of P wrt. the variable x, linearly applied to the semantics of Q. This syntactic derivative can be seen as a version of P, where exactly one call to the variable x that occurs during the computation of P is replaced with a call to Q: this explains why x is still free in $\frac{\partial P}{\partial x} \cdot Q$ in general. This provides a new approach of finite approximations of functions by a syntactical version of the Taylor Formula whose effect is to translate λ-calculus application $(P)Q$ into the differential λ-calculus in such a way that

$$(\lambda x.P)Q \text{ reduces to } \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n P}{\partial x^n} (\underbrace{Q,\dots,Q}_{\text{n times}}) \right) [0/x]\,. \tag{3}$$

The term of rank n in this formal sum corresponds to the part of the computation that uses the input Q exactly n times. Applying this transformation hereditarily to all the applications occurring in a λ-term, it becomes possible to turn it into an infinite sum of strongly normalizing *resource terms*, see [ER08].

Differentiation is deeply connected with *addition*, as it can already be seen in its definition Eq. (1) and in the syntactical Taylor expansion Eq. (3). As a result, the differential λ-calculus is always endowed with an unrestricted operation of sum between terms. Similarly, all categorical models of the differential λ-calculus and of differential LL (see [BCS06, BCS09, Man12]) were using categories where hom-sets have a structure of commutative monoid. The only available operational interpretation of such a sum being erratic choice, these formalisms are inherently non-deterministic. This is in sharp contrast with the uniformity property of the sum that can be observed in the syntax. For example, if the term of Eq. (3) reduces to a variable, then only one term of the sum is non-zero. The position n of this term gives the number of times the term Q is evaluated during the weak head reduction strategy as shown in [ER03]. Furthermore, many deterministic (or probabilistic) models of LL such as the ones based on the relational model (coherence spaces and probabilistic coherent spaces) feature morphisms in $\mathcal{L}(!X, Y)$ which are clearly analytic in some sense, although the hom-sets do not feature a commutative monoid structure.

Coherent differentiation. Recently, in [Ehr22b], it was observed that, in a setting where all coefficients are non-negative, differentiation survives to strong restrictions on the use of addition. Consider for instance a function $[0,1] \to [0,1]$ which is smooth on $[0,1)$ and all of whose iterated derivatives are everywhere ≥ 0[1]. If $x, u \in [0,1]$ are such that $x + u \in [0,1]$

[1] This actually implies that f is analytic.

then $f(x)+f'(x)u \leq f(x+u) \in [0,1]$ (this makes sense even if $f'(1) = \infty$, which can happen: take $f(x) = 1 - \sqrt{1-x}$). So if S is the set of all such pairs $\langle\!\langle x, u \rangle\!\rangle$ that we call *summable*, we can consider the function $\widetilde{\mathsf{D}}(f) : \langle\!\langle x, u \rangle\!\rangle \mapsto \left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot u \right\rangle\!\!\right\rangle$ as a map $S \to S$. This basic observation is generalized in [Ehr23] to a wide range of categorical models $\mathcal{L}$ of LL including coherence spaces, probabilistic coherence spaces *etc.* where hom-sets have only a *partially defined* addition. In these *summable categories*, S becomes an endofunctor $\mathcal{L} \to \mathcal{L}$ equipped with an additional structure which allows defining summability and (partial) finite sums in a very general way and turns out to induce a monad. Differentiation is then axiomatized as a distributive law $!\mathsf{S} \Rightarrow \mathsf{S}!$ between this monad (similar to the tangent bundle monad of a tangent category, see [Ros84]) and the resource comonad $!\_$ of the LL structure of the category[2] $\mathcal{L}$. Indeed, this distributive law allows one to extend S to $\mathcal{L}_!$, the Kleisli category of $!\_$, and this extension $\widetilde{\mathsf{D}} : \mathcal{L}_! \to \mathcal{L}_!$ inherits the monad structure of S. A category equipped with such a differentiation is then called a *coherent differential category*. It was also observed that the functor S often arises as $\mathbb{D} \multimap \_$ (the category $\mathcal{L}$ being most often symmetric monoidal closed, with internal hom of X and Y denoted as $X \multimap Y$), where $\mathbb{D} = 1 \,\&\, 1$ (1 is the unit of the tensor product of $\mathcal{L}$). This object $\mathbb{D}$ can be endowed with the structure of a comonoid from which the monad structure of S arise. Differentiation then boils down to the existence of a coalgebra structure $\mathbb{D} \to !\mathbb{D}$ on $\mathbb{D}$ which is compatible with its comonoid structure. This case where the coherent differential structure of a category arises in that way is called *elementary*.

Contributions of this work. The goal of the present article is to study the Taylor expansion in this setting of partial sums. We believe that this is a crucial step towards providing generic tools to define a denotational semantics of the differential λ-calculus and of the syntactical Taylor expansion in a much more general setting than the current state of the art of [Man12]. The starting idea is that for the same reason that $\widetilde{\mathsf{D}}f(x) = \left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x)\ u \right\rangle\!\!\right\rangle$ is defined, it should be possible to define directly a second order approximant.

$$\mathsf{T}f\,\langle\!\langle x, u_1, u_2 \rangle\!\rangle = \left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot u_1, \frac{1}{2}\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x) \cdot (u_1, u_1) + \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot u_2 \right\rangle\!\!\right\rangle$$

The term u_1 should be seen as a first order variation and the term u_2 as a second order variation. So $\mathsf{T}f\,\langle\!\langle x, u_1, u_2 \rangle\!\rangle$ gives the components (sorted by order) of the order 2 Taylor approximation of f on the variation $u_1 + u_2$. Because the coefficients are non-negative, this sum is lower than $f(x + u_1 + u_2)$, so it is well-defined. This idea should work for any order, and going to the limit, for an infinite amount of coefficient, yielding an operator which provides all the terms of the Taylor expansion.

The first step is to introduce an infinitary counterpart of the summability structure S. It turns out that S is not only a monad, but also a comonad. The monadic and comonadic structures interact well, turning S into a *bimonad* (Section 1). Surprisingly, the whole Taylor expansion operation is again a distributive law $!\mathsf{S} \Rightarrow \mathsf{S}!$ following the exact same properties as coherent differentiation, except that now S is infinitary in the sense that an "element" of $\mathsf{S}(X)$ is an $\mathbb{N}$-indexed family of elements of X whose infinitary sum is well-defined. This distributive law allows, as it is standard, to extend the functor S into a functor $\mathsf{T} : \mathcal{L}_! \to \mathcal{L}_!$ which inherits the monadic structure of S.

One more axiom is added, ensuring intuitively that the maps are analytic in the sense that they coincide with their Taylor expansion. This axiom was the missing piece to ensure that T inherits from S the structure of a bimonad. We call a category equipped with such

[2] Which by the way needs not be a fully-fledged LL model.

Taylor expansion an *analytic category* (Section 2). Again, in many concrete models of LL, the functor S is equal to $\mathbb{D} \multimap \_$ where this time $\mathbb{D} = 1 \,\&\, 1 \,\&\, \cdots$ ($\mathbb{N}$-indexed cartesian product). This object $\mathbb{D}$ can be endowed with the structure of a bimonoid that completely determines the bimonad structure of S. The analytic structure — that is, the aforementioned distributive law $!\mathsf{S} \Rightarrow \mathsf{S}!$ — then boils down to a coalgebra $\mathbb{D} \to !\mathbb{D}$ compatible with the bimonoid structure (Section 4).

In Section 5, we complement these general categorical considerations with concrete examples which are all models of LL where the exponential (the ! comonad) is free, that is, which are Lafont categories. The first example is based on a notion of coherence space discovered introduced in [Lam95], which seems deeply related to Scott semantics. The second one is based on nonuniform coherence spaces that were introduced in [BE01] and are known to have two different exponentials, one of them being the free exponential. In this setting, the free exponential provides an example of elementary analytic category, but we show that the non-free one is not an elementary analytic category, although it is an elementary coherent differential category in the sense of [Ehr23]. We also mention the fact that the usual Girard's coherence space, with their free exponential, are an elementary Taylor category, and last we deal with the case of probabilistic coherence spaces (introduced in [DE11]) whose only known exponential has been shown to be free. They provide yet another example of elementary analytic category where morphisms are analytic functions with non-negative real coefficients. In this quantitative setting, we compute explicitly the action of the T functor on morphisms induced by the $\mathbb{D}$ bimonoid, showing that it performs the expected Taylor expansion of morphisms.

Taylor expansion in cartesian closed categories. Because $\mathcal{L}_!$ is a cartesian closed category, it can be interesting to drift away from the SMCC structure of $\mathcal{L}$ by only looking at the structure induced in $\mathcal{L}_!$. This is what happened with differentiation. It was first categorically axiomatized in a typical LL setting with additive categories, introducing a notion of *differential categories* in [BCS06]. Differentiation was then carried to the setting of cartesian left-additive categories, introducing cartesian differential categories in [BCS09], leading to successful uses of differentiation outside the realm of LL. Unsurprisingly, the Kleisli categories of differential LL categories are instances of cartesian differential categories, but the latter are more general than the former and cover more examples of categories where differentiation is available. Similarly, left summability structures and *cartesian coherent differential categories* were introduced in [EW23] to axiomatize coherent differentiation directly in any cartesian category. They arise in particular as the Kleisli category of coherent differential categories, and are at the same time a conservative generalization of the cartesian differential categories.

Since analytic categories are very similar to coherent differential categories, it is possible to introduce in a very similar way a notion of *cartesian analytic category* (Section 3). We can then define the notion of a *cartesian closed analytic category*. This more direct axiomatization should provide the foundation for working on the denotational semantics of syntactical Taylor expansion, but also provide a categorical framework for Taylor expansion independent of LL.

Mates and distributive law. One contribution of this article is to exhibit the crucial role played by the mate construction (Section 8) in the elementary case, both in the setting of [Ehr23] and in the setting of this article. In the elementary case, the commutative bimonoid $\mathbb{D}$ induces a bimonad $\_ \otimes \mathbb{D}$. This bimonad is the left adjoint of $\mathsf{S} = (\mathbb{D} \multimap \_)$, so the mate construction induces a bimonad structure on S that turns out to be precisely the

one described in Section 1.4. Then the mate construction also induces a bijection between natural transformations $\partial : !\mathsf{S} \Rightarrow \mathsf{S}!$ and $\overline{\partial} : (!\_ \otimes \mathbb{D}) \Rightarrow !(\_ \otimes \mathbb{D})$, and it turns out that this bijection preserves distributive laws. It provides a crucial step when showing that Taylor expansion amount to a coalgebra on $\mathbb{D}$.

Related work. There might be connections between the work presented here and the recent article [KL23] where an account of Taylor expansion in differential LL is provided, based on the use of a resource exponential modality ! which has not only its standard comonad structure, but also a monad structure, thus turning it into a bimonad. In our setting, it is not the exponential modality which features a bimonad structure, but the infinitary summability functor S which does not at all play the same role: for instance, in LL models, the functor S preserves cartesian products whereas the ! functor turns the cartesian product into a tensor product. Another difference between the two approaches is that, being based on differential LL, [KL23] is based on additive categories whereas one of our main motivations is to deal with Taylor expansion in settings where addition is only partially defined. A more detailed analysis of the possible connections between the two approaches is definitely necessary.

Contents

1 Left ω-summability structures

Summability structures and *left summability structures* have been introduced respectively in [Ehr23] and [EW23]. Both are a categorical axiomatization of partial sums. More precisely, those categorical structures give to the hom-sets the structure of a finite counterpart of the notion of *partially additive monoid* introduced in [AM80]. The difference between summability structures and left summability structures is that in the former every morphism is linear with regard to the sum (we will call this property *additivity*), while this is not the case in the latter. Summability structures thus typically appear in models of LL $\mathcal{L}$, while left summability structures appear in the co-Kleisli categories of their exponential $\mathcal{L}_!$ or in other cartesian closed categories.

We introduce an infinitary counterpart of those structures. We work in the framework of left additive structures, because it is more general and is necessary for Section 3. Still, we will put a lot of emphasis on the properties of the category of additive morphisms, see Sections 1.2 and 1.4.

1.1 Left pre ω-summability structures

Let $\mathcal{C}$ be a category with objects $\mathbf{Obj}$ and hom-set $\mathcal{C}(X,Y)$ for any $X,Y \in \mathbf{Obj}$. We assume that any hom-set $\mathcal{C}(X,Y)$ contains a distinguished morphism $0^{X,Y}$ (usually X and Y are kept implicit) such that for any $f \in \mathcal{C}(Z,X)$, $0^{X,Y} \circ f = 0^{Z,Y}$.

Definition 1. An ω-summable pairing structure on a category $\mathcal{C}$ consists of:

- A map (functional class) $\mathsf{S} : \mathbf{Obj}(\mathcal{C}) \to \mathbf{Obj}(\mathcal{C})$ on objects;
- A family $(\pi_i)_{i=0}^{\infty}$ where π_i is a family of morphisms $\pi_{i,X} \in \mathcal{C}(\mathsf{S}X, X)$ such that for any X, the morphisms $(\pi_{i,X})_{i=0}^{\infty}$ are jointly monic. That is, for any $f,g \in \mathcal{C}(X,\mathsf{S}Y)$, if $\pi_{i,Y} \circ f = \pi_{i,Y} \circ g$ for any $i \in \mathbb{N}$, then $f = g$;
- A family of morphism $\sigma_X \in \mathcal{C}(\mathsf{S}X, X)$.

The object X will usually be kept implicit. We assume in what follows that $\mathcal{C}$ is equipped with an ω-summable pairing structure $(\mathsf{S}, (\pi_i)_{i=0}^{\infty}, \sigma)$.

Definition 2. A sequence of morphisms $(f_i)_{i=0}^{\infty}$ with $f_i \in \mathcal{C}(X,Y)$ is *summable* if there exists $\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} \in \mathcal{C}(X, \mathsf{S}Y)$ such that $\pi_i \circ \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} = f_i$. The joint monicity of the π_i's ensures that when $\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty}$ exists, it is unique. We call it the *witness* of the sum. Then we set $\sum_{i=0}^{\infty} f_i := \sigma \circ \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} \in \mathcal{C}(X,Y)$.

Remark 1. A more standard approach to notations would be to write the projections starting from the index 1 and not the index 0. The reason we proceed that way is that in Section 2, the element at position i in $\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty}$ can be seen as an order i variation or a degree of differentiation.

Proposition 1. *By definition, the morphisms $(\pi_i)_{i=0}^{\infty}$ are summable with witness $\langle\!\langle \pi_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{id}$ and sum $\sum_{i=0}^{\infty} \pi_i = \sigma$.*

Proposition 2 (Left additivity)**.** *Let $g_i \in \mathcal{C}(Y,Z)$ for any $i \in \mathbb{N}$ and $f \in \mathcal{C}(X,Y)$. If $(g_i)_{i=0}^{\infty}$ is summable, then $(g_i \circ f)_{i=0}^{\infty}$ is summable with witness $\langle\!\langle g_i \circ f \rangle\!\rangle_{i=0}^{\infty} = \langle\!\langle g_i \rangle\!\rangle_{i=0}^{\infty} \circ f$. Moreover, $\sum_{i=0}^{\infty}(g_i \circ f) = (\sum_{i=0}^{\infty} g_i) \circ f$.*

Proof. Let $w = \langle\!\langle g_i \rangle\!\rangle_{i=0}^{\infty} \circ f$. Then $\pi_i \circ w = g_i \circ f$ so w is a witness for $(g_i \circ f)_{i=0}^{\infty}$. Furthermore, $\sum_{i=0}^{\infty}(g_i \circ f) := \sigma \circ w = (\sum_{i=0}^{\infty} g_i) \circ f$. □

An important class of morphisms is that of additive morphisms, for which addition is compatible with composition on the right.

Definition 3. A morphism $h \in \mathcal{C}(Y,Z)$ is *additive* if $h \circ 0 = 0$ and if for any summable family $(f_i \in \mathcal{C}(X,Y))_{i=0}^{\infty}$, $(h \circ f_i)_{i=0}^{\infty}$ is summable and $\sum_{i=0}^{\infty}(h \circ f_i) = h \circ \sum_{i=0}^{\infty} f_i$.

Proposition 3. *A morphism h such that $h \circ 0 = 0$ is additive if and only if $(h \circ \pi_i)_{i=0}^{\infty}$ is summable with sum $h \circ \sigma$.*

Proof. For the forward implication, recall that $(\pi_i)_{i=0}^{\infty}$ is summable with sum σ. Thus, by definition of additivity, $(h \circ \pi_i)_{i=0}^{\infty}$ is summable with sum $h \circ \sigma$. For the reverse implication, assume that $(f_i)_{i=0}^{\infty}$ is summable. By assumption, $(h \circ \pi_i)_{i=0}^{\infty}$ is summable of sum $h \circ \sigma$, so Proposition 2 ensures that $(h \circ f_i)_{i=0}^{\infty} = (h \circ \pi_i \circ \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty})_{i=0}^{\infty}$ is summable with sum $h \circ \sigma \circ \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} = h \circ \sum_{i=0}^{\infty} f_i$. □

Definition 4. The ω-summable pairing structure $(\mathsf{S}, (\pi_i)_{i=0}^{\infty}, \sigma)$ is a *left pre ω-summability structure* if the projections $(\pi_i)_{i=0}^{\infty}$ and σ are additive. The ω-summable pairing structure $(\mathsf{S}, (\pi_i)_{i=0}^{\infty}, \sigma)$ is a *pre ω-summability structure* if all the morphisms in $\mathcal{C}$ are additive.

We now assume that $(\mathsf{S}, (\pi_i)_{i=0}^{\infty}, \sigma)$ is a left pre ω-summability structure. The additivity of the projections implies that the sum behaves well with respect to the operation $\langle\!\langle \rangle\!\rangle_{i=0}^{\infty}$

itself, and the additivity of σ ensures that the order of summation can be permuted.

Proposition 4. *Assume that for any $j \in \mathbb{N}$, $(f_{i,j})_{i=0}^{\infty}$ is a summable sequence of $\mathcal{C}(X,Y)$ such that $(\langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty})_{j=0}^{\infty}$ is summable. Then for any $i \in \mathbb{N}$, $(f_{i,j})_{j=0}^{\infty}$ is summable, $(\sum_{j=0}^{\infty} f_{i,j})_{i=0}^{\infty}$ is summable of witness $\langle\!\langle \sum_{j=0}^{\infty} f_{i,j} \rangle\!\rangle_{i=0}^{\infty} = \sum_{j=0}^{\infty} \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty}$. Furthermore,*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{i,j} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} f_{i,j}$$

Proof. By additivity of π_i, $\pi_i \circ \sum_{j=0}^{\infty} \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty} = \sum_{j=0}^{\infty} \pi_i \circ \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty} = \sum_{j=0}^{\infty} f_{i,j}$. By definition of summability, the equality above implies that $(\sum_{j=0}^{\infty} f_{i,j})_{i=0}^{\infty}$ is summable of witness $\sum_{j=0}^{\infty} \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty}$. Then its sum is equal to

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{i,j} := \sigma \circ \sum_{j=0}^{\infty} \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty} \overset{(\sigma \text{ additive})}{=} \sum_{j=0}^{\infty} \sigma \circ \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} f_{i,j}$$

□

Remark 2. In the proof above, the equality $\sigma \circ \sum_{j=0}^{\infty} \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty} = \sum_{j=0}^{\infty} \sigma \circ \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty}$ implicitly means that if the sum on the left-hand side of the equality is defined, then the sum on the right-hand side is also defined and both are equal. We will often use this kind of implicit formulation, as it makes proof much more concise and readable.

Proposition 5. *The family $(0)_{i=0}^{\infty}$ is summable of witness $\langle\!\langle 0 \rangle\!\rangle_{i=0}^{\infty} = 0$ and sum $\sum_{i=0}^{\infty} 0 = 0$. In particular, 0 is additive.*

Proof. On the one hand, $\pi_i \circ 0 = 0$ by additivity of π_i, so $(0)_{i=0}^{\infty}$ is summable of witness 0. On the other hand, $\sigma \circ 0 = 0$ by additivity of σ so $\sum_{i=0}^{\infty} 0 := \sigma \circ 0 = 0$. In particular, 0 is additive due to Proposition 3 because $0 \circ 0 = 0$ and $\langle\!\langle 0 \rangle\!\rangle_{i=0}^{\infty} = \langle\!\langle 0 \circ \pi_i \rangle\!\rangle_{i=0}^{\infty}$ so $(0 \circ \pi_i)_{i=0}^{\infty}$ is summable of sum $0 = 0 \circ \sigma$. □

1.2 The category of additive morphisms

We assume in this section that $\mathcal{C}$ is equipped with a left pre ω-summability structure $(\mathsf{S}, \vec{\pi}, \sigma)$.

Recall Proposition 3: a morphism h is additive if and only if $(h \circ \pi_i)_{i=0}^{\infty}$ is summable of sum $h \circ \sigma$. We define $\mathsf{S}h := \langle\!\langle h \circ \pi_i \rangle\!\rangle_{i=0}^{\infty}$, so that $\pi_i \circ \mathsf{S}h = h \circ \pi_i$ and $\sigma \circ \mathsf{S}h = h \circ \sigma$. We can easily check by joint monicity of the π_i that $\mathsf{S}h \circ \langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty} = \langle\!\langle h \circ x_i \rangle\!\rangle_{i=0}^{\infty}$, so $\mathsf{S}h$ consists in applying h in each coordinate.

Proposition 6. *For any X, the identity id_X is additive and $\mathsf{S}\mathsf{id}_X = \mathsf{id}_{\mathsf{S}X}$. For any $h \in \mathcal{C}(X,Y)$ and $h' \in \mathcal{C}(Y,Z)$, $h' \circ h$ is additive and $\mathsf{S}(h' \circ h) = \mathsf{S}h' \circ \mathsf{S}h$.*

Proof. The identity case is a straightforward consequence of Proposition 1 and Proposition 3. Assume that $h \in \mathcal{C}(X,Y)$ and $h' \in \mathcal{C}(Y,Z)$ are additive. Then $\pi_i \circ \mathsf{S}h' \circ \mathsf{S}h = h' \circ \pi_i \circ \mathsf{S}h = h' \circ h \circ \pi_i$ so $(h' \circ h \circ \pi_i)_{i=0}^{\infty}$ is summable of witness $\mathsf{S}h' \circ \mathsf{S}h$ and sum $\sigma \circ \mathsf{S}h' \circ \mathsf{S}h = h' \circ \sigma \circ \mathsf{S}h = h' \circ h \circ \sigma$. But $h' \circ h \circ 0 = h' \circ 0 = 0$ so by Proposition 3, $h' \circ h$ is additive and $\mathsf{S}(h' \circ h) = \mathsf{S}h' \circ \mathsf{S}h$. □

Definition 5. Define $\mathcal{C}^{\mathsf{add}}$ the category with the same objects and composition as $\mathcal{C}$ and whose morphisms are the additive morphisms of $\mathcal{C}$. This is a category thanks to Proposition 6, and there is a forgetful functor $\mathcal{U} : \mathcal{C}^{\mathsf{add}} \to \mathcal{C}$. Proposition 6 also ensures that S is a functor from $\mathcal{C}^{\mathsf{add}}$ to $\mathcal{C}$. Finally, the projections π_i and σ (as morphism in $\mathcal{C}$) are natural transformations $\mathsf{S} \Rightarrow \mathcal{U}$.

Notations 1. We will write the composition of $h \in \mathcal{C}^{\mathsf{add}}(X,Y)$ and $h' \in \mathcal{C}^{\mathsf{add}}(Y,Z)$ as $h'\,h$, since it corresponds to the composition of linear maps which are linear with regard to the sum.

We would like to turn S into an endofunctor on $\mathcal{C}^{\mathsf{add}}$. This is possible, assuming (Pair-Add) below. We will show in Proposition 10 that if $(\mathsf{S}, \vec{\pi}, \sigma)$ is a left summability structure (see Definition 13), then (Pair-Add) necessarily hold, so this is not a strong assumption.

Definition 6. The left pre ω-summability structure follows (Pair-Add) if whenever $(h_i)_{i=0}^{\infty}$ is a summable family of additive morphisms, then $\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty}$ is additive.

Lemma 1. *Under the assumptions of (Pair-Add) above, $\sum_{i=0}^{\infty} h_i = \sigma \langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty}$ is also additive.*

Proof. It is the composition of two additive morphisms so it is additive. □

Any left pre ω-summability structure on $\mathcal{C}$ trivially induce a pre ω-summability structure on $\mathcal{C}^{\mathsf{add}}$, since the projections and the sum are all additive. Furthermore, (Pair-Add) ensures that two morphisms of $\mathcal{C}^{\mathsf{add}}(X,Y)$ are summable in $\mathcal{C}^{\mathsf{add}}$ if and only if they are summable in $\mathcal{C}$. Besides, if h is additive then $h \circ \pi_i$ is additive. So by (Pair-Add), $\mathsf{S}h = \langle\!\langle h \circ \pi_i \rangle\!\rangle_{i=0}^{\infty}$ is also additive. This means that S turns into an endofunctor on $\mathcal{C}^{\mathsf{add}}$, and that the π_i and σ (as morphisms in $\mathcal{C}^{\mathsf{add}}$) are natural transformations $\mathsf{S} \Rightarrow \mathsf{Id}$.

Proposition 7. *The pairing of natural transformations in $\mathcal{C}^{\mathsf{add}}$ is also a natural transformation. More precisely, let (α^i) be a family of natural transformations $\alpha^i_X \in \mathcal{C}^{\mathsf{add}}(FX, GX)$. Then the family $(\langle\!\langle \alpha^i_X \rangle\!\rangle_{i=0}^{\infty} \in \mathcal{C}^{\mathsf{add}}(FX, \mathsf{S}GX))_X$ is a natural transformation $F \Rightarrow \mathsf{S}G$, and the family $(\sum_{i=0}^{\infty} \alpha^i_X \in \mathcal{C}^{\mathsf{add}}(FX, GX))_X$ is a natural transformation $F \Rightarrow G$.*

Proof. Assume that $F, G : \mathcal{D} \to \mathcal{C}^{\mathsf{add}}$ are two functors and $\alpha_X \in \mathcal{C}^{\mathsf{add}}(FX, GX)$ is a natural transformation. We want to show that $\langle\!\langle \alpha^i_X \rangle\!\rangle_{i=0}^{\infty} \in \mathcal{C}^{\mathsf{add}}(FX, \mathsf{S}GX)$ is a natural transformation $F \Rightarrow \mathsf{S}G$. That is, for all $f \in \mathcal{D}(X,Y)$, $\langle\!\langle \alpha^i_Y \rangle\!\rangle_{i=0}^{\infty} Ff = \mathsf{S}Gf \langle\!\langle \alpha^i_X \rangle\!\rangle_{i=0}^{\infty}$. But $\pi_i \langle\!\langle \alpha^i_Y \rangle\!\rangle_{i=0}^{\infty} Ff = \alpha^i_Y Ff = Gf\, \alpha^i_X = Gf\, \pi_i \langle\!\langle \alpha^i_X \rangle\!\rangle_{i=0}^{\infty} = \pi_i \mathsf{S}Gf \langle\!\langle \alpha^i_X \rangle\!\rangle_{i=0}^{\infty}$. By joint monicity of the π_i, we conclude that $(\langle\!\langle \alpha^i_X \rangle\!\rangle_{i=0}^{\infty})_X$ is natural. Furthermore, $\sum_{i=0}^{\infty} \alpha^i_X = \sigma_X \langle\!\langle \alpha^i_X \rangle\!\rangle_{i=0}^{\infty}$ is natural since σ is natural. □

If $(\mathsf{S}, (\pi_i)_{i=0}^{\infty}, \sigma)$ is a pre ω-summability structure, then $\mathcal{C} = \mathcal{C}^{\mathsf{add}}$, (Pair-Add) obviously holds, and every result stated in $\mathcal{C}^{\mathsf{add}}$ hold in $\mathcal{C}$. In fact, summability structure where initially axiomatized in [Ehr23] as an endofuctors S for which π_i and σ are natural. By Theorem 1 below, those two axiomatization are equivalent.

Theorem 1. *Let $(\mathsf{S}, (\pi_i)_{i=0}^{\infty}, \sigma)$ be an ω-summable pairing structure, as in Definition 1. The following are equivalent*

- *$(\mathsf{S}, (\pi_i)_{i=0}^{\infty}, \sigma)$ is a pre ω-summability structure on $\mathcal{C}$*
- *S is endowed with the structure of a functor for which the π_i and σ are natural transformations, and for any morphism $h \in \mathcal{C}(X,Y)$, $h \circ 0 = 0$.*

Proof. The forward direction is a consequence of the development of this section. Conversely, for any $h \in \mathcal{C}(X,Y)$, $h \circ 0 = 0$ by assumption, and the equation of naturality $\pi_i \circ \mathsf{S}h = h \circ \pi_i$ ensures that $\langle\!\langle h \circ \pi_i \rangle\!\rangle_{i=0}^{\infty}$ is summable with witness $\mathsf{S}h$ and sum $\sigma \circ \mathsf{S}h = h \circ \sigma$. So by Proposition 3, h is additive. □

1.3 Left ω-summability structure

Given an injection $\varphi : A \hookrightarrow B$ and a family $\vec{f} = (f_a)_{a\in A}$ we define a family $\vec{g} = \varphi_* \overrightarrow{f} = (g_b \in \mathcal{C}(Y,X))_{b\in B}$ by

$$g_b = \begin{cases} f_a & \text{if } b \in \varphi(A) \text{ and } \varphi(a) = b \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2. *The operation $\varphi \to \varphi_*$ is functorial, that is $\mathsf{Id}_* \overrightarrow{f} = \overrightarrow{f}$ and $\psi_*\varphi_* \overrightarrow{f} = (\psi\circ\varphi)_* \overrightarrow{f}$.*

Notations 2. A $\mathbb{N}$-indexed family $\vec{f} = (f_n \in \mathcal{C}(X,Y))_{n\in\mathbb{N}}$ is the same thing as a sequence $(f_n \in \mathcal{C}(X,Y))_{n=0}^{\infty}$. Whenever such a sequence is summable, we write $\langle\!\langle \vec{f} \rangle\!\rangle := \langle\!\langle f_n \rangle\!\rangle_{n=0}^{\infty}$ and $\sum \vec{f} := \sum_{i=0}^{\infty} f_i$.

Definition 7. (S-com) We say that the pre ω-summability structure is *commutative* if, for any set A, any $\overrightarrow{f} \in \mathcal{C}(X,Y)^A$ and any injections $\varphi, \psi : I \to \mathbb{N}$, $\varphi_* \overrightarrow{f}$ is summable iff $\psi_* \overrightarrow{f}$ is summable, and then $\sum \varphi_* \overrightarrow{f} = \sum \psi_* \overrightarrow{f}$.

Assume that the ω-summability structure satisfies (S-com). The following definition is meaningful by this assumption.

Definition 8. For any set A, a family $\overrightarrow{f} \in \mathcal{C}(X,Y)^A$ is *summable* if there is an injection $\varphi : A \hookrightarrow \mathbb{N}$ such that $\varphi_* \overrightarrow{f}$ is summable, and then we set $\sum \overrightarrow{f} = \sum_{a\in A} f_a = \sum \varphi_* \overrightarrow{f}$.

Remark 3. When $A = \mathbb{N}$, this notion of summability coincides with the already introduced notion of summability for sequences, thanks to (S-com). So the apparent clash of terminology is harmless. Notice also that if $\overrightarrow{f} \in \mathcal{C}(X,Y)^A$ is summable, then A is finite or countable.

Remark 4. One major difference between this general definition of A-indexed summability is that it does not make sense to speak of the witness of summability for a summable family $\overrightarrow{f} \in \mathcal{L}(X,Y)^A$, but only of $\varphi_* \overrightarrow{f}$ for a given injection $\varphi : A \to \mathbb{N}$.

Observe that the empty family is always summable of sum 0. Besides, by Lemma 3 below, for any bijection $\varphi : A \to A$, the family $(f_a)_{a\in A}$ is summable if and only if $(f_{\varphi(a)})_{a\in A}$ is summable, so the summability of a family does not depend on the choice of indexes.

Lemma 3. *If $\overrightarrow{f} = \mathcal{C}(X,Y)^A$ is an at most countable family of morphisms and $\varphi : A \to B$ is injective with B at most countable, then $\varphi_* \overrightarrow{f}$ is summable iff $\overrightarrow{f}$ is summable, and then $\sum \varphi_* \overrightarrow{f} = \sum \overrightarrow{f}$.*

Proof. Assume first that $\overrightarrow{f}$ is summable. Let $\psi : B \hookrightarrow \mathbb{N}$ be an injection, then $\psi\circ\varphi : A \to \mathbb{N}$ is an injection, hence $(\psi \circ \varphi)_* \overrightarrow{f} = \psi_*\varphi_* \overrightarrow{f}$ is summable from which it follows that $\varphi_* \overrightarrow{f}$ is summable with $\sum \varphi_* \overrightarrow{f} = \sum (\psi \circ \varphi)_* \overrightarrow{f} = \sum \overrightarrow{f}$.

Conversely, assume that $\varphi_* \overrightarrow{f}$ is summable, so let $\psi : B \to \mathbb{N}$ be an injection, we know that $\psi_*\varphi_* \overrightarrow{f} = (\psi \circ \varphi)_* \overrightarrow{f}$ is summable and since $\psi \circ \varphi$ is injective it follows that $\overrightarrow{f}$ is summable. Moreover, $\sum \overrightarrow{f} = \sum (\psi \circ \varphi)_* \overrightarrow{f} = \sum \varphi_* \overrightarrow{f}$. ∎

Lemma 4. *If $\overrightarrow{f} \in \mathcal{C}(X,Y)^A$ is summable, $g \in \mathcal{C}(U,X)$ and $h \in \mathcal{C}(Y,V)$ is additive, then $\overrightarrow{f'} = (h \circ f_a \circ g \in \mathcal{C}(U,V))_{a\in A}$ is summable with $\sum \overrightarrow{f}' = h \circ (\sum \overrightarrow{f}) \circ g$.*

Proof. This is a direct consequence of Proposition 2, of the additivity of h and of the definition of summability. ∎

Definition 9. (S-zero) We say that the pre ω-summability structure *has zero* if, for any $f \in \mathcal{C}(X,Y)$, the family $\overrightarrow{f} \in \mathcal{C}(X,Y)^{\mathbb{N}}$ such that $f_0 = f$ and $f_{n+1} = 0$ is summable and $\sum \overrightarrow{f} = f$.

Definition 10. (S-flat) A pre ω-summability structure is *flattenable* if $(\pi_i \circ \pi_j)_{(i,j)\in\mathbb{N}^2} \in \mathcal{C}(\mathsf{S}^2X, X)^{\mathbb{N}^2}$ is summable.

Definition 11. (S-ass) A pre ω-summability structure *is associative* if for any summable family $\overrightarrow{f} \in \mathcal{C}(X,Y)^{\mathbb{N}^2}$,

- for all $i \in \mathbb{N}$ the family $(f_{i,j})_{j\in\mathbb{N}}$ is summable;
- the family $(\sum_{j\in\mathbb{N}} f_{i,j})_{i\in\mathbb{N}}$ is summable;
- and we have

$$\sum_{(i,j)\in\mathbb{N}^2} f_{i,j} = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} f_{i,j}\,.$$

Definition 12. (S-wit) A pre ω-summability structure *has witnesses* if, for any family $\overrightarrow{h} = \mathcal{C}(X,\mathsf{S}Y)^{\mathbb{N}}$, if the family $\sigma \circ \overrightarrow{h} = (\sigma \circ h_i \in \mathcal{C}(X,Y))_{i\in\mathbb{N}}$ is summable, then $\overrightarrow{h}$ is summable.

Definition 13. A left ω-summability structure is a left pre ω-summability structure that satisfies (S-zero), (S-com), (S-flat), (S-ass) and (S-wit). An ω-summability structure is a pre ω-summability structure that satisfies (S-zero), (S-com), (S-flat), (S-ass) and (S-wit).

We assume that $\mathcal{C}$ is equipped with a left ω-summability structure $(\mathsf{S}, \overrightarrow{\pi}, \sigma)$. We show in Theorem 3 that the sum induced by this structure is a $\sum$-monoid following the terminology of [Hin13]. Those monoids are the same as the *partially additive monoids* introduced in [AM80], except that the limit axiom is dropped[3]. Then the *partially additive categories* of [AM80] are probably a particular instance of our partially left additive categories in which $\mathsf{S}X$ is equal to the countable coproduct of X with itself. We first show that (S-flat) and (S-wit) provides a necessary condition to (S-ass).

Lemma 5. *If $\overrightarrow{f} \in \mathcal{C}(X,Y)^{\mathbb{N}^2}$ satisfies*

- *for all $i \in \mathbb{N}$ the family $(f_{i,j})_{j\in\mathbb{N}}$ is summable,*
- *the family $(\sum_{j\in\mathbb{N}} f_{i,j})_{i\in\mathbb{N}}$ is summable*

then $\overrightarrow{f}$ is summable. By (S-ass), we then have $\sum_{(i,j)\in\mathbb{N}^2} f_{i,j} = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} f_{i,j}$.

Proof. Let $\overrightarrow{h} \in \mathcal{C}(X,\mathsf{S}Y)^{\mathbb{N}}$ be defined by $h_i = \langle\!\langle f_{i,j} \rangle\!\rangle_{i=0}^{\infty}$ for each $i \in \mathbb{N}$. We have assumed that $\sigma \circ \overrightarrow{h} = (\sum_{j\in\mathbb{N}} f_{i,j})_{i\in\mathbb{N}} \in \mathcal{C}(X,Y)^{\mathbb{N}}$ is summable and hence by (S-wit), the family $\overrightarrow{h}$ is summable. Let $H = \langle\!\langle \overrightarrow{h} \rangle\!\rangle \in \mathcal{C}(X,\mathsf{S}^2Y)$. By (S-flat) and Lemma 4, the family $(\pi_j \circ \pi_i \circ H)_{(i,j)\in\mathbb{N}^2}$ is summable, but $\pi_j \circ \pi_i \circ H = \pi_j \circ h_i = f_{i,j}$. So we have proven that $\overrightarrow{f}$ is summable. The last statement of the lemma results from a direct application of (S-ass) to $\overrightarrow{f}$. □

Theorem 2. *Let $(f_a \in \mathcal{C}(X,Y))_{a\in A}$ and let $p : A \to B$, assuming the sets A and B to be at most countable. Then $\overrightarrow{f}$ is summable iff for all $b \in B$ the family $(f_a)_{p(a)=b}$ is summable and the family $(\sum_{p(a)=b} f_a)_{b\in B}$ is summable. When these equivalent conditions hold, we have $\sum_{a\in A} f_a = \sum_{b\in B}\sum_{p(a)=b} f_a$.*

[3] This axiom will still be used when dealing with fixpoints, see Remark 6

Proof. Let $\varphi : B \to \mathbb{N}$ be an injection and let $(\varphi^b : p^{-1}(\{b\}) \to \mathbb{N})_{b\in B}$ be injections. Then the map $\psi : A \to \mathbb{N}^2$ given by $\psi(a) = (\varphi(p(a)), \varphi^{p(a)}(a))$ is an injection. For each $b \in B$ let $\overrightarrow{f^b} = (f_a)_{p(a)=b}$ be the restriction of $\overrightarrow{f}$ to $p^{-1}(\{b\})$. Let $\overrightarrow{g} = \psi_* \overrightarrow{f} = (g_{i,j} \in \mathcal{C}(X,Y))_{(i,j)\in\mathbb{N}^2}$.

Assume first that $\overrightarrow{f}$ is summable, so that $\psi_* \overrightarrow{f} = \overrightarrow{g}$ is summable. By (S-flat), we know that $(g_{i,j})_{j\in\mathbb{N}}$ is summable for each $i \in \mathbb{N}$ and $(\sum_{j\in\mathbb{N}} g_{i,j})_{i\in\mathbb{N}}$ is summable by (S-ass). If $i = \varphi(b)$ for a (uniquely determined) $b \in B$ we have $(g_{i,j})_{j\in\mathbb{N}} = \varphi^b_* \overrightarrow{f^b}$ so that $\overrightarrow{f^b}$ is summable and $\sum_{j\in\mathbb{N}} g_{i,j} = \sum_{p(a)=b} f_a$. If $i \notin \varphi(B)$ then $g_{i,j} = 0$ for all $j \in \mathbb{N}$. It follows that $(\sum_{j\in\mathbb{N}} g_{i,j})_{i\in\mathbb{N}} = \varphi_*(\sum_{p(a)=b} f_a)_{b\in B}$ and hence $(\sum_{p(a)=b} f_a)_{b\in B}$ is summable and sums to $\sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} g_{i,j} = \sum_{(i,j)\in\mathbb{N}^2} g_{i,j}$ by (S-ass), and this latter sum coincides with $\sum_{a\in A} f_a$ since $\overrightarrow{g} = \psi_* \overrightarrow{f}$.

Assume conversely that $\overrightarrow{f^b}$ is summable for all $b \in B$ and that $(\sum_{p(a)=b} f_a)_{b\in B}$ is summable. Then $\varphi^b_* \overrightarrow{f^b} = (g_{\varphi(b),j})_{j\in\mathbb{N}}$ is summable for all $b \in B$ and $g_{i,j} = 0$ if $i \notin \varphi(B)$; it follows that $(g_{i,j})_{j\in J}$ is summable for each $i \in \mathbb{N}$. Moreover,

$$\sum_{j\in\mathbb{N}} g_{i,j} = \begin{cases} \sum_{p(a)=b} f_a & \text{if } i = \varphi(b) \\ 0 & \text{if } i \notin \varphi(B) \end{cases}$$

and hence $(\sum_{j\in\mathbb{N}} g_{i,j})_{i\in\mathbb{N}} = \varphi_*(\sum_{p(a)=b} f_a)_{b\in B}$ and since $(\sum_{p(a)=b} f_a)_{b\in B}$ is summable it follows that $(\sum_{j\in\mathbb{N}} g_{i,j})_{i\in\mathbb{N}}$ is summable. By Lemma 5 it follows that $(g_{i,j})_{(i,j)\in\mathbb{N}^2}$ is summable and $\sum_{(i,j)\in\mathbb{N}^2} g_{i,j} = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} g_{i,j}$, that is $\overrightarrow{f}$ is summable (since $\overrightarrow{g} = \psi_* \overrightarrow{f}$) and $\sum_{a\in A} f_a = \sum_{b\in B} \sum_{p(a)=b} f_a$. ☐

Theorem 3. *The sum induced by $(\mathsf{S}, \vec{\pi}, \sigma)$ gives to $\mathcal{C}(X,Y)$ the structure of a $\sum$-monoid. That is:*

(1) Unary axiom: The family with only one element (f) is summable of sum f.

(2) Partition associativity axiom: for any $(B_i)_{i\in I}$ partition of A and $\vec{f} = (f_a \in \mathcal{C}(X,Y))_{a\in A}$, $\vec{f}$ is summable if and only if for all $i \in I$ the family $(f_b)_{b\in B_i}$ is summable and $(\sum_{b\in B_i} f_b)_{i\in I}$ is summable, and if one of those conditions is fulfilled, then

$$\sum \vec{f} = \sum_{i\in I} \sum_{b\in A_i} f_b \,.$$

Proof. The first statement is a direct consequence of (S-zero). The second statement is a consequence of Theorem 3, with $p : A \to I$ defined so that $p(a) = i$ if and only if $a \in B_i$. ☐

Remark 5. As discussed in [MA86, Hin13], the "only if" assumption of the partition associativity axiom in Theorem 3 that arises from (S-wit) is very strong, as it implies that the morphisms are positive: if $x + y = 0$ then $x = y = 0$. This leaves behind interesting models of LL in which the coefficients are not necessarily nonnegative, but in which maps are definitely analytics, such as Köthe spaces, see [Ehr02], or finiteness spaces, see [Ehr05].

For now, we keep this stronger and fundamentally positive axiomatization because it suits quite well with the situations which occur in the denotational semantics of programming languages and of proofs. A more general axiomatization of summability structure based on the partial commutative monoids of [Hin13] where the "only if" assumption is dropped is currently a work in progress. It should allow to recover the summability dynamic of both finiteness spaces and Köthe spaces.

Remark 6. In [EW23], additive morphisms only preserve finite summations (including 0), which corresponds to the usual algebraic notion of morphism of monoid. In our ω-summability setting, additivity means not only preservation of finite sums, but also of all infinite sums whose existence is prescribed by S. This means that additive morphisms also feature a property of *continuity* whose precise nature depends on the category $\mathcal{C}$ at stake.

A very interesting situation occurs when the ω-summability structure of $\mathcal{C}$ satisfies three additional properties

- a family $\overrightarrow{f} \in \mathcal{C}(X,Y)^{\mathbb{N}}$ is summable as soon as, for any *finite* set A and any injection $\varphi : A \to \mathbb{N}$, the family $\varphi_* \overrightarrow{f}$ is summable;
- the preorder relation $\leq$ defined on hom-sets of $\mathcal{C}$ by $f \leq g$ if $\exists h \in \mathcal{C}(X,Y)$ such that $g = f + h$ (where $+$ is the binary addition induced by $(\mathsf{S}, (\pi_i)_{i=0}^{\infty}, \sigma)$ on $\mathcal{C}(X,Y)$) is antireflexive, that is, is an order relation;
- If for any *finite* subset A' of A one has $\sum_{a\in A'} f_a \leq f$, then $\sum_{a\in A} f_a \leq f$.

The first condition turns the structure of $\sum$-monoid into a fully fledged partially additive monoid, and the two other conditions turn it into an additive domain, see Section 8.3 of [MA86]. When these conditions hold, each hom-set $\mathcal{C}(X,Y)$ is easily seen to be an ω-complete partial order (ordered by $\leq$ and having 0 as least element). The lub of $(\sum_{i=0}^{n} f_i)_{n\in\mathbb{N}}$ is given by $\sum_{i=0}^{\infty} f_i$. Then, additive morphisms commute with the corresponding lubs in the sense that $h \circ \sup_{i\in\mathbb{N}} g_i$ when $h \in \mathcal{C}(Y,Z)$ is additive and $\overrightarrow{g} \in \mathcal{C}(X,Y)^{\mathbb{N}}$ is a monotone sequence.

All the examples provided in Section 4.7 are instances of this situation and, for that reason, feature general fixpoint operators allowing to combine our coherent Taylor structures with general recursion in the spirit of [Ehr22a].

Proposition 8. *Let $(f_j \in \mathcal{C}(X,Y))_{j\in\mathbb{N}}$ and $(h_i \in \mathcal{C}(Y,Z))_{i\in\mathbb{N}}$ be summable and such that $\forall i \in \mathbb{N}$, h_i is additive. Then $(h_i \circ f_j)_{(i,j)\in\mathbb{N}^2}$ is summable and*

$$\sum_{(i,j)\in\mathbb{N}^2} h_i \circ f_j = (\sum_{i\in\mathbb{N}} h_i) \circ (\sum_{j\in\mathbb{N}} f_j)\,.$$

Proof. Let $(g_{i,j} = h_i \circ f_j)_{(i,j)\in\mathbb{N}^2}$. For each $i \in \mathbb{N}$ the family $(g_{i,j})_{j\in\mathbb{N}}$ is summable and sums to $h_i \circ \sum_{j\in\mathbb{N}} f_j$ by additivity of h_i. Furthermore, the family $(h_i \circ \sum_{j\in\mathbb{N}} f_j)_{i\in\mathbb{N}}$ is summable and sums to $(\sum \overrightarrow{h}) \circ (\sum \overrightarrow{f})$ by left additivity (Proposition 2). So by Lemma 5, the family $(g_{i,j})_{(i,j)\in\mathbb{N}^2}$ is summable, and we obtain the announced equation. □

Theorem 4. *There is a morphism $\mathsf{c} \in \mathcal{C}(\mathsf{S}^2X, \mathsf{S}^2X)$ such that for all $i,j \in \mathbb{N}$, one has $\pi_i \circ \pi_j \circ \mathsf{c} = \pi_j \circ \pi_i$.*

Proof. The family $(\pi_j \in \mathcal{C}(\mathsf{S}^2X, \mathsf{S}X))_{j\in\mathbb{N}}$ is summable (and sums to $\sigma_{\mathsf{S}X}$). Hence, for each $i \in \mathbb{N}$, the family $(\pi_i \circ \pi_j)_{j\in\mathbb{N}}$ is summable (and sums to $\pi_i \circ \sigma_{\mathsf{S}X}$) by additivity of π_i. But by left additivity, the family $(\pi_i \circ \sigma_{\mathsf{S}X})_{i\in\mathbb{N}}$ is summable (of sum $\sigma_X \circ \sigma_{\mathsf{S}X}$) and hence by (S-wit), the family $(\langle\!\langle \pi_i \circ \pi_j \rangle\!\rangle_{j\in\mathbb{N}})_{i\in\mathbb{N}} \in \mathcal{C}(\mathsf{S}^2X, \mathsf{S}X)^{\mathbb{N}}$ is summable, so that we can set $\mathsf{c} = \langle\!\langle \langle\!\langle \pi_i \circ \pi_j \rangle\!\rangle_{j\in\mathbb{N}} \rangle\!\rangle_{i\in\mathbb{N}} \in \mathcal{C}(\mathsf{S}^2X, \mathsf{S}^2X)$. We have $\pi_i \circ \mathsf{c} = \langle\!\langle \pi_i \circ \pi_j \rangle\!\rangle_{j\in\mathbb{N}}$ and hence $\pi_j \circ \pi_i \circ \mathsf{c} = \pi_i \circ \pi_j$ as required. □

As another illustration of these axioms, we have the following.

Theorem 5. *There is a morphism $\theta \in \mathcal{C}(\mathsf{S}^2X, \mathsf{S}X)$ such that, for all $i \in \mathbb{N}$, $\pi_i \circ \theta = \sum_{j=0}^{i} \pi_{i-j} \circ \pi_j$. For all $i \in \mathbb{N}$, there is a morphism $\iota_i \in \mathcal{C}(X, \mathsf{S}X)$ such that $\pi_j \circ \iota_i = \delta_{i,j}\mathsf{id}$ (and then $\sigma \circ \iota_i = \mathsf{id}$).*

Proof. By (S-flat), $(\pi_i \circ \pi_j)_{(i,j)\in\mathbb{N}^2}$ is summable. Let $B_n = \{(k, n-k) | k \in [\![0, n]\!]\}$. Then the B_n are a partition of $\mathbb{N}^2$, so by Theorem 3 the family $(\sum_{k=0}^{n} \pi_k \circ \pi_{n-k})_{n\in\mathbb{N}}$ is summable, which implies the existence of θ_X. Next consider the one-element family $\overrightarrow{f} = (f_a)_{a\in\{*\}}$ with $f_* = \mathsf{id}$. Let $\varphi : A \to \mathbb{N}$ defined by $\varphi(*) = 0$. Then $\varphi_* \overrightarrow{f}$ is summable of sum id by (S-zero). By (S-com), it implies that $\psi_* \overrightarrow{f}$ is summable of sum id, where ψ is defined as $\psi(*) = i$. Let $\iota_i = \langle\!\langle \psi_* \overrightarrow{f} \rangle\!\rangle \in \mathcal{C}(X, \mathsf{S}X)$. It satisfies the announced condition. □

Proposition 9. *There is a morphism $\mathsf{I} \in \mathcal{C}(\mathsf{S}X, \mathsf{S}^2X)$ such that $\pi_i \circ \pi_j \circ \mathsf{I} = \delta_{i,j}\pi_i$. That is, $\mathsf{I} = \langle\!\langle \iota_i \circ \pi_i \rangle\!\rangle_{i=0}^{\infty}$.*

Proof. Define $f_{i,j} = \delta_{i,j}\pi_i$. Then for all i, $(f_{i,j})_{j\in\mathbb{N}}$ is summable of sum π_i. But $(\pi_i)_{i\in\mathbb{N}}$ is summable so by (S-wit), $\mathsf{I} := \left\langle\!\!\left\langle \langle\!\langle f_{i,j} \rangle\!\rangle_{j=0}^{\infty} \right\rangle\!\!\right\rangle_{i=0}^{\infty}$ exists. Then $\pi_i \circ \pi_j \circ \mathsf{I} = f_{i,j} = \delta_{i,j}\pi_i$. □

1.4 The bimonad S

We now assume in this section that $(\mathsf{S}, \vec{\pi}, \sigma)$ is a left ω-summability structure. We want to study the structure induced in $\mathcal{C}^{\mathsf{add}}$. First, we show that the axiom (Pair-Add) necessarily hold.

Proposition 10. *A left ω-summability structure always follows (Pair-Add).*

Proof. Let $(h_i)_{i=0}^{\infty}$ be a summable sequence, with $h_i \in \mathcal{C}(X, Y)$ additive. First, $\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty} \circ 0 = \langle\!\langle h_i \circ 0 \rangle\!\rangle_{i=0}^{\infty} = \langle\!\langle 0 \rangle\!\rangle_{i=0}^{\infty} = 0$ by additivity of the h_i. So by Proposition 3, it suffices to prove that $(\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty} \circ \pi_j)_{j=0}^{\infty}$ is summable of sum $\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty} \circ \sigma$. But $(\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty} \circ \pi_j)_{j=0}^{\infty} = (\langle\!\langle h_i \circ \pi_j \rangle\!\rangle_{i=0}^{\infty})_{j=0}^{\infty}$ by Proposition 2. By Proposition 8, the family $(h_i \circ \pi_j)_{(i,j)\in\mathbb{N}^2}$ is summable, so by (S-wit), $\left\langle\!\!\left\langle \langle\!\langle h_i \circ \pi_j \rangle\!\rangle_{i=0}^{\infty} \right\rangle\!\!\right\rangle_{j=0}^{\infty}$ exists. Furthermore,

$$\begin{aligned}
\sigma \circ \left\langle\!\!\left\langle \langle\!\langle h_i \circ \pi_j \rangle\!\rangle_{i=0}^{\infty} \right\rangle\!\!\right\rangle_{j=0}^{\infty} &= \sum_{j=0}^{\infty} \langle\!\langle h_i \circ \pi_j \rangle\!\rangle_{i=0}^{\infty} \\
&= \left\langle\!\!\left\langle \sum_{j=0}^{\infty} h_i \circ \pi_j \right\rangle\!\!\right\rangle_{i=0}^{\infty} \quad \text{by Proposition 4} \\
&= \left\langle\!\!\left\langle h_i \circ \sum_{j=0}^{\infty} \pi_j \right\rangle\!\!\right\rangle_{i=0}^{\infty} \quad \text{by additivity of } h_i \\
&= \langle\!\langle h_i \circ \sigma \rangle\!\rangle_{i=0}^{\infty} = \langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty} \circ \sigma
\end{aligned}$$

It concludes the proof. □

Lemma 6. *The pre ω-summability $(\mathsf{S}, \vec{\pi}, \sigma)$ on $\mathcal{C}^{\mathsf{add}}$ follows (S-zero) and follows (S-com). Besides, a family $(h_a \in \mathcal{C}^{\mathsf{add}}(X, Y))_{a\in A}$ is summable in $\mathcal{C}^{\mathsf{add}}$ if and only if it is summable in $\mathcal{C}$, and $\sum_{a\in A} h_a$ is additive. Thus, the pre ω-summability structure also follow (S-ass) and (S-wit), and it is an ω-summability structure on $\mathcal{C}^{\mathsf{add}}$.*

Proof. This is a direct consequence of (Pair-Add) and the definition of the sum, using the fact that 0 is additive (Proposition 5). □

Note that for all functors $F, G : \mathcal{D} \to \mathcal{C}^{\mathsf{add}}$ for some category $\mathcal{D}$, 0 is a natural transformation $F \Rightarrow G$. Indeed, $0\,Ff = 0 = Gf\,0$ by additivity of Gf. This observation will be used in the proofs that follow.

Lemma 7. *If for all $a \in A$, $(\alpha^a_X)_X \in \mathcal{C}^{\mathsf{add}}(FX, GY)$ is a natural transformation $F \Rightarrow G$ such that for all X, $(\alpha^a_X)_{a\in A}$ is summable, then $(\sum_{a\in A} \alpha^a_X \in \mathcal{C}^{\mathsf{add}}(FX, GY))_X$ is a natural transformation.*

Proof. This is a direct consequence of Proposition 7 and the definition of the sum, using the fact that 0 is a natural transformation $F \Rightarrow G$. □

Proposition 11. *The families c, θ, ι_i and l are additive, and they are natural transformations: $\mathsf{c} : \mathsf{S}^2 \Rightarrow \mathsf{S}^2$, $\theta : \mathsf{S}^2 \Rightarrow \mathsf{S}$, $\iota_i : \mathsf{Id} \Rightarrow \mathsf{S}$ and $\mathsf{l} : \mathsf{S} \Rightarrow \mathsf{S}^2$.*

Proof. All the morphisms above are defined as pairing of 0, projections, and sums of projections. Those basic blocks are all additive and are natural transformations (the sum of projections is additive and natural thanks to Lemmas 6 and 7). So by Proposition 7 and (Pair-Add), c, θ, ι and l are all additive and are natural transformations. □

Proposition 12. *The tuple $\underline{\mathsf{S}} = (\mathsf{S}, \iota_0, \theta)$ is a monad on $\mathcal{C}^{\mathsf{add}}$.*

Proof. We have

$$\pi_i \, \theta_X \, \iota_{0,\mathsf{S}X} = \sum_{l+r=i} \pi_l \, \pi_r \, \iota_{0,\mathsf{S}X} = \pi_i \, \mathsf{Id} = \pi_i$$

$$\pi_i \, \theta \, \mathsf{S}\zeta_X = \sum_{l+r=i} \pi_l \, \pi_r \, \mathsf{S}\iota_{0,X} = \sum_{l+r=i} \pi_l \, \iota_{0,X} \, \pi_r = \pi_i$$

using the naturality of π_r. Hence, $\theta \, \iota_{0,\mathsf{S}X} = \theta \, \mathsf{S}\zeta_X = \mathsf{Id}$ by joint monicity of the π_i's.

Next we have

$$\begin{aligned}
\pi_i \, \theta_X \, \theta_{\mathsf{S}X} &= \sum_{l+r=i} \pi_l \, \pi_r \, \theta_{\mathsf{S}X} \\
&= \sum_{l+r=i} \pi_l \sum_{s+t=r} \pi_s \, \pi_t \\
&= \sum_{l+r+s=i} \pi_l \, \pi_r \, \pi_s
\end{aligned}$$

and

$$\begin{aligned}
\pi_i \, \theta_X \, \mathsf{S}\theta_X &= \sum_{l+r=i} \pi_l \, \pi_r \, \mathsf{S}\theta_X \\
&= \sum_{l+r=i} \pi_l \, \theta_X \, \pi_r \\
&= \sum_{l+r=i} \Big(\sum_{j+k=l} \pi_j \, \pi_k \Big) \, \pi_r \\
&= \sum_{j+k+r=i} \pi_j \, \pi_k \, \pi_k
\end{aligned}$$

and hence $\theta_X \, \theta_{\mathsf{S}X} = \theta_X \, \mathsf{S}\theta_X$. □

Proposition 13. *The tuple $\overline{\mathsf{S}} = (\mathsf{S}, \sigma, \mathsf{l})$ is a comonad on $\mathcal{C}^{\mathsf{add}}$.*

Proof. We have

$$\pi_i \, \mathsf{S}\sigma_X \, \mathsf{l}_X = \sigma_X \, \pi_i \, \mathsf{l}_X = \sigma_X \, \iota_i \, \pi_i = \pi_i$$

$$\pi_i \, \sigma_{\mathsf{S}X} \, \mathsf{l}_X = \pi_i \sum_{j=0}^{\infty} \iota_j \, \pi_j = \sum_{j=0}^{\infty} \pi_i \, \iota_j \, \pi_j = \pi_i$$

So by joint monicity of the π_i, $\mathsf{S}\sigma_X \, \mathsf{l}_X = \sigma_{\mathsf{S}X} \, \mathsf{l}_X = \mathsf{id}_X$. Next we have

$$\pi_i \, \pi_j \, \mathsf{S}\mathsf{l}_X \mathsf{l}_X = \pi_i \, \mathsf{l}_X \, \pi_j \, \mathsf{l}_X = \iota_i \, \pi_i \, \iota_j \, \pi_j = \begin{cases} \iota_i \, \pi_i \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

$$\pi_i \;\; \pi_j \, \mathsf{l}_{\mathsf{S}X} \, \mathsf{l}_X = \pi_i \, \iota_j \, \pi_j \, \mathsf{l} = \pi_i \, \iota_j \, \iota_j \, \pi_j = \begin{cases} \iota_i \, \pi_i \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

So by joint monicity of the $\pi_i \, \pi_j$ we have $\mathsf{S}\mathsf{l}_X \mathsf{l}_X = \mathsf{l}_{\mathsf{S}X} \, \mathsf{l}_X$, and we conclude. □

The monad structure and the comonad structure are compatible together, in the sense that they form a c-bimonad, recall Definition 73.

Lemma 8. *Here are some useful observations on* c*:* c *is involutive,* $\mathsf{c} = \langle\!\langle \mathsf{S}\pi_i \rangle\!\rangle_{i=0}^{\infty}$ *and* $\mathsf{S}\pi_i \, \mathsf{c} = \pi_i$.

Proof. First, $\pi_i \, \pi_j \, \mathsf{c} \, \mathsf{c} = \pi_j \, \pi_i \, \mathsf{c} = \pi_i \, \pi_i \, \mathsf{id}$ so by joint monicity of the $\pi_i \, \pi_j$, $\mathsf{c} \, \mathsf{c} = \mathsf{id}$. Furthermore, $\pi_i \, \pi_j \, \mathsf{c} = \pi_j \, \pi_i = \pi_i \, \mathsf{S}\pi_j$ so by joint monicity of the π_i, $\pi_j \, \mathsf{c} = \mathsf{S}\pi_j$. That is, $\mathsf{c} = \langle\!\langle \mathsf{S}\pi_j \rangle\!\rangle_{i=0}^{\infty}$. Finally, $\pi_i \, \mathsf{c} = \mathsf{S}\pi_i$ so using the fact that c is involutive, $\pi_i = \mathsf{S}\pi_i \, \mathsf{c}$. □

Proposition 14. *The natural transformation* c *is a distributive law* $\underline{\mathsf{S}}\,\overline{\mathsf{S}} \Rightarrow \overline{\mathsf{S}}\,\underline{\mathsf{S}}$ *and* $\overline{\mathsf{S}}\,\underline{\mathsf{S}} \Rightarrow \underline{\mathsf{S}}\,\overline{\mathsf{S}}$.

Proof. By Remark 37 it is only necessary to show that c is a distributive law $\underline{\mathsf{S}}\,\overline{\mathsf{S}} \Rightarrow \overline{\mathsf{S}}\,\underline{\mathsf{S}}$ since c is involutive. The first condition is that c is a distributive law $\underline{\mathsf{S}}\mathsf{S} \Rightarrow \mathsf{S}\underline{\mathsf{S}}$. It corresponds to the two diagrams below. For the sake of readability, we write $\underline{\mathsf{S}}$ instead of just S to make clear which part is playing the role of the monad.

$$\begin{array}{ccc} \mathsf{S} & & \\ {\scriptstyle \iota_0 \mathsf{S}} \downarrow & \searrow^{\mathsf{S}\iota_0} & \\ \underline{\mathsf{S}}\mathsf{S} & \xrightarrow[\mathsf{c}]{} & \mathsf{S}\underline{\mathsf{S}} \end{array} \qquad \begin{array}{ccccc} \underline{\mathsf{S}}^2\mathsf{S} & \xrightarrow{\underline{\mathsf{S}}\mathsf{c}} & \underline{\mathsf{S}}\mathsf{S}\underline{\mathsf{S}} & \xrightarrow{\mathsf{c}\underline{\mathsf{S}}} & \mathsf{S}\underline{\mathsf{S}}^2 \\ {\scriptstyle \theta \mathsf{S}} \downarrow & & & & \downarrow {\scriptstyle \mathsf{S}\theta} \\ \underline{\mathsf{S}}\mathsf{S} & & \xrightarrow[\mathsf{c}]{} & & \mathsf{S}\underline{\mathsf{S}} \end{array}$$

We show both diagram using the joint monicity of the $\pi_i \, \pi_j$.

$$\pi_i \, \pi_j \, \mathsf{c} \, \iota_0 = \pi_i \, \pi_i \, \iota_0 = \begin{cases} \pi_j \text{ if } i = 0 \\ 0 \text{ otherwise} \end{cases}$$

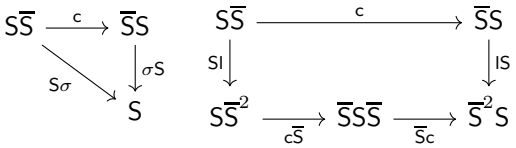
$$\pi_i \, \pi_j \, \mathsf{S}\iota_0 = \pi_i \, \iota_0 \, \pi_j = \begin{cases} \pi_j \text{ if } i = 0 \\ 0 \text{ otherwise} \end{cases}$$

So the left diagram holds.

$$\pi_i \, \pi_j \, \mathsf{c} \, \theta_{\mathsf{S}X} = \pi_j \, \pi_i \, \theta_{\mathsf{S}X} = \pi_j \left(\sum_{k=0}^{i} \pi_k \, \pi_{i-k} \right)$$

$$\begin{aligned}
\pi_i\,\pi_j\,\mathsf{S}\theta_X\,\mathsf{c}_{\mathsf{S}X}\,\mathsf{S}\mathsf{c}_X &= \pi_i\,\theta_X\,\pi_j\,\mathsf{c}_{\mathsf{S}X}\,\mathsf{S}\mathsf{c}_X \quad \text{naturality of } \pi_j \\
&= \pi_i\,\theta_X\,\mathsf{S}\pi_j\,\mathsf{S}\mathsf{c}_X \\
&= \pi_i\,\theta_X\,\mathsf{S}^2\pi_j \quad \text{functoriality of } \mathsf{S} \\
&= (\sum_{k=0}^{i} \pi_k\,\pi_{i-k})\,\mathsf{S}^2\pi_j \\
&= \pi_j\,(\sum_{k=0}^{i} \pi_k\,\pi_{i-k}) \quad \text{the sum is natural by Lemma 7}
\end{aligned}$$

So the right diagram holds. Next, we show that c is a distributive law $\mathsf{S}\overline{\mathsf{S}} \Rightarrow \overline{\mathsf{S}}\mathsf{S}$. It corresponds to the two diagrams below. For the sake of readability, we write $\overline{\mathsf{S}}$ instead of just S to make clear which part is playing the role of the comonad.



The left diagram is a consequence of the joint monicity of the π_i, using the naturality of σ, of π_i, and using the fact that $\mathsf{S}\pi_i\,\mathsf{c} = \pi_i$ (see Lemma 8).

$$\pi_i\,\sigma_{\mathsf{S}X}\,\mathsf{c} = \sigma_{\mathsf{S}X}\,\mathsf{S}\pi_i\,\mathsf{c} = \sigma_{\mathsf{S}X}\,\pi_i = \pi_i\,\mathsf{S}\sigma_X$$

The right diagram is a consequence of the joint monicity of $\pi_i\,\pi_j$.

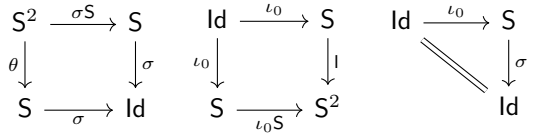
$$\begin{aligned}
\pi_i\,\pi_j\,\mathsf{S}\mathsf{c}_X\,\mathsf{c}_{\mathsf{S}X}\,\mathsf{S}\mathsf{l}_X &= \pi_i\,\mathsf{c}_X\,\pi_j\,\mathsf{c}_{\mathsf{S}X}\,\mathsf{S}\mathsf{l}_X \\
&= \mathsf{S}\pi_i\,\mathsf{S}\pi_j\,\mathsf{S}\mathsf{l}_X \\
&= \mathsf{S}(\pi_i\,\pi_j\,\mathsf{l}) \\
&= \begin{cases} \mathsf{S}0 = 0 \text{ if } i \neq j \\ \pi_i \text{ otherwise} \end{cases}
\end{aligned}$$

$$\pi_i\,\pi_j\,\mathsf{l}_{\mathsf{S}X}\,\mathsf{c}_X = \begin{cases} 0 \text{ if } i \neq j \\ \pi_i\,\mathsf{c} = \mathsf{S}\pi_i \text{ otherwise} \end{cases}$$

□

Theorem 6. *The monad* $\underline{\mathsf{S}}$ *and the comonad* $\overline{\mathsf{S}}$ *form a* c*-bimonad on* $\mathcal{C}^{\mathsf{add}}$.

Proof. The first three diagrams turning $\underline{\mathsf{S}}$ and $\overline{\mathsf{S}}$ into a c-bimonad are the following.



The left diagram holds thanks to the computation below that relies on (S-ass) and Proposition 8.

$$\sigma_X\,\theta_X = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\pi_k\,\pi_{n-k}\right) = \sum_{(i,j)\in\mathbb{N}^2}\pi_i\,\pi_j = \left(\sum_{i\in\mathbb{N}}\pi_i\right)\left(\sum_{j\in\mathbb{N}}\pi_j\right) = \sigma\,\sigma$$

The center diagram holds by a straightforward computation using the joint monicity of the $\pi_i\,\pi_j$ and some case analysis. The rightmost diagram holds by (S-zero). The last diagram is the following.

$$\begin{array}{ccccc} \mathsf{SS} & \xrightarrow{\theta} & \mathsf{S} & \xrightarrow{\mathsf{l}} & \mathsf{SS} \\ {\scriptstyle \mathsf{l}*\mathsf{l}}\downarrow & & & & \uparrow{\scriptstyle \theta*\theta} \\ \mathsf{SSSS} & & \xrightarrow[\mathsf{ScS}]{} & & \mathsf{SSSS} \end{array}$$

We use the joint monicity of the $\pi_i\,\pi_j$. We check the top path first.

$$\pi_i\,\pi_j\,\mathsf{l}_X\,\theta_X = \begin{cases} \pi_i\,\theta_X = \sum_{k=0}^{i} \pi_k\,\pi_{i-k} \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

For the bottom part, we first compute $\pi_i\,\pi_j\,(\theta * \theta)_X$.

$$\begin{aligned} \pi_i\,\pi_j\,(\theta*\theta)_X &= \pi_i\,\pi_j\,\mathsf{S}\theta_X\,\theta_{\mathsf{S}\mathsf{S}X} \\ &= \pi_i\,\theta_X\,\pi_j\,\theta_{\mathsf{S}\mathsf{S}X} \\ &= \left(\sum_{k=0}^{i} \pi_k\,\pi_{i-k}\right)\left(\sum_{k=0}^{j} \pi_l\,\pi_{j-l}\right) \\ &= \sum_{k\in[\![0,i]\!],l\in[\![0,j]\!]} \pi_k\,\pi_{i-k}\,\pi_l\,\pi_{j-l} \end{aligned}$$

Now observe that

$$\begin{aligned} \pi_{i_4}\,\pi_{i_3}\,\pi_{i_2}\,\pi_{i_1}\,\mathsf{Sc}_{\mathsf{S}X}\,(\mathsf{l}*\mathsf{l})_X &= \pi_{i_4}\,\pi_{i_3}\,\pi_{i_2}\,\pi_{i_1}\,\mathsf{Sc}_{\mathsf{S}X}\,\mathsf{SSl}_X\,\mathsf{l}_{\mathsf{S}X} \\ &= \pi_{i_4}\,\pi_{i_3}\,\pi_{i_2}\,\mathsf{c}_{\mathsf{S}X}\,\pi_{i_1}\,\mathsf{SSl}_X\,\mathsf{l}_{\mathsf{S}X} \\ &= \pi_{i_4}\,\pi_{i_2}\,\pi_{i_3}\,\pi_{i_1}\,\mathsf{SSl}_X\,\mathsf{l}_{\mathsf{S}X} \\ &= \pi_{i_4}\,\pi_{i_2}\,\mathsf{l}_X\,\pi_{i_3}\,\pi_{i_1}\,\mathsf{l}_{\mathsf{S}X} \\ &= \begin{cases} \pi_{i_2}\,\pi_{i_1} \text{ if } i_1 = i_3 \text{ and } i_2 = i_4 \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} \pi_i\,\pi_j\,(\theta*\theta)_X\,\mathsf{Sc}_{\mathsf{S}X}\,(\mathsf{l}*\mathsf{l})_X &= \left(\sum_{k\in[\![0,i]\!],l\in[\![0,j]\!]} \pi_k\,\pi_{i-k}\,\pi_l\,\pi_{j-l}\right)\mathsf{Sc}_{\mathsf{S}X}\,(\mathsf{l}*\mathsf{l})_X \\ &= \sum_{k\in[\![0,i]\!],l\in[\![0,j]\!]} \pi_k\,\pi_{i-k}\,\pi_l\,\pi_{j-l}\,\mathsf{Sc}_{\mathsf{S}X}\,(\mathsf{l}*\mathsf{l})_X \\ &= \begin{cases} \sum_{k=0}^{i} \pi_k\,\pi_{i-l} \text{ if } i = j \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

We conclude that the diagram commutes. □

1.5 Summability structure in models of LL

Assume that $\mathcal{L}$ is a category equipped with an ω-summability structure $(\mathsf{S}, \vec{\pi}, \sigma)$. Since $\mathcal{L}^{\mathsf{add}} = \mathcal{L}$, we write the composition of $f \in \mathcal{L}(X,Y)$ with $g \in \mathcal{L}(Y,Z)$ as $g\,f$. Then as seen in Section 1.4, S is a bimonad on $\mathcal{L}$. The category $\mathcal{L}$ is typically a model of LL, but it does not to be a full-fledged model, so we will detail any assumption in use.

1.5.1 Interaction with the monoidal structure

Assume that $\mathcal{L}$ is a symmetric monoidal category. We write $(\otimes, 1, \lambda, \rho, \alpha, \gamma)$ the symmetric monoidal structure. We show how the ω-summability structure should relate with this structure.

Definition 14. The ω-summability structure follows (S$\otimes$-dist) if for all objects X_0, X_1 $(\pi_i \otimes X_1)_{i=0}^{\infty} \in \mathcal{L}(\mathsf{S}X_0 \otimes X_1, X_0 \otimes X_1)$ and $(X_0 \otimes \pi_i \in \mathcal{L}(X_0 \otimes \mathsf{S}X_1, X_0 \otimes X_1))_{i=0}^{\infty}$ are summable, with respective sums $\sigma \otimes X_1$ and $X_1 \otimes \sigma$. Define

$$\varphi^0_{X_0,X_1} = \langle\!\langle \pi_i \otimes X_1 \rangle\!\rangle_{i=0}^{\infty} \in \mathcal{L}(\mathsf{S}X_0 \otimes X_1, \mathsf{S}(X_0 \otimes X_1)) \text{ so that } \sigma\, \varphi^0 = (\sigma \otimes X_1)$$

$$\varphi^1_{X_0,X_1} = \langle\!\langle X_0 \otimes \pi_i \rangle\!\rangle_{i=0}^{\infty} \in \mathcal{L}(X_0 \otimes \mathsf{S}X_1, \mathsf{S}(X_0 \otimes X_1)) \text{ so that } \sigma\, \varphi^1 = (X_0 \otimes \sigma)$$

Those are natural transformations thanks to Proposition 7.

Remark 7. Because the monoidal product is symmetric, only one of the assumption above is actually necessary: φ^1 can be defined from φ^0 as $\varphi^1 = \mathsf{S}\gamma^{\otimes}\, \varphi^0\, \gamma^{\otimes}$ and vice versa.

Proposition 15. *The ω-summability structure follows (S$\otimes$-dist) if and only if whenever $(f_i \in \mathcal{L}(X_0, Y_0))_{i=0}^{\infty}$ and $(g_i \in \mathcal{L}(X_1, Y_1))_{i=0}^{\infty}$ are summable, then for all $f \in \mathcal{L}(X_0, Y_0)$ and $g \in \mathcal{L}(X_1, Y_1)$, $(f_i \otimes g)_{i=0}^{\infty}$ is summable, $(f \otimes g_i)_{i=0}^{\infty}$ is summable, and*

$$\sum_{i=0}^{\infty} f_i \otimes g = \left(\sum_{i=0}^{\infty} f_i\right) \otimes g \qquad \sum_{i=0}^{\infty} f \otimes g_i = f \otimes \left(\sum_{i=0}^{\infty} g_i\right)$$

Proof. The converse direction is trivial, taking $f_i = \pi_i$ and $g_i = \pi_i$. For the forward direction, define $w = \varphi^0_{Y_0,Y_1}\, (\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} \otimes g)$. Then $\pi_i\, w = (\pi_i \otimes Y_1)\, (\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} \otimes g) = f_i \otimes g$. So $(f_i \otimes g)_{i=0}^{\infty}$ is summable of sum $\sigma\, w = (\sigma \otimes Y_1)\, (\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} \otimes g) = (\sum_{i=0}^{\infty} f_i) \otimes g$. We do the same for $(f \otimes g_i)_{i=0}^{\infty}$ and conclude. ☐

Proposition 16. *If $(f_i \in \mathcal{L}(X_0, Y_0))_{i\in\mathbb{N}}$ and $(g_j \in \mathcal{L}(X_1, Y_1))_{j\in\mathbb{N}}$ are summable, then $(f_i \otimes g_j)_{(i,j)\in\mathbb{N}^2}$ is summable and*

$$\sum_{(i,j)\in\mathbb{N}^2} f_i \otimes g_j = \left(\sum_{i\in\mathbb{N}} f_i\right) \otimes \left(\sum_{j\in\mathbb{N}} g_j\right)$$

Proof. Observe that $f_i \otimes g_j = (f_i \otimes Y_1)\, (X_0 \otimes g_j)$. By (S$\otimes$-dist), $(f_i \otimes Y_1)_{i\in\mathbb{N}}$ is summable of sum $(\sum_{i=0}^{\infty} f_i) \otimes Y_1$ and $(X_0 \otimes g_j)_{j\in\mathbb{N}}$ is summable of sum $X_0 \otimes (\sum_{j=0}^{\infty} g_j)$. So by Proposition 8, $(f_i \otimes g_j)_{(i,j)\in\mathbb{N}^2}$ is summable and

$$\sum_{(i,j)\in\mathbb{N}^2} f_i \otimes g_j = \left(\left(\sum_{i\in\mathbb{N}} f_i\right) \otimes Y_1\right) \left(X_0 \otimes \left(\sum_{j\in\mathbb{N}} g_j\right)\right) = \left(\sum_{i\in\mathbb{N}} f_i\right) \otimes \left(\sum_{j\in\mathbb{N}} g_j\right)$$

☐

Theorem 7. *The natural transformations φ^0 and φ^1 are a strength for the monad $\underline{\mathsf{S}}$ (see Definition 67). Furthermore, the monad $\underline{\mathsf{S}}$ equipped with φ^0 and φ^1 is commutative. The commutativity of the monad is a consequence of the commutation of the following diagram.*

$$\begin{array}{ccccc}
\mathsf{S}(\mathsf{S}X \otimes Y) & \xleftarrow{\varphi^1_{\mathsf{S}X,Y}} & \mathsf{S}X \otimes \mathsf{S}Y & \xrightarrow{\varphi^0_{X,\mathsf{S}Y}} & \mathsf{S}(X \otimes \mathsf{S}Y) \\
\downarrow{\scriptstyle \mathsf{S}\varphi^0_{X,Y}} & & & & \downarrow{\scriptstyle \mathsf{S}\varphi^1_{X,Y}} \\
\mathsf{S}^2(X \otimes Y) & & \xrightarrow[\mathsf{c}_{X\otimes Y}]{} & & \mathsf{S}^2(X \otimes Y)
\end{array}$$

Proof. Let us check that φ^1 is a strength.

$$\begin{array}{ccc} 1 \otimes \mathsf{S}X & \xrightarrow{\varphi^1} & \mathsf{S}(1 \otimes X) \\ & \searrow^{\lambda_{\mathsf{S}X}} & \downarrow{\scriptstyle \mathsf{S}\lambda_X} \\ & & \mathsf{S}X \end{array} \qquad \begin{array}{ccc} (X \otimes Y) \otimes \mathsf{S}Z & \xrightarrow{\varphi^1_{X\otimes Y,Z}} & \mathsf{S}((X \otimes Y) \otimes Z) \\ {\scriptstyle \alpha_{X,Y,\mathsf{S}Z}}\downarrow & & \downarrow{\scriptstyle \mathsf{S}\alpha_{X,Y,Z}} \\ X \otimes (Y \otimes \mathsf{S}Z) & \xrightarrow[X\otimes\varphi^1_{Y,Z}]{} X \otimes \mathsf{S}(Y \otimes Z) & \xrightarrow[\varphi^1_{X,Y\otimes Z}]{} \mathsf{S}(X \otimes (Y \otimes Z)) \end{array}$$

Those two diagrams above are just routine check by joint monicity of the π_i, their naturality (recall that $\mathcal{L} = \mathcal{L}^{\mathsf{add}}$) and the naturality of λ and α.

$$\begin{array}{ccc} X \otimes Y & & \\ {\scriptstyle X\otimes\iota_0}\downarrow & \searrow^{\iota_0} & \\ X \otimes \mathsf{S}Y & \xrightarrow[\varphi^1]{} & \mathsf{S}(X \otimes Y) \end{array} \qquad \begin{array}{ccc} X \otimes \mathsf{S}^2Y & \xrightarrow{\varphi^1_{X,\mathsf{S}Y}} \mathsf{S}(X \otimes \mathsf{S}Y) & \xrightarrow{\mathsf{S}\varphi^1_{X,Y}} \mathsf{S}^2(X \otimes Y) \\ {\scriptstyle X\otimes\theta_Y}\downarrow & & \downarrow{\scriptstyle \theta_{X\otimes Y}} \\ X \otimes \mathsf{S}Y & \xrightarrow[\varphi^1_{X,Y}]{} & \mathsf{S}(X \otimes Y) \end{array}$$

The left diagram is a consequence of the fact that $X \otimes 0 = 0$.

$$\pi_i \, \varphi^1_{X,Y} \, (X \otimes \iota_0) = X \otimes (\pi_i \, \iota_0) = \begin{cases} X \otimes 0 = 0 \text{ if } i \neq 0 \\ X \otimes Y \text{ otherwise} \end{cases} = \pi_i \, \iota_0$$

The right diagram is a consequence of the distributivity of the sum of $\otimes$.

$$\pi_i \, \theta_{X\otimes Y} \, \mathsf{S}\varphi^1_{X,Y} \, \varphi^1_{X,\mathsf{S}Y} = \sum_{k=0}^{i} \pi_k \, \pi_{i-k} \, \mathsf{S}\varphi^1_{X,Y} \, \varphi^1_{X,\mathsf{S}Y} = \sum_{k=0}^{i} \pi_k \, \varphi^1_{X,Y} \, \pi_{i-k} \, \varphi^1_{X,\mathsf{S}Y} = \sum_{k=0}^{i} (X \otimes \pi_k \, \pi_{i-k})$$

$$\pi_i \, \varphi^1 \, (X \otimes \theta_Y) = (X \otimes \pi_i) \, (X \otimes \theta_Y) = X \otimes (\sum_{k=0}^{i} \pi_k \, \pi_{i-k})$$

So the monad $\underline{\mathsf{S}}$ is strong. Finally, the monad is commutative if the diagram below commutes.

$$\begin{array}{ccccc} \mathsf{S}(\mathsf{S}X \otimes Y) & \xleftarrow{\varphi^1_{\mathsf{S}X,Y}} & \mathsf{S}X \otimes \mathsf{S}Y & \xrightarrow{\varphi^0_{X,\mathsf{S}Y}} & \mathsf{S}(X \otimes \mathsf{S}Y) \\ {\scriptstyle \mathsf{S}\varphi^0_{X,Y}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{S}\varphi^1_{X,Y}} \\ \mathsf{S}^2(X \otimes Y) & \xrightarrow[\theta_{X\otimes Y}]{} & \mathsf{S}(X \otimes Y) & \xleftarrow[\theta_{X\otimes Y}]{} & \mathsf{S}^2(X \otimes Y) \end{array} \tag{4}$$

This is a consequence of stronger property, which is as announced the commutation of the diagram below.

$$\begin{array}{ccccc} \mathsf{S}(\mathsf{S}X \otimes Y) & \xleftarrow{\varphi^1_{\mathsf{S}X,Y}} & \mathsf{S}X \otimes \mathsf{S}Y & \xrightarrow{\varphi^0_{X,\mathsf{S}Y}} & \mathsf{S}(X \otimes \mathsf{S}Y) \\ {\scriptstyle \mathsf{S}\varphi^0_{X,Y}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{S}\varphi^1_{X,Y}} \\ \mathsf{S}^2(X \otimes Y) & & \xrightarrow[\mathsf{c}_{X\otimes Y}]{} & & \mathsf{S}^2(X \otimes Y) \end{array}$$

It is proved by the joint monicity of the $\pi_i \, \pi_j$.

$$\pi_i \, \pi_j \, \mathsf{c} \, \mathsf{S}\varphi^0 \, \varphi^1 = \pi_j \, \pi_i \, \mathsf{S}\varphi^0 \, \varphi^1 = \pi_j \, \varphi^0 \, \pi_i \, \varphi^1 = (\pi_j \otimes Y) \, (\mathsf{S}X \otimes \pi_j) = \pi_j \otimes \pi_i$$

$$\pi_i \, \pi_j \, \mathsf{S}\varphi^1 \, \varphi^0 = \pi_i \, \varphi^1 \, \pi_j \, \varphi^0 = (X \otimes \pi_i) \, (\pi_j \otimes \mathsf{S}Y) = \pi_j \otimes \pi_i$$

□

As discussed in Section 7.3, the commutative monad $\underline{\mathsf{S}}$ is then a lax symmetric monoidal monad (see Definition 66). The natural transformation $\mathsf{L}_{X_0,X_1} \in \mathcal{L}(\mathsf{S}X_0 \otimes \mathsf{S}X_1, \mathsf{S}(X_0 \otimes X_1))$ is defined as the diagonal of the diagram in Eq. (4). Observe that

$$\mathsf{L}_{X_0,X_1} = \left\langle\!\!\left\langle \sum_{k=0}^{i} \pi_k \otimes \pi_{i-k} \right\rangle\!\!\right\rangle_{i=0}^{\infty}$$

It means that if $(f_i \in \mathcal{L}(X_0, Y_0))_{i\in\mathbb{N}}$ and $(g_j \in \mathcal{L}(X_1, Y_1))_{j\in\mathbb{N}}$ are summable, then

$$\mathsf{L}_{Y_0,Y_1}\left(\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} \otimes \langle\!\langle g_j \rangle\!\rangle_{j=0}^{\infty}\right) = \left\langle\!\!\left\langle \sum_{k=0}^{i} f_k \otimes g_{i-k} \right\rangle\!\!\right\rangle_{i=0}^{\infty}$$

If one intuitively see $\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty}$ as the power series $\sum_{i=0}^{\infty} f_i X^i$ and $\langle\!\langle g_i \rangle\!\rangle_{i=0}^{\infty}$ as the power series $\sum_{j=0}^{\infty} g_j X^j$, then $\mathsf{L}_{Y_0,Y_1}\left(\langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} \otimes \langle\!\langle g_j \rangle\!\rangle_{j=0}^{\infty}\right)$ behaves as the Cauchy product of the two power series.

1.5.2 Interaction with the closed structure

Assume now that (S⊗-dist) holds and that the category is closed with regard to its symmetric monoidal structure.

Definition 15. The category $\mathcal{L}$ is closed (with regard to $\otimes$) if for all object A and X, the pair of objects (A, X) of $\mathcal{L}$ has an internal hom $(A \multimap X, \mathsf{ev}_A)$. That is, there exists $\mathsf{ev}_A \in \mathcal{L}((A \multimap X) \otimes A, X)$ such that for any morphism $f \in \mathcal{L}(X \otimes A, Y)$, there exists a unique morphism $\mathsf{cur}_A(f) \in \mathcal{L}(X, A \multimap Y)$ such that

$$\mathsf{ev}_A(\mathsf{cur}_A(f) \otimes A) = f$$

Then $\mathsf{cur}_A : \mathcal{L}(X \otimes A, Y) \to \mathcal{L}(X, A \multimap Y)$ is a bijection whose inverse is given by

$$\mathsf{cur}_A^{-1}(g) = \mathsf{ev}_A\,(g \otimes A)$$

We chose to label cur_A and ev_A with the object A (as opposed to the objects X and Y that are always kept implicit) because it makes some situations clearer, but we will often keep the object A implicit and simply write cur and ev.

If A is such that an internal hom of (A, X) exists for all X, then there is an adjunction $\_ \otimes A \dashv A \multimap \_$, of unit $\mathsf{cur}_A(\mathsf{id}_{X\otimes A}) \in \mathcal{L}(X, A \multimap (X \otimes A))$ and co-unit $\mathsf{ev}_A \in \mathcal{L}((A \multimap X) \otimes A, X)$. The functor $A \multimap \_$ maps a morphism $f \in \mathcal{L}(X, Y)$ to a morphism $A \multimap f \in \mathcal{L}(A \multimap X, A \multimap Y)$ defined as $\mathsf{cur}_A(f\,\mathsf{ev}_A)$. Then cur_A and cur_A^{-1} are natural bijections.

$$\mathsf{cur}_A(f\,g\,(h \otimes A)) = (A \multimap f)\,\mathsf{cur}_A(g)\,h$$
$$\mathsf{cur}_A^{-1}((A \multimap f)\,g\,h) = f\,\mathsf{cur}_A^{-1}(g)\,(h \otimes A)$$

We can define a natural morphism

$$\varphi^{\multimap} := \mathsf{cur}((\mathsf{Sev})\,\varphi^0_{A\multimap X,A}) \in \mathcal{L}(\mathsf{S}(A \multimap X), A \multimap \mathsf{S}X)$$

Proposition 17. *We have* $(A \multimap \pi_i)\,\varphi^{\multimap} = \pi_i$ *and* $(A \multimap \sigma)\,\varphi^{\multimap} = \sigma$

Proof. Observe that

$$\begin{aligned}(A \multimap \pi_i)\,\varphi^{\multimap} &= \mathsf{cur}(\pi_i\,(\mathsf{Sev})\,\varphi^0_{A\multimap X,A}) \quad \text{by naturality of } \mathsf{cur}\\ &= \mathsf{cur}(\mathsf{ev}\,\pi_i\,\varphi^0_{A\multimap X,A}) \quad \text{by naturality of } \pi_i\\ &= \mathsf{cur}(\mathsf{ev}\,(\pi_i \otimes A))\\ &= \mathsf{cur}(\mathsf{cur}^{-1}(\pi_i)) = \pi_i\end{aligned}$$

The equality $(A \multimap \sigma)\,\varphi^{\multimap} = \sigma$ is proved similarly, using that $\sigma\,\varphi^0_{X_0,X_1} = (\sigma \otimes X_1)$ □

Proposition 18. *If $(\mathsf{cur}(f_i))_{i=0}^{\infty}$ is summable with $f_i \in \mathcal{C}(X \otimes A, Y)$, then $(f_i)_{i=0}^{\infty}$ is summable and $\sum_{i=0}^{\infty} \mathsf{cur}(f_i) = \mathsf{cur}(\sum_{i=0}^{\infty} f_i)$.*

Proof. Let $w = \mathsf{cur}^{-1}(\varphi^{\multimap}_X\ \langle\!|\mathsf{cur}(f_i)|\!\rangle_{i=0}^{\infty}) \in \mathcal{L}(X \otimes A, \mathsf{S}Y)$. Then

$$\begin{aligned}\pi_i\,w &= \mathsf{cur}^{-1}((Y \multimap \pi_i)\,\varphi^{\multimap}_X\ \langle\!|\mathsf{cur}(f_i)|\!\rangle_{i=0}^{\infty}) \quad \text{by naturality of } \mathsf{cur}^{-1}\\ &= \mathsf{cur}^{-1}(\pi_i\ \langle\!|\mathsf{cur}(f_i)|\!\rangle_{i=0}^{\infty}) \quad \text{by Proposition 17}\\ &= f_i\end{aligned}$$

So w is a witness for $(f_i)_{i=0}^{\infty}$, and $\sum_{i=0}^{\infty} f_i = \sigma\,w$. But

$$\begin{aligned}\sigma\,w &= \mathsf{cur}^{-1}((Y \multimap \sigma)\,\varphi^{\multimap}_X\ \langle\!|\mathsf{cur}(f_i)|\!\rangle_{i=0}^{\infty}) \quad \text{by naturality of } \mathsf{cur}^{-1}\\ &= \mathsf{cur}^{-1}(\sigma\ \langle\!|\mathsf{cur}(f_i)|\!\rangle_{i=0}^{\infty}) \quad \text{by Proposition 17}\\ &= \mathsf{cur}^{-1}(\sum_{i=0}^{\infty} \mathsf{cur}(f_i))\end{aligned}$$

So $\sum_{i=0}^{\infty} f_i = \mathsf{cur}^{-1}(\sum_{i=0}^{\infty} \mathsf{cur}(f_i))$. It implies that $\mathsf{cur}(\sum_{i=0}^{\infty} f_i) = \sum_{i=0}^{\infty} \mathsf{cur}(f_i)$. □

Definition 16. The ω-summability structure follows (S⊗-fun) if $\varphi^{\multimap}$ is an isomorphism.

The invertibility of $\varphi^{\multimap}$ provides an only if condition for Proposition 18.

Remark 8. Monicity is preserved by right adjoint functors, meaning that the $A \multimap \pi_i$ are jointly monic. This fact can also be checked by hand in a rather straightforward way using the fact that cur is a bijection and is natural. This will be useful in the proof of Proposition 19 below.

Proposition 19. *The following are equivalent:*

(1) (S⊗-fun) holds;

(2) $(A \multimap \pi_i)_{i=0}^{\infty}$ is summable;

(3) if $(f_i \in \mathcal{L}(X,Y))_{i=0}^{\infty}$ is summable then $(A \multimap f_i)_{i=0}^{\infty}$ is summable;

(4) if $(f_i \in \mathcal{L}(X \otimes A, Y))_{i=0}^{\infty}$ is summable then $(\mathsf{cur}(f_i))_{i=0}^{\infty}$ is summable.

And then $(\varphi^{\multimap})^{-1} = \langle\!|A \multimap \pi_i|\!\rangle_{i=0}^{\infty}$.

Proof. (1) ⇒ (2) : Assume that $\varphi^{\multimap}$ is an isomorphism. By Proposition 17, $(A \multimap \pi_i) \circ \varphi^{\multimap} = \pi_i$. Then $A \multimap \pi_i = \pi_i \circ (\varphi^{\multimap})^{-1}$. Thus, $(A \multimap \pi_i)_{i=0}^{\infty}$ is summable of witness $(\varphi^{\multimap})^{-1}$.

(2) ⇒ (1): Assume that $(A \multimap \pi_i)_{i=0}^{\infty}$ is summable. Then by Proposition 17

$$\pi_i \circ \langle\!|A \multimap \pi_i|\!\rangle_{i=0}^{\infty} \circ \varphi^{\multimap} = (A \multimap \pi_i) \circ \varphi^{\multimap} = \pi_i$$

$$(A \multimap \pi_i) \circ \varphi^{\multimap} \circ \langle\!|A \multimap \pi_i|\!\rangle_{i=0}^{\infty} = \pi_i \circ \langle\!|A \multimap \pi_i|\!\rangle_{i=0}^{\infty} = A \multimap \pi_i$$

We conclude that $\langle\!\langle A \multimap \pi_i \rangle\!\rangle_{i=0}^{\infty} \circ \varphi^{\multimap} = \mathsf{id}_{\mathsf{S}(A \multimap X)}$ by joint monicity of the π_i, and that $\varphi^{\multimap} \circ \langle\!\langle A \multimap \pi_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{id}_{A \multimap \mathsf{S}X}$ by joint monicity of the $A \multimap \pi_i$ (see Remark 8).

(2) $\Rightarrow$ (3): Assume that $(f_i)_{i=0}^{\infty}$ is summable. Observe that $A \multimap f_i = (A \multimap \pi_i) \circ (A \multimap \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty})$. Then, by assumption $(A \multimap \pi_i)_{i=0}^{\infty}$ is summable so $(A \multimap f_i)_{i=0}^{\infty}$ is summable by left additivity.

(3) $\Rightarrow$ (2): (2) is a particular case of case (3), taking $f_i = \pi_i$.

(3) $\Rightarrow$ (4): By naturality of cur, $\mathsf{cur}(f_i) = (A \multimap f_i) \circ \mathsf{cur}(\mathsf{id}_{X \& A})$. By assumption, if $(f_i)_{i=0}^{\infty}$ is summable then $(A \multimap f_i)_{i=0}^{\infty}$ is summable so by left additivity, $(\mathsf{cur}(f_i))_{i=0}^{\infty}$ is summable.

(4) $\Rightarrow$ (3): Recall that $A \multimap f_i = \mathsf{cur}(f_i \circ \mathsf{ev})$. If $(f_i)_{i=0}^{\infty}$ is summable then $(f_i \circ \mathsf{ev})_{i=0}^{\infty}$ is summable by left additivity, so $(\mathsf{cur}(f_i \circ \mathsf{ev}))_{i=0}^{\infty}$ is summable by assumption. It concludes the proof. □

Remark 9. We recognize the mate construction, see Remark 36: $\varphi^{\multimap}$ is the mate of φ^0 through the adjunction $\_ \otimes A \dashv A \multimap \_$ taking $L = R = \mathsf{S}$. It is showed in Proposition 40 and discussed in Section 7.3 that φ^0 is a distributive law $\mathsf{S}\_ \otimes A \Rightarrow \mathsf{S}(\_ \otimes A)$. So by Theorem 33, (S⊗-fun) is a necessary and sufficient condition to ensure that the adjunction $\_ \otimes A \dashv A \multimap \_$ extends to $\mathcal{L}_{\mathsf{S}}$, meaning that $\mathcal{L}_{\mathsf{S}}$ is a symmetric monoidal closed category.

1.5.3 Interaction with the cartesian product

Assume now that $\mathcal{L}$ is also a cartesian category (with finite or countable products). We write the terminal object $\top$ and the cartesian product of $(X_i)_{i \in I}$ as $\&_{i \in I} X_i$, following the notations of LL. The projections are written as $\mathsf{p}_i \in \mathcal{L}(\&_{i \in I} X_i, X_i)$, and the pairing of the $f_i \in \mathcal{C}(X, Y_i)$ as $\langle f_i \rangle_{i \in I} \in \mathcal{L}(X, \&_{i \in I} Y_i)$. Let t_X be the unique morphism of $\mathcal{L}(X, \top)$. Observe that $\mathsf{t}_X = 0^{X, \top}$.

Definition 17. (S-&) The ω-summability structure is compatible with the cartesian product if the morphism

$$\mathsf{c}_{\&} = \langle \mathsf{S}\mathsf{p}_i \rangle_{i \in I} \in \mathcal{L}(\mathsf{S}(\underset{i \in I}{\&} X_i), \underset{i \in I}{\&} \mathsf{S}X_i)$$

is an isomorphism. See Section 3.2 for a motivation of this definition on the more general setting of left additive categories. It is shown in Proposition 27 that $\mathsf{c}_{\&}^{-1} = \langle\!\langle \&_{i \in I} \pi_j \rangle\!\rangle_{j=0}^{\infty}$.

Then we can check that the monad $\underline{\mathsf{S}}$ on $\mathcal{L}$ equipped with the natural transformations ι_0 and $\mathsf{c}_{\&}^{-1}$ is a *lax symmetric monoidal monad* (see Definition 66) with regard to the symmetric monoidal structure induced by the cartesian product. The diagram can be checked by hand, but this is a consequence of a more general observation. As mentioned in [AHF18] in paragraph 2.3, any monad $\underline{M} = (M, \eta, \mu)$ on a cartesian category can be endowed with the structure of an oplax symmetric monoidal monad (see Remark 34) taking n^0 to be the unique element of $\mathcal{C}(M\top, \top)$ and $n^2_{X_0, X_1} := \langle M\mathsf{p}_0, M\mathsf{p}_1 \rangle \in \mathcal{C}(M(X_0 \& X_1), MX_0 \& MX_1)$. Then it turns out that if $n^0 = \eta_{\top}^{-1}$ and if n^2 is invertible, then $(\underline{M}, n^0, n^2)$ is de facto a lax monoidal monad.

This is precisely what happens here, since $\iota_{0,\top} = \langle\!\langle \mathsf{t}_X \rangle\!\rangle_{i=0}^{\infty} = \mathsf{t}_{\mathsf{S}X}^{-1}$. Indeed, $\mathsf{t}_{\mathsf{S}\top} \langle\!\langle \mathsf{t}_{\top} \rangle\!\rangle_{i=0}^{\infty} = \mathsf{id}_{\top}$ simply because $\top$ is final, and $\pi_i \langle\!\langle \mathsf{t}_{\top} \rangle\!\rangle_{i=0}^{\infty} \mathsf{t}_{\mathsf{S}\top} \in \mathcal{C}(\mathsf{S}\top, \top)$ is necessarily equal to π_i (also because $\top$ is final), so by joint monicity of the π_i, $\langle\!\langle \mathsf{t}_{\top} \rangle\!\rangle_{i=0}^{\infty} \circ \mathsf{t}_{\mathsf{S}\top} = \mathsf{id}_{\mathsf{S}\top}$.

Remark 10. This oplax structure $\mathsf{c}_{\&}$ actually comes from the adjunction $\Delta \dashv \_ \,\&\, \_$ (where Δ is the diagonal functor defined in Definition 62) and the mate construction of Section 8 applied on the natural transformation $\mathsf{id} : \Delta M \Rightarrow (M \times M)\Delta$. As shown in Theorem 33, the invertibility of $\mathsf{c}_{\&}$ is then a necessary and sufficient condition to extend the adjunction

$\Delta \dashv \_ \,\&\, \_$ to the Kleisli category $\mathcal{L}_{\mathsf{S}}$, meaning that it is a cartesian category ($\top$ is final since $\iota_0 \in \mathcal{C}^{\mathsf{add}}(\top, \mathsf{S}\top)$ is an isomorphism).

As discussed in Section 7.3 (taking the symmetric monoidal structure to be the one generated by the cartesian structure &), it means that $\underline{\mathsf{S}}$ is a commutative monad where the two strength are defined as

$$\psi^0 = \mathsf{c}_{\&}^{-1}\left(\mathsf{T}X_0 \,\&\, \iota_0\right) = \langle\!\langle \pi_0 \,\&\, X_1, \pi_1 \,\&\, 0, \pi_2 \,\&\, 0, \ldots \rangle\!\rangle \in \mathcal{C}(\mathsf{S}X_0 \,\&\, X_1, \mathsf{S}(X_0 \,\&\, X_1))$$

$$\psi^1 = \mathsf{c}_{\&}^{-1}\left(\iota_0 \,\&\, \mathsf{T}X_1\right) = \langle\!\langle X_0 \,\&\, \pi_0, 0 \,\&\, \pi_1, 0 \,\&\, \pi_2, \ldots \rangle\!\rangle \in \mathcal{C}(X_0 \,\&\, \mathsf{S}X_1, \mathsf{S}(X_0 \,\&\, X_1))$$

1.5.4 Interaction with the resource comonad

Here, we only assume that $\mathcal{L}$ is equipped with a resource comonad.

Definition 18. A *resource comonad* is a tuple $(!, \mathsf{der}, \mathsf{dig}, \mathsf{m}^0, \mathsf{m}^2)$ where $(!, \mathsf{der}, \mathsf{dig})$ is a comonad on $\mathcal{L}$ (with counit der, called *dereliction*, and comultiplication dig called *digging*) and $(\mathsf{m}^0, \mathsf{m}^2)$ is a strong symmetric monoidal structure on the functor $!\_$ from the SMC $(\mathcal{L}, \&, \top)$ to the SMC $(\mathcal{L}, \otimes, 1)$ (see Definition 64) satisfying some coherence diagram that we will not recall here, see for instance [Mel09]. Recall in particular that $\mathsf{m}^0 \in \mathcal{L}(1, !\top)$ is an iso and $\mathsf{m}^2_{X,Y} \in \mathcal{L}(!X \otimes !Y, !(X \,\&\, Y))$ is a natural isomorphism.

Let $\mathcal{L}_!$ be the co-Kleisli category of this comonad. It has the same object as $\mathcal{L}$ and $\mathcal{L}_!(X,Y) = \mathcal{L}(!X, Y)$. The identity in this category is defined as $\mathsf{der}_X \in \mathcal{L}_!(X,X)$. The composition of $f \in \mathcal{L}_!(X,Y)$ with $g \in \mathcal{L}_!(Y,Z)$ is defined as $g \circ f = g\,!f\,\mathsf{dig}_X$. There is a functor $\mathsf{Der} : \mathcal{L} \to \mathcal{L}_!$ defined as $\mathsf{Der}\,X = X$ and $\mathsf{Der}\,h = h\,\mathsf{der}$.

Proposition 20. *For any $f \in \mathcal{L}_!(X,Y)$ and $h \in \mathcal{L}(Y,Z)$, $\mathsf{Der}\,h \circ f = h\,f$. For any $g \in \mathcal{L}_!(Y,Z)$ and $h \in \mathcal{L}(X,Y)$, $g \circ \mathsf{Der}\,h = g\,!h$*

Proof. By naturality of der and triangle identity of the comonad, $\mathsf{Der}\,h \circ f = h\,\mathsf{der}_Y\,!f\,\mathsf{dig}_X = h\,f\,\mathsf{der}_{!X}\,\mathsf{dig}_X = h\,f$. By triangle identity of the comonad, $g \circ \mathsf{Der}\,h = g\,!h\,!\mathsf{der}_X\,\mathsf{dig}_X = g\,!h$. □

We show that the ω-summability structure on $\mathcal{L}$ induces a left ω-summability structure on $\mathcal{L}_!$.

Proposition 21. *The structure $(\mathsf{S}, (\mathsf{Der}\,\pi_i)_{i=0}^\infty, \mathsf{Der}\,\sigma)$ is a left pre ω-summability structure on $\mathcal{L}_!$, $\mathsf{Der}\,\langle\!\langle h_i \rangle\!\rangle_{i=0}^\infty = \langle\!\langle \mathsf{Der}\,h_i \rangle\!\rangle_{i=0}^\infty$, and $\mathsf{Der}\sum_{i=0}^\infty h_i = \sum_{i=0}^\infty \mathsf{Der}\,h_i$. In particular, for any $h \in \mathcal{L}(X,Y)$, $\mathsf{Der}\,h$ is additive.*

Proof. For any $f \in \mathcal{L}_!(X, \mathsf{S}Y)$, $\mathsf{Der}\,\pi_i \circ f = \pi_i\,f$ so the $\mathsf{Der}\,\pi_i$ are jointly monic and $(f_i \in \mathcal{L}_!(X,Y))_{i=0}^\infty$ is summable in $\mathcal{L}_!$ if and only $(f_i \in \mathcal{L}(!X,Y))_{i=0}^\infty$ is summable in $\mathcal{L}$. Also, $\mathsf{Der}\,\sigma \circ \langle\!\langle f_i \rangle\!\rangle_{i=0}^\infty = \sigma\,\langle\!\langle f_i \rangle\!\rangle_{i=0}^\infty$ so the sums agree. Furthermore, $\mathsf{Der}\,\langle\!\langle h_i \rangle\!\rangle_{i=0}^\infty = \langle\!\langle \mathsf{Der}\,h_i \rangle\!\rangle_{i=0}^\infty$ by joint monicity of the $\mathsf{Der}\,\pi_i$ since $\mathsf{Der}\,\pi_i \circ \langle\!\langle \mathsf{Der}\,h_i \rangle\!\rangle_{i=0}^\infty = \pi_i\,\langle\!\langle \mathsf{Der}\,h_i \rangle\!\rangle_{i=0}^\infty = \mathsf{Der}\,h_i = \mathsf{Der}\,(\pi_i\,\langle\!\langle h_i \rangle\!\rangle_{i=0}^\infty) = \mathsf{Der}\,\pi_i \circ \mathsf{Der}\,\langle\!\langle h_i \rangle\!\rangle_{i=0}^\infty$. Then $\mathsf{Der}\,\sigma \circ \langle\!\langle \mathsf{Der}\,h_i \rangle\!\rangle_{i=0}^\infty = \mathsf{Der}\,\sigma \circ \mathsf{Der}\,\langle\!\langle h_i \rangle\!\rangle_{i=0}^\infty = \mathsf{Der}\,(\sum_{i=0}^\infty h_i)$. In particular, if $h \in \mathcal{L}(X,Y)$ then $\mathsf{Der}\,h$ is additive. Indeed, $\mathsf{Der}\,h \circ 0 = h\,0 = 0$ and because $(h\,\pi_i)_{i=0}^\infty$ is summable of sum $h \circ \sigma$, then $(\mathsf{Der}\,h \circ \mathsf{Der}\,\pi_i)_{i=0}^\infty = (\mathsf{Der}\,(h\,\pi_i))_{i=0}^\infty$ is summable of witness $\mathsf{Der}\,\mathsf{S}h$ and sum $\mathsf{Der}\,(h\,\sigma) = \mathsf{Der}\,h \circ \mathsf{Der}\,\sigma$. By Proposition 3 it means that $\mathsf{Der}\,h$ is additive.

Then $\mathsf{Der}\,\pi_i$ and $\mathsf{Der}\,\sigma$ are all additive so $(\mathsf{S}, (\mathsf{Der}\,\pi_i)_{i=0}^\infty, \mathsf{Der}\,\sigma)$ is a left pre ω-summability structure on $\mathcal{L}_!$. □

As we observed, $(f_i \in \mathcal{L}_!(X,Y))_{i=0}^\infty$ is summable in $\mathcal{L}_!$ if and only $(f_i \in \mathcal{L}(!X,Y))_{i=0}^\infty$ is summable in $\mathcal{L}$, so this structure follows (S-com). It immediately follows that a family $(f_a \in$

$\mathcal{L}_!(X,Y))_{a\in a}$ is summable in $\mathcal{L}_!$ if and only if $(f_a \in \mathcal{L}(!X,Y))_{a\in A}$ is summable in $\mathcal{L}$, and the sums are equal. As a result, $(\mathsf{S}, (\mathsf{Der}\,\pi_i)_{i=0}^\infty, \mathsf{Der}\,\sigma)$ follows (S-ass), (S-zero) and (S-wit). It also follows from the fact that $\mathsf{Der}\,\langle\!\langle h_i \rangle\!\rangle_{i=0}^\infty = \langle\!\langle \mathsf{Der}\,h_i \rangle\!\rangle_{i=0}^\infty$ that if $(h_a \in \mathcal{L}(X,Y))_{a\in A}$ is summable in $\mathcal{L}$ then $(\mathsf{Der}\,h_a \in \mathcal{L}_!(X,Y))_{a\in a}$ is summable in $\mathcal{L}_!$ and $\sum_{a\in A} \mathsf{Der}\,h_a = \mathsf{Der}\sum_{a\in A} h_a$. As a result $(\mathsf{S}, (\mathsf{Der}\,\pi_i)_{i=0}^\infty, \mathsf{Der}\,\sigma)$ follows (S-flat) and is a left ω-summability structure on $\mathcal{L}_!$.

Remark 11. Let $\mathsf{S}^{\mathsf{add}}$ be the functor on $\mathcal{L}_!^{\mathsf{add}}$ induced by the ω-summability structure $(\mathsf{S}, (\mathsf{Der}\,\pi_i)_{i=0}^\infty, \mathsf{Der}\,\sigma)$ on $\mathcal{L}_!$, so that there is no ambiguity with the functor S on $\mathcal{L}$. In the proof of Proposition 21 we saw that for any $h \in \mathcal{L}(X,Y)$, $(\mathsf{Der}\,h \circ \mathsf{Der}\,\pi_i)_{i=0}^\infty$ is summable of witness $\mathsf{Der}\,\mathsf{S}h$. It means by definition of $\mathsf{S}^{\mathsf{add}}$ that

$$\mathsf{S}^{\mathsf{add}}(\mathsf{Der}\,h) = \mathsf{Der}\,(\mathsf{S}h)$$

so $\mathsf{S}^{\mathsf{add}}$ is an extension of S to $\mathcal{L}_!^{\mathsf{add}}$.

Definition 19. An ω-summable resource category $\mathcal{L}$ is a cartesian and a symmetric monoidal category equipped with a resource comonad, as well as an ω-summability structure following (S$\otimes$-dist) and (S-&).

We do not assume (S$\otimes$-fun) in the definition above as it is not crucial to define Taylor expansion, so any use of it will be made explicit.

2 Taylor expansion as a distributive law in models of LL

We have all the necessary tool to axiomatize Taylor expansion in models of LL. Assume that $\mathcal{L}$ is an infinitary summable resource category. In particular, we do not need to assume (S$\otimes$-fun) for this section to make sense. As seen in Section 1.5.4, $\mathcal{L}_!$ is then endowed with a left ω-summability structure $(\mathsf{S}, (\mathsf{Der}\,\pi_i)_{i=0}^\infty, \mathsf{Der}\,\sigma)$. We provide the intuitions first on what the Taylor expansion operator should look like.

2.1 Motivation

The idea behind differential LL is that a morphism $f \in \mathcal{L}_!(X,Y) = \mathcal{L}(!X,Y)$ can be seen as some kind of analytic map between some kind of vector spaces associated with X and with Y. Let us recall what is an analytic map. A map $f : E \to F$ between two Banach spaces is differentiable in a point x if its variation around x can be approximated by a continuous linear map $\frac{\mathsf{d}f}{\mathsf{d}x}(x) \in \mathcal{L}(E,F)$ called the differential of f at x, that is

$$f(x+u) = f(x) + \frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u + o(\|u\|) \tag{5}$$

If f is regular enough, the map $x \mapsto \frac{\mathsf{d}f}{\mathsf{d}x}(x)$ going from E to $\mathcal{L}(E,F)$ is also differentiable so that for any x there exists a map $\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x) \in \mathcal{L}(E,\mathcal{L}(E,F))$ called the second order differential. Repeating the process yields an n-order differential $\frac{\mathsf{d}^n f}{\mathsf{d}^n x}(x) \in \mathcal{L}(E,\mathcal{L}(E,(\ldots,\mathcal{L}(E,F)\ldots)))$ which can also be seen as an n linear map $E\times\cdots\times E \to F$. These iterated differentials allow approximating f around a point x by a map which is polynomial of n:

$$f(x+u) = \sum_{k=0}^{n} \frac{1}{k!}\frac{\mathsf{d}^k f}{\mathsf{d}^k x}(x)(u,\ldots,u) + o(\|u\|^n)$$

A map is analytic if it is equal to the limit of its successive approximations, that is

$$f(x+u) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathsf{d}^n f}{\mathsf{d}^n x}(x)(u, \dots, u) \tag{6}$$

The series in the equation above is called the *Taylor series* of f at x.

The point of coherent differentiation is to generalize the ideas of differentiation to a setting where addition is not necessarily a total operation. In coherent differentiation, $\mathcal{L}$ is a summable resource category, that is, a category equipped with a binary counterpart of our ω-summability structure (see [Ehr23]) that induces a left summability structure on $\mathcal{L}_!$ (see [EW23]). Then all the sums in Eq. (5) must sums in the sense of this induced left summability structure. This means that we should see the differential of $f \in \mathcal{L}_!(X, Y)$ as a morphism $\widetilde{\mathsf{D}} f \in \mathcal{L}_!(\mathsf{S}X, \mathsf{S}Y)$ that intuitively maps a summable pair $\langle\!\langle x, u \rangle\!\rangle$ to a summable pair $\left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot u \right\rangle\!\!\right\rangle$. We will use the notation $\mathsf{D}f \langle\!\langle x, u \rangle\!\rangle := \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot u$ and this operator D corresponds to corresponding operator in the (left-additive) cartesian differential categories of [BCS09].

It turns out that this point of view comes with real benefits, because the chain rule of the differential calculus then correspond to the functoriality of $\widetilde{\mathsf{D}}$ on $\mathcal{L}_!$, and the other rules (Leibniz, Schwarz, linearity of the derivative) correspond to the naturality of $\mathsf{Der}\,\iota_0$, $\mathsf{Der}\,\theta$, $\mathsf{Der}\,\mathsf{l}$ and $\mathsf{Der}\,\mathsf{c}$ with respect to $\widetilde{\mathsf{D}}$. An equationnal account of those observations can be found in the long version of [EW23][4]. Note that the axiomatization of differentiation as a functor for which ι_0, θ, l and c are natural is very similar to the axiomatization of *tangent categories* in [CC14].

The map $\widetilde{\mathsf{D}}^2 f$ can be seen as the following map.

$$\widetilde{\mathsf{D}}^2 f \langle\!\langle \langle\!\langle x, u \rangle\!\rangle, \langle\!\langle v, w \rangle\!\rangle \rangle\!\rangle = \left\langle\!\!\left\langle \left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot u \right\rangle\!\!\right\rangle, \left\langle\!\!\left\langle \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot v, \frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)(u, v) + \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot w \right\rangle\!\!\right\rangle \right\rangle\!\!\right\rangle . \tag{7}$$

Note that the rightmost component $\pi_1 \pi_1 \widetilde{\mathsf{D}}^2 f = \mathsf{DD}f$ does not only contain the second order derivative $\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)(u, v)$, but also the term $\frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot w$. This happens because $\mathsf{DD}f$ is the *total* derivative of $\mathsf{D}f$, that is its derivative with regard to both of its coordinates at the same time, whereas $\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)$ is only the partial derivative of $\mathsf{D}f$ with respect to its first argument.

One specificity of coherent differentiation is that the second order derivative $\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)(u, v)$ that appears Eq. (7) requires u and v to be summable. This is in sharp contrast with what we want to do for Taylor expansion in Eq. (6): there is no reason for u to be summable with itself, but $\frac{1}{n!} \frac{\mathsf{d}^n f}{\mathsf{d}^n x}(x)(u, \dots, u)$ is well-defined nonetheless thanks to the sharply decreasing coefficient $\frac{1}{n!}$ in front of the derivative. This phenomenon does not seem to be taken easily into account by the coherent differential setting.

So instead of defining $\widetilde{\mathsf{D}}$ only as the first order development $\left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x) \ \cdot u \right\rangle\!\!\right\rangle$, we define it directly as whole Taylor expansion operator. Let us start with what a functor implementing a second order development would look like. Let us introduce a functor T as follows (using a notion of ternary summability structure that should exist, as mentioned in Remark 14).

$$\mathsf{T}f \langle\!\langle x, u_1, u_2 \rangle\!\rangle = \left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot u_1, \frac{1}{2} \frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)(u_1, u_1) + \frac{\mathsf{d}f}{\mathsf{d}x}(x) \cdot u_2 \right\rangle\!\!\right\rangle \tag{8}$$

[4] In a setting where the category $\mathcal{C}$ considered can be any cartesian category, and not necessarily the Kleisli category of a model of LL

The term u_1 can be seen as a first order variation and the term u_2 as a second order variation. So $\mathsf{T}f\,\langle\!\langle x, u_1, u_2\rangle\!\rangle$ gives the components (graded by orders) of the best order 2 approximation of f on the variation $u_1 + u_2$ (and guarantees that the sums involved are well-defined). We can recover the usual order 2 development of f taking $u_2 = 0$

$$\mathsf{T}f\,\langle\!\langle x, u, 0\rangle\!\rangle = \left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u, \frac{1}{2}\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)(u,u)\right\rangle\!\!\right\rangle .$$

The term $\frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u_2$ is still crucial though, and comes from the Faà di Bruno formula. Indeed, one can check that

$$\begin{aligned}
&\mathsf{T}g(\mathsf{T}f\,\langle\!\langle x, u, 0\rangle\!\rangle)\\
&= \mathsf{T}g\left\langle\!\!\left\langle f(x), \frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u, \frac{1}{2}\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)(u,u)\right\rangle\!\!\right\rangle\\
&= \left\langle\!\!\left\langle g(f(x)), \frac{\mathsf{d}g}{\mathsf{d}x}(x)\cdot\frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u, \frac{1}{2}\frac{\mathsf{d}^2 g}{\mathsf{d}^2 x}(f(x))\cdot\left(\frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u, \frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u\right) + \frac{1}{2}\frac{\mathsf{d}g}{\mathsf{d}x}(x)\cdot\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)(u,u)\right\rangle\!\!\right\rangle
\end{aligned}$$

We recognize here second order terms in the development of $g \circ f$, thanks to the chain rule and the second order chain rule (formalized in the Faà di Bruno formula).

$$\frac{\mathsf{d}^2(g\circ f)}{\mathsf{d}^2 x}(x)(u,u) = \frac{\mathsf{d}^2 g}{\mathsf{d}^2 x}(x)\left(\frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u, \frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u\right) + \frac{\mathsf{d}g}{\mathsf{d}x}(x)\cdot\frac{\mathsf{d}^2 f}{\mathsf{d}^2 x}(x)(u,u)$$

So we have

$$\mathsf{T}g(\mathsf{T}f\,\langle\!\langle x, u, 0\rangle\!\rangle) = \mathsf{T}(g\circ f)\,\langle\!\langle x, u, 0\rangle\!\rangle .$$

This means that the compositionality of the Taylor expansion and the Faà di Bruno formula should be related to the functoriality of T.

Similar computations can be performed for all finite orders n instead of 2, and ultimately for an infinite sequence of terms, possibly of all finite degrees. Let $\mathcal{M}(n) := \{m \in \mathcal{M}_{\text{fin}}(\mathbb{N}^*) | \sum_{i\in\mathbb{N}^*} i\,m(i) = n\}$ (where $\mathcal{M}_{\text{fin}}(\mathbb{N}^*)$ is the set of finite multisets of elements of $\mathbb{N}^*$, see Section 5.1 for the notations). Define $\mathsf{T}f$ as the map

$$\mathsf{T}f\,\langle\!\langle x_i\rangle\!\rangle_{i=0}^{\infty} = \left\langle\!\!\left\langle \sum_{m\in\mathcal{M}(n)} \frac{1}{m!}\frac{\mathsf{d}^{\#m} f}{\mathsf{d}^{\#m} x}(x_0)\cdot\vec{x}_m\right\rangle\!\!\right\rangle_{n=0}^{\infty} \tag{9}$$

where $m! = \prod_{i\in\mathsf{supp}(m)} m(i)!$, $\#m = \sum_{i\in\mathsf{supp}(m)} m(i)$ and

$$\vec{x}_m = (\underbrace{x_1,\ldots,x_1}_{m(1)\text{ times}},\ldots,\underbrace{x_i,\ldots,x_i}_{m(i)\text{ times}},\ldots,\underbrace{x_n,\ldots,x_n}_{m(n)\text{ times}})$$

The term x_i should be seen as an order i infinitesimal $x_i\,\epsilon^i$, so that $\mathsf{T}f\,\langle\!\langle x_i\rangle\!\rangle_{i=0}^{\infty}$ contains the components sorted by order of the Taylor expansion of f at $x_0 + \epsilon\,x_1 + \epsilon^2\,x_2 + \cdots$.

The case $k = n$ and $m = [1,\ldots,1]$ gives the value $\frac{\mathsf{d}^n f}{\mathsf{d}^n x}(x_0)(u_1,\ldots,u_1)$, so we can recover all the terms of the Taylor expansion of f by erasing all the higher order variations.

$$\mathsf{T}f\,\langle\!\langle x, u, 0, \ldots\rangle\!\rangle = \left\langle\!\!\left\langle \frac{1}{n!}\frac{\mathsf{d}^n f}{\mathsf{d}^n x}(x)(u,\ldots,u)\right\rangle\!\!\right\rangle_{n=0}^{\infty} .$$

Again, the other cases are still very relevant as they allow to recover the compositionality of the Taylor expansion

$$\mathsf{T}g(\mathsf{T}f\,\langle\!\langle x, u, 0, \ldots\rangle\!\rangle) = \left\langle\!\!\left\langle \sum_{m\in\mathcal{M}(n)} \frac{1}{m!}\frac{\mathsf{d}^{\#m} g}{\mathsf{d}^{\#m} x}(x)\cdot\vec{y}_m\right\rangle\!\!\right\rangle_{n=0}^{\infty}$$

where $\vec{y} = (\frac{1}{n!}\frac{\mathsf{d}^n f}{\mathsf{d}^n x}(x)\cdot(u,\dots,u))_{n=0}^{\infty}$. We recognize above the terms of the Taylor expansion of $g \circ f$ using the Faà di Bruno formula

$$\frac{\mathsf{d}^n(g\circ f)}{\mathsf{d}^n x}(x)\cdot(u,\dots,u) = \sum_{m\in\mathcal{M}(n)} \frac{n!}{m!}\frac{\mathsf{d}^{\#m} g}{\mathsf{d}^{\#m} x}(x)\cdot\vec{y}_m\,.$$

So we have

$$\mathsf{T}g(\mathsf{T}f\,\langle\!\langle x,u,0,\dots\rangle\!\rangle) = \mathsf{T}(g\circ f)\,\langle\!\langle x,u,0,\dots\rangle\!\rangle$$

which means that the compositionality of the Taylor expansion and the Faà di Bruno formula are closely related to the functoriality of T.

Finally, if f is linear then $\frac{\mathsf{d}f}{\mathsf{d}x}(x)\cdot u = f(u)$ and $\frac{\mathsf{d}^n f}{\mathsf{d}^n x}(x)(u,\dots,u) = 0$. So for any $f \in \mathcal{L}_!(X,Y)$ that is linear (that is, $f = \mathsf{Der}\,\overline{f}$ for some $\overline{f} \in \mathcal{L}(X,Y)$) we should have

$$\mathsf{T}f(\langle\!\langle x_i\rangle\!\rangle_{i=0}^{\infty}) = \langle\!\langle f(x_i)\rangle\!\rangle_{i=0}^{\infty}$$

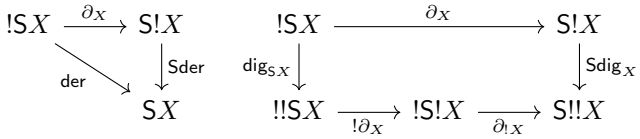
That is, $\mathsf{T}f = \mathsf{Der}\,\mathsf{S}\overline{f}$. So T should extend the functor S to $\mathcal{L}_!$, see Section 6 for a generic definition.

2.2 The axioms of Taylor expansion

Let $\mathcal{L}$ be an ω-summable resource category. As motivated above, Taylor expansion should be seen as a functor T on $\mathcal{L}_!$ that extends S to $\mathcal{L}_!$. It is showed in Section 6 that this notion of extension is deeply tied to the notion of distributive laws. So Taylor expansion should be a natural transformation $\partial : !\mathsf{S} \Rightarrow \mathsf{S}!$ following the axioms below.

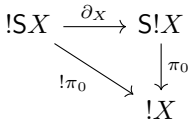
Remark 12. These axioms are exactly the same as the axioms of coherent differentiation [Ehr23], except that the summability structure is now infinitary. Their meaning in coherent differentiation is well understood, see also [EW23]. They should have a similar meaning in this new setting of Taylor expansion, but the underlying combinatorics is more complicated and still slightly unclear. What we know for now is that these axioms indeed hold in our examples from LL for the exact same reasons that the axioms of coherent differentiation hold[5], and that the functor T involved indeed correspond to the intuitive formula given in Eq. (9), see Theorem 24.

(∂-chain)



This axiom means that ∂ is a distributive law (see Section 6) between the functor S and the comonad !. By Theorem 27, this axiom means that S can be extended to a functor T on $\mathcal{L}_!$ defined as $\mathsf{T}f = (\mathsf{S}f)\,\partial$. This functor T corresponds to the operator motivated in Section 2.1. So this axiom should be understood as the higher order chain rule (Faà di Bruno formula).

(∂-local)



[5] Note however that some models admit a coherent differentiation but not such coherent Taylor expansion, see Section 5.4

This axiom means that π_0 is a morphism of distributive law between ∂ and id. By Theorem 28, this axiom means that π_0 extends to a natural transformation $\mathsf{T} \Rightarrow \mathsf{Id}$ on $\mathcal{L}_!$. Note that this axiom is requested for π_0 and not for the π_i with $i > 0$. As noted in [Ehr23], differentiation breaks the symmetry between the components of the functor S.

(∂-add)
$$\begin{array}{ccc} !X & & !\mathsf{S}^2X \xrightarrow{\partial_{\mathsf{S}X}} \mathsf{S}!\mathsf{S}X \xrightarrow{\mathsf{S}\partial_X} \mathsf{S}^2!X \\ {\scriptstyle !\iota_0}\downarrow \quad \searrow^{\iota_0} & & {\scriptstyle !\theta_X}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle \theta_{!X}} \\ !\mathsf{S}X \xrightarrow[\partial_X]{} \mathsf{S}!X & & !\mathsf{S}X \xrightarrow[\partial_X]{} \mathsf{S}!X \end{array}$$

This axiom means that ι_0 is a morphism of distributive law between id and ∂, and that θ is a morphism of distributive law between the composition of ∂ with itself and ∂. By Theorem 28, this axiom means that ι_0 and θ extend to natural transformations $\mathsf{Der}\,\iota_0 : \mathsf{Id} \Rightarrow \mathsf{T}$ and $\mathsf{Der}\,\theta : \mathsf{T}^2 \Rightarrow \mathsf{T}$ on $\mathcal{L}_!$, meaning that T inherits the monad structure of S. Dually, this axiom also mean that $!\_$ extends to a functor $\widehat{!}\_$ on the Kleisli category $\mathcal{L}_\mathsf{S}$ of S, and (∂-chain) together with Theorem 25 ensures that $\widehat{!}\_$ inherits the structure of comonad of $!\_$. This axiom should be seen as the additivity of the iterated derivatives in each of their coordinates.

(∂-Schwarz)
$$\begin{array}{ccccc} !\mathsf{S}^2X & \xrightarrow{\partial_{\mathsf{S}X}} & \mathsf{S}!\mathsf{S}X & \xrightarrow{\mathsf{S}\partial_X} & \mathsf{S}^2!X \\ {\scriptstyle !\mathsf{c}_X}\downarrow & & & & \downarrow{\scriptstyle \mathsf{c}_{!X}} \\ !\mathsf{S}^2X & \xrightarrow[\partial_{\mathsf{S}X}]{} & \mathsf{S}!\mathsf{S}X & \xrightarrow[\mathsf{S}\partial_X]{} & \mathsf{S}^2!X \end{array}$$

This axiom means that c is a morphism of distributive law (again for the composition of ∂ with itself). By Theorem 28, this axiom also means that c extends to a natural transformation $\mathsf{Der}\,\mathsf{c} : \mathsf{T}^2 \Rightarrow \mathsf{T}^2$ on $\mathcal{L}_!$. It can be interpreted as a higher order equivalent of the Schwarz theorem that states that the second order derivative is symmetric.

The next diagram is not among the axioms given in [Ehr23], but as discussed Section 5.1 of the long version of [EW23] it should have been a part of it.

(∂-lin)
$$\begin{array}{ccccc} !\mathsf{S}X & & \xrightarrow{\qquad\partial_X\qquad} & & \mathsf{S}!X \\ {\scriptstyle !!_X}\downarrow & & & & \downarrow{\scriptstyle !_{!X}} \\ !\mathsf{S}^2X & \xrightarrow[\partial_{\mathsf{S}X}]{} & \mathsf{S}!\mathsf{S}X & \xrightarrow[\mathsf{S}\partial_X]{} & \mathsf{S}^2!X \end{array}$$

This axiom means that l is a morphism of distributive law between ∂ and the composition of ∂ with itself. By Theorem 28, this axiom also means that l extends to a natural transformation $\mathsf{Der}\,\mathsf{l} : \mathsf{T} \Rightarrow \mathsf{T}^2$ on $\mathcal{L}_!$. Together with (∂-add), this axiom means that the derivatives are not only additive in their individual coordinates, but also T-linear, see Definition 22[6]

(∂-&)
$$\begin{array}{ccccc} !\mathsf{S}X \otimes !\mathsf{S}Y & \xrightarrow{\partial_X \otimes \partial_Y} & \mathsf{S}!X \otimes \mathsf{S}!Y & \xrightarrow{\mathsf{L}_{!X,!Y}} & \mathsf{S}(!X \otimes !Y) \\ {\scriptstyle \mathsf{m}^2_{\mathsf{S}X,\mathsf{S}Y}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{Sm}^2_{X,Y}} \\ !(\mathsf{S}X \,\&\, \mathsf{S}Y) & \xrightarrow[!\mathsf{c}_\&^{-1}]{} & !\mathsf{S}(X \,\&\, Y) & \xrightarrow[\partial_{X\&Y}]{} & \mathsf{S}!(X \,\&\, Y) \end{array}$$

This axiom means that m^2 is a morphism of distributive law between the composition of ∂ with L, and the composition of ∂ with $(\mathsf{c}_\&)^{-1}$ (recall that the structure of a lax symmetric

[6] This explains the clash of terminology with [Ehr23], in which the axiom (∂-lin) corresponds to our axiom (∂-add)

monoidal monad can be seen as a kind of distributive law). By Theorem 26, it means that m^2 extends to a natural transformation $\widehat{!}\_\widehat{\otimes}\widehat{!}\_ \Rightarrow \widehat{!}(\_\widehat{\&}\_)$. This diagram means that if a morphism is multilinear in the sense of linear logic, then it is T-multilinear in the sense of Definition 30. More on this in Section 3.4.

Remark 13. In [Ehr23], the diagram below was stated as part of the axiom (∂-&).

$$\begin{array}{ccccc} !\mathsf{S}\top & & \xrightarrow{\quad\partial_\top\quad} & & \mathsf{S}!\top \\ {\scriptstyle !0}\downarrow & & & & \downarrow{\scriptstyle \mathsf{S}(\mathsf{m}^0)^{-1}} \\ !\top & \xrightarrow[(\mathsf{m}^0)^{-1}]{} & 1 & \xrightarrow[\iota_0]{} & \mathsf{S}1 \end{array}$$

It turns out that this diagram always holds (assuming (∂-add)). Indeed, we know that $!0 = !\mathsf{t}_{\mathsf{S}\top}$ is invertible of inverse $!\iota_0$, see Section 1.5.3. So we can prove the diagram with the diagram chase below. The commutation (a) is (∂-add) and the commutation (b) is the naturality of ι_0.

$$\begin{array}{ccccc} !\mathsf{S}\top & & \xrightarrow{\quad\partial_\top\quad} & & \mathsf{S}!\top \\ {\scriptstyle !\iota_0}\uparrow & (a) & \overset{\iota_0}{\nearrow} & & \downarrow{\scriptstyle \mathsf{S}(\mathsf{m}^0)^{-1}} \\ !\top & \xrightarrow[(\mathsf{m}^0)^{-1}]{} & 1 & \xrightarrow[\iota_0]{} & \mathsf{S}1 \end{array} \qquad (b)$$

Proposition 22. *The left diagram always hold, and the right diagram is a consequence of (∂-&). See Definition 42 for a definition of* wk *and* ctr*.*

$$\begin{array}{ccc} !\mathsf{S}X & \xrightarrow{\partial_X} & \mathsf{S}!X \\ {\scriptstyle \mathsf{wk}_{\mathsf{S}X}}\downarrow & & \downarrow{\scriptstyle \mathsf{S}\mathsf{wk}_X} \\ 1 & \xrightarrow[\iota_0]{} & \mathsf{S}1 \end{array} \qquad \begin{array}{ccccc} !\mathsf{S}X & & \xrightarrow{\qquad\partial_X\qquad} & & \mathsf{S}!X \\ {\scriptstyle \mathsf{ctr}_{\mathsf{S}X}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{S}\mathsf{ctr}_X} \\ !\mathsf{S}X\otimes!\mathsf{S}X & \xrightarrow[\partial_X\otimes\partial_X]{} & \mathsf{S}!X\otimes\mathsf{S}!X & \xrightarrow[\mathsf{L}_{!X,!X}]{} & \mathsf{S}(!X\otimes!X) \end{array}$$

Proof. This is a straightforward consequence of the diagram of Remark 13 and (∂-&), unfolding the definition of wk and ctr and using naturality. □

The left diagram means that wk is a morphism of distributive law ($\iota_0 : !1 \Rightarrow \mathsf{S}1$ is the distributive law associated to the extension of the constant functor 1 on $\mathcal{L}$ as the constant functor 1 on $\mathcal{L}_\mathsf{S}$). The right diagram means that ctr is a morphism of distributive law between ∂ and the composition of ∂ with L. By Theorem 26, those diagrams mean that the contraction and weakening extends to $\mathcal{L}_\mathsf{S}$. The result of Proposition 22 is not surprising then, since the weakening and the contraction on $\mathcal{L}_\mathsf{S}$ can also be defined directly from m^0 and the extension of m^2 to $\mathcal{L}_\mathsf{S}$.

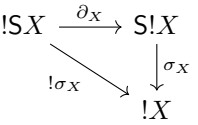
Definition 20. A *Taylor expansion* in an ω-summable resource category is a natural transformation $\partial : !\mathsf{S} \Rightarrow \mathsf{S}!$ following (∂-chain), (∂-local), (∂-add), (∂-Schwarz), (∂-lin), (∂-local).

Remark 14. It should be possible to define in a uniform way a notion of n-ary summability structure S_n for any $n \in \mathbb{N} \cup \{\infty\}$, as a summability structure which has only projections π_i for $i \in [\![0, n]\!]$. Then the summability structures of [Ehr23] would be a particular instance for which $n = 1$, and the ω-summability structure would be a particular instance for which $n = \infty$. It should induce a notion of n-ary summable resource category for any n.

Then a Taylor expansion in an n-ary summable resource category would simply be a distributive law $!\mathsf{S}_n \Rightarrow \mathsf{S}_n!$ defined exactly in the same way as above. This operation should be seen as an order n Taylor approximation. Then the coherent differentiation of [Ehr23] would be a particular case in which $n = 1$, and the Taylor expansion in our article a particular case in which $n = \infty$.

We have not stated yet the fact that the morphisms in the category are analytic, that is, equal to their Taylor expansion. This is the role of the axiom (∂-analytic) below.

(∂-analytic)



This axiom means that σ is a morphism of distributive law between ∂ and id. By Theorem 28, it means that σ extends to a natural transformation $\mathsf{Der}\,\sigma : \mathsf{T} \Rightarrow \mathsf{Id}$. Together with (∂-lin), they allow to extend the comonad $\overline{\mathsf{S}}$ to $\mathcal{L}_!$. The axioms (∂-lin) and (∂-analytic) can also be interpreted as the fact that ∂ is a distributive law between the functor $!\_$ and the comonad $\overline{\mathsf{S}}$. Recall that in Section 2.1 the Taylor expansion of f at x on variation u could be defined as $\mathsf{Der}\,(\sigma) \circ \mathsf{T}f \circ \langle\!\langle x, u, 0, \ldots \rangle\!\rangle$. So the naturality of $\mathsf{Der}\,(\sigma)$ implies that $\mathsf{Der}\,(\sigma) \circ \mathsf{T}f \circ \langle\!\langle x, u, 0, \ldots \rangle\!\rangle = f \circ \mathsf{Der}\,(\sigma) \circ \langle\!\langle x, u, 0, \ldots \rangle\!\rangle = f \circ (x + u)$. This exactly corresponds to the property that f is equal to its Taylor expansion.

Definition 21. An *analytic category* is an ω-summable resource category equipped with a Taylor expansion that follows (∂-analytic).

Remark 15. The purpose of our choice of terminology is to make a clear distinction between the infintary setting of the present article and the finitary settings of [Ehr23, EW23], this is why we prefer to speak directly of Taylor category: it is a category where any morphism has a Taylor expansion (involving all its higher derivatives), which is a morphism of the same category. In such categories, the morphisms however are not necessarily equal to the infinite sum of all the terms of their Taylor expansion. This Taylor expansion is provided by the functor T, which by the way is much richer than a mere Taylor expansion and is strongly related to the Faà di Bruno formula, by need of functoriality.

We use the adjective "analytic" for the situation where any morphism is equal to the sum of all the terms of its Taylor expansion, following the standard mathematical terminology, with the slight difference that, in Analysis, analyticity is a local concept whereas here, it is a global condition (btw. we have nothing like a topology which would allow to make it local).

Although the notion of order n coherent differential category makes perfectly sense, an order n Taylor category should not be very interesting, as it would mean that the morphisms are all polynomials of degree lower than n.

Remark 16. Except for (∂-local) (that is about the structure of the functor T) and (∂-&) (that is discussed in Section 3.4), the axioms of coherent differentiation are exactly the necessary and sufficient conditions to extend the structure of the bimonad S to $\mathcal{L}_!$: ∂ is a distributive law between the comonad $!\_$ and the functor S by (∂-chain), a distributive law between the functor $!\_$ and the monad $\underline{\mathsf{S}}$ by (∂-add), a distributive law between the functor $!\_$ and the comonad $\overline{\mathsf{S}}$ by (∂-lin) and (∂-analytic), and c is a morphism of distributive law by (∂-Schwarz). It means that Taylor expansion essentially act as a bimonad.

In [KL23], Taylor expansion is framed in some models of differential LL as a monad structure on $!\_$ turning it into a bimonad. This bimonad seems to be quite different from our bimonad S, but a closer comparison should be investigated.

3 Cartesian analytic categories

We provide a direct axiomatization of Taylor expansion in any category $\mathcal{C}$. We show how this expansion should interact with the cartesian closed structure of $\mathcal{C}$, whenever it has one. Typically, $\mathcal{C} = \mathcal{L}_!$ for some model $\mathcal{L}$ of LL, but the point of this axiomatization is that it is more general, more compact, and does not depend at all on LL. We assume throughout this section that $\mathcal{C}$ is equipped with a left ω-summability structure $(\mathsf{S}, \vec{\pi}, \sigma)$.

3.1 Taylor expansion in a left additive category

The axiomatization of Taylor expansion and analyticity on $\mathcal{C}$ is very similar to the coherent differentiation of [EW23], except that left summability structure is now infinitary and that σ is natural.

Definition 22. Let T be a map on morphisms such that for any $f \in \mathcal{C}(X,Y)$, $\mathsf{T}f \in \mathcal{C}(\mathsf{S}X, \mathsf{S}Y)$. A morphism h is T-linear if it is additive and if $\pi_i \circ \mathsf{T}h = h \circ \pi_i$. That is, $\mathsf{T}h = \mathsf{S}h$.

Definition 23. An (infinitary) Taylor expansion on $\mathcal{C}$ is a map on morphisms T such that for any $f \in \mathcal{C}(X,Y)$, $\mathsf{T}f \in \mathcal{C}(\mathsf{S}X, \mathsf{S}Y)$ and such that:

- (T-chain) T is a functor
- (T-local) π_0 is a natural transformation
- (T-proj-lin) The projections π_i are T-linear
- (T-sum-lin) σ and 0 are T-linear
- (T-add) $\iota_0 \in \mathcal{C}(X, \mathsf{T}X)$ and $\theta \in \mathcal{C}(\mathsf{T}^2X, \mathsf{T}X)$ are natural transformations
- (T-lin) $\mathsf{l} \in \mathcal{C}(\mathsf{T}X, \mathsf{T}^2X)$ is a natural transformation
- (T-Schwarz) $\mathsf{c} \in \mathcal{C}(\mathsf{T}^2X, \mathsf{T}^2X)$ is a natural transformation

Again, assuming a suitable notion of n-ary summability structure, it should be possible to define an order n Taylor expansion. The operator $\mathsf{T}f$ would perform the order n Taylor approximation of f.

Definition 24. An analytic structure on $\mathcal{C}$ is a Taylor expansion such that $\sigma \in \mathcal{C}(\mathsf{T}X, X)$ is natural. We call this property (T-analytic).

We only assume in what follows that T is a map on morphism such that for any $f \in \mathcal{C}(X,Y)$, $\mathsf{T}f \in \mathcal{C}(\mathsf{S}X, \mathsf{S}Y)$. Any use of the axioms of Taylor expansion will be made explicit.

Proposition 23. *Assuming (T-chain) and (T-proj-lin), if $(f_i)_{i=0}^{\infty}$ is summable then $(\mathsf{T}f_i)_{i=0}^{\infty}$ is summable and $\langle\!\langle \mathsf{T}f_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{c} \circ \mathsf{T} \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty}$.*

Proof. We have $\pi_i \circ \mathsf{c} \circ \mathsf{T} \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{S}\pi_i \circ \mathsf{T} \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{T}\pi_i \circ \mathsf{T} \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{T}(\pi_i \circ \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty}) = \mathsf{T}f_i$ using (T-proj-lin) and (T-chain). □

Proposition 24. *The axiom (T-chain) ensures that the composition of two T-linear morphism is also T-linear and that id is T-linear.*

Proof. If $h \in \mathcal{C}(X,Y)$ and $h' \in \mathcal{C}(Y,Z)$ are T-linear, then $h' \circ h$ is additive by Proposition 6. Furthermore, $\pi_i \circ \mathsf{T}(h' \circ h) = \pi_i \circ \mathsf{T}h' \circ \mathsf{T}h = h' \circ \pi_i \circ \mathsf{T}h = h' \circ h \circ \pi_i$ so $h' \circ h$ is T-linear. □

Definition 25. Let $\mathcal{C}^{\mathsf{lin}}$ be the category with the same objects as $\mathcal{C}$ and whose morphisms are the T-linear morphisms. The identity and the composition are the same as in $\mathcal{C}$. Observe that $\mathcal{C}^{\mathsf{lin}}$ is a sub category of $\mathcal{C}^{\mathsf{add}}$.

Then (T-chain), (T-proj-lin) and (T-sum-lin) ensure that $(\mathsf{S}, \vec{\pi}, \sigma)$ is a pre ω-summability structure on $\mathcal{C}^{\mathsf{lin}}$. Besides, they ensure that a sequence of morphisms $(f_i \in \mathcal{C}^{\mathsf{lin}}(X,Y))_{i=0}^{\infty}$ is summable if and only if it is summable in $\mathcal{C}$, see Proposition 25 below. It implies that $(\mathsf{S}, \vec{\pi}, \sigma)$ is an ω-summability structure on $\mathcal{C}^{\mathsf{lin}}$.

Proposition 25. *Assuming (T-proj-lin) and (T-chain), if the sequence $(h_i \in \mathcal{C}(X,Y))_{i=0}^{\infty}$ is summable and h_i is T-linear, then $\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty}$ is T-linear. Assuming (T-sum-lin), $\sum_{i=0}^{\infty} h_i$ is also T-linear. Then, for any family $(h_a)_{a\in A}$ of T-linear morphisms, $\sum_{a\in A} h_a$ is T-linear.*

Proof. We know that $\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty}$ is additive thanks to (Pair-Add), and $\mathsf{T}\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{c}\circ\langle\!\langle \mathsf{T}h_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{c} \circ \langle\!\langle \mathsf{S}h_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{S}\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty}$ (the last equality can be easily checked by joint monicity of the $\pi_i \circ \pi_j$); so $\langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty}$ is T-linear. Then, by (T-sum-lin), (T-chain) and Proposition 24, $\sum_{i=0}^{\infty} h_i = \sigma \circ \langle\!\langle h_i \rangle\!\rangle_{i=0}^{\infty}$ is also T-linear. Finally, the last statement hold by definition of the generalized sums. □

Corollary 1. *Assuming (T-chain), (T-proj-lin) and (T-sum-lin), ι_i, θ, c, l and σ are all T-linear*

Proof. They are all pairings of T-linear morphisms and sums of T-linear morphisms, so they are T-linear by Proposition 7 above. □

As a result, $\mathsf{T}\iota_i = \mathsf{S}\iota_i$, $\mathsf{T}\theta = \mathsf{S}\theta$, $\mathsf{Tc} = \mathsf{Sc}$, $\mathsf{Tl} = \mathsf{Sl}$ and $\mathsf{T}\sigma = \mathsf{S}\sigma$. In particular, all the diagrams turning S into a bimonad also hold when replacing S by T. So the axioms of a analytic structure except for (T-local) are exactly the conditions allowing turn T into a bimonad that extends to $\mathcal{C}$ the bimonad S on $\mathcal{C}^{\mathsf{lin}}$.

3.2 Interaction with the cartesian structure

We assume that $\mathcal{C}$ is a cartesian category, equipped with a left ω-summability structure $(\mathsf{S}, \vec{\pi}, \sigma)$. The notations on the cartesian product will be the same as the one of Section 1.5. This section is a straightforward adaption of the work of [EW23] to the setting where the summability structure is infinitary. For the rest of this section, any set I is considered to be universally quantified over the sets such that $\&_{i\in I}$ is well-defined. In particular, the category may have countable products or not.

3.2.1 Left ω-summability structure and cartesian product

First, observe that $\langle\!\langle \mathsf{t}_\top \rangle\!\rangle_{i=0}^{\infty} \in \mathcal{C}(\top, \mathsf{S}\top)$ is an isomorphism of inverse $0 = \mathsf{t}_{\mathsf{S}\top}$. Indeed, $\mathsf{t}_{\mathsf{S}\top} \circ \langle\!\langle \mathsf{t}_\top \rangle\!\rangle_{i=0}^{\infty} = \mathsf{id}_\top$ simply because $\top$ is final, and $\pi_i \circ \langle\!\langle \mathsf{t}_\top \rangle\!\rangle_{i=0}^{\infty} \circ \mathsf{t}_{\mathsf{S}\top} \in \mathcal{C}(\mathsf{S}\top, \top)$ is necessarily equal to π_i (also because $\top$ is final), so by joint monicity of the π_i, $\langle\!\langle \mathsf{t}_\top \rangle\!\rangle_{i=0}^{\infty} \circ \mathsf{t}_{\mathsf{S}\top} = \mathsf{id}_{\mathsf{S}\top}$.

Definition 26. The left ω-summability structure is compatible with the cartesian product if the projections p_i are additive and if $\mathsf{c}_\& := \langle \mathsf{S}\mathsf{p}_i \rangle_{i\in I} \in \mathcal{C}(\mathsf{S}\,\&_{i\in I} X_i, \&_{i\in I} \mathsf{S}X_i)$ is an isomorphism ($\mathsf{c}_\&$ is well-defined because the projections are additive)

Let us break down this definition in more details. The additivity of the projection implies that the sum on pairs is the coordinate wise sum.

Proposition 26. *The following are equivalent*

- *For all $i \in I$, p_i is additive*
- *$\langle 0 \rangle_{i\in I} = 0$ and if $(\langle f_j^i \rangle_{i\in I} \in \mathcal{C}(X, \&_{i\in I} X_i))_{j=0}^{\infty}$ is summable, then for all $i \in I$, $(f_j^i)_{j=0}^{\infty}$ is summable and $\sum_{j=0}^{\infty} \langle f_j^i \rangle_{i\in I} = \langle \sum_{j=0}^{\infty} f_j^i \rangle_{i\in I}$*

Proof. Assume that the p_i are additive. Then $\mathsf{p}_i \circ 0 = 0 = \mathsf{p}_i \circ \langle 0 \rangle_{i \in I}$. Thus, by joint monicity of the p_i, $0 = \langle 0 \rangle_{i \in I}$. Furthermore, assume that $(\langle f_j^i \rangle_{i \in I})_{j=0}^{\infty}$ is summable. Then by additivity of p_i, the sequence $(\mathsf{p}_i \circ \langle f_j^i \rangle_{i \in I})_{j=0}^{\infty} = (f_j^i)_{j=0}^{\infty}$ is summable and $\sum_{j=0}^{\infty} f_j^i = \mathsf{p}_i \circ \sum_{j=0}^{\infty} \langle f_j^i \rangle_{i \in I}$. So the joint monicity of the p_i implies that $\sum_{j=0}^{\infty} \langle f_j^i \rangle_{i \in I} = \langle \sum_{j=0}^{\infty} f_j^i \rangle_{i \in I}$.

Conversely, if $\langle 0 \rangle_{i \in I} = 0$ then $\mathsf{p}_i \circ 0 = \mathsf{p}_i \circ \langle 0 \rangle_{i \in I} = 0$. Furthermore, let $(f_j \in \mathcal{C}(X, \&_{i \in I} Y_i))_{j=0}^{\infty}$ be a summable sequence of morphisms. We can write $(f_j)_{j=0}^{\infty} = (\langle \mathsf{p}_i \circ f_j \rangle_{i \in I})_{j=0}^{\infty}$. Then by assumption, for all $i \in I$, $(\mathsf{p}_i \circ f_j)_{j=0}^{\infty}$ is summable and $\sum_{j=0}^{\infty} f_j = \langle \sum_{j=0}^{\infty} \mathsf{p}_i \circ f_j \rangle_{i \in I}$ so $\mathsf{p}_i \circ \sum_{j=0}^{\infty} f_j = \sum_{j=0}^{\infty} \mathsf{p}_i \circ f_j$. It means that p_i is additive. □

Corollary 2. *If the projections are additive, then $\&_{i \in I} 0 = 0$ and for any $(\&_{i \in I} f_j^i \in \mathcal{C}(\&_{i \in I} X_i, \&_{i \in I} Y_i))_{j=0}^{\infty}$ summable, then for all $i \in I$, $(f_j^i)_{j=0}^{\infty}$ is summable and $\sum_{j=0}^{\infty} \&_{i \in I} f_j^i = \&_{i \in I} \sum_{j=0}^{\infty} f_j^i$.*

Proof. This is a direct consequence of Proposition 26 using the fact that $\&_{i \in I} f_i = \langle f_i \circ \mathsf{p}_i \rangle_{i \in I}$ and the left additivity of the sum. □

We now assume that the projections are additive. Then the fact that $\mathsf{c}_{\&}$ is an isomorphism provides an only if condition for the summability of pairs: it suffices that each of their coordinates are individually summable.

Proposition 27. *The following assertions are equivalent*

(1) $\mathsf{c}_{\&}$ is an isomorphism;

(2) $(\&_{i \in I} \pi_j)_{j=0}^{\infty}$ is summable;

(3) if for any $i \in I$, $(f_j^i \in \mathcal{C}(X_i, Y_i))_{j=0}^{\infty}$ is summable, then $(\&_{i \in I} f_j^i)_{j=0}^{\infty}$ is summable;

(4) if for any $i \in I$, $(f_j^i \in \mathcal{C}(X, Y_i))_{j=0}^{\infty}$ is summable, then $(\langle f_j^i \rangle_{i \in I})_{j=0}^{\infty}$ is summable.

And when one of those holds, $\mathsf{c}_{\&}^{-1} = \langle\!\langle \&_{i \in I} \pi_j \rangle\!\rangle_{j=0}^{\infty}$.

To sum up, the left ω-summability structure is compatible with the cartesian product if and only if the following property hold: $\langle 0 \rangle_{i \in I} = 0$, $(\langle f_j^i \rangle_{i \in I} \in \mathcal{C}(X, \&_{i \in I} X_i))_{j=0}^{\infty}$ is summable if and only if for all $i \in I$, $(f_j^i)_{j=0}^{\infty}$ is summable, and $\sum_{j=0}^{\infty} \langle f_j^i \rangle_{i \in I} = \langle \sum_{j=0}^{\infty} f_j^i \rangle_{i \in I}$. We now assume that the left ω-summability structure is compatible with the cartesian product.

Proposition 28. *If $h_i \in \mathcal{C}(X, Y_i)$ are additive, then $\langle h_i \rangle_{i \in I}$ is additive. If $h_i' \in \mathcal{C}(X_i, Y_i)$ are additive, then $\&_{i \in I} h_i'$ is additive. In particular, $\mathcal{C}^{\mathsf{add}}$ is a cartesian category.*

Proof. First, $\langle h_i \rangle_{i \in I} \circ 0 = \langle h_i \circ 0 \rangle_{i \in I} = \langle 0 \rangle_{i \in I} = 0$. Furthermore, assume that $(f_j)_{j=0}^{\infty}$ is summable. For any i, h_i is additive so $(h_i \circ f_j)_{j=0}^{\infty}$ is summable. By Proposition 27, $(\langle h_i \circ f_j \rangle_{i \in I})_{j=0}^{\infty} = (\langle h_i \rangle_{i \in I} \circ f_j)_{j=0}^{\infty}$ is then summable, of sum $\sum_{j=0}^{\infty} (\langle h_i \rangle_{i \in I} \circ f_j) = \langle \sum_{j=0}^{\infty} h_i \circ f_j \rangle_{i \in I} = \langle h_i \circ \sum_{j=0}^{\infty} f_j \rangle_{i \in I} = \langle h_i \rangle_{i \in I} \circ \sum_{j=0}^{\infty} f_j$, because h_i is additive. We conclude that $\langle h_i \rangle_{i \in I}$ is additive. As a result, $\&_{i \in I} h_i' = \langle h_i' \circ \mathsf{p}_i \rangle_{i \in I}$ is also additive. Thus, the cartesian product $\&$ on $\mathcal{C}$ restricts to a cartesian product on $\mathcal{C}^{\mathsf{add}}$, and $\top$ is terminal in $\mathcal{C}^{\mathsf{add}}$ because $\mathsf{t}_X = 0 \in \mathcal{C}(X, \top)$ is additive, so $\mathcal{C}^{\mathsf{add}}$ is cartesian. □

Then $(\mathsf{c}_{\&})_{X_0, X_1} = \langle \mathsf{S}\mathsf{p}_0, \mathsf{S}\mathsf{p}_1 \rangle \in \mathcal{C}(\mathsf{S}(X_0 \,\&\, X_1), \mathsf{S}X_0 \,\&\, \mathsf{S}X_1)$ is a natural transformation in $\mathcal{C}^{\mathsf{add}}$. Besides, it is invertible in $\mathcal{C}$ of inverse $\langle\!\langle \pi_i \,\&\, \pi_i \rangle\!\rangle_{i=0}^{\infty}$. This inverse is additive by (Pair-Add) and Proposition 28, so $\mathsf{c}_{\&}$ is a natural isomorphism in $\mathcal{C}^{\mathsf{add}}$. It means that the observations of Section 1.5.3 hold in $\mathcal{C}^{\mathsf{add}}$: the monad $\underline{\mathsf{S}}$ on $\mathcal{C}^{\mathsf{add}}$ equipped with ι_0 and $\mathsf{c}_{\&}^{-1}$ is a lax symmetric monoidal monad, it is equipped with two strengths ψ^0 and ψ^1, and the

adjunction $\Delta_{\&} \dashv \_ \,\&\, \_$ extend to the Kleisli category of S, meaning that it is a cartesian category.

3.2.2 Differential structure and cartesian product

We assume that $\mathcal{C}$ is equipped with a left ω-summability structure $(\mathsf{S}, \vec{\pi}, \sigma)$ that is compatible with the cartesian product.

Definition 27. A Taylor expansion is compatible with the cartesian product if the projections p_i are T-linear. A *cartesian analytic category* is a cartesian category equipped with an analytic structure that is compatible with the cartesian product.

Proposition 29. *Similarly to Proposition 23,* $\langle \mathsf{T} f_i \rangle_{i\in I} = \mathsf{c}_{\&} \circ \mathsf{T}\langle f_i \rangle_{i\in I}$

Proof. We have $\mathsf{p}_i \circ \mathsf{c}_{\&} \circ \mathsf{T}\langle f_i \rangle_{i\in I} = \mathsf{S}\mathsf{p}_i \circ \mathsf{T}\langle f_i \rangle_{i\in I} = \mathsf{T}\mathsf{p}_i \circ \mathsf{T}\langle f_i \rangle_{i\in I} = \mathsf{T}(\mathsf{p}_i \circ \langle f_i \rangle_{i\in I}) = \mathsf{T} f_i$, and we conclude by joint monicity of the p_i. □

Proposition 30. *If $h_i \in \mathcal{C}(X, Y_i)$ is a collection of T-linear morphisms, then $\langle h_i \rangle_{i\in I}$ is T-linear. If $h'_i \in \mathcal{C}(X_i, Y_i)$ is a collection of T-linear morphisms, then $\&_{i\in I} h'_i$ is T-linear. In particular, $\mathcal{C}^{\mathsf{lin}}$ is a cartesian category.*

Proof. First, we know that $\langle h_i \rangle_{i\in I}$ is additive thanks to Proposition 28. Then, $\mathsf{T}\langle h_i \rangle_{i\in I} = \mathsf{c}_{\&}^{-1} \circ \langle \mathsf{T} h_i \rangle_{i\in I} = \mathsf{c}_{\&}^{-1} \circ \langle \mathsf{S} h_i \rangle_{i\in I}$ by Proposition 29. So $\pi_j \circ \mathsf{T}\langle h_i \rangle_{i\in I} = (\&_{i\in I}\, \pi_j) \circ \langle \mathsf{S} h_i \rangle_{i\in I}$ using that $\mathsf{c}_{\&}^{-1} = \langle\!\langle \&_{i\in I}\, \pi_j \rangle\!\rangle_{j=0}^{\infty}$ (Proposition 27). But $\&_{i\in I}\, \pi_j \circ \langle \mathsf{S} h_i \rangle_{i\in I} = \langle \pi_j \circ \mathsf{S} h_i \rangle_{i\in I} = \langle h_i \circ \pi_j \rangle_{i\in I} = \langle h_i \rangle_{i\in I} \circ \pi_j$. So $\pi_j \circ \mathsf{T}\langle h_i \rangle_{i\in I} = \langle h_i \rangle_{i\in I} \circ \pi_j$. We conclude that $\mathsf{T}\langle h_i \rangle_{i\in I} = \mathsf{S}\langle h_i \rangle_{i\in I}$ by joint monicity of the π_j. The T-linearity of $\&_{i\in I}\, h'_i$ then follows directly from above thanks to the fact that $\&_{i\in I}\, h'_i = \langle h'_i \circ \mathsf{p}_i \rangle_{i\in I}$ and that the composition of two T-linear morphism is also T-linear. We conclude that $\mathcal{C}^{\mathsf{lin}}$ is cartesian because $\mathsf{t}_X = 0$ is also T-linear by (T-sum-lin), so $\top$ is also final in $\mathcal{C}^{\mathsf{lin}}$. □

The T-linearity of the projections imply that $\mathsf{c}_{\&} = \langle \mathsf{S}\mathsf{p}_i \rangle_{i\in I} = \langle \mathsf{T}\mathsf{p}_i \rangle_{i\in I} \in \mathcal{C}(\mathsf{T}\,\&_{i\in I}\, X_i, \&_{i\in I}\, \mathsf{T} X_i)$ is a natural transformation. Since $\mathsf{c}_{\&}$ is invertible, it is then a natural isomorphism and $\mathsf{c}_{\&}^{-1} \in \mathcal{C}(\&_{i\in I}\, \mathsf{T} X_i, \mathsf{T}\,\&_{i\in I}\, X_i)$ is natural. Besides, both $\mathsf{c}_{\&}$ and $\mathsf{c}_{\&}^{-1}$ are T-linear, thanks to Propositions 27 and 30. Then for the same reasons that $(\underline{\mathsf{S}}, \iota_0, \mathsf{c}_{\&}^{-1})$ is a lax symmetric monoidal monad in $\mathcal{C}^{\mathsf{add}}$, $(\underline{\mathsf{T}}, \iota_0, \mathsf{c}_{\&}^{-1})$ is a lax symmetric monoidal monad on $\mathcal{C}$.

Remark 17. As in Remark 10, the invertibility of $\mathsf{c}_{\&}$ ensures by Theorem 33 that the adjunction $\Delta \dashv \_ \,\&\, \_$ extends to the Kleisli category of $\underline{\mathsf{T}}$, $\mathcal{C}_{\underline{\mathsf{T}}}$. So $\mathcal{C}_{\underline{\mathsf{T}}}$ is cartesian (again, ι_0 is an isomorphism in $\mathcal{C}(\top, \mathsf{T}\top)$ so $\top$ is terminal in $\mathcal{C}_{\underline{\mathsf{T}}}$).

As seen in Section 7.3 (taking the symmetric monoidal structure to be the one generated by the cartesian structure &), it means that T is a commutative monad where the two strength are defined as

$$\Psi^0 = \mathsf{c}_{\&}^{-1} \circ (\mathsf{T} X_0 \,\&\, \iota_0) = \langle\!\langle \pi_0 \,\&\, X_1, \pi_1 \,\&\, 0, \pi_2 \,\&\, 0, \ldots \rangle\!\rangle \in \mathcal{C}(\mathsf{T} X_0 \,\&\, X_1, \mathsf{T}(X_0 \,\&\, X_1))$$

$$\Psi^1 = \mathsf{c}_{\&}^{-1} \circ (\iota_0 \,\&\, \mathsf{T} X_1) = \langle\!\langle X_0 \,\&\, \pi_0, 0 \,\&\, \pi_1, 0 \,\&\, \pi_2, \ldots \rangle\!\rangle \in \mathcal{C}(X_0 \,\&\, \mathsf{T} X_1, \mathsf{T}(X_0 \,\&\, X_1))$$

Those strengths are T-linear, and coincides with the strengths ψ^0 and ψ^1 associated to the lax monoidal monad $\underline{\mathsf{S}}$ on $\mathcal{C}^{\mathsf{add}}$.

Definition 28. Let $f \in \mathcal{C}(X_0 \,\&\, X_1, Y)$. Define $\mathsf{T}_0 f = \mathsf{T} f \circ \Psi^0 \in \mathcal{C}(\mathsf{T} X_0 \,\&\, X_1, \mathsf{T} Y)$ and $\mathsf{T}_1 f = \mathsf{T} f \circ \Psi^1 \in \mathcal{C}(X_0 \,\&\, \mathsf{T} X_1, \mathsf{T} Y)$.

Intuitively, the strength maps a summable family $\langle\!\langle x_i \rangle\!\rangle_{i=0}^{\infty}$ and an element y to the family $\langle\!\langle (x_0, y), (x_1, 0), (x_2, 0), \ldots \rangle\!\rangle$. So $\mathsf{T}_0 f$ performs the Taylor expansion of f on this family. In particular, the coefficient at position 1 should be seen as $\frac{\mathsf{d} f}{\mathsf{d} x_0, y}(x_0, y) \cdot (x_1, 0) = \partial_0 f(x_0, y) \cdot x_1$

where ∂_0 is the partial derivative of f with regard to its first argument. So $\mathsf{T}_0, \mathsf{T}_1$ are the infinitary counterpart of the notion of partial derivatives: the Taylor expansion of f is computed only with regard to a variation on one input.

This theory of partial derivative behaves very nicely and is crucial for the development of a syntax. We refer the reader to [EW23] for a development of this theory in the framework of coherent differentiation.

Definition 29. A morphism $f \in \mathcal{C}(Y_0 \,\&\, Y_1, Z)$ is additive in its first coordinate if $f \circ (0 \,\&\, Y_1) = 0$ and if for all sequence $(h_i \in \mathcal{C}(X, Y_0))_{i=0}^{\infty}$ summable, then $(f \circ (h_i \,\&\, Y_1))_{i=0}^{\infty}$ is summable and

$$\sum_{i=0}^{\infty} f \circ (h_i \,\&\, Y_1) = f \circ ((\sum_{i=0}^{\infty} h_i) \,\&\, Y_1)$$

We define in a very similar way the notion of morphism additive in their second coordinate. A morphism is bi-additive if it is additive in both of its coordinates.

Proposition 31. *A morphism $f \in \mathcal{C}(Y_0 \,\&\, Y_1, Z)$ such that $f \circ (0 \,\&\, Y_1) = 0$ is additive in its first coordinate if and only $(f \circ (\pi_i \,\&\, Y_1))_{i=0}^{\infty}$ is summable of sum $f \circ (\sigma \,\&\, Y_1)$. A similar result hold for morphisms additive in their second coordinate.*

Proof. Very similar to the proof of Proposition 3. □

Definition 30. A morphism $h \in \mathcal{C}(X_0 \,\&\, X_1, Y)$ is T-linear in its first coordinate if it is additive in that coordinate and if $\pi_i \circ \mathsf{T}_0 h = h \circ (\pi_i \& X_1)$. It is linear in its second coordinate if it is additive in that coordinate and if $\pi_i \circ \mathsf{T}_1 h = h \circ (X_0 \,\&\, \pi_i)$. A morphism is T-bilinear if it is linear in both of its coordinates.

Those notions can be generalized to arbitrary finite product, defining a strength

$$\Psi^i = \mathsf{c}_{\&} \circ (\mathsf{id}_{X_0} \,\&\, \cdots \,\&\, \iota_0 \,\&\, \cdots \,\&\, \mathsf{id}_{X_n}) \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, \mathsf{T}X_i \,\&\, \cdots \,\&\, X_n, \mathsf{T}(X_0 \,\&\, \cdots \,\&\, X_n)).$$

This induces a Taylor expansion with regard to only one parameter $\mathsf{T}_i f = f \circ \Psi^i \in \mathcal{C}(X_0 \,\&\, \cdots \,\&\, \mathsf{T}X_i \,\&\, \cdots \,\&\, X_n)$. It is then possible to define a notion of multi-additive morphism and a notion of T-multilinear morphism. The latter play an important role in [EW23].

3.3 Compatibility with the cartesian closed structure

We assume that $\mathcal{C}$ is a cartesian category, equipped with an ω-summability structure $(\mathsf{S}, \vec{\pi}, \sigma)$ and a Taylor expansion compatible with the cartesian product. We assume that $\mathcal{C}$ is closed with regard to $\&$. That is, for all objects X and A there exists an object $A \Rightarrow X$ and $\mathsf{Ev} \in \mathcal{C}((A \Rightarrow X) \,\&\, A, X)$ such that for any morphism $f \in \mathcal{C}(X \,\&\, A, Y)$, there exists a unique morphism $\mathsf{Cur}(f) \in \mathcal{C}(X, A \Rightarrow Y)$ such that

$$\mathsf{Ev}(\mathsf{Cur}(f) \,\&\, A) = f$$

Then $\mathsf{Cur} : \mathcal{C}(X \,\&\, A, Y) \to \mathcal{C}(X, A \Rightarrow Y)$ is a bijection whose inverse is given by

$$\mathsf{Cur}^{-1}(g) = \mathsf{Ev} \circ (g \,\&\, A)$$

The closure can be seen as an adjunction $\_ \,\&\, A \dashv A \Rightarrow \_$ for any object A, of unit $\mathsf{Cur}(\mathsf{id}_{X \& A}) \in \mathcal{C}(X, A \Rightarrow (X \,\&\, A))$ and co-unit $\mathsf{Ev} \in \mathcal{C}((A \Rightarrow X) \,\&\, A, X)$. The functor $A \Rightarrow \_$ maps a morphism $f \in \mathcal{C}(X, Y)$ to a morphism $A \Rightarrow f \in \mathcal{C}(A \Rightarrow X, A \Rightarrow Y)$ defined as $\mathsf{Cur}(f \circ \mathsf{Ev})$. Then Cur and Cur^{-1} are natural bijections. That is, Eq. (10) below holds

$$\begin{aligned} \mathsf{Cur}(f \circ g \circ (h \,\&\, A)) &= (A \Rightarrow f) \circ \mathsf{Cur}(g) \circ h \\ \mathsf{Cur}^{-1}((A \Rightarrow f) \circ g \circ h) &= f \circ \mathsf{Cur}^{-1}(g) \circ (h \,\&\, A) \end{aligned} \tag{10}$$

3.3.1 Left ω-summability structure and closure

Definition 31. An ω-summability structure is compatible with the closure if Ev is additive in its first coordinate, and if

$$\mathsf{c}_{\Rightarrow} := \mathsf{Cur}\left\langle\!\left\langle \mathsf{Ev} \circ (\pi_i \,\&\, A)\right\rangle\!\right\rangle_{i=0}^{\infty} \in \mathcal{C}(\mathsf{S}(A \Rightarrow X), A \Rightarrow \mathsf{S}X)$$

is an isomorphism.

Let us break down this definition in more details. This is very similar to what happens with the cartesian product in Section 3.2.1. The linearity of Ev means intuitively that if a family of maps is summable, then it is summable point wise.

Proposition 32. *The following are equivalent*

(1) Ev *is additive in its first coordinate;*

(2) $\mathsf{Cur}^{-1}(0) = 0$ *and for any* $(g_i \in \mathcal{C}(X, A \Rightarrow Y))_{i=0}^{\infty}$*, if* $(g_i)_{i=0}^{\infty}$ *is summable then* $(\mathsf{Cur}^{-1}(g_i))_{i=0}^{\infty}$ *is summable and* $\sum_{i=0}^{\infty} \mathsf{Cur}^{-1}(g_i) = \mathsf{Cur}^{-1}(\sum_{i=0}^{\infty} g_i)$*;*

(3) $\mathsf{Cur}(0) = 0$ *and for any* $(f_i \in \mathcal{C}(X \,\&\, A, Y))_{i=0}^{\infty}$*, if* $(\mathsf{Cur}(f_i))_{i=0}^{\infty}$ *is summable then* $(f_i)_{i=0}^{\infty}$ *is summable and* $\sum_{i=0}^{\infty} \mathsf{Cur}(f_i) = \mathsf{Cur}(\sum_{i=0}^{\infty} f_i)$*.*

Proof. (1) $\Leftrightarrow$ (2): By definition, $\mathsf{Cur}^{-1}(f) = \mathsf{Ev} \circ (f \,\&\, A)$ so the left additivity of Ev precisely corresponds to (2).

(2) $\Leftrightarrow$ (3): This is a straightforward proof using the fact that Cur and Cur^{-1} are inverse of each other.

□

Corollary 3. *Assume that* Ev *is additive in its first coordinate and that* $(A \Rightarrow h_i \in \mathcal{C}(A \Rightarrow X, A \Rightarrow Y))_{i=0}^{\infty}$ *is summable. Then* $(h_i \in \mathcal{C}(X, Y))_{i=0}^{\infty}$ *is summable and* $\sum_{i=0}^{\infty}(A \Rightarrow h_i) = A \Rightarrow (\sum_{i=0}^{\infty} h_i)$*.*

Proof. Recall that $A \Rightarrow h_i = \mathsf{Cur}(h_i \circ \mathsf{Ev})$ so by Proposition 32, if $(A \Rightarrow h_i)_{i=0}^{\infty}$ is summable then $(h_i \circ \mathsf{Ev})_{i=0}^{\infty}$ is summable. But $h_i = h_i \circ \mathsf{Ev} \circ (\mathsf{Cur}(\mathsf{id}) \,\&\, A)$ so by left additivity, $(h_i)_{i=0}^{\infty}$ is summable and by left additivity again, $\sum_{i=0}^{\infty}(h_i \circ \mathsf{Ev}) = (\sum_{i=0}^{\infty} h_i) \circ \mathsf{Ev}$ so

$$\sum_{i=0}^{\infty}(A \Rightarrow h_i) = \sum_{i=0}^{\infty} \mathsf{Cur}(h_i \circ \mathsf{Ev}) = \mathsf{Cur}(\sum_{i=0}^{\infty}(h_i \circ \mathsf{Ev})) = \mathsf{Cur}((\sum_{i=0}^{\infty} h_i) \circ \mathsf{Ev}) = A \Rightarrow (\sum_{i=0}^{\infty} h_i)$$

□

The invertibility of $\mathsf{c}_{\Rightarrow}$ on the other hand provides an "only if" condition for Proposition 32. It means that a family of maps is summable if and only if it is summable point-wise, and the sum of the maps is the point-wise sum.

Lemma 9. $(A \Rightarrow \pi_i) \circ \mathsf{c}_{\Rightarrow} = \pi_i$*.*

Proof. The point is to use the naturality of Cur.

$$\begin{aligned}
(A \Rightarrow \pi_i) \circ \mathsf{c}_{\Rightarrow} &= (A \Rightarrow \pi_i) \circ \mathsf{Cur}\left\langle\!\left\langle \mathsf{Ev} \circ (\pi_i \,\&\, A)\right\rangle\!\right\rangle_{i=0}^{\infty} \\
&= \mathsf{Cur}(\pi_i \circ \left\langle\!\left\langle \mathsf{Ev} \circ (\pi_i \,\&\, A)\right\rangle\!\right\rangle_{i=0}^{\infty}) && \text{By naturality of } \mathsf{Cur} \\
&= \mathsf{Cur}(\mathsf{Ev} \circ (\pi_i \,\&\, A)) \\
&= \mathsf{Cur}(\mathsf{cur}^{-1}(\pi_i)) = \pi_i
\end{aligned}$$

□

Remark 18. Monicity is preserved by right adjoint functors, meaning that the $A \Rightarrow \pi_i$ are jointly monic. This fact can also be checked by hand in a rather straightforward way using the fact that Cur is a bijection and is natural. This will be useful in the proof of Proposition 33 below.

Proposition 33. *The following assertions are equivalent*

(1) $\mathsf{c}_{\Rightarrow}$ *is an isomorphism;*

(2) $(A \Rightarrow \pi_i \in \mathcal{C}(A \Rightarrow \mathsf{S}X, A \Rightarrow X))_{i=0}^{\infty}$ *is summable;*

(3) for any sequence $(f_i \in \mathcal{C}(X,Y))_{i=0}^{\infty}$ *summable,* $(A \Rightarrow f_i)_{i=0}^{\infty}$ *is summable*

(4) for any sequence $(f_i \in \mathcal{C}(X \,\&\, A, Y))_{i=0}^{\infty}$ *summable,* $(\mathsf{cur}(f_0))_{i=0}^{\infty}$ *is summable.*

Then, $\langle\!\langle A \Rightarrow \pi_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{c}_{\Rightarrow}^{-1}$.

Proof. (1) $\Rightarrow$ (2): Assume that $\mathsf{c}_{\Rightarrow}$ is an isomorphism, of inverse w. By Lemma 9, $(A \Rightarrow \pi_i) \circ \mathsf{c}_{\Rightarrow} = \pi_i$. Then $A \Rightarrow \pi_i = \pi_i \circ w$. Thus, $(A \Rightarrow \pi_i)_{i=0}^{\infty}$ is summable of witness w.

(2) $\Rightarrow$ (1): Assume that $(A \Rightarrow \pi_i)_{i=0}^{\infty}$ is summable. Then by Lemma 9

$$\pi_i \circ \langle\!\langle A \Rightarrow \pi_i \rangle\!\rangle_{i=0}^{\infty} \circ \mathsf{c}_{\Rightarrow} = (A \Rightarrow \pi_i) \circ \mathsf{c}_{\Rightarrow} = \pi_i$$

$$(A \Rightarrow \pi_i) \circ \mathsf{c}_{\Rightarrow} \circ \langle\!\langle A \Rightarrow \pi_i \rangle\!\rangle_{i=0}^{\infty} = \pi_i \circ \langle\!\langle A \Rightarrow \pi_i \rangle\!\rangle_{i=0}^{\infty} = A \Rightarrow \pi_i$$

We conclude that $\langle\!\langle A \Rightarrow \pi_i \rangle\!\rangle_{i=0}^{\infty} \circ \mathsf{c}_{\Rightarrow} = \mathsf{id}_{\mathsf{S}(A \Rightarrow X)}$ by joint monicity of the π_i, and that $\mathsf{c}_{\Rightarrow} \circ \langle\!\langle A \Rightarrow \pi_i \rangle\!\rangle_{i=0}^{\infty} = \mathsf{id}_{A \Rightarrow \mathsf{S}X}$ by joint monicity of the $A \Rightarrow \pi_i$ (see Remark 18).

(2) $\Rightarrow$ (3): Assume that $(f_i)_{i=0}^{\infty}$ is summable. Observe that $A \Rightarrow f_i = (A \Rightarrow \pi_i) \circ (A \Rightarrow \langle\!\langle f_i \rangle\!\rangle_{i=0}^{\infty})$. Then, by assumption $(A \Rightarrow \pi_i)_{i=0}^{\infty}$ is summable so $(A \Rightarrow f_i)_{i=0}^{\infty}$ is summable by left additivity.

(3) $\Rightarrow$ (2): (2) is a particular case of case (3), taking $f_i = \pi_i$.

(3) $\Rightarrow$ (4): By naturality of Cur, $\mathsf{Cur}(f_i) = (A \Rightarrow f_i) \circ \mathsf{cur}(\mathsf{id}_{X \& A})$. By assumption, if $(f_i)_{i=0}^{\infty}$ is summable then $(A \Rightarrow f_i)_{i=0}^{\infty}$ is summable so by left additivity, $(\mathsf{Cur}(f_i))_{i=0}^{\infty}$ is summable.

(4) $\Rightarrow$ (3): Recall that $A \Rightarrow f_i = \mathsf{Cur}(f_i \circ \mathsf{Ev})$. If $(f_i)_{i=0}^{\infty}$ is summable then $(f_i \circ \mathsf{Ev})_{i=0}^{\infty}$ is summable by left additivity, so $(\mathsf{Cur}(f_i \circ \mathsf{Ev}))_{i=0}^{\infty}$ is summable by assumption. It concludes the proof. □

To sum up, the left ω-summability structure is compatible with the closure if and only if the following property hold: $\mathsf{Cur}(0) = 0$ and for any family of $f_i \in \mathcal{C}(X \,\&\, A, Y)$, $(f_i)_{i=0}^{\infty}$ is summable if and only if $(\mathsf{Cur}(f_i))_{i=0}^{\infty}$ is summable, and $\sum_{i=0}^{\infty} \mathsf{Cur}(f_i) = \mathsf{Cur}(\sum_{i=0}^{\infty} f_i)$.

Proposition 34. *If* $h \in \mathcal{C}(X \,\&\, A, Y)$ *is additive in its first coordinate, then* $\mathsf{Cur}(h)$ *is additive. If* $h' \in \mathcal{C}(X, Y)$ *is additive, then* $A \Rightarrow h'$ *is additive.*

Proof. First, $\mathsf{Cur}(h) \circ 0 = \mathsf{Cur}(h \circ (0 \& A)) = \mathsf{Cur}(0) = 0$ by naturality of Cur and by additivity of h in its first coordinate. Furthermore, if $(f_i)_{i=0}^{\infty}$ is summable then $(h \circ (f_i \,\&\, A))_{i=0}^{\infty}$ is summable by left additivity of h, so $(\mathsf{Cur}(h) \circ f_i)_{i=0}^{\infty} = (\mathsf{Cur}(h \circ (f_i \,\&\, A)))_{i=0}^{\infty}$ is summable by Proposition 33, and

$$\sum_{i=0}^{\infty} \mathsf{Cur}(h \circ (f_i \,\&\, A)) = \mathsf{Cur}(\sum_{i=0}^{\infty} h \circ (f_i \,\&\, A)) = \mathsf{Cur}(h \circ ((\sum_{i=0}^{\infty} f_i) \,\&\, A)) = \mathsf{Cur}(h) \circ (\sum_{i=0}^{\infty} f_i)$$

so h is additive. Then, $A \Rightarrow h' = \mathsf{Cur}(h' \circ \mathsf{Ev})$. It is easy to check that $h' \circ \mathsf{Ev}$ is left additive (it is the composition of a left additive morphism with an additive one). So by what precedes, $A \Rightarrow h'$ is additive. □

3.3.2 Differential structure and closure

Definition 32. A Taylor expansion that is compatible with the cartesian product is compatible with the closure if $\mathsf{Ev} \in \mathcal{C}((A \Rightarrow X) \,\&\, A, X)$ is linear in its first coordinate.

We now assume that the Taylor expansion is compatible with the closure. It implies that $\pi_i \circ \mathsf{T}_0\mathsf{Ev} = \mathsf{Ev} \circ (\pi_i \,\&\, A)$ so

$$\mathsf{c}_\Rightarrow := \mathsf{Cur}\left\langle\!\left\langle \mathsf{Ev} \circ (\pi_i \,\&\, \mathsf{id}_A)\right\rangle\!\right\rangle_{i=0}^{\infty} = \mathsf{Cur}(\mathsf{TEv} \circ \Psi^0_{A\Rightarrow X,A}) \in \mathcal{C}(\mathsf{T}(A \Rightarrow X), A \Rightarrow \mathsf{T}X)$$

so $\mathsf{c}_\Rightarrow$ is a natural transformation $\mathsf{T}(A \Rightarrow \_) \Rightarrow (A \Rightarrow \mathsf{T}\_)$. We recognize above the mate construction, see Remark 36: $\mathsf{c}_\Rightarrow$ is the mate of Ψ^0 through the adjunction $\_ \,\&\, A \dashv A \Rightarrow \_$. It is showed in Proposition 40 and discussed in Section 7.3 that Ψ^0 is a distributive law $\mathsf{T}\_ \,\&\, A \Rightarrow \mathsf{T}(\_ \,\&\, A)$. So by Theorem 33, the invertibility of $\mathsf{c}_\Rightarrow$ is a necessary and sufficient condition to ensure that the adjunction $\_ \,\&\, A \dashv A \Rightarrow \_$ extends to $\mathcal{C}_{\underline{\mathsf{T}}}$, meaning that $\mathcal{C}_{\underline{\mathsf{T}}}$ is a cartesian closed category.

Definition 33. A *cartesian closed analytic category* is a cartesian closed category equipped with a left ω-summability and an analytic structure compatible with the cartesian product and the closure.

3.4 Cartesian closed analytic categories arising from LL

Let $\mathcal{L}$ be an analytic category, see Definition 21. Then as seen in Section 1.5.4, $(\mathsf{S}, (\mathsf{Der}\,\pi_i)_{i=0}^{\infty}, \mathsf{Der}\,\sigma)$ is a left ω-summability structure on $\mathcal{L}_!$, $\mathsf{Der}\,\langle\!\langle h_i\rangle\!\rangle_{i=0}^{\infty} = \langle\!\langle \mathsf{Der}\,h_i\rangle\!\rangle_{i=0}^{\infty}$ and $\mathsf{Der}\sum_{i=0}^{\infty} h_i = \sum_{i=0}^{\infty}\mathsf{Der}\,h_i$. This latter observation ensures that the morphisms defined from $(\mathsf{S}, (\mathsf{Der}\,\pi_i)_{i=0}^{\infty}, \mathsf{Der}\,\sigma)$ in Theorems 4 and 5 and Proposition 9 are respectively equal to $\mathsf{Der}\,\mathsf{c}$, $\mathsf{Der}\,\theta$, $\mathsf{Der}\,\iota_0$ and $\mathsf{Der}\,\mathsf{l}$. Then as discussed in Section 2, the analytic structure induces a functor T on $\mathcal{L}_!$ for which $\mathsf{Der}\,\pi_0$, $\mathsf{Der}\,\mathsf{c}$, $\mathsf{Der}\,\theta$, $\mathsf{Der}\,\iota_0$, $\mathsf{Der}\,\mathsf{l}$ and $\mathsf{Der}\,\sigma$ are natural transformations. Then by Proposition 35 below, $\mathsf{Der}\,\pi_i$, $\mathsf{Der}\,\sigma$, $\mathsf{Der}\,0 = 0$ are all T-linears, so $\mathcal{L}_!$ inherits from $\mathcal{L}$ a analytic structure.

Definition 34. A morphism $f \in \mathcal{L}_!(!X, Y)$ is *linear* if $f = \mathsf{Der}\,(h)$ for some $h \in \mathcal{L}(X, Y)$.

Proposition 35. *By (∂-chain), every linear morphism is also* T*-linear.*

Proof. Let $f = \mathsf{Der}\,h \in \mathcal{L}_!(X, Y)$. By Proposition 21, $\mathsf{Der}\,h$ is additive. Furthermore, (∂-chain) ensures that T extends S, meaning that $\mathsf{T}(\mathsf{Der}\,h) = \mathsf{Der}\,(\mathsf{S}h)$. Then, $\mathsf{Der}\,\pi_i \circ \mathsf{T}(\mathsf{Der}\,h) = \mathsf{Der}\,\pi_i \circ \mathsf{Der}\,(\mathsf{S}h) = \mathsf{Der}\,(\pi_i\,\mathsf{S}h) = \mathsf{Der}\,(h\,\pi_i) = \mathsf{Der}\,h \circ \mathsf{Der}\,\pi_i$ so $\mathsf{Der}\,h$ is T-linear. □

Remark 19. There are three successive layers of linearity: additivity, T-linearity and linearity. Linearity implies T-linearity and T-linearity implies additivity. As discussed in [BCS09], additivity does not necessarily imply T-linearity. The link between T-linearity and linearity should be investigated further, drawing inspirations from the works on whether a cartesian differential category is the co-Kleisli category of some model of linear logic.

The category $\mathcal{L}_!$ is cartesian. The cartesian product $\&_{i\in I} X_i$ is the same as the one in $\mathcal{L}$, and the projections are $\mathsf{Der}\,\mathsf{p}_i$. We can check using the joint monicity of the $\mathsf{Der}\,\mathsf{p}_i$ that $\mathsf{Der}\,h_0 \,\&\, \mathsf{Der}\,h_1 = \mathsf{Der}\,(h_0 \,\&\, h_1)$, meaning that the functor $\_ \,\&\, \_ : \mathcal{L}^2 \to \mathcal{L}$ extend to a functor $\_ \,\&\, \_ : \mathcal{L}_!^2 \to \mathcal{L}_!$. Besides, $\mathsf{Der}\,h_0 \,\&\, X_1 = \mathsf{Der}\,h_0 \,\&\, \mathsf{Der}\,X_1 = \mathsf{Der}\,(h_0 \,\&\, X_1)$ and similarly $X_0 \,\&\, \mathsf{Der}\,h_1 = \mathsf{Der}\,(X_0 \,\&\, h_1)$.

By Proposition 35 above, the projections are T-linear (in particular, they are additive). Besides, the morphism introduced in Definition 26 is equal to

$$\langle \mathsf{S}^{\mathsf{add}}(\mathsf{Der}\,\mathsf{p}_i)\rangle_{i\in I} = \langle \mathsf{Der}\,(\mathsf{S}\mathsf{p}_i)\rangle_{i\in I} = \mathsf{Der}\,\langle \mathsf{S}\mathsf{p}_i\rangle_{i\in I} = \mathsf{Der}\,\mathsf{c}_\&$$

Thus by (S-&), it is invertible (of inverse $\mathsf{Der}\,\mathsf{c}_{\&}^{-1}$). Thus, $\mathcal{L}_!$ is a cartesian analytic category. Also,

$$\Psi^0 := \mathsf{Der}\,\mathsf{c}_{\&}^{-1} \circ (\mathsf{T}X_0 \,\&\, \mathsf{Der}\,\iota_0) = \mathsf{Der}\,\mathsf{c}_{\&}^{-1} \circ \mathsf{Der}\,(\mathsf{S}X_0 \,\&\, \iota_0) = \mathsf{Der}\,(\mathsf{c}_{\&}^{-1}\,(\mathsf{S}X_0 \,\&\, \iota_0))$$

So $\Psi^0 = \mathsf{Der}\,\psi^0$ and similarly $\Psi^1 = \mathsf{Der}\,\psi^1$.

The axiom (∂-&) that has not played a role yet essentially ensures that a bilinear morphism is T-bilinear. As shown in [Ehr22a], the axiom (∂-&) implies the commutations below.

$$\begin{array}{ccccc} !\mathsf{S}X \otimes !Y & \xrightarrow{\partial_X \otimes !Y} & S!X \otimes !Y & \xrightarrow{\varphi^0_{!X,!Y}} & S(!X \otimes !Y) \\ {\scriptstyle \mathsf{m}^2{}_{\mathsf{S}X,Y}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{S}\mathsf{m}^2{}_{X,Y}} \\ !(\mathsf{S}X \,\&\, Y) & \xrightarrow[!\psi^0]{} & !\mathsf{S}(X \,\&\, Y) & \xrightarrow[\partial_{X\&Y}]{} & \mathsf{S}!(X \,\&\, Y) \end{array} \tag{11}$$

$$\begin{array}{ccccc} !X \otimes !\mathsf{S}Y & \xrightarrow{!X \otimes \partial_Y} & !X \otimes \mathsf{S}!Y & \xrightarrow{\varphi^1} & \mathsf{S}(!X \otimes !Y) \\ {\scriptstyle \mathsf{m}^2{}_{X,\mathsf{S}Y}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{S}\mathsf{m}^2{}_{X,Y}} \\ !(X \,\&\, \mathsf{S}Y) & \xrightarrow[!\psi^1_{X,Y}]{} & !\mathsf{S}(X \,\&\, Y) & \xrightarrow[\partial_{X\&Y}]{} & \mathsf{S}!(X \,\&\, Y) \end{array} \tag{12}$$

Those diagrams are not surprising. For example, Eq. (11) means that m^2 is a morphism between the distributive law $!\mathsf{S}\_ \otimes !Y \Rightarrow \mathsf{S}(!\_ \otimes !Y)$ and the distributive law $!(\mathsf{S}\_ \,\&\, Y) \Rightarrow \mathsf{S}!(\_ \,\&\, Y)$. Those are similar to the distributive laws involved in (∂-&) except that they use the strengths ψ^0 and φ^0 instead of the lax monoidalities $\mathsf{c}_{\&}^{-1}$ and L.

Definition 35. A morphism $f \in \mathcal{L}_!(X \,\&\, Y, Z)$ is bilinear if there exists $h \in \mathcal{L}(X \otimes Y, Z)$ such that

$$f = \; !(X \,\&\, Y) \xrightarrow{(\mathsf{m}^2)^{-1}} !X \otimes !Y \xrightarrow{\mathsf{der}\otimes\mathsf{der}} X \otimes Y \xrightarrow{\;h\;} Z$$

It is linear in its first coordinate if there exists $h \in \mathcal{L}(X \otimes !Y, Z)$ such that

$$f = \; !(X \,\&\, Y) \xrightarrow{(\mathsf{m}^2)^{-1}} !X \otimes !Y \xrightarrow{\mathsf{der}\otimes !Y} X \otimes !Y \xrightarrow{\;h\;} Z$$

We can define similarly what is a morphism linear in its second coordinate. Observe that a bilinear morphism is linear in both of its coordinate.

Proposition 36. *A morphism linear in a coordinate is* T*-linear in that coordinate. A bilinear morphism is* T*-bilinear.*

Proof. Assume that f is linear in its first coordinate. $f = h\,(\mathsf{der}\otimes !Y)\,(\mathsf{m}^2{}_{X,Y})^{-1}$. Then for any $g \in \mathcal{L}(A, X)$

$$\begin{aligned} f \circ (\mathsf{Der}\,g \,\&\, Y) &= f \circ \mathsf{Der}\,(g \,\&\, Y) \\ &= h\,(\mathsf{der}\otimes !Y)\,(\mathsf{m}^2{}_{X,Y})^{-1}\,!(g \,\&\, Y) \quad \text{by Proposition 20} \\ &= h\,(\mathsf{der}\otimes !Y)\,(!g \otimes !Y)\,(\mathsf{m}^2{}_{X,Y})^{-1} \quad \text{by naturality of } \mathsf{m}^2 \\ &= h\,(g \otimes !Y)\,(\mathsf{der}_{\mathsf{S}X} \otimes !Y)\,(\mathsf{m}^2{}_{X,Y})^{-1} \quad \text{by naturality of } \mathsf{der} \end{aligned}$$

So $f \circ (0 \,\&\, Y) = f \circ (\mathsf{Der}\,0 \,\&\, Y) = h\,(0 \otimes !Y)\,(\mathsf{der}_{\mathsf{S}X} \otimes !Y)\,(\mathsf{m}^2{}_{X,Y})^{-1} = 0$ by (S⊗-dist), by additivity in the morphisms in $\mathcal{L}$, and by left additivity. Moreover,

$$f \circ (\mathsf{Der}\,\pi_i \,\&\, Y) = h\,(\pi_i \otimes !Y)\,(\mathsf{der}_{\mathsf{S}X} \otimes !Y)\,(\mathsf{m}^2{}_{X,Y})^{-1}$$

so by (S⊗-dist), by additivity in the morphisms in $\mathcal{L}$, and by left additivity, the sequence $(f \circ (\mathsf{Der}\,\pi_i \,\&\, Y))_{i=0}^{\infty}$ is summable of sum

$$h\,(\sigma\otimes !Y)\,(\mathsf{der}_{\mathsf{S}X}\otimes !Y)\,(\mathsf{m}^2{}_{X,Y})^{-1} = f \circ (\mathsf{Der}\,\sigma \,\&\, Y)$$

Thus f is additive in its first coordinate. Furthermore,

$$\begin{aligned}
\mathsf{T}_0 f &= \mathsf{T}f \circ \mathsf{Der}\,(\psi^0_{X,Y}) \\
&= \mathsf{S}f\,\partial_{X\&Y}\,!\psi^0_{X,Y} \quad \text{by definition and Proposition 20} \\
&= \mathsf{S}h\,\mathsf{S}(\mathsf{der}_X\otimes !Y)\,\mathsf{Sm}^2{}^{-1}_{X,Y}\,\partial_{X\&Y}\,!\psi^0_{X,Y} \\
&= \mathsf{S}h\,\mathsf{S}(\mathsf{der}_X\otimes !Y)\,\varphi^0_{!X,!Y}\,(\partial_X\otimes !Y)\,(\mathsf{m}^2{}_{\mathsf{S}X,Y})^{-1} \quad \text{by Eq. (11)} \\
&= \mathsf{S}h\,\varphi^0_{X,!Y}\,(\mathsf{Sder}_X\otimes !Y)\,(\partial_X\otimes !Y)\,(\mathsf{m}^2{}_{\mathsf{S}X,Y})^{-1} \quad \text{by naturality of } \varphi^0 \\
&= \mathsf{S}h\,\varphi^0_{X,!Y}\,(\mathsf{der}_{\mathsf{S}X}\otimes !Y)\,(\mathsf{m}^2{}_{\mathsf{S}X,Y})^{-1} \quad \text{by } (\partial\text{-chain})
\end{aligned}$$

In particular, $\mathsf{T}_0 f$ is also linear in its first coordinate, and

$$\begin{aligned}
\mathsf{Der}\,\pi_i \circ \mathsf{T}_0 f &= \pi_i\,\mathsf{S}h\,\varphi^0_{X,!Y}\,(\mathsf{der}_{\mathsf{S}X}\otimes !Y)\,(\mathsf{m}^2{}_{\mathsf{S}X,Y})^{-1} \quad \text{by what precedes and Proposition 20} \\
&= h\,\pi_i\,\varphi^0_{X,!Y}\,(\mathsf{der}_{\mathsf{S}X}\otimes !Y)\,(\mathsf{m}^2{}_{\mathsf{S}X,Y})^{-1} \quad \text{by naturality of } \pi_i \\
&= h\,(\pi_i\otimes !Y)\,(\mathsf{der}_{\mathsf{S}X}\otimes !Y)\,(\mathsf{m}^2{}_{\mathsf{S}X,Y})^{-1} \\
&= h\,(\mathsf{der}_X\otimes !Y)\,(!\pi_i\otimes !Y)\,(\mathsf{m}^2{}_{\mathsf{S}X,Y})^{-1} \quad \text{by naturality of } \mathsf{der} \\
&= h\,(\mathsf{der}_X\otimes !Y)\,(\mathsf{m}^2{}_{X,Y})^{-1}\,!(\pi_i \,\&\, Y) \quad \text{by naturality of } \mathsf{m}^2 \\
&= f \circ \mathsf{Der}\,(\pi_i \,\&\, Y) \quad \text{by Proposition 20} \\
&= f \circ (\mathsf{Der}\,\pi_i \,\&\, Y)
\end{aligned}$$

So f is T-linear in its first coordinate. A similar proof based on Eq. (12) show that if f is linear in its second coordinate then it is T-linear in that coordinate. Finally, applying both results on a bilinear morphism show that it is T-bilinear. □

Assume that $\mathcal{L}$ is closed with regard to $\otimes$, and that (S⊗-fun) holds. The category $\mathcal{L}_!$ is closed with regard to its cartesian product. The internal hom-set of (A, X) is given by $(A \Rightarrow X, \mathsf{Ev})$ where $A \Rightarrow X = !A \multimap X$ and

$$\mathsf{Ev} = \; !((!A \multimap X) \,\&\, A) \xrightarrow{(\mathsf{m}^2{}_{!A\multimap X,A})^{-1}} !(!A \multimap X)\otimes !A \xrightarrow{\mathsf{der}_{!A\multimap X}\otimes !A} (!A \multimap X)\otimes !A \xrightarrow{\quad\mathsf{ev}\quad} X$$

If $f \in \mathcal{L}_!(X \,\&\, A, Y)$ then $f\,\mathsf{m}^2{}_{X,A} \in \mathcal{L}(!X\otimes !A, Y)$ and $\mathsf{Cur}(f) = \mathsf{cur}(f\,\mathsf{m}^2{}_X) \in \mathcal{L}_!(X, !A \multimap Y)$.

Observe that Ev is linear in its first coordinate, so it is additive and T-linear in that coordinate, thanks to Proposition 36. Furthermore, for any sequence $(f_i \in \mathcal{L}_!(X \,\&\, A, Y))_{i=0}^{\infty}$ that is summable, $(f_i\,\mathsf{m}^2)_{i=0}^{\infty}$ is summable by left additivity, so $(\mathsf{Cur}(f_i))_{i=0}^{\infty}$ is summable by (S⊗-fun) and Proposition 19. By Proposition 33, it implies that $\mathsf{c}_{\Rightarrow}$ is an iso, so the ω-summability structure and the analytic structure are compatible with the closure. To summarize everything, we have proved the following result.

Theorem 8. *For any analytic category $\mathcal{L}$ that is closed and that follows (S⊗-fun), $\mathcal{L}_!$ is a cartesian closed analytic category.*

4 The elementary theory

We focus now on a more specific situation which seems quite common in models of Linear Logic, and where the summability functor S is induced by a specific object, namely the $\mathbb{N}$-indexed cartesian product $\mathbb{D}$ of the unit 1 of the tensor product. Because of its conceptual simplicity and of its ubiquity, we call this situation *elementary*. In this case, the structure can be described very simply as a !-coalgebra structure on $\mathbb{D}$. We also provide three examples and one non-example of this situation.

All the categories $\mathcal{L}$ under consideration are assumed to have zero-morphisms, which means that they are enriched over the monoidal category of pointed sets. Let us spell out explicitly this assumption. For any two objects of $\mathcal{L}$, we assume that $\mathcal{L}(X,Y)$ has a distinguished element $0_{X,Y}$ (we will most often omit the subscripts), and the following equations hold:

$$0\, f = 0 \quad f\, 0 = 0$$

If $u \in \{0,1\}$ and $f \in \mathcal{L}(X,Y)$, we define $uf \in \mathcal{L}(X,Y)$ as the morphism which is $0_{X,Y}$ if $u = 0$ and f if $u = 1$.

4.1 The bimonoid of degrees

Let $\mathcal{L}$ be a symmetric monoidal category (SMC). We stick to the following conventions, which are standard in Linear Logic: the monoidal product is $\otimes$, the monoidal unit is 1, the canonical isos are $\lambda \in \mathcal{L}(1 \otimes X, X)$, $\rho \in \mathcal{L}(X \otimes 1, X)$, $\alpha \in \mathcal{L}((X_1 \otimes X_2) \otimes X_3, X_1 \otimes (X_2 \otimes X_3))$ and $\gamma \in \mathcal{L}(X_1 \otimes X_2, X_2 \otimes X_1)$. We will add subscripts when there are ambiguities as to the objects on which these isos are acting.

Notations 3. Let A be a set and P be a predicate on A, we define a family $([P(a)] \in \mathcal{L}(1,1))_{a \in A}$ by $[P(a)] = \mathsf{Id}_1$ if $P(a)$ holds and $[P(a)] = 0$ otherwise.

When a family of objects $(X_i)_{i \in I}$ admits a cartesian product in $\mathcal{L}$, we use $\&_{i \in I} X_i$ to denote this product and $(\mathsf{p}_i \in \mathcal{L}(\&_{j \in I} X_j, X_i))$ to denote the associated projections. Given morphisms $(f_i \in \mathcal{L}(X, Y_i))_{i \in I}$, we use $\langle f_i \rangle_{i \in I}$ for the unique element of $\mathcal{L}(Y, \&_{i \in I} X_i)$ such that $\mathsf{p}_i \langle f_j \rangle_{j \in I} = f_i$ for all $i \in I$.

We assume that all finite and countable cartesian products of 1 do exist in $\mathcal{L}$.

Definition 36. We set $\mathbb{D} = \&_{i \in \mathbb{N}} 1 = \overbrace{1 \,\&\, 1 \,\&\, \cdots}^{\mathbb{N}}$. The object $\mathbb{D}$ is called *the object of degrees* of $\mathcal{L}$.

We can define *injections* into $\mathbb{D}$ by $(\overline{\pi}_i = \langle \delta_{j,i} \mathsf{Id} \rangle_{j \in \mathbb{N}} \in \mathcal{L}(1, \mathbb{D}))_{i \in \mathbb{N}}$, in other words $\overline{\pi}_i$ is characterized by

$$\mathsf{p}_j\, \overline{\pi}_i = \begin{cases} \mathsf{Id} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{L}$ is enriched in pointed sets, we have also to assume zero-morphisms to be absorbing for the monoidal product:

$$f \otimes 0 = 0 \quad 0 \otimes f = 0\,.$$

We do not need the SMC $\mathcal{L}$ to be closed, but we nevertheless require some internal homs $(X \multimap Y, \mathsf{ev})$ to exist, see Definition 15. Remember that if $f \in \mathcal{L}(X_2, X_1)$ and $g \in \mathcal{L}(Y_1, Y_2)$,

if the pairs (X_i, Y_i) have internal homs $(X_i \multimap Y_i, \mathsf{ev}_{X_i})$ for $i = 1, 2$ then it is possible to define $f \multimap g \in \mathcal{L}(X_1 \multimap Y_1, X_2 \multimap Y_2)$, turning $\multimap$ into a functor $\mathcal{L}^{\mathsf{op}} \times \mathcal{L} \to \mathcal{L}$. Explicitly $f \multimap g = \mathsf{cur}_{X_2}(g\, \mathsf{ev}_{X_1}\, ((X_1 \multimap Y_1) \otimes f))$.

If $Y_1 = Y_2 = Y$ and $g = \mathsf{id}_Y$ then we obtain $f \multimap Y = \mathsf{cur}_{X_2}(\mathsf{ev}_{X_1}((X_1 \multimap Y) \otimes f)) \in \mathcal{L}(X_1 \multimap Y, X_2 \multimap Y)$. Then for any $h \in \mathcal{L}(Z \otimes X_1, Y)$,

$$(f \multimap Z)\, \mathsf{cur}_{X_1}(h) = \mathsf{cur}_{X_2}(h\, (Z \otimes f)) \tag{13}$$

Lemma 10. *Assume that (X, Y) and (X', Y) have internal homs and let $\overrightarrow{f} \in \mathcal{L}(X', X)^I$ be such that, for all object Z, the morphisms $(Z \otimes f_i)_{i \in I}$ are jointly epic. Then the morphisms $(f_i \multimap Y \in \mathcal{L}(X \multimap Y, X' \multimap Y))_{i \in I}$ are jointly monic.*

Proof. Let $g_1, g_2 \in \mathcal{L}(Z, X \multimap Y)$ be such that, for all $i \in I$, one has $(f_i \multimap Y)\, g_1 = (f_i \multimap Y)\, g_2$ for all $i \in I$ and $j \in \{1, 2\}$. We have by Eq. (13) $(f_i \multimap Y)\, g_j = \mathsf{cur}(\mathsf{cur}^{-1}(g_j)\, (Z \otimes f_i))$ and hence $\mathsf{cur}^{-1}(g_1)\, (Z \otimes f_i) = \mathsf{cur}^{-1}(g_1)\, (Z \otimes f_i)$ so that $\mathsf{cur}^{-1}(g_1) = \mathsf{cur}^{-1}(g_2)$ by our assumption on the f_i's and hence $g_1 = g_2$. □

Notice that $(1 \multimap X, \mathsf{ev}_1)$ always exists: we can take $1 \multimap X = X$, $\mathsf{ev}_1 = \rho$ and $\mathsf{cur}_1(f) = f\, \rho^{-1}$.

Assume that the internal hom $(\mathbb{D} \multimap X, \mathsf{ev})$ exists for all object X of $\mathcal{L}$.

In particular, we can define the functor $\mathsf{S} = (\mathbb{D} \multimap \_) : \mathcal{L} \to \mathcal{L}$, equipped with the natural transformations $\pi_i = (\overline{\pi}_i \multimap X) \in \mathcal{L}(\mathsf{S}X, X)$ and $\sigma = (\Delta \multimap X) \in \mathcal{L}(\mathsf{S}X, X)$.

Remark 20. More explicitly, $\pi_i = (\overline{\pi}_i \multimap X) = \mathsf{cur}_1(\mathsf{ev}_{\mathbb{D}}\, ((\mathbb{D} \multimap X) \otimes \overline{\pi}_i)) = \mathsf{ev}_{\mathbb{D}}\, ((\mathbb{D} \multimap X) \otimes \overline{\pi}_i)\, \rho^{-1}$ and similarly $\sigma = \mathsf{ev}_{\mathbb{D}}\, ((\mathbb{D} \multimap X) \otimes \Delta)\, \rho^{-1}$.

Definition 37. An elementary pre-ω-summable category is a symmetric monoidal category $\mathcal{L}$ with zero morphisms, where the tensor unit 1 has an ω-cartesian product $(\mathbb{D}, \overrightarrow{\mathsf{p}})$ such that

- all internal homs $(\mathbb{D} \multimap X, \mathsf{ev})$ exist;
- for any object X of $\mathcal{L}$, the morphisms $(X \otimes \overline{\pi}_i)_{i \in \mathbb{N}}$ are jointly epic.

Remark 21. Upon taking $X = 1$, the second condition implies that the $(\overline{\pi}_i)_{i \in \mathbb{N}}$ are jointly epic. Conversely, if the SMC $\mathcal{L}$ is closed, the joint epicity of the $(\overline{\pi}_i)_{i \in \mathbb{N}}$ imply this second condition. Assume indeed that $\mathcal{L}$ is closed and that the $(\overline{\pi}_i)_{i \in \mathbb{N}}$ are jointly epic. Let $f, g \in \mathcal{L}(X \otimes \mathbb{D}, Y$ be such that $(f\, (X \otimes \overline{\pi}_i)) = g\, (X \otimes \overline{\pi}_i))_{i \in \mathbb{N}}$. By naturality of γ, we get $(f\, \gamma\, (\overline{\pi}_i \otimes X)) = g\, \gamma\, (\overline{\pi}_i \otimes X))_{i \in \mathbb{N}}$ and hence $(\mathsf{cur}(f\, \gamma)\, \overline{\pi}_i = \mathsf{cur}(g\, \gamma)\, \overline{\pi}_i)_{i \in \mathbb{N}}$ so that $\mathsf{cur}(f\, \gamma) = \mathsf{cur}(g\, \gamma)$ and hence $f = g$.

Lemma 11. *If $\mathcal{L}$ is an elementary pre-ω-summable category, then $(\mathsf{S}, \overrightarrow{\pi}, \sigma)$ is a pre ω-summability structure on $\mathcal{L}$*

Proof. We only have to check that the π_i's are jointly monic, it suffices to apply Lemma 10 to the $\overline{\pi}_i$'s. □

Lemma 12. *If $\mathcal{L}$ is an elementary pre-ω-summable category, then for any $n \in \mathbb{N}$ the family of morphisms $(X \otimes \overline{\pi}_{i_1} \otimes \cdots \otimes \overline{\pi}_{i_n})_{\overrightarrow{i} \in \mathbb{N}^n}$ is jointly epic.*

Proof. Induction on $n \geq 1$. The base case is just our assumption that $\mathcal{L}$ is an elementary pre-ω-summable category. Assume that the property holds for n and let $f, g \in \mathcal{L}(X \otimes \mathbb{D}^{\otimes(n+1)})$ be such that $f\, (X \otimes \overline{\pi}_{i_1} \otimes \cdots \otimes \overline{\pi}_{i_n} \otimes \overline{\pi}_{i_{n+1}}) = g\, (X \otimes \overline{\pi}_{i_1} \otimes \cdots \otimes \overline{\pi}_{i_n} \otimes \overline{\pi}_{i_{n+1}})$ for all $\overrightarrow{i} \in \mathbb{N}^{n+1}$.

For $i \in \mathbb{N}$ let $f_i \in \mathcal{L}(X \otimes \mathbb{D}^{\otimes n})$ be defined as

$$X \otimes \mathbb{D}^{\otimes n} \xrightarrow{(\rho)^{-1}} X \otimes \mathbb{D}^{\otimes n} \otimes 1 \xrightarrow{X \otimes \mathbb{D}^{\otimes n} \otimes \overline{\pi}_i} X \otimes \mathbb{D}^{\otimes n} \otimes \mathbb{D} \xrightarrow{f} Y$$

and similarly for g_i. By inductive hypothesis we have $f_i = g_i$ for all $i \in \mathbb{N}$ and hence $f = g$ since $\mathcal{L}$ is an elementary pre-ω-summable category. □

Saying that $\overrightarrow{f} = (f_i \in \mathcal{L}(X, Y))_{i \in \mathbb{N}}$ is summable in this pre ω-summability structure means thanks to Eq. (13) that there is $h \in \mathcal{L}(X \otimes \mathbb{D}, Y)$ such that, for all $i \in \mathbb{N}$, the following diagram commutes

$$\begin{array}{ccc} X \otimes 1 & \xrightarrow{X \otimes \overline{\pi}_i} & X \otimes \mathbb{D} \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f_i} & Y \end{array}$$

We set $\langle\!\langle \overrightarrow{f} \rangle\!\rangle^{\otimes} = h$ since this h is unique when it exists by the fact that the $\overline{\pi}_i$'s a jointly epic, so that $\langle\!\langle \overrightarrow{f} \rangle\!\rangle = \mathsf{cur}\langle\!\langle \overrightarrow{f} \rangle\!\rangle^{\otimes}$.

And by Eq. (13) again the sum is given by

$$\sum \overrightarrow{f} = \langle\!\langle \overrightarrow{f} \rangle\!\rangle^{\otimes} (X \otimes \Delta) \rho^{-1} \in \mathcal{L}(X, Y) .$$

Lemma 13. *Let $\overrightarrow{f} \in \mathcal{L}(X, 1)^{\mathbb{N}}$. The family of morphisms $(\overline{\pi}_i f_i)_{i \in \mathbb{N}} \in \mathcal{L}(X, \mathbb{D})^{\mathbb{N}}$ is summable and has $\langle \overrightarrow{f} \rangle \in \mathcal{L}(X, \mathbb{D})$ as sum.*

Proof. Let $h = \langle h_i \rangle_{i \in \mathbb{N}} \in \mathcal{L}(X \otimes \mathbb{D}, \mathbb{D})$ where h_i is the following composition of morphisms

$$X \otimes \mathbb{D} \xrightarrow{X \otimes \mathsf{p}_i} X \otimes 1 \xrightarrow{\rho} X \xrightarrow{f_i} 1$$

We have $h (X \otimes \overline{\pi}_j) = \langle f_i \rho (X \otimes (\mathsf{p}_i \overline{\pi}_j)) \rangle_{i \in \mathbb{N}} = \langle \delta_{i,j} f_i \rangle_{i \in \mathbb{N}} \rho = \langle \delta_{i,j} f_j \rangle_{i \in \mathbb{N}} \rho = \overline{\pi}_j f_j \rho$ which shows that family $(\overline{\pi}_i f_i)_{i \in \mathbb{N}}$ is summable. Next we have $\sum_{i \in \mathbb{N}} \overline{\pi}_i f_i = h (X \otimes \Delta) \rho^{-1} = \langle f_i \rho (X \otimes (\mathsf{p}_i \Delta)) \rangle_{i \in \mathbb{N}} \rho^{-1} = \langle f_i \rho (X \otimes \mathsf{Id}_1) \rangle_{i \in \mathbb{N}} \rho^{-1} = \langle \overrightarrow{f} \rangle$. □

Lemma 14. *Let $I, J \subseteq \mathbb{N}$ and $\psi : J \to I$ be a function. There is a unique $\chi(I, \psi, J) \in \mathcal{L}(\mathbb{D}, \mathbb{D})$ such that*

$$\mathsf{p}_j \chi(I, \psi, J) \overline{\pi}_i = \begin{cases} \mathsf{Id}_1 & \text{if } j \in J \text{ and } \psi(j) = i \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The morphism $\chi(I, \psi, J) = \langle f_j \rangle_{j \in \mathbb{N}}$ where $f_j = \overline{\pi}_j [j \in J] \mathsf{p}_{\psi(j)}$ for all $j \in \mathbb{N}$ satisfies the required condition, and uniqueness results from the joint monicity of the $\overline{\pi}_i$'s. □

Lemma 15. *Let $I, J \subseteq \mathbb{N}$ and $\eta : I \to J$ be a bijection. Then $\chi(I, \eta^{-1}, J) \in \mathcal{L}(\mathbb{D}, \mathbb{D})$ satisfies, for all $i \in \mathbb{N}$,*

$$\chi(I, \eta^{-1}, J) \overline{\pi}_i = \begin{cases} \overline{\pi}_{\eta(i)} & \text{if } i \in I \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Immediate from Lemma 14. □

We set $\chi(I) = \chi(I, \mathsf{Id}, I) \in \mathcal{L}(\mathbb{D}, \mathbb{D})$.

Lemma 16. *If $\eta : I \to J$ is a bijection then $\chi(I, \eta^{-1}, J)\,\Delta = \chi(J)\,\Delta$.*

Proof. Using the same notations as in the proof of Lemma 14, we have $\chi(I, \eta, J)\,\Delta = \langle f_j \rangle_{j\in\mathbb{N}}\,\Delta = \langle f_j\,\Delta\rangle_{j\in\mathbb{N}}$, but

$$f_j\,\Delta = \begin{cases} \mathsf{p}_{\eta^{-1}(j)}\,\Delta & \text{if } j \in J \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \mathsf{Id}_1 & \text{if } j \in J \\ 0 & \text{otherwise} \end{cases}$$

which proves our contention. □

Lemma 17. *If $\mathcal{L}$ is an elementary pre-ω-summable category, then $(\mathsf{S}, \overrightarrow{\pi}, \sigma)$ satisfies (S-zero) and (S-com).*

Proof. We check first (S-zero) so let $f \in \mathcal{L}(X, Y)$ and let $\overrightarrow{f} \in \mathcal{L}(X, Y)^{\mathbb{N}}$ be such that $f_0 = f$ and $f_{n+1} = 0$. Let $h = f\,\rho\,(X \otimes \mathsf{p}_0) : X \otimes \mathbb{D} \to Y$, we have $h\,(X \otimes \overline{\pi}_n) = \delta_{n,0} f\,\rho = f_n\,\rho$ and hence $\overrightarrow{f}$ is summable with $\langle\!\langle \overrightarrow{f} \rangle\!\rangle^{\otimes} = h$. We have $\sum \overrightarrow{f} = h\,(X \otimes \Delta)\,\rho^{-1} = f\,\rho(X \otimes (\mathsf{p}_0\,\Delta))\,\rho^{-1} = f$ since $\mathsf{p}_0\,\Delta = \mathsf{Id}_1$.

We check now (S-com). Let A be a set and $\overrightarrow{f} \in \mathcal{L}(X, Y)^A$ and $\varphi, \psi : A \to \mathbb{N}$ be injections. Assume that $\varphi_* \overrightarrow{f}$ is summable, so let $h = \langle\!\langle \varphi_* \overrightarrow{f} \rangle\!\rangle^{\otimes}$, that is $h : X \otimes \mathbb{D} \to Y$ satisfies, for all $i \in \mathbb{N}$,

$$h\,(X \otimes \overline{\pi}_i)\,(\rho)^{-1} = \begin{cases} f_{\varphi^{-1}(i)} & \text{if } n \in \varphi(A) \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\varphi \circ \psi^{-1}$ is a bijection $\psi(A) \to \varphi(A)$, so let $h' = h\,(X \otimes k)$ where

$$k = \chi(\psi(A), (\varphi \circ \psi^{-1})^{-1}, \varphi(A))\,,$$

we have, for all $j \in \mathbb{N}$,

$$\begin{aligned} h'\,(X \otimes \overline{\pi}_j)\,(\rho)^{-1} &= h\,(X \otimes (k\,\overline{\pi}_j))\,(\rho)^{-1} \\ &= \begin{cases} h\,(X \otimes (\overline{\pi}_{(\varphi\circ\psi^{-1})(j)}))\,\rho^{-1} & \text{if } j \in \psi(A) \\ 0 & \text{otherwise} \end{cases} \quad \text{by Lemma 15} \\ &= \begin{cases} f_{\psi^{-1}(j)} & \text{if } j \in \psi(A) \\ 0 & \text{otherwise} \end{cases} \quad \text{since } h = \langle\!\langle \varphi_* \overrightarrow{f} \rangle\!\rangle^{\otimes} \\ &= \psi_* \overrightarrow{f} \end{aligned}$$

which shows that $\psi_* \overrightarrow{f}$ is summable. We prove that $\sum \psi_* \overrightarrow{f} = \sum \varphi_* \overrightarrow{f}$. We have

$$\begin{aligned} \sum \psi_* \overrightarrow{f} &= h'\,(X \otimes \Delta)\,\rho^{-1} \\ &= h\,(X \otimes (k\,\Delta))\,\rho^{-1} \\ &= h\,(X \otimes (\chi(\varphi(A))\,\Delta))\,\rho^{-1} \quad \text{by Lemma 16} \\ &= h\,(X \otimes \chi(\varphi(A)))\,(X \otimes \Delta)\,\rho^{-1} \end{aligned}$$

but $h\,(X \otimes \chi(\varphi(A))) = h$ by joint epicity of the $X \otimes \overline{\pi}_i$'s: we have indeed

$$\begin{aligned} h\,(X \otimes \chi(\varphi(A)))\,(X \otimes \overline{\pi}_i) &= h\,(X \otimes (\chi(\varphi(A))\,\overline{\pi}_i)) \\ &= \begin{cases} h\,(X \otimes \overline{\pi}_i) & \text{if } i \in \varphi(A) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} f_{\varphi^{-1}(i)}\,\rho & \text{if } i \in \varphi(A) \\ 0 & \text{otherwise} \end{cases} \quad \text{by definition of } h\,, \end{aligned}$$

and hence $\sum \psi_* \overrightarrow{f} = h\,(X \otimes \Delta)\,\rho^{-1} = \sum \varphi_* \overrightarrow{f}$. □

Lemma 18. *Let $\mathcal{L}$ be an elementary pre-ω-summable category. Let $\overrightarrow{f} \in \mathcal{L}(X,Y)^{\mathbb{N}^2}$ be summable. Then for all $i \in \mathbb{N}$, the family $\overrightarrow{f(i)} = (f_{i,j})_{j\in\mathbb{N}} \in \mathcal{L}(X,Y)^{\mathbb{N}}$ is summable. Moreover, the family $(\sum \overrightarrow{f(i)} = \sum_{j\in\mathbb{N}} f_{i,j})_{i\in\mathbb{N}}$ is summable, and we have $\sum_{i\in\mathbb{N}} \sum \overrightarrow{f(i)} = \sum_{(i,j)\in\mathbb{N}^2} f_{i,j}$. In other words, the property (S-ass) holds.*

Proof. By our assumption about $\overrightarrow{f}$, we can take an arbitrary injection $\varphi : \mathbb{N}^2 \to \mathbb{N}$, and we know that $\varphi_* \overrightarrow{f}$ is summable so let $h = \langle\!\langle \varphi_* \overrightarrow{f} \rangle\!\rangle^{\otimes}$, in other words, for all $n \in \mathbb{N}$,

$$h\,(X \otimes \overline{\pi}_n) = \begin{cases} f_{\varphi^{-1}(n)}\,\rho & \text{if } n \in I \\ 0 & \text{otherwise} \end{cases}$$

where $I = \varphi(\mathbb{N}^2) \subseteq \mathbb{N}$.

Let $i \in \mathbb{N}$, $I_i = \varphi(\{i\} \times \mathbb{N}) \subseteq I$ and $k_i = \chi(I_i) \in \mathcal{L}(\mathbb{D},\mathbb{D})$. Let $\psi^i : \mathbb{N} \to \mathbb{N}^2$ be the injection given by $\psi^i(j) = (i,j)$ so that $\varphi \circ \psi^i$ is an injection $\mathbb{N} \to I_i$. Define $\overrightarrow{g} = (\varphi \circ \psi^i)_* \overrightarrow{f(i)} \in \mathcal{L}(X,Y)^{\mathbb{N}}$. We have, for all $n \in \mathbb{N}$,

$$h\,(X \otimes (k_i\,\overline{\pi}_n)) = \begin{cases} h\,(X \otimes \overline{\pi}_n) & \text{if } n \in I_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} f_{\varphi^{-1}(n)}\,\rho = f_{(i,(\varphi\circ\psi^i)^{-1}(n))}\,\rho & \text{if } n \in I_i \\ 0 & \text{otherwise,} \end{cases}$$

that is $h\,(X \otimes k_i) = \langle\!\langle \overrightarrow{g} \rangle\!\rangle^{\otimes}$ (by the definitions of I_i and of ψ^i), and hence $\overrightarrow{g}$ is summable, so that $\overrightarrow{f(i)}$ is summable for each $i \in \mathbb{N}$. Let $h_i = \langle\!\langle \overrightarrow{f(i)} \rangle\!\rangle^{\otimes} \in \mathcal{L}(X \otimes \mathbb{D}, Y)$ so that $\sum \overrightarrow{f(i)} = h_i\,(X \otimes \Delta)\rho^{-1}$.

Now we prove that $(\sum \overrightarrow{f(i)})_{i\in\mathbb{N}}$ is summable. Let $\psi : I \to \mathbb{N}$ which maps $n \in I$ to the first component of the pair $\varphi^{-1}(n)$. In other words, for $n \in I$, $\psi(n)$ is the unique $i \in \mathbb{N}$ such that $n \in I_i$ (by injectivity of φ, the set I is the disjoint union of the I_i's). Let $k' = \chi(\mathbb{N}, \psi, I) \in \mathcal{L}(\mathbb{D},\mathbb{D})$ (see Lemma 14), we have $\mathsf{p}_j\,k'\,\overline{\pi}_i = [j \in I \text{ and } \psi(j) = i]$ and notice that we also have $\mathsf{p}_j\,k_i\,\Delta = [j \in I \text{ and } \psi(j) = i]$ since $k_i = \chi(I_i)$. Therefore, $k'\,\overline{\pi}_i = k_i\,\Delta \in \mathcal{L}(1,\mathbb{D})$ for all $i \in \mathbb{N}$. It follows that $h\,(X \otimes (k'\,\overline{\pi}_i)) = h\,(X \otimes (k_i\,\Delta))$ and we know that $h\,(X \otimes k_i) = \langle\!\langle (\varphi \circ \psi^i)_* \overrightarrow{f(i)} \rangle\!\rangle^{\otimes}$ so that $h\,(X \otimes (k_i\,\Delta))\,\rho^{-1} = \sum (\varphi \circ \psi^i)_* \overrightarrow{f(i)}$. It follows that $(\sum (\varphi \circ \psi^i)_* \overrightarrow{f(i)})_{i\in\mathbb{N}}$ is summable, with $h(X \otimes k') = \langle\!\langle \sum (\varphi \circ \psi^i)_* \overrightarrow{f(i)} \rangle\!\rangle^{\otimes}_{i\in\mathbb{N}}$, but $\sum (\varphi \circ \psi^i)_* \overrightarrow{f(i)} = \sum \overrightarrow{f(i)}$ by Lemma 17 which ends the proof that $(\sum \overrightarrow{f(i)})_{i\in\mathbb{N}}$ is summable.

We have $\sum_{i\in\mathbb{N}} \sum \overrightarrow{f(i)} = h\,(X \otimes (k'\,\Delta))\,\rho^{-1}$ and $\sum_{(i,j)\in\mathbb{N}^2} f_{i,j} = h\,(X \otimes \Delta)\,\rho^{-1}$. Notice that $k'\,\Delta = \chi(I)\,\Delta$ by Lemma 16. We conclude the proof by observing that $h\,(X \otimes \chi(I)) = h$ since $h = \langle\!\langle \varphi_* \overrightarrow{f} \rangle\!\rangle^{\otimes}$ and $I = \varphi(\mathbb{N}^2)$. □

Definition 38. (𝔻-flat) An elementary pre ω-summable category is *flattenable* if, for any injection $\varphi : \mathbb{N}^2 \to \mathbb{N}$, there is a morphism $k \in \mathcal{L}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ such that, for all $i, j, n \in \mathbb{N}$

$$k\,\overline{\pi}_n = \begin{cases} (\overline{\pi}_i \otimes \overline{\pi}_j)\,\rho^{-1} & \text{if } \varphi(i,j) = n \\ 0 & \text{otherwise.} \end{cases}$$

Notice that k is then unique by Lemma 12.

Lemma 19. *If elementary pre ω-summable category satisfies (𝔻-flat) then the associated pre ω-summable category satisfies (S-flat).*

Proof. Let $\varphi : \mathbb{N} \to \mathbb{N}^2$ be an injection, we must show that $\varphi_* \overrightarrow{f} \in \mathcal{L}(\mathsf{S}^2 X, X)^{\mathbb{N}}$ is summable, where $\overrightarrow{f} = ((\pi_i)_X\,(\pi_j)_{\mathsf{S}X})_{(i,j)\in\mathbb{N}^2}$. We have $\overline{\pi}_i \otimes \overline{\pi}_j \multimap X \in \mathcal{L}((\mathbb{D} \otimes \mathbb{D} \multimap X), (1 \otimes 1 \multimap X))$ and there are canonical[7] natural isomorphisms

$$\epsilon_X \in \mathcal{L}(\mathsf{S}^2 X = (\mathbb{D} \multimap (\mathbb{D} \multimap X)), \mathbb{D} \otimes \mathbb{D} \multimap X) \text{ and } \nu_X \in \mathcal{L}(1 \otimes 1 \multimap X, X)$$

such that $\pi_i\,\pi_i = \nu_X\,(\overline{\pi}_i \otimes \overline{\pi}_j \multimap X)\,\epsilon_X$. Let $k \in \mathcal{L}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ defined from φ using (𝔻-flat). Let $h' \in \mathcal{L}((\mathbb{D} \otimes \mathbb{D} \multimap X) \otimes \mathbb{D}, X)$ be the following composition of morphisms

$$(\mathbb{D} \otimes \mathbb{D} \multimap X) \otimes \mathbb{D} \xrightarrow{(\mathbb{D}\otimes\mathbb{D}\multimap X)\otimes k} (\mathbb{D} \otimes \mathbb{D} \multimap X) \otimes \mathbb{D} \otimes \mathbb{D} \xrightarrow{\mathsf{ev}} X$$

In the sequel, we leave implicit some instances of the isos λ and ρ to increase readability. Setting $Y = (\mathbb{D} \otimes \mathbb{D} \multimap X)$ we have, by definition of k,

$$h'\,(Y \otimes \overline{\pi}_n) = \begin{cases} \mathsf{ev}\,(Y \otimes (\overline{\pi}_i \otimes \overline{\pi}_j)) & \text{if } \varphi(i,j) = n \\ 0 & \text{otherwise} \end{cases}$$

so that $\mathsf{cur}(h') \in \mathcal{L}(\mathbb{D} \otimes \mathbb{D} \multimap X, \mathsf{S}X)$ satisfies

$$\begin{aligned} \pi_n\,\mathsf{cur}(h') &= (\overline{\pi}_n \multimap X)\,\mathsf{cur}(h') \\ &= \mathsf{cur}(h'\,(Y \otimes \overline{\pi}_n)) \\ &= \begin{cases} \mathsf{cur}(\mathsf{ev}\,(Y \otimes (\overline{\pi}_i \otimes \overline{\pi}_j))) & \text{if } \varphi(i,j) = n \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \overline{\pi}_i \otimes \overline{\pi}_j \multimap X & \text{if } \varphi(i,j) = n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

so that $h = \mathsf{cur}(h')\,\epsilon_X \in \mathcal{L}(\mathsf{S}^2 X, \mathsf{S}X)$ satisfies $\pi_n\,h = (\varphi_* \overrightarrow{f})_n$ which shows that $(\pi_i\,\pi_j)_{(i,j)\in\mathbb{N}^2}$ is summable. □

Definition 39. (𝔻-wit) An elementary pre ω-summability structure *has witnesses* if, for any $\overrightarrow{h} \in \mathcal{L}(X \otimes \mathbb{D}, Y)^{\mathbb{N}}$, if $\overrightarrow{h}\,(X \otimes \Delta)\,\rho^{-1} \in \mathcal{L}(X, Y)^{\mathbb{N}}$ is summable, then $\overrightarrow{h}$ is summable.

Lemma 20. *If elementary pre ω-summable category satisfies (𝔻-wit) then the associated pre ω-summable category satisfies (S-wit).*

Proof. This is immediate, actually (𝔻-wit) is a straightforward reformulation of (S-wit) in the elementary setting. □

[7] In the sense that they can be expressed using only the SMC structure of $\mathcal{L}$ and the fact that for all object Y the internal hom $(\mathbb{D} \multimap Y, \mathsf{ev})$ exists.

Definition 40. An elementary ω-summable category is a symmetric monoidal category $\mathcal{L}$ with zero morphisms, where the tensor unit 1 has an ω-cartesian product $(\mathbb{D}, \overrightarrow{\mathsf{p}})$ such that

- all internal homs $(\mathbb{D} \multimap X, \mathsf{ev})$ do exist;
- for any object X of $\mathcal{L}$, the morphisms $(X \otimes \overline{\pi}_i)_{i\in\mathbb{N}}$ are jointly epic;
- the conditions (𝔻-flat) and (𝔻-wit) are satisfied.

Remark 22. As already noticed in [Ehr23], being an elementary ω-summable category is a *property* of the SMC $\mathcal{L}$, and not an additional structure.

The following result summarizes what we have proven so far.

Theorem 9. *Any elementary ω-summable category has an ω-summability structure $(\mathsf{S}, \overrightarrow{\pi}, \sigma)$ with $\mathsf{S} = (\mathbb{D} \multimap \_)$, $\pi_i = (\overline{\pi}_i \multimap X)$ and $\sigma = (\Delta \multimap X)$.*

From now on we assume that $\mathcal{L}$ is an elementary ω-summable category.

Proposition 37. *If $\overrightarrow{f} \in \mathcal{L}(X_0, Y_0)^{\mathbb{N}}$ is summable, then $\overrightarrow{f} \otimes X_1 = (f_i \otimes X_1)_{i=0}^{\infty}$ is summable, and we have $\langle\!\langle \overrightarrow{f} \otimes X_1 \rangle\!\rangle^{\otimes} = (\langle\!\langle \overrightarrow{f} \rangle\!\rangle^{\otimes} \otimes X_1)\,(X \otimes \gamma)$ and $\sum(\overrightarrow{f} \otimes X_1) = (\sum \overrightarrow{f}) \otimes X_1$.*

If $\overrightarrow{f} \in \mathcal{L}(X_1, Y_1)^{\mathbb{N}}$ is summable, then $X_0 \otimes \overrightarrow{f} = (X_0 \otimes f_i)_{i=0}^{\infty}$ is summable, and we have $\langle\!\langle X_0 \otimes \overrightarrow{f} \rangle\!\rangle^{\otimes} = (X_0 \otimes \langle\!\langle \overrightarrow{f} \rangle\!\rangle^{\otimes})$ and $\sum(X_0 \otimes \overrightarrow{f}) = X_0 \otimes (\sum \overrightarrow{f})$.

So the ω-summability structure follows (S⊗-dist).

The proof is trivial.

4.2 The comonoid structure of $\mathbb{D}$

We equip $\mathbb{D}$ with a comonoid structure. The counit is the projection $\mathsf{p}_0 = \mathsf{p}_0 \in \mathcal{L}(\mathbb{D}, 1)$. We define a comultiplication $\widetilde{\theta} \in \mathcal{L}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ which satisfies $\widetilde{\theta}\,\overline{\pi}_n = \sum_{i=0}^{n} \overline{\pi}_i \otimes \overline{\pi}_{n-i}$. To this end we use Theorem 5 which yields $\theta_{\mathbb{D}\otimes\mathbb{D}} \in \mathcal{L}(\mathbb{D} \multimap (\mathbb{D} \multimap \mathbb{D} \otimes \mathbb{D}), \mathbb{D} \multimap \mathbb{D} \otimes \mathbb{D})$ and hence $\mathsf{cur}^{-1}(\theta_{\mathbb{D}\otimes\mathbb{D}}) \in \mathcal{L}((\mathbb{D} \multimap (\mathbb{D} \multimap \mathbb{D} \otimes \mathbb{D})) \otimes \mathbb{D}, \mathbb{D} \otimes \mathbb{D})$. But we have $\lambda \in \mathcal{L}(1 \otimes \mathbb{D} \otimes \mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ whence $\mathsf{cur}(\mathsf{cur}(\lambda)) \in \mathcal{L}(1, \mathbb{D} \multimap (\mathbb{D} \multimap \mathbb{D} \otimes \mathbb{D}))$ and hence

$$\widetilde{\theta} = \mathsf{cur}^{-1}(\theta_{\mathbb{D}\otimes\mathbb{D}})\,(\mathsf{cur}(\mathsf{cur}(\lambda)) \otimes \mathbb{D})\,\lambda^{-1} \in \mathcal{L}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$$

Given $n \in \mathbb{N}$, we know that $(\pi_i\,\pi_{n-i})_{i=0}^{n}$ is summable and that $\pi_n\,\theta = \sum_{i=0}^{n} \pi_i\,\pi_{n-i}$. From this fact and from the fact that $(\pi_i)_X = (\overline{\pi}_i \multimap X)$, by standard computations using only the SMC structure of $\mathcal{L}$ and the fact that all internal homs $\mathbb{D} \multimap X$ exist, we can easily deduce that the family $(\overline{\pi}_i \otimes \overline{\pi}_{n-i} \in \mathcal{L}(1 \otimes 1, \mathbb{D} \otimes \mathbb{D}))_{i=0}^{n}$ is summable and that $\widetilde{\theta}\,\overline{\pi}_n\,\lambda_1 = \sum_{i=0}^{n} \overline{\pi}_i \otimes \overline{\pi}_{n-i}$.

Notice that the joint epicity of the $\overline{\pi}_n$'s shows that $\widetilde{\theta}$ is uniquely characterized by these equations.

Lemma 21. *The triple $(\mathbb{D}, \mathsf{p}_0, \widetilde{\theta})$ is a commutative comonoid.*

Proof. We prove first that

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\widetilde{\theta}} & \mathbb{D} \otimes \mathbb{D} \\ & {\scriptstyle \rho^{-1}} \searrow & \downarrow {\scriptstyle \mathbb{D}\otimes\mathsf{p}_0} \\ & & \mathbb{D} \otimes 1 \end{array}$$

so let $n \in \mathbb{N}$, we have

$$(\mathbb{D} \otimes \mathsf{p}_0)\,\widetilde{\theta}\,\overline{\pi}_n = (\mathbb{D} \otimes \mathsf{p}_0)\sum_{i=0}^{n}(\overline{\pi}_i \otimes \overline{\pi}_{n-i})\lambda^{-1}$$
$$= \sum_{i=0}^{n}(\overline{\pi}_i \otimes (\mathsf{p}_0\,\overline{\pi}_{n-i}))\lambda^{-1}$$

but $\mathsf{p}_0\,\overline{\pi}_{n-i} = \mathsf{p}_0\,\overline{\pi}_{n-i} = \delta_{n-i,0}\mathsf{Id}_1$ and hence $(\mathbb{D} \otimes \mathsf{p}_0)\,\widetilde{\theta}\,\overline{\pi}_n = (\overline{\pi}_n \otimes \mathsf{Id}_1)\,{\lambda_1}^{-1} = {\lambda_{\mathbb{D}}}^{-1}$. So the diagram commutes by joint epicity of the $\overline{\pi}_n$'s.

Next we prove that

$$\begin{array}{ccccc}
\mathbb{D} & \xrightarrow{\widetilde{\theta}} & \mathbb{D} \otimes \mathbb{D} & \xrightarrow{\widetilde{\theta}\otimes\mathbb{D}} & (\mathbb{D} \otimes \mathbb{D}) \otimes \mathbb{D} \\
\downarrow{\scriptstyle \widetilde{\theta}} & & & & \downarrow{\scriptstyle \alpha} \\
\mathbb{D} \otimes \mathbb{D} & & \xrightarrow{\mathbb{D}\otimes\widetilde{\theta}} & & \mathbb{D} \otimes (\mathbb{D} \otimes \mathbb{D})
\end{array}$$

so let $n \in \mathbb{N}$, we have

$$\begin{aligned}
\alpha\,(\widetilde{\theta} \otimes \mathbb{D})\,\widetilde{\theta}\,\overline{\pi}_n &= \alpha\,(\widetilde{\theta} \otimes \mathbb{D})\,(\sum_{i=0}^{n}\overline{\pi}_i \otimes \overline{\pi}_{n-i})\,\lambda^{-1} \\
&= \alpha\,(\sum_{i=0}^{n}(\widetilde{\theta}\,\overline{\pi}_i) \otimes \overline{\pi}_{n-i})\,\lambda^{-1} \\
&= \alpha\,(\sum_{i=0}^{n}(\sum_{j=0}^{i}\overline{\pi}_j \otimes \overline{\pi}_{i-j}) \otimes \overline{\pi}_{n-i})\,(\lambda^{-1} \otimes 1)\,\lambda^{-1} \\
&= \alpha\,(\sum_{\substack{i,j,k\in\mathbb{N}\\ i+j+k=n}}(\overline{\pi}_i \otimes \overline{\pi}_j) \otimes \overline{\pi}_k)\,(\lambda^{-1} \otimes 1)\,\lambda^{-1} \quad \text{by Proposition 37} \\
&= (\sum_{\substack{i,j,k\in\mathbb{N}\\ i+j+k=n}}\overline{\pi}_i \otimes (\overline{\pi}_j \otimes \overline{\pi}_k))\,(1 \otimes \lambda^{-1})\,\lambda^{-1} \\
&= (\mathbb{D} \otimes \widetilde{\theta})\,\widetilde{\theta}\,\overline{\pi}_n \quad \text{by a similar computation}
\end{aligned}$$

and the announced diagram commutes by joint epicity of the $\overline{\pi}_n$'s.

One proves similarly that

$$\begin{array}{ccc}
\mathbb{D} & \xrightarrow{\widetilde{\theta}} & \mathbb{D} \otimes \mathbb{D} \\
& \searrow^{\widetilde{\theta}} & \downarrow{\scriptstyle \gamma} \\
& & \mathbb{D} \otimes \mathbb{D}
\end{array}$$

using the fact that $\gamma\,(\sum_{i=0}^{n}\overline{\pi}_i \otimes \overline{\pi}_{n-i})\lambda^{-1} = (\sum_{i=0}^{n}\overline{\pi}_{n-i} \otimes \overline{\pi}_i)\gamma\,\lambda^{-1}$ and that $\gamma\,\lambda^{-1} = \lambda^{-1}$ because $\lambda_1 = \rho_1$. □

Next we equip $\mathbb{D}$ with a commutative monoid structure. First we have $\Delta = \Delta \in \mathcal{L}(1, \mathbb{D})$. Next for each $i \in \mathbb{N}$ we have $\lambda\,(\mathsf{p}_i \otimes \mathsf{p}_i) \in \mathcal{L}(\mathbb{D} \otimes \mathbb{D}, 1)$, and we set $\widetilde{\mathsf{I}} = \langle \lambda\,(\mathsf{p}_i \otimes \mathsf{p}_i)\rangle_{i\in\mathbb{N}} \in \mathcal{L}(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})$. Notice that $\widetilde{\mathsf{I}}\,(\overline{\pi}_i \otimes \overline{\pi}_j) = \delta_{i,j}\overline{\pi}_i\lambda_1 = \delta_{i,j}\overline{\pi}_i\rho_1$ for all $i, j \in \mathbb{N}$, which fully characterizes $\widetilde{\mathsf{I}}$ by Lemma 12.

Lemma 22. *The triple* $(\mathbb{D}, \Delta, \widetilde{\mathsf{I}})$ *is a commutative monoid in* $\mathcal{L}$.

Proof. We prove

$$\begin{array}{ccc} \mathbb{D} \otimes 1 & \xrightarrow{\mathbb{D}\otimes\Delta} & \mathbb{D} \otimes \mathbb{D} \\ & \searrow^{\rho} & \downarrow{\scriptstyle \widetilde{\mathsf{l}}} \\ & & \mathbb{D} \end{array}$$

so let $i \in \mathbb{N}$, we have

$$\begin{aligned} \widetilde{\mathsf{l}}\,(\mathbb{D} \otimes \Delta)\,(\overline{\pi}_i \otimes 1) &= \widetilde{\mathsf{l}}\,(\overline{\pi}_i \otimes \Delta) \\ &= \widetilde{\mathsf{l}}\,(\overline{\pi}_i \otimes (\sum_{j\in\mathbb{N}} \overline{\pi}_j)) \\ &= \widetilde{\mathsf{l}} \sum_{j\in\mathbb{N}} \overline{\pi}_i \otimes \overline{\pi}_j \\ &= \sum_{j\in\mathbb{N}} \delta_{i,j} \overline{\pi}_i \rho_1 \\ &= \rho_{\mathbb{D}}\,(\overline{\pi}_i \otimes 1) \quad \text{by naturality of } \rho \end{aligned}$$

which proves the commutation by Lemma 12.

Next we prove

$$\begin{array}{ccccc} (\mathbb{D} \otimes \mathbb{D}) \otimes \mathbb{D} & \xrightarrow{\alpha} & \mathbb{D} \otimes (\mathbb{D} \otimes \mathbb{D}) & \xrightarrow{\mathbb{D}\otimes\widetilde{\mathsf{l}}} & \mathbb{D} \otimes \mathbb{D} \\ \downarrow{\scriptstyle \widetilde{\mathsf{l}}\otimes\mathbb{D}} & & & & \downarrow{\scriptstyle \widetilde{\mathsf{l}}} \\ \mathbb{D} \otimes \mathbb{D} & & \xrightarrow{\qquad\qquad \widetilde{\mathsf{l}} \qquad\qquad} & & \mathbb{D} \end{array}$$

so let $i, j, k \in \mathbb{N}$, we have

$$\begin{aligned} \widetilde{\mathsf{l}}\,(\mathbb{D} \otimes \widetilde{\mathsf{l}})\,\alpha\,((\overline{\pi}_i \otimes \overline{\pi}_j) \otimes \overline{\pi}_k) &= \widetilde{\mathsf{l}}\,(\mathbb{D} \otimes \widetilde{\mathsf{l}})\,(\overline{\pi}_i \otimes (\overline{\pi}_j \otimes \overline{\pi}_k)) \\ &= \widetilde{\mathsf{l}}\,(\overline{\pi}_i \otimes (\delta_{j,k} \overline{\pi}_j\,\lambda)) \\ &= \delta_{j,k} \widetilde{\mathsf{l}}\,(\overline{\pi}_i \otimes \overline{\pi}_j)\,(1 \otimes \lambda) \\ &= \delta_{j,k} \delta_{i,j} \overline{\pi}_i\,\lambda\,(1 \otimes \lambda) \\ &= \widetilde{\mathsf{l}}\,(\widetilde{\mathsf{l}} \otimes \mathbb{D}\,)\,((\overline{\pi}_i \otimes \overline{\pi}_j) \otimes \overline{\pi}_k) \end{aligned}$$

by a similar computation, so the diagram commutes by Lemma 12. One proves similarly that

$$\begin{array}{ccc} \mathbb{D} \otimes \mathbb{D} & \xrightarrow{\gamma} & \mathbb{D} \otimes \mathbb{D} \\ & \searrow^{\widetilde{\mathsf{l}}} & \downarrow{\scriptstyle \widetilde{\mathsf{l}}} \\ & & \mathbb{D} \end{array}$$

using again the fact that $\lambda_1 = \rho_1$. □

Theorem 10. *The tuple* $(\mathbb{D}, \Delta, \widetilde{\mathsf{l}}, \mathsf{p}_0, \widetilde{\theta})$ *is a bicommutative bimonoid.*

Proof. First we have

$$\begin{array}{ccc} 1 & \xrightarrow{\Delta} & \mathbb{D} \\ & \searrow^{\mathsf{Id}} & \downarrow{\scriptstyle \mathsf{p}_0} \\ & & 1 \end{array}$$

since $\mathsf{p}_0\,\Delta = \mathsf{p}_0\,\Delta = \mathsf{Id}_1$.

Next we prove that

$$\begin{array}{ccccc} \mathbb{D}\otimes\mathbb{D} & \xrightarrow{\widetilde{\mathsf{I}}} & \mathbb{D} & \xrightarrow{\widetilde{\theta}} & \mathbb{D}\otimes\mathbb{D} \\ {\scriptstyle \widetilde{\theta}\otimes\widetilde{\theta}}\downarrow & & & & \uparrow{\scriptstyle \widetilde{\mathsf{I}}\otimes\widetilde{\mathsf{I}}} \\ (\mathbb{D}\otimes\mathbb{D})\otimes(\mathbb{D}\otimes\mathbb{D}) & & \xrightarrow{\gamma_{2,3}} & & (\mathbb{D}\otimes\mathbb{D})\otimes(\mathbb{D}\otimes\mathbb{D}) \end{array}$$

where $\gamma_{2,3}$ is defined using the canonical isomorphisms α and γ of the SMC structure of $\mathcal{L}$, and is characterized by $\gamma_{2,3}\,((\overline{\pi}_{i_1}\otimes\overline{\pi}_{i_2})\otimes(\overline{\pi}_{i_3}\otimes\overline{\pi}_{i_4})) = (\overline{\pi}_{i_1}\otimes\overline{\pi}_{i_3})\otimes(\overline{\pi}_{i_2}\otimes\overline{\pi}_{i_4})$. We have

$$\widetilde{\theta}\,\widetilde{\mathsf{I}}\,(\overline{\pi}_i\otimes\overline{\pi}_j) = \delta_{i,j}\widetilde{\theta}\,\overline{\pi}_i\,\lambda = \delta_{i,j}(\sum_{k=0}^{i}\overline{\pi}_k\otimes\overline{\pi}_{i-k})\,\lambda$$

and

$$\begin{aligned} &(\widetilde{\mathsf{I}}\otimes\widetilde{\mathsf{I}})\,\gamma_{2,3}\,(\widetilde{\theta}\otimes\widetilde{\theta})\,(\overline{\pi}_i\otimes\overline{\pi}_j) \\ &\quad= (\widetilde{\mathsf{I}}\otimes\widetilde{\mathsf{I}})\,\gamma_{2,3}\,((\sum_{i_1+i_2=i}\overline{\pi}_{i_1}\otimes\overline{\pi}_{i_2})\otimes(\sum_{j_1+j_2=j}\overline{\pi}_{j_1}\otimes\overline{\pi}_{j_2}))\,(\lambda^{-1}\otimes\lambda^{-1}) \\ &\quad= (\widetilde{\mathsf{I}}\otimes\widetilde{\mathsf{I}})\,\gamma_{2,3}\,(\sum_{\substack{i_1+i_2=i\\ j_1+j_2=j}}(\overline{\pi}_{i_1}\otimes\overline{\pi}_{i_2})\otimes(\overline{\pi}_{j_1}\otimes\overline{\pi}_{j_2}))\,(\lambda^{-1}\otimes\lambda^{-1}) \\ &\quad= (\widetilde{\mathsf{I}}\otimes\widetilde{\mathsf{I}})\,(\sum_{\substack{i_1+i_2=i\\ j_1+j_2=j}}(\overline{\pi}_{i_1}\otimes\overline{\pi}_{j_1})\otimes(\overline{\pi}_{i_2}\otimes\overline{\pi}_{j_2}))\,\gamma_{2,3}\,(\lambda^{-1}\otimes\lambda^{-1}) \\ &\quad= (\sum_{\substack{i_1+i_2=i\\ j_1+j_2=j}}\delta_{i_1,j_1}\delta_{i_2,j_2}\overline{\pi}_{i_1}\otimes\overline{\pi}_{i_2})\,(\lambda\otimes\lambda)\,\gamma_{2,3}\,(\lambda^{-1}\otimes\lambda^{-1}) \end{aligned}$$

which ends the proof that $\widetilde{\theta}\,\widetilde{\mathsf{I}}\,(\overline{\pi}_i\otimes\overline{\pi}_j) = (\widetilde{\mathsf{I}}\otimes\widetilde{\mathsf{I}})\,\gamma_{2,3}\,(\widetilde{\theta}\otimes\widetilde{\theta})\,(\overline{\pi}_i\otimes\overline{\pi}_j)$ for all $i,j\in\mathbb{N}$ upon observing that if $\delta_{i_1,j_1}\delta_{i_2,j_2}=1$ then $i=i_1+i_2=j_1+j_2=j$ and $(\lambda_1\otimes\lambda_1)\,\gamma_{2,3}\,(\lambda_1^{-1}\otimes\lambda_1^{-1}) = \mathsf{Id}_{(1\otimes1)\otimes(1\otimes1)}$. The diagram commutes by Lemma 12. □

4.3 Correspondence between the bimonad S and the bimonoid structure

We can define a functor $\mathsf{S}_\otimes : \mathcal{L}\to\mathcal{L}$ by $\mathsf{S}_\otimes X = X\otimes\mathbb{D}$ and similarly on morphisms: $\mathsf{S}_\otimes f = f\otimes\mathbb{D}\in\mathcal{L}(\mathsf{S}_\otimes X,\mathsf{S}_\otimes Y)$ if $f\in\mathcal{L}(X,Y)$. The bimonoid structure of $\mathbb{D}$ induces straightforwardly a bimonad structure on this functor. The comonad structure $\overline{\mathsf{S}_\otimes}$ is given by

$$\mathsf{S}_\otimes X = X\otimes\mathbb{D}\xrightarrow{X\otimes\mathsf{p}_0} X\otimes 1\xrightarrow{\rho} X$$

$$\mathsf{S}_\otimes X = X\otimes\mathbb{D}\xrightarrow{X\otimes\widetilde{\theta}} X\otimes(\mathbb{D}\otimes\mathbb{D})\xrightarrow{\alpha}(X\otimes\mathbb{D})\otimes\mathbb{D} = \mathsf{S}_\otimes^2 X$$

The monad structure $\underline{\mathsf{S}_\otimes}$ is given by

$$X\xrightarrow{\rho^{-1}} X\otimes 1\xrightarrow{X\otimes\Delta} X\otimes\mathbb{D} = \mathsf{S}_\otimes X$$

$$\mathsf{S}_\otimes^2 X = (X\otimes\mathbb{D})\otimes\mathbb{D}\xrightarrow{\alpha} X\otimes(\mathbb{D}\otimes\mathbb{D})\xrightarrow{X\otimes\widetilde{\mathsf{I}}} X\otimes\mathbb{D}$$

and the distributive law is (keeping the associativity isomorphisms implicit)

Bimonoid $\mathbb{D}$		Bimonad $\mathsf{S}_\otimes$		Bimonad S
Monoid unit Δ		Monad unit $(X \otimes \Delta)\,\rho^{-1}$		Comonad unit σ
Monoid multiplication $\widetilde{\mathsf{I}}$		Monad sum $(X \otimes \widetilde{\mathsf{I}})\,\alpha$		Comonad sum I
Comonoid unit p_0	$\Longleftrightarrow$	Comonad unit $\rho\,(X \otimes \mathsf{p}_0)$	$\overset{\text{mates}}{\Longleftrightarrow}$	Monad unit ι_0
Comonoid multiplication $\widetilde{\theta}$		Comonad sum $\alpha\,(X \otimes \widetilde{\theta})$		Monad sum θ
Monoid commutativity γ		Distributive law $X \otimes \gamma$		Distributive law c
Projection $\mathsf{p}_i \in \mathcal{L}(\mathbb{D}, 1)$		$\rho\,(X \otimes \mathsf{p}_i) \in \mathcal{L}(X \otimes \mathbb{D}, X)$		Injection ι_i
Injection $\overline{\pi}_i \in \mathcal{L}(1, \mathbb{D})$		$(X \otimes \overline{\pi}_i)\,\rho^{-1} \in \mathcal{L}(X, X \otimes \mathbb{D})$		Projection π_i

Figure 1: Bimonoid and bimonad relations

$$\mathsf{S}^2_\otimes X = X \otimes \mathbb{D} \otimes \mathbb{D} \xrightarrow{X \otimes \gamma} X \otimes \mathbb{D} \otimes \mathbb{D}$$

Now the functor $\mathsf{S} = (\mathbb{D} \multimap \_) : \mathcal{L} \to \mathcal{L}$ is the right adjoint of $\mathsf{S}_\otimes$ and hence, as shown in Section 8.2, S inherits from the mate construction a bimonad structure which is exactly the same as the one described in Section 1.4. The different constructions are summarized in Section 4.3. Let us write down the details.

By Remark 36 instantiated in $L \dashv R = \_ \otimes \mathbb{D} \dashv \mathbb{D} \multimap \_$ and $L' \dashv R' = \mathsf{Id} \dashv \mathsf{Id}$, the mate construction maps a natural transformation $\lambda_X \in \mathcal{L}(X, X \otimes \mathbb{D})$ to a natural transformation $\mu_X \in \mathcal{L}(\mathbb{D} \multimap X, X)$ defined as $\mu = \mathsf{ev}\ \lambda$. It follows from the explicit formula of Remark 20 that $\pi_i \in \mathcal{L}(\mathbb{D} \multimap X, X)$ is the mate of $(X \otimes \overline{\pi}_i)\,\rho^{-1}$ and that $\sigma \in \mathcal{L}(\mathbb{D} \multimap X, X)$ is the mate of $(X \otimes \Delta)\,\rho^{-1}$

By Remark 36 instantiated in $L \dashv R = \mathsf{Id} \dashv \mathsf{Id}$ and $L' \dashv R' = \_ \otimes \mathbb{D} \dashv \mathbb{D} \multimap \_$, the mate construction maps a natural transformation $\lambda_X \in \mathcal{L}(X \otimes \mathbb{D}, X)$ to a natural transformation $\mu_X \in \mathcal{L}(X, \mathbb{D} \multimap X)$ defined as $\mu = \mathsf{cur}(\lambda)$. By a standard computation involving Eq. (13), or by using the fact that π_i is the mate of $(X \otimes \overline{\pi}_i)\,\rho^{-1}$ and the fact that the mate construction is compositional (see Section 8.1), we can show that

$$\pi_i\,\mu = \lambda\,(X \otimes \overline{\pi}_i)\,\rho^{-1}$$

Thus, by joint monicity of the π_i, the mate of $\rho\,(X \otimes \mathsf{p}_i) \in \mathcal{L}(X \otimes \mathbb{D}, X)$ is necessarily $\iota_i \in \mathcal{L}(X, \mathbb{D} \multimap X)$.

By Remark 36 instantiated in $L \dashv R = \_ \otimes \mathbb{D} \dashv \mathbb{D} \multimap \_$ and $L' \dashv R' = (\_ \otimes \mathbb{D}) \otimes \mathbb{D} \dashv \mathbb{D} \multimap \mathbb{D} \multimap \_$, the mate construction maps a natural transformation $\lambda_X \in \mathcal{L}((X \otimes \mathbb{D}) \otimes \mathbb{D}, X \otimes \mathbb{D})$ to a natural transformation $\mu_X \in \mathcal{L}(\mathbb{D} \multimap X, \mathbb{D} \multimap \mathbb{D} \multimap X)$ defined as $\mu = \mathsf{cur}(\mathsf{cur}(\mathsf{ev}\ \lambda_{\mathbb{D} \multimap X}))$. By a standard computation involving Eq. (13), or by using the fact that π_i is the mate of $(X \otimes \overline{\pi}_i)\,\rho^{-1}$ and the fact that the mate construction is compositional (see Section 8.1),

$$\pi_i\,\pi_j\,\mu = \mathsf{ev}\,\lambda\,((\mathbb{D} \multimap X) \otimes \overline{\pi}_i \otimes \overline{\pi}_j)\,\rho^{-1}\,\rho^{-1}$$

But ev is an iso since $\mathsf{cur}(\mathsf{ev}) = \mathsf{id}$. So $\pi_i\,\pi_j\,\mu = \pi_k = \mathsf{ev}\,((\mathbb{D} \multimap X) \otimes \overline{\pi}_k)\,\rho^{-1}$ if and only if $\lambda\,((\mathbb{D} \multimap X) \otimes \overline{\pi}_i \otimes \overline{\pi}_j) = (\mathbb{D} \multimap X) \otimes \overline{\pi}_k$ and $\pi_i\,\pi_j\,\mu = 0$ if and only if $\lambda\,((\mathbb{D} \multimap X) \otimes \overline{\pi}_i \otimes \overline{\pi}_j)\,\rho^{-1} = 0$. Thus, by joint monicity of the $\pi_i\,\pi_j$, the mate of $(X \otimes \widetilde{\mathsf{I}})\,\alpha$ is necessarily I.

By Remark 36 instantiated in $L \dashv R = (\_ \otimes \mathbb{D}) \otimes \mathbb{D} \dashv \mathbb{D} \multimap \mathbb{D} \multimap \_$, and $L' \dashv R' = \_ \otimes \mathbb{D} \dashv \mathbb{D} \multimap \_$, the mate construction maps a natural transformation $\lambda_X \in \mathcal{L}(X \otimes \mathbb{D}, (X \otimes \mathbb{D}) \otimes \mathbb{D})$ to a natural transformation $\mu_X \in \mathcal{L}(\mathbb{D} \multimap \mathbb{D} \multimap X, \mathbb{D} \multimap X)$ defined as $\mu = \mathsf{cur}(\mathsf{ev}\,(\mathsf{ev} \otimes \mathbb{D})\,\lambda)$.

By a similar computation as the ones above, one can show that

$$\pi_i\,\mu = \mathsf{ev}\,(\mathsf{ev}\otimes\mathbb{D})\,\lambda\,((\mathbb{D}\multimap\mathbb{D}\multimap X)\otimes\overline{\pi}_i)\,\rho^{-1}$$

In particular, $\pi_i\,\mu = \sum_{k=0}^{i}\pi_k\,\pi_{i-k}$ if and only if $\lambda\,((\mathbb{D}\multimap\mathbb{D}\multimap X)\otimes\overline{\pi}_i) = (\mathbb{D}\multimap\mathbb{D}\multimap X)\otimes\sum_{k=0}^{i}(\overline{\pi}_k\otimes\overline{\pi}_{i-k})$, thanks to the computations below.

$$\sum_{k=0}^{i}\pi_k\pi_{i-k} = \sum_{k=0}^{i}\left(\mathsf{ev}\,(\mathsf{ev}\otimes\mathbb{D})\,((\mathbb{D}\multimap\mathbb{D}\multimap X)\otimes\overline{\pi}_k\otimes\overline{\pi}_{i-k})\,\rho^{-1}\rho^{-1}\right)\quad\text{SMC computation}$$

$$= \mathsf{ev}\,(\mathsf{ev}\otimes\mathbb{D})\left((\mathbb{D}\multimap\mathbb{D}\multimap X)\otimes\sum_{k=0}^{i}(\overline{\pi}_k\otimes\overline{\pi}_{i-k})\right)\rho^{-1}\rho^{-1}$$

(left) additivity and (S⊗-dist)

It means that the mate of $\alpha\,(X\otimes\widetilde{\theta})$ is θ. [8]

Finally, by Remark 36 instantiated in $L\dashv R = (\_\otimes\mathbb{D})\otimes\mathbb{D}\dashv\mathbb{D}\multimap\mathbb{D}\multimap\_$, and $L'\dashv R' = (\_\otimes\mathbb{D})\otimes\mathbb{D}\dashv\mathbb{D}\multimap\mathbb{D}\multimap\_$, the mate construction maps a natural transformation $\lambda_X\in\mathcal{L}((X\otimes\mathbb{D})\otimes\mathbb{D},(X\otimes\mathbb{D})\otimes\mathbb{D})$ to a natural transformation $\mu_X\in\mathcal{L}(\mathbb{D}\multimap\mathbb{D}\multimap X,\mathbb{D}\multimap\mathbb{D}\multimap X)$ defined as $\mu = \mathsf{cur}(\mathsf{cur}(\mathsf{ev}\,(\mathsf{ev}\otimes\mathbb{D})\,\lambda))$. We can show with similar argument as above that $\pi_i\,\pi_i\,\mu = \pi_j\,\pi_i$ if and only if $\lambda\,((\mathbb{D}\multimap\mathbb{D}\multimap X)\otimes\overline{\pi}_i\otimes\overline{\pi}_j) = (\mathbb{D}\multimap\mathbb{D}\multimap X)\otimes\overline{\pi}_j\otimes\overline{\pi}_i$, so the mate of $X\otimes\gamma$ must be c.

Definition 41. We define a natural transformation (up to associativity)

$$\overline{\mathsf{L}}_{X_0,X_1} = \quad X_0\otimes X_1\otimes\mathbb{D}\xrightarrow{X_0\otimes X_1\otimes\widetilde{\theta}}X_0\otimes X_1\otimes\mathbb{D}\otimes\mathbb{D}\xrightarrow{X_0\otimes\gamma\otimes\mathbb{D}}X_0\otimes\mathbb{D}\otimes X_1\otimes\mathbb{D}$$

That is, $\overline{\mathsf{L}}_{X_0,X_1}\in\mathcal{L}(\mathsf{S}_{\otimes}(X_0\otimes X_1),\mathsf{S}_{\otimes}X_0\otimes\mathsf{S}_{\otimes}X_1)$

The natural transformation $\overline{\mathsf{L}}$ is characterized by the equation $\overline{\mathsf{L}}_{X_0,X_1}\,(X_0\otimes X_1\otimes\overline{\pi}_n) = \sum_{k=0}^{n}(X_0\otimes\overline{\pi}_k\otimes X_1\otimes\overline{\pi}_{n-k})$. Applying the mate construction of Section 8 with $H = K = \_\otimes\_$ and taking $L'\dashv R' = \_\otimes\mathbb{D}\dashv\mathbb{D}\multimap\mathbb{D}\multimap\_$ and $L\dashv R = (\_\otimes\mathbb{D})\times(\_\otimes\mathbb{D})\dashv(\mathbb{D}\multimap\_)\times(\mathbb{D}\multimap\_)$ (this is the product of the adjunction $\_\otimes\mathbb{D}\dashv\mathbb{D}\multimap\_$ with itself, see Section 8.4) yield a natural transformation

$$\alpha = \mathsf{cur}((\mathsf{ev}\otimes\mathsf{ev})\,\overline{\mathsf{L}}_{\mathbb{D}\multimap X_0,\mathbb{D}\multimap X_1})\in\mathcal{L}((\mathbb{D}\multimap X_0)\otimes(\mathbb{D}\multimap X_1),\mathbb{D}\multimap(X_0\otimes X_1))$$

A computation similar to the other computations of this section show that $\pi_i\,\alpha = \sum_{k=0}^{i}\pi_i\otimes\pi_{i-k}$ so $\alpha = \mathsf{L}$ (where L is defined in Section 1.5.1). Recall that $(\underline{\mathsf{S}},\iota_0,\mathsf{L})$ is a lax monoidal monad. By Theorem 32, $(\overline{\mathsf{S}_{\otimes}},\rho\,(\mathsf{id}\otimes\mathsf{p}_0),\overline{\mathsf{L}})$ is then an oplax monoidal comonad. This oplax structure is the one associated to the strengths (up to associativity)

$$X_0\otimes\gamma_{X_1,\mathbb{D}}\in\mathcal{L}(\mathsf{S}_{\otimes}(X_0\otimes X_1),\mathsf{S}_{\otimes}X_0\otimes X_1)$$
$$\mathsf{id}\in\mathcal{L}(\mathsf{S}_{\otimes}(X_0\otimes X_1),X_0\otimes\mathsf{S}_{\otimes}X_1)$$

Those strengths were implicitly used when defining $\overline{\mathsf{L}}$ in the first place. The mates of those two natural transformations are φ^0 and φ^1 respectively. Observe in particular that $X_0\otimes\gamma$ is a natural isomorphism, so φ^0 is an isomorphism (the mate of an isomorphism is an isomorphism).

[8] In fact, the existence of $\widetilde{\theta}$ was proved implicitly by taking the left mate of θ, so this is a somewhat redundant argument

Proposition 38. *If $\mathcal{L}$ is monoidal closed, then (S$\otimes$-fun) holds.*

Proof. As seen above, φ^0 is an iso, so its mate $\varphi^{-\circ} = \mathsf{cur}((\mathsf{Sev})\,\varphi^0_{A\multimap X,A})$ through the adjunction $\_ \otimes A \dashv A \multimap \_$ is also an iso. □

Assume that $\mathcal{L}$ is cartesian. There is a natural transformation

$$\langle \mathsf{p}_0 \otimes \mathbb{D}, \mathsf{p}_1 \otimes \mathbb{D}\rangle \in \mathcal{L}((X_0 \,\&\, X_1) \otimes \mathbb{D}, (X_0 \otimes \mathbb{D}) \,\&\, (X_1 \otimes \mathbb{D}))$$

Applying the mate construction of Section 8 with $H = K = \_ \,\&\, \_$ and the same two adjunctions as above yield a natural transformation

$$\alpha = \mathsf{cur}((\mathsf{ev} \,\&\, \mathsf{ev})\,\langle \mathsf{p}_0 \otimes \mathbb{D}, \mathsf{p}_1 \otimes \mathbb{D}\rangle) \in \mathcal{L}((\mathbb{D} \multimap X_0) \,\&\, (\mathbb{D} \multimap X_1), \mathbb{D} \multimap (X_0 \,\&\, X_1))$$

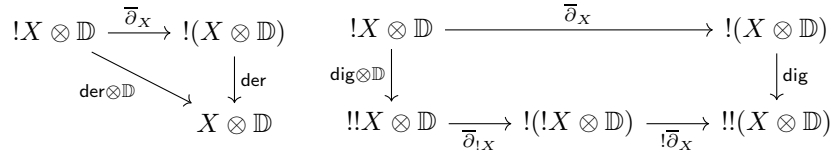
Again, a computation involving the SMC structure show that $\pi_i\,\alpha = \pi_i \,\&\, \pi_i$ so $\alpha = \mathsf{c}^{-1}_{\&}$ by Proposition 27. So Proposition 39 below holds.

Proposition 39. *If $\mathcal{L}$ is cartesian, then the elementary ω-summability structure follows (S-&).*

4.4 A mate to the distributive law ∂

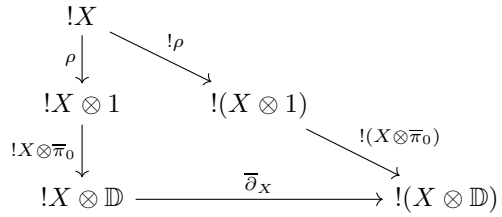
Let $\mathcal{L}$ be an elementary ω-summable category. We assume moreover that $\mathcal{L}$ is cartesian and is equipped with a *resource comonad*, see Definition 18, so that it is a summable resource category. By applying the results of Section 8.3, we can show that the mate construction induce a bijection between the distributive laws of Section 2 $\partial : !\mathsf{S} \Rightarrow \mathsf{S}!$ and a distributive law $\overline{\partial} : \mathsf{S}_{\otimes}! \Rightarrow !\mathsf{S}_{\otimes}$. It means that in the elementary case, Taylor expansion can be directly expressed as the existence of such $\overline{\partial} : !\_ \otimes \mathbb{D} \Rightarrow !(\_ \otimes \mathbb{D})$ with the following properties.

($\overline{\partial}$-chain)



The commutation ($\overline{\partial}$-chain) mean that $\overline{\partial}$ is a distributive law between the functor $\mathsf{S}_{\otimes}$ and the comonad $!\_$. By Corollary 6, ($\overline{\partial}$-chain) holds if and only if ∂ is a distributive law between the functor S and the comonad $!\_$, that is, if (∂-chain) holds.

($\overline{\partial}$-local)



The commutation ($\overline{\partial}$-local) mean that the natural transformation $(X \otimes \overline{\pi}_0)\,\rho^{-1} \in \mathcal{L}(X, X \otimes \mathbb{D})$ is a morphism of distributive law. By Corollary 8, it is one if and only if π_0 (its mate) is a morphism of distributive law, that is, if (∂-local) commutes.

$$
\begin{array}{ccc}
!X \otimes \mathbb{D} & \xrightarrow{\overline{\partial}_X} & !(X \otimes \mathbb{D}) \\
{\scriptstyle !X\otimes \mathsf{p}_0}\downarrow & & \downarrow{\scriptstyle !(X\otimes \mathsf{p}_0)} \\
!X \otimes 1 & & !(X \otimes 1) \\
 & {\scriptstyle \rho}\searrow & \downarrow{\scriptstyle !\rho} \\
 & & !X
\end{array}
\qquad
\begin{array}{ccccc}
!X \otimes \mathbb{D} & & \xrightarrow{\qquad\overline{\partial}_X\qquad} & & !(X \otimes \mathbb{D}) \\
{\scriptstyle !X\otimes \widetilde{\theta}}\downarrow & & & & \downarrow{\scriptstyle !(X\otimes \widetilde{\theta})} \\
!X \otimes (\mathbb{D} \otimes \mathbb{D}) & & & & !(X \otimes (\mathbb{D} \otimes \mathbb{D})) \\
{\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle !\alpha} \\
(!X \otimes \mathbb{D}) \otimes \mathbb{D} & \xrightarrow[\overline{\partial}_X\otimes\mathbb{D}]{} & !(X \otimes \mathbb{D}) \otimes \mathbb{D} & \xrightarrow[\overline{\partial}_{X\otimes\mathbb{D}}]{} & !((X \otimes \mathbb{D}) \otimes \mathbb{D})
\end{array}
$$

(∂̄-add)

The commutation (∂̄-add) mean that $\overline{\partial}$ is a distributive law between the functor ! and the comonad structure on $\mathsf{S}_\otimes$ described in Section 4.3. By Corollary 9, (∂̄-add) holds if and only if ∂ is a distributive law between the functor $!\_$ and the comonad $\overline{\mathsf{S}}$, that is if (∂-add) holds.

In the next diagram, the use of α is kept implicit.

$$
\begin{array}{ccccc}
!X \otimes \mathbb{D} \otimes \mathbb{D} & \xrightarrow{\overline{\partial}_X\otimes\mathbb{D}} & !(X \otimes \mathbb{D}) \otimes \mathbb{D} & \xrightarrow{\overline{\partial}_{X\otimes\mathbb{D}}} & !(X \otimes \mathbb{D} \otimes \mathbb{D}) \\
{\scriptstyle !X\otimes\gamma}\downarrow & & & & \downarrow{\scriptstyle !(X\otimes\alpha)} \\
!X \otimes \mathbb{D} \otimes \mathbb{D} & \xrightarrow[\overline{\partial}_X\otimes\mathbb{D}]{} & !(X \otimes \mathbb{D}) \otimes \mathbb{D} & \xrightarrow[\overline{\partial}_{X\otimes\mathbb{D}}]{} & !(X \otimes \mathbb{D} \otimes \mathbb{D})
\end{array}
$$

(∂̄-Schwarz)

The commutation (∂̄-Schwarz) mean that $X \otimes \gamma$ is a morphism of distributive law. By Corollary 8, it is one if and only if c (its mate) is a morphism of distributive law, that is, if (∂-Schwarz) hold.

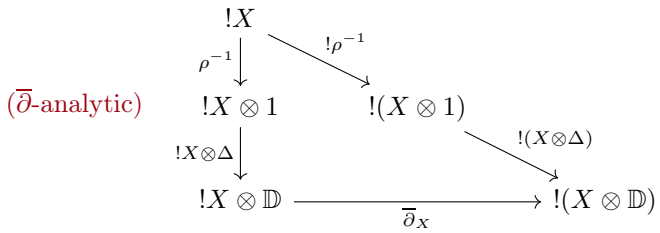
$$
\begin{array}{ccccc}
!X_0 \otimes !X_1 \otimes \mathbb{D} & \xrightarrow{\overline{\mathsf{L}}} & !X_0 \otimes \mathbb{D} \otimes !X_1 \otimes \mathbb{D} & \xrightarrow{\overline{\partial}\otimes\overline{\partial}} & !(X_0 \otimes \mathbb{D}) \otimes !(X_1 \otimes \mathbb{D}) \\
{\scriptstyle \mathsf{m}^2\otimes\mathbb{D}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{m}^2} \\
!(X_1 \,\&\, X_2) \otimes \mathbb{D} & \xrightarrow[\overline{\partial}]{} & !((X_0 \,\&\, X_1) \otimes \mathbb{D}) & \xrightarrow[!\langle \mathsf{p}_0\otimes\mathbb{D}, \mathsf{p}_1\otimes\mathbb{D}\rangle]{} & !((X_0 \otimes \mathbb{D}) \,\&\, (X_1 \otimes \mathbb{D}))
\end{array}
$$

(∂̄-&)

The commutation (∂̄-&) mean that m^2 is a morphism of distributive laws between the composition of $\overline{\partial}$ with $\overline{\mathsf{L}}$, and the composition of $\overline{\partial}$ with $\langle \mathsf{p}_0 \otimes \mathbb{D}, \mathsf{p}_1 \otimes \mathbb{D}\rangle$. But as we saw in Section 4.3, the mate of $\overline{\mathsf{L}}$ is L and the mate of $\langle \mathsf{p}_0 \otimes \mathbb{D}, \mathsf{p}_1 \otimes \mathbb{D}\rangle$ is $\mathsf{c}_\&^{-1}$. By compositionality of the mate construction and Corollary 5, (∂̄-&) holds if and only if (∂-&) holds.

$$
\begin{array}{ccccc}
(!X \otimes \mathbb{D}) \otimes \mathbb{D} & \xrightarrow{\overline{\partial}_X\otimes\mathbb{D}} & !(X \otimes \mathbb{D}) \otimes \mathbb{D} & \xrightarrow{\overline{\partial}_{X\otimes\mathbb{D}}} & !((X \otimes \mathbb{D}) \otimes \mathbb{D}) \\
{\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle !\alpha} \\
!X \otimes (\mathbb{D} \otimes \mathbb{D}) & & & & !(X \otimes (\mathbb{D} \otimes \mathbb{D})) \\
{\scriptstyle !X\otimes\widetilde{\mathsf{l}}}\downarrow & & & & \downarrow{\scriptstyle !(X\otimes\widetilde{\mathsf{l}})} \\
!X \otimes \mathbb{D} & & \xrightarrow[\overline{\partial}_X]{} & & !(X \otimes \mathbb{D})
\end{array}
$$

(∂̄-lin)



Together, the commutations ($\overline{\partial}$-lin) and ($\overline{\partial}$-analytic) mean that $\overline{\partial}$ is a distributive law between the functor !_ and the monad structure on $\mathsf{S}_\otimes$ described in Section 4.3. By Corollary 10, they hold if and only if ∂ is a distributive law between the functor !_ and the comonad $\overline{\mathsf{S}}$, that is, if and only if (∂-lin) and (∂-analytic) hold. The following result summarizes what we have proved in this section.

Theorem 11. *Let $\partial : !\mathsf{S} \Rightarrow \mathsf{S}!$ and $\overline{\partial} : \mathsf{S}_\otimes ! \Rightarrow !\mathsf{S}_\otimes$ be mates. Then ∂ follow (∂-chain), (∂-local), (∂-add), (∂-Schwarz), (∂-&), (∂-lin) and (∂-analytic) if and only if $\overline{\partial}$ follows ($\overline{\partial}$-chain), ($\overline{\partial}$-local), ($\overline{\partial}$-add), ($\overline{\partial}$-Schwarz), ($\overline{\partial}$-&), ($\overline{\partial}$-lin) and ($\overline{\partial}$-analytic).*

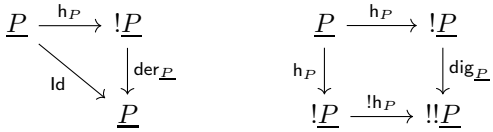
4.5 The Taylor coalgebra structure of $\mathbb{D}$

An important structure can be derived from the cartesian structure of $\mathcal{L}$ and its resource comonad: a lax symmetric monoidality of the functor !, from the SMC $(\mathcal{L}, \otimes, 1)$ to itself. More precisely we have a morphism $\mu^0 \in \mathcal{L}(1, !1)$ and a natural transformation $\mu^2_{X,Y} \in \mathcal{L}(!X \otimes !Y, !(X \otimes Y))$ which satisfy some coherence diagrams, and can be defined as the following compositions of morphisms

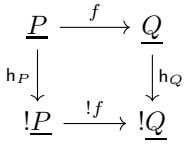
$$1 \xrightarrow{\mathsf{m}^0} !\top \xrightarrow{\mathsf{dig}_\top} !!\top \xrightarrow{!(\mathsf{m}^0)^{-1}} !1$$

$$!X \otimes !Y \xrightarrow{\mathsf{m}^2_{X,Y}} !(X \,\&\, Y) \xrightarrow{\mathsf{dig}_{X\&Y}} !!(X \,\&\, Y) \xrightarrow{!((\mathsf{m}^2_{X,Y})^{-1})} !(!X \otimes !Y) \xrightarrow{!(\mathsf{dig}_X \otimes \mathsf{dig}_Y)} !(X \otimes Y)$$

Particularly important is the associated Eilenberg-Moore category $\mathcal{L}^!$ whose objects are the coalgebras of !, that is, the pairs $P = (\underline{P}, \mathsf{h}_P)$ where $\underline{P}$ is an object of P and $\mathsf{h}_P \in \mathcal{L}(\underline{P}, !\underline{P})$ makes the two following diagrams commute



In this category, an element of $\mathcal{L}^!(P, Q)$ is an $f \in \mathcal{L}(\underline{P}, \underline{Q})$ such that



It is easy to check that $(1, \mu^0)$ is an object of $\mathcal{L}^!$ that we simply denote as 1 (so that $\mathsf{h}_1 = \mu^0$) and that, given objects P_1 and P_2 of $\mathcal{L}^!$, the object $\underline{P_1} \otimes \underline{P_2}$ can be equipped with a !-coalgebra structure defined as

$$\underline{P_1} \otimes \underline{P_2} \xrightarrow{\mathsf{h}_{P_1} \otimes \mathsf{h}_{P_2}} !\underline{P_1} \otimes !\underline{P_2} \xrightarrow{\mu^2} !(\underline{P_1} \otimes \underline{P_2})$$

We use $P_1 \otimes P_2$ to denote this coalgebra, so that $\mathsf{h}_{P_1 \otimes P_2} = \mu^2 (\mathsf{h}_{P_1} \otimes \mathsf{h}_{P_2})$.

Each object P of $\mathcal{L}^!$ can be equipped with a weakening morphism $\mathsf{w}_P \in \mathcal{L}^!(P, 1)$ and a contraction morphism $\mathsf{c}_P \in \mathcal{L}^!(P, P \otimes P)$ which can be defined as follows

$$\underline{P} \xrightarrow{\mathsf{h}_P} !\underline{P} \xrightarrow{!\mathsf{t}} !\top \xrightarrow{(\mathsf{m}^0)^{-1}} 1$$

$$\underline{P} \xrightarrow{\mathsf{h}_P} !\underline{P} \xrightarrow{!\langle \mathsf{Id}_{\underline{P}}, \mathsf{Id}_{\underline{P}} \rangle} !(\underline{P} \,\&\, \underline{P}) \xrightarrow{(\mathsf{m}^2)^{-1}} !\underline{P} \otimes !\underline{P} \xrightarrow{\mathsf{der} \otimes \mathsf{der}} \underline{P} \otimes \underline{P}$$

Theorem 12. *For any object P of $\mathcal{L}^!$, the triple $(P, \mathsf{w}_P, \mathsf{c}_P)$ is a commutative comonoid in $\mathcal{L}^!$.*

Given $(f_i \in \mathcal{L}^!(P_i, Q_i))_{i=1,2}$, it is easy to check that $f_1 \otimes f_2 \in \mathcal{L}^!(P_1 \otimes P_2, Q_1 \otimes Q_2)$. And therefore if $(f_i \in \mathcal{L}^!(P, Q_i))_{i=1,2}$, one can define $\langle f_1, f_2 \rangle^{(!)} = (f_1 \otimes f_2)\, \mathsf{c}_Q \in \mathcal{L}^!(Q, P_1 \otimes P_2)$. We can also define projections $\mathsf{p}_i^{(!)} \in \mathcal{L}^!(P_1 \otimes P_2, P_i)$, for instance the first projection is simply

$$\underline{P_1} \otimes \underline{P_2} \xrightarrow{\underline{P_1} \otimes \mathsf{w}_{P_2}} \underline{P_1} \otimes 1 \xrightarrow{\rho} \underline{P_1}$$

Theorem 13. *The category $\mathcal{L}^!$ is cartesian, with 1 as terminal object and $(P_1, \otimes P_2, \mathsf{p}_1^{(!)}, \mathsf{p}_2^{(!)})$ as cartesian product of P_1 and P_2. Given $(f_i \in \mathcal{L}^!(P, Q_i))_{i=1,2}$, $\langle f_1, f_2 \rangle^{(!)} = (f_1 \otimes f_2)\, \mathsf{c}_Q \in \mathcal{L}^!(Q, P_1 \otimes P_2)$ is the unique morphism such that $\mathsf{p}_i^{(!)} \langle f_1, f_2 \rangle^{(!)} = f_i$ for $i = 1, 2$.*

This is a non-trivial result, for which we refer to [Mel09].

For any object X of $\mathcal{L}$, the pair $\mathsf{E}X = (!X, \mathsf{dig}_X)$ is an object of $\mathcal{L}^!$. This defines a functor $\mathsf{E} : \mathcal{L} \to \mathcal{L}^!$ which maps $f \in \mathcal{L}(X, Y)$ to $!f \in \mathcal{L}^!(\mathsf{E}X, \mathsf{E}Y)$ as easily checked. The coalgebra $\mathsf{E}X$ is the cofree coalgebra generated by X in the sense that, for any object P of $\mathcal{L}^!$ and any $f \in \mathcal{L}(\underline{P}, X)$, there is exactly one morphism $f^! \in \mathcal{L}^!(P, \mathsf{E}X)$ such that $\mathsf{der}_X\, f^! = f$.

The "image" of this functor is a full subcategory which can be described, up to equivalence, as the Kleisli category of the ! comonad.

Definition 42. We write $\mathsf{wk}_X \in \mathcal{L}(!X, 1)$ and $\mathsf{ctr}_X \in \mathcal{L}(!X, !X \otimes !X)$ for the weakening and contraction associated to this free coalgebra $(!X, \mathsf{dig}_X)$. In other words $\mathsf{wk}_X = \mathsf{w}_{\mathsf{E}X}$ and $\mathsf{ctr}_X = \mathsf{c}_{\mathsf{E}X}$.

Definition 43. An analytic coalgebra on $\mathcal{L}$ is a morphism $\widetilde{\partial} \in \mathcal{L}(\mathbb{D}, !\mathbb{D})$ such that $(\mathbb{D}, \widetilde{\partial})$ is an object of $\mathcal{L}^!$, the four structure maps of the bimonoid $(\mathbb{D}, \Delta, \widetilde{\mathsf{l}}, \mathsf{p}_0, \widetilde{\theta})$ are morphisms in $\mathcal{L}^!$, and such that $\overline{\pi}_0$ is a morphism in $\mathcal{L}^!$. When such analytic coalgebra is given, and when there are no possible ambiguities, we simply use $\mathbb{D}$ to denote the coalgebra $(\mathbb{D}, \widetilde{\partial})$.

Let us make these conditions more explicit. The fact that $(\mathbb{D}, \widetilde{\partial})$ is an object of $\mathcal{L}^!$ means that the two following diagrams commute

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\widetilde{\partial}} & !\mathbb{D} \\ & \searrow^{\mathsf{Id}_{\mathbb{D}}} & \downarrow{\scriptstyle \mathsf{der}_{\mathbb{D}}} \\ & & \mathbb{D} \end{array} \qquad \begin{array}{ccc} \mathbb{D} & \xrightarrow{\widetilde{\partial}} & !\mathbb{D} \\ {\scriptstyle \widetilde{\partial}}\downarrow & & \downarrow{\scriptstyle \mathsf{dig}_{\mathbb{D}}} \\ !\mathbb{D} & \xrightarrow{!\widetilde{\partial}} & !!\mathbb{D} \end{array}$$

The fact that $\mathsf{p}_0 \in \mathcal{L}^!(\mathbb{D}, 1)$ means that

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\widetilde{\partial}} & !\mathbb{D} \\ {\scriptstyle \mathsf{p}_0}\downarrow & & \downarrow{\scriptstyle !\mathsf{t}} \\ 1 & \xrightarrow{\mathsf{m}^0} & !\top \end{array}$$

and the fact that $\widetilde{\theta} \in \mathcal{L}^!(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ means that

$$\begin{array}{ccccc} \mathbb{D} & & \xrightarrow{\qquad\qquad\widetilde{\partial}\qquad\qquad} & & !\mathbb{D} \\ {\scriptstyle \widetilde{\theta}}\downarrow & & & & \downarrow{\scriptstyle !\widetilde{\theta}} \\ \mathbb{D} \otimes \mathbb{D} & \xrightarrow{\widetilde{\partial}\otimes\widetilde{\partial}} & !\mathbb{D} \otimes !\mathbb{D} & \xrightarrow{\mu^2} & !(\mathbb{D} \otimes \mathbb{D}) \end{array}$$

The fact that $\Delta \in \mathcal{L}^!(1, \mathbb{D})$ and $\overline{\pi}_0 \in \mathcal{L}^!(1, \mathbb{D})$ mean that the following diagrams commute

$$\begin{array}{ccc} 1 & \xrightarrow{\mu^0} & !1 \\ {\scriptstyle \overline{\pi}_0}\downarrow & & \downarrow{\scriptstyle !\overline{\pi}_0} \\ \mathbb{D} & \xrightarrow{\widetilde{\partial}} & !\mathbb{D} \end{array} \qquad\qquad \begin{array}{ccc} 1 & \xrightarrow{\mu^0} & !1 \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle !\Delta} \\ \mathbb{D} & \xrightarrow{\widetilde{\partial}} & !\mathbb{D} \end{array}$$

and the fact that $\widetilde{\mathsf{l}} \in \mathcal{L}^!(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})$ means

$$\begin{array}{ccccc} \mathbb{D} & & \xrightarrow{\qquad\qquad\widetilde{\partial}\qquad\qquad} & & !\mathbb{D} \\ {\scriptstyle \widetilde{\mathsf{l}}}\uparrow & & & & \uparrow{\scriptstyle !\widetilde{\mathsf{l}}} \\ \mathbb{D} \otimes \mathbb{D} & \xrightarrow{\widetilde{\partial}\otimes\widetilde{\partial}} & !\mathbb{D} \otimes !\mathbb{D} & \xrightarrow{\mu^2} & !(\mathbb{D} \otimes \mathbb{D}) \end{array}$$

Theorem 14. *We have* $\mathsf{p}_0 = \mathsf{w}_{\mathbb{D}} \in \mathcal{L}^!(\mathbb{D}, 1)$ *and* $\widetilde{\theta} = \mathsf{c}_{\mathbb{D}} \in \mathcal{L}^!(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$.

Proof. This is a direct consequence of Definition 43; the first equation results from the fact that 1 is the terminal object of $\mathcal{L}^!$, and the first equation implies that $\mathsf{p}_i^{(!)}\,\widetilde{\theta} = \mathsf{Id}_{\mathbb{D}}$ for $i = 1, 2$ which entails $\widetilde{\theta} = \langle \mathsf{Id}_{\mathbb{D}}, \mathsf{Id}_{\mathbb{D}} \rangle^{(!)} = \mathsf{c}_{\mathbb{D}}$. □

Theorem 15. *Any analytic coalgebra* $\widetilde{\partial}$ *induces a natural transformation*

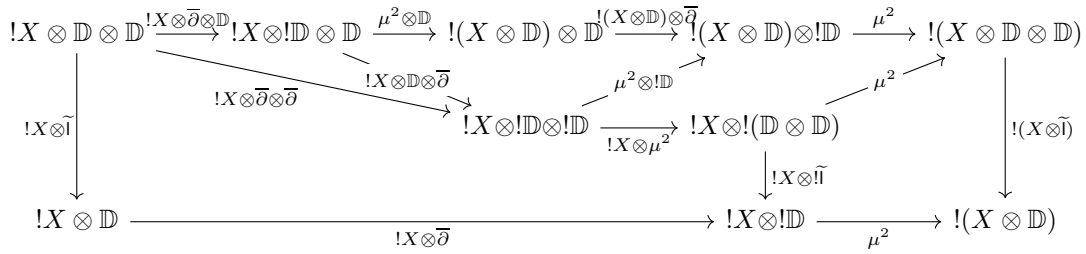
$$\overline{\partial} = \quad !X \otimes \mathbb{D} \xrightarrow{!X \otimes \widetilde{\partial}} !X \otimes !\mathbb{D} \xrightarrow{\mu^2} !(X \otimes \mathbb{D})$$

that follows ($\overline{\partial}$*-chain), (*$\overline{\partial}$*-local), (*$\overline{\partial}$*-add), (*$\overline{\partial}$*-Schwarz), (*$\overline{\partial}$*-&), (*$\overline{\partial}$*-lin) and (*$\overline{\partial}$*-analytic).*

Proof. We show that $\overline{\partial}$ follows ($\overline{\partial}$-analytic) with the diagram chase below. The only crucial argument involved is the fact that $\Delta \in \mathcal{L}^!(1, \mathbb{D})$.

$$\begin{array}{ccccc} !X & & \xrightarrow{!\rho^{-1}} & & !(X \otimes 1) \\ {\scriptstyle !X\otimes 1}\downarrow & & & & \uparrow{\scriptstyle \mu^2} \\ !X \otimes 1 & & \xrightarrow[!X\otimes\mu^0]{} & & !X \otimes !1 \\ {\scriptstyle !X\otimes\Delta}\downarrow & & & & \downarrow{\scriptstyle !X\otimes !\Delta} \\ !X \otimes \mathbb{D} & & \xrightarrow[!X\otimes\overline{\partial}]{} & !X \otimes !\mathbb{D} & \xrightarrow[\mu^2]{} !(X \otimes \mathbb{D}) \end{array} \qquad !(X\otimes\Delta)$$

We show that $\overline{\partial}$ follows ($\overline{\partial}$-lin) with the diagram chase below (any use of α is kept implicit). The only crucial argument involved is the fact that $\widetilde{\mathsf{l}} \in \mathcal{L}^!(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})$.



The other computations are similar and can be found in Theorem 19 of [Ehr23] (they do not involve any argument on the summability structure, so they directly carry to our setting). Note that the proofs of ($\overline{\partial}$-Schwarz) and ($\overline{\partial}$-&) do not rely on any of the assumptions on $\widetilde{\partial}$. □

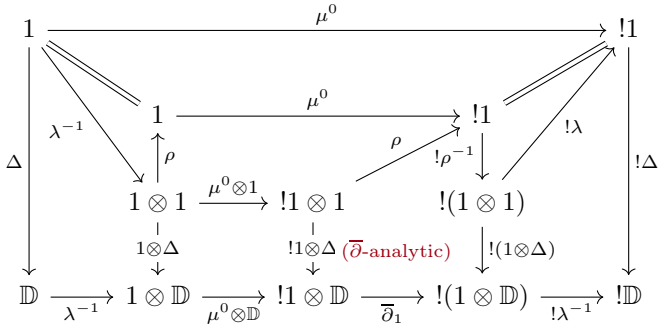
Theorem 16. *Any natural transformation $\overline{\partial} : \mathsf{S}_\otimes ! \Rightarrow !\mathsf{S}_\otimes$ that follows ($\overline{\partial}$-chain), ($\overline{\partial}$-local), ($\overline{\partial}$-add), ($\overline{\partial}$-Schwarz), ($\overline{\partial}$-&), ($\overline{\partial}$-lin) and ($\overline{\partial}$-analytic) induces an analytic coalgebra given by*

$$\widetilde{\partial} = \quad \mathbb{D} \xrightarrow{\lambda^{-1}} 1 \otimes \mathbb{D} \xrightarrow{\mu^0 \otimes \mathbb{D}} !1 \otimes \mathbb{D} \xrightarrow{\overline{\partial}_1} !(1 \otimes \mathbb{D}) \xrightarrow{!\lambda} !\mathbb{D}$$

Furthermore, $\overline{\partial} = \quad !X \otimes \mathbb{D} \xrightarrow{!X \otimes \widetilde{\partial}} !X \otimes !\mathbb{D} \xrightarrow{\mu^2} !(X \otimes \mathbb{D})$.

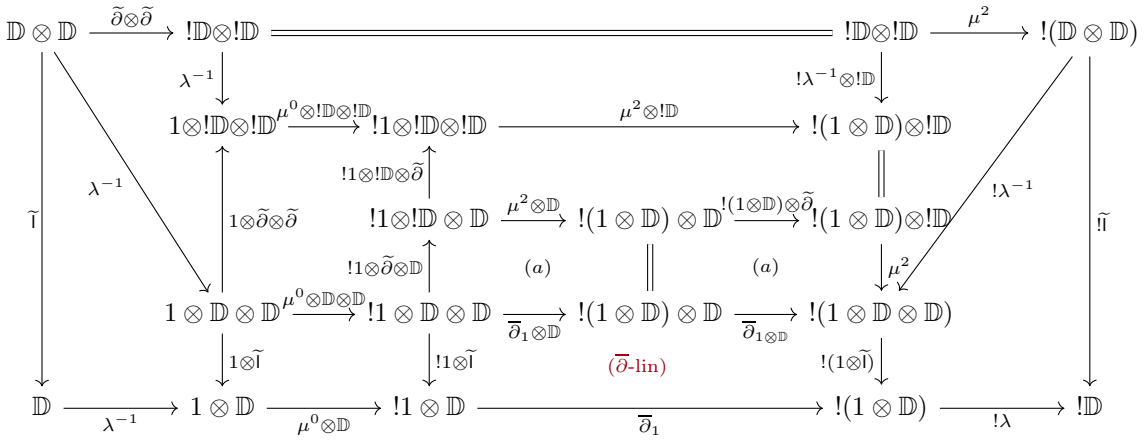
Proof. The fact that $\overline{\partial} = \mu^2 (!X \otimes \overline{\partial})$ can be found in Theorem 17 of [Ehr23]. Again, no argument on the summability structure are used so the proof carry directly to our setting.

We prove that $\Delta \in \mathcal{L}^!(1, \mathbb{D})$ with the following diagram chase. The only crucial argument involved is ($\overline{\partial}$-analytic).



We prove that $\widetilde{\mathsf{l}} \in \mathcal{L}^!(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})$ with the following diagram chase. The proof involves

($\overline{\partial}$-lin) and the fact that $\overline{\partial} = \mu^2 \, (!X \otimes \overline{\partial})$ (commutation $(*)$).



The other computations rely on similar arguments and can be found in [Ehr23]. □

Corollary 4. *The constructions of Theorem 15 and Theorem 16 are a bijection. Thus, for any elementary summable resource category, there is a bijective correspondence between analytic structures following (∂-analytic) and analytic coalgebras.*

Proof. We already know from Theorem 16 that if $\widetilde{\partial} := !\lambda \, \overline{\partial}_1 \, (\mu^0 \otimes \mathbb{D}) \, \lambda^{-1}$ then $\overline{\partial} = \mu^2 \, (!X \otimes \widetilde{\partial})$. Conversely, assume that $\overline{\partial} := \mu^2 \, (!X \otimes \widetilde{\partial})$. Then we can check by a straightforward computation that $\widetilde{\partial} = !\lambda \, \overline{\partial}_1 \, (\mu^0 \otimes \mathbb{D}) \, \lambda^{-1}$. So the constructions of Theorem 15 and Theorem 16 are a bijection. Then, we know from Theorem 11 that the existence of $\overline{\partial}$ following ($\overline{\partial}$-chain), ($\overline{\partial}$-local), ($\overline{\partial}$-add), ($\overline{\partial}$-Schwarz), ($\overline{\partial}$-&), ($\overline{\partial}$-lin) and ($\overline{\partial}$-analytic) is equivalent to the existence of an analytic structure ∂. □

4.6 A remarkable isomorphism

In this section we assume that $\mathcal{L}$ is an elementary ω-summable category equipped with a analytic coalgebra.

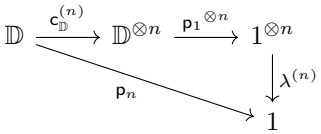
Given an object P of $P^!$, let $\mathsf{c}^{(n)} \in \mathcal{L}^!(P, P^{\otimes n})$ be the n-ary version of the comultiplication of the comonoid P, so that $\mathsf{c}^{(0)} = \mathsf{w}$, $\mathsf{c}^{(1)} = \mathsf{Id}$ and $\mathsf{c}^{(2)} = \mathsf{c}$.

In the case where P is the coalgebra $(\mathbb{D}, \widetilde{\partial})$, we know by Theorem 14 that $\mathsf{c}_{\mathbb{D}} = \widetilde{\theta}$ and hence we have

$$\mathsf{c}_{\mathbb{D}}^{(n)} \, \overline{\pi}_k \, \lambda^{(n)} = \sum_{\substack{\overrightarrow{i} \in \mathbb{N}^k \\ i_1 + \cdots + i_k = n}} \overline{\pi}_{i_1} \otimes \cdots \otimes \overline{\pi}_{i_k}$$

where $\lambda^{(n)} \in \mathcal{L}(1^{\otimes n}, 1)$ is the unique canonical isomorphism induced by the SMC structure of $\mathcal{L}$.

Lemma 23. *For all $n \in \mathbb{N}$ we have*



Proof. Given $k \in \mathbb{N}$, we have

$$\begin{aligned}\lambda^{(n)}\, \mathsf{p}_1{}^{\otimes n}\, \mathsf{c}_{\mathbb{D}}^{(n)}\, \overline{\pi}_k &= \lambda^{(n)}\, \mathsf{p}_1{}^{\otimes n}\Big(\sum_{\substack{\overrightarrow{i} \in \mathbb{N}^n \\ i_1+\cdots+i_n=k}} \overline{\pi}_{i_1} \otimes \cdots \otimes \overline{\pi}_{i_n} \Big)\, (\lambda^{(n)})^{-1} \\ &= \delta_{k,n} \mathsf{Id}_1 \\ &= \mathsf{p}_n\, \overline{\pi}_k\end{aligned}$$

so that the diagram commutes by joint epicity of the $\overline{\pi}_k$'s. □

Definition 44. For all object X, we define a natural morphism $\mathsf{der}_X^n \in \mathcal{L}(!X, X^{\otimes n})$ as

$$!X \xrightarrow{\mathsf{c}_{\mathsf{E}X}^{(n)}} (!X)^{\otimes n} \xrightarrow{\mathsf{der}_X{}^{\otimes n}} X^{\otimes n}$$

and call it *n-ary generalized dereliction*.

For each $n \in \mathbb{N}$ we define $\mathsf{deg}^{(n)} = \lambda^{(n)}\, \mathsf{der}_1^n \in \mathcal{L}(!1, 1)$, and then we define $\mathsf{deg} = \langle \mathsf{deg}^{(n)} \rangle_{n\in\mathbb{N}} \in \mathcal{L}(!1, \mathbb{D})$. In other words, thanks to Lemma 13, we have

$$\mathsf{deg} = \sum_{k=0}^{\infty} \overline{\pi}_k\, \lambda^{(k)}\, \mathsf{der}_1{}^{\otimes n}\, \mathsf{c}_{\mathsf{E}1}^{(n)}\,. \tag{14}$$

Conversely, we define $\mathsf{deg}' \in \mathcal{L}(\mathbb{D}, !1)$ as

$$\mathbb{D} \xrightarrow{\widetilde{\partial}} !\mathbb{D} \xrightarrow{!\mathsf{p}_1} !1$$

Lemma 24. $\mathsf{deg}\, \mathsf{deg}' = \mathsf{Id}_{\mathbb{D}}$.

Proof. Given $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathsf{deg}^{(n)}\, \mathsf{deg}' &= \lambda^{(n)}\, \mathsf{der}_1{}^{\otimes n}\, \mathsf{c}_{\mathsf{E}1}^{(n)}\, !\mathsf{p}_1\, \widetilde{\partial} \\ &= \lambda^{(n)}\, \mathsf{der}_1{}^{\otimes n}\, !\mathsf{p}_1{}^{\otimes n}\, \mathsf{c}_{\mathsf{E}\mathbb{D}}^{(n)}\, \widetilde{\partial} \\ &= \lambda^{(n)}\, \mathsf{p}_1{}^{\otimes n}\, \mathsf{der}_{\mathbb{D}}{}^{\otimes n}\, \mathsf{c}_{\mathsf{E}\mathbb{D}}^{(n)}\, \widetilde{\partial} \\ &= \lambda^{(n)}\, \mathsf{p}_1{}^{\otimes n}\, \mathsf{c}_{\mathbb{D}}^{(n)} \\ &= \mathsf{p}_n\end{aligned}$$

by Lemma 23 and hence $\mathsf{deg}\, \mathsf{deg}' = \langle \mathsf{deg}^{(n)} \rangle_{n\in\mathbb{N}}\, \mathsf{deg}' = \langle \mathsf{deg}^{(n)}\, \mathsf{deg}' \rangle_{n\in\mathbb{N}} = \langle \mathsf{p}_n \rangle_{n\in\mathbb{N}} = \mathsf{Id}_{\mathbb{D}}$. □

Definition 45. The resource category $\mathcal{L}$ is *finitary* if the generalized derelictions $(\mathsf{der}_1^n)_{n\in\mathbb{N}}$ are jointly monic, that is deg is monic.

The intuition is that the "points" of $!1$ contain only a finite amount of information. This is typically the case when the definition of the exponential is based on multisets.

Theorem 17. *If the elementary analytic category $\mathcal{L}$ is finitary, then $\mathsf{deg}'\, \mathsf{deg} = \mathsf{Id}_1$. The coalgebras objects $\mathbb{D}$ and $!1$ are isomorphic in $\mathcal{L}^!$.*

Proof. We have $\mathsf{deg}\, \mathsf{deg}'\, \mathsf{deg} = \mathsf{deg}$ by Lemma 24 and hence $\mathsf{deg}'\, \mathsf{deg} = \mathsf{Id}$ by monicity of deg. The second statement results from the fact that clearly $\mathsf{deg}' \in \mathcal{L}^!(\mathbb{D}, !1)$. □

Remark 23. So when $\mathcal{L}$ is a finitary elementary analytic category which is closed, the objects $!1 \multimap X$ and $\mathsf{S}X$ are isomorphic, meaning that a morphism f from 1 to X in $\mathcal{L}_!$ is the same thing as a summable family $\overrightarrow{x}$ of elements of X: f can be considered as a power series on the object 1 of scalars, whose coefficients are the x_i's, that is $f(t) = \sum_{n=0}^{\infty} t^n x_n$.

More generally, if $f \in \mathcal{L}_!(X,Y) = \mathcal{L}(!X, Y)$, we can define $h \in \mathcal{L}(!X \otimes !1, Y)$ as the following composition of morphism

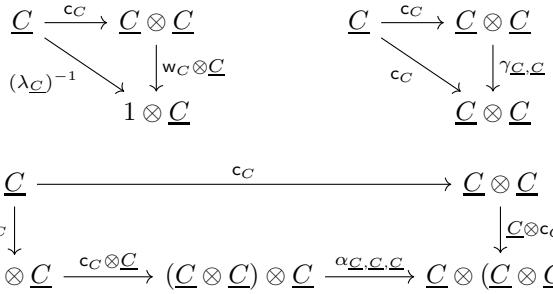
$$!X \otimes !1 \xrightarrow{\mu^2} !(X \otimes 1) \xrightarrow{!\rho} !X \xrightarrow{f} Y$$

which can be seen as a two parameter analytic function which, by the isomorphism between 1 and $\mathbb{D}$, can be considere as a summable family $(h_n \in \mathcal{L}_!(X,Y))_{n\in\mathbb{N}}$. This means intuitively that we can write $f(tx) = \sum_{n=0}^{\infty} t^n h_n(x)$, that is, h_n is the n-homogeneous component of f which can be considered as a "polynomial" morphism $X \to Y$. This morphism can also be obtained as $\pi_n\, \mathsf{T}(f)\, !\iota_1$, using the Taylor functor.

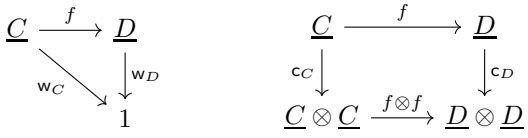
Remark 24. It seems very likely that the free exponential of a Lafont SMC (see Section 4.7) is finitary, although we don't know how to prove it yet.

4.7 The case of Lafont resource categories

If $\mathcal{L}$ is an SMC, one defines the category $\mathcal{L}^{\otimes}$ of commutative $\otimes$-comonoids: an object of this category is a triple $C = (\underline{C}, \mathsf{w}_C, \mathsf{c}_C)$ where the counit $\mathsf{w}_C \in \mathcal{L}(\underline{C}, 1)$ and the comultiplication $\mathsf{c}_C \in \mathcal{L}(\underline{C}, \underline{C} \otimes \underline{C})$ satisfy the following commutations



A morphism from a comonoid C to a comonoid D is an $f \in \mathcal{L}(\underline{C}, \underline{D})$ such that the two following diagrams commute



and identities and composition are defined as in $\mathcal{L}$.

There is an obvious forgetful functor $\mathsf{U} : \mathcal{L}^{\otimes} \to \mathcal{L}$ which maps C to $\underline{C}$ and acts as the identity on morphisms.

Definition 46. The SMC $\mathcal{L}$ is a *Lafont category* if the functor U has a right adjoint.

Theorem 18. *Any cartesian Lafont SMC $\mathcal{L}$ has a canonical structure of resource category.*

This is a standard result, see for instance [Mel09]. It means that $\mathcal{L}$ is endowed with a particular resource structure $(!, \mathsf{der}, \mathsf{dig}, \mathsf{m}^0, \mathsf{m}^2)$ that we describe now. The resource modality which arises in that way is often called the *free exponential* of $\mathcal{L}$.

Let $\mathsf{C} : \mathcal{L} \to \mathcal{L}^{\otimes}$ be the right adjoint of U. By this adjunction, the functor $\mathsf{U} \circ \mathsf{C} : \mathcal{L} \to \mathcal{L}$ has a structure of comonad: this is our resource comonad $(!, \mathsf{der}, \mathsf{dig})$.

For the Seely isomorphisms, we must first notice the following property.

Theorem 19. *The category $\mathcal{L}^{\otimes}$ is cartesian, with terminal object $(1, \mathsf{Id}_1, \lambda_1 = \rho_1)$ and cartesian product of C_1 and C_2 the triple $(\underline{C_1} \otimes \underline{C_2}, \mathsf{w}, \mathsf{c})$ where the unit and the multiplication are given by*

$$\underline{C_1} \otimes \underline{C_2} \xrightarrow{\mathsf{w}_{C_1} \otimes \mathsf{w}_{C_2}} 1 \otimes 1 \xrightarrow{\lambda \otimes \lambda} 1$$

$$(\underline{C_1} \otimes \underline{C_2}) \otimes (\underline{C_1} \otimes \underline{C_2}) \xrightarrow{\gamma_{2,3}} (\underline{C_1} \otimes \underline{C_1}) \otimes (\underline{C_2} \otimes \underline{C_2}) \xrightarrow{\mathsf{c}_{C_1} \otimes \mathsf{c}_{C_2}} \underline{C_1} \otimes \underline{C_2}$$

where $\gamma_{2,3}$ is defined using the coherence isos of the SM structure of $\mathcal{L}$. The first projection is

$$\underline{C_1} \otimes \underline{C_2} \xrightarrow{\mathsf{w}_{C_1} \otimes \underline{C_2}} 1 \otimes \underline{C_2} \xrightarrow{\lambda} \underline{C_2}$$

and similarly for the second one.

As a right adjoint, the functor C preserves cartesian products, and the Seely isomorphisms embody this preservation.

Theorem 20. *A Lafont elementary ω-summable category has a Taylor structure.*

Proof. This is an immediate consequence of the fact that $(\mathbb{D}, \mathsf{p}_0, \widetilde{\theta})$ is a commutative comonoid and of the fact that p_0, $\widetilde{\theta}$, Δ, $\widetilde{!}$ and $\overline{\pi}_0$ are comonoid morphisms. □

Remark 25. So any Lafont ω-summable category is automatically an elementary analytic category. It turns out that all the concrete instances of elementary analytic categories that we know for the time being are of this kind.

Remark 26. There are Lafont categories that are not ω-summable, such as Köthe spaces ([Ehr02]) or finiteness spaces ([Ehr05]). As mentioned in Remark 5, it might be possible to give weaker axioms to our ω-summability structures in order to capture those models.

5 Examples of elementary analytic categories

5.1 Some notations

If A is a set, a (finite) multiset of elements of A is a function $m : A \to \mathbb{N}$ whose support $\mathsf{supp}(m) = \{a \in A \mid m(a) \neq 0\}$ is a finite set. Intuitively, $m(a)$ is the number of occurrences of a in m, and we write $a \in m$ if $a \in \mathsf{supp}(m)$. We use $[\,]$ for the empty multiset such that $\mathsf{supp}([\,]) = \emptyset$. We use $\mathcal{M}_{\mathrm{fin}}(A)$ for the set of all finite multisets of all elements of A, that we consider as a commutative monoid (actually it is the free commutative monoid generated by A), whose operation is denoted additively: if $m_1, \ldots, m_n \in \mathcal{M}_{\mathrm{fin}}(A)$, then we write $m_1 + \cdots + m_n$ for their pointwise sum. If $\overrightarrow{a} = (a_1, \ldots, a_k) \in A^k$, we use $[\overrightarrow{a}] = [a_1, \ldots, a_k]$ for the element m of $[A]$ which contains the elements of $\overrightarrow{a}$, taking multiplicities into account, that is $m(a)$ is the number of $i \in \{1, \ldots, k\}$ such that $a_i = a$. The size, or cardinality, of a multiset m is $\#m = \sum_{a \in A} m(a) \in \mathbb{N}$. We use $\mathcal{M}^+_{\mathrm{fin}}(A)$ for the set of all $m \in \mathcal{M}_{\mathrm{fin}}(A)$ such that $\#m > 0$. We set $m! = \prod_{a \in A} m(a)! \in \mathbb{N}$ and call this number the factorial of m. For any $\overrightarrow{m} \in \mathcal{M}_{\mathrm{fin}}(A)^n$, the quotient $\mathsf{mn}(\overrightarrow{m}) = \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!}$ is an integer and is called the *multinomial coefficient* of $\overrightarrow{m}$.

If $u \in R^A$ where R is a semiring (with standard algebraic notations) and $m \in \mathcal{M}_{\text{fin}}(A)$ we set $u^m = \prod_{a \in A} u_a^{m(a)}$. If $u(1), \ldots, u(k) \in R^A$, the usual multinomial formula generalizes to

$$\Big(\sum_{i=1}^{k} u(i)\Big)^m = \sum_{\substack{\overrightarrow{m} \in \mathcal{M}_{\text{fin}}(A)^k \\ m_1 + \cdots + m_k = m}} \mathsf{mn}(\overrightarrow{m}) \prod_{i=1}^{k} u(i)^{m_i}.$$

5.2 Strict coherence spaces

Our first example is based on a notion of coherence space introduced in [Lam95] and which also arise naturally in the setting of Indexed Linear Logic [BE01]. The nice feature of this model, from the point of view of coherent differentiation, is that it has a non-trivial and very simple notion of summability.

Definition 47. A *strict coherence space* (SCS) is a structure $E = (|E|, \frown_E)$ where $|E|$ is a set and $\frown_E$ is a binary symmetric relation on $|E|$.

Definition 48. A *clique* of a strict coherence space E is a subset x of $|E|$ such that $\forall a, a' \in X \; a \frown_E a'$. We use $\mathsf{Cl}(E)$ for the set of all cliques of E.

Definition 49. If E and F are SCS, we define a SCS $E \multimap F$ by $|E \multimap F| = |E| \times |F|$ and $(a,b) \frown_{E \multimap F} (a',b')$ if $a \frown_E a' \Rightarrow b \frown_F b'$.

Obviously $\mathsf{Id}_E = \{(a,a) \mid a \in |E|\} \in \mathsf{Cl}(E \multimap E)$ and if $s \in \mathsf{Cl}(E \multimap F)$ and $t \in \mathsf{Cl}(F \multimap G)$ then the relational composition $t\,s = \{(a,c) \in |E| \times |G| \mid \exists b \in |B| \; (a,b) \in s \text{ and } (b,c) \in t\}$ belongs to $\mathsf{Cl}(E \multimap G)$.

Definition 50. The category **Scs** has the SCS as objects, and $\mathbf{Scs}(E,F) = \mathsf{Cl}(E \multimap F)$, with identities defined as diagonal relations and composition as relational composition.

Definition 51. The *dual* of E is $E^\perp$, the SCS whose web is $|E|$ and $a \frown_{E^\perp} a'$ is $a \smile_E a'$, which means that $\neg(a \frown_E a')$.

The SCS 1 is defined by $|1| = \{*\}$ and $* \frown_1 *$. So that $\mathsf{Cl}(1) = \{\emptyset, \{*\}\}$. Then one define $\bot = 1^\perp$ so that $|\bot| = \{*\}$, with $* \smile_\bot *$ so that $\mathsf{Cl}(\bot) = \{\emptyset\}$. If E_1 and E_1 are SCS, we set $E_1 \otimes E_2 = (E_1 \multimap E_2^\perp)^\perp$, that is $(a_1,a_2) \frown_{E_1 \otimes E_2} (a_1',a_2')$ if $a_i \frown_{E_i} a_i'$ for $i = 1,2$.

Lemma 25. *If* $(s_i \in \mathbf{Scs}(E_i,F_i))_{i=1,2}$ *then* $s_1 \otimes s_2 \subseteq |E_1 \otimes E_2| \times |F_1 \otimes F_2|$ *defined by* $s_1 \otimes s_2 = \{((a_1,a_2),(b_1,b_2)) \mid (a_i,b_i) \in s_i \text{ for } i = 1,2\}$ *belongs to* $\mathbf{Scs}(E_1 \otimes E_2, F_1 \otimes F_2)$. *The operation* $\otimes$ *defined in that way is a functor* $\mathbf{Scs}^2 \to \mathbf{Scs}$.

The proof is straightforward. This bifunctor, together with its neutral element 1, turns **Scs** into an SMC, by taking the following coherence isos: $\rho_E = \{((a,*),a) \mid a \in |E|\} \in \mathbf{Scs}(E \otimes 1, E)$, $\lambda_E = \{((*,a),a) \mid a \in |E|\} \in \mathbf{Scs}(1 \otimes E, E)$, $\alpha_{E_1,E_2,E_3} = \{(((a_1,a_2),a_3),(a_1,(a_2,a_3)) \mid a_i \in |E_i| \text{ for } i = 1,2,3\} \in \mathbf{Scs}((E_1 \otimes E_2) \otimes E_3, E_1 \otimes (E_2 \otimes E_3))$ and $\gamma_{E_1,E_2} = \{((a_1,a_2),(a_2,a_1)) \mid a_i \in |E_i| \text{ for } i = 1,2\}$.

This SCM is closed, with internal hom of E and F the pair $(E \multimap F, \mathsf{ev})$ where $\mathsf{ev} = \{(((a,b),a),b) \mid a \in |E| \text{ and } b \in |F|\}$. Let us check that $\mathsf{ev} \in \mathbf{Scs}((E \multimap F) \otimes E, F)$, so let $a, a' \in |E|$ and $b, b' \in |F|$ with $((a,b),a) \frown_{(E \multimap F) \otimes E} ((a',b'),a')$. This implies $a \frown_E a'$ and also $(a,b) \frown_{E \multimap F} (a',b')$. Therefore, we have $b \frown_F b'$ as required. The transpose of $s \in \mathbf{Scs}(G \otimes E, F)$ is $\mathsf{cur}(s) = \{(c,(a,b)) \mid ((c,a),b) \in s\} \in \mathbf{Scs}(G, E \multimap F)$.

The SMCC **Scs** is $*$-autonomous with $\bot$ as dualizing object. This means that the morphism $\mathsf{cur}(\mathsf{ev}\,\gamma_{E \multimap \bot, E}) \in \mathcal{L}(E, (E \multimap \bot) \multimap \bot)$ is an iso. This iso can also be expressed more simply by observing that there is a simple iso in $\mathbf{Scs}(E \multimap \bot, E^\perp)$, namely the relation $\{((a,*),a) \mid a \in |E|\}$ which is actually a bijection from $|E \multimap \bot|$ to $|E^\perp|$.

The category **Scs** is cartesian. If $(E_i)_{i\in I}$ is a family of such objects, then we define $\&_{i\in I} E_i$ as the SCS whose web is $|\&_{i\in I} E_i| = \bigcup_{i\in I}\{i\}\times|E_i|$ and whose coherence relation is defined by saying that $(i,a) \frown_{\&_{i\in I} E_i} (i',a')$ if $i = i' \Rightarrow a \frown_{E_i} a'$. Indeed, the relations $(\mathsf{p}_i = \{((i,a),a) \mid i\in I \text{ and } a\in|E_i|\})_{i\in I}$ satisfy $\mathsf{p}_i \in \mathbf{Scs}(\&_{j\in I} E_j, E_i)$ and for any family $(s_i \in \mathbf{Scs}(F,E_i))_{i\in I}$ there is exactly one morphism $\langle s_i\rangle_{i\in I} \in \mathbf{Scs}(F,\&_{i\in I} E_i)$ such that $\forall j\in I\ \mathsf{p}_j\,\langle s_i\rangle_{i\in I} = s_j$, namely

$$\langle s_i\rangle_{i\in I} = \{(b,(i,a)) \mid i\in I \text{ and } (b,a)\in s_i\}$$

so the p_i's are the projections of this cartesian product.

By $*$-autonomy, the category **Scs** is also cocartesian with coproduct $\bigoplus_{i\in I} E_i = (\&_{i\in I} E_i^\perp)^\perp$ whose web is $|\bigoplus_{i\in I} E_i| = \bigcup_{i\in I}\{i\}\times|E_i|$ and coherence is given by $(i,a) \frown_{\bigoplus_{j\in I} E_j} (i',a')$ if $i = i'$ and $a \frown_{E_i} a'$. The corresponding injections are $(\overline{\mathsf{p}}_i = \{(a,(i,a)) \mid i\in I \text{ and } a\in|E_i|\} \in \mathbf{Scs}(E_i, \bigoplus_{j\in J} E_j))_{i\in I}$.

In the special case where $I = \emptyset$, the product of the empty family is the terminal object $\top = (\emptyset,\emptyset)$ and the coproduct of the empty family is the initial object $0 = (\emptyset,\emptyset) = \top$.

Theorem 21. *The SMC* **Scs** *is a Lafont category.*

Proof. Easy verification. □

So the category **Scs** can be equipped with a *free* resource modality $(!, \mathsf{der}, \mathsf{dig}, \mathsf{m}^0, \mathsf{m}^2)$, and it is easy to check that this resource modality can be described as follows (this is part of the proof of Theorem 21). First $|!E| = \mathcal{M}_{\mathrm{fin}}(|E|)$ is the set of finite multisets of elements of $|E|$, and $m \frown_{!E} m'$ if $a \frown_E a'$ for all $a\in m$ and $a'\in m'$. If $s\in\mathbf{Scs}(E,F)$ then

$$!s = \{([a_1,\dots,a_n],[b_1,\dots,b_n]) \mid n\in\mathbb{N} \text{ and } ((a_i,b_i)\in s)_{i=1}^n\}$$

and one checks easily that $!s \in \mathbf{Scs}(!E,!F)$: let $(m,p),(m',p')\in !s$, we must prove that $(m,p) \frown_{!E\multimap !F} (m',p')$, so assume that $m \frown_{!E} m'$ and let us prove that $p \frown_{!F} p'$. Let $b\in p$ and $b'\in p'$. There are $a\in m$ and $a'\in m'$ such that $(a,b),(a',b')\in s$. We have $a\frown_E a'$ and $(a,b)\frown_{E\multimap F}(a',b')$ and hence $b\frown_F b'$.

The counit and comultiplication of the comonad are

$$\mathsf{der}_E = \{([a],a) \mid a\in|E|\}$$
$$\mathsf{dig}_E = \{(m_1+\dots+m_n,[m_1,\dots,m_n]) \mid n\in\mathbb{N} \text{ and } (m_i\in\mathcal{M}_{\mathrm{fin}}(|E|))_{i=1}^n\}\,.$$

It is obvious that $\mathsf{der}_E\in\mathcal{L}(!E,E)$, let us check that $\mathsf{dig}_E\in\mathcal{L}(!E,!!E)$ so let $(m,M),(m',M')\in\mathsf{dig}_E$ and assume that $m\frown_{!E} m'$, we must prove that $M\frown_{!E} M'$. Let $p\in M$ and $p'\in M'$ so that $m = p+m_1$ and $m' = p'+m_1'$ for some multisets m_1 and m_1'. Since $m\frown_{!E} m'$ we have $p\frown_{!E} p'$.

The Seely isomorphisms are $\mathsf{m}^0 = \{(*,[\,])\}$ and $\mathsf{m}^2_{E_1,E_2} = \{((m_1,m_2), 1\cdot m_1 + 2\cdot m_2) \mid (m_1,m_2)\in|!E_1|\times|!E_1|\}$ where $i\cdot[a_1,\dots,a_n] = [(i,a_1),\dots,(i,a_n)]$. We have $\mathsf{m}^0\in\mathcal{L}(1,!\top)$ because $[\,]\frown_{!\top}[\,]$, and m^0 is an iso because $*\frown_1 *$. The fact that the relation $\mathsf{m}^2_{E_1,E_2}$ is a bijection is obvious, and it is easy to check that it is an isomorphism in $\mathbf{Scs}(!E_1\otimes !E_2, !(E_1\,\&\,E_2))$. Assume for instance that $(m_1,m_2)\frown_{!E_1\otimes !E_2}(m_1',m_2')$ and let us prove that $p = 1\cdot m_1 + 2\cdot m_2 \frown_{!(E_1\&E_2)} p' = 1\cdot m_1' + 2\cdot m_2'$ so let $(i,c)\in p$ and $(i',c')\in p'$, one must check that $(i,c)\frown_{E_1\&E_2}(i',c')$, so assume that $i = i'$. Then, by definition of p and p' we have $c\in m_i$ and $c'\in m_i'$ and hence $c\frown_{E_i} c'$.

The induced symmetric lax monoidal structure (μ^0, μ^2) is easily proven to be

$$\mu^0 = \{(*, k[*]) \mid k \in \mathbb{N}\} \in \mathbf{Scs}(1, !1)$$
$$\mu^2_{E,F} = \{(([a_1, \ldots, a_k], [b_1, \ldots, b_k]), [(a_1, b_1), \ldots, (a_k, b_k)] \mid k \in \mathbb{N},\ a_1, \ldots, a_k \in |E| \text{ and } b_1, \ldots, b_k \in |F|\} \in \mathbf{Scs}(!E \otimes !F, !(E \otimes F))\,.$$

Remark 27. Although formally similar to Girard's coherence spaces [Gir87], SCS have quite different properties and are closer to Scott semantics based on cpo's and continuous functions than to Berry stable semantics. As an example, representing the type of booleans by $1 \oplus 1$ with $\mathsf{tt} = (1, *)$ and $\mathsf{ff} = (2, *)$, we can define a "parallel or" morphism

$$\mathsf{por} = \{(([\mathsf{tt}], [\,]), \mathsf{tt}), (([\,], [\mathsf{tt}]), \mathsf{tt}), (([\mathsf{ff}], [\mathsf{ff}]), \mathsf{ff})\} \in \mathbf{Scs}(!(1 \oplus 1) \otimes !(1 \oplus 1), 1 \oplus 1))$$

which is a clique because $\mathsf{tt} \frown_{1\oplus 1} \mathsf{tt}$. So SCS are compatible with this form of non-determinism (there is no deterministic implementation of this por morphism), and nevertheless implement a non-trivial form of coherence since for instance $\{\mathsf{tt}, \mathsf{ff}\} \notin \mathsf{Cl}(1 \oplus 1) = \{\emptyset, \{\mathsf{tt}\}, \{\mathsf{ff}\}\}$.

The category **Scs** has zero morphisms: take $0_{E,F} = \emptyset \in \mathbf{Scs}(E, F)$.

The object $\mathbb{D} = \mathop{\&}_{i\in\mathbb{N}} 1$ can be described as follows (up to trivial iso): $|\mathbb{D}| = \mathbb{N}$ and $\forall i, j \in \mathbb{N}\ i \frown_{\mathbb{D}} j$.

The injections $\overline{\pi}_i \in \mathbf{Scs}(1, \mathbb{D})$ are easy to describe: $\overline{\pi}_i = \{(*, i)\}$.

Lemma 26. **Scs** *is an elementary pre ω-summable category.*

Proof. Since **Scs** is an SMCC, this amounts by Remark 21 to saying that the $\overline{\pi}_i$'s are jointly epic which is obvious since, given $s \in \mathbf{Scs}(\mathbb{D}, E)$ we clearly have $s = \{(i, a) \mid s\,\overline{\pi}_i \neq \emptyset\}$. □

Let E be an SCS, then the SCS $E \otimes \mathbb{D}$ has $|E| \times \mathbb{N}$ as web, and one has $(a, n) \frown_{E\otimes\mathbb{D}} (a', n')$ iff $a \frown_E a'$.

Lemma 27. *A family of morphisms $\overrightarrow{s} \in \mathbf{Scs}(E, F)^{\mathbb{N}}$ is summable iff $\bigcup_{i\in\mathbb{N}} s_i \in \mathbf{Scs}(E, F)$, and if this is the case then $\sum \overrightarrow{s} = \bigcup_{i\in\mathbb{N}} s_i$.*

Proof. Saying that $\overrightarrow{s}$ is summable means that there is $t \in \mathbf{Scs}(E \otimes \mathbb{D}, F)$ such that $s_i = \{(a, b) \in |E \multimap F| \mid ((a, i), b) \in t\}$. If this is the case then, given $(a, b) \in s_i$ and $(a', b') \in s_{i'}$, if $a \frown_E a'$ then $(a, i) \frown_{E\otimes\mathbb{D}} (a', i')$ and hence $b \frown_F b'$. It follows that $\bigcup_{i\in\mathbb{N}} s_i \in \mathbf{Scs}(E, F)$. Conversely, if $\bigcup_{i\in\mathbb{N}} s_i \in \mathbf{Scs}(E, F)$ then $t = \{((a, i), b) \mid i \in \mathbb{N} \text{ and } (a, b) \in s_i\} \in \mathbf{Scs}(E \otimes \mathbb{D}, F)$ as easily checked, and then we have $s_i = \{(a, b) \in |E \multimap F| \mid ((a, i), b) \in t\}$ for each $i \in \mathbb{N}$. If such a t exists, then, by definition, $\sum \overrightarrow{s} = \{(a, b) \mid \exists i\ ((a, i), b) \in t\} = \bigcup_{i\in\mathbb{N}} s_i$. □

Lemma 28. **Scs** *satisfies (𝔻-flat).*

Proof. Let $\varphi : \mathbb{N}^2 \to \mathbb{N}$ be an injection and let $k = \{(\varphi(i, j), (i, j)))\} \subseteq \mathbb{N} \times (\mathbb{N} \times \mathbb{N})$. It is obvious that $k \in \mathbf{Scs}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$ since $(i, j) \frown_{\mathbb{D}\otimes\mathbb{D}} (i', j')$ for any $(i, j), (i', j') \in \mathbb{N}^2$. This morphism obviously satisfies the required property. □

Lemma 29. **Scs** *satisfies (𝔻-wit).*

Proof. Let $\overrightarrow{t} \in \mathbf{Scs}(\mathbb{D}, E)^{\mathbb{N}}$ be such that the family $(t_i\,\Delta \in \mathbf{Scs}(1, E))_{i\in\mathbb{N}}$ is summable, that is

$$t = \bigcup_{i\in\mathbb{N}} t_i\,\Delta = \{(*, a) \mid \exists i, j \in \mathbb{N}\ (j, a) \in t_i)\} \in \mathbf{Scs}(1, E)\,,$$

we contend that $\overrightarrow{t}$ is summable. This amounts to proving that $u \in \mathbf{Scs}(\mathbb{D} \otimes \mathbb{D}, E)$ where $u = \{((j,i),a) \mid (j,a) \in t_i\}$ since then we will have $(t_i = u\,(\mathbb{D} \otimes \overline{\pi}_i)\,\rho^{-1})_{i \in \mathbb{N}}$. If $((i,j),a),((i',j'),a') \in u$ we have $(j,a) \in t_i$ and $(j',a') \in t_{i'}$ and hence $(*,a),(*,a') \in t$ so that $a \frown_E a'$. It follows that $u \in \mathbf{Scs}(\mathbb{D} \otimes \mathbb{D}, E)$. □

So we have proven the following.

Theorem 22. **Scs** *is a Lafont elementary ω-summable category.*

The comonoid structure of $\mathbb{D}$ is given by

$$\mathsf{p}_0 = \{(0,*)\} \in \mathbf{Scs}(\mathbb{D}, 1)$$
$$\widetilde{\theta} = \{(n,(i,j)) \mid n,i,j \in \mathbb{N} \text{ and } n = i+j\} \in \mathbf{Scs}(\mathbb{D}, \mathbb{D} \otimes \mathbb{D})$$

and its monoid structure is

$$\Delta = \{(*,n) \mid n \in \mathbb{N}\} \in \mathbf{Scs}(1, \mathbb{D})$$
$$\widetilde{\mathsf{l}} = \{((i,i),i)) \mid i \in \mathbb{N}\} \in \mathbf{Scs}(\mathbb{D} \otimes \mathbb{D}, \mathbb{D})\,.$$

We describe the associated functor $\mathsf{S} = (\mathbb{D} \multimap \_) : \mathbf{Scs} \to \mathbf{Scs}$. First, $\mathsf{S}E$ has $\mathbb{N} \times |E|$ as web, and $(i,a) \frown_{\mathsf{S}E} (i',a')$ iff $a \frown_E a'$. It follows that we have the following order isomorphism

$$\mathsf{Cl}(\mathsf{S}E) \simeq \{\overrightarrow{x} \in \prod_{i \in \mathbb{N}} \mathsf{Cl}(E_i) \mid \bigcup_{i \in \mathbb{N}} x_i \in \mathsf{Cl}(E)\}\,.$$

And we have $\pi_i = \{((i,a),a) \mid a \in |E|\}$ and $\sigma = \{((i,a),a) \mid i \in \mathbb{N} \text{ and } a \in |E|\}$. Given $s \in \mathbf{Scs}(E,F)$, we have $\mathsf{S}(s) = \{((i,a),(i,b)) \mid i \in \mathbb{N} \text{ and } (a,b) \in s\}$.

We have seen that S has a bimonad structure induced by the bimonoid structure of $\mathbb{D}$, see Figure 1. This structure is quite easy to describe. The unit of the monad is $\iota_0 = \{(a,(0,a)) \mid a \in |E|\} \in \mathbf{Scs}(E, \mathsf{S}E)$ and the multiplication of the monad is $\theta = \{((i,(j,a)),(i+j,a)) \mid i,j \in \mathbb{N} \text{ and } a \in |E|\} \in \mathbf{Scs}(\mathsf{S}^2E, \mathsf{S}E)$. The unit of the comonad is $\sigma = \{((i,a),a) \mid i \in \mathbb{N} \text{ and } a \in |E|\} \in \mathbf{Scs}(\mathsf{S}E, S)$ and the multiplication of the comonad is $\mathsf{l} = \{((i,a),(i,(i,a))) \mid i \in \mathbb{N} \text{ and } a \in |E|\} \in \mathbf{Scs}(\mathsf{S}E, \mathsf{S}^2E)$. Last the distributive law of the bimonad is $\mathsf{c} = \{((i,(j,a)),(j,(i,a))) \mid i,j \in \mathbb{N} \text{ and } a \in |E|\} \in \mathbf{Scs}(\mathsf{S}^2E, \mathsf{S}^2E)$.

By Theorem 20, $\mathbb{D}$ has a Taylor structure, that is, a !-coalgebra structure $\widetilde{\partial} \in \mathbf{Scs}(\mathbb{D}, !\mathbb{D})$ which is given by

$$\widetilde{\partial} = \{(n,[i_1,\dots,i_k]) \mid k \in \mathbb{N} \text{ and } i_1,\dots,i_k \in \mathbb{N} \text{ with } i_1 + \cdots + i_k = n\}\,.$$

This is simply due to the fact that the k-ary version $\mathbb{D} \to \mathbb{D}^{\otimes k}$ is the relation $\{(n,(i_1,\dots,i_k)) \mid i_1 + \cdots + i_k = n\}$.

As seen in Corollary 4, this coalgebra structure induces the distributive law $\partial_E = \mathsf{cur}(\partial'_E) \in \mathbf{Scs}(!\mathsf{S}E, \mathsf{S}!E)$ where $\partial'_E \in \mathbf{Scs}(!(\mathbb{D} \multimap E) \otimes \mathbb{D}, !E)$ is

$$!(\mathbb{D} \multimap E) \otimes \mathbb{D} \xrightarrow{!(\mathbb{D} \multimap E) \otimes \widetilde{\partial}} !(\mathbb{D} \multimap E) \otimes !\mathbb{D} \xrightarrow{\mu^2} !((\mathbb{D} \multimap E) \otimes \mathbb{D}) \xrightarrow{!\mathsf{ev}} !E$$

so that

$$\partial_E = \{([(i_1,a_1),\dots,(i_k,a_k)],(i_1 + \cdots + i_k,[a_1,\dots,a_k])) \mid k \in \mathbb{N},\ i_1,\dots,i_k \in \mathbb{N} \text{ and } a_1,\dots,a_k \in |E|\}\,.$$

Remember that the extension of S to the Kleisli category $\mathbf{Scs}_!$ is the functor $\mathsf{T} : \mathbf{Scs}_! \to \mathbf{Scs}_!$ which maps an object E to $\mathsf{S}E$ (the object of summable families) and a morphism $s \in \mathbf{Scs}_!(E,F)$ to

$$\begin{aligned}\mathsf{T}(s) &= (\mathsf{S}s)\,\partial_E \\ &= \{([(i_1,a_1),\dots,(i_k,a_k)],(i_1+\dots+i_k,b)) \\ &\qquad\qquad \mid k\in\mathbb{N},\ i_1,\dots,i_k\in\mathbb{N} \text{ and } ([a_1,\dots,a_k],b)\in s\}\,.\end{aligned}$$

The morphism $\iota_1 = (\mathsf{p}_1 \multimap E) \in \mathbf{Scs}(E,\mathsf{S}E)$ satisfies $\iota_1 = \{(a,(1,a)) \mid a \in |E|\}$ so that $!\iota_1 = \{([a_1,\dots,a_k],[(1,a_1),\dots,(1,a_k)]) \mid k\in\mathbb{N} \text{ and } a_1,\dots,a_k\in|E|\} \in \mathbf{Scs}(!E,!\mathsf{S}E)$. It follows that $\mathsf{T}(s)\,!\iota_1 = \{([a_1,\dots,a_k],(k,b)) \mid ([a_1,\dots,a_k],b)\in s\} \in \mathbf{Scs}_!(E,\mathsf{S}F)$ is such that $\pi_k\,\mathsf{T}(s)\,!\iota_1 \in \mathbf{Scs}_!(E,F)$ is the k-homogeneous component of s, that is, the set of all $(m,b)\in s$ such that the size of the multiset m is k.

Notice last that the resource modality is easily checked to be finitary in the sense of Definition 45 so that we know that $\mathsf{deg} \in \mathbf{Scs}(!1,\mathbb{D})$ is an iso by Theorem 17. This can also be checked directly: $|!1| = \{k[*] \mid k\in\mathbb{N}\} \simeq \mathbb{N}$ and we have $k[*] \frown_{!1} k'[*]$ for all $k,k'\in\mathbb{N}$ since $* \frown_1 *$.

5.3 Girard's coherence spaces

Just as in [Ehr23], one can show that the usual Girard's coherence spaces (CS) have an analytic coalgebra. Remember that such a coherence space is a pair $E = (|E|, \frown_E)$ where $|E|$ is a set (the web) and $\frown_E$ is a binary, reflexive and symmetric relation on $|E|$. A clique of E is a subset x of $|E|$ such that $\forall a,a'\in x\ a\frown_E a'$. Given CS E and F, one defines a CS $E\multimap F$ by $|E\multimap F|$ and $(a,b)\frown_{E\multimap F}(a',b')$ if $a\frown_E a' \Rightarrow (b\frown_F b'$ and $b=b'\Rightarrow a=a')$, and the category $\mathbf{Coh}$ has CS as objects, and $\mathbf{Coh}(E,F) = \mathsf{Cl}(E\multimap F)$, identity morphisms and composition being defined as in $\mathbf{Scs}$.

The category $\mathbf{Coh}$ is doubtlessly the most popular model of Linear Logic, and is a Lafont category with $!E$ defined as follows: $|!E| = \{[a_1,\dots,a_n] \mid n\in\mathbb{N} \text{ and } \{a_1,\dots,a_n\}\in\mathsf{Cl}(E)\}$. This is a major difference between all the other models presented in this section, where $|!E| = \mathcal{M}_{\mathrm{fin}}(|E|)$: one often says that the CS exponential is *uniform*, whereas the exponentials of the other models are *non-uniform*. As far as we know, it is not possible to equip the category $\mathbf{Coh}$ with a non-uniform exponential. One has also to be careful with the definition of the action of this functor on morphisms: given $s\in\mathbf{Coh}(E,F)$, one takes $!s = \{([a_1,\dots,a_k],[b_1,\dots,b_k]) \mid k\in\mathbb{N},\ \{a_1,\dots,a_k\}\in\mathsf{Cl}(E) \text{ and } (a_i,b_i)\in s \text{ for all } i\}$.

The category $\mathbf{Coh}$ is easily seen to be elementarily ω-summable. The object of degrees $\mathbb{D}$ satisfies $|\mathbb{D}| = \mathbb{N}$ and $\forall i,j\in\mathbb{N}\ i\frown_{\mathbb{D}} j$. Therefore, $\mathsf{S}E$ satisfies $|\mathsf{S}E| = \mathbb{N}\times|E|$ with $(i,a)\frown_{\mathsf{S}E}(i',a')$ if $a\frown_E a'$ and $i\neq i'\Rightarrow a\neq a'$. It follows that $\mathsf{Cl}(\mathsf{S}E) \simeq \{\overrightarrow{x}\in\mathsf{Cl}(E)^{\mathbb{N}} \mid \bigcup_{i\in\mathbb{N}} x_i \in \mathsf{Cl}(E) \text{ and } i\neq j\Rightarrow x_i\cap x_j = \emptyset\}$. The canonical analytic coalgebra is $\widetilde{\partial} = \{(n,[i_1,\dots,i_k]) \in \mathbb{N}\times\mathcal{M}_{\mathrm{fin}}(\mathbb{N}) \mid k\in\mathbb{N} \text{ and } i_1+\dots+i_k = n\}$. The induced Taylor functor $\mathsf{T} : \mathbf{Coh}_! \to \mathbf{Coh}_!$ maps a CS E to $\mathsf{S}E$, and if $s\in\mathbf{Coh}_!(E,F)$ then $\mathsf{T}(s)\in\mathbf{Coh}_!(\mathsf{S}E,\mathsf{S}F)$ is given by

$$\begin{aligned}\mathsf{T}(s) = \{([(i_1,a_1),\dots,(i_k,a_k)],(n,b)) \mid k\in\mathbb{N},\\ \{(i_1,a_1),\dots,(i_k,a_k)\}\in\mathsf{Cl}(\mathsf{S}E),\ i_1+\dots+i_k = n \text{ and } ([a_1,\dots,a_k],b)\in s\}\,.\end{aligned}$$

So if $s = \{([a,a],b)\}$ is a simple "quadratic" morphism, for having $\{(i_1,a),(i_2,a)\}\in\mathsf{S}E$, we need $i_1 = i_2$. It follows that

$$\mathsf{T}(s) = \{([(i,a),(i,a)],(2i,b)) \mid i\in\mathbb{N}\}$$

whereas in SCS, we had $\mathsf{T}(s) = \{([(i_1,a),(i_2,a)],(i_1+i_2,b)) \mid i_1,i_2 \in \mathbb{N}\}$. The precise meaning of this difference between the actions of the T functor in the uniform setting of coherence spaces and in the non-uniform one of SCS has still to be fully understood.

5.4 Nonuniform coherence spaces

Formally, nonuniform coherence spaces (NUCS) can be considered as a refinement of SCS, but they have quite different properties, being much closer to Girard's coherence spaces and to the stable semantics. In particular the relation por of Remark 27 is rejected by NUCS.

We refer to [Ehr23], Section 6.1 for a detailed presentation of NUCS, we just recall the basic definitions.

Definition 52. A NUCS is a tuple $E = (|E|, \frown_E, \smile_E)$ where $|E|$ is a set (the web of E) and $\frown_E$ (strict coherence) and $\smile_E$ (strict incoherence) are disjoint binary symmetric relations on $|E|$. The relation $\equiv_E = |E|^2 \setminus (\frown_E \cup \smile_E)$ (which is also symmetric) is called neutrality and the large coherence and incoherence relations are defined as $\subset_E = \frown_E \cup \equiv_E$ and $\asymp_E = \smile_E \cup \equiv_E$. A clique of a NUCS E is a subset x of $|E|$ such that $\forall a,a' \in x\ a \subset_E a'$, and we use $\mathsf{Cl}(E)$ for the set of all cliques of E.

The dual of a NUCS E is $E^\perp = (|E|, \smile_E, \frown_E)$ and one defines $E \multimap F$ by stipulating that $|E \multimap F| = |E| \times |F|$ and by providing the large coherence relation and the neutrality: $(a,b) \equiv_{E\multimap F} (a',b')$ if $a \equiv_E a'$ and $b \equiv_F b'$, and $(a,b) \subset_{E\multimap F} (a',b')$ if

$$a \subset_E a' \Rightarrow (b \subset_F b' \text{ and } b \equiv_E b' \Rightarrow a \equiv_E a')\,.$$

Then the category **Nucs** has the NUCS as objects and $\mathbf{Nucs}(E,F) = \mathsf{Cl}(E \multimap F)$, identity morphisms and composition being defined as in the category **Scs**. The definition of the SMC structure of **Nucs** is completely similar to that of **Scs** as well as the proof that the category **Nucs** (with dualizing object $\bot = 1 = (*,\emptyset,\emptyset)$) is $*$-autonomous and cartesian. Notice that we have here an important difference between **Nucs** and **Scs**: in the latter category, the objects 1 and $\bot = 1^\perp$ are not isomorphic.

Nucs is proven to be an elementary ω-summable category exactly as **Scs**. The induced functor $\mathsf{S} : \mathbf{Scs} \to \mathbf{Scs}$ is such that $|\mathsf{S}E| = \mathbb{N} \times |E|$ and $(i,a) \frown_{\mathsf{S}E} (i',a')$ if $a \frown_E a'$ and $(i,a) \equiv_{\mathsf{S}E} (i',a')$ if $i = i'$ and $a \equiv_E a'$, and this functor acts on morphisms exactly as in the setting of SCS. This means that a family $(x_i \in \mathsf{Cl}(E))_{i\in\mathbb{N}}$ is summable iff $a \frown_E a'$ as soon as $a \in x_i$ and $a' \in x_{i'}$ when $i \neq i'$. For instance, for a family $(x_i \in \mathsf{Cl}(1))_{i\in\mathbb{N}}$ to be summable, we need all the x_i to be empty but possibly one (which is then equal to $\{*\}$).

As shown in [Bou11], the SMC **Nucs** is a Lafont category, the induced resource modality is $(!_\mathsf{b}, \mathsf{der}, \mathsf{dig}, \mathsf{m}^0, \mathsf{m}^2)$ where $|!_\mathsf{b}E| = \mathcal{M}_{\mathrm{fin}}(|E|)$ and $m \subset_{!_\mathsf{b}E} m'$ if $\forall a \in m\, \forall a' \in m'\ a \subset_E a'$ and $m \equiv_{!_\mathsf{b}E} m'$ if $m \subset_E m'$ and $m = [a_1,\dots,a_n]$ and $m' = [a'_1,\dots,a'_n]$ with $a_i \equiv_E a'_i$ for $i = 1,\dots,n$ and der, dig, m^0 and m^2 are defined as in **Scs**.

So by Theorem 20, exactly as for **Scs**, $\mathbb{D}$ has an analytic coalgebra, that is, a $!_\mathsf{b}$-coalgebra structure which is defined exactly as in **Scs**.

One interesting feature of **Nucs** is that it admits another resource modality $!_\mathsf{be}$ whose structure morphisms der, dig, m^0 and m^2 are, again, defined as in **Scs**. This exponential was actually the first one discovered for NUCS because it arises naturally in the setting of Indexed Linear Logic, see [BE01] where NUCS were introduced as a particular example of denotational models based on phase semantics. One has $|!_\mathsf{be}E| = |E|$ and, given $m = [a_1,\dots,a_n], m' = [a_{n+1},\dots,a_k]$, one has $m \subset_{!_\mathsf{be}E} m'$ if $\forall i,j \in \{1,\dots,k\}\ i \neq j \Rightarrow a_i \subset_E a_j$ and $m \frown_{!_\mathsf{be}E} m'$ if $m \subset_{!_\mathsf{be}E} m'$ and $\exists i \in \{1,\dots,k\}\, \forall j \in \{1,\dots,k\}\ i \neq j \Rightarrow a_i \frown_E a_j$.

Let us describe $!_{\mathsf{be}}1$: we have $|!_{\mathsf{be}}1| = \{i[*] \mid i \in \mathbb{N}\} \simeq \mathbb{N}$. Next $0 \frown_{!_{\mathsf{be}}1} 1$ and $i \equiv_{!_{\mathsf{be}}1} j$ as soon as $i + j \geq 2$. But $\mathbb{D}$ is characterized by $|\mathbb{D}| = \mathbb{N}$ and $i \equiv_{\mathbb{D}} i$ and $i \frown_{\mathbb{D}} j$ when $i \neq j$, and therefore $\mathbb{D}$ and $!_{\mathsf{be}}1$ are not isomorphic, and since $!_{\mathsf{be}}$ is easily seen to be finitary, the only possibility is that this resource modality $!_{\mathsf{be}}$ has no analytic coalgebra $\widetilde{\partial}$ and thus no analytic structure ∂.

Remark 28. On the other hand, setting $\mathbb{D}_2 = 1 \,\&\, 1$, it is not hard to check that we have a coalgebra structure $\delta \in \mathbf{Nucs}(\mathbb{D}_2, !_{\mathsf{be}}\mathbb{D}_2)$ given by

$$\delta = \{(i, [i_1, \ldots, i_k]) \mid i, i_1, \ldots, i_k \in \{0, 1\} \text{ and } i = i_1 + \cdots + i_k\}$$

so that **Nucs**, equipped with the $!_{\mathsf{be}}$ resource modality, is an elementary model of coherent differentiation in the sense of [Ehr23], which shows that the analytic structure is strictly stronger than the coherent differential structure.

5.5 Probabilistic coherence spaces

This last example is in some sense similar to both **Scs** and **Nucs**, additionally featuring nonnegative real coefficients.

Given an at most countable set A and $u, u' \in \overline{\mathbb{R}_{\geq 0}}^A$, we set $\langle u \mid u' \rangle = \sum_{a \in A} u_a u'_a \in \overline{\mathbb{R}_{\geq 0}}$ where $\overline{\mathbb{R}_{\geq 0}}$ is the completed half real line. Given $P \subseteq \overline{\mathbb{R}_{\geq 0}}^A$, we define $P^\perp \subseteq \overline{\mathbb{R}_{\geq 0}}^A$ as

$$P^\perp = \{u' \in \overline{\mathbb{R}_{\geq 0}}^A \mid \forall u \in P \ \langle u \mid u' \rangle \leq 1\}\,.$$

Observe that if P satisfies $\forall a \in A \, \exists x \in P \ x_a > 0$ and $\forall a \in A \, \exists m \in \mathbb{R}_{\geq 0} \forall x \in P \ x_a \leq m$ then $P^\perp \in \mathbb{R}_{\geq 0}^I$ and $P^\perp$ satisfies the same two properties that we call *local boundedness* which can also be rephrased as

$$\forall a \in A \quad 0 < \sup_{x \in P} x_a < \infty\,.$$

A probabilistic pre-coherence space (pre-PCS) is a pair $X = (|X|, \mathsf{P}X)$ where $|X|$ is a set and $\mathsf{P}X \subseteq \overline{\mathbb{R}_{\geq 0}}^{|X|}$ satisfies $\mathsf{P}X^{\perp\perp} = \mathsf{P}X$. A probabilistic coherence space (PCS) is a pre-PCS X such that $\mathsf{P}X$ is locally bounded.

Given a PCS X and $x \in \mathsf{P}X$ we set $\|x\|_X = \sup_{x' \in \mathsf{P}X^\perp} \langle x \mid x' \rangle \in [0, 1]$. This operation obeys the usual properties of a norm: $\|x\| = 0 \Rightarrow x = 0$, $\|x_0 + x_1\| \leq \|x_0\| + \|x_1\|$ and $\|\lambda x\| = \lambda \|x\|$ for all $\lambda \in [0, 1]$.

Remark 29. Given $x \in \mathsf{P}X$ and $a \in |X|$ we use the notations x_a or $x(a)$ for the corresponding element of $\mathbb{R}_{\geq 0}$, depending on the context. In some situations x_i can denote an element of $\mathsf{P}X$ and in such a situation we will prefer the notation $x_i(a)$ to denote the a-component of x_i to avoid the ugly x_{ia}.

Given $t \in \overline{\mathbb{R}_{\geq 0}}^{A \times B}$ considered as a matrix (where A and B are at most countable sets) and $u \in \overline{\mathbb{R}_{\geq 0}}^A$, we define $t \cdot u \in \overline{\mathbb{R}_{\geq 0}}^B$ by $(t \cdot u)_b = \sum_{a \in A} t_{a,b} u_a$ (usual formula for applying a matrix to a vector), and if $s \in \overline{\mathbb{R}_{\geq 0}}^{B \times C}$ we define the product[9] $s\,t \in \overline{\mathbb{R}_{\geq 0}}^{A \times C}$ of the matrix s and t as usual by $(s\,t)_{a,c} = \sum_{b \in B} t_{a,b} s_{b,c}$. This is an associative operation.

Let X and Y be PCSs, a morphism from X to Y is a matrix $t \in \mathbb{R}_{\geq 0}^{|X| \times |Y|}$ such that $\forall x \in \mathsf{P}X \ t \cdot x \in \mathsf{P}Y$. It is clear that the identity (diagonal) matrix is a morphism from X

[9] We write this product in the reverse order wrt. the usual algebraic conventions on matrices, because it is the notion of composition in our category, and we respect the standard order of factors when writing a composition in a category. This is a well known and unfortunate mismatch of conventions.

to X and that the matrix product of two morphisms is a morphism and therefore, PCSs equipped with this notion of morphism form a category **Pcoh**.

The condition $t \in \mathbf{Pcoh}(X,Y)$ is equivalent to $\forall x \in \mathsf{P}X\, \forall y' \in \mathsf{P}Y^\perp\ \langle t \cdot x \mid y' \rangle \leq 1$ and observe that $\langle t \cdot x \mid y' \rangle = \langle t \mid x \otimes y' \rangle$ where $(x \otimes y')_{(a,b)} = x_a y'_b$. We define $X \multimap Y = (|X| \times |Y|, \{t \in \mathbb{R}_{\geq 0}^{|X \multimap Y|} \mid \forall x \in \mathsf{P}X\ t \cdot x \in \mathsf{P}Y\})$: this is a pre-PCS by this observation, and checking that it is indeed a PCS is easy.

We define then $X \otimes Y = (X \multimap Y^\perp)^\perp$; this is a PCS which satisfies $\mathsf{P}(X \otimes Z) = \{x \otimes z \mid x \in \mathsf{P}X \text{ and } z \in \mathsf{P}Z\}^{\perp\perp}$ where $(x \otimes z)_{(a,c)} = x_a z_c$. Then it is easy to see that we have equipped in that way the category **Pcoh** with a symmetric monoidal structure for which it is $*$-autonomous with the dualizing object $\perp = 1 = (\{*\}, [0,1])$, which coincides with the unit of $\otimes$. The $*$-autonomy follows easily from the observation that $(X \multimap \perp) \simeq X^\perp$.

Lemma 30. *Given $s,t \in \mathbf{Pcoh}(X_1 \otimes \cdots \otimes X_k, Y)$, if for all $(x_i \in \mathsf{P}X_i)_{i=1}^k$ one has $s \cdot (x_1 \otimes \cdots \otimes x_k) = t \cdot (x_1 \otimes \cdots \otimes x_k)$ then $s = t$.*

The category **Pcoh** is cartesian: if $(X_j)_{j \in J}$ is an at most countable family of PCSs, then $(\with_{j \in J} X_j, (\mathsf{p}j)_{j \in J})$ is the cartesian product of the X_j's, with $|\with_{j \in J} X_j| = \bigcup_{j \in J} \{j\} \times |X_j|$, $(\mathsf{p}j)_{(k,a),a'} = 1$ if $j = k$ and $a = a'$ and $(\mathsf{p}j)_{(k,a),a'} = 0$ otherwise, and $x \in \mathsf{P}(\with_{j \in J} X_j)$ if $\mathsf{p}j \cdot x \in \mathsf{P}X_j$ for each $j \in J$ (for $x \in \mathbb{R}_{\geq 0}^{|\with_{j \in J} X_j|}$). Given $(t_j \in \mathbf{Pcoh}(Y, X_j))_{j \in J}$, the unique morphism $t = \langle t_j \rangle_{j \in J} \in \mathbf{Pcoh}(Y, \with_{j \in J} X_j)$ such that $\mathsf{p}j\, t = t_j$ is simply defined by $t_{b,(j,a)} = (t_j)_{a,b}$. The dual operation $\bigoplus_{j \in J} X_j$, which is a coproduct, is characterized by $|\bigoplus_{j \in J} X_j| = \bigcup_{j \in J} \{j\} \times |X_j|$ and $x \in \mathsf{P}(\bigoplus_{j \in J} X_j)$ if $x \in \mathsf{P}(\with_{j \in J} X_j)$ and $\sum_{j \in J} \|\mathsf{p}j \cdot x\|_{X_j} \leq 1$.

As to the exponentials, one sets $|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and $\mathsf{P}(!X) = \{x^! \mid x \in \mathsf{P}X\}^{\perp\perp}$ where, given $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$, $x^!_m = x^m = \prod_{a \in |X|} x_a^{m(a)}$. A morphism $t \in \mathbf{Pcoh}(!X, Y) = \mathsf{P}(!X \multimap Y)$ is completely characterized by the associated analytic function

$$\begin{aligned} \widehat{t} : \mathsf{P}X &\to \mathsf{P}Y \\ x &\mapsto t \cdot x^! = \sum_{m \in !|X|, b \in |Y|} t_{m,b} x^m\, \mathsf{e}_b\,. \end{aligned}$$

Lemma 31. *Let $t \in \mathbb{R}_{\geq 0}^{|!X_1 \otimes \cdots \otimes !X_k \multimap Y|}$. One has $t \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, Y)$ iff for all $(x_i \in \mathsf{P}X_i)_{i=1}^k$ one has $t \cdot (x_1^! \otimes \cdots \otimes x_k^!) \in \mathsf{P}Y$.*

Lemma 32. *If $s,t \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, Y)$ satisfy $s \cdot (x_1^! \otimes \cdots \otimes x_k^!) = t \cdot (x_1^! \otimes \cdots \otimes x_k^!)$ for all $(x_i \in \mathsf{P}X_i)_{i=1}^k$ then $s = t$.*

This very useful property uses crucially the local boundedness property of PCSs.

Then given $t \in \mathbf{Pcoh}(X,Y)$, we explain now how to define $!t \in \mathbf{Pcoh}(!X, !Y)$. Let $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$ and $p \in \mathcal{M}_{\mathrm{fin}}(|Y|)$. We use $\mathsf{L}(m,p)$ for the set of all $r \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y|)$ such that

$$\forall a \in |X|\ m(a) = \sum_{b \in |Y|} r(a,b) \quad \text{and} \quad \forall b \in |Y|\ p(b) = \sum_{a \in |X|} r(a,b)\,.$$

Notice that if $r \in \mathcal{M}_{\mathrm{fin}}(|X| \times |Y|)$ then $\#r = \#m = \#p$ so that $\mathsf{L}(m,p)$ is non-empty iff $\#m = \#p$. When $r \in \mathsf{L}(n,p)$ we set

$$\begin{bmatrix} p \\ r \end{bmatrix} = \prod_{b \in |Y|} \frac{p(b)!}{\prod_{a \in |X|} r(a,b)!}$$

which belongs to $\mathbb{N} \setminus \{0\}$; this is a generalized multinomial coefficient. Then we have

$$(!t)_{m,p} = \sum_{r \in \mathsf{L}(m,p)} \begin{bmatrix} p \\ r \end{bmatrix} t^r$$

where we recall that $t^r = \prod_{(a,b) \in |X| \times |Y|} t_{a,b}^{r(a,b)}$. The main feature of this definition is that for all $x \in \mathsf{P}X$ one has $\widehat{!t}(x) = !t \cdot x^! = (t \cdot x)^!$. This property fully characterizes $!t$. The comonad structure is given by $\mathsf{der}_X \in \mathbb{R}_{\geq 0}^{|!X \multimap X|}$ given by $(\mathsf{der}_X)_{m,a} = \delta_{m,[a]}$ so that $\forall x \in \mathsf{P}X \ \mathsf{der}_X \cdot x^! = x \in \mathsf{P}X$ and therefore $\mathsf{der}_X \in \mathsf{P}(!X, X)$. Similarly, one defines $\mathsf{dig}_X \in \mathbb{R}_{\geq 0}^{|!X \multimap !!X|}$ by $(\mathsf{dig}_X)_{(m,[m_1,\ldots,m_n])} = \delta_{m,m_1+\cdots+m_n}$ so that $\forall x \in \mathsf{P}X \ \mathsf{dig}_X \cdot x^! = x^{!!}$ and hence, again, $\mathsf{dig}_X \in \mathsf{P}(!X, !!X)$. The equations required to prove that $(!\_, \mathsf{der}, \mathsf{dig})$ is indeed a comonad result from Lemma 32. For instance, let $t \in \mathsf{P}(X, Y)$, we have $(\mathsf{dig}_Y\, !t) \cdot x^! = \mathsf{dig}Y \cdot (t \cdot x^!) = \mathsf{dig}_X \cdot (t \cdot x)^! = (t \cdot x)^{!!}$ and $(!!t\, \mathsf{dig}_X) \cdot x^! = !!t \cdot (\mathsf{dig}_X \cdot x^!) = !!t \cdot x^{!!} = (!t \cdot x^!)^! = (t \cdot x)^{!!}$ which shows that dig is a natural transformation. As another example, we have $(\mathsf{dig}!X\, \mathsf{dig}_X) \cdot x^! = \mathsf{dig}!X \cdot x^{!!} = x^{!!!}$ and $(!\mathsf{dig}_X\, \mathsf{dig}_X) \cdot x^! = !\mathsf{dig}_X \cdot x^{!!} = (\mathsf{dig}_X \cdot x^!)^! = (x^{!!})^! = x^{!!!}$ and hence $\mathsf{dig}!X\, \mathsf{dig}_X = !\mathsf{dig}_X\, \mathsf{dig}_X$ which is one of the required comonad commutations. The others are proven similarly.

The monoidality Seely isomorphisms $\mathsf{m}^0 \in \mathbf{Pcoh}(1, !\top)$ and $\mathsf{m}^2_{X_1,X_2} \in \mathbf{Pcoh}(!X_1 \otimes !X_2, !(X_1 \& X_2))$ are given by $\mathsf{m}^0_{*,[]} = 1$ and $\mathsf{m}^2_{((m_1,m_2),m)} = \delta_{1 \cdot m_1 + 2 \cdot m_2, m}$ where, for a multiset $m = [a_1, \ldots, a_k]$ we set $i \cdot m = [(i, a_1), \ldots, (i, a_k)]$, see Section 5.1. It is obvious that m^0 is an iso. To check that $\mathsf{m}^2_{X_1,x_2}$ is a morphism we use Lemma 31: let $x_i \in \mathsf{P}X_i$ for $i = 1, 2$, one has $\mathsf{m}^2_{X_1,X_2} \cdot (x_1^! \otimes x_2^!) = \langle x_1, x_2 \rangle^! \in \mathsf{P}!(X_1 \,\&\, X_2)$. Conversely, defining $s \in \mathbb{R}_{\geq 0}^{!(X_1 \& X_2) \multimap (!X_1 \otimes !X_2)}$ by $s_{m,(m_1,m_2)} = \delta_{1 \cdot m_1 + 2 \cdot m_2, m}$ we have $s \cdot \langle x_1, x_2 \rangle^! = x_1^! \otimes x_2^! \in \mathsf{P}(!X_1 \otimes !X_2)$ for all $x_i \in \mathsf{P}X_i$ $(i = 1, 2)$, and hence $s \in \mathbf{Pcoh}(!(X_1 \,\&\, X_2), (!X_1 \otimes !X_2))$. It is obvious that s is the inverse of $\mathsf{m}^2_{X_1,X_2}$ which is therefore an iso in **Pcoh**. Proving that it is natural and that it satisfies all the required commutations for turning **Pcoh** into a model of LL is routine (using crucially Lemma 32).

The induced lax monoidality $\mu^k \in \mathbf{Pcoh}(!X_1 \otimes \cdots \otimes !X_k, !(X_1 \otimes \cdots \otimes X_k))$ is such that $(\mu^k)_{(m_1,\ldots,m_k),m} = 1$ if $m = [(a_1^1, \ldots, a_k^1), \ldots, (a_1^n, \ldots, a_k^n))]$ and $(m_i = [a_i^1, \ldots, a_i^n])_{i=1}^k$, and $(\mu^k)_{(m_1,\ldots,m_k),m} = 0$ otherwise.

Theorem 23. *The SMC* **Pcoh** *is a Lafont category.*

Proof. This is the object of [CEPT17]. ☐

5.5.1 Elementary analytic structure of Pcoh

The category **Pcoh** has zero-morphisms (we have the 0 matrix in $\mathbf{Pcoh}(X, Y)$ for any two objects X and Y).

The object $\mathbb{D} = \&_{i \in \mathbb{N}} 1$ can be described as $|\mathbb{D}| = \mathbb{N}$ and $\mathsf{P}\mathbb{D} = \{x \in \mathbb{R}_{\geq 0}^{\mathbb{N}} \mid \forall i \in \mathbb{N} \ x_i \in [0, 1]\}$. The morphisms $(\overline{\pi}_i \in \mathbf{Pcoh}(1, \mathbb{D}))_{i \in \mathbb{N}}$ are characterized by $\overline{\pi}_i \cdot u = u\mathsf{e}_i$ for $u \in \mathsf{P}1 = [0, 1]$. These morphisms are jointly epic because, for any $t \in \mathbf{Pcoh}(\mathbb{D}, X)$ and $x \in \mathsf{P}\mathbb{D}$ one has $t \cdot x = \sum_{i \in \mathbb{N}} x_i (t \cdot \mathsf{e}_i)$.

Given an at most countable set A, let $\lambda_\infty A = \&_{a \in A} 1$, that is $\lambda_\infty A$ is the PCS whose web is A, and $\mathsf{P}(\lambda_\infty A) = [0, 1]^A$, and let $\lambda_1 A = \bigoplus_{a \in A} 1$, that is $\lambda_1 A$ is the PCS whose web is A, and $\mathsf{P}(\lambda_1 A) = \{x \in \mathbb{R}_{\geq 0}^A \mid \sum_{a \in A} x_a \leq 1\}$. We have $\lambda_1 A = \lambda_\infty A^\perp$ and $\lambda_\infty A = \lambda_1 A^\perp$.

Lemma 33. $\lambda_\infty A \otimes \lambda_\infty B = \lambda_\infty (A \times B)$. *In particular* $\mathbb{D} \otimes \mathbb{D} = \lambda_\infty (\mathbb{N} \times \mathbb{N})$.

Proof. Since $(\lambda_\infty A \otimes \lambda_\infty B)^\perp = (\lambda_\infty A \multimap (\lambda_\infty B)^\perp) = (\lambda_\infty A \multimap \lambda_1 B)$, an element of $\mathsf{P}(\lambda_\infty A \otimes \lambda_\infty B)^\perp$ is a $t \in \mathbb{R}_{\geq 0}^{A\times B}$ such that $t \cdot \sum_{a\in A} \mathsf{e}_a \in \mathsf{P}(\lambda_1 B)$ which means that $\sum_{(a,b)\in A\times B} t_{a,b} \leq 1$, that is $t \in \lambda_1(A \times B)$ which proves our contention. ☐

Let $\varphi : \mathbb{N}^2 \to \mathbb{N}$ be an injection, and let $k \in \mathbb{R}_{\geq 0}^{|\mathbb{D}\multimap\mathbb{D}\otimes\mathbb{D}|}$ be defined by $k_{l,(i,j)} = \delta_{l,\varphi(i,j)}$, it results from Lemma 33 that $k \in \mathbf{Pcoh}(\mathbb{D}, \mathbb{D}\otimes\mathbb{D})$. So **Pcoh** satisfies (ℙ-flat).

An element of $\mathbf{Pcoh}(\mathbb{D}, X)$ is an $s \in \mathbb{R}_{\geq 0}^{\mathbb{N}\times|X|}$ such that $\sum_{i\in\mathbb{N}} s \cdot \mathsf{e}_i \in \mathsf{P}X$, that is, a family $(x(i) \in \mathsf{P}X)_{i\in\mathbb{N}}$ is summable iff $\sum_{i\in\mathbb{N}} x(i) \in \mathsf{P}X$. The fact that **Pcoh** satisfies (ℙ-wit) follows easily from this observation. Let indeed $(s(i) \in \mathbf{Pcoh}(\mathbb{D}, X))_{i\in\mathbb{N}}$ be such that $(\sum_{j\in\mathbb{N}} s(i) \cdot \mathsf{e}_j \in \mathsf{P}X)_{i\in\mathbb{N}}$ is summable, that is $(\sum_{i,j\in\mathbb{N}} s(i)_{j,a})_{a\in|X|} \in \mathsf{P}X$. Let then $t \in \mathbb{R}_{\geq 0}^{(\mathbb{N}\times\mathbb{N})\times|X|}$ be given by $t_{(i,j),a} = s(i)_{j,a}$, we have $t \in \mathbf{Pcoh}(\mathbb{D}\otimes\mathbb{D}, X)$ by Lemma 33 and $t\,(\overline{\pi}_i\otimes\mathbb{D})\,\lambda^{-1} = s(i)$ for each $i \in \mathbb{N}$ which shows that $(s(i))_{i\in\mathbb{N}}$ is summable.

So **Pcoh** is an elementary ω-summable category.

The bimonoid structure of $\mathbb{D}$ is identical to that of $\mathbb{D}$ in **Scs**, replacing relations with $\{0,1\}$-valued matrices. For instance $\widetilde{\theta} \in \mathbf{Pcoh}(\mathbb{D}, \mathbb{D}\otimes\mathbb{D})$ is the element of $\mathbb{R}_{\geq 0}^{\mathbb{N}\times(\mathbb{N}\times\mathbb{N})}$ such that $\widetilde{\theta}_{(n,(i,j))} = \delta_{n,i+j}$ and $\Delta \in \mathbf{Pcoh}(1, \mathbb{D})$ is the element of $\mathbb{R}_{\geq 0}^{\{*\}\times\mathbb{N}}$ such that $\Delta_{*,i} = 1$ for all $i \in \mathbb{N}$ (the diagonal morphism of $\mathbb{D} = \mathop{⅋}_{i\in\mathbb{N}} 1$).

By Theorem 23 the category **Pcoh** has a canonical Taylor structure $\widetilde{\partial} \in \mathbf{Pcoh}(\mathbb{D}, !\mathbb{D})$, which is given by $\widetilde{\partial}_{n,[i_1,\dots,i_k]} = \delta_{n,i_1+\dots+i_k}$.

This coalgebra structure induces the distributive law $\partial_X = \mathsf{cur}(\partial'_X) \in \mathbf{Scs}(!\mathsf{S}X, \mathsf{S}!X)$ where $\partial'_X \in \mathbf{Scs}(!(\mathbb{D}\multimap X)\otimes\mathbb{D}, !X)$ is

$$!(\mathbb{D}\multimap X)\otimes\mathbb{D} \xrightarrow{!(\mathbb{D}\multimap X)\otimes\widetilde{\partial}} !(\mathbb{D}\multimap X)\otimes !\mathbb{D} \xrightarrow{\mu^2} !((\mathbb{D}\multimap X)\otimes\mathbb{D}) \xrightarrow{!\mathsf{ev}} !X$$

One checks very easily that

$$(\mu^2\,(!(\mathbb{D}\multimap X)\otimes\widetilde{\partial}))_{([(i_1,a_1),\dots,(i_k,a_k)],n),q} = \begin{cases} 1 & \text{if } q = [((a_1,i_1),j_1),\dots,((a_k,i_k),j_k)] \\ & \text{with } j_1,\dots,j_k \in \mathbb{N} \text{ such that } j_1+\dots+j_k = n \\ 0 & \text{otherwise,} \end{cases}$$

observe that this morphism $\mu^2\,(!(\mathbb{D}\multimap X)\otimes\widetilde{\partial})$ has only 0 and 1 coefficients.

Notice that $\mathsf{ev} \in \mathbf{Pcoh}((\mathbb{D}\multimap X)\otimes\mathbb{D}, X)$ is characterized by $\mathsf{ev}_{((i,a),j),b} = \delta_{i,j}\delta_{a,b}$ and hence if $!\mathsf{ev}_{q,m} \neq 0$ with $q \in \mathcal{M}_{\mathrm{fin}}((\mathbb{N}\times|X|)\times\mathbb{N})$ and $m \in \mathcal{M}_{\mathrm{fin}}(|X|)$, we must have $m = [a_1,\dots,a_k]$ and $q = [((i_1,a_1),i_1),\dots,((i_k,a_k),i_k)]$ for some $k \in \mathbb{N}$, $a_1,\dots,a_k \in |X|$ and $i_1,\dots,i_k \in \mathbb{N}$. For such multisets m and q, the set $\mathsf{L}(q,m)$ has exactly one element $r \in \mathcal{M}_{\mathrm{fin}}(((\mathbb{N}\times|X|)\times\mathbb{N})\times|X|)$ such that $\mathsf{ev}^r \neq 0$, namely $r = [(((i_1,a_1),i_1),a_1),\dots,(((i_k,a_k),i_k),a_k)]$, and we have therefore

$$!\mathsf{ev}_{q,m} = \begin{bmatrix} m \\ r \end{bmatrix} = \prod_{a\in|X|} \frac{m(a)!}{\prod_{i\in\mathbb{N}} p(i,a)!} = \frac{m!}{p!} \in \mathbb{N}$$

where $p \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N}\times|X|)$ is defined by $p(i,a) = q((i,a),i) = r(((i,a),i),a)$. It follows that, for $p \in \mathcal{M}_{\mathrm{fin}}(\mathbb{N}\times|X|)$, $n \in \mathbb{N}$ and $m = [a_1,\dots,a_k] \in \mathcal{M}_{\mathrm{fin}}(|X|)$, we have

$$\partial_{p,(n,m)} = \begin{cases} \frac{m!}{p!} & \text{if } p = [(i_1,a_1),\dots,(i_k,a_k)] \text{ with } i_1+\dots+i_k = n \\ 0 & \text{otherwise.} \end{cases}$$

Remark 30. It is interesting, and a bit puzzling, to observe that the numerical coefficients associated with the computation of derivatives (the 42 in the derivative $42x^{41}$ of x^{42}) are generated, in the definition of ∂, by the exponential combined with the evaluation map when computing $!\mathsf{ev}$ — which btw. is a purely LL morphism, not using any differential structure —, and not by the $\widetilde{\partial}$ morphism itself, which seems to be "the truly differential part" of this definition

It follows that the functor $\mathsf{T} : \mathbf{Pcoh}_! \to \mathbf{Pcoh}_!$, which maps a PCS X to $\mathsf{S}X$ (such that $|\mathsf{S}X| = \mathbb{N} \times |X|$ and $\mathsf{P}(\mathsf{S}X) \simeq \{\overrightarrow{x} \in \mathsf{P}X^{\mathbb{N}} \mid \sum_{i=0}^{\infty} x(i) \in \mathsf{P}X\}$) maps a morphism $s \in \mathbf{Pcoh}_!(X,Y)$ to $t = \mathsf{T}(s) \in \mathbf{Pcoh}_!(\mathsf{S}X, \mathsf{S}Y)$ which is given by

$$\mathsf{T}(s)_{[(i_1,a_1),\dots,(i_k,a_k)],(n,b)} = \delta_{n,i_1+\cdots+i_k} \frac{[a_1,\dots,a_k]!}{[(i_1,a_1),\dots,(i_k,a_k)]!} s_{[a_1,\dots,a_k],b}\,.$$

Notice that a multiset $p = [(i_1,a_1),\dots,(i_k,a_k)]$ can be written *in a unique way* $p = j_1 \cdot m_1 + \cdots + j_l \cdot m_l$ where $j_1 < \cdots < j_l \in \mathbb{N}$ and $(m_i \in \mathcal{M}^+_{\mathrm{fin}}(|X|))_{i=1}^l$ are such that $m_1 + \cdots + m_l = m = [a_1,\dots,a_k]$. The condition that the m_i's are non-empty is crucial for this enumeration to be bijective: the only way to get $p = [\,]$ is by choosing $l = 0$. It follows that the associated function $\widehat{\mathsf{T}(s)} : \mathsf{P}(\mathsf{S}X) \to \mathsf{P}(\mathsf{S}Y)$ satisfies, for each $\overrightarrow{x} \in \mathsf{P}(\mathsf{S}X)$ and $(n,b) \in |\mathsf{S}Y|$

$$\widehat{\mathsf{T}(s)}(\overrightarrow{x})_{n,b} = \sum_{l=0}^{\infty} \sum_{\overrightarrow{m} \in \mathcal{M}^+_{\mathrm{fin}}(|X|)^l} \sum_{\substack{j_1<\cdots<j_l \in \mathbb{N} \\ j_1\#m_1+\cdots+j_l\#m_l=n}} \mathsf{mn}(\overrightarrow{m}) s_{m_1+\cdots+m_l,b} x(j_1)^{m_1} \cdots x(j_l)^{m_l}$$

where we recall that $\mathsf{mn}(\overrightarrow{m}) = \frac{(m_1+\cdots+m_l)!}{m_1!\cdots m_l!} \in \mathbb{N}$ is the multinomial coefficient of the sequence $\overrightarrow{m}$ of multisets.

For $n = 0$, there are two ways to fulfill the condition $j_1\#m_1 + \cdots + j_l\#m_l = n$: either with $l = 0$, or with $l = 1$ and $j_1 = 0$, and then m_1 can be any element of $\mathcal{M}^+_{\mathrm{fin}}(|X|)$. Therefore, we have

$$\widehat{\mathsf{T}(s)}(\overrightarrow{x})_{0,b} = s_{[\,],b} + \sum_{m \in \mathcal{M}^+_{\mathrm{fin}}(|X|)} s_{m,b} x(0)^m = \widehat{s}(x(0))_b$$

so that $\widehat{\mathsf{T}(s)}(\overrightarrow{x})(0) = \widehat{s}(x(0))$.

The map $\widehat{s}$ has derivatives of all orders; more precisely, for $d \in \mathbb{N}^+$ and $x, u^1, \dots, u^d \in \mathsf{P}X$ such that $x + u^1 + \cdots + u^d \in \mathsf{P}X$, the d-th derivative computed at x and d-linearly applied to $(u^1,\dots,u^d)$ is $\widehat{s}^{(d)}(x)(u^1,\dots,u^d) \in \mathsf{P}Y$ given by

$$\widehat{s}^{(d)}(x)(u^1,\dots,u^d)_b = \sum_{m \in \mathcal{M}_{\mathrm{fin}}(|X|)} \sum_{\overrightarrow{a} \in |X|^d} \frac{(m + [\overrightarrow{a}])!}{m!} s_{m+[\overrightarrow{a}],b} x^m u^1_{a_1} \cdots u^d_{a_d}$$

by iterating the computation shown in [Ehr22b] and [Ehr23] for the first order derivative.

Theorem 24. *For all* $n \in \mathbb{N}^+$

$$\widehat{\mathsf{T}(s)}(\overrightarrow{x})_{n,b} = \sum_{k=0}^{\infty} \sum_{\overrightarrow{n} \in (\mathbb{N}^+)^k} \sum_{\substack{0<i_1<\cdots<i_k \in \mathbb{N} \\ n_1 i_1+\cdots+n_k i_k=n}} \frac{1}{\prod_{i=1}^k n_i!} \widehat{s}^{(n_1+\cdots+n_k)}(x(0))(\overline{x(i_1)}^{n_1}, \dots, \overline{x(i_k)}^{n_k})$$

where $\overline{x}^l$ *is the list of repeated arguments* $\overbrace{x,\dots,x}^{n\times}$.

The right-hand expression above is the same as Equation (9) upon identifying a pair $(\overrightarrow{i}, \overrightarrow{n}) \in \mathbb{N}^k \times (\mathbb{N}^+)^k$ such that $0 < i_1 < \cdots < i_k$ with the element m of $\mathcal{M}_{\text{fin}}(\mathbb{N}^+)$ such that $\mathsf{supp}(m) = \{i_1, \dots, i_k\}$, which maps i_j to n_j for $j = 1, \dots, k$. We explained in Section 2.1 how this formula indeed corresponds to a compositional formulation of Taylor expansion, thanks to the Faà di Bruno formula.

Proof. Let v be the right-hand expression, we have

$$\begin{aligned}
v &= \sum_{k=0}^{\infty} \sum_{\overrightarrow{n} \in (\mathbb{N}^+)^k} \sum_{\substack{0<i_1<\cdots<i_k \in \mathbb{N} \\ n_1 i_1 + \cdots + n_k i_k = n}} \frac{1}{\prod_{i=1}^k n_i!} \sum_{m \in \mathcal{M}_{\text{fin}}(|X|)} \\
&\qquad \sum_{\overrightarrow{a} \in |X|^{n_1+\cdots+n_k}} \frac{(m + [\overrightarrow{a}])!}{m!} s_{m+[\overrightarrow{a}],b} x(0)^m \prod_{j=1}^{n_1} x(i_1)_{a_j} \prod_{j=n_1+1}^{n_2} x(i_2)_{a_j} \cdots \prod_{j=n_1+\cdots+n_{k-1}+1}^{n_1+\cdots+n_k} x(i_k)_{a_j} \\
&= \sum_{k=0}^{\infty} \sum_{\overrightarrow{n} \in (\mathbb{N}^+)^k} \sum_{\substack{0<i_1<\cdots<i_k \in \mathbb{N} \\ n_1 i_1 + \cdots + n_k i_k = n}} \frac{1}{\prod_{i=1}^k n_i!} \sum_{m \in \mathcal{M}_{\text{fin}}(|X|)} \\
&\qquad \sum_{\substack{\overrightarrow{m} \in \mathcal{M}_{\text{fin}}(|X|)^k \\ (\#m_i = n_i)_{i=1}^k}} \frac{(m + m_1 + \cdots + m_k)!}{m!} s_{m+m_1+\cdots+m_k,b} x(0)^m x(i_1)^{m_1} \cdots x(i_k)^{m_k} \prod_{i=1}^k \frac{n_i!}{m_i!}
\end{aligned}$$

because $\frac{n_i!}{m_i!}$ is the number of enumerations of the elements of m_i (taking multiplicities into account). So we have

$$\begin{aligned}
v &= \sum_{k=0}^{\infty} \sum_{\overrightarrow{n} \in (\mathbb{N}^+)^k} \sum_{\substack{0<i_1<\cdots<i_k \in \mathbb{N} \\ n_1 i_1 + \cdots + n_k i_k = n}} \sum_{m \in \mathcal{M}_{\text{fin}}(|X|)} \\
&\qquad \sum_{\substack{\overrightarrow{m} \in \mathcal{M}_{\text{fin}}(|X|)^k \\ (\#m_i = n_i)_{i=1}^k}} \mathsf{mn}(m, \overrightarrow{m}) s_{m+m_1+\cdots+m_k,b} x(0)^m x(i_1)^{m_1} \cdots x(i_k)^{m_k} \\
&= \sum_{k=0}^{\infty} \sum_{m \in \mathcal{M}^+_{\text{fin}}(|X|)} \sum_{\overrightarrow{m} \in \mathcal{M}^+_{\text{fin}}(\overrightarrow{X})^k} \sum_{\substack{0<i_1<\cdots<i_k \in \mathbb{N} \\ \#m_1 i_1 + \cdots + \#m_k i_k = n}} \mathsf{mn}(m, \overrightarrow{m}) s_{m+m_1+\cdots+m_k,b} x(0)^m x(i_1)^{m_1} \cdots x(i_k)^{m_k} \\
&\qquad + \sum_{k=0}^{\infty} \sum_{\overrightarrow{m} \in \mathcal{M}^+_{\text{fin}}(\overrightarrow{X})^k} \sum_{\substack{0<i_1<\cdots<i_k \in \mathbb{N} \\ \#m_1 i_1 + \cdots + \#m_k i_k = n}} \mathsf{mn}([\,], \overrightarrow{m}) s_{m_1+\cdots+m_k,b} x(i_1)^{m_1} \cdots x(i_k)^{m_k} \\
&= \sum_{k=1}^{\infty} \sum_{\overrightarrow{m} \in \mathcal{M}^+_{\text{fin}}(\overrightarrow{X})^k} \sum_{\substack{i_1=0<i_2<\cdots<i_k \in \mathbb{N} \\ \#m_1 i_1 + \cdots + \#m_k i_k = n}} \mathsf{mn}(m, \overrightarrow{m}) s_{m+m_1+\cdots+m_k,b} x(i_1)^{m_1} \cdots x(i_k)^{m_k} \\
&\qquad + \sum_{k=0}^{\infty} \sum_{\overrightarrow{m} \in \mathcal{M}^+_{\text{fin}}(\overrightarrow{X})^k} \sum_{\substack{0<i_1<\cdots<i_k \in \mathbb{N} \\ \#m_1 i_1 + \cdots + \#m_k i_k = n}} \mathsf{mn}([\,], \overrightarrow{m}) s_{m_1+\cdots+m_k,b} x(i_1)^{m_1} \cdots x(i_k)^{m_k} \\
&= \widehat{\mathsf{T}(s)}(\overrightarrow{x}) . \square
\end{aligned}$$

It follows in particular that $\widehat{\mathsf{T}(s)}(x, 0, 0, \dots) = (\widehat{s}(x), 0, 0, \dots)$ and that, as motivated in

Section 2.1, we recover all the terms of the usual Taylor expansion by taking

$$\widehat{\mathsf{T}(s)}(x,u,0,0,\dots) = \big(\frac{1}{n!}\widehat{s}^{(n)}(x)(\overline{u}^n)\big)_{n\in\mathbb{N}}$$

and the models makes it explicit that these terms are summable, because x and u are.

6 Annex: distributive laws

The goal of this section is to study under which condition it is possible to extend some structure on a category to the Kleisli category of a monad or a comonad on this category.

If $\underline{M} = (M,\eta,\mu)$ is a monad on $\mathcal{C}$, then its Kleisli category $\mathcal{C}_{\underline{M}}$ is the category with the same objects as $\mathcal{C}$ and such that $\mathcal{C}_{\underline{M}}(X,Y) = \mathcal{C}(X,MY)$. The identity is given by $\eta_X \in \mathcal{C}_{\underline{M}}(X,X)$ and the composition of $f \in \mathcal{C}_{\underline{M}}(X,Y)$ with $g \in \mathcal{C}_{\underline{M}}(Y,Z)$ is defined as $g \circ_{\underline{M}} f = \mu_Y \circ Mg \circ f$. There is a functor $K_{\underline{M}} : \mathcal{C} \to \mathcal{C}_{\underline{M}}$ such that $K_{\underline{M}}X = X$ and for any $f \in \mathcal{C}(X,Y)$, $K_{\underline{M}}f = \eta \circ f \in \mathcal{C}_{\underline{M}}(X,Y)$. Then for any $f \in \mathcal{C}_{\underline{M}}(X',X)$, $h \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}_{\underline{M}}(Y,Y')$,

$$K_{\underline{M}}h \circ_{\underline{M}} f = Mh \circ f \qquad g \circ_{\underline{M}} K_{\underline{M}}h = g \circ h \tag{15}$$

These equations are obtained by a straightforward computation using naturality and the triangle identities of the monad.

The co-Kleisli category of a comonad $\overline{D}$ is defined in a dual way. Let $\underline{M}_1$ and $\underline{M}_2$ be two monads (or two comonads) on a category $\mathcal{C}_1$ and $\mathcal{C}_2$. We will write $\mathcal{C}_{\underline{M}_1}$ and $\mathcal{C}_{\underline{M}_2}$ the Kleisli category of $\underline{M}_1$ and $\underline{M}_2$ respectively (in order to avoid overloading the indexes).

Definition 53. A functor $\widehat{F} : \mathcal{C}_{\underline{M}_1} \to \mathcal{C}_{\underline{M}_2}$ is an extension of the functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ if the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \\ {\scriptstyle K_{\underline{M}_1}}\downarrow & & \downarrow{\scriptstyle K_{\underline{M}_2}} \\ \mathcal{C}_{\underline{M}_1} & \xrightarrow[\widehat{F}]{} & \mathcal{C}_{\underline{M}_2} \end{array}$$

That is, for any $X \in \mathbf{Obj}(\mathcal{C}_1)$, $\widehat{F}X = FX$, and for any $f \in \mathcal{C}_1(X,Y)$, $\widehat{F}K_{\underline{M}_1}f = K_{\underline{M}_2}Ff$.

Definition 54. Let $F,G : \mathcal{C}_1 \to \mathcal{C}_2$ with respective extensions $\widehat{F},\widehat{G} : \mathcal{C}_{\underline{M}_1} \to \mathcal{C}_{\underline{M}_2}$. Let $\alpha : F \Rightarrow G$ be a natural transformation. Then $K_{\underline{M}_2}\alpha_X \in \mathcal{C}_{\underline{M}_2}(\widehat{F}X,\widehat{G}X)$. We say that α extends to $\widehat{F}$ and $\widehat{G}$ if $K_{\underline{M}_2}\alpha$ is a natural transformation $\widehat{F} \Rightarrow \widehat{G}$.

This notion of structure extension is deeply related to the notion of distributive laws. Distributive law were introduced in [Str72]. See [PW02] for a more accessible yet comprehensive introduction in the general 2-categorical setting.

6.1 Distributive law between a monad and a functor

Throughout this section, we always assume that $\underline{M}_1 = (M_1,\eta^1,\mu^1)$ is a monad on a category $\mathcal{C}_1$, $\underline{M}_2 = (M_2,\eta^2,\mu^2)$ is a monad on a category $\mathcal{C}_2$, and $\underline{M}_3 = (M_3,\eta^3,\mu^3)$ is a monad on a category $\mathcal{C}_3$.

Definition 55. A natural transformation $\lambda : F\underline{M}_1 \Rightarrow \underline{M}_2F$ is a *distributive law* of F on

two monads $\underline{M}_1$ and $\underline{M}_2$ if the following two diagrams commute.

$$\begin{array}{ccc} F & & \\ {\scriptstyle F\eta^1}\downarrow & \searrow^{\eta^2 F} & \\ FM_1 & \xrightarrow[\lambda]{} & M_2F \end{array} \qquad \begin{array}{ccccc} FM_1^2 & \xrightarrow{\lambda M_1} & M_2FM_1 & \xrightarrow{M_2\lambda} & M_2^2F \\ {\scriptstyle F\mu^1}\downarrow & & & & \downarrow{\scriptstyle \mu^2 F} \\ FM_1 & & \xrightarrow[\lambda]{} & & M_2F \end{array}$$

Remark 31. We write $F\underline{M}_1 \Rightarrow \underline{M}_2F$ instead of $FM_1 \Rightarrow M_2$ because it makes obvious what part is playing the role of the monad. We will keep this convention through the article.

Definition 56. Let $\lambda^F : F\underline{M}_1 \Rightarrow \underline{M}_2F$ and $\lambda^G : G\underline{M}_1 \Rightarrow \underline{M}_2G$ two distributive laws. A morphism between λ^F and λ^G is a natural transformation $\alpha : F \Rightarrow G$ such that the diagram below commutes.

$$\begin{array}{ccc} FM_1 & \xrightarrow{\lambda^F} & M_2F \\ {\scriptstyle \alpha M_1}\downarrow & & \downarrow{\scriptstyle M_2\alpha} \\ GM_1 & \xrightarrow[\lambda^G]{} & M_2G \end{array}$$

The theorem below is a well known result, that is proved in [PW02] for example.

Theorem 25. *Any extension $\widehat{F} : \mathcal{C}_{\underline{M}_1} \to \mathcal{C}_{\underline{M}_2}$ induces a distributive law $\lambda : F\underline{M}_1 \Rightarrow \underline{M}_2F$. This law is given by the image of $\mathsf{id}_{M_1X} \in \mathcal{C}_{\underline{M}_1}(M_1X, X)$ by $\widehat{F}$*

$$\lambda_X := \widehat{F}(\mathsf{id}_{M_1X}) \in \mathcal{C}_{\underline{M}_2}(FM_1X, FX) = \mathcal{C}_2(FM_1X, M_2FX).$$

Conversely, any distributive law $\lambda : F\underline{M}_1 \Rightarrow \underline{M}_2F$ induces an extension $\widehat{F} : \mathcal{C}_{\underline{M}_1} \to \mathcal{C}_{\underline{M}_2}$ that maps an object X to $\widehat{F}X := FX$ and a morphism $f \in \mathcal{C}_{\underline{M}_1}(X, Y) = \mathcal{C}_1(X, M_1Y)$ to

$$\widehat{F}f := \quad FX \xrightarrow{Ff} FM_1Y \xrightarrow{\lambda^F_Y} M_2FY \quad \in \mathcal{C}_{\underline{M}_2}(FX, FY)$$

Those two constructions are inverse of each other, so there is a bijection between extensions and distributive laws.

The result below is also proved in [PW02], but it seems to be less considered in the literature. Although the proof is simple, this result will be crucial in the development of this article.

Theorem 26. *Let $\lambda^F : F\underline{M}_1 \Rightarrow \underline{M}_2F$ and $\lambda^G : G\underline{M}_1 \Rightarrow \underline{M}_2G$ be two distributive laws and let $\widehat{F}, \widehat{G}$ be their associated extensions. Then a natural transformation $F \Rightarrow G$ is a morphism between the distributive laws λ^F and λ^G if and only if it extends to $\widehat{F}$ and $\widehat{G}$.*

Proof. The naturality of $K_{\underline{M}}\alpha$ means that for any $f \in \mathcal{C}_{\underline{M}}(X, Y)$,

$$\widehat{G}f \circ_{\underline{M}_2} K_{\underline{M}_2}(\alpha_X) = K_{\underline{M}_2}(\alpha_Y) \circ_{\underline{M}_2} \widehat{F}f$$

By Eq. (15) and by definition of $\widehat{F}$ and $\widehat{G}$ from λ^F and λ^G, this is equivalent to the equation

$$\lambda^G_Y \circ Gf \circ \alpha_X = M_2\alpha_Y \circ \lambda^F_Y \circ Ff$$

By naturality of α in $\mathcal{C}$, this is equivalent to the equation

$$\lambda^G_Y \circ \alpha_Y \circ Ff = M_2\alpha_Y \circ \lambda^F_Y \circ Ff$$

It follows easily that $K_{\underline{M}}\alpha$ is natural if and only if α is a morphism of distributive law. □

Remark 32. We can define a category **Ext** whose objects are the monads $\underline{M}_1$ on small categories[10], and whose morphisms $\mathbf{Ext}(\underline{M}_1, \underline{M}_2)$ are the pairs $(F, \widehat{F})$ where $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor and $\widehat{F} : \mathcal{C}_{\underline{M}_1} \to \mathcal{C}_{\underline{M}_2}$ is an extension of F. The identity arrow is the pair $(\mathsf{Id}_{\mathcal{C}_1}, \mathsf{Id}_{\mathcal{C}_{\underline{M}_1}}) \in \mathbf{Ext}(\underline{M}_1, \underline{M}_1)$. The composition of $(F, \widehat{F}) \in \mathbf{Ext}(\underline{M}_1, \underline{M}_2)$ with $(G, \widehat{G}) \in \mathbf{Ext}(\underline{M}_2, \underline{M}_3)$ is defined as the pair $(GF, \widehat{G}\widehat{F}) \in \mathbf{Ext}(\underline{M}_1, \underline{M}_3)$ (we can check that $\widehat{G}\widehat{F}$ is an extension of GF). In fact, **Ext** is a 2-category. A 2-cell $(F, \widehat{F}) \Rightarrow (G, \widehat{G})$ is a natural transformation $\alpha : F \Rightarrow G$ that extends to $\widehat{F}$ and $\widehat{G}$ in the sense of Definition 54.

We can also define a category $\mathbf{Mon}^*$ whose objects are also the monads $\underline{M}$ on small categories. A morphism between $\underline{M}_1$ and $\underline{M}_2$ consists in a pair (F, λ^F) where $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor and $\lambda^F : F\underline{M}_1 \Rightarrow \underline{M}_2 F$ is a distributive law. The identity on $\underline{M}_1$ is the pair $(\mathsf{Id}, \mathsf{id}M_1)$ where $\mathsf{Id} : \mathcal{C}_1 \to \mathcal{C}_1$ is the identity functor and $(\mathsf{id}M_1)_X = \mathsf{id}_{M_1 X} \in \mathcal{C}_1(M_1X, M_1X)$. Given $(F, \lambda^F) \in \mathbf{Mon}^*(\underline{M}_1, \underline{M}_2)$ and $(G, \lambda^G) \in \mathbf{Mon}^*(\underline{M}_2, \underline{M}_3)$, we can define their composition as (GF, λ^{GF}) where λ^{GF} is defined with the following pasting diagram.

$$\begin{array}{ccccc}
\mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 & \xrightarrow{G} & \mathcal{C}_3 \\
{\scriptstyle M_1}\downarrow & \swarrow_{\lambda^F} & \downarrow{\scriptstyle M_2} & \swarrow_{\lambda^G} & \downarrow{\scriptstyle M_3} \\
\mathcal{C}_1 & \xrightarrow[F]{} & \mathcal{C}_2 & \xrightarrow[G]{} & \mathcal{C}_3
\end{array}$$

In other word, $\lambda^{GF} = \lambda^G F \circ G\lambda^F$. That is,

$$\lambda^{GF} := \; GFM_1 \xrightarrow{G\lambda^F} GM_2F \xrightarrow{\lambda^G F} M_3GF$$

This category $\mathbf{Mon}^*$ is a 2-category. Its 2-cells are the morphisms of distributive laws.

Then we can check that if $\widehat{F}$ is the extension of F associated to λ^F and $\widehat{G}$ is the extension of G associated to λ^G, then λ^{GF} defined above is the distributive law associated to the extension $\widehat{G}\widehat{F}$. In other word, the bijection of Theorem 25 is an isomorphism between the categories $\mathbf{Mon}^*$ and **Ext**. Besides, Theorem 26 ensures that this isomorphism is an isomorphism of 2-categories.

6.2 Distributive law between a comonad and a functor

Similar results stand for comonads. Throughout this section, we always assume that $\overline{D}_1 = (D_1, \epsilon^1, \delta^1)$ is a comonad on a category $\mathcal{C}_1$, $\overline{D}_2 = (D_2, \epsilon^2, \delta^2)$ is a comonad on a category $\mathcal{C}_2$, and $\overline{D}_3 = (D_3, \epsilon^3, \delta^3)$ is a comonad on a category $\mathcal{C}_3$. The notion of structure extension is also related to a distributive law, except that this time around the natural transformation is the dual of the one of the previous section and is of shape $\overline{D}_2 F \Rightarrow F\overline{D}_1$.

Definition 57. A natural transformation $\lambda : \overline{D}_2 F \Rightarrow F\overline{D}_1$ is a *distributive law* of F on two comonads $\overline{D}_1$ and $\overline{D}_2$ if the following two diagrams commute.

$$\begin{array}{ccc}
D_2F & \xrightarrow{\lambda} & FD_1 \\
& {\scriptstyle \epsilon^2 F}\searrow & \downarrow{\scriptstyle F\epsilon^1} \\
& & F
\end{array}
\qquad\qquad
\begin{array}{ccccc}
D_2F & & \xrightarrow{\qquad\lambda\qquad} & & FD_1 \\
{\scriptstyle \delta^2 F}\downarrow & & & & \downarrow{\scriptstyle F\delta^2} \\
D_2^2F & \xrightarrow[D_2\lambda]{} & D_2FD_1 & \xrightarrow[\lambda D_2]{} & FD_1^2
\end{array}$$

[10] For foundational issues

Definition 58. Let $\lambda^F : \overline{D}_2 F \Rightarrow F\overline{D}_1$ and $\lambda^G : \overline{D}_2 G \Rightarrow G\overline{D}_1$ two distributive laws. A morphism between λ^F and λ^G is a natural transformation $\alpha : F \Rightarrow G$ such that the diagram below commutes.

$$\begin{array}{ccc} D_2F & \xrightarrow{\lambda^F} & FD_1 \\ {\scriptstyle D_2\alpha}\downarrow & & \downarrow{\scriptstyle \alpha D_1} \\ D_2G & \xrightarrow[\lambda^G]{} & GD_1 \end{array}$$

Theorem 27. *Any extension $\widehat{F} : \mathcal{C}_{\overline{D}_1} \to \mathcal{C}_{\overline{D}_2}$ induces a distributive law $\lambda : \overline{D}_2 F \Rightarrow F\overline{D}_1$. This law is given by the image of $\mathsf{id}_{D_1X} \in \mathcal{C}_{\overline{D}_1}(X, D_1X)$ by $\widehat{F}$*

$$\lambda_X := \widehat{F}(\mathsf{id}_{D_1X}) \in \mathcal{C}_{\overline{D}_2}(FX, FD_1X) = \mathcal{C}_2(D_2FX, FD_1X).$$

Conversely, any distributive law $\lambda : \overline{D}_2 F \Rightarrow F\overline{D}_1$ induces an extension $\widehat{F} : \mathcal{C}_{\overline{D}_1} \to \mathcal{C}_{\overline{D}_2}$ that maps an object X to $\widehat{F}X := FX$ and a morphism $f \in \mathcal{C}_{\overline{D}_1}(X, Y) = \mathcal{C}_1(D_1X, Y)$ to

$$\widehat{F}f := \; D_2FX \xrightarrow{\lambda_X} FD_1X \xrightarrow{Ff} FY \;\; \in \mathcal{C}_{\underline{M}_2}(FX, FY)$$

Those two constructions are inverse of each other, so there is a bijection between extensions and distributive laws.

Theorem 28. *Let $\lambda^F : \overline{D}_2 F \Rightarrow F\overline{D}_1$ and $\lambda^G : \overline{D}_2 G \Rightarrow G\overline{D}_1$ be two distributive laws and let $\widehat{F}, \widehat{G}$ be their associated extensions. Then a natural transformation $F \Rightarrow G$ is a morphism between the distributive laws λ^F and λ^G if and only if it extends to $\widehat{F}$ and $\widehat{G}$.*

Remark 33. We can again define a 2-category **coExt** in the exact same way as **Ext**, except that the objects are now comonads. We can also define a 2-category $\mathbf{coMon}^*$ similar to $\mathbf{Mon}^*$ whose objects are the comonads $\overline{D}$ on small categories, whose morphisms are the pairs (F, λ^F) where $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor and $\lambda^F : \overline{D}_2 F \Rightarrow F\overline{D}_1$ is a distributive law, and whose 2-cells are the morphisms of distributive laws. The composition of $(F, \lambda^F) \in \mathbf{coMon}^*(\overline{D}_1, \overline{D}_2)$ with $(G, \lambda^G) \in \mathbf{coMon}^*(\overline{D}_2, \overline{D}_3)$ is defined as (GF, λ^{GF}) where λ^{GF} is defined as the following pasting diagram.

$$\begin{array}{ccccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 & \xrightarrow{G} & \mathcal{C}_3 \\ {\scriptstyle D_1}\downarrow & \swarrow{\scriptstyle \lambda^F} & {\scriptstyle D_2}\downarrow & \swarrow{\scriptstyle \lambda^G} & \downarrow{\scriptstyle D_3} \\ \mathcal{C}_1 & \xrightarrow[F]{} & \mathcal{C}_2 & \xrightarrow[G]{} & \mathcal{C}_3 \end{array}$$

In other word, $\lambda^{GF} = G\lambda^F \circ \lambda^G F$. That is,

$$\lambda^{GF}_X := \; D_3GF \xrightarrow{\lambda^G F} GD_2F \xrightarrow{G\lambda^F} GFD_1 \; .$$

Again, we can check that if $\widehat{F}$ is the extension of F associated to λ^F and $\widehat{G}$ is the extension of G associated to λ^G, then λ^{GF} defined above is the distributive law associated to the extension $\widehat{G}\widehat{F}$. In other word, the bijection of Theorem 27 is an isomorphism between the categories $\mathbf{coMon}^*$ and **coExt**. Besides, Theorem 28 ensures that this isomorphism is an isomorphism of 2-categories.

Distributive law	Role of H and K	Type of extension
$FH \Rightarrow KF$	Monads	Extension of F to Kleisli
$FH \Rightarrow KF$	Comonads	Liftings of F to co-Eilenberg-Moore
$KF \Rightarrow FH$	Monads	Liftings of F to Eilenberg-Moore
$KF \Rightarrow FH$	Comonads	Extension of F to co-Kleisli

Figure 2: Configurations for distributive laws

6.3 Distributive law between a monad and a comonad

Let $\underline{M} = (M, \eta, \mu)$ be a monad on a category $\mathcal{C}$, and $\overline{D} = (D, \epsilon, \delta)$ be a comonad on $\mathcal{C}$.

Definition 59. A distributive law of the comonad $\overline{D}$ on the monad $\underline{M}$ is a natural transformation $\lambda : \overline{D}\underline{M} \Rightarrow \underline{M}\overline{D}$ that is both a distributive law $\lambda : D\underline{M} \Rightarrow \underline{M}D$ (Definition 55) and a distributive law $\lambda : \overline{D}M \Rightarrow M\overline{D}$ (Definition 57).

Note that the following assertions are equivalent

- λ is a distributive law of the comonad $\overline{D}$ on the monad $\underline{M}$
- λ is a distributive law $D\underline{M} \Rightarrow \underline{M}D$ and ϵ and δ are morphisms of distributive laws
- λ is a distributive law $\overline{D}M \Rightarrow M\overline{D}$ and η and μ are morphisms of distributive laws

By Theorems 27 and 28, a distributive law between a monad and a comonad ensures that $\underline{M}$ extends to a monad $\widehat{\underline{M}}$ on $\mathcal{C}_{\overline{D}}$. By Theorems 25 and 26, it also ensures that $\overline{D}$ extends to a comonad $\widehat{\overline{D}}$ on $\mathcal{C}_{\underline{M}}$. We can also check that $(\mathcal{C}_{\overline{D}})_{\widehat{\underline{M}}} = (\mathcal{C}_{\underline{M}})_{\widehat{\overline{D}}}$, so a distributive law between a monad and a comonad allows to combine them in arbitrary ways.

6.4 Distributive laws and lifting to Eilenberg-Moore category

There are two other notions of distributive law, by taking the dual direction of Definitions 55 and 57.

Definition 60. A natural transformation $\lambda : \underline{M}_2 F \Rightarrow F\underline{M}_1$ is a *distributive law* of F on two monads $\underline{M}_1$ and $\underline{M}_2$ if two diagrams analogue to Definition 55 commute. A natural transformation $\lambda : F\overline{D}_1 \Rightarrow \overline{D}_2 F$ is a *distributive law* of F on two comonads $\overline{D}_1$ and $\overline{D}_2$ if two diagrams analogue to Definition 57 commute. Morphism of distributive law are defined in the same way as Definition 56

Those notions of distributive law are tied to the notion of lifting of a functor and natural transformations to the Eilenberg-Moore categories $\mathcal{C}^{\underline{M}_1}$ and $\mathcal{C}^{\underline{M}_2}$. A lifting of a functor $\mathcal{C}_1 \to \mathcal{C}_2$ is a functor $\overline{F} : \mathcal{C}^{\underline{M}_1} \to \mathcal{C}^{\underline{M}_2}$ such that the diagram below commutes.

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \\ {\scriptstyle \mathcal{U}}\uparrow & & \uparrow{\scriptstyle \mathcal{U}} \\ \mathcal{C}^{\underline{M}_1} & \xrightarrow[\overline{F}]{} & \mathcal{C}^{\underline{M}_2} \end{array}$$

Where $\mathcal{U}$ is the forgetful functor. More details on [PW02]. We can summarize the different configurations for a distributive law, depending on the direction of the natural transformation and the choice between monads and comonads, see Fig. 2.

There is also a notion of a distributive law of the monad $\underline{M}$ on the comonad $\overline{D}$ similar to the one of Definition 59. This time, such distributive laws are associated to lifting of the monad $\underline{M}$ to the Eilenberg-Moore category of $\overline{D}$, and lifting of $\overline{D}$ to the Eilenberg-Moore

category of $\underline{M}$.

Definition 61. A natural transformation $\underline{M}\overline{D} \Rightarrow \overline{D}\underline{M}$ is a distributive law of the monad $\underline{M}$ on the comonad $\overline{D}$ if it is both a distributive law $\underline{M}D \Rightarrow D\underline{M}$, and a distributive law $M\overline{D} \Rightarrow \overline{D}M$ in the sense of Definition 60.

7 Annex: symmetric monoidal monads and distributive law

An important concept in the theory of monads (and in our present article) is the concept of lax (symmetric) monoidal functor and lax (symmetric) monoidal monad. It is important because if $\mathcal{L}$ is a (symmetric) monoidal category and $\underline{M}$ is a monad on $\mathcal{L}$, then $\mathcal{L}_{\underline{M}}$ inherits from $\mathcal{L}$ the structure of a (symmetric) monoidal category. As expected, this notion is then deeply connected to the notion of distributive laws, and we discuss this connection in this section. Lax monoidal structures can also be expressed in terms of strength, see [Koc70, Koc72]. We detail this process as the notion of strength crucially allow us to define a theory of Taylor expansion with regard to only one parameter in Section 3.2.

7.1 Distributive laws on product categories

Let us recall first some fact and notation about product categories. The category **Cat** is a cartesian category, with terminal object the category **1** which contains one object and one morphism, and whose cartesian product is defined as in Definition 62 below.

Definition 62. Given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$, the product category $\mathcal{C}_1 \times \mathcal{C}_2$ is the category whose objects are the pairs (X_1, X_2) with $X_1 \in \mathbf{Obj}(\mathcal{C}_1)$ and $X_2 \in \mathbf{Obj}(\mathcal{C}_2)$ and whose morphisms are the pairs (f_1, f_2) with $f_1 \in \mathcal{C}_1(X_1, Y_1)$ and $f_2 \in \mathcal{C}_2(X_2, Y_2)$.

For any functors $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ and $F_2 : \mathcal{C}_2 \to \mathcal{D}_2$, we can define the functor $F_1 \times F_2 : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}_1 \times \mathcal{D}_2$ by $(F_1 \times F_2)(X_1, X_2) := (F_1X_1, F_2X_2)$ and $(F \times F_2)(f_1, f_2) := (F_1f_1, F_2f_2)$.

Given $F_1 : \mathcal{C} \to \mathcal{D}_1$ and $F_2 : \mathcal{C} \to \mathcal{D}_2$, we can define $\langle F_1, F_2\rangle : \mathcal{C} \to \mathcal{D}_1 \times \mathcal{D}_2$ by $\langle F_1, F_2\rangle X = (F_1X, F_2X)$ and $\langle F_1, F_2\rangle f = (F_1f, F_2f)$. For any category $\mathcal{C}$, we can define the functor $\Delta^{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by $\Delta^{\mathcal{C}}(X) = (X, X)$ and $\Delta^{\mathcal{C}} f = (f, f)$. That is, $\Delta^{\mathcal{C}} = \langle \mathsf{Id}, \mathsf{Id}\rangle$.

Given the functors $F_1, G_1 : \mathcal{C}_1 \to \mathcal{D}_1$ and $F_2, G_2 : \mathcal{C}_2 \to \mathcal{D}_2$ and two natural transformation $\alpha_1 : F_1 \Rightarrow G_1$ and $\alpha_2 : F_2 \Rightarrow G_2$, we can define the natural transformation $(\alpha_1, \alpha_2) : F_1 \times F_2 \Rightarrow G_1 \times G_2$ by $(\alpha_1, \alpha_2)_{(X_1,X_2)} := (\alpha_{1,X_1}, \alpha_{2,X_2})$.

Definition 63. Given a monad $\underline{M}_1 = (M_1, \eta^1, \mu^1)$ on $\mathcal{C}_1$ and a monad $\underline{M}_2 = (M_2, \eta^2, \mu^2)$ on $\mathcal{C}_2$, we can define the monad $\underline{M}_1 \times \underline{M}_2$ on $\mathcal{C}_1 \times \mathcal{C}_2$ whose unit is (η^1, η^2) and whose sum is (μ^1, μ^2). We can check that $(\mathcal{C}_1 \times \mathcal{C}_2)_{\underline{M}_1 \times \underline{M}_2} = \mathcal{C}_{\underline{M}_1} \times \mathcal{C}_{\underline{M}_2}$.

Lemma 34. *The functor $\Delta^{\mathcal{C}_{\underline{M}}} : \mathcal{C}_{\underline{M}} \to \mathcal{C}_{\underline{M}} \times \mathcal{C}_{\underline{M}}$ is an extension of $\Delta^{\mathcal{C}}$ (recall that $\mathcal{C}_{\underline{M}} \times \mathcal{C}_{\underline{M}} = (\mathcal{C} \times \mathcal{C})_{M \times \underline{M}}$). Its associated distributive law is $\mathsf{id}\langle M, M\rangle : \langle M, M\rangle \Rightarrow \langle M, M\rangle$ (observe that $\Delta^{\mathcal{C}} M = (M \times M)\Delta^{\mathcal{C}} = \langle M, M\rangle$).*

Proof. Straightforward computation. □

Lemma 35. *Let $F : \mathcal{C}_1 \to \mathcal{D}_1$ and $G : \mathcal{C}_2 \to \mathcal{D}_2$. We assume that $\mathcal{C}_1, \mathcal{D}_1, \mathcal{C}_2, \mathcal{D}_2$ are all equipped with a respective monad that we keep implicit. Assume that $\widehat{F}$ is an extension of F associated to the distributive law λ^F and $\widehat{G}$ is an extension of G associated to the distributive law λ^G. Then $\widehat{F} \times \widehat{G}$ is an extension of (F, G) whose associated distributive law is (λ^F, λ^G).*

Proof. Straightforward computation. □

Lemma 36. *Let $F : \mathcal{C} \to \mathcal{D}_1$ and $G : \mathcal{C} \to \mathcal{D}_2$. We assume that $\mathcal{C}, \mathcal{D}_1, \mathcal{D}_2$ are all equipped with a respective monad that we keep implicit. Assume that $\widehat{F}$ is an extension of F associated to the distributive law λ^F and $\widehat{G}$ is an extension of G associated to the distributive law λ^G. Then $\langle \widehat{F}, \widehat{G} \rangle$ is an extension of $\langle F, G \rangle$ whose associated distributive law is (λ^F, λ^G).*

Proof. Observe that $\langle \widehat{F}, \widehat{G} \rangle = (\widehat{F} \times \widehat{G})\Delta^{\mathcal{C}_{\underline{M}}}$. By Lemmas 34 and 35 we know that $\widehat{F} \times \widehat{G}$ is an extension of $F \times G$ and that $\Delta^{\mathcal{C}_{\underline{M}}}$ is an extension of $\Delta^{\mathcal{C}}$, so by compositionality $\langle \widehat{F}, \widehat{G} \rangle$ is an extension of $(F \times G)\Delta^{\mathcal{C}} = \langle F, G \rangle$ and its distributive law is given by the composition of the two respective distributive laws. □

Lemma 37. *Let $\underline{M}$ be a monad on $\mathcal{C}$. For any object A of $\mathcal{C}$, the constant endofunctor $A^{\mathcal{C}_{\underline{M}}} : \mathcal{C}_{\underline{M}} \to \mathcal{C}_{\underline{M}}$ is an extension of the constant endofunctor $A : \mathcal{C} \to \mathcal{C}$. Its associated distributive law is $\eta \in \mathcal{C}(A, MA)$*

Proof. The fact that $A^{\mathcal{C}_{\underline{M}}}$ is an extension of A is immediate. Its associated distributive law is $A^{\mathcal{C}_{\underline{M}}}(\mathsf{id}_{MX}) = \mathsf{id}_A^{\mathcal{C}_{\underline{M}}} = \eta$. □

7.2 Lax monoidal functor and lax monoidal monads

Definition 64. A lax monoidal functor from a monoidal category $(\mathcal{L}, \otimes, 1)$ to another monoidal category $(\mathcal{L}', \bullet, I)$ is a tuple (F, m^0, m^2) where $F : \mathcal{L} \to \mathcal{L}'$ is a functor, $m^0 \in \mathcal{L}'(I, F1)$ and $m^2_{X,Y} \in \mathcal{L}'(FX \bullet FY, F(X \otimes Y))$ is a natural transformation such that the diagrams below commute.

$$\begin{array}{ccccc} FX \bullet I & \xrightarrow{FX \bullet m^0} & FX \bullet F1 & \xrightarrow{m^2} & F(X \otimes 1) \\ {\scriptstyle \rho^\bullet}\downarrow & & & & \downarrow{\scriptstyle F\rho^\otimes} \\ FX & = & = & = & FX \end{array} \qquad \begin{array}{ccccc} I \bullet FX & \xrightarrow{m^0 \bullet FX} & F1 \bullet FX & \xrightarrow{m^2} & F(1 \otimes X) \\ {\scriptstyle \lambda^\bullet}\downarrow & & & & \downarrow{\scriptstyle F\lambda^\otimes} \\ FX & = & = & = & FX \end{array}$$

$$\begin{array}{ccccc} (FX \bullet FY) \bullet FZ & \xrightarrow{m^2 \bullet FZ} & F(X \otimes Y) \bullet FZ & \xrightarrow{m^2} & F((X \otimes Y) \otimes Z) \\ {\scriptstyle \alpha^\bullet}\downarrow & & & & \downarrow{\scriptstyle F\alpha^\otimes} \\ FX \bullet (FY \bullet FZ) & \xrightarrow[FX \bullet m^2]{} & FX \bullet F(Y \otimes Z) & \xrightarrow[m^2]{} & F(X \otimes (Y \otimes Z)) \end{array}$$

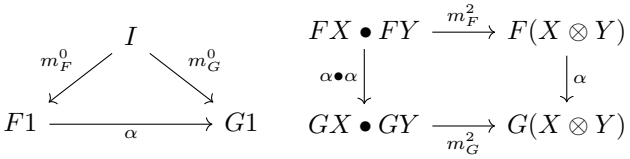
A lax monoidal functor from a symmetric monoidal category $(\mathcal{L}, \otimes, 1)$ to another symmetric monoidal category $(\mathcal{L}', \bullet, I)$ (with respective symmetries $\gamma^\otimes$ and $\gamma^\bullet$) is *symmetric* if the additional diagram below commutes.

$$\begin{array}{ccc} FX \bullet FY & \xrightarrow{m^2_{X,Y}} & F(X \otimes Y) \\ {\scriptstyle \gamma^\bullet_{FX,FY}}\downarrow & & \downarrow{\scriptstyle F\gamma^\otimes_{X,Y}} \\ FY \bullet FX & \xrightarrow[m^2_{Y,X}]{} & F(Y \otimes X) \end{array}$$

A strong (symmetric) monoidal functor is a lax (symmetric) monoidal functor such that m^0 is an iso, and m^2 is a natural isomorphism.

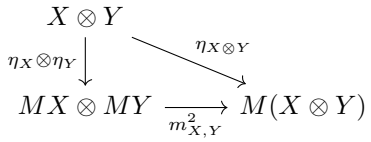
Definition 65. A monoidal natural transformation between two monoidal functors (F, m^0_F, m^2_F) and (G, m^0_G, m^2_G) is a natural transformation $\alpha : F \Rightarrow G$ such that the following diagram

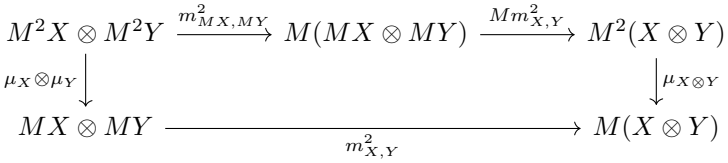
commutes.



There is in fact a 2-category in which the objects are the (small) monoidal categories, the morphisms are the lax monoidal functors, and the 2-cells are the monoidal natural transformations. The composition of two lax monoidal functors is given by the composition of the functor and a suitable composition of their associated natural transformations. The composition of 2-cells is the same as the composition in **Cat**.

Definition 66. A lax (symmetric) monoidal monad on a (symmetric) monoidal category $(\mathcal{L}, \otimes, 1)$ is the data of (M, η, μ, m^0, m^2) such that (M, m^0, m^2) is a lax (symmetric) monoidal functor from $(\mathcal{L}, \otimes, 1)$ to itself, (M, η, μ) is a monad on $\mathcal{L}$, and such that η and μ are monoidal natural transformations. Unfolding the diagram, this means that $m^0 = \eta_1 \in \mathcal{L}(1, M1)$ and the following diagrams commute.





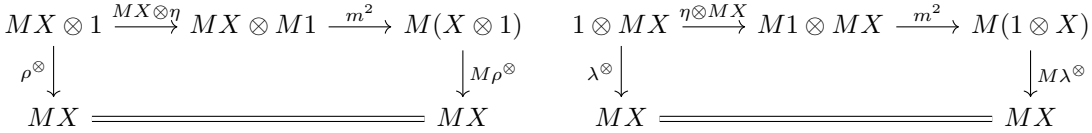
A lax monoidal monad is symmetric if its underlying monoidal functor is symmetric.

Lax (symmetric) monoidal monads are a well studied notion because they are related to the extension of the (symmetric) monoidal structure to the Kleisli category.

Theorem 29. *If $\underline{M}$ is a (symmetric) monoidal monad on $\mathcal{L}$, then the structure of (symmetric) monoidal category of $\mathcal{L}$ extends to $\mathcal{L}_{\underline{M}}$.*

It turns out that lax monoidal monads are an example of distributive law. This shed light on Theorem 29 above. The diagram of Definition 66 indeed corresponds to the fact that the natural transformation $m^2 : (\_ \otimes \_)(M \times M) \Rightarrow M(\_ \otimes \_)$ is a distributive law of the functor $\_ \otimes \_$ on the monads $\underline{M} \times \underline{M}$ and $\underline{M}$. By Theorem 25, it means that $\_ \otimes \_$ extends to a functor $\_ \widehat{\otimes} \_ : \mathcal{L}_{\underline{M}} \times \mathcal{L}_{\underline{M}} \to \mathcal{L}_{\underline{M}}$.

The two diagram below that are part of the assumption that M is a (symmetric) monoidal functor can be interpreted as the fact that $\lambda^{\otimes}$ and $\rho^{\otimes}$ are morphisms of distributive law. Indeed, see Proposition 40 below.



Proposition 40. *For any objects X and Y, the functor $X\widehat{\otimes}\_$ is an extension of $X\otimes\_$ with associated distributive law $m^2\circ(\eta_X\otimes M\mathsf{id})$. Similarly, the functor $\_\widehat{\otimes}Y$ is an extension of $\_\otimes Y$ with associated distributive law $m^2\circ(M\mathsf{id}\otimes\eta_Y)$.*

Proof. Observe that $1\otimes\_ = (\_\otimes\_)\langle 1,\mathsf{Id}\rangle$ and $1\widehat{\otimes}\_ = (\_\widehat{\otimes}\_)\langle 1^{\mathcal{L}_{\underline{M}}},\mathsf{Id}\rangle$. We conclude by compositionality of extensions and distributive laws, using the results of Lemma 36 and Lemma 37. □

Similarly, the two other diagrams mean that $\alpha^\otimes$ and $\gamma^\otimes$ are morphisms of distributive laws. By Theorem 26 this ensures that they extend to natural transformations $K_{\underline{M}}(\lambda^\otimes) : 1\widehat{\otimes}\_ \Rightarrow \mathsf{Id}^{\mathcal{C}_{\underline{M}}}$, $K_{\underline{M}}(\rho^\otimes) : \_\widehat{\otimes}1 \Rightarrow \mathsf{Id}^{\mathcal{C}_{\underline{M}}}$, $K_{\underline{M}}(\alpha^\otimes) : (\_\widehat{\otimes}\_)\widehat{\otimes}\_ \Rightarrow \_\widehat{\otimes}(\_\widehat{\otimes})$ and $K_{\underline{M}}(\gamma^\otimes) : \__1\widehat{\otimes}\__2 \Rightarrow \__2\widehat{\otimes}\__1$. This is why $\mathcal{L}_{\underline{M}}$ inherits from $\mathcal{L}$ the structure of a (symmetric) monoidal category.

Remark 34. Dually, there is a notion of a (symmetric) oplax monoidal functor from a monoidal category $(\mathcal{L},\otimes,1)$ to another monoidal category $(\mathcal{L}',\bullet,I)$. It is a tuple (F,n^0,n^2) where $F:\mathcal{C}_1\to\mathcal{C}_2$ is a functor, $n^0\in\mathcal{C}_2(F1,I)$ and $n^2_{X,Y}\in\mathcal{C}_2(F(X\otimes Y),FX\bullet FY)$ is a natural transformation, and such that diagrams similar to the ones of Definition 64 commute. There is also a notion of oplax monoidal monad (also called Hopf Monad). In the same way that lax monoidal monads are related to the distributive laws of Definition 55, Hopf Monad are related to the distributive laws of Definition 60, as observed in [Wis08]. This is not surprising then that a monad $\underline{M}$ on $\mathcal{L}$ is a Hopf monads if and only if the symmetric monoidal structure of $\mathcal{L}$ lifts to the Eilenberg-Moore category of $\underline{M}$, see [Moe02].

Finally, an oplax monoidal comonad is a comonad $\overline{D}$ such that the functor D is oplax monoidal, and such that ϵ,δ are monoidal natural transformations. This yield a definition very similar to the one of Definition 66.

7.3 Commutative monad

It is well-known that symmetric monoidal monads are the same as commutative monads, see [Koc70, Koc72]. Let us recall what is a commutative monad. Let $\underline{M}$ be a monad on a symmetric monoidal category $\mathcal{L}$.

Definition 67. A *left strength* for $\underline{M}$ is a natural transformation $\mathsf{t}_{X,Y}\in\mathcal{L}(X\otimes MY, M(X\otimes Y))$ subject to commutation with the monoidal structure

$$\begin{array}{ccc} 1\otimes MX & \xrightarrow{\mathsf{t}_{1,X}} & M(1\otimes X) \\ & {\scriptstyle\lambda^\otimes_{MX}}\searrow & \downarrow{\scriptstyle M\lambda^\otimes_X} \\ & & MX \end{array} \qquad \begin{array}{ccccc} (X\otimes Y)\otimes MZ & & \xrightarrow{\qquad\mathsf{t}_{X\otimes Y,Z}\qquad} & & M((X\otimes Y)\otimes Z) \\ \downarrow{\scriptstyle\alpha^\otimes_{X,Y,MZ}} & & & & \downarrow{\scriptstyle M\alpha^\otimes_{X,Y,Z}} \\ X\otimes(Y\otimes MZ) & \xrightarrow[X\otimes\mathsf{t}_{Y,Z}]{} & X\otimes M(Y\otimes Z) & \xrightarrow[\mathsf{t}_{X,Y\otimes Z}]{} & M(X\otimes(Y\otimes Z)) \end{array}$$

and commutation with the monad structure

$$\begin{array}{ccc} X\otimes Y & & \\ {\scriptstyle X\otimes\eta_Y}\downarrow & \searrow{\scriptstyle\eta_{X\otimes Y}} & \\ X\otimes MY & \xrightarrow[\mathsf{t}_{X,Y}]{} & M(X\otimes Y) \end{array} \qquad \begin{array}{ccccc} X\otimes M^2Y & \xrightarrow{\mathsf{t}_{X,MY}} & M(X\otimes MY) & \xrightarrow{M\mathsf{t}_{X,Y}} & M^2(X\otimes Y) \\ {\scriptstyle X\otimes\mu_Y}\downarrow & & & & \downarrow{\scriptstyle\mu_{X\otimes Y}} \\ X\otimes MY & & \xrightarrow[\mathsf{t}_{X,Y}]{\qquad\qquad} & & M(X\otimes Y) \end{array}$$

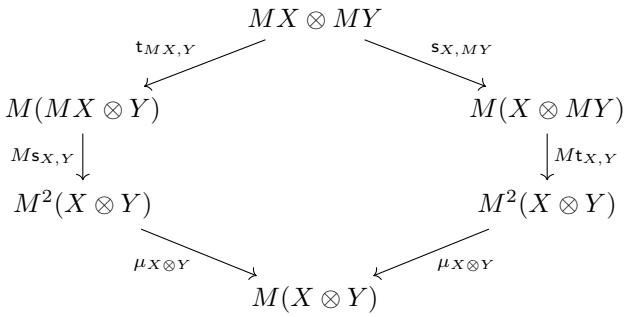
A *right strength* is a natural transformation $\mathsf{s}_{X,Y}\in\mathcal{L}(MX\otimes Y, M(X\otimes Y))$ subject to similar commutations. A *strength*[11] is the combination of a left strength and a right strength such

[11] This terminology has nothing to do with the use of the word *strong* in strong monoidal functors

that the two induced map in $\mathcal{L}(X \otimes MY \otimes Z, M(X \otimes Y \otimes Z))$ obtained by combining t twice or s twice agree (up to associativity of $\otimes$).

Remark 35. When the monoidal category is symmetric, any left strength t induces a right strength $\mathsf{s}_{X,Y} = M\gamma^{\otimes}_{Y,X} \circ \mathsf{t}_{Y,X} \circ \gamma^{\otimes}_{X,Y}$ and vice versa.

Definition 68. A *commutative monad* is a monad equipped with a strength (t, s) such that the following diagram commutes.



Any commutative monad is a (symmetric) monoidal monad, (see [Koc70]), defining $m^2 \in \mathcal{L}(MX \otimes MY, M(X \otimes Y))$ as the diagonal of the square above. Conversely, any (symmetric) monoidal monad is a commutative monad, see [Koc72]. The strengths are defined as

$$\mathsf{t}_{X,Y} := \; X \otimes MY \xrightarrow{\eta_X \otimes MY} MX \otimes MY \xrightarrow{m^2_{X,Y}} M(X \otimes Y)$$

$$\mathsf{s}_{X,Y} := \; MX \otimes Y \xrightarrow{MX \otimes \eta_Y} MX \otimes MY \xrightarrow{m^2_{X,Y}} M(X \otimes Y)$$

Recall from Proposition 40 that $\mathsf{t}_{X,\_}$ is the distributive law associated to the extension $X\widehat{\otimes}\_$ of $X \otimes \_$, and $\mathsf{s}_{\_,Y}$ is the distributive law associated to the extension $\_\widehat{\otimes}Y$ of $\_ \otimes Y$. So the equivalence between (symmetric) monoidal monads and commutative monads can also be understood as the fact that an extension $\_\widehat{\otimes}\_$ is the same as two extensions $X\widehat{\otimes}\_$ and $\_\widehat{\otimes}Y$ compatible together that can be combined through the bifunctor lemma.

8 Annex: adjunction, mates and distributive law

The goal of this section is to describe the mate construction, and how it relates distributive laws together. The main application of this section is in Section 4, where the functor S developed in Section 1 is equal to $\mathbb{D} \multimap \_$, because of the central adjunction $\_ \otimes \mathbb{D} \dashv \mathbb{D} \multimap \_$.

8.1 Adjunctions and the mate construction

Definition 69. Two functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are adjoint if there exists two natural transformations $\eta : \mathsf{Id}_{\mathcal{C}} \Rightarrow RL$ and $\epsilon : LR \Rightarrow \mathsf{Id}_{\mathcal{D}}$ such that the compositions

$$L \xrightarrow{L\eta} LRL \xrightarrow{\epsilon L} L \qquad R \xrightarrow{\eta R} RLR \xrightarrow{R\epsilon} R$$

are the identity natural transformations. Those two equations are called the triangle identities. The functor L is called the left adjoint, R is called the right adjoint, η is called the unit, and ϵ is called the co-unit. We write an adjunction $\eta, \epsilon : L \dashv R$.

An adjunction can also be characterized in terms of the existence of two natural bijections $\Phi_{X,Y} : \mathcal{D}(LX, Y) \to \mathcal{C}(X, RY)$ and $\Psi_{X,Y} : \mathcal{C}(X, RY) \to \mathcal{D}(LX, Y)$ inverse of each other. There are defined respectively as $\Phi_{X,Y}(f) = Rf \circ \eta_X$ and $\Psi_{X,Y}(g) = \epsilon_Y \circ Lg$. Conversely, the unit and co-unit can be obtained from Φ and Ψ taking $\eta_X = \Phi_{X,LX}(\mathsf{id}_{LX})$ and $\epsilon = \Psi_{RX,X}(\mathsf{id}_{RX})$.

Definition 70. There exists a category whose objects are the (small) categories, and whose morphisms are the adjoint pairs. The identity is the trivial adjoint pair $\mathsf{id}, \mathsf{id} : \mathsf{Id} \dashv \mathsf{Id}$. The composition of an adjoint pair $\eta_1, \epsilon_1 : L_1 \dashv R_1$ with an adjoint pair $\eta_2, \epsilon_2 : L_2 \dashv R_2$ is defined as the adjoint pair $\eta, \epsilon : L_2 L_1 \dashv R_1 R_2$ where

$$\eta := \ \mathsf{Id} \xrightarrow{\eta_1} R_1 L_1 \xrightarrow{R_1 \eta_2 L_1} R_1 R_2 L_2 L_1$$

$$\epsilon := \ L_2 L_1 R_2 R_1 \xrightarrow{L_2 \epsilon_1 R_2} L_2 R_2 \xrightarrow{\epsilon_2} \mathsf{Id}.$$

It is straightforward to check that η and ϵ follow the triangle equalities. If Φ^i, Ψ^i are the natural bijections associated to $\eta_i, \epsilon_i : L_i \dashv R_i$, then the natural bijections associated to $\eta, \epsilon : L_2 L_1 \dashv R_1 R_2$ are $\Phi(f) = \Phi^1(\Phi^2(f))$ and $\Psi(g) = \Psi^2(\Psi^1(g))$.

We assume throughout this section that we have the following adjunctions, of respective units η, η', η'' and respective counits $\epsilon, \epsilon', \epsilon''$.

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D} \qquad \mathcal{C}' \underset{R'}{\overset{L'}{\rightleftarrows}} \mathcal{D}' \qquad \mathcal{C}'' \underset{R''}{\overset{L''}{\rightleftarrows}} \mathcal{D}'' \ .$$

Proposition 41. *Let $H : \mathcal{C} \to \mathcal{C}'$ and $K : \mathcal{D} \to \mathcal{D}'$ two functors. There exists a bijection called the* mate construction *between the natural transformations $\lambda : L'H \Rightarrow KL$ and the natural transformations $\mu : HR \Rightarrow R'K$. This bijection is given by the following pasting diagrams.*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ {\scriptstyle L}\downarrow & \swarrow{\scriptstyle \lambda} & \downarrow{\scriptstyle L'} \\ \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' \end{array} \quad \mapsto \quad \begin{array}{ccccccc} \mathcal{D} & \xrightarrow{R} & \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & = & \mathcal{C}' \\ \| & \swarrow{\scriptstyle \epsilon} & {\scriptstyle L}\downarrow & \swarrow{\scriptstyle \lambda} & {\scriptstyle L'}\downarrow & \swarrow{\scriptstyle \eta'} & \| \\ \mathcal{D} & = & \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' & \xrightarrow[R']{} & \mathcal{C}' \end{array}$$

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ {\scriptstyle R}\downarrow & \nearrow{\scriptstyle \mu} & \downarrow{\scriptstyle R'} \\ \mathcal{C} & \xrightarrow[H]{} & \mathcal{C}' \end{array} \quad \mapsto \quad \begin{array}{ccccccc} \mathcal{C} & \xrightarrow{L} & \mathcal{D} & \xrightarrow{K} & \mathcal{D}' & = & \mathcal{D}' \\ \| & \nearrow{\scriptstyle \eta} & {\scriptstyle R}\downarrow & \nearrow{\scriptstyle \mu} & {\scriptstyle R'}\downarrow & \nearrow{\scriptstyle \epsilon'} & \| \\ \mathcal{C} & = & \mathcal{C} & \xrightarrow[H]{} & \mathcal{C}' & \xrightarrow[L']{} & \mathcal{D}' \end{array}$$

In other word, λ and μ are mates if they are defined from each other by $\mu = R'K\epsilon \circ R'\lambda R \circ \eta' HR$ and $\lambda = \epsilon' KL \circ L'\mu L \circ L'H\eta$.

$$\mu = \ HR \xrightarrow{\eta' HR} R'L'HR \xrightarrow{R'\lambda R} R'KLR \xrightarrow{R'K\epsilon} R'K$$

$$\lambda = \ L'H \xrightarrow{L'H\eta} L'HRL \xrightarrow{L'\mu L} L'R'KL \xrightarrow{\epsilon' KL} KL$$

Remark 36. Observe that if $\Phi'_{X,Y} : \mathcal{D}'(L'X, Y) \to \mathcal{C}'(X, R'Y)$ is the natural bijection associated to $\eta', \epsilon' : L' \dashv R'$, then $\mu_X = \Phi'_{HRX,X}(K\epsilon_X \circ \lambda_{RX})$. This alternative definition of the right mate will be useful latter on when considering the mate construction applied through the adjunction $\_ \otimes A \dashv A \multimap \_$.

One particular case is when $\mathcal{C} = \mathcal{C}'$, $\mathcal{D} = \mathcal{D}'$ and $H = K = \mathsf{Id}$ so that there is a bijection between the natural transformations $\lambda : L' \Rightarrow L$ and $\mu : R \Rightarrow R'$.

$$\mu = \; R \xrightarrow{\eta' R} R'L'R \xrightarrow{R'\lambda R} R'LR \xrightarrow{R'\epsilon} R'$$

$$\lambda = \; L' \xrightarrow{L'\eta} L'RL \xrightarrow{L'\mu L} L'R'L \xrightarrow{\epsilon' L} L$$

Then $\mu_X = \Phi'_{RX,X}(\epsilon_X \circ \lambda_{RX})$.

One can define a double category associated to those constructions. A definition of double category can be found in [KS74]. Let us unfold this definition in our setting. We can define a category of *horizontal morphisms* by taking the category **Cat** whose objects are the (small) categories and whose morphisms are the functor. We can also define a category of *vertical morphisms* by taking the category Definition 70. Then we can define a *square* as two vertical morphisms $L \dashv R$ and $L' \dashv R'$, two horizontal morphisms H, K and a natural transformation $\lambda : L'H \Rightarrow KL$. The fact that those are a square become apparent when writing λ with a pasting diagram.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ {\scriptstyle L}\downarrow & \swarrow_{\lambda} & \downarrow{\scriptstyle L'} \\ \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' \end{array}$$

There are identity squares defined below, both in the horizontal and vertical directions. We can check that the left square instantiated in $L = R = \mathsf{Id}$ is the same as the right square instantiated in $H = \mathsf{Id}$.

$$\begin{array}{ccc} \mathcal{C} & = & \mathcal{C} \\ {\scriptstyle L}\downarrow & \swarrow_{\mathsf{id}} & \downarrow{\scriptstyle L} \\ \mathcal{D} & = & \mathcal{D} \end{array} \qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ \| & \swarrow_{\mathsf{id}} & \| \\ \mathcal{C} & \xrightarrow[H]{} & \mathcal{C}' \end{array}$$

It is then possible to compose two squares horizontally or vertically, using pasting diagrams.

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & \xrightarrow{H'} & \mathcal{C}'' \\ {\scriptstyle L}\downarrow & \swarrow_{\lambda} & {\scriptstyle L'}\downarrow & \swarrow_{\lambda'} & \downarrow{\scriptstyle L''} \\ \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' & \xrightarrow[K']{} & \mathcal{D}'' \end{array} \qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ {\scriptstyle L_1}\downarrow & \swarrow_{\lambda_1} & \downarrow{\scriptstyle L'_1} \\ \mathcal{D} & \xrightarrow{-K\to} & \mathcal{D}' \\ {\scriptstyle L_2}\downarrow & \swarrow_{\lambda_2} & \downarrow{\scriptstyle L'_2} \\ \mathcal{E} & \xrightarrow[J]{} & \mathcal{E}' \end{array}$$

That is, the horizontal composition of the two squares is given by the natural transformation

$$K'\lambda \circ \lambda' H : \; L''H'H \xrightarrow{\lambda' H} K'L'H \xrightarrow{K'\lambda \cdot} K'KL$$

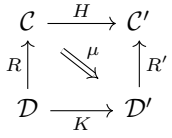
And the vertical composition of the two squares is given by the natural transformation

$$\lambda_2 L_1 \circ L'_2 \lambda_1 : \; L'_2 L'_1 H \xrightarrow{L'_2 \lambda_1} L'_2 K L_1 \xrightarrow{\lambda_2 L_1 \cdot} J L_2 L_1$$

By property of pasting diagrams, we know that the horizontal/vertical identity squares are neutral with regard to the horizontal/vertical composition, that the horizontal and vertical compositions are associative, and that the squares follow the interchange law of double categories (that is, all the possible ways to compose blocks of multiple squares are equivalent).

Similarly, there is a double category with the same horizontal and vertical morphisms as above, but where the squares are natural transformations $\mu : HR \Rightarrow R'K$. So a square

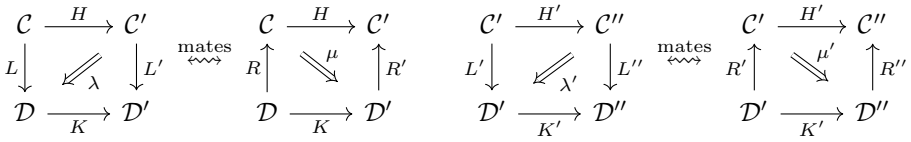
corresponds to the following pasting diagram.



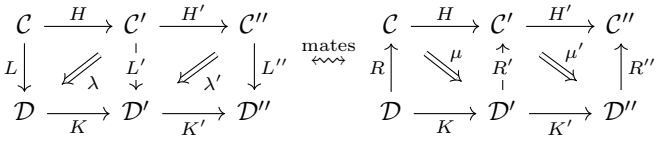
The composition of squares is also given by the composition of the pasting diagrams, and the identity squares are similar.

Theorem 30 ([KS74])**.** *The bijection of mate is an isomorphism of double categories between the two double categories described above.*

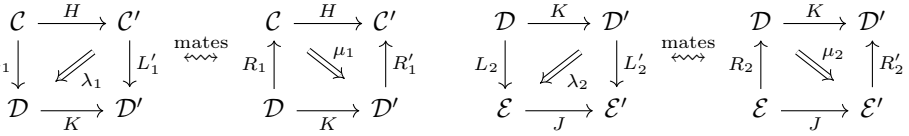
The fact that it is an isomorphism means that if the following squares are mate



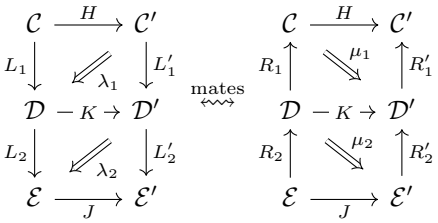
then their horizontal compositions are also mates.



Furthermore, if the following squares are mate



then their vertical compositions are also mates.



Here is a reformulation of the results that does not use pasting diagrams.

Proposition 42. *If $\lambda : L'H \Rightarrow KL$, $\mu : HR \Rightarrow R'K$ are mates and if $\lambda' : L''H' \Rightarrow K'L'$, $\mu' : H'R' \Rightarrow R''K'$ are mates, then their horizontal compositions*

$$K'\lambda \circ \lambda' H : \; L''H'H \xrightarrow{\lambda' H} K'L'H \xrightarrow{K'\lambda} K'KL$$

$$\mu' K \circ H'\mu : \; H'HR \xrightarrow{H'\mu} H'R'K \xrightarrow{\mu' K} R''K'K$$

are also mates.

Proposition 43. *If* $\lambda_1 : L_1'H \Rightarrow KL_1$, $\mu_1 : HR_1 \Rightarrow R_1'K$ *are mates and if* $\lambda_2 : L_2'K \Rightarrow JL_2$, $\mu_2 : KR_2 \Rightarrow R_2'J$ *are mates, then their vertical compositions*

$$\lambda_2 L_1 \circ L_2'\lambda_1 : \; L_2'L_1'H \xrightarrow{L_2'\lambda_1} L_2'KL_1 \xrightarrow{\lambda_2 L_1} JL_2L_1$$

$$R_1'\mu_2 \circ \mu_1 R_2 : \; HR_1R_2 \xrightarrow{\mu_1 R_2} R_1'KR_2 \xrightarrow{R_1'\mu_2} R_1'R_2'J$$

are also mates.

Definition 71. We call **Adj** the double category where vertical morphisms are adjunction, horizontal morphisms are functors, and squares are pairs (λ, μ) of mates. By Theorem 30, it is isomorphic to the two categories described above.

8.2 Mate construction between a monad and a comonad

Assume that $\eta, \epsilon : L \dashv R$. Then as seen in Definition 70, there is an adjunction $\mathsf{id}, \mathsf{id} : \mathsf{Id} \dashv \mathsf{Id}$ and an adjunction $LL \dashv RR$ of unit $R\eta L \circ \eta$ and counit $\epsilon \circ L\epsilon R$. Then the mate construction induce a bijection between the natural transformations $\eta^R : \mathsf{Id} \Rightarrow R$ and the natural transformations $\epsilon^L : L \Rightarrow \mathsf{Id}$, given by

$$\epsilon^L = \; L \xrightarrow{L\eta^R} LR \xrightarrow{\epsilon} \mathsf{Id} \qquad \eta^R = \; \mathsf{Id} \xrightarrow{\eta} RL \xrightarrow{R\epsilon^L} R$$

It also induces a bijection between the natural transformations $\mu^R : R^2 \Rightarrow R$ and the natural transformations $\delta^L : L \Rightarrow L^2$ given by

$$\delta^L = \; L \xrightarrow{L\eta} LRL \xrightarrow{LR\eta L} LRRLL \xrightarrow{L\mu^R LL} LRLL \xrightarrow{\epsilon LL} LL$$

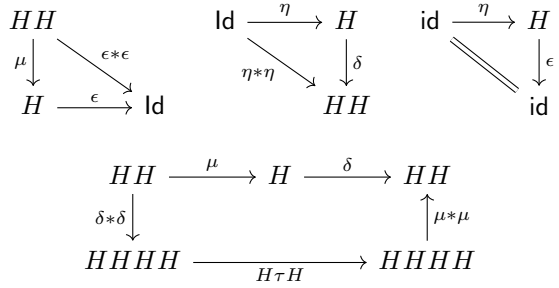
$$\mu^R = \; RR \xrightarrow{\eta RR} RLRR \xrightarrow{R\delta^L RR} RLLRR \xrightarrow{RL\epsilon R} RLR \xrightarrow{R\epsilon} R$$

The compositionality of the mate construction ensures that (R, η^R, μ^R) is a monad if and only if $(L, \epsilon^L, \delta^L)$ is a comonad. It means that the mate construction induces a bijection between the monad structures on R and the comonad structures on L.

Definition 72. A monad $\underline{R}$ and a comonad $\underline{L}$ such that $L \dashv R$ are called *mates* if their structure are related through the bijection defined above.

Similarly, the mate construction induces a bijection between the comonad structures on R and the monad structures on L. A comonad $\underline{R}$ and a monad $\underline{L}$ are called mate if their structure are related through this bijection.

Definition 73 ([MW11])**.** Assume that $\overline{H} = (H, \eta, \mu)$ is a monad on $\mathcal{C}$ and $\underline{H} = (H, \epsilon, \delta)$ is a comonad on $\mathcal{C}$. Then $(H, \eta, \mu, \epsilon, \delta)$ is a τ-bimonad if τ is a distributive law $\overline{H}\underline{H} \Rightarrow \underline{H}\overline{H}$ of the monad $\overline{H}$ on the comonad $\underline{H}$ (see Definition 59) and a distributive law $\underline{H}\overline{H} \Rightarrow \overline{H}\underline{H}$ of the comonad $\underline{H}$ on the monad $\overline{H}$ (see Definition 61), and if the diagrams below commute.



Recall that $*$ is defined as the horizontal composition of natural transformations: if $F, G : \mathcal{C} \to \mathcal{D}$, $F', G' : \mathcal{D} \to \mathcal{E}$, $\alpha : F \Rightarrow G$ and $\beta : F' \Rightarrow G'$, then $\beta * \alpha : GF \Rightarrow G'F'$ is defined as $(\beta * \alpha)_X := \beta_{F'X} \circ G\alpha_X = G'\alpha_X \circ \beta_{FX}$.

Remark 37. The four diagram making τ is a distributive law $\overline{H}\underline{H} \Rightarrow \underline{H}\overline{H}$ are exactly the same as the diagrams making τ a distributive law $\underline{H}\overline{H} \Rightarrow \overline{H}\underline{H}$, except that the arrows involving τ are reversed. In particular, if τ is involutive then any of the two assumptions implies the other.

Theorem 31 (7.7 of [MW11])**.** *The mate construction induces a bijection between bimonads structure on L and bimonad structures on R.*

Proof. We already know that the mate construction relates monads structure on L with comonad structures on R, and comonad structures on L with monads structures on R. The compositionality of the mate construction then allows to prove that it preserves the commutation of the bimonad diagrams. It then suffices to prove that it relates distributive law $\overline{L}\underline{L} \Rightarrow \underline{L}\overline{L}$ with distributive laws $\underline{R}\overline{R} \Rightarrow \overline{R}\underline{R}$, and distributive laws $\underline{L}\overline{L} \Rightarrow \overline{L}\underline{L}$ with distributive laws $\overline{R}\underline{R} \Rightarrow \underline{R}\overline{R}$. The proof of this observation can be found in section 7.5 of [MW11], and is very similar to the development of Section 8.3. □

8.3 Application to distributive laws

The mate construction relates distributive laws and morphism of distributive laws together. The reason is that they preserve compatibility, in the sense below. It is possible that the results of this section and Section 8.5 can be seen as a particular instance of doctrinal adjunction, see [Kel74].

Definition 74. A pair of natural transformation $\alpha : H_1 \Rightarrow H_2$, $\beta : K_1 \Rightarrow K_2$ is compatible with the squares $\lambda_1 : L'H_1 \Rightarrow K_1L$ and $\lambda_2 : L'H_2 \Rightarrow K_2L$ if the diagram (1) below commutes. Similarly, it is compatible with the squares $\mu_1 : H_1R \Rightarrow R'K_1$ and $\mu_2 : H_2R \Rightarrow R'K_2$ if the diagram (2) below commutes.

$$(1)\quad \begin{array}{ccc} L'H_1 & \xrightarrow{\lambda_1} & K_1L \\ {\scriptstyle L'\alpha}\downarrow & & \downarrow{\scriptstyle \beta L} \\ L'H_2 & \xrightarrow[\lambda_2]{} & K_2L \end{array} \qquad (2)\quad \begin{array}{ccc} H_1R & \xrightarrow{\mu_1} & R'K_1 \\ {\scriptstyle \alpha R}\downarrow & & \downarrow{\scriptstyle R'\beta} \\ H_2R & \xrightarrow[\mu_2]{} & R'K_2 \end{array}$$

Proposition 44. *If λ_1, μ_1 are mate and λ_2, μ_2 are mate, then (α, β) is compatible with λ_1 and λ_2 if and only if it is compatible with μ_1 and μ_2.*

Proof. The proof is a straightforward computation using the definition of mate, but we can also do a more generic proof by making full use of the isomorphism of Theorem 30. The compatibility of (α, β) with λ_1 and λ_2 can be expressed as the equality (1) of pasting diagram. Similarly, the compatibility of (α, β) with μ_1 and μ_2 can be expressed as the equality (2) of pasting diagrams.

$$(1)\quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{H_1} & \mathcal{C}' \\ {\scriptstyle L}\downarrow & \swarrow_{\lambda_1} & \downarrow{\scriptstyle L'} \\ \mathcal{D} & \xrightarrow[K_1]{} & \mathcal{D}' \\ \| & \swarrow_{\beta} & \| \\ \mathcal{D} & \xrightarrow[K_2]{} & \mathcal{D}' \end{array} = \begin{array}{ccc} \mathcal{C} & \xrightarrow{H_1} & \mathcal{C}' \\ \| & \swarrow_{\alpha} & \| \\ \mathcal{C} & \xrightarrow[H_2]{} & \mathcal{C}' \\ {\scriptstyle L}\downarrow & \swarrow_{\lambda_2} & \downarrow{\scriptstyle L'} \\ \mathcal{D} & \xrightarrow[K_2]{} & \mathcal{D}' \end{array} \qquad (2)\quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{H_1} & \mathcal{C}' \\ {\scriptstyle R}\uparrow & \searrow_{\mu_1} & \uparrow{\scriptstyle R'} \\ \mathcal{D} & \xrightarrow[K_1]{} & \mathcal{D}' \\ \| & \searrow_{\beta} & \| \\ \mathcal{D} & \xrightarrow[K_2]{} & \mathcal{D}' \end{array} = \begin{array}{ccc} \mathcal{C} & \xrightarrow{H_1} & \mathcal{C}' \\ \| & \searrow_{\alpha} & \| \\ \mathcal{C} & \xrightarrow[H_2]{} & \mathcal{C}' \\ {\scriptstyle R}\uparrow & \searrow_{\mu_2} & \uparrow{\scriptstyle R'} \\ \mathcal{D} & \xrightarrow[K_2]{} & \mathcal{D}' \end{array}$$

By assumption, λ_1 and μ_1 are mates. Besides, β is mate with itself through the identity adjunction. Thus, by Theorem 30, the two squares on the left-hand side of the equalities are mate of each other. Similarly, the two squares at the right-hand side of the equalities are mate of each other. Because the mate construction is a bijection, it implies that the two equality are equivalent. □

Corollary 5. *Assume that L can be endowed with a comonad structure $\overline{L}$ and R with a monad structure $\underline{R}$. Let F, G be two endofunctors on $\mathcal{C}$. Assume that $\lambda^F : \overline{L}F \Rightarrow F\overline{L}$ and $\mu^F : F\underline{R} \Rightarrow \underline{R}F$ are mates and distributive laws, and that $\lambda^G : \overline{L}G \Rightarrow G\overline{L}$ and $\mu^G : G\underline{R} \Rightarrow \underline{R}G$ are mates and distributive laws. Then $\alpha : F \Rightarrow G$ is a morphism from λ^F to λ^G if and only if it is a morphism from μ^F to μ^G. The same hold when L is endowed with the structure of a monad $\underline{L}$ and R with the structure of a comonad $\overline{R}$ instead.*

Proof. The diagrams corresponding to the fact that α is a morphism of distributive law are the following.

$$\begin{array}{ccc} LF & \xrightarrow{\lambda^F} & FL \\ {\scriptstyle L\alpha}\downarrow & & \downarrow{\scriptstyle \alpha L} \\ LG & \xrightarrow[\lambda^G]{} & GL \end{array} \qquad \begin{array}{ccc} FR & \xrightarrow{\mu^F} & RF \\ {\scriptstyle \alpha R}\downarrow & & \downarrow{\scriptstyle R\alpha} \\ GR & \xrightarrow[\mu^G]{} & RG \end{array}$$

They correspond respectively to the fact that (α, α) is compatible with λ^F and λ^G, and to the fact that (α, α) is compatible with μ^F and μ^G, so they are equivalent by Proposition 44. □

Remark 38. We will see in Corollary 9 and Corollary 10 that whenever $\overline{L}$ (respectively $\underline{L}$) is the mate of $\underline{R}$ (respectively $\overline{R}$), then λ^F is a distributive law if and only if μ^F is a distributive law, and λ^G is a distributive law if and only if μ^G is a distributive law, so the assumption of the result above is not strong.

Corollary 6. *Assume that $L' = L$ and $R' = R$, that H, K can be equipped with a comonad structure $\overline{H} = (H, \epsilon^H, \delta^H)$ and $\overline{K} = (K, \epsilon^K, \delta^K)$, and that $\lambda : LH \Rightarrow KL$ and $\mu : HR \Rightarrow RK$ are mate. Then $\lambda : L\overline{H} \Rightarrow \overline{K}L$ is a distributive law (in the sense of Definition 60) if and only if $\mu : \overline{H}R \Rightarrow R\overline{K}$ is a distributive law (in the sense of Definition 57).*

Proof. Notice that $\mu : \overline{H}R \Rightarrow R\overline{K}$ is a distributive law as in Definition 57 if and only if (ϵ^H, ϵ^K) is compatible with μ and id, and if (δ^H, δ^K) is compatible with μ and the horizontal composite $\mu K \circ H\mu$. Similarly, $\lambda : L\overline{H} \Rightarrow \overline{K}L$ is a distributive law in the sense of Definition 60 if and only if (ϵ^H, ϵ^K) is compatible with λ and id, and if (δ^H, δ^K) is compatible with μ and the horizontal composite $K\lambda \circ \lambda H$. But id and id are mate, λ and μ are mates, and $K\lambda \circ \lambda H$ and $\mu K \circ H\mu$ are mates thanks to Proposition 42. So by Proposition 44, μ is a distributive law if and only if λ is a distributive law. □

Corollary 7. *Assume that H, K can be equipped with a monad structure $\underline{H}$ and $\underline{K}$. Assume that $\lambda : LH \Rightarrow KL$ and $\mu : HR \Rightarrow RK$ are mate. Then $\lambda : L\underline{H} \Rightarrow \underline{K}L$ is a distributive law (in the sense of Definition 55) if and only if $\mu : \underline{H}R \Rightarrow R\underline{K}$ is a distributive law (in the sense of Definition 60).*

Proof. Same proof as in Corollary 6. □

Definition 75. A pair of natural transformations $\alpha : L_2 \Rightarrow L_1$ and $\alpha' : L_2' \Rightarrow L_1'$ is compatible with the squares $\lambda_1 : L_1'H \Rightarrow KL_1$ and $\lambda_2 : L_2'H \Rightarrow KL_2$ if the diagram (1) below commutes.

A pair of natural transformations $\beta : R_1 \Rightarrow R_2$ and $\beta' : R'_1 \Rightarrow R'_2$ is compatible with the squares $\mu_1 : HR_1 \Rightarrow R'_1K$ and $\mu_2 : HR_2 \Rightarrow R'_2K$ if the diagram (2) below commutes.

$$(1)\quad \begin{array}{ccc} L'_1H & \xrightarrow{\lambda_1} & KL_1 \\ {\scriptstyle \alpha' H}\uparrow & & \uparrow{\scriptstyle K\alpha} \\ L'_2H & \xrightarrow[\lambda_2]{} & KL_2 \end{array} \qquad (2)\quad \begin{array}{ccc} HR_1 & \xrightarrow{\mu_1} & R'_1K \\ {\scriptstyle H\beta}\downarrow & & \downarrow{\scriptstyle \beta' K} \\ HR_2 & \xrightarrow[\mu_2]{} & R'_2K \end{array}$$

Proposition 45. *Assume that α, β are mates, that α', β' are mates, that λ_1, μ_1 are mates and that λ_2, μ_2 are mates. Then (α, α') is compatible with λ_1 and λ_2 if and only if (β, β') is compatible with μ_1 and μ_2.*

Proof. The compatibility conditions rewrite as the equality of the following pasting diagrams.

$$(1)\quad \begin{array}{ccccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & = & \mathcal{C}' \\ {\scriptstyle L_1}\downarrow & \swarrow_{\lambda_1} & {\scriptstyle L'_1}\downarrow & \swarrow_{\alpha'} & \downarrow{\scriptstyle L'_2} \\ \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' & = & \mathcal{C}' \end{array} = \begin{array}{ccccc} \mathcal{C} & = & \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ {\scriptstyle L_1}\downarrow & \swarrow_{\alpha} & {\scriptstyle L_2}\downarrow & \swarrow_{\lambda_2} & \downarrow{\scriptstyle L'_2} \\ \mathcal{D} & = & \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' \end{array}$$

$$(2)\quad \begin{array}{ccccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & = & \mathcal{C}' \\ {\scriptstyle R_1}\uparrow & \searrow^{\mu_1} & {\scriptstyle R'_1}\uparrow & \searrow^{\beta'} & \uparrow{\scriptstyle R'_2} \\ \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' & = & \mathcal{C}' \end{array} = \begin{array}{ccccc} \mathcal{C} & = & \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ {\scriptstyle R_1}\uparrow & \searrow^{\beta} & {\scriptstyle R_2}\uparrow & \searrow^{\mu_2} & \uparrow{\scriptstyle R'_2} \\ \mathcal{D} & = & \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' \end{array}$$

By Theorem 30 and assumption, the two pasting diagrams on the left-hand side of the equalities are mate, and the two pasting diagrams on the right-hand side of the equalities are mates. Because the mate construction is a bijection, the two equalities are equivalent. □

Remark 39. Using pasting diagrams shed light on why Definition 74 and Definition 75 are very similar yet different. They consist in the same kind of equation except that the first one is vertical, and the second one is horizontal. This also explain why an alternative proof consisting in unfolding the definition of the mate construction is very straightforward in Proposition 44, but not straightforward at all in the case of Proposition 45. The reason is that the functoriality of the mate construction for the vertical composition holds almost by definition of the mate, whereas the functoriality of the mate construction for the horizontal composition involves a non-trivial computation that would get duplicated multiple times in the proof.

Corollary 8. *Assume that $L'_i = L_i$ and $R'_i = R_i$, that H, K can be equipped with a comonad structure $\overline{H}$ and $\overline{K}$, and that $\lambda_i : L_i\overline{H} \Rightarrow \overline{K}L_i$ and $\mu_i : \overline{H}R_i \Rightarrow R_i\overline{K}$ are distributive laws and are mate. Assume that $\alpha : L_2 \Rightarrow L_1$ and $\beta : R_1 \Rightarrow R_2$ are mates. Then α is a morphism of distributive law (from λ_2 to λ_1) if and only if β is a morphism of distributive law (from μ_1 to μ_2). The same property hold when taking a comonad structure $\underline{H}$ and $\underline{K}$ instead.*

Proof. The commutation making α a morphism of distributive law is the same as the one expressing that the pair (α, α) is compatible with the squares $\lambda_1 : L_1\overline{H} \Rightarrow \overline{K}L_1$ and $\lambda_2 : L_2\overline{H} \Rightarrow \overline{K}L_2$. The commutation making β a morphism of distributive law is the same as the one expressing that the pair (β, β) is compatible with the squares $\mu_1 : \overline{H}R_1 \Rightarrow R_1\overline{K}$ and $\mu_2 : \overline{H}R_2 \Rightarrow R_2\overline{K}$. We conclude by Proposition 45. □

Corollary 9. *Assume that $\overline{L} = (L, \epsilon, \delta)$ is a comonad and $\underline{R} = (R, \eta, \mu)$ is a monad on the same category $\mathcal{C}$ and that they are mate. Assume that $\overline{L'} = (L', \epsilon', \delta')$ is a comonad*

and $\underline{R'} = (R', \eta', \mu')$ *is a monad on the same category* $\mathcal{C}'$ *and that they are mate. Let* $F : \mathcal{C} \to \mathcal{C}'$*. Assume that* $\lambda : L'F \Rightarrow FL$ *and* $\mu : FR \Rightarrow R'F$ *are mate. Then* $\lambda : \overline{L'}F \Rightarrow F\overline{L}$ *is a distributive law if and only if* $\mu : F\underline{R} \Rightarrow \underline{R'}F$ *is a distributive law.*

Proof. λ is a distributive law if the following diagrams commute.

$$\begin{array}{ccc} L'F & \xrightarrow{\lambda} & FL \\ {\scriptstyle \epsilon' F}\downarrow & & \downarrow{\scriptstyle F\epsilon} \\ F & = & F \end{array} \qquad \begin{array}{ccccc} L'F & & \xrightarrow{\quad\lambda\quad} & & FL \\ {\scriptstyle \delta' F}\downarrow & & & & \downarrow{\scriptstyle F\delta} \\ L'L'F & \xrightarrow[L'\lambda]{} & L'FL & \xrightarrow[\lambda L]{} & FLL \end{array}$$

The diagram on the left can be interpreted as the fact that (ϵ, ϵ') is compatible with λ and id. The diagram on the right can be interpreted as the fact that (δ, δ') is compatible with the composite $\lambda L \circ L'\lambda$ and λ. On the other hand, μ is a distributive law is the following diagrams commute.

$$\begin{array}{ccc} F & = & F \\ {\scriptstyle F\eta}\downarrow & & \downarrow{\scriptstyle \eta' F} \\ FR & \xrightarrow[\mu]{} & R'F \end{array} \qquad \begin{array}{ccccc} FRR & \xrightarrow{\mu R} & R'FR & \xrightarrow{R'\mu} & R'R'F \\ {\scriptstyle F\mu}\downarrow & & & & \downarrow{\scriptstyle \mu' F} \\ FR & & \xrightarrow[\mu]{\qquad\qquad} & & R'F \end{array}$$

The diagram on the left can be interpreted as the fact that (η, η') is compatible with μ and id. The diagram on the right can be interpreted as the fact that (μ, μ') is compatible with the composite $R'\mu \circ \mu R$ and μ. By Proposition 43, $\lambda L' \circ L'\lambda$ and $R'\mu \circ \mu R$ are mate. So we can apply Proposition 45 to conclude that those two assumption are equivalent. □

Corollary 10. *Under the same assumption as in Corollary 9 except that* $\underline{L}, \underline{L'}$ *are monads and* $\overline{R}, \overline{R'}$ *are comonads,* $\lambda : \underline{L'}H \Rightarrow K\underline{L}$ *is a distributive law if and only if* $\mu : H\overline{R} \Rightarrow \overline{R'}K$ *is a distributive law.*

Proof. The proof is the same as Corollary 9. □

8.4 Application to lax and oplax structures

Any adjunctions $\eta_1, \epsilon_1 : L_1 \dashv R_1$ and $\eta_2, \epsilon_2 : L_2 \dashv R_2$, induces an adjunction $(\eta_1, \eta_2), (\epsilon_1, \epsilon_2) : L_1 \times L_2 \dashv R_1 \times R_2$.

Proposition 46. *If* $\lambda_1 : L_1'H_1 \Rightarrow K_1L_1$ *and* $\mu_1 : H_1R_1 \Rightarrow R_1'K_1$ *are mates, and if* $\lambda_2 : L_2'H_2 \Rightarrow K_2L_2$ *and* $\mu_2 : H_2R_2 \Rightarrow R_2'K_2$ *are mates then*

$$(\lambda_1, \lambda_2) : (L_1' \times L_2')(H_1 \times H_2) \Rightarrow (K_1 \times K_2)(L_1 \times L_2)$$
$$(\mu_1, \mu_2) : (H_1 \times H_2)(R_1 \times R_2) \Rightarrow (R_1' \times R_2')(K_1 \times K_2)$$

are mates.

Proof. Straightforward computation. □

Remark 40. It is very likely that the observation above means that **Adj** is monoidal, for a suitable notion of monoidal double category.

Assume that $(\mathcal{C}, \otimes, 1)$ and $(\mathcal{D}, \bullet, I)$ are (symmetric) monoidal categories. The adjunction $L \dashv R$ induces a bijection between the morphisms $m^0 \in \mathcal{C}(1, RI)$ and the morphisms $n^0 \in \mathcal{D}(LI, 1)$. This can also be seen as an instance of the mate construction when taking $H = K$

to be the constant functor 1 on $\mathcal{C}$. The mate construction applied on the bifunctor $\_\otimes\_ : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and taking the adjunctions to be $L\dashv R$ and $L\times L\dashv R\times R$ induces a bijection between natural transformations $n^2_{X,Y}\in\mathcal{D}(L(X\otimes Y), LX\bullet LY)$ and $m^2_{X,Y}\in\mathcal{C}(RX\otimes RY, R(X\bullet Y))$. The result below is well known, and is also a result of doctrinal adjunction, see [Kel74].

Theorem 32. *The bijection above induces a bijection between lax monoidal structures on R and oplax monoidal structures on L. Furthermore, if R is endowed with a monad structure $\underline{R}=(R,\eta,\mu)$ and L with a comonad structure $\overline{L}=(L,\epsilon,\delta)$ that are mate, $\underline{R}$ is a lax symmetric monoidal monad if and only if $\overline{L}$ is an oplax symmetric monoidal comonad.*

Proof. The two diagram below are equivalent.

$$\begin{array}{ccccc} LX\bullet I & \xrightarrow{LX\bullet m^0} & LX\bullet L1 & \xrightarrow{m^2} & L(X\otimes 1)\\ {\scriptstyle\rho^\bullet}\downarrow & & & & \downarrow{\scriptstyle L\rho^\otimes}\\ LX & = & = & = & LX \end{array}\qquad \begin{array}{ccccc} R(X\bullet I) & \xrightarrow{n^2} & RX\otimes RI & \xrightarrow{RX\otimes n^0} & RX\otimes 1\\ {\scriptstyle R\rho^\bullet}\downarrow & & & & \downarrow{\scriptstyle \rho^\otimes}\\ RX & = & = & = & RX \end{array}$$

Indeed, they express respectively that $(\rho^\bullet,\rho^\otimes)$ is compatible with $m^2\circ(LX\bullet m^0)$ and id, and that $(\rho^\bullet,\rho^\otimes)$ is compatible with $(RX\otimes n^0)\circ n^2$ and id. But $m^2\circ(LX\bullet m^0)$ and $(RX\otimes n^0)\circ n^2$ are mates by compositionality of the mate construction and by Proposition 46. The other equivalences are similar, so (R,m^0,m^2) is lax monoidal if and only if (L,n^0,n^2) is oplax monoidal.

Furthermore, if $\underline{R}$ is a monad and $\overline{L}$ is a comonad, $n^0=\epsilon$ if and only if $m^0=\eta$ because ϵ and η are mates. Finally, the two diagrams of Definition 66 mean that m^2 is a distributive law. By Corollary 9, it is one if and only if n^2 is a distributive law, and the associated diagrams are exactly the ones turning $\overline{L}$ into an oplax monoidal comonad. □

8.5 Extension of an adjunction

Let $\eta,\epsilon : L\dashv R$ be an adjunction, with $L:\mathcal{C}\to\mathcal{D}$ and $R:\mathcal{D}\to\mathcal{C}$. Let $\underline{M}_1=(M_1,\eta^1,\mu^1)$ be a monad on $\mathcal{C}$, and $\underline{M}_2=(M_2,\eta^2,\mu^2)$ a monad on $\mathcal{D}$.

Definition 76. The adjunction $\eta,\epsilon : L\dashv R$ extends to the Kleisli categories $\mathcal{C}_{\underline{M}_1}$ and $\mathcal{D}_{\underline{M}_2}$ if there exists $\widehat{L}:\mathcal{C}_{\underline{M}_1}\to\mathcal{D}_{\underline{M}_2}$ and $\widehat{R}:\mathcal{D}_{\underline{M}_2}\to\mathcal{C}_{\underline{M}_1}$ extensions of L and R respectively, such that η and ϵ extend to natural transformations $K_{\underline{M}_1}\eta : \mathsf{Id}_{\mathcal{C}_{\underline{M}_1}}\Rightarrow\widehat{R}\widehat{L}$ and $K_{\underline{M}_2}\epsilon:\widehat{L}\widehat{R}\Rightarrow\mathsf{Id}_{\mathcal{D}_{\underline{M}_2}}$. This induces and adjunction $K_{\underline{M}_1}\eta, K_{\underline{M}_2}\epsilon:\widehat{L}\dashv\widehat{R}$.

Proposition 47. *The adjunction $\eta,\epsilon : L\dashv R$ extends to the Kleisli categories $\mathcal{C}_{\underline{M}_1}$ and $\mathcal{D}_{\underline{M}_2}$ if and only if there exists two distributive laws $\lambda^L : L\underline{M}_1\Rightarrow\underline{M}_2L$ and $\lambda^R : R\underline{M}_2\Rightarrow\underline{M}_1R$ such that the following diagrams commute.*

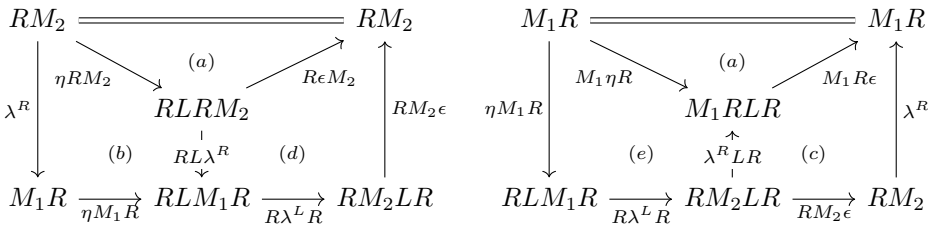
$$\begin{array}{ccccc} M_1 & = & = & = & M_1\\ {\scriptstyle\eta M_1}\downarrow & & & & \downarrow{\scriptstyle M_1\eta}\\ RLM_1 & \xrightarrow[R\lambda^L]{} & RM_2L & \xrightarrow[\lambda^RL]{} & M_1RL \end{array}\qquad \begin{array}{ccccc} LRM_2 & \xrightarrow{L\lambda^R} & LM_1R & \xrightarrow{\lambda^LR} & M_2LR\\ {\scriptstyle\epsilon M_2}\downarrow & & & & \downarrow{\scriptstyle M_2\epsilon}\\ M_2 & = & = & = & M_2 \end{array} \tag{16}$$

Proof. By Theorem 25, the existence of the two distributive laws is a necessary and sufficient condition to ensure the existence of the two extension $\widehat{L}$ and $\widehat{R}$. The diagrams of Eq. (16) mean that η and ϵ are morphisms of distributive laws for the distributive law associated to the extensions $\widehat{L}\widehat{R}$ and $\widehat{R}\widehat{L}$, so by Theorem 25, it is equivalent to the fact that η and ϵ extend to natural transformations $K_{\underline{M}_1}\eta$ and $\mathcal{C}_{\underline{M}_2}\epsilon$. □

Proposition 48. *Let $\lambda^L : LM_1 \Rightarrow M_2L$ and $\lambda^R : RM_2 \Rightarrow M_1R$ be two natural transformations (we do not assume that they are distributive laws). Then the diagrams of Eq. (16) commute if and only if λ^R is the inverse of the mate μ^L of λ^L:*

$$\mu^L = \; M_1R \xrightarrow{\eta M_1 R} RLM_1R \xrightarrow{R\lambda^L R} RM_2LR \xrightarrow{RM_2\epsilon} RM_2$$

Proof. Assume that the diagrams of Eq. (16) commute. We show that λ^R is the inverse of μ^L with the following diagram chase.



Commutation (a) is the triangle identities of the adjunction, (b) is the naturality of η, (c) is the naturality of λ^R, and (d) and (e) are the diagram of Eq. (16).

The converse direction is a similar computation. □

Theorem 33. *The adjunction $\eta, \epsilon : L \dashv R$ extends to the Kleisli categories $\mathcal{C}_{\underline{M}_1}$ and $\mathcal{D}_{\underline{M}_2}$ if and only if there exists $\lambda^L : LM_1 \Rightarrow M_2L$ and $\lambda^R : RM_2 \Rightarrow M_1R$ two natural transformations such that $\lambda^L : L\underline{M}_1 \Rightarrow \underline{M}_2L$ is a distributive law and λ^R is the inverse of its mate.*

Proof. By Corollary 7, the mate μ^L of λ^L is a distributive law $\underline{M}_1R \Rightarrow R\underline{M}_2$. Since λ^R is the inverse of μ^L, it must be a distributive law $\lambda^R : R\underline{M}_2 \Rightarrow \underline{M}_1R$. Furthermore, by Proposition 48 the diagrams of Eq. (16) hold, so by Proposition 47 the adjunction extends to the Kleisli categories.

□

9 Conclusion

We have developed a theory of Taylor expansion in categories which are not necessarily additive. The main motivations for this work are first that Taylor expansion has been shown to be a useful tool in the analysis of functional programs, see for instance [BM20], and second that most concrete denotational models of such languages (such as coherence spaces, probabilistic coherence spaces etc) feature only a *partial* addition of morphisms for the very good reason that full additivity prevents denotational models from accounting for the determinism of computations. For instance, the values $\mathtt{tt}$ and $\mathtt{ff}$ of the object of booleans should not be summable in a deterministic model. In the very same line of idea, the uniformity of the Taylor expansion observed in [ER08] seems to be closely related to the summability constraints observed in coherence spaces and non-uniform coherence spaces, and accounts syntactically for to the fundamental determinism of the λ-calculus (Church-Rosser and Standardization theorems) and of the execution of terms in the Krivine machine.

It turns out that all the categorical axiomatizations of denotational models which account for the Taylor expansion of morphisms, and are most often based on differential LL, make the assumption that homsets are monoids where infinite summations are possible for the

obvious reason that infinite summations are an essential ingredient of Taylor expansion. We have shown in this paper that this strong form of additivity is not a fatality: Taylor expansion can also exist in settings where only a partial version of (finite and infinite) addition is available.

Differentiation was already accommodated in such partially additive categories in [Ehr23] and the approach developed here follows a similar pattern. One main difference is that we had to develop a more subtle notion of infinitary (countable) summability. Beyond this main difference, the resulting theory of coherent Taylor expansion is strikingly similar to that of coherent differentiation — and not essentially more complicated —, with one additional nice feature: the resulting Taylor functor is not only a monad (just as the tangent functor in the tangent categories of [Ros84]) but also a comonad. This comonad structure, and more precisely the naturality of its counit, reflects the fact that nonlinear morphisms coincide with their Taylor expansion, expressing abstractly that they are analytic.

We have developed this theory in a LL setting of resource categories, where the analytic structure arises as a distributive law wrt. the resource comonad, and also in general cartesian categories, following the main ideas of [EW23].

This first denotational investigation of coherent Taylor expansion is a strong incentive for developing now a syntactic analysis of this operation, which might be similar to the coherent differential PCF of [Ehr22a], this will be the object of further work. Another natural question is whether this coherent Taylor expansion has an associated resource calculus, just like Taylor expansion in the setting of differential LL, see [ER08].

The connection between coherent differentiation and coherent Taylor expansion also deserves further study: such a study might be based on the observation in Section 5 that there are simple models of LL which accommodate coherent differentiation but not coherent Taylor expansion.

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