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NONCOMMUTATIVE SYMMETRIC FUNCTIONS AND LAGRANGE INVERSION II: NONCROSSING PARTITIONS AND THE FARAHAT-HIGMAN ALGEBRA

JEAN-CHRISTOPHE NOVELLI AND JEAN-YVES THIBON

ABSTRACT. We introduce a new pair of mutually dual bases of noncommutative symmetric functions and quasi-symmetric functions, and use it to derive generalizations of several results on the reduced incidence algebra of the lattice of noncrossing partitions. As a consequence, we obtain a quasi-symmetric version of the Farahat-Higman algebra.

1. INTRODUCTION

By the *Lagrange series*, we shall mean the (unique) formal power series

$$(1) \quad g(t) = \sum_{n \geq 0} g_n t^n$$

solving the functional equation

$$(2) \quad g(t) = f(tg(t)) = \sum_{n \geq 1} f_n t^n g(t)^n \text{ where } f(t) = \sum_{n \geq 0} f_n t^n, f_0 = 1.$$

Besides its numerous applications in enumerative combinatorics, where the f_n are specified numbers, the *generic* Lagrange series (where the f_n are indeterminates) is of great interest in algebraic combinatorics. Specifically, if one interprets the f_n as the homogenous symmetric functions $f_n = h_n(X)$, the symmetric function $g_n(X)$

- (i) is the Frobenius characteristic of the representation of the symmetric group \mathfrak{S}_n on the set PF_n of parking functions of length n [11];
- (ii) provides an isomorphism between the reduced incidence Hopf algebra \mathcal{H}_{NC} of the family of lattices of noncrossing partitions and the Hopf algebra Sym of symmetric functions by sending the class y_n of $[\mathbf{0}_{n+1}, \mathbf{1}_{n+1}]$ to g_n [7];
- (iii) provides an isomorphism between the Farahat-Higman algebra of symmetric groups and symmetric functions by identifying¹ the reduced classes c_μ with the dual basis of $g^\mu := g_{\mu_1} \cdots g_{\mu_r}$ [15, 9, 10].

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¹Actually, Macdonald uses the equivalent basis $h_\mu^*(X) = g^\mu(-X)$.

The Lagrange series has a natural noncommutative version, already apparent in the original version of Raney's combinatorial proof [21]: if in (2) the f_n are interpreted as non-commuting variables, including f_0 , g_n becomes the sum of all Łukasiewicz words of length $n + 1$: writing for short $f_{i_1 i_2 \dots}$ for $f_{i_1} f_{i_2} \dots$,

$$(3) \quad g_0 = f_0, \quad g_1 = f_{10}, \quad g_2 = f_{200} + f_{110}, \quad g_3 = f_{3000} + f_{2100} + f_{2010} + f_{1200} + f_{1110}, \dots$$

i.e., the Polish codes for plane rooted trees on n vertices (obtained by reading the arities of the nodes in prefix order). These words also encode in a natural way various Catalan sets. Setting $f_i = a^i b$, we obtain Dyck words (with an extra b at the end). Seeing $f_{i_1 i_2 \dots i_r}$ as encoding the nondecreasing word $1^{i_1} 2^{i_2} \dots r^{i_r}$, we obtain a nondecreasing parking function, which can itself be decoded as a noncrossing partition, whose blocks are encoded by their minimal elements repeated as many times as the lengths of the blocks.



For example, the word f_{2100} encodes the plane tree , the Dyck word $aababb \cdot b$, the nondecreasing parking function 112 and the noncrossing partition 13|2.

The noncommutative Lagrange series can be interpreted as a noncommutative symmetric function: keeping the functional equation (2), we set $f_n = S_n = S_n(A)$ with $f_0 = 1$, and obtain

$$(4) \quad g_0 = 1, \quad g_1 = S_1, \quad g_2 = S_2 + S^{11}, \quad g_3 = S_3 + 2S^{21} + S^{12} + S^{111}, \dots$$

and we may ask whether there are analogues for these noncommutative symmetric functions of Properties (i), (ii), (iii).

Point (i) has been dealt with in [17]: $g_n(A)$ is the noncommutative Frobenius characteristic of the natural representation of the 0-Hecke algebra $H_n(0)$ on parking functions. Various consequences of this fact, including a noncommutative q -Lagrange formula and generalisations to (k, ℓ) -parking functions have been derived there. Other applications have been given in [18, 19, 13].

The aim of the present paper is to investigate points (ii) and (iii) in the noncommutative setting. Introducing the multiplicative basis $g^I := g_{i_1} \dots g_{i_r}$ of \mathbf{Sym} , and computing its coproduct and antipode, we obtain natural noncommutative versions of these results.

Definition 1.1. *The ordered type of a noncrossing partition is the composition formed by the length of its blocks, ordered by increasing values of their minima. Its reduced ordered type is the composition obtained from its ordered type by subtracting 1 to its components and removing the zeros.*

We define the ordered cycle type and the reduced ordered cycle type of a permutation similarly.

We have then the following interpretation of the coproduct:

Theorem 1.2. *The coefficient a_{IJ} in the coproduct*

$$(5) \quad \Delta g_n = \sum_{I, J} a_{IJ} g^I \otimes g^J$$

is equal to the number of noncrossing partitions π of $[n+1]$ of reduced ordered type I , and whose (right) Kreweras complement π' has reduced ordered type J .

This implies a quasi-symmetric refinement of Macdonald's realization of the graded Farahat-Higman algebra. For a composition $I = (i_1, \dots, i_r)$, define the canonical permutation σ_I as the permutation of $\mathfrak{S}_{|I|+r}$ whose nontrivial cycles are

$$(6) \quad (12 \dots i_1 + 1)(i_1 + 2 \dots i_1 + i_2 + 2) \dots (i_1 + \dots + i_{r-1} + r - 1 \dots i_1 + \dots + i_r + r).$$

Corollary 1.3. *Let $c_I \in QSym$ be the dual basis of g^I . The coefficient a_{JK}^I in the product*

$$(7) \quad c_J c_K = \sum_I a_{JK}^I c_I$$

is equal to the number of minimal factorizations $\sigma_I = \alpha\beta$ of the canonical permutation σ_I of reduced cycle type I with α of reduced cycle type J and β of reduced cycle type K .

While this gives back the result of Macdonald by summing over compositions with the same underlying partitions, the a_{JK}^I only count factorizations of particular permutations, and this result is rather to be interpreted as providing a noncommutative version of the reduced incidence algebra of the lattices of noncrossing partitions.

For example, Δg_5 contains the terms $7g^{12} \otimes g^{11}$ and $11g^{21} \otimes g^{11}$, so any 6-cycle in \mathfrak{S}_n has $18 = 7 + 11$ factorizations into permutations of cycle types (321^{n-5}) and (221^{n-4}) , but the refined coefficients 7 and 11 are only meaningful for the particular 6-cycle (123456) .

Theorem 1.2 is a noncommutative analogue of the main result of [7], which establishes an isomorphism of Hopf algebras between the reduced incidence algebra \mathcal{H}_{NC} of noncrossing partitions and symmetric functions. Another result of [7] is a combinatorial description of the antipode of \mathcal{H}_{NC} . This amounts to computing $g(-X)$ in the basis g^μ .

Rather than working with the antipode, we shall work with the automorphism $S_n(A) \mapsto S_n(-A) = (-1)^n \Lambda_n(A)$, and prove the equivalent result

Theorem 1.4. *Define coefficients a_I by*

$$(8) \quad g_n(-A) = \sum_{I \models n} (-1)^{\ell(I)} a_I g^I$$

Then,

$$(9) \quad a_I = \sum_{J \leq 2I} \langle M_J, g_{2n} \rangle = \langle E_{2I}, g_{2n} \rangle$$

where E is the so-called essential basis of quasi-symmetric functions. It is equal to the number of sylvester classes of words of evaluation $2\bar{I}$ [18, 19] or alternatively, to the number of parking quasi-ribbons of shape $(2I)^\sim$ [17], and also to the number of nondecreasing parking functions of type $2I + 1^r := (2i_1 + 1, \dots, 2i_r + 1)$, which is the same as the number of plane trees whose arities of internal nodes read in prefix order form the composition $2I + 1^r$.

For example, the term $5g^{21}$ in

$$(10) \quad g_3(-A) = -g_3 + 5g^{21} + 3g^{12} - 12g^{111}$$

corresponds to the 5 parking quasi-ribbons of shape $(42)^\sim = 12111$ which are

$$(11) \quad 1|23|4|5|6, 1|22|4|5|6, 1|22|3|5|6, 1|22|3|4|6, 1|22|3|4|5,$$

and the term $3g^{12}$ corresponds to the 3 parking quasi-ribbons of shape $(24)^\sim = 11121$

$$(12) \quad 1|2|3|45|6, 1|2|3|44|6, 1|2|3|44|5.$$

The term $5g^{21}$ corresponds also to the 5 sylvester classes of evaluation 24, which are those of the words

$$(13) \quad 112222, 211222, 221122, 222112, 222211,$$

which can be read by filling the sectors of the plane trees of skeleton 53 as in [12].

The coefficient \tilde{a}_I in the antipode

$$(14) \quad \tilde{\omega}(g_n) = \sum_{I \models n} (-1)^{\ell(I)} \tilde{a}_I g^I$$

also has an explicit, but more complicated interpretation.

$$(15) \quad \langle c_I, \tilde{\omega}(g_n) \rangle = (-1)^n \sum_{J \models n} \langle V_I, g^J \rangle \langle M_{\bar{J}}, g \rangle.$$

The factor $\langle M_{\bar{J}}, g \rangle$ is a number of nondecreasing parking functions and the $\langle V_I, g^J \rangle$ count parking quasi-ribbons and have all the same sign. This is therefore a cancellation-free combinatorial formula.

For example,

$$(16) \quad \tilde{\omega}(g_3) = -12g^{111} + 4g^{12} + 4g^{21} - g^3$$

and the contributions to the coefficient of g^{21} are

$$(17) \quad \langle V_{21}, g_3 \rangle \langle M_3, g \rangle = -3 \times 1$$

where the factor -3 counts the parking quasi-ribbons $11|2, 11|3, 12|3$, and

$$(18) \quad \langle V_{21}, g^{21} \rangle \langle M_{12}, g \rangle = -1 \times 1$$

where, dualizing, $\langle V_{21}, g^{21} \rangle = \langle V_2 \otimes V_1, g_2 \otimes g_1 \rangle = -1 \times 1$.

Similarly, the contributions to the coefficient of g^{12} are

$$(19) \quad \langle V_{12}, g_3 \rangle \langle M_3, g \rangle = -2 \times 1,$$

the -2 counts the parking quasi-ribbons $1|22, 1|23$, and

$$(20) \quad \langle V_{12}, g^{12} \rangle \langle M_{21}, g \rangle = -1 \times 2,$$

where, dualizing, $\langle V_{12}, g^{12} \rangle = \langle V_1 \otimes V_2, g_1 \otimes g_2 \rangle = -1 \times 1$.

Byproducts. The proofs of the aforementioned results rely on a couple of elementary combinatorial properties that we have not been able to find in the literature and appear to be of independent interest.

First, given a binary tree T and its infix labeling (*i.e.*, its corresponding binary search tree), we describe a straightforward algorithm for visiting cyclically its nodes (*i.e.*, going from the node labelled i to that labelled $i + 1 \pmod n$): move one step down the right branch of i (if i is at its bottom, then go to the top of the branch), then move one step up the current left branch (again, if i is at its top, go to the bottom of the branch). This property is easily proved by observing that any such tree hides two permutations whose product is the standard long cycle, see Note 4.10.

Second, we prove that a noncrossing partition can be reconstructed from its ordered type and the ordered type of its right Kreweras complement, and provide an algorithm doing this. This property is the key ingredient in the calculation of Δg_n , see Theorem 4.11.

This paper is a continuation of [17], to which the reader is referred for background and notation.

2. THE LAGRANGE BASES OF **Sym** AND $QSym$

2.1. The Lagrange basis in **Sym.** The Lagrange series in $\mathbf{Sym}(A)$ is defined by

$$(21) \quad g(A) = 1 + \sum_{n \geq 1} S_n(A)g(A)^n$$

and we denote by g_n its homogenous component of degree n . If $X = (x_i)$ is a sequence of mutually commuting variables, $g_n(X)$ becomes an ordinary symmetric function. It is equal to $h_n^*(-X)$ where $*$ is Macdonald's involution [15, Ex. 24 p. 36].

As mentioned in the introduction, it was shown in [17] that

$$(22) \quad g_n(A) = \sum_{\pi \in \text{NDPF}(n)} S^{\text{Ev}(\pi)},$$

where NDPF is the set of nondecreasing parking functions and $\text{Ev}(\pi)$ is the *evaluation* of π , that is, the ordered sequence of number of occurrences of i in π for $i \geq 1$. Since by convention $S_0 = 1$, we can replace $\text{Ev}(\pi)$ by the *packed evaluation*, or *type* $t(\pi)$ of π , which is the composition obtained by removing the zeros in $\text{Ev}(\pi)$.

For example, there are five nondecreasing parking functions: 111, 112, 113, 122, and 123. Forgetting the trailing zeroes, their respective evaluations are respectively 3, 21, 201, 12, and 111, so that,

$$(23) \quad g_3 = S_3 + 2S^{21} + S^{12} + S^{111}.$$

Since g_n begins with a term S_n , their products $g^I = g_{i_1} \dots g_{i_r}$ are triangular on the S^I hence form a basis of **Sym**.

Since the coefficient of S^J in g_n is the number of nondecreasing parking functions of type J , or equivalently the number of noncrossing partitions of ordered type J , the coefficient of S^J in g^I is the number of nondecreasing parking functions of type

J having breakpoints at the descents of I , or the number of noncrossing partitions of ordered type J finer than the interval partition of type I .

Ordering compositions in reverse lexicographic order, *e.g.*, [3, 21, 12, 111] for $n = 3$, the matrix of the g^J on the S^I is

$$(24) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

2.2. A related basis. This combinatorial description suggests to introduce another basis

$$(25) \quad f^I := \sum_{J \geq I} (-1)^{\ell(I) - \ell(J)} g^J,$$

where $J \geq I$ means that J is finer than I , or that the descents of I are descents of J .

The transition matrix from the f to the S is much simpler: the coefficients are nonnegative integers and each nondecreasing parking function contributes to the row indexed by its type and to the column indexed by its breakpoints.

For example, at $n = 3$, the combinatorial description and the matrix from f to S are as follows:

$$(26) \quad \begin{pmatrix} 111 & & & \\ 112 & 113 & & \\ & & 122 & \\ & & & 123 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ . & . & 1 \\ . & . & . & 1 \end{pmatrix}$$

This basis will be investigated in a separate paper in relation to the quasi-symmetric Farahat-Higman algebra.

2.3. The dual Lagrange basis in $QSym$. We denote by $c_I \in QSym$ the dual basis of (g^I) .

By definition of the duality between **Sym** and $QSym$ the transpose of the matrix in (24) is the matrix of the monomial quasi-symmetric functions M_I in the basis c_J .

2.4. Some other relevant properties. The expansions of g_n on the bases S^I , Λ^I and R_I are given in [17]. It is also proved in this reference that g is invariant under the involution $S^I \mapsto S^{I^\sim}$, and that $g(-A)$ satisfies the functional equation

$$(27) \quad g(-A)^{-1} = \sum_{n \geq 0} S_n(A) g(-A)^n,$$

that is, $g_n(-A)$ is the image of $S_n(A)$ by the antipode of the Hopf algebra of non-commutative formal diffeomorphisms of [3].

2.5. The k -Lagrange series. We shall also need the series $g^{(k)}$, defined by the functional equation

$$(28) \quad g^{(k)} = \sum_{n \geq 0} S_n [g^{(k)}]^{kn}.$$

It can also be defined as $g^{(k)} = \phi_k(g)$, where ϕ_k is the adjoint of the power-sum plethysm operator $\psi^k : M_I \mapsto M_{kI}$ on $QSym$.

Recall that a k -parking function is a word over the positive integers whose nondecreasing rearrangement $a_1 a_2 \cdots a_n$ satisfies $a_i \leq k(i - 1) + 1$. Its k -evaluation $\text{Ev}_k(\mathbf{a})$ is essentially the classical evaluation of a word but we will here define it as the number of occurrences of all letters from 1 to $kn + 1$. In particular, the k -evaluation of a nonempty k -parking function ends with a sequence of at least k zeros.

Indeed, we have proved in [17] that the solution of (28) where S_0 is an indeterminate is

$$(29) \quad g^{(k)} = \sum_{\pi \in \text{NDPF}^{(k)}} S^{\text{Ev}_k(\pi)},$$

where $\text{NDPF}^{(k)}$ stands for nondecreasing k -parking functions.

In particular, if one sends S_0 to 1, the coefficient of S^I in $g_n^{(k)}$ of degree n is the number of nondecreasing k -parking functions of type I .

For example, setting $h = \phi_2(g) = g^{(2)}$,

$$(30) \quad h_0 + h_1 + h_2 + h_3 + \cdots = S_0 + S_1(h_0 + h_1 + h_2 + \cdots)^2 + g_2(h_0 + h_1 + \cdots)^4 + g_3(h_0 + \cdots)^6$$

yields, by iterated substitutions

$$(31) \quad h_0 = 1, \quad h_1 = S_1, \quad h_2 = S^2 + 2S^{11}, \quad h_3 = S_3 + 4S^{21} + 2S^{12} + 5S^{111},$$

The 2-parking functions of size 3 are

$$(32) \quad 111, 112, 113, 114, 115, 122, 133, 123, 124, 125, 134, 135,$$

and one can check that their types indeed encode the expansion of h_3 .

The k -evaluations of k -parking functions are generalized Łukasiewicz words. They are the words $w_1 \dots w_{kn+1}$ of length $kn + 1$ whose partial sums $k(w_1 + \cdots + w_i) - i$ are always nonnegative except when $i = kn + 1$ where the sum becomes strictly negative. This property is easily translated in terms of generalized Dyck paths: send w_i to w_i times the step $(1, k)$ followed by a step $(1, -1)$. The conditions on the evaluations mean that the path stays weakly above the axis on all steps but the last.

2.6. Connecting the k -Lagrange series. There is a simple but useful connection between k -parking functions and $(k - 1)$ -parking functions. Let \mathbf{a} be a k -parking function and let \mathbf{a}' be its largest prefix that is a $(k - 1)$ -parking function.

In terms of evaluations, this means that $\text{Ev}(\mathbf{a}')$ is a prefix of $\text{Ev}(\mathbf{a})$. If \mathbf{a}' is considered as a $(k - 1)$ -parking function, the corresponding path ends at height -1 , and ends at height $k - 1$ as a k -parking function. Now, since each downstep decrements the height by one, one can cut the remainder of the path of \mathbf{a} the first time it reaches each height from $k - 2$ down to 0. One then gets a total of k (possibly one-downstep) paths, all encoding a k -parking function. Conversely, given a $(k - 1)$ -parking function

of length i and a list of i k -parking functions, one obtains a k -parking function by concatenating their evaluations.

For example, consider the 2-parking function 12246911141717. Its evaluation (up to 21) is

$$(33) \quad 120101001010010020000$$

Its largest 1-parking prefix is 1224. It is of size 4 and its evaluation (as a 1-parking function) is 12010. We then remove the prefix and cut the remainder into four parts as

$$(34) \quad 1001010010020000 = 100.10100.100.20000,$$

that all are evaluations of 2-parking functions.

Thus, a k -parking function can be uniquely decomposed as $\mathbf{a} = \mathbf{a}'\mathbf{b}_1 \cdots \mathbf{b}_i$ where \mathbf{a}' is its maximal $(k-1)$ -parking prefix of length i and the \mathbf{b}_j are k -parking functions. This translates into the following functional equation:

Lemma 2.1. *The series $g^{(k)} := \phi_k(g)$ satisfies*

$$(35) \quad g^{(k)} = \sum_{n \geq 0} g_n^{(k-1)} [g^{(k)}]^n.$$

■

For example, setting $h = \phi_2(g) = g^{(2)}$,

$$(36) \quad h_0 + h_1 + h_2 + h_3 + \cdots = g_0 + g_1(h_0 + h_1 + h_2 + \cdots) + g_2(h_0 + h_1 + \cdots)^2 + g_3(h_0 + \cdots)^3$$

yields

$$(37) \quad \begin{aligned} h_0 &= 1, & h_1 &= g_1 = S_1, & h_2 &= g_1 h_1 + g_2 = S^2 + 2S^{11}, \\ h_3 &= g_1 h_2 + 2g_2 h_1 + g_3 = S_3 + 4S^{21} + 2S^{12} + 5S^{111}, \end{aligned}$$

where g_i is replaced by its expansion on the S^I as in (4).

2.7. Base change from S to g . To compute the change of basis from S to g , we proceed as in [13]. In this reference, “noncommutative free cumulants” K_n are defined by the functional equation

$$(38) \quad \sigma_1 = \sum_{n \geq 0} K_n \sigma_1^n$$

and it is proved that

$$(39) \quad K(A) = \sum_{n \geq 0} K_n(A) = g(-A)^{-1}.$$

Setting $g(A) = \sigma_1(B)$, we see that $K_n(B) = S_n(A)$ and that

$$(40) \quad K_n = \sum_I k_I S^I \Leftrightarrow S_n = \sum_I k_I g^I.$$

To expand S_n on the basis g^I , we can therefore apply the recipe given in [13, Eq. (50)]: start from the expansion of g_{n-1} on the elementary basis, as given in [17], and replace each Λ^I by $g^{i_1+1, i_2, \dots, i_r} - g^{1I}$.

For example, starting with

$$(41) \quad g_3 = \Lambda^3 - 3\Lambda^{21} - 2\Lambda^{12} + 5\Lambda^{1111},$$

this substitution yields

$$(42) \quad S_4 = (g^4 - g^{13}) - 3(g^{31} - g^{121}) - 2(g^{22} - g^{112}) + 5(g^{211} - g^{1111}).$$

The first values are

$$\begin{aligned} S_1 &= g_1 \\ S_2 &= g_2 - g^{11} \\ S_3 &= g_3 - 2g^{21} - g^{12} + 2g^{111} \\ S_4 &= g_4 - 3g^{31} - g^{13} - 2g^{22} + 5g^{211} + 3g^{121} + 2g^{112} - 5g^{1111} \end{aligned}$$

and one checks that the inverse matrix of (24) is indeed

$$(43) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -1 & -1 & 1 \end{pmatrix}$$

2.8. An involution. It is not immediate that there is an analogue of Macdonald's star involution in the noncommutative setting. Indeed, g does not commute with the S_n , and writing (21) in the (ambiguous) form $g = \sigma_g(A)$ does not allow to conclude that $g^{-1} = \lambda_{-g}(A)$. However, this relation does hold, and we have:

Proposition 2.2. *The algebra automorphism $F \mapsto \tilde{F}$ defined on the elementary symmetric functions by*

$$(44) \quad \Lambda_n \mapsto \tilde{\Lambda}_n = g_n$$

*is an involution of **Sym**.*

Proof – The noncommutative free cumulants being given by

$$(45) \quad K_n(A) = \bar{g}_n(-A) \quad \text{where} \quad \bar{g}(A) := g(A)^{-1},$$

we have therefore

$$(46) \quad \sigma_1(-A) = \sum_{n \geq 0} \bar{g}_n(A) \sigma_1(-A)^n$$

so that

$$(47) \quad (-1)^n \Lambda_n(A) = S_n(-A) = g_n|_{S_k \mapsto \bar{g}_k} = \sum_{I \models n} c_I \bar{g}^I \quad \text{where} \quad g_n =: \sum_{I \models n} c_I S^I,$$

and if on the one hand we define coefficients b_J^I by

$$(48) \quad (-1)^{|I|} \Lambda^I = \sum_J b_J^I S^J,$$

then $\bar{g}^I = \sum_J b_J^I g^J$, and

$$(49) \quad \Lambda_n = \sum_{I, J} c_I b_J^I g^J.$$

But on the other hand

$$(50) \quad (-1)^n \sum_{I, J} c_I b_J^I \Lambda^J = (-1)^n \sum_I c_I (-1)^{|I|} S^I = g_n. \quad \blacksquare$$

2.9. Proof of Theorem 1.4. With this at hand, we can compute the first values of $g_n(-A)$:

$$\begin{aligned} g_1(-A) &= -g_1 \\ g_2(-A) &= -g_2 + 3g^{11} \\ g_3(-A) &= -g_3 + 5g^{21} + 3g^{12} - 12g^{111} \\ g_4(-A) &= -g_4 + 7g^{31} + 5g^{22} + 3g^{13} - 25g^{211} - 18g^{121} - 12g^{112} + 55g^{1111}. \end{aligned}$$

One can observe that the sum of the absolute values of the coefficients build up the sequence 1,4,21,126,... [23, A003168] and that the coefficients refine the triangle [23, A102537], which occurs in [19, Sec. 5.3]. This suggests that the coefficient of $\pm g^I$ should count sylvester classes of packed words of evaluation $2\bar{I}$.

We propose to show

$$(51) \quad g_n(-A) = \sum_{I \models n} (-1)^{\ell(I)} \left(\sum_{J \leq 2I} \langle M_J, g \rangle \right) g^I = \sum_{I \models n} (-1)^{\ell(I)} \left(\sum_{J \leq I} \langle M_J, \phi_2(g) \rangle \right) g^I,$$

where ϕ_2 is the adjoint of $\psi^2 : M_I \mapsto M_{2I}$ [19]. Equivalently, we want to prove that

$$(52) \quad g_n(-A) = \sum_{I \models n} \langle M_I, \phi_2(g(-A)) \rangle g^I.$$

We start from the expansion

$$(53) \quad g_n(-A) = (-1)^n \sum_{I \models n} \langle M_I, g \rangle \Lambda^I.$$

Let V_I be the dual basis of Λ^I . According to the previous considerations, we can write

$$\begin{aligned} g_n(-A) &= (-1)^n \sum_{I \models n} \langle M_I, g \rangle \sum_{I_1 \models i_1, \dots, I_r \models i_r} \langle V_{I_1}, g \rangle \cdots \langle V_{I_r}, g \rangle g^{I_1 I_2 \cdots I_r} \\ (54) \quad &= (-1)^n \sum_{J \models n} g^J \sum_{I \leq J} \langle M_I, g \rangle \langle \Delta^r V_J, g_{i_1} \otimes \cdots \otimes g_{i_r} \rangle \\ &= (-1)^n \sum_{J \models n} \left(\sum_{I \leq J} \langle M_I, g \rangle \langle V_J, g^I \rangle \right) g^J. \end{aligned}$$

We are thus reduced to show

$$(55) \quad \langle M_I, \phi_2(g(-A)) \rangle = (-1)^n \sum_{J \leq I} \langle V_I, g^J \rangle \langle M_J, g \rangle.$$

Summing the right-hand sides multiplied by S^I yields

$$\begin{aligned} \sum_I (-1)^{|I|} \sum_{J \leq I} \langle V_I, g^J \rangle \langle M_J, g \rangle S^I &= \sum_I \sum_{J \leq I} \langle V_I, g^J \rangle \langle M_J, g \rangle \Lambda^I(-A) \\ &= \sum_J \langle M_J, g \rangle \sum_{I \geq J} \langle V_I, g^J \rangle \Lambda^I(-A) \\ &= \sum_J \langle M_J, g \rangle g^J(-A). \end{aligned}$$

Doing the same with the left-hand sides, we have finally to show that

$$(56) \quad \phi_2(g(-A)) = \sum_J \langle M_J, g \rangle g^J(-A),$$

or equivalently, that

$$(57) \quad h := \sum_J \langle M_J, g \rangle g^J$$

satisfies

$$(58) \quad h = \sum_{n \geq 0} g_n h^n.$$

which follows from Lemma 2.1, since h defined as above is obtained by substituting g_n to S_n in g .

This concludes the proof of Theorem 1.4.

2.10. Another argument. Instead of Lemma 2.1, we can rely upon the tilde involution. This leads to a different combinatorial interpretation of the coefficients.

Recall that

$$(59) \quad \begin{aligned} g(-A)^{-1} &= \sum_{n \geq 0} S_n(A) g(-A)^n \Leftrightarrow g(-A) = 1 - \sum_{n \geq 1} S_n(A) g(-A)^{n+1} \\ &:= \sum_{n \geq 0} S_n(B) g(-A)^n, \end{aligned}$$

setting $S_1(B) = 0$ and $S_n(B) = -S_{n-1}(A)$ for $n \geq 2$. Hence, the coefficient of S^I in $g(-A)$ is equal to

$$(60) \quad \langle M_I, g(-A) \rangle = (-1)^{\ell(I)} \langle M_{I+1^r}, g(A) \rangle,$$

where $I + 1^r = (i_1 + 1, \dots, i_r + 1)$. Applying the involution $\tilde{\Lambda}_n = g_n$, and setting $h = g(-A)$, we have

$$(61) \quad \tilde{h} = \sum_{n \geq 0} \widetilde{S_n(-A)} \tilde{h}^n = \sum_{n \geq 0} (-1)^n \tilde{\Lambda}_n \tilde{h}^n = \sum_{n \geq 0} g_n(-\tilde{h})^n.$$

This is, up to signs, the functional equation for $g^{(2)} = \phi_2(g)$, so that

$$(62) \quad g_n(-A) = (-1)^n \tilde{g}_n^{(2)}.$$

Hence, the coefficient of g^I in $g_n(-A)$ is

$$(63) \quad \langle c_I, g_n(-A) \rangle = (-1)^n \langle V_I, g_n^{(2)} \rangle = (-1)^n \langle V_{2I}, g_{2n} \rangle$$

for which a combinatorial interpretation in terms of parking quasi-ribbons is given in [17]:

$$(64) \quad g_n(A) = \sum_{I \models n} (-1)^{n-\ell(I)} c_I \Lambda^I,$$

where c_I is the number of parking quasi-ribbons of shape I .

We can also give a third combinatorial interpretation of $g(-A)$. The dual basis of Λ^I is

$$(65) \quad V_I = (-1)^{n-\ell(I)} \sum_{J \leq I} M_J$$

so that the coefficient of g^I in $g(-A)$ is equal to $\langle V_{2I}, g \rangle$, hence, replacing A by $-A$, to the coefficient of $(-1)^{2n} S^{2I} = S^{2I}$ in $g(-A)$.

We have seen that the coefficient δ_I of S^I in g and the coefficient λ_I of S^I in $g(-A)$ are related by

$$(66) \quad \lambda_I = (-1)^{\ell(I)} \delta_{i_1+1, i_2+1, \dots, i_p+1}.$$

We have therefore for the absolute value of the coefficient of g^I in $g(-A)$

$$(67) \quad \sum_{J \leq 2I} \langle M_J, g \rangle = \langle M_{2I+1^r}, g \rangle$$

which is the number of nondecreasing parking functions of type $(2i_1 + 1, \dots, 2i_r + 1)$, or equivalently, to the number of plane trees whose arities of the internal nodes read in infix order yield this composition.

2.11. The antipode of g . The antipode $\tilde{\omega}(g)$ can be obtained by a slight adaptation of the argument of Section 2.10.

Let $h = \tilde{\omega}(g) = \overline{g(-A)}$. Then,

$$(68) \quad \tilde{h} = \sum_{n \geq 0} \tilde{h}^n (-1)^n g_n(A)$$

which is, up to signs,

$$(69) \quad f = \sum_{n \geq 0} f^n g_n$$

whose solution is $f = \chi(g^{(2)})$, where χ is the involution $g^I \mapsto g^{\bar{I}}$. Hence,

$$(70) \quad \tilde{\omega}(g_n) = (-1)^n \widetilde{\chi(g_n^{(2)})}.$$

The coefficient de g^I in $\tilde{\omega}(g_n)$ is therefore

$$(71) \quad \langle c_I, \tilde{\omega}(g_n) \rangle = (-1)^n \sum_{J \models n} \langle V_I, g^J \rangle \langle M_{\bar{J}}, g \rangle.$$

The factor $\langle M_{\bar{J}}, g \rangle$ is a number of nondecreasing parking functions, and $\langle V_I, g^J \rangle$ counts parking quasi-ribbons with a common sign. This is therefore a cancellation-free combinatorial formula.

To compute the antipode of g_n :

- Express $g_n^{(2)}$ on the basis g^I

$$(72) \quad g_3^{(2)} = g_3 + 2g^{21} + g^{12} + g^{111}$$

- apply the involution $\chi : g^I \mapsto g^{\bar{I}}$ and multiply by $(-1)^n$

$$(73) \quad (-1)^3 \chi(g_3^{(2)}) = -(g_3 + 2g^{12} + g^{21} + g^{111})$$

- then expand it on the basis Λ^I

$$(74) \quad (-1)^3 \chi(g_3^{(2)}) = -(\Lambda^3 - 4\Lambda^{21} - 4\Lambda^{12} + 12\Lambda^{111})$$

- and finish by applying the tilde involution $\Lambda^I \mapsto g^I$

$$(75) \quad \tilde{\omega}(g_3) = -(g_3 - 4g^{21} - 4g^{12} + 12g^{111}).$$

3. EXPANSION OF THE COPRODUCT OF g_n ON THE BASIS g^I

3.1. Background: the Hopf algebra of nondecreasing parking functions.

One can also rewrite (22) in **Sym** as

$$(76) \quad g_n = \sum_I \delta_I S^I,$$

where δ_I is the number of nondecreasing parking functions of type I .

For example,

$$(77) \quad g_3 = S_3 + 2S^{21} + S^{12} + S^{111}$$

is obtained from 111, 112, 113, 122, 123.

We have defined in [20] an algebra **PQSym** based on symbols \mathbf{F}_a , where \mathbf{a} runs over all parking functions. One can show that **PQSym** has a Hopf subalgebra **CQSym** whose basis is defined by

$$(78) \quad \mathbf{P}^\pi = \sum_{\mathbf{a} \uparrow = \pi} \mathbf{F}_a,$$

where π is any nondecreasing parking function and the sum runs over all parking functions with the same nondecreasing rearrangement π .

If one denotes by $t(\pi)$ the packed evaluation of π , which coincides with the ordered type of the noncrossing partition encoded by π , then, the map $\phi : \mathbf{P}^\pi \mapsto S^{t(\pi)}$ is an epimorphism of Hopf algebras from **CQSym** to **Sym** [20], and

$$(79) \quad g = \phi(G), \text{ where } G := \sum_{\mathbf{a} \in \text{PF}} \mathbf{F}_a = \sum_{\pi \in \text{NDPF}} \mathbf{P}^\pi$$

is the formal sum of all parking functions.

For example,

$$(80) \quad G_3 = \mathbf{P}^{111} + \mathbf{P}^{112} + \mathbf{P}^{113} + \mathbf{P}^{122} + \mathbf{P}^{123},$$

so that one recovers (23) and (77) by sending \mathbf{P}^π to $S^{t(\pi)}$.

Thus, $\Delta g = (\phi \otimes \phi)(\Delta G)$ and one can get Δg from ΔG which is simpler, since, as we shall see shortly, it has an intermediate multiplicity-free expression.

3.2. Computation of the coproduct in CQSym. The coproduct in CQSym in the \mathbf{P} basis is given by

$$(81) \quad \Delta \mathbf{P}^\pi = \sum_{\substack{\pi=(uv)\uparrow \\ u,v \text{ nondecreasing}}} \mathbf{P}^{\text{Park}(u)} \otimes \mathbf{P}^{\text{Park}(v)}$$

where Park denotes the operation of *parkization* as described in [20], and the sum runs over all nondecreasing words u, v such that the nondecreasing rearrangement of uv is π .

For example,

$$(82) \quad \begin{aligned} \Delta \mathbf{P}^{1124} &= 1 \otimes \mathbf{P}^{1124} + \mathbf{P}^1 \otimes (\mathbf{P}^{112} + \mathbf{P}^{113} + \mathbf{P}^{123}) + \mathbf{P}^{11} \otimes \mathbf{P}^{12} \\ &+ \mathbf{P}^{12} \otimes (\mathbf{P}^{11} + 2\mathbf{P}^{12}) + (\mathbf{P}^{112} + \mathbf{P}^{113} + \mathbf{P}^{123}) \otimes \mathbf{P}^1 + \mathbf{P}^{1124} \otimes 1. \end{aligned}$$

Now, as an intermediate computation, we could “forget” to parkize u and v and write the coproduct of \mathbf{P}^π as the sum of all terms $\mathbf{P}^u \otimes \mathbf{P}^v$, over all pairs on nondecreasing words such that $uv = \pi$. This amounts to making the convention $\mathbf{P}^w = \mathbf{P}^{\text{Park}(w)}$ for an arbitrary nondecreasing word w .

For example, with this convention, the coproduct $\Delta \mathbf{P}^{1124}$ becomes

$$(83) \quad \begin{aligned} \Delta \mathbf{P}^{1124} &= 1 \otimes \mathbf{P}^{1124} + \mathbf{P}^1 \otimes \mathbf{P}^{124} + \mathbf{P}^2 \otimes \mathbf{P}^{114} + \mathbf{P}^4 \otimes \mathbf{P}^{112} \\ &+ \mathbf{P}^{11} \otimes \mathbf{P}^{24} + \mathbf{P}^{12} \otimes \mathbf{P}^{14} + \mathbf{P}^{14} \otimes \mathbf{P}^{12} + \mathbf{P}^{24} \otimes \mathbf{P}^{11} \\ &+ \mathbf{P}^{112} \otimes \mathbf{P}^4 + \mathbf{P}^{114} \otimes \mathbf{P}^2 + \mathbf{P}^{124} \otimes \mathbf{P}^1 + \mathbf{P}^{1124} \otimes 1. \end{aligned}$$

Note 3.1. With this convention, if one forgets to parkize all terms, this expression of ΔG_n becomes multiplicity-free, since a term $\mathbf{P}^u \otimes \mathbf{P}^v$ can only come from a $\Delta \mathbf{P}^\pi$ where π is obtained by sorting $u \cdot v$.

In other words, ΔG_n is the sum of terms $\mathbf{P}^u \otimes \mathbf{P}^v$, over all pairs on nondecreasing words such that uv is a parking function.

Define $G^I = G_{i_1} \cdots G_{i_r}$. We shall prove that ΔG is actually a sum of terms $G^I \otimes G^J$.

3.3. Profiles of nondecreasing words. Any nondecreasing word w admits a minimal factorization into shifted parking functions

$$(84) \quad w = w_1 w_2 \cdots w_k$$

i.e., each w_i is obtained by shifting a parking function \mathbf{a}_i by some integer b_i , which we write as $w_i = (\mathbf{a}_i)_{b_i}$ and each w_i is of maximal length.

For example,

$$(85) \quad \begin{aligned} w &= 2336799 = (1225688)_1 = (122)_1 \cdot 6799 \\ &= (122)_1 \cdot (1244)_5 = (122)_1 \cdot (12)_5 \cdot 99 \\ &= (122)_1 \cdot (12)_5 \cdot (11)_8, \end{aligned}$$

so that w decomposes as

$$(86) \quad 2336799 = 233 \cdot 67 \cdot 99$$

and the a_i s and the b_i s can be read above.

Definition 3.2. The profile $\text{pf}(w)$ of a word w is the pair $\binom{s}{c} = \binom{s_1 s_2 \dots s_k}{c_1 c_2 \dots c_k}$, where s_i is the first letter of w_i , that is, $1 + b_i$ and c_i its length.

We shall say a biword is a profile if there is a word w whose profile is that biword.

On our example, $\text{pf}(w) = \binom{269}{322}$.

There is a simple characterization of profiles:

Lemma 3.3. A biword $\binom{s_1 \dots s_k}{c_1 \dots c_k}$ is a profile iff $s_{i+1} > s_i + c_i$ for all $i \in [1, k - 1]$.

Proof – Let w be a nondecreasing word. Decompose it as $w_1 \dots w_k$ as above, and let c_i be the length of w_i .

Since w_i and w_{i+1} are different blocks of the decomposition of w , after shifting the suffix of w starting with w_i , its largest prefix which is a parking function will be exactly w_i . So the first letter of w_{i+1} has to be far enough from the first letter of w_i , more precisely, $s_{i+1} - s_i$ has to be strictly greater than their distance in the word which is c_i , whence the condition.

Conversely, given a biword satisfying the required conditions, it is easy to exhibit a word with that profile:

$$(87) \quad w = s_1^{c_1} s_2^{c_2} \dots s_k^{c_k}.$$

■

3.4. Biprofiles of pairs of nondecreasing words. Given two nondecreasing words u and v , we define their *biprofile* as the pair $(\text{pf}(u), \text{pf}(v))$.

Lemma 3.4. Let (u, v) be a pair of nondecreasing words of respective profiles $S = \binom{s_1 \dots s_k}{c_1 \dots c_k}$ and $T = \binom{t_1 \dots t_\ell}{d_1 \dots d_\ell}$.

Rearrange the biword $\binom{s_1 \dots s_k t_1 \dots t_\ell}{c_1 \dots c_k d_1 \dots d_\ell}$ as a joint profile, so that the top line is weakly increasing. If some $s_i = t_j$, put the biletter of s_i to the left of the one of t_j and write the result as $\binom{x_1 \dots x_{k+\ell}}{y_1 \dots y_{k+\ell}}$.

Then, the concatenation uv is a parking function iff

$$(88) \quad \forall m \in [1, k + \ell], \quad x_m \leq y_1 + \dots + y_{m-1} + 1.$$

In that case, we say that the biprofile is a parking biprofile.

Note that in general, the joint profile is not a profile.

For example, let $u = 2336799$ and $v = 11$. Then the concatenation of their profiles gets reordered as

$$(89) \quad \binom{2691}{3222} = \binom{1269}{2322}.$$

We have the inequalities

$$(90) \quad x_1 = 1 \leq 1, \quad x_2 = 2 \leq 3, \quad x_3 = 6 \leq 6, \quad x_4 = 9 > 8,$$

so that uv is not a parking function and indeed, there are only 7 values smaller than or equal to 8 in uv .

One can also check that if u is the same and $v = 116 = 11 \cdot 6$ then the joint profile is $\binom{12669}{23212}$, all inequalities are satisfied and uv is indeed a parking function.

Proof – Let us write as before $u = (u_1) \cdots (u_k)$ and $v = (v_1) \cdots (v_\ell)$. Then rearrange uv as blocks matching the rearranged concatenated biword in the statement:

$$(91) \quad w = (a_1)(a_2) \dots (a_{k+\ell})$$

where the (a_i) run over all factors of both u and v (the i -th block a_i comes from u if x_i is some s_i). Writing the blocks a_i as words, (91) becomes

$$(92) \quad w = (w_{z_0+1} \dots w_{z_1})(w_{z_1+1} \dots w_{z_2}) \dots (w_{z_{k+\ell-1}+1} \dots w_{z_{k+\ell}}),$$

where $z_i = y_1 + \cdots + y_i$ with the convention $z_0 = 0$.

Let us now assume that some $x_m > z_{m-1} + 1$. In this case, $w_{z_{m-1}+1} = x_m > z_{m-1} + 1$ and it has no letter strictly smaller on its right since the x_i are weakly increasing and among each block, the values are weakly increasing too. So w cannot be a parking function: it has less than z_{m-1} values smaller than or equal to $z_{m-1} + 1$.

Conversely, assume that all $x_m \leq z_{m-1} + 1$. In that case, each letter beginning a block satisfies $w_{z_{m-1}+1} \leq z_{m-1} + 1$. Now, any w_{z_i+j} with $j \leq y_i$ is at most $z_i + j$ since the subword $w_{z_i+1} \dots w_{z_i+j}$ is a nondecreasing parking function shifted by a fixed value and $w_{z_i+1} \leq z_i + 1$. So w is a parking function ($w_j \leq j$ for all j), and so is uv , since it is a rearrangement of w . This concludes the proof of the statement.

Note that w is not in general nondecreasing but it satisfies nonetheless $w_i \leq i$ for all i . ■

The lemma shows that whether uv is a parking function or not depends only on the biprofile of (u, v) , so that

Corollary 3.5. *If u and v are nondecreasing words such that uv is a parking function, then any pair (u', v') of nondecreasing words with the same biprofile as (u, v) is also such that $u'v'$ is a parking function.*

For example, consider the biprofile

$$(93) \quad \binom{25}{22} \text{ and } \binom{1}{3}.$$

There are four different choices for u : 2255, 2256, 2355, and 2356 and five choices for v : 111, 112, 113, 122, and 123. One can check that all 20 crossed concatenations are parking functions: write down any $w = v.u$ and observe that even if w is not always weakly increasing, $w_i \leq i$ for all i .

3.5. Regrouping terms in ΔG_n . We can now regroup the terms $\mathbf{P}^u \otimes \mathbf{P}^v$ in the “unparkized” multiplicity-free expression of ΔG_n according to their biprofiles, and write

$$(94) \quad \Delta G_n = \sum_{\binom{s}{c}, \binom{t}{d}} \sum_{\substack{\text{pf}(u) = \binom{s}{c} \\ \text{pf}(v) = \binom{t}{d}}} \mathbf{P}^u \otimes \mathbf{P}^v$$

where the sum runs over the parking biprofiles.

Now, given a parking biprofile $\binom{s}{c}, \binom{t}{d}$ and identifying w with $\text{Park}(w)$, each sum

$$(95) \quad \sum_{\substack{\text{pf}(u)=\binom{s}{c} \\ \text{pf}(v)=\binom{t}{d}}} \mathbf{P}^{\text{Park}(u)} \otimes \mathbf{P}^{\text{Park}(v)}$$

contributes exactly one term $G^c \otimes G^d$ to ΔG . Indeed, given the profile $\binom{s}{c}$, the list of nondecreasing words having that profile gives as parkized exactly all parking functions of size c_1 concatenated with all parking functions of size c_2 , *etc.*, so that we get a term G^c .

Continuing the example from Equation (93), the term corresponding to its biprofile is $G^{22} \otimes G^3$.

Finally,

Theorem 3.6. ΔG_n is the sum of all $G^I \otimes G^J$, where I, J run over the bottom elements of all pairs (b_1, b_2) of parking biprofiles of size n . ■

Note in particular that if one swaps the profiles, one still has a parking biprofile, which reflects the fact that Δ is cocommutative.

We shall also represent a profile as the minimal lexicographic nondecreasing word associated with it.

For example, the biprofile $\binom{2\ 6\ 9}{3\ 2\ 2}$ is now 2226699. With this notation, the parking biprofiles of size 3 correspond to the following pairs of words:

$$(96) \quad \begin{aligned} &(111, \emptyset), (11, 1), (11, 2), (11, 3), (22, 1), (13, 1), (13, 2), \\ &(1, 11), (2, 11), (3, 11), (1, 22), (1, 13), (2, 13), (\emptyset, 111). \end{aligned}$$

so that sending a word to its packed evaluation,

$$(97) \quad \Delta G_3 = G_3 \otimes 1 + (4G_2 + 2G^{11}) \otimes G_1 + G_1 \otimes (4G_2 + 2G^{11}) + 1 \otimes G_3.$$

Note 3.7. Note that the number of terms in ΔG_3 is $C_4 = 14$ (Catalan numbers), and in general ΔG_n has C_{n+1} terms. A first easy but not very satisfactory explanation goes as follows: since ΔG_n is a sum of positive terms and that each G_n is sent to the usual g_n when taking the commutative image from **Sym** to *Sym*, each term gives rise to one term of Δg_n . Since this coproduct is known to have Catalan terms, so does ΔG_n .

We will provide a complete combinatorial proof of this same result in the Appendix through a bijection between parking biprofiles, pairs of "compatible" compositions and then Motzkin paths. It is also possible to make a simple bijection between pairs of compositions and nondecreasing parking functions but since this bijection does not provide any combinatorial insight, we will only sketch it (see Note 4.8).

4. COMBINATORIAL INTERPRETATIONS OF ΔG_n

In the commutative case, it is known [7] that

$$(98) \quad \Delta g_n = \sum_{\pi \in \text{NC}_{n+1}} g^{\alpha(\pi)} \otimes g^{\alpha(K(\pi))}$$

where $\alpha(\pi)$ is the reduced type of π and K stands for the right Kreweras complement. We shall now see that this expression can be extended to the noncommutative case, replacing the type by the ordered type.

4.1. From parking biprofiles to pairs of compositions. We have seen that the coproduct of g_n (or G_n , its pre-image in **CQSym**) can be expanded in the basis $g^I \otimes g^J$ and that the terms are parametrized by parking biprofiles.

We shall now encode a profile p by an integer composition I .

Definition 4.1. Let $p = \binom{s_1 \dots s_k}{c_1 \dots c_k}$ be a profile, and let $n \geq s_k + c_k$. Define $C : p \mapsto I$ as follows:

- If $s_1 = 1$ then $I = (1 + c_1, I')$ where I' is the composition associated with the profile $\binom{s'_2 \dots s'_k}{c_2 \dots c_k}$ where $s'_i = s_i - c_1 - 1$.
- If $s_1 \neq 1$ then $I = (1, I')$ where I' is the composition associated with the profile $\binom{s'_1 \dots s'_k}{c_1 \dots c_k}$ where $s'_i = s_i - 1$.

Then, define $C_n(p)$ as the composition of n obtained by adding $n - s_k - c_k$ ones at the end of I .

For example, with $n = 12$,

$$(99) \quad C \left(\binom{269}{221} \right) = 1, \quad C \left(\binom{158}{221} \right) = 1, 3, \quad C \left(\binom{25}{21} \right) = 1, 3, 1, \quad C \left(\binom{14}{21} \right) \\ = 1, 3, 1, 3, \quad C \left(\binom{1}{1} \right) = 1, 3, 1, 3, 2,$$

and finally $C_{12}(p) = (1, 3, 1, 3, 2, 1, 1)$. Similarly,

$$(100) \quad C_{10} \left(\binom{16}{31} \right) = (4, 1, 2, 1, 1, 1).$$

Note 4.2. Thanks to Lemma 3.3, we know that a profile satisfies $s_{i+1} > s_i + c_i$. Thus, at each step of the previous algorithm, the s'_i are positive integers, so that one indeed gets an integer composition in the end.

Moreover, before adding ones at the end of I , one easily checks that I was a composition of $s_k + c_k$ so that I itself is a composition of n .

The map C is easily inverted:

Lemma 4.3. Let $I = (i_1, \dots, i_k)$ be a composition of n . Define a map C' by

$$(101) \quad C'(i_1, \dots, i_k) = \left(\begin{array}{ccc} d_1 & \dots & d_k \\ i_1 - 1 & \dots & i_k - 1 \end{array} \right),$$

removing the biletters when the bottom letter is 0 and where $d_j = 1 + i_1 + \dots + i_{j-1}$.

Then C' is the inverse map of C .

For example, with $I = (4, 1, 2, 1, 1, 1)$, one gets $D = (1, 5, 6, 8, 9, 10)$, so that $C'(I) = \binom{16}{31}$ and $n = 4 + 1 + 2 + 1 + 1 + 1 = 10$.

It will be useful to represent I as a sequence of dots separated by bars, such as

$$(102) \quad (4, 1, 2, 1, 1, 1) \iff \dots|\cdot|\cdot|\cdot|\cdot|.$$

On this representation, one easily reads $C'(I)$, and also the lexicographically minimal word u with profile $C'(I)$: write an integer equal to the position of the beginning of the block on each dot that is not immediately followed by a bar. On the example, we get

$$(103) \quad \textit{Afirsteasybutnotverysatisfactoryexplanationgoesasfollows}\dots|\cdot|\cdot|\cdot|\cdot| \implies 111\cdot|6\cdot|\cdot|\cdot|.$$

which indeed encodes $\binom{16}{31}$ and also its minimal word 1116.

Proof – By induction on the number of biletters of p . Let $I = C_n(p)$. If the first part of I is not 1, then we had $s_1 = 1$, its number of occurrences c_1 being exactly $i_1 - 1$ by definition, so C' records the correct biletter at the beginning of its image. The inductive definitions of the s' and the d are shifted in the same way from I to I' , which ensures that $C'(I')$ is the remaining part of p by induction.

If I begins with a 1, then s_1 was not 1 and $d_1 = 1$ appears through C' with a 0 at its bottom, so the biletter does not appear in $C'(I)$. As before, the s' and the d change in the same way from I to I' , so $C'(I')$ will translate as p by induction. ■

Now, given a parking biprofile $p = \binom{s}{c}$, $q = \binom{t}{d}$, we map it to a pair of compositions by computing $C_n(p)$ and $C_n(q)$ with $n = 1 + c_1 + \dots + c_k + d_1 + \dots + d_\ell$. Let us also denote this map by C . Note that condition (88) ensures that n is greater than both $s_k + c_k$ and $t_\ell + d_\ell$ so the map is well-defined and we get two compositions of n .

For example,

$$(104) \quad C \left(\binom{269}{221}, \binom{16}{31} \right) = ((1, 3, 1, 3, 2), (4, 1, 2, 1, 1, 1)).$$

Definition 4.4. *A pair of compositions is compatible if it is in the image of C , that is, the image of a parking biprofile.*

Note 4.5. Both I and J are compositions of the same integer n . Moreover, the number of parts of I is $n - (c_1 + \dots + c_k)$ whereas the number of J is accordingly $n - (d_1 + \dots + d_\ell)$, so that their total number of parts is $n + 1$.

Not all pairs satisfying this condition are compatible, but we shall see that I and its mirror conjugate \tilde{I} always are.

Lemma 4.6. *A pair of compositions (I, J) of the same integer n is compatible iff their total number of parts is $n + 1$ and if the word z obtained by sorting the concatenation of the descent sets of I and J satisfies $z_\ell \geq \ell$ for all its values.*

For example, given the pair $I = (1, 3, 1, 3, 2)$ and $J = (4, 1, 2, 1, 1, 1)$, the concatenation of their descents is $[1, 4, 4, 5, 5, 7, 8, 8, 9]$ and it satisfies the conditions of the statement.

As a counterexample, consider the pair $I = (1, 2, 1, 1, 2)$ and $J = (2, 1, 4)$. The sorted concatenation of their descents is $[1, 2, 3, 3, 4, 5]$ and $z_4 = 3 < 4$ so that it does not satisfy the conditions of the statement and indeed, $uv = 261444$ has only two letters smaller than or equal to 3.

Proof – We shall analyse the way in which the action of C' on uv depends on z . To this aim, we shall represent a pair of compositions by two sequences of dots separated by bars.

If $z_\ell \geq \ell$, there are at most $\ell - 1$ bars among the $(\ell - 1)$ first dots in the encodings of both I and J . So there are at least $2\ell - 2 - (\ell - 1) = \ell - 1$ values smaller than $\ell - 1$ in uv , and the parking constraint is fulfilled for $\ell - 1$.

So, if $z_\ell \geq \ell$ for all ℓ , then $C'(I, J)$ is a parking function.

Conversely, if some $z_\ell < \ell$, consider the smallest one z_k . Then, z_{k-1} was at least $k - 1$ but since the word z is weakly increasing, we must have $z_{k-1} = z_k = k - 1$. In other words, both compositions I and J have a bar after $k - 1$ dots and there are also $k - 2$ bars in total to the left of both these bars. So among the $k - 1$ dots on both lines of I and J , exactly $(2k - 2) - k = k - 2$ do not have a dot immediately after them. Moreover, all the dots after the k -th dot cannot be decoded as a value strictly smaller than k since both I and J have blocks beginning at position $k - 1$. So there are exactly $k - 2$ values smaller than $k - 1$ in uv and so uv is not a parking function. ■

At this point, we have mapped bijectively the parking biprofiles to particular pairs of compositions, and provided a characterization of the first easy but not very satisfactory explanation goes as follows. We can now use these results to provide an alternative description of the coproduct of G_n .

Lemma 4.7. *Through the bijection C , the map sending a parking biprofile to $G^c \otimes G^d$ translates as the map sending a pair of compositions (I, J) to $G^{i_1-1, \dots, i_r-1} \otimes G^{j_1-1, \dots, j_r-1}$ and removing the zeroes.*

Proof – Immediate by definition of C . ■

Here follows the whole list of compatible pairs of compositions of size 4:

$$(105) \quad \begin{aligned} &(4, 1111), (31, 211), (31, 121), (31, 112), (22, 211), (22, 121), (211, 31), \\ &(211, 22), (211, 13), (13, 211), (121, 31), (121, 22), (112, 31), (1111, 4). \end{aligned}$$

and one can then check the expression of G_3 of (97) by sending each composition I to G^{i_1-1, \dots, i_r-1} .

Note 4.8. We shall provide in the Appendix (Section 7) a meaningful bijection proving that pairs of compositions are enumerated by Catalan numbers but we can provide a very simple but dumb one that also proves that: given a pair (I, J) of n of respective descent sets (d_1, \dots, d_k) and (d'_1, \dots, d'_ℓ) , sort the word w given by the concatenation of the $2 * d_i - 1$ with the $2 * d'_i$ and the value $2n - 1$. Now compute w' where $w'_{n+1-i} = n + i - w_i$.

This is a bijection from the pairs of compositions to their image set since both set are easily revertible. And it is an exercise to check that w' is a nondecreasing parking function and conversely that any parking function gives rise to a valid pair of compositions.

Given the pair (13132, 412111), one gets the descents sets (1, 4, 5, 8) and (4, 5, 7, 8, 9) hence the word

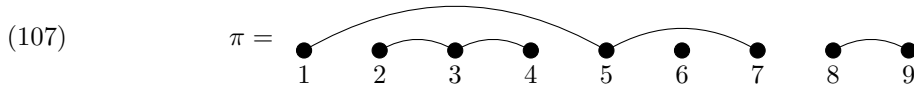
$$(106) \quad w = 1, 7, 8, 9, 10, 14, 15, 16, 18, 19 \quad \text{and} \quad w' = 1, 1, 2, 2, 2, 5, 5, 5, 5, 10$$

that is indeed a nondecreasing parking function.

4.2. From pairs of compositions to noncrossing partitions.

4.2.1. *Noncrossing partitions and permutations.* Recall that a noncrossing partition π can be interpreted as a permutation w_π whose cycles are the blocks of π read in increasing order. The *right Kreweras complement* [14] $\pi' = K(\pi)$ can then be defined as the noncrossing partition such that $w_{\pi'} = w_\pi^{-1}\gamma_n$, where $\gamma_n = (123\dots n)$ is the canonical long cycle. The permutations w_π are called *noncrossing permutations*.

For example, given the noncrossing partition π ,



we get

(108) $w_\pi = [(1, 5, 7)(2, 3, 4)(8, 9)] = [5, 3, 4, 2, 7, 6, 1, 9, 8]$

so that

(109) $w_{\pi'} = [7, 4, 2, 3, 1, 6, 5, 9, 8].[2, 3, 4, 5, 6, 7, 8, 9, 1]$
 $= [4, 2, 3, 1, 6, 5, 9, 8, 7] = [(1, 4)(5, 6)(7, 9)]$

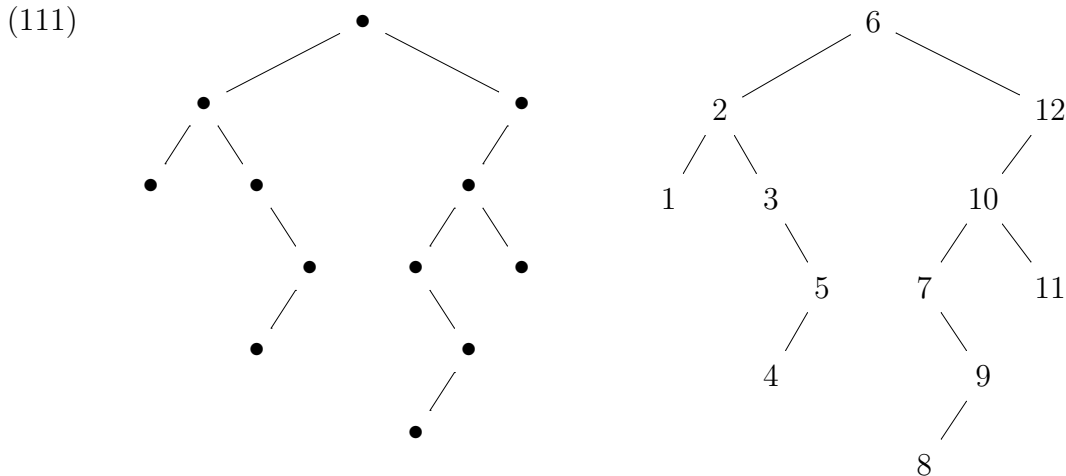
and π' is



The *canonical ordering* of a permutation is the list of its cycles in increasing order of their minimal elements.

4.2.2. *Planar binary trees and the Kreweras complement.* There are many bijections between noncrossing partitions and binary trees. But actually, on a binary tree t , one can directly read *two* noncrossing partitions π', π'' .

Let φ be the map sending a tree T to a pair (π', π'') as follows : label the nodes of T in infix order, so as to obtain a binary search tree:



Then the blocks of π' are the sets of labels of the left branches of T :

$$(112) \quad \pi' = \begin{array}{cccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$

and the blocks of π'' are those of its right branches:

$$(113) \quad \pi'' = \begin{array}{cccccccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}$$

Both π' and π'' are obviously noncrossing partitions. Moreover, if one traverses the tree in infix order and records the labels of each branch the first time it is encountered (that is, by its smallest value), both partitions π' and π'' come up with their canonical ordering. It is also easy to see that φ is also bijective since one can easily rebuild T from either π' or π'' . This means that one of the elements should be fully recoverable from the other, or, in other words, that they have a direct link with one another, and indeed:

Interpreting π' and π'' as permutations,

$$(114) \quad \pi' = [(126), (3), (4, 5), (7, 10, 12), (8, 9), (11)] = [2, 6, 3, 5, 4, 1, 10, 9, 8, 12, 11, 7]$$

and

$$(115) \quad \pi'' = [(1), (2, 3, 5), (4), (6, 12), (7, 9), (8), (10, 11)] = [1, 3, 5, 4, 2, 12, 9, 8, 7, 11, 10, 6]$$

and one can check that

$$(116) \quad \pi' \pi'' = [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1]$$

so that π'' is the right Kreweras complement of π' .

It is easy to see that this is true in general:

Lemma 4.9. *Let T be a binary tree and let $\varphi(T) = (\pi', \pi'')$. Then π'' is the right Kreweras complement of π' .*

Proof – The property holds for trees with at most 2 nodes and also for trees with no right or left branches since in these cases, either π' or π'' is the cycle γ_n and the other is the identity.

Assume by induction that the property holds for trees with at most $n - 1$ nodes. Let T be a tree whose left subtree T_L has $k - 1$ nodes and whose right subtree T_R has $n - k$ nodes. Its root has therefore label k . By induction hypothesis (or the special case mentioned above), the product of the cycles of the tree T'_L with root k , left subtree T_L and an empty right subtree is the cycle $\sigma'_1 = (1 \cdots k - 1k)$. Similarly, the product of the cycles of the tree T'_R with root k , empty left subtree and right subtree T_R is $\sigma'_2 = (k k + 1 \cdots n)$. The complete product is therefore $\sigma'_1 \sigma'_2 = (12 \cdots n)$. ■

Note 4.10. Lemma 4.9 amounts to saying that, inside a binary search tree, one gets from the position of i to the position of $i + 1$ modulo its number of nodes by

- moving one step down its right branch (and cycling if i is at the bottom of it),

- moving then one step up its left branch (and cycling if it is at the top of it).

This property is easily checked, and provides an alternative proof of the lemma.

4.2.3. *From trees and permutations to pairs of compositions.* As we have seen, reading the left branches and the right branches in infix order, the blocks of the partitions come naturally ordered with respect to their minima in increasing order, so that the compositions recording their lengths are the ordered types of π' and π'' .

It turns out that the tree T , and therefore π' and π'' can be unambiguously reconstructed from this pair of compositions.

Theorem 4.11. *Let $t(\pi)$ denote the ordered type of a noncrossing partition. The map*

$$(117) \quad \tau : \pi \mapsto (t(\pi), t(K(\pi)))$$

is injective.

The map τ goes from a noncrossing partition to a pair of compositions. Since one can easily go from a noncrossing partition to a tree, we shall also write τ as the map sending a tree to a pair of compositions and it is that map, sending a tree T to the lengths of the ordered types of $\varphi(T)$, that we will prove injective.

Let us consider the following backwards algorithm:

Algorithm 4.12. *Input: a pair of compositions $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$ obtained as the ordered lengths of the respective left and right branches of a tree.*

We shall build a tree one branch at each step. When gluing a branch on a node, mark this node.

Create a left branch of i_1 nodes. Then glue a right branch of j_1 nodes at the first unmarked node in infix order (in that case, it is the leftmost node since no one was marked yet).

Then move to the first unmarked node in infix order, which can be either a left or a right child (if it is the root, consider it as a left child), and create a new branch in the opposite direction (e.g., right if it is a left child) of the corresponding size, using the next unused part of I or of J depending on the direction. Iterate until there are no unmarked nodes left.

An example of this algorithm with $I = 312321$ and $J = 1312212$ is given on Fig. 1. Note that when a part is 1, we just mark the leftmost node and add no new node.

Proposition 4.13. *If I and J are the ordered lengths of the left and right branches of a tree T , then Algorithm 4.12 rebuilds T from I and J .*

Proof – Let T be a tree and let $\tau(T) = (I, J)$.

Apply Algorithm 4.12 to I and J . We shall prove by induction that after step k the partial tree is exactly the tree T' consisting of the first k left and/or right branches of T .

This is true at steps $k = 0$ and $k = 1$. Assume that this is true until step k and add a new (left or right) branch to T' as described in Algorithm 4.12. Let T'' be

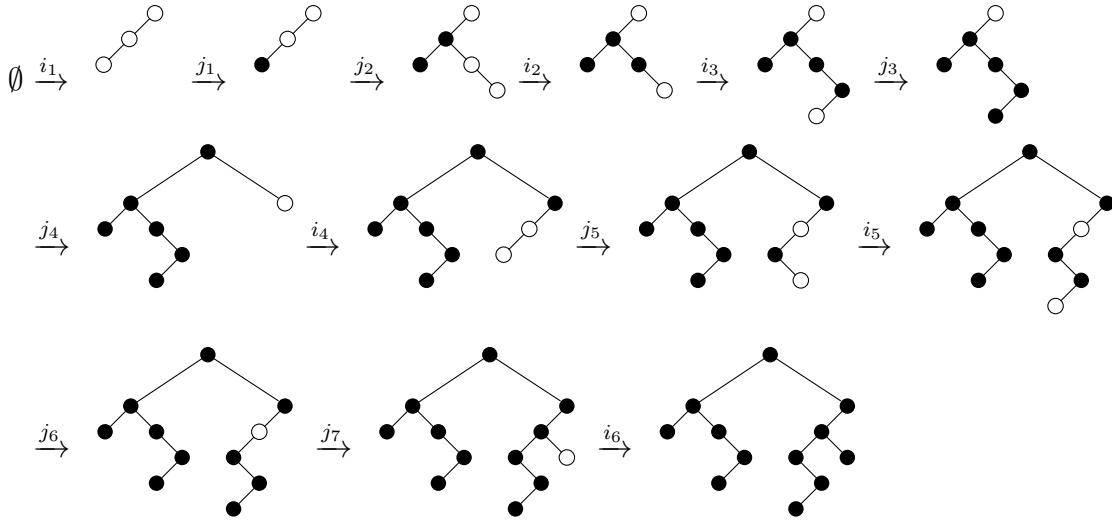


FIGURE 1. Algorithm 4.12 applied to the image of the tree in Equation (111). Unmarked nodes are white.

the resulting tree. Note that if T' is not equal to T , there are necessarily unmarked nodes so that step $k + 1$ is well-defined.

By construction, this branch has been added to the leftmost unmarked node x in infix order. By definition of this order, all marked nodes strictly before x will all be read in the same order as in T' , all read before x and its added branch. So the first $k + 1$ branches associated with T' are the first k branches of T followed by the added new branch. ■

Proof – [of the theorem] Thanks to Lemma 4.13, the map τ from binary trees to pairs of compositions induces a bijection with its image set. ■

We finally need to characterize the image set of τ .

Lemma 4.14. *Let T be a tree and $\tau(T) = (I, J)$.*

Then I and J are compatible in the sense of Definition 4.4.

Proof – First of all, it is well-known that if π' and π'' satisfy $\pi'\pi'' = \gamma_n$ and $\ell(\pi') + \ell(\pi'') = n$, then their total number of cycles is $n + 1$. So I and J , being the images of two such permutations have a total of n descents.

Now, let us consider two compositions I and J whose total number of descents is n . Sort these descents, and let $d = d_1 \dots d_n$ be the corresponding word. Either they satisfy the criterion or there is a smallest value k such that $d_k < k$. Since $d_{k-1} \geq k-1$, both $d_{k-1} = d_k = k-1$ and I and J both have a descent in $k-1$.

Let $I' = (i_1, \dots, i_{k_1})$ and $J' = (j_1, \dots, j_{k_2})$ be the prefixes of I and J such that $i_1 + \dots + i_{k_1} = j_1 + \dots + j_{k_2} = k-1$. Then I' and J' are both compositions of $k-1$ whose total number of descents is $k-2$, since we do not take into account their last descent. Moreover, their own descents are the first d_k s so I' and J' are compatible. By induction, they correspond therefore to a tree T' .

Let us now apply Algorithm 4.12 to I and J . Since the algorithm proceeds step by step, if it used part i_{k_1+1} or part j_{k_2+1} before going through I' and J' fully, the algorithm would have failed on the pair (I', J') , which is not the case. So it ends with T' at this exact step, there is no unmarked node left and the algorithm stops.

So, by induction, the algorithm fails if I and J are not compatible. Since we know that the algorithm succeeds with the images of the binary trees, it means that the image set of τ is included in the set of compatible pairs of compositions. But both sets are Catalan sets (see Notes 3.7 and 4.8) so they coincide. ■

4.2.4. *Conclusion of the proof of Theorem 1.2.* We have successfully expressed ΔG_n as a sum over parking biprofiles $G^c \otimes G^d$, then mapped parking biprofiles to compatible pairs of compositions, and proved that such pairs record the ordered lengths of left and right branches of binary trees. Such pairs of compositions in turn coincide with the reduced types of $(\pi, K(\pi))$ for $\pi \in \text{NC}_{n+1}$. This concludes the proof of Theorem 1.2. ■

Note 4.15. On the interpretation of Δg_n in terms of noncrossing partitions and their Kreweras complement, it is not apparent that Δ is cocommutative, since this operation is not an involution. It would then be interesting, given a noncrossing partition π of $[n + 1]$ of reduced ordered type I whose Kreweras complement π' has reduced ordered type J , to build a noncrossing partition π' of $[n + 1]$ of reduced ordered type J whose Kreweras complement π' has reduced ordered type I .

The known involutions on noncrossing partitions, iterations of Kreweras and the same up to reversal defined by Simion and Ullman in [22], do not have this property.

For example, with $I = (4, 1, 1, 1, 2, 2, 1)$ and $J = (3, 1, 3, 1, 2, 2)$, the map would exchange

(118) $p := [4, 2, 3, 5, 9, 6, 8, 7, 1, 12, 11, 10]$ and $p' := [7, 2, 4, 5, 3, 6, 8, 1, 12, 11, 10, 9]$

of respective cycles

(119) $c_0 = [(1, 4, 5, 9), (2), (3), (6), (7, 8), (10, 12), (11)]$ and $c'_0 = [(1, 7, 8), (2), (3, 4, 5), (6), (9, 12), (10, 11)]$

whose Kreweras complements have as cycles

(120) $c_1 = [(1, 2, 3), (4), (5, 6, 8), (7), (9, 12), (10, 11)]$ and $c'_1 = [(1, 2, 5, 6), (3), (4), (7), (8, 12), (9, 11), (10)].$

5. COPRODUCT OF g_n IN THE COMMUTATIVE CASE

The calculation of Δg_n is easier in the commutative case. We have seen in Lemma 4.9 that one can read a noncrossing partition π and its Kreweras complement on a binary tree. This information is enough to compute the coproduct of $g(X)$.

Indeed, $g(X + Y)$ satisfies the functional equation

$$(121) \quad g(X + Y) = \sigma_{g(X+Y)}(X + Y) = \sigma_{g(X+Y)}(X)\sigma_{g(X+Y)}(Y)$$

$$(122) \quad = \sum_{p \geq 0} h_p(X)g(X + Y)^p \sum_{q \geq 0} h_q(Y)g(X + Y)^q,$$

i.e., the right-hand side factorizes. This is not true anymore in the noncommutative case, and we had to rely upon a different argument, based on the possibility to reconstruct π and $K(\pi)$ from their ordered types.

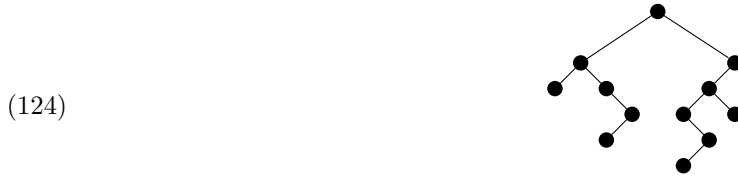
In the commutative case, we shall show that this equation coincides with that of the generating series of binary trees by lengths of the left and right branches. The argument is similar to the one used by Goulden and Jackson in their proof of Macdonald's formula for the top connexion coefficients [9, 10]. This provides an alternative (and simpler) proof of the result of [7].

For a binary tree t , set

$$(123) \quad w(t; u, v) = \prod_{\ell \in L(t)} u_{e(\ell)} \prod_{r \in R(t)} v_{e(r)},$$

where $L(t)$ and $R(t)$ are respectively the sets of left and right branches of t , and $e(b)$ denotes the number of edges in a branch b (with the convention $u_0 = v_0 = 1$).

For example, we have $w(t; u, v) = u_2^2 u_1^2 \cdot v_2 v_1^3$ on the following tree



Let W be the generating series

$$(125) \quad W(u, v) = \sum_{t \in \text{BT}} w(t; u, v) = 1 + u_1 + v_1 + u_2 + 3u_1 v_1 + v_2 + \dots$$

Denote by t_L and t_R the left and right subtrees of t , and let U, V be the generating series of the trees whose right (resp. left) subtree is empty:

$$(126) \quad U = \sum_{t_R = \emptyset} w(t; u, v), \quad V = \sum_{t_L = \emptyset} w(t; u, v).$$

These series satisfy the system

$$(127) \quad \begin{cases} V &= \sum_{n \geq 0} v_n U^n \\ U &= \sum_{n \geq 0} u_n V^n, \end{cases}$$

and classifying trees by length of the left branch of the root, we can write

$$(128) \quad W = V + u_1 V^2 + u_2 V^3 + u_3 V^4 + \dots = UV.$$

Recall that the commutative symmetric Lagrange series solves the equation

$$(129) \quad t = \frac{u}{\sigma_u(X)} \iff u = tg(t; X) = \sum_{n \geq 0} g_n(X) t^{n+1}.$$

If in (127) we set $u_n = g_n(X)$ and $v_n = g_n(Y)$, the system becomes, multiplying the first equation by U and the second one by V

$$(130) \quad \begin{cases} UV &= \sum_{n \geq 0} g_n(Y)U^{n+1} \\ UV &= \sum_{n \geq 0} g_n(X)V^{n+1}, \end{cases}$$

whence

$$(131) \quad UV = Ug(U; Y) \Leftrightarrow U = \frac{UV}{\sigma_{UV}(Y)} \Leftrightarrow V = \sigma_{UV}(Y)$$

and similarly

$$(132) \quad UV = Vg(U; X) \Leftrightarrow V = \frac{UV}{\sigma_{UV}(X)} \Leftrightarrow U = \sigma_{UV}(X)$$

Therefore,

$$(133) \quad W = UV = \sigma_{UV}(X + Y)$$

which is precisely the functional equation of Δg .

This proves Equation (98).

6. APPLICATION TO THE REDUCED INCIDENCE HOPF ALGEBRA OF NONCROSSING PARTITIONS

In the commutative case, the calculation of Δg_n and $g_n(-X)$, of which we have given new proofs, have important applications to the combinatorics of noncrossing partitions. Although the results of this Section are known, it seems appropriate to take the opportunity of giving a streamlined account of the theory in the light of the previous considerations.

The reduced incidence Hopf algebra \mathcal{H}_{NC} of the hereditary family of lattices NC_n is the vector space spanned by isomorphism classes of intervals of the NC_n for $n \geq 1$. The order is defined by $\pi \leq \pi'$ if π is finer than π' . The minimal element $\mathbf{0}_n$ is the partition into singletons, and the maximal element $\mathbf{1}_n$ is the partition with one block. As is well-known [24], any such interval is isomorphic to a Cartesian product of complete lattices NC_k . An interval $[\mathbf{0}_n, \pi]$ is isomorphic to $\prod_{B \in \pi} \text{NC}_{|B|}$, and an interval $[\pi, \mathbf{1}_n]$ is isomorphic to $[\mathbf{0}_n, K(\pi)]$. Finally, if $\sigma = \{B_1, \dots, B_r\}$, $[\pi, \sigma]$ is isomorphic to $\prod_i [\pi \cap B_i, \mathbf{1}_{B_i}]$.

The product of \mathcal{H}_{NC} is the Cartesian product. Thus, \mathcal{H}_{NC} is the polynomial algebra on the variables $y_n = [\text{NC}_{n+1}]$, and the coproduct is defined as

$$(134) \quad \delta y_n = \sum_{\pi \in \text{NC}_{n+1}} [\mathbf{0}_{n+1}, \pi] \otimes [\pi, \mathbf{1}_{n+1}].$$

One of the main results of [7] shows that $y_n \mapsto g_n(-X)$ is an isomorphism of Hopf algebras from \mathcal{H}_{NC} to Sym . This is precisely what we have just proved (using $g_n(X)$ instead) by a different method.

Another result of [7], which has been reproved by a different method in [6] is equivalent to the computation of the antipode $g_n(-X)$ in Sym . It is implied by

our calculation of $g(-A)$ (in [7] the coefficients are interpreted as counting polygon dissections, but our formula is cancellation-free as well, and produces the same coefficients).

It should be noted that these calculations imply a great deal of classical results about noncrossing partitions. In particular, the multiplicative functions on noncrossing partitions are the characters of \mathcal{H}_{NC} . Such a function ϕ is completely determined by its values $a_n = \phi(y_n)$ on the generators.

Using the above isomorphism, we can set $y_n = g_n$, and ϕ is entirely determined by the formal series (the Nica-Speicher Fourier transform [16])

$$(135) \quad \Phi(t) = \phi(g(t)) = \sum_{n \geq 0} a_n t^n.$$

Since $g(t) = \sum_{n \geq 0} t^n h_n g(t)^n$, we have

$$(136) \quad \Phi(t) = \sum_{n \geq 0} t^n \phi(h_n) \phi(g(t))^n = \sum_{n \geq 0} \alpha_n t^n \Phi(t)^n, \text{ with } \alpha_n = \phi(h_n).$$

Let ψ be another multiplicative function such that $\psi(g_n) = b_n$ and $\psi(h_n) = \beta_n$, and $\Psi(t) = \psi(g(t))$. Their convolution $\eta = \phi \star \psi$ is determined by

$$(137) \quad H(t) = \phi \star \psi(g(t)) = (\phi \otimes \psi) \Delta g(t)$$

$$(138) \quad = \left(\sum_{k \geq 0} \alpha_k t^k H(t)^k \right) \left(\sum_{\ell \geq 0} \beta_\ell t^\ell H(t)^\ell \right)$$

$$(139) \quad = \sum_{n \geq 0} t^n \left(\sum_{k+\ell=n} \alpha_k \beta_\ell \right) H(t)^n.$$

Thus, convolution corresponds to the ordinary product of formal series

$$(140) \quad \hat{\phi}(t) = \sum_{n \geq 0} \alpha_n t^n, \quad \hat{\psi}(t) = \sum_{n \geq 0} \beta_n t^n, \quad \hat{\eta}(t) = \hat{\phi}(t) \hat{\psi}(t) = \sum_{n \geq 0} \gamma_n t^n$$

since $H(t)$ satisfies

$$(141) \quad H(t) = \sum_{n \geq 0} \gamma_n t^n H(t)^n.$$

As an illustration, the Möbius function of the NC_{n+1} is the convolution inverse of the ζ function, which is defined by $\zeta([\sigma, \pi]) = 1$ if $\sigma \leq \pi$ and 0 otherwise. It is therefore characterized by $\zeta([0_{n+1}, 1_{n+1}]) = \zeta(y_n) = 1$.

If $\zeta(h_n) = \alpha_n$, then

$$(142) \quad Z(t) = \frac{1}{1-t} = \sum_{n \geq 0} \zeta(h_n) t^n Z(t)^n = \sum_{n \geq 0} \alpha_n \left(\frac{t}{1-t} \right)^n$$

yields $\hat{\zeta}(t) = 1 + t$. Hence,

$$(143) \quad \hat{\mu}(t) = \frac{1}{1+t} \text{ and } M(t) = \frac{1}{1+tM(t)},$$

so that

$$(144) \quad M(t) = \frac{-1 + \sqrt{1+4t}}{2t}.$$

One can also count intervals and multichains. Set $\zeta_k = \zeta^{\star k}$. Then, $\hat{\zeta}_k(t) = (1+t)^k$. Hence $Z_k(t)$ satisfies

$$(145) \quad Z_k(t) = (1 + tZ_k(t))^k,$$

or alternatively

$$(146) \quad X_k(t) = 1 + tX_k(t)^k \text{ with } X_k(t) = 1 + tZ_k(t),$$

and we recover the fact that multichains of length k are in bijection with $(k+1)$ -ary trees [5].

In [4], Edelman obtains a formula for the number of chains with prescribed ranks $0_{n+1} < \pi_1 < \dots < \pi_r < \pi_{r+1} = 1_{n+1}$. To derive it, one can compute

$$(147) \quad \psi := \varphi_{u_1} \star \varphi_{u_2} \star \dots \star \varphi_{u_{r+1}},$$

where $\varphi_u(g_n) = u^n$. Then, $\hat{\varphi}_u(t) = 1 + tu$ and

$$(148) \quad \hat{\psi}(t) = (1 + tu_1)(1 + tu_2) \cdots (1 + tu_{r+1}) = \lambda_t(U).$$

Lagrange inversion yields

$$(149) \quad \psi(g_n) = \frac{1}{n+1} e_n[(n+1)U]$$

and extracting the coefficient of a monomial, we obtain the number of chains such that $\text{rk}(\pi_i) - \text{rk}(\pi_{i-1}) = s_i$ is equal to

$$(150) \quad \frac{1}{n+1} \binom{n+1}{s_1} \binom{n+1}{s_2} \cdots \binom{n+1}{s_{r+1}}.$$

This is

$$(151) \quad \frac{1}{n+1} [u_1^{s_1} u_2^{s_2} \cdots u_{r+1}^{s_{r+1}}] \lambda_1 [u_1 + \cdots + u_{r+1}]^{n+1}$$

which is equal to the coefficient of m_μ in $\omega(g)$, where μ is the partition obtained by reordering the s_i , *i.e.*, to the scalar product $\langle e_\mu, g \rangle$.

This last expression can be interpreted in terms of the Farahat-Higman algebra. Let c_μ be the dual basis of g^μ in Sym (*i.e.* $c_\mu(-X)$ is what is denoted by g_μ in Macdonald's book [15, Ex. 24-25, p. 131-133]). Then, the elementary symmetric functions are

$$(152) \quad e_k = \sum_{\kappa \vdash k} c_\kappa.$$

Indeed,

$$(153) \quad \langle e_k, g^\kappa \rangle = \prod_i \frac{e_{\kappa_i}[\kappa_i + 1]}{\kappa_i + 1} = 1$$

for all $\kappa \vdash k$. Thus, it represents the sum of all permutations which can be written as a minimal product of k transpositions. Identifying NC_{n+1} with the interval $[id_{n+1}, (12 \cdots n + 1)]$ of the Cayley graph of \mathfrak{S}_{n+1} as in [2], noncrossing partitions are identified with the permutations lying on the minimal paths between the identity and the full cycle, the rank being the transposition length. If $\mu = (s_1, \dots, s_{r+1})$, the scalar product $\langle e_\mu, g_n \rangle$ is equal to the coefficient of c_n in the product $e_{s_1} s_{s_2} \cdots e_{s_{r+1}}$, hence to the number of factorisations of the full cycle into a product of permutations minimally factorisable into s_1, s_2, \dots transpositions, that is, to the number of chains of noncrossing partitions with the prescribed ranks.

As another example, since $c_n = M_n = p_n$, we can recover a result of Biane [2]: the number of minimal factorizations of an n -cycle into a product of cycles of orders a_1, \dots, a_r is the coefficient of c_{n-1} in the product $c_{a_1-1} c_{a_2-1} \cdots c_{a_r-1}$, that is,

$$(154) \quad \langle p_{a_1-1} \cdots p_{a_r-1}, g_{n-1}(X) \rangle = \langle p_{a_1-1} \cdots p_{a_r-1}, \frac{1}{n} h_{n-1}(nX) \rangle = n^{r-1}.$$

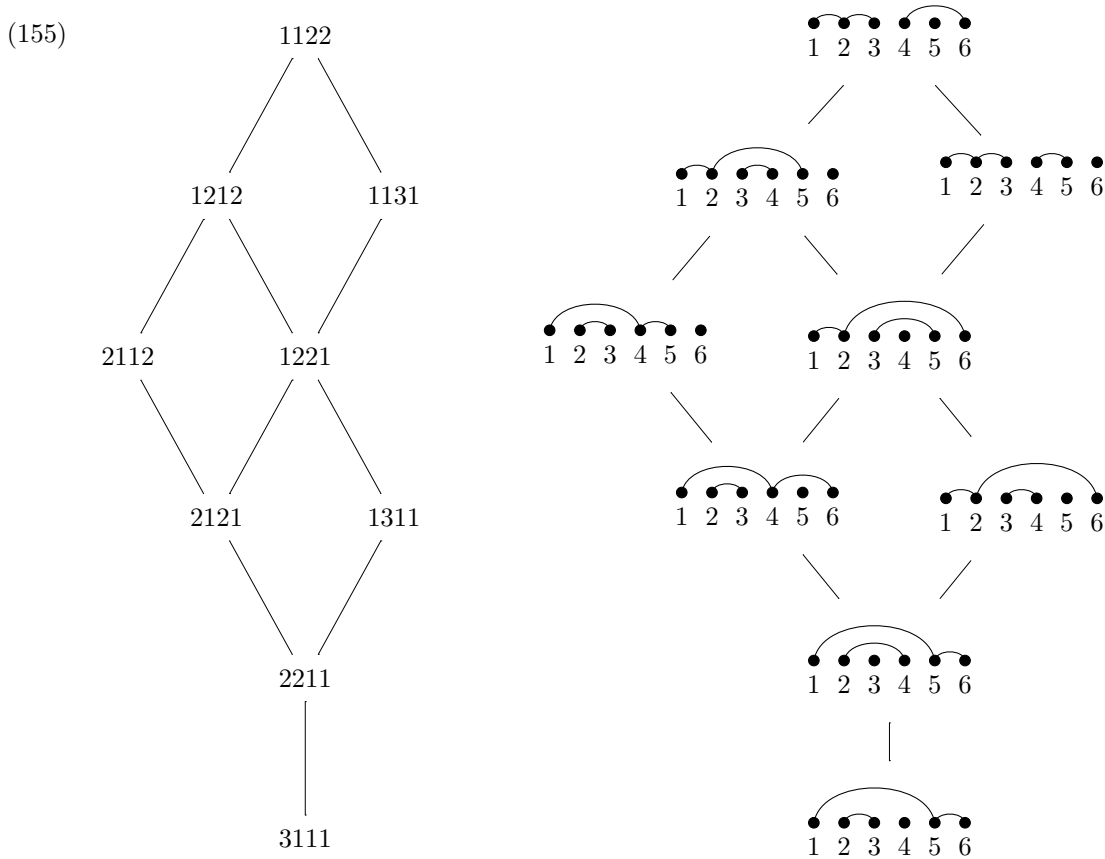
7. APPENDIX

7.1. Generating compatible pairs of compositions. Given a composition I , the list of compositions J compatible with I can be computed as follows.

The composition whose descent set is the complement of the descent set of I is \bar{I} , the mirror conjugate of I . Then, since we required that the sorted concatenation of the descent sets of I and J form a word greater componentwise than the sorted concatenation of I and \bar{I} , the J that are compatible with I are those obtained from \bar{I} by iterating the following process: given $C = (c_1, \dots, c_n)$, for any $i > 1$ such that $c_i > 1$, change C into C' by adding 1 to c_{i-1} and subtracting 1 to c_i .

In particular, the set of compositions compatible with I is equipped with a natural order, its top element being \bar{I} and its bottom element being $(|I| - k + 1, 1^{k-1})$ if k is the length of \bar{I} .

For example, with $I = 321$, its reverse conjugate is 1122 and the whole list of possibilities for J contains 9 elements which can be drawn on the following diagram



7.2. Descent words and Motzkin paths. We have seen that the compatible compositions are those whose concatenation of descent sets are greater than $1 \dots n$. The map sending compatible compositions to such descent words is of course highly non injective and its image set consists in the sorted words s such that $s_i \geq i$, $s_i \leq n$, and no value can be taken more than twice. Let us denote by S_n this set of words.

For example, with $n = 4$, we get the word 123 eight times, and all other words 133, 223 and 233 twice each, for a total of 14. For general n , the number of pairs of compositions with a given word s as image is obviously 2^k where k is the number of values used only once in s . Moreover, if one counts the number of words by their number of doubled letters (so that the first column is 1 and represents $s = 1 \dots n$ with no repeated letters), we find the following triangle:

$$(156) \quad \begin{array}{cccccc} & & & & & 1 \\ & & & & & 1 \\ & & & & & 1 & 3 \\ & & & & & 1 & 6 & 2 \\ & & & & & 1 & 10 & 10 \\ & & & & & 1 & 15 & 30 & 5 \\ & & & & & 1 & 21 & 70 & 35 \\ & & & & & 1 & 28 & 140 & 140 & 14 \end{array}$$

which is Sequence A055151 of [23], the triangular array of Motzkin paths of length n and with k up steps.

In one wants to see how powers of 2 come into play, one has to represent the table as follows:

$$(157) \quad \begin{array}{ccccccccc} & & & & & & & & & 1 \\ & & & & & & & & & 1 \\ & & & & & & & & & 1 & 1 \\ & & & & & & & & & 3 & 1 \\ & & & & & & & & & 2 & 6 & 1 \\ & & & & & & & & & 10 & 10 & 1 \\ & & & & & & & & & 5 & 30 & 15 & 1 \\ & & & & & & & & & 35 & 70 & 21 & 1 \\ & & & & & & & & & 14 & 140 & 140 & 28 & 1 \end{array}$$

Here, column k corresponds to the number of words appearing 2^k times. For example, the fifth line reads $2 \cdot 2^0 + 6 \cdot 2^2 + 2^4 = 42$.

We shall prove that S_n is indeed equinumerous with Motzkin paths, even with the extra parameter introduced above, but it will be easier to work with the set S'_n defined as the image of S_n by the map

$$(158) \quad w = w_1 \dots w_n \mapsto (n+1-w_n) \dots (n+1-w_1).$$

The condition on the words of S_n translates in S'_n as $w_i \leq i$, so that the w_i are parking functions. Now, the classical bijection between nondecreasing parking functions and noncrossing partitions sends a noncrossing partition c to the nondecreasing word where i appears as many times as the cardinality of the i -th part of c .


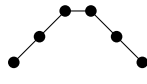
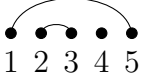

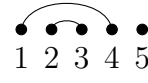
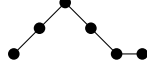
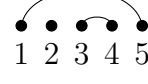
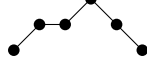
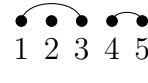
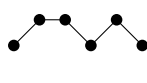
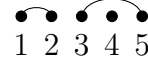
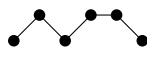
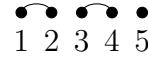
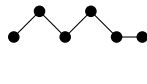
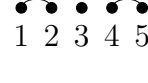
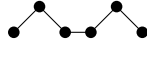
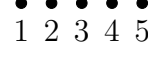
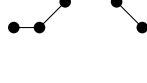
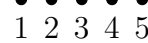
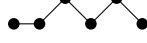
So S'_n corresponds to the noncrossing partitions with parts at most 2. Read such a noncrossing partition from left to right and draw an up step if we begin a part with two elements, a down step if we close such a part, and a horizontal step if we have a singleton. This is the natural bijection between these particular noncrossing partitions and Motzkin paths. Moreover, the statistic of the number of repeated letters is sent to the number of parts with two elements in the noncrossing partition, and then to the number of up steps in the Motzkin path.

We then have

Proposition 7.1. *The set S'_n and Motzkin paths M_n of n are equinumerous and the statistic of the number of repeated letters in S' corresponds to the number of up steps in M_n .*

For example, here is the whole of sublist S_5 consisting in words with two pairs of repeated letters (10 elements) and their successive images by the bijections.

(159)

34455	↔	11223	↔		↔	
24455	↔	11224	↔		↔	
14455	↔	11225	↔		↔	
33455	↔	11233	↔		↔	
22455	↔	11244	↔		↔	
23355	↔	11334	↔		↔	
13355	↔	11335	↔		↔	
22355	↔	11344	↔		↔	
33445	↔	12233	↔		↔	
22445	↔	12244	↔		↔	

Following [23], there is a simple formula for $|S_{n,k}|$, the number of elements of S_n with k repeated values: $|(S_{n,k})| = \binom{n}{2k} C_k$, so that the cardinality of the set of compatible pairs of compositions is

$$(160) \quad \sum_{k \geq 0} 2^{n-2k} \binom{n}{2k} C_k = C_{n+1},$$

thanks to Touchard, cited by several authors on the Catalan webpage of [23]. So we have proved by a simple and meaningful bijection that indeed ΔG_n has Catalan terms.

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