

STRUCTURE OF HYPERBOLIC POLYNOMIAL AUTOMORPHISMS OF \mathbb{C}^2 WITH DISCONNECTED JULIA SETS

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ABSTRACT. For a hyperbolic polynomial automorphism of \mathbb{C}^2 with a disconnected Julia set, and under a mild dissipativity condition, we give a topological description of the components of the Julia set. Namely, there are finitely many “quasi-solenoids” that govern the asymptotic behavior of the orbits of all non-trivial components. This can be viewed as a refined Spectral Decomposition for a hyperbolic map, as well as a two-dimensional version of the (generalized) Branner-Hubbard theory in one-dimensional polynomial dynamics. An important geometric ingredient of the theory is a John-like property of the Julia set in the unstable leaves.

CONTENTS

1. Introduction	1
2. Preliminaries and notation	6
3. External rays	11
4. Stable total disconnectedness	17
5. Classification of semi-local components of K^+ and J^+	19
6. Components of J and K	27
7. Complements	30
8. Non-divergence of holonomy and applications	34
Appendix A. The core of a quasi-solenoid	38
Appendix B. Continuity of affine structure	40
References	41

1. INTRODUCTION

1.1. Preamble on hyperbolic dynamics. The classical *Spectral Decomposition* of a hyperbolic (Axiom A) real diffeomorphism f of a compact manifold (developed by Smale, Anosov, Sinai, Bowen, and others) provides us with a rather complete topological picture of its dynamics. Namely, the non-wandering set $\Omega(f)$ is decomposed into finitely many *basic sets*, each of which modeled on an irreducible Markov chain. Among these basic sets there are several *attractors* that govern the asymptotic behavior of generic points of

the manifold. This picture has become a prototype for numerous other settings, including one-dimensional, non-invertible, holomorphic, partially or non-uniformly hyperbolic dynamical systems.

In the context of complex polynomial automorphisms of \mathbb{C}^2 , hyperbolic maps arise naturally as perturbations of one-dimensional hyperbolic polynomials. They were first studied in the late 1980s by Hubbard and Oberste-Vorth [24, 25] who showed that their topological structure can be fully described in terms of the original one-dimensional maps, whose Julia set and attracting cycles get perturbed to the basic sets of f (see also Fornæss-Sibony [19]).

Computer experiments indicate that, though hyperbolicity is not a prevalent phenomenon in dimension two, there should exist still plenty of non-perturbative examples. The first such candidate (a quadratic Hénon map with two co-existing attracting cycles) was proposed by Hubbard; it was further investigated by Oliva in his thesis [38]. However, it is a challenging problem, which requires computer assistance, to prove the hyperbolicity of a particular example, and this one still remains unconfirmed. Some time later, Ishii justified the hyperbolicity of several other non-perturbative Hénon maps: see [26, 27, 28] (of course, along with each such example comes an open set of hyperbolic parameters).

A systematic theory of hyperbolic polynomial automorphisms of \mathbb{C}^2 was launched by Bedford and Smillie in the early 1990's, relying notably on methods from Pluripotential Theory. In particular, they showed in [3] that any such a map only has one non-trivial basic set, its Julia set $J(f)$, while all others are just attracting cycles. Further combinatorial study of hyperbolic Hénon maps was carried out by Ishii and Smillie [29].

In this paper we will reveal a finer structure of the Julia set, related to its connected components, that leads to a finer “spectral decomposition”. Namely, under mild dissipativity assumptions, we will show that there are finitely many *quasi-solenoids* that govern the asymptotic behavior of all non-trivial components. Some of these quasi-solenoids are *tame* (i.e. lie on the boundary of the basins of some attracting cycles), while others might be *queer* (we do not know whether they actually exist).

Let us conclude this preamble by suggesting a potentially important role that hyperbolic maps may play in the Hénon story. They are not only interesting simple models for the general non-uniformly hyperbolic situation, but they may also be seen as “germs” for a Renormalization Theory which would lead to self-similarity features of the parameter spaces. In this respect, renormalizing hyperbolic Hénon maps around quasi-solenoids would be the beginning of this story.

1.2. One-dimensional prototype. Understanding the topological structure of the Julia set is one of the most basic problems in holomorphic dynamics. For polynomials in one variable, Fatou and Julia proved that the connectivity properties of the Julia set are dictated by the dynamical behavior of critical points. When the critical points do not escape, the Julia set J is connected; on the contrary, if all critical points do escape, J is a Cantor set. If J is connected and locally connected, the theory of external rays of Douady and Hubbard [13] and the theory of geodesic laminations of Thurston [44] give a topological model for the Julia set as the quotient of the circle by an equivalence relation which records the landing pattern of external rays. When the Julia set of a polynomial is disconnected, it admits uncountably many components, and one challenge

is to characterize when a component is non-trivial (i.e. not a point) in terms of the induced dynamics on the set of components. It turns out that this happens when and only when this component is preperiodic to a component containing a critical point: this is due to Branner and Hubbard [9] for cubic polynomials, and Qiu and Yin [41] in the general case (based upon the Kahn-Lyubich machinery [30, 31]). Then one may describe non-trivial periodic components by realizing them as Julia sets of connected polynomial-like maps and using the Douady and Hubbard Straightening Theorem [14].

In the hyperbolic case, the above theory is much easier and had belonged to folklore of the field:

Theorem 1.1. *Let p be a hyperbolic polynomial in \mathbb{C} , with a disconnected Julia set. Then the filled Julia set K has uncountably many components, and only countably of them are non-trivial. Any non-trivial component is preperiodic, and there are finitely many periodic components, each of which containing an attracting periodic point.*

Note that this is really a statement about polynomials: there are examples of hyperbolic rational maps on \mathbb{P}^1 whose Julia sets are Cantor sets of circles [37].

1.3. Main result. In this article we address similar issues in the setting of polynomial automorphisms of \mathbb{C}^2 . Let f be a polynomial automorphism of \mathbb{C}^2 with non-trivial dynamics: by this we mean for instance that the algebraic degree of the iterates f^n tend to infinity (see below §2.1 for more details on this). Its Julia set $J = J_f$ is the set of points at which both $(f^n)_{n \geq 0}$ and $(f^{-n})_{n \geq 0}$ are not locally normal. We also classically denote by K^+ (resp. K^-), the set of points with bounded forward (resp. backward) orbits, $K = K^+ \cap K^-$ and $J^\pm = \partial K^\pm$, so that $J = J^+ \cap J^-$. The complex Jacobian $\text{Jac } f$ is a non-zero constant. Thus, replacing f by f^{-1} if necessary, without loss of generality we assume from now on that $|\text{Jac } f| \leq 1$.

In this context, the connected vs. disconnected dichotomy for the Julia set was studied by Bedford and Smillie [6], who proved that the connectedness of J , or equivalently of K , is equivalent to the non-existence of “unstable critical points”, which are defined as tangencies between certain dynamically defined foliations. (Recall that f has no critical point in the usual sense, but these unstable critical points play the same role as escaping critical points in dimension one.) Bedford and Smillie also showed that when J is connected, there is a well-defined family of external rays along unstable manifolds, parameterized by a “solenoid at infinity”, which is the inverse limit of the dynamical system defined by $z \mapsto z^d$ on the unit circle.

To proceed further and try to extend the Douady-Hubbard description of the Julia set in terms of the combinatorics of external rays, given our current state of knowledge, we need to assume that f is uniformly hyperbolic. Recall from [3] that f is said to be *hyperbolic* if J is a hyperbolic set, which must then be of saddle type. In this case, f satisfies Smale’s Axiom A in \mathbb{C}^2 , and the Fatou set is the union of finitely many basins of attraction. (See [27] for an introductory account to this topic, which also discusses some combinatorial/topological models for Julia sets.)

By using the convergence of unstable external rays, it was shown in [7] that if f is hyperbolic and J is connected, then J can be described as a finite quotient of the solenoid at infinity. A non-trivial consequence of the results of [5],[6] and [7] is that in

this case f cannot be conservative, that is, $|\text{Jac } f| < 1$ (see [7, Cor. A.3]; recall that we assume $|\text{Jac } f| \leq 1$ here). An alternate argument for this fact was given by the first-named author in [15], where it is shown that a hyperbolic automorphism f with connected Julia set must possess an attracting periodic point, so in particular $|\text{Jac } f| < 1$. Surprisingly enough, the existence of an attracting point does not seem to follow easily from the description of J as a quotient of the solenoid.

In this article we focus on the disconnected case. A motivating question is the following conjecture from [15].

Conjecture 1.2. *Let f be a dissipative and hyperbolic automorphism of \mathbb{C}^2 , without attracting points. Then J is a Cantor set.*

Our main result is an essentially complete generalization of Theorem 1.1 in two dimensions, under a mild dissipativity assumption.

Main Theorem. *Let f be a hyperbolic polynomial automorphism of \mathbb{C}^2 , with a disconnected Julia set, and such that $|\text{Jac } f| < 1/\deg f$. Then there are uncountably many components of J , which can be of three (mutually exclusive) types:*

- (1) *point;*
- (2) *leafwise bounded;*
- (3) *or quasi-solenoid.*

Quasi-solenoidal components are periodic and there are only finitely many of them. Any component of type (2) is wandering and converges to a quasi-solenoidal one under forward iteration. The components of K are classified accordingly.

Under an additional assumption (NDH) on the behavior of stable holonomy between components, any quasi-solenoidal component of K contains an attracting periodic point.

Here $\deg f$ refers to the dynamical degree of f , which is the growth rate of algebraic degree under iteration (see §2.1). By definition, a component of J is *leafwise bounded* if it is a relatively bounded subset of some unstable manifold; this implies that its topology is that of a full plane continuum, properly embedded in \mathbb{C}^2 . A *quasi-solenoid* is a connected component with local product structure, which is totally disconnected in the stable direction and locally connected and leafwise unbounded in the unstable direction (see Definition 6.2). Components of type (2) are analogous to strictly preperiodic components in dimension 1; note however that by the local product structure of J there are uncountably many of them. Countability is restored by saturating with semi-local stable manifolds (see Theorem 5.20). The meaning of the (NDH) assumption will be explained below.

1.4. Outline. Let us discuss some of the main ideas of the proof, which occupies the most part of the paper. First, the assumption on the Jacobian is used to guarantee that *the slices of J (resp. K) by stable manifolds are totally disconnected*. It is reminiscent of the stronger *substantial dissipativity* assumption $|\text{Jac } f| < 1/(\deg f)^2$ used in [17, 34, 35]. We could indeed use substantial dissipativity and Wiman's Theorem in the style of these papers to achieve stable total disconnectivity. However, hyperbolicity allows for a Hausdorff dimension calculation which gives a better bound on the Jacobian (see Section 4).

The key step of the finiteness property in the main theorem is an analysis of geometry of the unstable slices of J and K . Using external rays, we first show in Section 3 that the complement of K along unstable manifolds satisfies a weak version of the *John property*. This property implies that the components of $K \cap W^u$ are locally connected, and that locally there are only finitely many components of diameter bounded from below.

This finiteness is used to get a classification of *semi-local components* of J^+ and K^+ . By this we mean that we fix a large bidisk \mathbb{B} (in adapted coordinates) in which J^+ and K^+ are vertical-like objects, and we look at components of $J^+ \cap \mathbb{B}$ (resp. $K^+ \cap \mathbb{B}$). We prove that *these semi-local components behave like components of J (resp. K) for one-dimensional polynomials*: only countably many of them are non-trivial, that is, not reduced to vertical submanifolds, and any non-trivial such component is preperiodic. Besides the finiteness induced by the John-like property, this relies on a key *homogeneity property* of such a semi-local component: either all its unstable slices are “thin”, or all of them are “thick”. To prove this *thin-thick dichotomy* we show that if a semi-local component admits a thin unstable slice, then by a careful choice of \mathbb{B} we can arrange that the stable foliation of this semi-local component is transverse to $\partial\mathbb{B}$. It follows that this component has a global product structure in \mathbb{B} (see Section 5 for details).

If C is a non-trivial component of J , it is easy to see that the ω -limit set of C must be contained in one of the finitely many thick semi-local components of J^+ . We show that it must have local product structure, hence be a quasi-solenoidal component of J . The main step is the following: for large $m \neq n$, by the expansion in the unstable direction, the unstable slices of $f^m(C)$ and $f^n(C)$ have a diameter bounded from below, so if $x_n \in f^n(C)$ is close to $x_m \in f^m(C)$, by the finiteness given by the John-like property, $f^n(C)$ and $f^m(C)$ must correspond one to the other under local stable holonomy. Furthermore, such a quasi-solenoidal component must coincide with the limit set of its semi-local component in J^+ , and the finiteness of the number of attractors follows (see Section 6).

To get a complete generalization of the one-dimensional situation, it remains to show that such a quasi-solenoidal component must “enclose” some attracting periodic point. Unfortunately, all our attempts towards this result stumbled over the following issue: if $x, y \in J$ are such that $y \in W^s(x)$, the stable holonomy induces a local homeomorphism $J \cap W_{\text{loc}}^u(x) \rightarrow J \cap W_{\text{loc}}^u(y)$. The point is that it might not be the case in general that this local homeomorphism can be continued along paths in $J \cap W^u(x)$, even when $J \cap W^u(x)$ is a relatively bounded subset of $W^u(x)$. (Compare with the Reeb phenomenon for foliations, illustrated in Figure 1.) This is a well-known difficulty in hyperbolic dynamics, which was encountered for instance in the classification of Anosov diffeomorphisms (see §8.1 for a short discussion). If this continuation property holds – this is the Non-Divergence of Holonomy (NDH) property referred to in the main theorem – then we can indeed conclude that non-trivial periodic components of K contain attracting orbits (see Section 8, in particular Theorem 8.4). This yields in particular a conditional proof of Conjecture 1.2. Let us also note that a simple instance where the NDH property holds is when the stable lamination of J^+ is transverse to $\partial\mathbb{B}$ (for some choice of \mathbb{B}), a property which can be checked in practice on specific examples.

In the course of the paper, we also establish a number of complementary facts, which do not enter into the proof of the main theorem: the existence of an external ray landing at every point of J (see Theorem 3.4); the structure of attracting basins (see § 7.2); a

simple topological model for the dynamics on Julia components (see § 7.3); the topological transitivity of quasi-solenoids (see Theorem 8.7). In Appendix A we sketch the construction of the *core* of a quasi-solenoidal component, which aims at describing its topological structure.

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2. PRELIMINARIES AND NOTATION

2.1. Vocabulary of complex Hénon maps. If $\mathbb{B} = D \times D$ is a bidisk, we denote by $\partial^v \mathbb{B} = \partial D \times D$ (resp. $\partial^h \mathbb{B} = D \times \partial D$) the vertical (resp. horizontal) boundary. An object in \mathbb{B} is horizontal if it intersects $\partial \mathbb{B}$ only in $\partial^v \mathbb{B}$, and likewise for vertical objects. A closed horizontal submanifold is a branched cover of finite degree over the first projection.

Let us collect some standard facts and notation (see [21, 3, 2, 19]). If f is a polynomial diffeomorphism of \mathbb{C}^2 with non-trivial dynamics, then by making a polynomial change of coordinates we may assume that f is a composition of complex Hénon mappings $(z, w) \mapsto (p_i(z) + a_i w, a_i z)$. In particular $\deg(f^n) = (\deg f)^n$ for every $n \geq 0$. We fix such coordinates from now on. As it is customary in this area of research, we will often abuse terminology and simply refer to f as a *complex Hénon map*. The degree of f is $d = \prod \deg(p_i) \geq 2$ and the relation $\deg(f^n) = d^n$ holds so that d coincides with the so-called *dynamical degree* of f .

In these adapted coordinates, there exists $R > 0$ such that for the bidisk $\mathbb{B} := D(0, R)^2$, we have that $f(\mathbb{B}) \cap \mathbb{B}$ (resp. $f^{-1}(\mathbb{B}) \cap \mathbb{B}$) is horizontally (resp. vertically) contained in \mathbb{B} and the points of $\partial^v(\mathbb{B})$ (resp. $\partial^h(\mathbb{B})$) escape under forward (resp. backward) iteration.

- ▶ K^\pm is the set of points with bounded forward orbits under $f^{\pm 1}$ and $K = K^+ \cap K^-$. Note that K^+ is vertical in \mathbb{B} and $f(\mathbb{B} \cap K^+) \subset K^+$. Similarly, K^- is horizontal and $f^{-1}(\mathbb{B} \cap K^-) \subset K^-$.
- ▶ $J^\pm = \partial K^\pm$ are the forward and backward Julia sets. If f is dissipative then $K^- = J^-$.
- ▶ $J = J^+ \cap J^-$ is the Julia set.

Following [6], we say that f is *unstably disconnected* if for some (and hence any) saddle periodic point p , $W^u(p) \cap K^+$ admits a compact component (relative to the topology induced by the biholomorphism $W^u(p) \simeq \mathbb{C}$), and unstably connected otherwise. If f is unstably disconnected, then it admits an *unstable transversal* Δ^u , that is a relatively compact domain in $W^u(p)$ which is a horizontal submanifold in \mathbb{B} : indeed pick a bounded Jordan domain $U \subset W^u(p)$ containing a compact component of $W^u(p) \cap K^+$ such that $\partial U \cap K^+ = \emptyset$ and iterate it forward.

2.2. Hyperbolicity and local product structure. Throughout the paper we assume that f is hyperbolic on J (hence Axiom A on \mathbb{C}^2 by [3]), with hyperbolic splitting $T\mathbb{C}^2|_J = E^u \oplus E^s$. Then there exists a continuous Riemannian metric $|\cdot|$ on J and constants $s < 1 < u$ such that for any $x \in J$, and any $v \in E^u(x) \setminus \{0\}$, $|Df_x \cdot v| \geq u|v|$ (resp. for any $v \in E^s(x)$, $|Df_x \cdot v| \leq s|v|$). By [16], it is enough to assume that f is hyperbolic on J^* , where J^* is the closure of saddle periodic points (and a posteriori one deduces that $J = J^*$).

In this situation the local stable and unstable manifolds of points of J have local uniform geometry: there exists a uniform $r > 0$ such that for every $x \in J$, $W^u(x)$ (resp. $W^s(x)$) is of size r at x , in the sense that it contains a graph of slope at most 1 over a disk of radius r in $E^u(x)$ (resp. $E^s(x)$). The reader is referred to [8, 1] for a detailed study of this notion. We denote by $W_\delta^{s/u}(x)$ the local stable/unstable manifold of radius δ at x , which is by definition the component of $W^{s/u}(x)$ in $B(x, \delta)$. When the precise size does not matter, we simply denote them by $W_{\text{loc}}^{s/u}$. Slightly reducing the expansion constant u if necessary, given two points z, z' in some local unstable manifold $W_\delta^u(x)$, there is a uniform constant C such that $d(f^{-n}(z), f^{-n}(z')) \leq Cu^{-n}$, for all $n \geq 0$.

There exists $\delta > 0$ and a neighborhood \mathcal{N} of J such that the restriction to \mathcal{N} of the family local stable/unstable manifolds of radius δ is a lamination, denoted by $\mathcal{W}^{u/s}$. The Julia set has local product structure so there is a covering by topological bidisks Q (flow boxes) such that the laminations $\mathcal{W}^{u/s}$ are trivial in Q and

$$J \cap Q \simeq (W_Q^s(x) \cap J) \times (W_Q^u(x) \cap J) = (W_Q^s(x) \cap J^-) \times (W_Q^u(x) \cap J^+).$$

It is shown in [3] that the family of global stable and unstable manifolds of points of J also has a lamination structure, which will be denoted by $\mathcal{W}^{s/u}$. More precisely, in the dissipative case, \mathcal{W}^s is a lamination of J^+ is laminated by stable manifolds and the other hand, \mathcal{W}^u is a lamination of $J^- \setminus \{a_1, \dots, a_N\}$, where $\{a_1, \dots, a_N\}$ is the finite set of attracting periodic points of f . No unstable leaf extends across an attracting point, even as a singular analytic set: indeed an unstable leaf is biholomorphic to \mathbb{C} , therefore such an extension would yield a submanifold of \mathbb{C}^2 biholomorphic to a (possibly singular) copy of \mathbb{P}^1 , which is impossible.

Under additional dissipativity assumptions, it was shown in [35] that the stable lamination \mathcal{W}^s in \mathbb{B} can be extended to a C^1 foliation in some neighborhood of J^+ : see Lemma 5.7 below.

Let us conclude this paragraph with a useful elementary result.

Lemma 2.1. *If f is hyperbolic, every holomorphic disk contained in K^+ is either contained in the Fatou set or in the stable manifold of a point of J .*

Proof. Indeed, if Δ is a disk contained in K^+ then Δ is a Fatou disk, i.e. $(f^n|_\Delta)_{n \geq 0}$ is a normal family. Now there are two possibilities: either Δ is contained in $\text{Int}(K^+)$ hence in the Fatou set, or it intersects J^+ . In the latter case, either Δ is contained in a stable leaf or by [2, Lem. 6.4], Δ must have a transversal intersection with some unstable manifold, so by the Inclination Lemma it is not a Fatou disk, which is a contradiction. \square

2.3. Affine structure. Global stable and unstable manifolds are uniformized by \mathbb{C} , so they admit a natural affine structure. Since any automorphism of \mathbb{C} is affine, f acts affinely on leaves. In particular there is a well defined notion of a round disk, which is f -invariant. Likewise, the Euclidean distance is well-defined in the leaves, up to a multiplicative constant.

For any $x \in J$ we choose a uniformization $\psi_x^u : \mathbb{C} \xrightarrow{\sim} W^u(x)$ such that $\psi_x^u(0) = x$ and $|(\psi_x^u)'(0)| = 1$

Lemma 2.2. *The family of uniformizations $(\psi_x^u)_{x \in J}$ is continuous up to rotations, that is, if $x_n \rightarrow x$ then $(\psi_{x_n}^u)$ is a normal family and its cluster values are of the form $\psi_x^u(e^{i\theta} \cdot)$.*

Proof. The result follows from the continuity of the affine structure on the unstable leaves (see Theorem B.1). \square

It is unclear whether the assignment $J \ni x \mapsto \psi_x^u$ can be chosen to be continuous, that is, if a consistent choice of rotation factor $e^{i\theta}$ can be made. This can be done locally but there might be topological obstructions to extend the continuity to J . Notice that the (ψ_x^u) provide a normalization for the leafwise Euclidean distance. The normalized Euclidean distance on $W^u(x)$ will be denoted by d_x^u . If $C \subset W^u(x)$, its diameter with respect to d_x^u will be denoted by Diam_x . By Lemma 2.2, d_x^u varies continuously with x . For $R > 0$ we let $D_x^u(x, R) := \psi_x^u(D(0, R))$.

By construction, f is a uniformly expanding linear map in these affine coordinates, that is $f \circ \psi_x^u = \psi_{f(x)}^u(\lambda_x^u \cdot)$, with $|\lambda_x^u| = \|df|_{E_x^u}\|$. By hyperbolicity there is a positive constant C such that for every $x \in J$,

$$(1) \quad \left| \prod_{i=0}^{n-1} \lambda_{f^i(x)}^u \right| \geq C u^n,$$

where $u > 1$ was defined in §2.2.

By the Koebe Distortion Theorem there exists a uniform $r > 0$ such that the $D^u(x, r)$ are contained in the flow boxes (see e.g. [8, Lemma 3.7]). By the local bounded geometry of the leaves, the distance induced by the affine structure on the $D^u(x, r)$ is equivalent to that induced by the ambient Hermitian structure. Then, iterating finitely many times we can promote this result on the $D^u(x, R)$ for every given $R > 0$.

All the above discussion holds for stable manifolds, with superscripts u replaced by s .

2.4. Connected and semi-local components. For every $x \in J$ (or more generally $x \in K^+ \cap \mathbb{B}$) we denote by $K_{\mathbb{B}}^+(x)$ the connected component of x in $K^+ \cap \mathbb{B}$, which is a vertical subset of \mathbb{B} . It follows from the Hénon-like property that $f(K_{\mathbb{B}}^+(x)) \subset K_{\mathbb{B}}^+(f(x))$, thus f induces a (non-invertible) dynamical system on the set of connected components of $K^+ \cap \mathbb{B}$. The same discussion applies to components of $J^+ \cap \mathbb{B}$. More generally, for any closed connected subset $C \subset J$ (resp. $C \subset K$), we define $J_{\mathbb{B}}^+(C)$ (resp. $K_{\mathbb{B}}^+(C)$) to be the connected component of C in $J^+ \cap \mathbb{B}$ (resp. $K^+ \cap \mathbb{B}$). Of course for $x \in C$, $J_{\mathbb{B}}^+(x) = J_{\mathbb{B}}^+(C)$ holds. A related concept is $W_{\mathbb{B}}^s(x)$, the component of $\mathbb{B} \cap W^s(x)$ containing x . If we set $W_{\mathbb{B}}^s(C) = \bigcup_{x \in C} W_{\mathbb{B}}^s(x)$ then $W_{\mathbb{B}}^s(C)$ is contained in $K_{\mathbb{B}}^+(C)$ but this

inclusion may be strict. This phenomenon may happen when for some $x \in C$, $W_{\mathbb{B}}^s(x)$ is tangent to $\partial\mathbb{B}$ (see Figure 1).

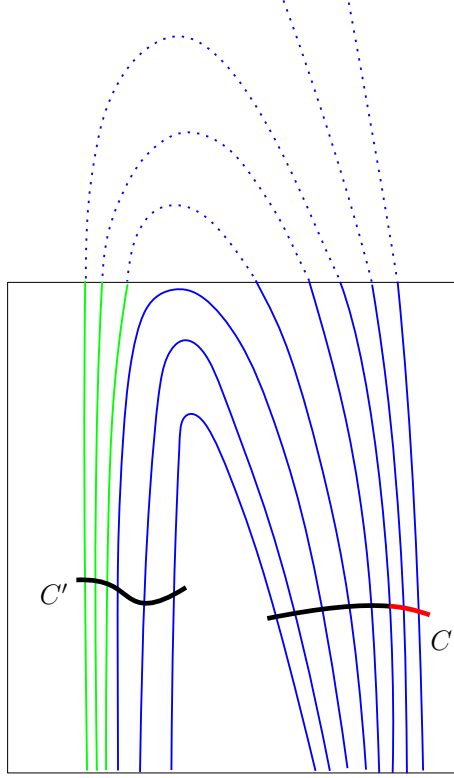


FIGURE 1. Discontinued holonomy. The green components belong to $K_{\mathbb{B}}^+(C)$ but not to $W_{\mathbb{B}}^s(C)$ (in blue). The red part of C cannot be followed under stable holonomy to C' due to a Reeb-like phenomenon.

For $x \in K$, we denote by $K^s(x)$ (resp. $K^u(x)$) the connected component of $K \cap W^s(x) = K^- \cap W^s(x)$ (resp. $K \cap W^u(x) = K^+ \cap W^u(x)$) containing x , and also $K(x)$ its connected component in K . For $x \in J$, we define $J^s(x)$, $J^u(x)$ and $J(x)$ similarly. More generally, if needed, we use the notation $\text{Comp}_E(x)$ for the connected component of x in a set E .

We use the subscript ‘i’ to denote topological operations (interior, closure, etc.) relative to the intrinsic topology in stable/unstable manifolds.

Lemma 2.3. *Assume that f is hyperbolic. Then every connected component of $K^+ \cap \mathbb{B}$ has a connected boundary, which is a component of $J^+ \cap \mathbb{B}$.*

Proof. Observe that if p is an interior point of $K^+ \cap L$, where L is a horizontal line, then it belongs to a Fatou disk. Since L is not contained in J^+ , by Lemma 2.1, we get that $p \in \text{Int}(K^+)$. This implies that for every $x \in K^+ \cap \mathbb{B}$, $\partial K_{\mathbb{B}}^+(x) \subset \bigcup_{t \in \mathbb{D}} \partial_{L_t}(K_{\mathbb{B}}^+(x) \cap L_t)$, where $L_t = \mathbb{D} \times \{t\}$ and ∂_{L_t} refers to the boundary in L_t . The converse inclusion is obvious, so $\partial K_{\mathbb{B}}^+(x) \cap \mathbb{B} = \bigcup_{t \in \mathbb{D}} \partial_{L_t}(K_{\mathbb{B}}^+(x) \cap L_t)$. Since $K_{\mathbb{B}}^+(x) \cap L_t$ is compact and

polynomially convex, and obviously $K_{\mathbb{B}}^+(x) = \bigcup_{t \in \mathbb{D}} K_{\mathbb{B}}^+(x) \cap L_t$, this means that $K_{\mathbb{B}}^+(x)$ is obtained from $\partial K_{\mathbb{B}}^+(x) \cap \mathbb{B}$ by filling the holes of all components of $\partial_L(K_{\mathbb{B}}^+(x) \cap L)$ in every horizontal line. Now assume $\partial K_{\mathbb{B}}^+(x) \cap \mathbb{B}$ is disconnected, so we can write it as $B_1 \cup B_2$, where each B_i is relatively open and $B_1 \cap B_2 = \emptyset$. In every horizontal slice L , $B_i \cap L$ must be a union of components of $\partial_L(K_{\mathbb{B}}^+(x) \cap L)$. For $i = 1, 2$, let \hat{B}_i be the set obtained by filling the holes of B_i in each horizontal line in \mathbb{B} . The previous discussion shows that $K_{\mathbb{B}}^+(x) = \hat{B}_1 \cup \hat{B}_2$, where the \hat{B}_i are relatively open in $K_{\mathbb{B}}^+(x)$ and disjoint. This is a contradiction, therefore $\partial K_{\mathbb{B}}^+(x) \cap \mathbb{B}$ is connected.

For the second statement, simply observe that if $D \subset J^+ \cap \mathbb{B}$ is a connected set such that $\partial K_{\mathbb{B}}^+(x) \cap \mathbb{B} \subset D$, then D is contained in $K_{\mathbb{B}}^+(x)$ and also in ∂K^+ so $D \subset \partial K_{\mathbb{B}}^+(x) \cap \mathbb{B}$ and we are done. \square

2.5. Basic properties of leafwise components. Here we assume that f is a hyperbolic and dissipative complex Hénon map. The following result is well-known.

Lemma 2.4. *For every $x \in K$ we have $\text{Int}_i(K^u(x)) \subset \text{Int}(K^+)$ and $\partial_i(K^u(x)) \subset J$. In particular if $\text{Int}_i(K^u(x))$ is non-empty, each of its components is contained in an attracting basin. Likewise $\text{Int}_i K^s(x) = \emptyset$ and $J^s(x) = K^s(x)$.*

Proof. Indeed, since stable and unstable manifolds cannot coincide along some open set, if Δ is a disk contained in $K^u(x)$, it follows from Lemma 2.1 that $\Delta \subset \text{Int}(K^+)$, and the remaining conclusions follow. \square

For x in J , $K^u(x)$ may be bounded or unbounded for the intrinsic (leafwise) topology. By the maximum principle, $K^u(x)$ is polynomially convex, so if $K^u(x)$ (or equivalently $J^u(x)$) is leafwise bounded, then $K^u(x)$ is simply the polynomially convex hull of $J^u(x)$ (i.e. is obtained by filling in the leafwise bounded components of the complement).

Lemma 2.5. *Given $x \in K$, in the following properties we have $(iv) \Leftrightarrow (iii) \Rightarrow (ii) \Leftrightarrow (i)$:*

- (i) $K^u(x)$ is leafwise bounded;
- (ii) $J^u(x)$ is leafwise bounded;
- (iii) $W_{\mathbb{B}}^u(x)$ is leafwise bounded;
- (iv) $W_{\mathbb{B}}^u(x)$ is a closed horizontal submanifold of \mathbb{B} .

Furthermore if (ii) holds, then (iii) holds for $f^n(x)$ for sufficiently large n .

Proof. The implication (i) \Rightarrow (ii) follows directly from the fact that $J^u(x) = \partial_i K^u(x)$. Now assume that $J^u(x)$ is leafwise bounded. Working in $W^u(x) \simeq \mathbb{C}$, we have that $K^u(x)$ is a closed connected polynomially convex set and $J^u(x)$ is a bounded connected component of $\partial_i K^u(x)$. Since every point of $J^u(x)$ lies on the boundary of $W^u(x) \setminus K^+$ (for the intrinsic topology), the compact set obtained by filling the holes of $J^u(x)$ must be $K^u(x)$, so the converse implication holds.

Since $K^u(x) \subset W_{\mathbb{B}}^u(x)$, obviously (iii) implies (i). Conversely, $K^u(x)$ is the decreasing intersection of the sequence of components of x in $W^u(x) \cap f^{-n}(\mathbb{B})$. Hence, if $K^u(x)$ is leafwise bounded it follows that $\text{Comp}_{W^u(x) \cap f^{-n}(\mathbb{B})}(x)$ is leafwise bounded for large enough n , and so does $W^u(f^n(x)) \cap \mathbb{B}$.

Recall that for every x , $W^u(x)$ is an injectively immersed copy of \mathbb{C} , whose image is a leaf of the lamination of $J^- \setminus \{a_1, \dots, a_N\}$. Here the a_i are the attracting points, and a leaf never extends to a submanifold in the neighborhood of a_i ⁽¹⁾. In particular, J^- is laminated near $\partial\mathbb{B}$. If $W_{\mathbb{B}}^u(x)$ is leafwise bounded, then it is of the form $\psi_x^u(\Omega)$, where Ω is some bounded open set in \mathbb{C} . Since ψ^u extends to a neighborhood of $\bar{\Omega}$, $W_{\mathbb{B}}^u(x)$ is a properly embedded submanifold of \mathbb{B} , which extends to a neighborhood of $\bar{\mathbb{B}}$. So (iii) implies (iv). Finally, if (iv) holds, since J^- is a lamination near $\partial\mathbb{B}$, we see that $W_{\mathbb{B}}^u(x)$ extends to a submanifold S in a neighborhood of $\bar{\mathbb{B}}$. Then $W_{\mathbb{B}}^u(x)$ is relatively compact in $S \subset W^u(x)$ so if Ω is such that $\psi_x^u(\Omega) = W_{\mathbb{B}}^u(x)$ then Ω is relatively compact in \mathbb{C} , and (iii) follows. \square

3. EXTERNAL RAYS

In this section we study external rays along the unstable lamination (i.e. along J^-) for a hyperbolic complex Hénon map. The existence and convergence properties of external rays were studied in the unstably connected case in [6, 7]. Recall that when $|\text{Jac}(f)| < 1$, unstable connectedness is equivalent to the connectedness of J . The results that we prove here do not rely on any unstable connectivity or dissipativity assumption, nevertheless what we have in mind is the case of a dissipative unstably disconnected map.

3.1. Escaping from K^+ along an external ray. By definition, an *unstable external ray* (simply called “external rays” in the following) is a piecewise smooth continuous path contained in a leaf $W^u(x)$ of the unstable lamination, which is a union of gradient lines of $G^+|_{W^u(x)}$ outside the (leafwise locally finite) set of critical points of $G^+|_{W^u(x)}$. As usual we assume that G^+ is strictly monotone along external rays (which will be considered as ascending or descending depending on the context). We do not prescribe rules for the behavior of rays hitting critical points, so in particular there is no attempt at defining a notion of “external map”.

In the next proposition the length of curves is relative to the ambient metric in \mathbb{C}^2 . We show that external rays ascend fairly quickly.

Proposition 3.1. *Let f be a hyperbolic polynomial automorphism of \mathbb{C}^2 of dynamical degree $d > 1$. For every $r_1 < r_2$ there exists $\ell(r_1, r_2)$ such that for every $x \in J^- \setminus K^+$ such that if $G^+(x) = r_1$, any external ray through x reaches $\{G^+ = r_2\}$ along a path whose length is bounded by $\ell(r_1, r_2)$. In addition $\ell(r_1, r_2)$ is bounded by a function $\bar{\ell}(r_2)$ depending only on r_2 . Furthermore $\ell(r_1, r_2) \rightarrow 0$ when $r_1 \rightarrow r_2$ and $\bar{\ell}(r_2) = O(r_2^\alpha)$ when $r_2 \rightarrow 0$, for some $\alpha > 0$.*

Remark 3.2. Notice that no dissipativity is assumed here so the result holds along stable leaves as well.

Proof. Start with $r_1 = 1$ and $r_2 = d$. In $J^- \cap \{1 \leq G^+ \leq d\}$ the leaves of W^u have uniform geometry and no leaf of W^u is contained in an equipotential hypersurface of the form $\{G^+ = C\}$, in particular unstable critical points have uniform order. Thus by compactness and continuity of G^+ , we infer the existence of uniform δ_0 and ℓ_0 such that for

¹Indeed otherwise this would induce a compactification of unstable manifolds, yielding an embedding of \mathbb{P}^1 into \mathbb{C}^2 .

every $x \in J^- \cap \{1 \leq G^+ \leq d\}$, any external ray through x of length ℓ_0 reaches $\{G^+ = r\}$ with $r \geq G^+(x) + \delta_0$. By concatenating such pieces of rays, we deduce the conclusion of the proposition for $r_1 = 1$ and $r_2 = d$ (and $\ell(1, d) \leq (d-1)\ell_0/\delta_0$). Pulling back finitely many times and concatenating again, we get a similar conclusion for $\{r_0 \leq G^+ \leq d\}$ for any fixed r_0 .

Let us now fix r_0 such that $\{0 < G \leq dr_0\} \cap J^-$ is contained in $W_{\text{loc}}^u(J)$. Any piece of external ray between the levels $\{G^+ = r_0/d^n\}$ and $\{G^+ = r_0/d^{n-1}\}$ is the pull-back of a piece of external ray in $\{r_0 \leq G^+ \leq dr_0\}$. Thus by concatenation it follows that any external ray starting from $\{G^+ = r_0/d^n\}$ reaches $\{G^+ = r_0\}$ along a path of length bounded by $\leq C\ell(r_0, dr_0) \sum_{k=1}^n u^{-k}$, where u is the expansion constant introduced in §2.2.

This proves the existence of the functions $\ell(r_1, r_2)$ and $\bar{\ell}(r_2)$

The same ideas imply immediately that $\ell(r_1, r_2) \rightarrow 0$ when $r_1 \rightarrow r_2$. For the last statement simply note that for every $r_1 < r_2 \leq r_0$,

$$\ell(r_1 < r_2) \leq C \sum_{k=k_0}^{\infty} u^{-k} = O(u^{-k_0})$$

where k_0 is the greatest integer such that $r_0 d^{-k_0} \geq r_2$, therefore $\ell(r_1 < r_2) = O(r_2^\alpha)$, with $\alpha = \frac{\log u}{\log d}$. \square

It is easy to deduce from these ideas that all (descending) external rays land. However, since there is no well defined external map, the characterization of the set of landing points does not seem to follow directly from this landing property.

Corollary 3.3 (John-Hölder property). *There exists a constant $\alpha > 0$ such that for any sufficiently small $\eta > 0$, for any $x \in J^- \setminus K^+$ sufficiently close to K^+ , there exists a path of length at most $O(\eta^\alpha)$ in $W^u(x) \setminus K^+$ joining x to a point η -far from K^+ .*

Proof. By the previous proposition, there exists a path of length $O(r^{\alpha_1})$ joining x to a point y such that $G^+(y) = r$. Now the Green function is Hölder continuous (see [19]) and that $K^+ = \{G^+ = 0\}$, so $d(x, K^+) \geq Cr^{\alpha_2}$. The result follows. \square

This John-Hölder property has deep consequences for the topology of $K^+ \cap W^u(x)$, which will play an important role in the paper. Intuitively it means that there cannot exist long “channels” between local components of K^+ .

This property is strongly reminiscent of the so-called John condition for plane domains, which have been much studied in one-dimensional dynamics, in relation with non-uniform hyperbolicity (see e.g. [12, 23]). In the Hénon context, it was shown in [7] that for unstably connected hyperbolic maps, the components of $W^u(x) \setminus K^+$ satisfy the John property. It is very likely that using the continuity of affine structure along unstable leaves, their arguments can be adapted to the disconnected case as well: this would upgrade Corollary 3.3 to the actual John condition. One advantage of this weaker property is that it makes no reference to the affine structure of the leaves, so it is more flexible and may be adapted to semi-local situations (e.g. Hénon-like maps).

3.2. Accesses and landing.

Theorem 3.4. *Let f be a hyperbolic polynomial automorphism of \mathbb{C}^2 with dynamical degree $d > 1$.*

- (1) *For every $x \in J$, $D^u(x, 1) \setminus K^+$ admits finitely many connected components, and at least one of these components contains x in its closure.*
- (2) *For any component Ω of $D^u(x, 1) \setminus K^+$ such that $\bar{\Omega} \ni x$ there is an external ray landing at x through Ω .*

For the proof, it is convenient to work in the affine coordinates given by the unstable parameterizations. We work in the disks $D^u(x, 1)$ and measure path length relative to the normalized affine metric, which is equivalent to the ambient one.

Proof. The first observation is that $D^u(x, 1) \setminus K^+$ contains x in its closure: otherwise x would lie in the leafwise interior of K^+ , thus contradicting Lemma 2.4. Furthermore, by the maximum principle, if $y \in D^u(x, 1) \setminus K^+$ is arbitrary, the component of y in $D^u(x, 1) \setminus K^+$ reaches the boundary of $D^u(x, 1)$.

We claim that there exists $\eta_1 > 0$ such that for any $x \in J$ and any component Ω of $D^u(x, 1) \setminus K^+$ such that $\Omega \cap D^u(x, 1/4) \neq \emptyset$, then:

$$\sup G^+|_{D^u(x, 1/2) \cap \Omega} \geq \eta_1.$$

This follows directly from Proposition 3.1: indeed there exists $\eta_1 > 0$ such that any point of $J^- \setminus K^+$ reaches $\{G^+ = \eta_1\}$ along a path of length $1/4$. By the Hölder continuity of G^+ , we infer that any such component Ω contains a disk of radius $C\eta_1^\alpha$, so there are finitely many of them.

In particular if (x_n) is a sequence in $D^u(x, 1) \setminus K^+$ converging to x , infinitely many of them must belong to the same component Ω of $D^u(x, 1) \setminus K^+$, which shows that $\bar{\Omega}$ contains x . This proves assertion (1) of the theorem.

Fix now a component Ω of $D^u(x, 1) \setminus K^+$ such that $\bar{\Omega} \ni x$. Let η_1 be as above and fix ε such that $\varepsilon < \eta_1/d$ and $\ell(\varepsilon, d\varepsilon) < \min(1/2, (u-1)/2)$ where $\ell(\cdot)$ is as in Proposition 3.1 and the constant u was defined in §2.2. We do the following construction: for every point $y \in \{G^+ = \varepsilon\} \cap \bar{D}^u(x, 1/2)$, we consider all ascending external rays emanating from y until they reach $\{G^+ = d\varepsilon\}$. The lengths of the corresponding rays is not larger than $\ell(\varepsilon, d\varepsilon)$. These are the rays of 0th generation and we denote by E_0 the set of their endpoints ⁽²⁾, which by the assumption on $\ell(\varepsilon, d\varepsilon)$ is contained in $\{G^+ = d\varepsilon\} \cap D^u(x, 1)$. We note that E_0 is a closed set because it is the ending point set of a compact family of external rays. Since $\varepsilon < \eta_1/d$, E_0 has non-empty intersection with Ω .

Performing the same construction in $D^u(f(x), 1)$ we obtain a set of rays of 0th generation in that disk, which connect $\{G^+ = \varepsilon\} \cap \bar{D}^u(f(x), 1/2)$ to $\{G^+ = d\varepsilon\}$, and their endpoints lie in

$$\{G^+ = d\varepsilon\} \cap \bar{D}^u\left(f(x), \frac{1}{2} + \ell(\varepsilon, d\varepsilon)\right).$$

²Recall that since we do not prescribe the behavior of external rays at critical points of G^+ there is no reason that external rays fill up the whole unstable lamination, so E_0 could be smaller than $\{G^+ = d\varepsilon\}$

The pull-backs of these rays by f have their endpoints in

$$\{G^+ = \varepsilon\} \cap \overline{D^u} \left(x, \frac{1}{u} \left(\frac{1}{2} + \ell(\varepsilon, d\varepsilon) \right) \right) \subset \{G^+ = \varepsilon\} \cap D^u \left(x, \frac{1}{2} \right),$$

by the assumption on $\ell(\varepsilon, d\varepsilon)$. These are the rays of 1st generation in $D^u(x, 1)$. We define $E_1 \subset E_0$ to be the closed set of points for which we can concatenate a ray of 0th generation with a ray of 1st generation to descend all the way to $\{G^+ = \varepsilon/d\}$. Notice that $f(\Omega) \cap D^u(f(x), 1)$ is not necessarily connected, so it is a union of components of $D^u(f(x), 1) \setminus K^+$, and since $\overline{f(\Omega)} \ni f(x)$, at least one of these components reaches $D^u(f(x), 1/2)$, so it contains rays of 0th generation. This shows that E_1 has non-empty intersection with Ω .

Continuing inductively this construction, we obtain a decreasing sequence (E_n) of closed subsets in $\{G^+ = d\varepsilon\} \cap D^u(x, 1)$, each of which intersecting Ω . If $e \in \bigcap_n E_n \cap \Omega$, then there is a ray through e (hence in Ω) converging to K^+ , whose part in $\{\varepsilon d^{-n-1} \leq G^+ \leq \varepsilon d^{-n}\}$ is the pull-back under f^n of a piece of external ray in $D^u(f^n(x), 1)$. Therefore this ray lands at x , and the proof of assertion (2) is complete. \square

Remark 3.5. The existence of a convergent external ray along any access to a saddle periodic point can be obtained exactly as in the 1-dimensional case (see [18]), without assuming uniform hyperbolicity. In that case the Denjoy-Carleman-Ahlfors Theorem is used instead of the John-Hölder property to guarantee the finiteness of the number of local components.

3.3. Topology of $K^+ \cap W^u$. In this section we review the consequences of Corollary 3.3 for the topology of unstable components of K^+ .

Theorem 3.6. *Let f be a hyperbolic Hénon map. Then for every $x \in J$:*

- (i) every component of $K^+ \cap W^u(x)$ (resp. $J^+ \cap W^u(x)$) is locally connected;
- (ii) for any smoothly bounded domain $\Omega \subset W^u(x)$, for every $\delta > 0$, $K^+ \cap \Omega$ (resp. $J^+ \cap \Omega$) admits at most finitely many components of diameter larger than δ .

As before this follows from [7] when f is unstably connected (see Theorems 3.5 and 5.6 there), so we focus on the unstably disconnected case. In this case it is known that $K^+ \cap W^u(x)$ has uncountably many point components (see [7, Thm 3.1]). Using (ii) we can be more precise:

Corollary 3.7. *Let f be hyperbolic and unstably disconnected. Then for every $x \in J$, all but at most countably many components of $K^+ \cap W^u(x)$ are points.*

Let us stress that the conclusions of the theorem follow solely from Corollary 3.3 together with some elementary topological considerations. Remark also that the assumption that Ω has smooth boundary in (ii) is necessary: indeed otherwise it could cut a component of K^+ in infinitely many parts of large diameter (think e.g. of the closed unit square cut out by some comb-like domain).

Part or all of Theorem 3.6 is presumably known to specialists, however for completeness we provide some details. Let us first define a notion of “fast escaping from a compact set”.

Definition 3.8. Let Ω be a smoothly bounded domain in \mathbb{C} and K be a closed subset in $\Omega \subset \mathbb{C}$. We say that K satisfies the *fast escaping property* in Ω if there exists an increasing continuous function ℓ with $\ell(0) = 0$ such that for any sufficiently small $\eta > 0$ and any $x \notin K$, there exists a path $\gamma : [0, 1] \rightarrow \Omega \setminus K$ of length at most $\ell(\eta)$ such that $\gamma(0) = x$ and $d(\gamma(1), K) \geq \eta$.

Corollary 3.3 asserts that if f is hyperbolic, then for every $x \in J$, and any leafwise bounded domain $\Omega \subset W^u(x)$, $K^+ \cap W^u(x)$ satisfies the fast escaping property in Ω with $\ell(\eta) = c\eta^\alpha$. Note that both properties (i) and (ii) in Theorem 3.6 are local in $W^u(x)$ so the choice of ambient or leafwise topology or metric is harmless.

The following lemma takes care of item (ii) of the theorem.

Lemma 3.9. *Let K be a closed subset of a smoothly bounded domain $\Omega \subset \mathbb{C}$, satisfying the fast escaping property. Then for every $\delta > 0$, there are at most finitely many components of K (resp. of $\text{Int}(K)$, of ∂K) of diameter greater than δ .*

Proof. We first prove the result for K and $\text{Int}(K)$ and then explain how to modify the proof to deal with ∂K . Let us first assume that Ω is the unit square Q , and denote by π_1 and π_2 the coordinate projections of Q . Assume by contradiction that there are infinitely many components $(C_i)_{i \geq 0}$ of K with diameter $\geq \delta$. Then there exists $\pi \in \{\pi_1, \pi_2\}$ such that infinitely many C_i satisfy $\text{Diam}(\pi(C_i)) \geq \delta/2$. Therefore there is an interval I of length $\delta/4$ such that for infinitely many i , C_i disconnects the strip $\pi^{-1}(I)$, and we conclude that $\pi^{-1}(I) \setminus \bigcup C_i$ has infinitely many connected components U_j going all the way across the strip. (Notice that the U_j may contain other points of K .) Let c be the center point of I . Since the C_i are distinct components of K , for each j there exists a point x_j in $U_j \cap \pi^{-1}(c)$ which does not belong to K . If η is chosen such that $\ell(\eta) = \delta/20$ we infer from the fast escaping property that for every j , U_j contains a disk of radius η , which is the desired contradiction.

For $\text{Int}(K)$ the argument is identical except that instead of c we take a small open interval I' about c and argue that if the C_i are distinct components of $\text{Int}(K)$, there exists $x_j \in U_j \cap \pi^{-1}(I')$ which does not belong to K .

In the general case, take a square Q such that $\Omega \Subset Q$ and replace K by $K' = \overline{K} \cap \overline{\Omega}$. Let us check that K' satisfies the fast escaping property in Q . Indeed, if $x \in Q \setminus K'$ we have either $x \in \Omega$, $x \in \partial\Omega$ or $x \in Q \setminus \overline{\Omega}$. In the first case we take the path γ given by the fast escaping property of K in Ω . In the second case, any small ball B about x intersects $\Omega \setminus K$, and we simply take a path starting from some $x' \in B \cap (\Omega \setminus K)$. Finally in the last case we use the fact that $\overline{\Omega}$ has the fast escaping property in Q .

By the first part of the proof we conclude that K' has finitely many components of diameter $\geq \delta$. Since any component of K (resp. $\text{Int}(K)$) is contained in a component of K' (resp. $\text{Int}(K')$), we are done.

The proof that ∂K admits only finitely many components of diameter greater than δ goes exactly along the same lines. We assume that there are infinitely many components C_i of ∂K disconnecting the strip $\pi^{-1}(I)$, so that $\pi^{-1}(I) \setminus \bigcup C_i$ also has infinitely many components U_j . The difference with the previous case is that some of these components may be completely included in K . We modify the argument as follows. Denote by U'_j the components completely included in K and by U''_j the remaining ones. We claim

that there are infinitely many U_j'' 's. Indeed since the C_i are components of ∂K , two components of the form U_j' must be separated by a component of the form U_j'' . So there are infinitely many such components. Then we take a small open interval $I' \subset I$ containing c and we repeat this argument, to obtain that there are infinitely many j 's such that $U_j'' \cap \pi^{-1}(I')$ contains a point x_j that does not belong to K . Then we proceed with the proof as in the previous case, by constructing infinitely many disjoint disks of radius η in Q to get a contradiction. \square

Proof of (i) in Theorem 3.6. Since $J^+ \cap W^u(x) = \partial_i(K^+ \cap W^u(x))$, general topology implies that local connectivity of $J^+ \cap W^u(x)$ implies that of $K^+ \cap W^u(x)$ (see [33, §49.III]) so it is enough to focus on J^+ . For convenience we plug in some dynamical information. Since f is unstably disconnected, it admits an unstable transversal Δ^u , that is a horizontal disk of finite degree in \mathbb{B} contained in some unstable manifold (of a periodic saddle point, say). For every $x \in J$, $W^s(x)$ intersects Δ^u : this easily follows from the density of $W^s(x)$ in J^+ and the local product structure. Fix $y \in W^s(x) \cap \Delta^u$. By using the local holonomy along the stable lamination $W_{\text{loc}}^u(x) \rightarrow W_{\text{loc}}^u(y)$, we see that $J^+ \cap W^u(x)$ is locally connected at x if and only if $J^+ \cap W^u(y)$ is locally connected at y . Therefore it is enough to show that $J^+ \cap \Delta^u$ is locally connected. Since $K^+ \cap \Delta^u$ is polynomially convex and compactly contained in Δ^u it follows that $\Omega := \Delta^u \setminus K^+$ is connected and $J^+ \cap \Delta^u = \partial\Omega$. Likewise every component of $\partial\Omega$ is of the form ∂A , where A is a component of $\Delta^u \cap K^+$. For such a component, by Carathéodory's Theorem local connectivity of ∂A is equivalent to that of A , which is of course equivalent to local connectivity of A at every point of its boundary. Let us fix $x_0 \in \partial A$: to complete the proof we have to show that A is locally connected at x_0 .

Assume by contradiction that A is not locally connected at x_0 . Then for small $\varepsilon > 0$ such that if C denotes the component of $A \cap \overline{B}(x_0, \varepsilon)$, then $x_0 = \lim x_n$, where x_n belongs to $A \setminus C$. Without loss of generality we can assume that $x_n \in B(x_0, \varepsilon/2)$. Let $C_n = \text{Comp}_{A \cap \overline{B}(x_0, \varepsilon)}(x_n)$, which by definition is disjoint from C . Passing to a subsequence if necessary, we may assume that the C_n are disjoint (the construction here is similar to that of convergence continua in [33, §49.VI]). Since C and the C_n intersect $\partial B(x_0, \varepsilon)$, their diameter is bounded from below by some $\delta > 0$. From this point the proof is similar to that of Lemma 3.9: we can find an orthogonal projection π such that C and the C_n cross the strip $\pi^{-1}(I)$ horizontally and $\pi^{-1}(I) \setminus (C \cup \bigcup C_n)$ admits infinitely many connected components U_j going all the way across the strip. If $\pi^{-1}(c)$ denotes the center line of the strip, for every j , $\pi^{-1}(c) \cap U_j$ has non-trivial intersection with Ω , and the fast escaping property of Ω gives a contradiction as before. \square

3.4. Complement: John-Hölder property in basins. We illustrate the comments from § 3.1 on the versatility of the John-Hölder property by sketching a proof of the following result.

Theorem 3.10. *Let f be a hyperbolic polynomial automorphism of \mathbb{C}^2 , and \mathcal{B} be an attracting basin. Then the John-Hölder property holds in \mathcal{B} , i.e. for any component Ω of $\mathcal{B} \cap W^u(x)$ there exists η_0 depending only on Ω such that for any $y \in \Omega$ sufficiently close to J , there exists a path in of length $O(\eta^\alpha)$ in $W^u(x)$ joining y to a point η -far from J .*

Remark 3.11. A difference between this result and Corollary 3.3 is that in Corollary 3.3 the constant η_0 is independent of the component of $W^u(x) \setminus K^+$, because G^+ reaches arbitrary large values in each component. Here the situation is different because $\mathcal{B} \cap W^u(x)$ typically has (infinitely) many small components, so how far we can get from the boundary really depends on the component.

Proof. For convenience we present a proof which is purposely close to that of Proposition 3.1 and Corollary 3.3. Replace f some iterate so that \mathcal{B} is the basin of attraction of a fixed point a with multipliers λ_1, λ_2 , with $|\lambda_2| \leq |\lambda_1|$. There exists a biholomorphism $\phi : \mathcal{B} \rightarrow \mathbb{C}^2$, which conjugates the dynamics to that of the triangular map $(z_1, z_2) \mapsto (\lambda_1 z_1 + r(z_2), \lambda_2 z_2)$, where r is a polynomial which is non-zero only when there is a resonance $\lambda_2 \neq \lambda_1^j$ between the eigenvalues (see [43]). Introduce the function

$$\tilde{H}(z_1, z_2) = |z_1 - r(z_2/\lambda_2)|^2 + |z_2|^{2\alpha}, \text{ where } \alpha = \frac{\log \lambda_1}{\log \lambda_2} \geq 1$$

and put $H = \tilde{H} \circ \phi$. This is a smooth strictly psh function on \mathcal{B} which satisfies $H \circ f = |\lambda_1|^2 H$. To get a better analogy with the previous case we may consider H^{-1} which satisfies $H^{-1} \circ f = |\lambda_1|^{-2} H^{-1}$, and tends to zero when approaching J . The restriction of this function to any local unstable disk in $\mathcal{B} \setminus \{a\}$ is non-constant and one easily checks that its set of critical points is discrete.

Arguing in Proposition 3.1, we define a family of rays in \mathcal{B} by considering gradient lines of H (or equivalently H^{-1}) along \mathcal{W}^u , first in the fundamental domain $\{|\lambda_1|^2 \leq H^{-1} \leq 1\}$ and then in $\{0 < H^{-1} \leq 1\}$ by pulling back. It follows that for every component Ω of $\mathcal{B} \cap W^u(x)$, for every $0 < r_1 < r_2 < \max_{\Omega} |H^{-1}|$, and any $y \in \Omega$ such that $H^{-1}(y) = r_1$, there exists a ray of length $\ell(r_1, r_2) = O(r_2^\alpha)$ joining y to a point of $\{H^{-1} = r_2\}$.

To conclude the argument we need to adapt the proof of Corollary 3.3, which relies on the Hölder continuity of the Green function. Instead we use an argument based on uniform hyperbolicity. Indeed, let $x \in J$ and $y \in W_{\text{loc}}^u(x)$ be such that $d^u(x, y) = \varepsilon$. We want to show that $H^{-1}(y) \lesssim \varepsilon^\alpha$ for some α . By the expansion along unstable manifolds and the local uniform geometry it takes at most $N \leq C |\log \varepsilon|$ iterates to map y into a given compact subset of \mathcal{B} . Hence

$$H^{-1}(y) = |\lambda_1|^{2N} H^{-1}(f^N(y)) \leq C |\lambda_1|^{2N} \leq C |\lambda_1|^{2C |\log \varepsilon|} = C \varepsilon^{-2C \log |\lambda_1|}$$

and we are done. \square

4. STABLE TOTAL DISCONNECTEDNESS

We say that f (or J) is *stably totally disconnected* if for every $x \in J$, $W^s(x) \cap J^-$ is totally disconnected. Note that since J has local product structure with respect to the stable and unstable laminations, $W^s(x) \cap J = W^s(x) \cap J^-$.

Proposition 4.1. *Let f be a hyperbolic Hénon map. The following assertions are equivalent.*

- (i) *Every leaf of the stable lamination in \mathbb{B} is a vertical submanifold of finite degree.*

- (ii) *The leaves of the stable lamination in \mathbb{B} are vertical submanifolds of uniformly bounded degree.*
 (iii) *For every x in J , $J^s(x) = K^s(x) = \{x\}$, that is, f is stably totally disconnected.*

Note that dissipativity is not required here, so this result holds in the unstable direction as well.

Proof. The implication (ii) \Rightarrow (i) is obvious and its converse (i) \Rightarrow (ii) follows from the semi-continuity properties of the degree and is identical to [35, Lemma 5.1]. To prove that (iii) \Rightarrow (i) we use Lemma 2.5 for the stable lamination: indeed if $J^s(x)$ is a point for every x , then all 4 conditions of Lemma 2.5 are equivalent, and the equivalence of properties (ii) and (iii) there yield the result. Finally, (ii) \Rightarrow (iii) does not require hyperbolicity and was established in [15, Prop. 2.14]. For convenience, let us recall the argument: for every vertical disk D of degree $\leq k$, and every component D' of $D \cap f(\mathbb{B})$, the modulus of the annulus $D \setminus D'$ is bounded below by $m = m(k) > 0$, and for every $x \in J$ there is an infinite nest of such annuli surrounding the component of x in $W^s(x) \cap J$. So $W^s(x) \cap J$ is totally disconnected and we are done. \square

A way to ensure the boundedness of the degrees of semi-local stable manifolds originates in [17] and relies on Wiman's theorem for entire functions. The following result is contained in [35].

Proposition 4.2. *Let f be a hyperbolic Hénon map such that $|\text{Jac } f| \leq d^{-2}$. Then f is stably totally disconnected.*

Proof (sketch). Fix $x \in J$ and $v \in E^s(x)$. Uniform hyperbolicity together with the assumption on the Jacobian imply that $\|df_x^n(v)\| \leq Cs^n$, where $s < d^{-2}$. Denote as before ψ_x^s the normalized stable parameterization. It follows that $f^n \circ \psi_x^s(\cdot) = \psi_{f^n(x)}^s(\lambda_n \cdot)$, where $|\lambda_n| \leq Cs^n$. Then from the relation

$$G^- \circ \psi_x^s(\lambda_n^{-1}\zeta) = d^n G^- \circ \psi_{f^n(x)}^s(\zeta)$$

we deduce that $G^- \circ \psi_x^s$ is a subharmonic function of order smaller than $1/2$ and Wiman's theorem implies that $\text{Comp}_{(\psi_x^s)^{-1}(\mathbb{B})}(x)$ is a bounded domain in \mathbb{C} , thus $W_{\mathbb{B}}^s(x)$ has bounded vertical degree and we are done. \square

Another idea, which was communicated to us by Pierre Berger, is to use a Hausdorff dimension argument to prove directly that stable slices of J are totally disconnected. Indeed the Hausdorff dimension of stable slices of J^- can be estimated using thermodynamic formalism for hyperbolic maps. This turns out to give a better bound on the Jacobian.

Proposition 4.3. *Let f be a hyperbolic Hénon map such that $|\text{Jac } f| < d^{-1}$. Then f is stably totally disconnected.*

Proof. Since J is a locally maximal hyperbolic set and the dynamics along stable manifolds is conformal, there is an exact formula for the Hausdorff dimension of $J \cap W_{\text{loc}(x)}^s$

for any $x \in J^-$, given by:

$$(2) \quad \delta^s := \dim_H (J \cap W_{\text{loc}}^s(x)) = \frac{h_{\kappa^s}(f)}{-\int \log |df|_{E^s(x)}|d\kappa^s(x)}$$

(see Pesin's book [39, Thm 22.1]; this goes back to the work of Manning and McCluskey [36]), where κ^s is a certain invariant measure (the unique equilibrium state associated to $\delta^s \log |df|_{E^s}$) and $h_{\kappa^s}(f)$ is its measure theoretic entropy. By the variational principle we have that $h_{\kappa^s}(f) \leq \log d$. On the other hand the Lyapunov exponent in the denominator in the right hand side of (2) is bounded below by $|\log |\text{Jac } f|| > \log d$. Therefore we conclude that $\dim_H (J \cap W_{\text{loc}}^s(x)) < 1$ from which it follows that $J \cap W_{\text{loc}}^s(x)$ is totally disconnected. \square

Question 4.4. *Is a dissipative hyperbolic Hénon map always stably totally disconnected?*

5. CLASSIFICATION OF SEMI-LOCAL COMPONENTS OF K^+ AND J^+

Throughout this section, f is a dissipative and hyperbolic complex Hénon map of degree d with a disconnected Julia set (or equivalently, f is unstably disconnected). We assume moreover that f is stably totally disconnected. The results of §4 imply that this holds whenever $|\text{Jac } f| < 1/d$. We fix a large bidisk \mathbb{B} as before, and our purpose is to classify the connected components of $J^+ \cap \mathbb{B}$ and study the induced dynamics on this set of components.

5.1. Geometric preparations. We start with some general lemmas about vertical submanifolds in a bidisk. We define the angle $\angle(v, w)$ between two complex directions v and w at $x \in \mathbb{C}^2$ to be their distance in $\mathbb{P}(T_x \mathbb{C}^2) \simeq \mathbb{P}^1$ relative to the Fubini-Study metric induced by the standard Hermitian structure of $T_x \mathbb{C}^2 \simeq \mathbb{C}^2$.

Lemma 5.1. *Let M be a vertical submanifold in $\mathbb{D} \times \mathbb{D}$, and let $a \in \mathbb{D}$ and $r > 0$ such that M has no horizontal tangency in $\mathbb{D} \times D(a, 2r)$. Then there exists a universal constant C_0 such that for any $x \in \mathbb{D} \times D(a, r)$, the angle between $T_x M$ and the horizontal direction is bounded from below by $C_0 r$.*

Proof. If M has no horizontal tangency in $\mathbb{D} \times D(a, 2r)$, then $M \cap (\mathbb{D} \times D(a, 2r))$ is the union of $\deg(M)$ vertical graphs. Let Γ be one of these graphs. Then $\varphi := \pi_1 \circ (\pi_2|_{\Gamma})^{-1}$ maps $D(a, 2r)$ into $2\mathbb{D}$ and $\Gamma = \{(\varphi(w), w), w \in D(a, 2r)\}$. By the Cauchy estimate, we get that $|\varphi'| \leq 2/r$ on $D(a, r)$ and the result follows. \square

A typical use of this result is by taking the contrapositive: if a vertical submanifold M in $\mathbb{D} \times \mathbb{D}$ has a near horizontal tangency in $\mathbb{D} \times D(a, r)$, then it has an actual horizontal tangency in $\mathbb{D} \times D(a, 2r)$. Let us denote by $[e_1] \in \mathbb{P}(T\mathbb{C}^2)$ the horizontal direction.

Corollary 5.2. *Let M be a vertical submanifold in $\mathbb{D} \times \mathbb{D}$ which extends as a vertical submanifold to $\mathbb{D} \times (3/2)\mathbb{D}$. There exists a universal constant C_1 such that if for some $a \in \mathbb{D}$, there exists $x \in M \cap (\mathbb{D} \times \{a\})$ such that $\angle(T_x M, [e_1]) < \theta$, then there exists $a' \in (3/2)\mathbb{D}$ such that $|a - a'| < C_1 \theta$ and M is tangent to $\mathbb{D} \times \{a'\}$.*

For the sake of completeness let us also state a slightly stronger result:

Corollary 5.3. *Let M be a vertical submanifold in $\mathbb{D} \times \mathbb{D}$ of degree at most k which extends as a vertical submanifold to $\mathbb{D} \times r_0 \mathbb{D}$ for some $r_0 > 1$ (say $r_0 = 3/2$). There exists a function $h = h_k$ such that $h(\theta) \rightarrow 0$ as $\theta \rightarrow 0$ with the following property: if $x \in M$ is such that the angle between $T_x M$ and the horizontal direction is bounded by $\theta \ll 1$ then there exists $x' \in M$ with $d(x, x') \leq h(\theta)$ such that M has a horizontal tangency at x' .*

Proof. Indeed, letting $a = \pi_2(x)$, and applying Corollary 5.2 we see that the connected component of M containing x in $D(a, C_1 \theta) \times \mathbb{D}$ cannot be a vertical graph, so it admits a horizontal tangency. Furthermore, an easy compactness argument shows that the diameter of a connected component of $M \cap D(a, r) \times \mathbb{D}$ is bounded by $h_k(r)$ with $h_k(r) \rightarrow 0$ as $r \rightarrow 0$. The result follows. \square

Remark 5.4. It is likely that $h_k(r) = O(r^{1/k})$ but the precise argument needs to be found.

The following result is a precise version of the Reeb stability theorem (see [11]) which is specialized to our setting.

Lemma 5.5. *Let $x_0 \in J$ be such that $W_{\mathbb{B}}^s(x_0)$ is transverse to $\partial \mathbb{B}$. Then there exists δ depending only on $\min_{y \in W_{\mathbb{B}}^s(x_0) \cap \partial \mathbb{B}} \angle(T_y W_{\mathbb{B}}^s(x_0), [e_1])$ such that if $\tau \subset J^u(x_0)$ is a connected compact set containing x_0 , of diameter less than δ , then for every $x \in \tau$, $W^s(x)$ is transverse to $\partial \mathbb{B}$, $\deg W_{\mathbb{B}}^s(x) = \deg W_{r\mathbb{B}}^s(x_0)$ and $\bigcup_{x \in \tau} W_{\mathbb{B}}^s(x)$ is homeomorphic to $\tau \times W_{\mathbb{B}}^s(x_0)$.*

Note that it is slightly abusing to say that $W_{\mathbb{B}}^s(x)$ is transverse to $\partial(\mathbb{B})$ since $W_{\mathbb{B}}^s(x)$ precisely stops at $\partial \mathbb{B}$. Of course $W_{\mathbb{B}}^s(x)$ extends to a neighborhood of \mathbb{B} and what we mean is transversality for this extension.

Remark 5.6. Later on we will use this lemma with $r\mathbb{B}$ instead of \mathbb{B} for $1 \leq r \leq 2$ (see Proposition 5.12). It will be important there that the constant δ is uniform with $r \in [1, 2]$, which easily follows from the proof.

Proof. Set $\theta = \min_{y \in W_{\mathbb{B}}^s(x_0) \cap \partial \mathbb{B}} \angle(T_y W_{\mathbb{B}}^s(x_0), [e_1])$. The stable lamination in a neighborhood of $\bar{\mathbb{B}}$ is covered by finitely many flow boxes. So there exists $r > 1$ depending only on θ such that $W_{r\mathbb{B}}^s(x_0)$ is transverse to $\partial(r\mathbb{B})$. Since the stable leaves in \mathbb{B} are simply connected, we can apply a local version of the Reeb stability theorem (see [11, Prop. 11.4.8]) which asserts that when $\tau \subset J \cap W^u(x_0)$ is sufficiently small, for $x \in \tau$, by local triviality of the stable lamination, the domain $W_{r\mathbb{B}}^s(x_0) \subset W^s(x_0)$ can be lifted to a domain $D_x \subset W^s(x)$, and the collection $\{D_x, x \in \tau\}$ is topologically a product. Since $W_{\mathbb{B}}^s(x_0)$ is transverse to $\partial \mathbb{B}$, $W_{\mathbb{B}}^s(x_0) \subset W_{r\mathbb{B}}^s(x_0)$ is a smoothly bounded domain and, reducing τ if necessary, the transversality persists, $\text{Comp}_{D_x \cap \mathbb{B}}(x)$ varies continuously and $\bigcup_{x \in \tau} W_{\mathbb{B}}^s(x)$ is a product. Finally, if we fix any horizontal line, say close to $\partial \mathbb{B}$ by transversality and continuity, its number of intersection points with $W_{\mathbb{B}}^s(x)$ is constant, hence the statement on the degree.

What remains to be seen is why the size of the allowed transversal τ depends only on the minimal angle θ . This follows from the mechanism of Reeb stability. What we need to know is how far we can push x in τ so as to keep the transversality between $W_{\mathbb{B}}^s(x)$

and $\partial\mathbb{B}$. Pick $y \in W_{\mathbb{B}}^s(x_0) \cap \partial\mathbb{B}$. Understanding how a neighborhood of y in $W^s(x_0)$ evolves when the base point $x \in \tau$ changes depends on the choice of a path γ joining x_0 to y in $W^s(x_0)$ and of a covering of γ by a chain of overlapping plaques. (Recall that by definition a *plaque* is the intersection between a leaf and a flow box.) Notice first that there is a uniform control of the length of a such a path γ : for instance we can take an external ray and apply Proposition 3.1 (see Remark 3.2). So the length of a minimal chain of plaques joining x_0 to y is uniformly bounded, and there exists $\delta = \delta(\theta)$ such that if $\text{Diam}_{x_0}(\tau) < \delta$, then the continuation of the plaque containing y remains transverse to $\partial\mathbb{B}$. Finally, the number of plaques required to cover $\partial W_{\mathbb{B}}^s(x_0)$ depends basically on the volume of $W_{r\mathbb{B}}^s(x_0)$ for some $r > 1$, which in turn depends only on the degree of $W_{r'\mathbb{B}}^s(x_0)$ for some $r' > r$. By Proposition 4.1 this degree is uniformly bounded. So the number of plaques is uniformly bounded and we are done. \square

We will also need the following extension lemma.

Lemma 5.7 ([35, Prop. 5.8]). *There exists a neighborhood \mathcal{N} of $J^+ \cap \mathbb{B}$ such that the stable lamination \mathcal{W}^s extends to a C^1 foliation of \mathcal{N} .*

Observe that in [35] it is assumed that $|\text{Jac } f| < d^{-2}$ but what is really needed for extending the stable lamination is the boundedness of the vertical degree which holds in our setting (cf. Proposition 4.1). The C^1 regularity of the holonomy will not be used in the paper.

Using this extension lemma, we can extend Lemma 5.5 to a statement about an open neighborhood of $W_{\mathbb{B}}^s(x_0)$ with exactly the same proof.

Lemma 5.8. *Let $x_0 \in J$ be such that $W_{\mathbb{B}}^s(x_0)$ is transverse to $\partial\mathbb{B}$. Then there exists δ depending only on $\min_{y \in W_{r\mathbb{B}}^s(x_0) \cap \partial\mathbb{B}} \angle \left(T_y W_{\mathbb{B}}^s(x_0), [e_1] \right)$ such that for every $x \in D^u(x_0, \delta)$, $W^s(x)$ is transverse to $\partial\mathbb{B}$, $\deg \mathcal{W}_{\mathbb{B}}^s(x) = \deg W_{\mathbb{B}}^s(x_0)$ and $\bigcup_{x \in D_{x_0}^u(x_0, \delta)} W_{\mathbb{B}}^s(x)$ is homeomorphic to $D_{x_0}^u(x_0, \delta) \times W_{\mathbb{B}}^s(x_0)$.*

5.2. Thin and thick components. In this section we study the geometry of the components of $J^+ \cap \mathbb{B}$. The arguments rely mostly on the geometry of the stable lamination, not on the dynamics of f . One main result is that thin components of $K^+ \cap \mathbb{B}$ have a simple leaf structure (Proposition 5.12). It follows that for a given component of $J^+ \cap \mathbb{B}$, either all its unstable slices are small, or all of them are large (Proposition 5.13). Together with the results of §3.3 this leads to a description and some regularity properties of components of $J^+ \cap \mathbb{B}$ and $K^+ \cap \mathbb{B}$.

We start with a simple case.

Proposition 5.9. *If $x \in J$ is such that $K^u(x) = J^u(x) = \{x\}$ then $K_{\mathbb{B}}^+(x) = J_{\mathbb{B}}^+(x) = W_{\mathbb{B}}^s(x)$.*

Proof. As observed above the inclusion $W_{\mathbb{B}}^s(x) \subset K_{\mathbb{B}}^+(x)$ is obvious. For the converse inclusion, observe that for every $n \in \mathbb{Z}$, $K^u(f^n(x)) = \{f^n(x)\}$. For $n \geq 1$, consider a small loop $\gamma_n \subset W^u(f^n(x))$ around $f^n(x)$ that is disjoint from K^+ . By the local product structure we can extend it to a germ of 3-manifold $\tilde{\gamma}_n$ transverse to $W^u(f^n(x))$, disjoint from K^+ , and of size uniformly bounded from below in the stable direction. Since $W_{2\mathbb{B}}^s(x)$

has finite vertical degree in $2\mathbb{B}$, it admits finitely many horizontal tangencies, so we can fix $1 \leq r \leq 2$ such that $W_{r\mathbb{B}}^s$ is transverse to $\partial(r\mathbb{B})$. Then by the Inclination Lemma, for large n , $f^{-n}(\tilde{\gamma}_n)$ contains a small “tube” around $W_{r\mathbb{B}}^s(x)$ whose boundary is disjoint from K^+ . It follows that $K_{r\mathbb{B}}^+(x) = W_{r\mathbb{B}}^s(x)$, hence $K_{\mathbb{B}}^+(x) \subset W_{r\mathbb{B}}^s(x) \cap \mathbb{B}$. Finally $W_{r\mathbb{B}}^s(x) \cap \mathbb{B}$ has finitely many components, and one of them is $W_{\mathbb{B}}^s(x)$, so $K_{\mathbb{B}}^+(x) = W_{\mathbb{B}}^s(x)$. \square

Here is a first interesting consequence.

Corollary 5.10. *All but countably many components of $K^+ \cap \mathbb{B}$ are vertical submanifolds.*

Proof. Fix a global unstable transversal Δ^u in \mathbb{B} . Then every component of $K^+ \cap \mathbb{B}$ intersects Δ^u . Indeed, for any such component C , ∂C is contained in J^+ so it contains stable manifolds. Stable manifolds in \mathbb{B} are vertical and of finite degree, so they have non-trivial (transverse) intersection with Δ^u . Now if C is non-trivial, that is, not reduced to a vertical submanifold, then by Proposition 5.9, any component of $C \cap \Delta^u$ is non-trivial, and the result follows from Corollary 3.7. \square

Another case where $J_{\mathbb{B}}^+(x)$ is easily understood is when stable leaves are transverse to $\partial\mathbb{B}$.

Proposition 5.11. *Assume that $J^u(x)$ is a leafwise bounded component such that for every $y \in J^u(x)$, $W_{\mathbb{B}}^s(y)$ is transverse to $\partial\mathbb{B}$. Then*

$$(3) \quad J_{\mathbb{B}}^+(x) = \bigcup_{y \in J^u(x)} W_{\mathbb{B}}^s(y).$$

Note that this result is not true if the transversality assumption is omitted (see Figure 1 for a visual explanation).

Proof. Let C be defined by the right hand side of (3). Since the $W_{\mathbb{B}}^s(y)$, $y \in J^u(x)$, are transverse to $\partial\mathbb{B}$, they vary continuously with y . It follows that C is a closed connected set. To show that $C = J_{\mathbb{B}}^+(x)$, it is convenient to use the extension of the stable lamination to a neighborhood of $J^+ \cap \mathbb{B}$ (given in Lemma 5.7). Let (U_n) be a basis of open neighborhoods of $J^u(x)$ in $W^u(x)$ such that for every n , $\partial U_n \cap J = \emptyset$. For every $\delta > 0$, U_n is contained in the δ -neighborhood of $J^u(x)$ for large n . Thus, by Lemma 5.8 the leaves issued from U_n are transverse to $\partial\mathbb{B}$ and stay close to C . Let \tilde{U}_n be the saturation of U_n in the extended foliation. Then (\tilde{U}_n) is a basis of neighborhoods of C in \mathbb{B} such that $\partial\tilde{U}_n$ is disjoint from J^+ . We conclude that $C = J_{\mathbb{B}}^+(x)$. \square

The structure of $J_{\mathbb{B}}^+(x)$ is not so easy to describe without this transversality assumption. Still, the argument can (almost) be salvaged if $J^u(x)$ is small enough. This will be a key property in the following.

Proposition 5.12. *There exists $\delta_1 > 0$ such that if $x \in J$ is such that $\text{Diam}_x(J^u(x)) \leq \delta_1$, then there exists $1 \leq r \leq 2$ such that for every $y \in J^u(x)$, $W_{r\mathbb{B}}^s(y)$ is transverse to $\partial(r\mathbb{B})$ and $J^u(x)$ can be followed under holonomy along $W_{r\mathbb{B}}^s(x)$. In particular $J_{r\mathbb{B}}^+(x)$ is*

homeomorphic to $J^u(x) \times W_{r\mathbb{B}}^s(x)$ and

$$(4) \quad J_{\mathbb{B}}^+(x) \subset J_{r\mathbb{B}}^+(x) = W_{r\mathbb{B}}^s(J^u(x)) \subset W_{2\mathbb{B}}^s(J^u(x)) = \bigcup_{y \in J^u(x)} W_{2\mathbb{B}}^s(y).$$

Recall that Diam_x denotes the diameter relative to the normalized leafwise metric d_x^u induced by the affine structure. By polynomial convexity, if $K^u(x)$ is leafwise bounded, then $J^u(x) = \partial_i K^u(x)$ so $\text{Diam}_x(K^u(x)) = \text{Diam}_x(J^u(x))$. Recall from §2.3 that by the Koebe Distortion Theorem, the ambient distance d and the leafwise Euclidean distance d_x^u are equivalent in a small neighborhood of x , with universal bounds, i.e. in some neighborhood of x in $W^u(x)$ we have $d/2 \leq d_x^u \leq 2d$. In particular if $\text{Diam}_x(J^u(x))$ is small enough then $\text{Diam}(J^u(x))$ and $\text{Diam}(K^u(x))$ are comparable to $\text{Diam}_x(J^u(x))$ (where Diam denotes the ambient diameter).

Proof of Proposition 5.12. Recall that every leaf of the stable lamination in $3\mathbb{B}$ is a vertical disk of degree bounded by D , so by the Riemann-Hürwitz formula it admits at most $D - 1$ horizontal tangencies. For $k = 0, \dots, D$, let $r_k = 1 + \frac{k}{D}$, and fix $\theta < \frac{C_0}{8D}$, where C_0 is as in Lemma 5.1. Let $x \in J$ be arbitrary. By the pigeonhole principle, there exists $k \in \{0, \dots, D - 1\}$ such that $W_{2\mathbb{B}}^s(x)$ has no horizontal tangency in $r_{k+1}\mathbb{B} \setminus r_k\mathbb{B}$. So by Lemma 5.1 (scaled to $2\mathbb{B}$ and applied to any a such that $|a| = R(r_k + r_{k+1})/2$, where R is the radius of \mathbb{B}) we infer that

$$\min_{y \in \partial(r'_k\mathbb{B})} \angle(T_y W_{\mathbb{B}}^s(x_0), [e_1]) \geq \theta, \text{ where } r'_k = \frac{r_k + r_{k+1}}{2}.$$

Therefore, by Lemma 5.5 and Remark 5.6 there exists δ_1 depending only on θ , hence ultimately only on D , hence on f , such that if $\text{Diam}_x(J^u(x)) \leq \delta_1$, then for every $y \in J^u(x)$, $W_{r'_k\mathbb{B}}^s$ is transverse to $\partial(r'_k\mathbb{B})$ and $W_{r'_k\mathbb{B}}^s(J^u(x))$ is topologically a product. This completes the proof of the first part of the proposition. From this point, the description of $J_{2\mathbb{B}}^+(x)$ in (4) directly follows from Proposition 5.11. \square

It follows from this analysis that if C is a semi-local component of J^+ , then either all its unstable slices are large or all of them are small.

Proposition 5.13. *There exists $0 < \delta_1 \leq \delta_2$ such that for every component C of $J^+ \cap \mathbb{B}$ the following alternative holds:*

- (i) either for every $x \in C \cap J$, $\text{Diam}_x J^u(x) \leq \delta_2$;
- (ii) or for every $x \in C \cap J$, $\text{Diam}_x J^u(x) > \delta_1$.

In addition if (i) holds then C satisfies the conclusions of Proposition 5.12.

Referring to this dichotomy in the following, we will say that a component is *thin* (resp. *thick*) if it satisfies (i) (resp. (ii)). We stress that the Proposition asserts that a component is thick as soon as *one* of its unstable slices has intrinsic diameter larger than δ_2 . As seen before (see e.g. Corollary 5.10), if Δ^u is an unstable transversal, every semi-local component of J^+ intersects Δ^u , so from Theorem 3.6 we immediately deduce:

Corollary 5.14. *There are only finitely many thick components of $J^+ \cap \mathbb{B}$.*

Proposition 5.13 is a direct consequence of the following lemma.

Lemma 5.15. *Let δ_1 be as in Proposition 5.12. There exists $\delta_2 \geq \delta_1$ such that if x is such that $\text{Diam}_x(J^u(x)) \leq \delta_1$, then for every $y \in J_{\mathbb{B}}^+(x) \cap J$, $\text{Diam}_y(J^u(y)) \leq \delta_2$.*

Proof. Indeed by Proposition 5.12, if $\text{Diam}_x(J^u(x)) \leq \delta_1$, then any point in $J_{\mathbb{B}}^+(x)$ can be joined to $y \in J^u(x)$ by a path contained in $W_{2\mathbb{B}}^s(y)$. Furthermore, as explained in the proof of Lemma 5.5, the plaque-length of such a γ is uniformly bounded. The bound on $\text{Diam}_y(J^u(y))$ then follows from the uniform continuity of holonomy along bounded paths in the stable lamination. \square

Remark 5.16. The argument of Propositions 5.12 and 5.13 makes no use of the fact that $J^u(x)$ is a component of $J \cap W^u(x)$. Thus the same statements hold for the saturation by semi-local stable leaves of any (say closed) subset X of an unstable manifold: if its diameter of X is small enough then, changing the bidisk \mathbb{B} if necessary, the saturation \hat{X} of X by semi-local stable manifolds is a product and all the stable slices of \hat{X} have a small diameter.

Proposition 5.17. *Let Δ^u be an unstable transversal in \mathbb{B} . For every connected component C of $J^+ \cap \mathbb{B}$ (resp. $K^+ \cap \mathbb{B}$), $C \cap \Delta^u$ admits finitely many connected components.*

Proof. Let us first discuss the case of components of $J^+ \cap \mathbb{B}$. For thick components, the result follows immediately from Corollary 5.14, so we may assume that C is thin. As already seen, C intersects Δ^u . Pick $x \in C \cap \Delta^u$, in particular $x \in J$. Since C is thin, for some $1 \leq r \leq 2$, $W_{r\mathbb{B}}^s(x)$ is transverse to $\partial(r\mathbb{B})$ and by Proposition 5.12, $J^u(x)$ can be followed under holonomy along $W_{r\mathbb{B}}^s(x)$. Since $W_{r\mathbb{B}}^s(x)$ and Δ^u have finitely many intersection points, we infer that $J_{r\mathbb{B}}^+(x) \cap \Delta^u$ has finitely many connected components. Finally, $J_{\mathbb{B}}^+(x) = C$ coincides with the component of $J_{r\mathbb{B}}^+(x) \cap \mathbb{B}$ containing x , so $C \cap \Delta^u$ is a union of connected components of $J_{r\mathbb{B}}^+(x) \cap \Delta^u$ and we conclude that there are finitely many of them.

We now discuss components of $K^+ \cap \mathbb{B}$. Recall from Lemma 2.3 that for such a component C , ∂C is a component of $J^+ \cap \mathbb{B}$. Assume first that such a component A is thin. Given $x \in A \cap \Delta^u$, $J^u(x)$ can be followed under holonomy along $W_{r\mathbb{B}}^s(x)$ for some $1 \leq r \leq 2$. If the polynomial hull of $J^u(x)$ is non-empty, then it has a small diameter and it can be followed by holonomy in $r\mathbb{B}$ along the extended foliation just as in Proposition 5.12 and it is topologically a product. It follows that $C \cap \Delta^u$ is the polynomial hull of $J_{\mathbb{B}}^+(x) \cap \Delta^u$ and it has finitely many components. On the other hand, if every component of ∂C is thick, then $\partial C \cap \Delta^u$ is contained in the finitely many components of $K^+ \cap \Delta^u$ of diameter greater than some δ , and so is $C \cap \Delta^u$. This concludes the proof. \square

We conclude this subsection by giving a general description of components of $J^+ \cap \mathbb{B}$. Fix an unstable transversal Δ^u . Let $x \in J \cap \Delta^u$ and consider $W_{\mathbb{B}}^s(J^u(x)) = \bigcup_{y \in J^u(x)} W_{\mathbb{B}}^s(y)$. If every $W_{\mathbb{B}}^s(y)$ is transverse to $\partial\mathbb{B}$ then by Proposition 5.11, $W_{\mathbb{B}}^s(J^u(x)) = J_{\mathbb{B}}^+(x)$. In the general case we define a relation between components of $J^+ \cap \Delta^u$ by declaring that $C_1 \leftrightarrow C_2$ if and only if there exists $x \in C_1$ such that $W_{\mathbb{B}}^s(x) \cap C_2 \neq \emptyset$ (or equivalently there exists $(x_1, x_2) \in C_1 \times C_2$ such that $W_{\mathbb{B}}^s(x_1) = W_{\mathbb{B}}^s(x_2)$). Then extend this relation to an equivalence relation (still denoted by \leftrightarrow) by allowing finite chains

C_1, \dots, C_n . Finally we define

$$\widehat{W}_{\mathbb{B}}^s(J^u(x)) := \bigcup_{C \leftrightarrow J^u(x)} \bigcup_{y \in C} W_{\mathbb{B}}^s(y).$$

Proposition 5.18. *For any $x \in J$, $J_{\mathbb{B}}^+(x)$ coincides with $\widehat{W}_{\mathbb{B}}^s(J^u(x))$.*

Proof. By Proposition 5.17, $J_{\mathbb{B}}^+(x) \cap \Delta^u$ admits finitely many connected components $(C_i)_{i \in I}$. Every point $z \in J_{\mathbb{B}}^+(x)$ belongs to some $W_{\mathbb{B}}^s(y)$, $y \in \Delta^u$, and necessarily y belongs to some C_i , say C_{i_0} . Furthermore, if $z' \in J_{\mathbb{B}}^+(x)$ is close to z , by the continuity of stable manifolds, there exists $y' \in \Delta^u$ close to y such that $z' \in W_{\mathbb{B}}^s(y')$. Since the C_i are at positive distance from each other it follows that y' belongs to C_{i_0} . In other words, $W_{\mathbb{B}}^s(C_{i_0})$ is relatively open in $J_{\mathbb{B}}^+(x)$. Clearly $W_{\mathbb{B}}^s(C_{i_0})$ is connected, and even arcwise connected since by Theorem 3.6 C_{i_0} is locally connected. Thus the $W_{\mathbb{B}}^s(C_i)$ realize a finite cover of $J_{\mathbb{B}}^+(x)$ by connected open sets, which are contained in or disjoint from $J_{\mathbb{B}}^+(x)$. Define a non-oriented graph on I by joining i and j whenever $W_{\mathbb{B}}^s(C_i) \cap W_{\mathbb{B}}^s(C_j) \neq \emptyset$. If we fix i_0 such that $W_{\mathbb{B}}^s(C_{i_0}) \subset J_{\mathbb{B}}^+(x)$, it follows that $J_{\mathbb{B}}^+(x) = \bigcup_{i \in I_0} W_{\mathbb{B}}^s(C_i)$ where I_0 is the component of i_0 in the graph. This is exactly the announced description. \square

Let us point out the following interesting consequence of the proof:

Corollary 5.19. *Every connected component of $J^+ \cap \mathbb{B}$ (resp. $K^+ \cap \mathbb{B}$) is locally connected.*

Proof. Given a component $J_{\mathbb{B}}^+(x)$ of $J^+ \cap \mathbb{B}$, with notation as in the previous proof, $(W_{\mathbb{B}}^s(C_i))_{i \in I}$ is a finite cover of $J_{\mathbb{B}}^+(x)$ by locally connected and relatively open sets: local connectedness follows. If now C is a component of $K^+ \cap \mathbb{B}$, we saw in the proof of Proposition 5.17 that ∂C is a finite union of components of $J^+ \cap \mathbb{B}$, therefore ∂C is locally connected. General topology then implies that C is locally connected and we are done. \square

5.3. Induced dynamics on the set of components of J^+ . We still consider a uniformly hyperbolic dissipative Hénon map, with a disconnected and stably totally disconnected Julia set, and fix a large bidisk \mathbb{B} as before. Since f maps $K^+ \cap \mathbb{B}$ (resp. $J^+ \cap \mathbb{B}$) into itself, it induces a dynamical system on the set of its connected components. Recall that a component is said *non-trivial* if it is not reduced to a vertical submanifold.

Theorem 5.20. *Let f be dissipative and hyperbolic with a disconnected and stably totally disconnected Julia set and $\mathbb{B} \subset \mathbb{C}^2$ be a large bidisk. Then $K^+ \cap \mathbb{B}$ (resp. $J^+ \cap \mathbb{B}$) admits uncountably many components, at most countably many of which being non-trivial. Any non-trivial connected component of $K^+ \cap \mathbb{B}$ (resp. $J^+ \cap \mathbb{B}$) is preperiodic, and there are finitely many non-trivial periodic components.*

Remark 5.21. Notice a periodic component of $K^+ \cap \mathbb{B}$ can be trivial, that is, a vertical submanifold. Since it is mapped into itself by some f^N in this case we conclude that it is of the form $W_{\mathbb{B}}^s(x)$ for some saddle periodic point x .

Lemma 5.22. *The function $y \mapsto \text{Diam}_y(J^u(y))$ (resp. $y \mapsto \text{Diam}_y(K^u(y))$) is upper semi-continuous on J . In particular if $y_n \rightarrow y_\infty$ and $(\text{Diam}_{y_n}(K^u(y_n)))$ is unbounded, then $K^u(y_\infty)$ is leafwise unbounded, and likewise for J^u .*

Proof. Recall that $\text{Diam}_y(J^u(y)) = \text{Diam}_y(K^u(y))$ for every $y \in J$ (including the case where it is infinite) so it is enough to deal with $K^u(y)$. Assume first that the y_n belong to the same local leaf and $y_n \rightarrow y_\infty$. If $K^u(y_\infty)$ is leafwise bounded, we can consider a closed loop γ enclosing it and disjoint from K^+ . Then for large enough n , γ also encloses $K^u(y_n)$, and any cluster value of this sequence for the Hausdorff topology is a continuum contained in K^+ and containing y_∞ . It follows that

$$\limsup_{n \rightarrow \infty} \text{Diam}_{y_\infty}(K^u(y_n)) \leq \text{Diam}_{y_\infty}(K^u(y_\infty))$$

hence

$$\limsup_{n \rightarrow \infty} \text{Diam}_{y_n}(K^u(y_n)) \leq \text{Diam}_{y_\infty}(K^u(y_\infty)),$$

as desired. Of course if $K^u(y_\infty)$ is leafwise unbounded, the inequality is obvious.

Assume now that the y_n belong to different local leaves. As before, the case where $K^u(y_\infty)$ is leafwise unbounded is obvious. If $K^u(y_\infty)$ is leafwise bounded, again we consider a closed loop γ enclosing it and disjoint from K^+ . In addition we can assume that $\text{Diam}_{y_\infty}(\gamma)$ is arbitrary close to $\text{Diam}_{y_\infty}(K^u(y_\infty))$. When $y_n \rightarrow y_\infty$, γ can be lifted to a loop $\tilde{\gamma}_n$ in $W^u(y_n)$, with roughly the same diameter (here we use the continuity of the leafwise distance d_y^u), and $K^u(y_n)$ is enclosed in $\tilde{\gamma}_n$. The semi-continuity of the diameter follows. \square

Proof of Theorem 5.20. Fix an unstable transversal Δ^u , and recall that any component of $K^+ \cap \mathbb{B}$ (resp. $J^+ \cap \mathbb{B}$) intersects Δ^u . By [6, Thm 7.1], $J^+ \cap \Delta^u$ admits uncountably many point components, thus the first assertion of the theorem follows from Proposition 5.9. Then Corollary 5.10 asserts that at most countably many components are non-trivial.

Let $x \in J^+ \cap \Delta^u$ and assume that $J_{\mathbb{B}}^+(x)$ (or equivalently $K_{\mathbb{B}}^+(x)$) is non-trivial. Since Δ^u is a global transversal, $J^u(x)$ is leafwise bounded. For $n \geq 0$, $J^u(x_n) = f^n(J^u(x))$ where $x_n = f^n(x)$, and by (1),

$$(5) \quad \text{Diam}_{x_n}(J^u(x_n)) \geq Cu^n \text{Diam}_x(J^u(x)) \xrightarrow{n \rightarrow \infty} \infty.$$

Let x_∞ be any accumulation point of (x_n) . By Lemma 5.22, $J^u(x_\infty)$ is leafwise unbounded, and so does $K^u(x_\infty)$.

By local product structure, for large n , the holonomy along the stable lamination defines a projection

$$D^u(x_n, 3/2) \cap J^+ \rightarrow D^u(x_\infty, 2) \cap J^+$$

which we simply denote by π^s . It is Lipschitz (see Lemma 5.7) and a homeomorphism onto its image. Notice that $\pi^s(D^u(x_n, 3/2) \cap J^+)$ contains $D^u(x_\infty, 1) \cap J^+$ for large n . For large n , $J^u(x_n)$ intersects the boundary of $D^u(x_n, 3/2)$, so the sets $J^u(\pi^s(x_n))$ define a sequence of components of $J^+ \cap D^u(x_\infty, 1)$ of diameter bounded from below. From Theorem 3.6 we infer that this sequence is finite. Let us denote by C_j , $j = 1, \dots, N$ these components. By the Pigeonhole Principle there exist $n \neq n'$ such that $\pi_s(x_n)$ and $\pi_s(x_{n'})$ belong to the same C_j , thus x_n and $x_{n'}$ belong to the local stable saturation of C_j . Therefore the sequence $(J_{\mathbb{B}}^+(x_n))$ is eventually periodic, and so is $(K_{\mathbb{B}}^+(x_n))$.

Consider now a non-trivial periodic component C of $J^+ \cap \mathbb{B}$. Then it is of the form $J_{\mathbb{B}}^+(x)$ for some $x \in \Delta^u \cap J^+$. The previous argument shows that there are points

$x' \in C \cap J$ such that $J^u(x')$ is leafwise unbounded. By Proposition 5.13, the components of the slices $J_{\mathbb{B}}^+ \cap \Delta^u$ have diameter uniformly bounded from below (here we use the fact that for every $x \in \Delta^u \cap J^+$, the distance d_x^u is uniformly comparable to the ambient distance on Δ^u). Thus, by Theorem 3.6 only finitely many such components can arise and we conclude that C belongs to a finite set of components. The corresponding result for components of $K^+ \cap \mathbb{B}$ follows from Lemma 2.3. \square

Remark 5.23. Using techniques similar to those of §5.2 it is easily seen that any component of $K^+ \cap \mathbb{B}$ has finitely many preimages. In other words, the induced dynamical system on components of $K^+ \cap \mathbb{B}$ is finite-to-1. Indeed assume by contradiction that C is a component such that $f^{-1}(C) \cap \mathbb{B}$ has infinitely many preimages C_i . Then by Theorem 3.6, for some i , $C^i \cap \Delta^u$ has a component of small diameter. Therefore by pushing forward, there is some $x \in C \cap J$ such that $\text{Diam}_x(J^u(x))$ is small, that is, $J_{\mathbb{B}}^+(x)$ (or equivalently $K_{\mathbb{B}}^+(x)$) is thin. But it is easy to show that a thin component admits finitely many preimages, and we arrive at the desired contradiction. \square

6. COMPONENTS OF J AND K

We keep the same setting as before, that is, f is a uniformly hyperbolic dissipative Hénon map, with a disconnected and stably totally disconnected Julia set. In this section, we complete the proof of the main theorem by classifying the connected components of J and K .

We start with an easy fact. Recall the notation $E(x) = \text{Comp}_E(x)$.

Proposition 6.1. *If $x \in J$ is such that $J^u(x)$ is leafwise bounded then $J(x) = J^u(x)$.*

Proof. First, $J^u(x)$ is a connected set such that $x \in J^u(x) \subset J$ so it is contained in $J(x)$. To prove the converse statement, let (U_n) be a sequence of open neighborhoods of $J^u(x)$ in $W^u(x)$ decreasing to $J^u(x)$ and such that $\partial_i U_n \cap J = \emptyset$. Since $J^s(x) = \{x\}$, for every n any sufficiently small loop γ about x in $W^s(x)$ can be propagated along U_n to yield an open set \tilde{U}_n such that $\partial \tilde{U}_n = \emptyset$. Note that we did not prove any extension result for the unstable lamination, so we cannot simply say that we propagate γ by using some “unstable holonomy”. On the other hand we can simply use the inclination lemma, by pushing forward a small thickening of $f^{-n}(\gamma)$ as a 3 manifold transverse to $W^s(f^{-n}(x))$. Finally, for every n , $\partial \tilde{U}_n$ is relatively open and closed in J , so it contains $J(x)$ and we conclude that $J(x) = J^u(x)$. \square

To understand the structure of periodic components of J , let us introduce a definition.

Definition 6.2. A *quasi-solenoid* is a saddle hyperbolic set such that $f^k(\Lambda) = \Lambda$ for some k and:

- ▶ Λ is connected;
- ▶ Λ has local product structure;
- ▶ for every $x \in \Lambda$, $\Lambda \cap W^u(x)$ is leafwise unbounded and locally connected, and $\Lambda \cap W^s(x)$ is totally disconnected.

Observe that in this definition we do not require that $\Lambda \cap W_{\text{loc}}^s(x)$ is a Cantor set. In other words, we allow for isolated points in a stable transversal (this phenomenon will be ruled out later under appropriate hypotheses, see Theorem 8.7).

Theorem 6.3. *Let f be dissipative and hyperbolic with a disconnected and stably totally disconnected Julia set and \mathbb{B} be as above. Let C be a periodic component of $J^+ \cap \mathbb{B}$ and k be its period. Then $\Lambda := \bigcap_{n \geq 0} f^{kn}(C)$ is a point or a quasi-solenoid, and it is a connected component of J .*

Proof. Replacing f by some iterate, we may assume C is invariant, that is, $k = 1$. If C is a vertical manifold, it follows from Remark 5.21 that Λ is a point, and the other properties follow easily, so the interesting case is when C is non-trivial. Then, arguing in the proof of Theorem 5.20, by (5), C contains points such that $\text{Diam}_x(J^u(x))$ is arbitrary large, so it is thick in the sense of Proposition 5.13. Define $\Lambda := \bigcap_{n \geq 0} f^n(C) = \bigcap_{n \geq 0} \overline{f^n(C)}$. Since by assumption $f(C) \subset C$, Λ is a decreasing intersection of compact connected sets. Hence Λ is an invariant connected hyperbolic set contained in J , and $f(\Lambda) = \Lambda$. Let us show that it is a connected component of J . For this, let Λ' be the connected component of Λ in J . By definition $\Lambda \subset \Lambda'$. Since Λ' is connected and contained in $J^+ \cap \mathbb{B}$, it must be contained in C . Furthermore since $f(\Lambda) = \Lambda$, and f permutes the components of J , we have that $f(\Lambda') = \Lambda'$, hence for every $n \geq 1$, $f^{-n}(\Lambda') \subset C$, and we conclude that $\Lambda' \subset \bigcap_{n \geq 0} f^n(C) = \Lambda$, as was to be shown.

We claim that for every $x \in \Lambda$, $J^u(x)$ is leafwise unbounded. Indeed for every $x \in \Lambda$, we have that $x = f^n(x_{-n})$ with $x_{-n} = f^{-n}(x) \in C$ and since C is thick, $\text{Diam}_{x_{-n}}(J^u(x_{-n}))$ is uniformly bounded from below, and the result follows.

By Lemma 3.9, for every $x \in \Lambda$, there are only finitely many components of $J \cap D^u(x, 1)$ intersecting $\partial_i D^u(x, 1)$ and $D^u(x, 1/2)$. A simple compactness argument using the holonomy invariance of J^+ shows that this number is uniformly bounded, therefore there exists a uniform $\delta > 0$ such that leafwise unbounded components of J^+ intersecting $D^u(x, 1/2)$ are δ -separated in $D^u(x, 1)$ relative to the distance d_x^u (or equivalently, relative to the ambient one). From this we deduce that for every $x \in \Lambda$, there exists $\delta > 0$ such that Λ coincides with $J^u(x)$ in $W_\delta^u(x)$, and it follows from Theorem 3.6 that Λ is locally connected in the unstable direction.

Let us show that Λ has local product structure. For this, let $y_1, y_2 \in \Lambda$ be close (i.e. $d(y_1, y_2) \ll \delta$), denote by $\pi^s : W_{\text{loc}}^u(y_1) \rightarrow W_{\text{loc}}^u(y_2)$ the projection along stable leaves, and let $z_2 = \pi^s(y_1)$. Since $J^u(y_1)$ and $J^u(y_2)$ are leafwise unbounded, if $d(y_1, y_2)$ is small enough, $J^u(z_2)$ intersects $\partial_i D^u(y_2, 1)$, and so does $J^u(y_2)$. By definition of δ , it follows that $J^u(y_2) = J^u(z_2)$, hence y_2 and z_2 belong to the same connected component of J . In particular, z_2 belongs to C . Since f^{-1} contracts distances along unstable manifolds, and respects connected components of J , we can repeat this argument with $f^{-n}(y_2)$ and $f^{-n}(z_2)$ for any $n \geq 0$ and we conclude that $z_2 \in \Lambda$, as was to be shown. \square

Theorem 6.4. *Let f be dissipative and hyperbolic with a disconnected and stably totally disconnected Julia set. Then every component of J is either*

- (1) a point;
- (2) or of the form $J^u(x)$ with $J^u(x)$ non-trivial and leafwise bounded;
- (3) or a periodic quasi-solenoid.

In addition:

- (i) *There are finitely many quasi-solenoidal components*
- (ii) *Every periodic component of J is either a point or a quasi-solenoid.*
- (iii) *Every non-trivial component of J is attracted by a quasi-solenoid. More precisely, given a non-trivial component C for every $\delta > 0$ there exists n such that $f^{kn}(C) \subset W_\delta^s(\Lambda)$, where Λ is a quasi-solenoid of period k .*

Note that in assertion (ii), the uniformity of n as a function of δ is not a direct consequence of the fact that $\omega(C) \subset \Lambda$.

Proof. To establish the announced trichotomy, by Proposition 6.1 it is enough to show that if C is a component such that for some $x \in C$, $J^u(x)$ is leafwise unbounded, then C is a periodic quasi-solenoid. Note that for every $n \geq 1$, $J^u(f^{-n}(x))$ is leafwise unbounded. Therefore the component of $f^{-n}(x)$ in $J^+ \cap \mathbb{B}$ is thick in the sense of Proposition 5.13, and by Corollary 5.14, $J_{\mathbb{B}}^+(f^{-n}(x))$ belongs to a finite set of semi-local components. Thus there exists a component C^+ of $J^+ \cap \mathbb{B}$ and an infinite sequence n_i such that $f^{-n_i}(x) \in C^+$, hence C^+ is periodic of some period k and reversing time we get that $J^u(x)$ is included in $\Lambda := \bigcap_{n \geq 0} f^{kn}(C^+)$. By Theorem 6.3, Λ is a quasi-solenoid and $J(x) = C = \Lambda$.

Since there are only finitely many periodic semi-local components of J^+ , this argument shows that J has only finitely many solenoidal components.

For assertion (ii), let C be a periodic component of J which is not reduced to a point, and let $x \in C$. Without loss of generality we assume C is fixed. Expansion in the unstable direction shows that if $J^u(x)$ is leafwise bounded, then $J^u(x) = \{x\}$, which is a contradiction. Thus by the first part of the proof, C is a quasi-solenoid.

To prove (iii), let C be a non-trivial component of J , and for some large bidisk \mathbb{B} , let C^+ be the component of $J^+ \cap \mathbb{B}$ containing C . Then by Theorem 5.20 C^+ is ultimately periodic (with preperiod k), thus by Theorem 6.3, $\bigcap_{n \geq 0} f^{kn}(C^+)$ is a periodic quasi-solenoid Λ . This shows that C is attracted by Λ in the sense that for large n , $f^{kn}(C)$ is contained in a δ -neighborhood of Λ . To get the more precise statement that $f^{kn}(C) \subset W_\delta^s(\Lambda)$, we have to show that $W_\delta^s(\Lambda)$ is relatively open in $C^+ \cap J$. The argument is the same as for the local product structure: since large leafwise components of J are separated by some uniform distance and C is thick, if $x \in C \cap J$ is sufficiently close to $y \in \Lambda$, $W_{\text{loc}}^s(x) \cap W_{\text{loc}}^u(y)$ must belong to a large component of $W_{\text{loc}}^u(y) \cap J$, therefore it belongs to $J^u(y)$, and we are done. \square

Remark 6.5. Leafwise bounded components of J are locally connected, as follows from Theorem 3.6. On the other hand a quasi-solenoid is not locally connected, since it locally has the structure of a Cantor set times a (locally) connected set.

The following result says that there is a 1-1 correspondence between components of K and J , so that the previous theorems yield a description of components of K as well.

Proposition 6.6. *Every component of K contains a unique component of J .*

For polynomials in one variable, the analogous statement is the fact that every component of K has a connected boundary, which follows from polynomial convexity. Here, components of K have empty interior so this has to be formulated differently.

Proof. Every component of K contains a point of J , for otherwise it would be contained in $\text{Int}(K^+)$, so it is of the form $K(x)$ for some $x \in J$. If $J(x) = \{x\}$ the result is obvious. Now assume that $J^u(x)$ is leafwise bounded. By Lemma 2.4, $K^u(x)$ is obtained by filling the holes of $J^u(x)$ in $W^u(x) \simeq \mathbb{C}$, so $J^u(x)$ is equal to the intrinsic boundary of $K^u(x)$ and the result follows.

The most interesting case is when $J(x)$ is a quasi-solenoid. Replacing f by f^k for some $k \geq 1$, we may assume that $J(x)$ is fixed. We proved in Theorem 6.3 that $J(x) = \bigcap_{n \geq 0} f^n(J_{\mathbb{B}}^+(x))$. The very same proof shows that $K(x) = \bigcap_{n \geq 0} f^n(K_{\mathbb{B}}^+(x))$. By Lemma 2.3, $\partial K_{\mathbb{B}}^+(x)$ contains a unique component of $J_{\mathbb{B}}^+(x)$ (namely, its boundary), and we conclude by arguing that if $K(x)$ contained two distinct components $J(x)$ and $J(y)$ of J , then $K_{\mathbb{B}}^+(x)$ would contain $J_{\mathbb{B}}^+(x)$ and $J_{\mathbb{B}}^+(y)$, which must be distinct because $\bigcap_{n \geq 0} f^n(J_{\mathbb{B}}^+(x)) \neq \bigcap_{n \geq 0} f^n(J_{\mathbb{B}}^+(y))$, and this is impossible. \square

7. COMPLEMENTS

We keep the setting as in Sections 5 and 6. Here we prove a number of complementary facts which do not enter into the proof of the main theorem, so we sometimes allow the presentation to be a little sketchy.

7.1. Transitivity. A desirable property of quasi-solenoids is transitivity, or chain transitivity. At this stage we are not able to show that quasi-solenoidal components are transitive, but let us already explain a partial result in this direction. The full statement will be obtained in Theorem 8.7 under an additional assumption.

Proposition 7.1. *If Λ is a quasi-solenoidal component of J of period k , there exists a quasi-solenoid $\Lambda' \subset \Lambda$ of period $k\ell$, which is saturated by unstable components (that is, if $x \in \Lambda'$ then $J^u(x) \subset \Lambda'$), with the property that $f^{k\ell}|_{\Lambda'}$ is topologically mixing. In addition, stable slices of Λ' are Cantor sets and for every periodic point $p \in \Lambda'$, $\Lambda' = \overline{J^u(p)}$.*

This proposition follows from general facts from hyperbolic dynamics. Let us recall some basics. Recall that if Λ is a compact hyperbolic set with local product structure, then by Smale's Spectral Decomposition Theorem (see e.g. [46, §4.2]), the non-empty closed invariant subset

$$\Omega := \mathcal{C}(f|_{\Lambda}) = \overline{\text{Per}(f|_{\Lambda})}$$

(where by definition $\mathcal{C}(f|_{\Lambda})$ is the chain recurrent set of $f|_{\Lambda}$) admits a decomposition of the form $\Omega = \Omega_1 \cup \dots \cup \Omega_N$. The Ω_i are called the basic pieces. They are closed (and hence relatively open in Ω), f induces a permutation on the basic pieces and if q is the least integer such that $f^q(\Omega_i) = \Omega_i$, then $f^q|_{\Omega_i}$ is topologically mixing. In addition, Ω and the Ω_i have local product structure.

Proof. For notational simplicity replace f^k by f so that $k = 1$. Consider the ω -limit set $\omega(\Lambda) = \bigcup_{x \in \Lambda} \omega(x)$. Since a limit point is non-wandering, it is chain recurrent, so $\omega(\Lambda) \subset \Omega$. Conversely, since any periodic point is an ω -limit point, we see that $\text{Per}(f|_{\Lambda}) \subset \omega(\Lambda)$, hence $\Omega \subset \omega(\Lambda)$ and $\omega(\Lambda) = \Omega$. Then the Shadowing Lemma implies that $\Lambda \subset W^s(\Omega) = \bigcup_{x \in \Omega} W^s(x)$. Fix a small $\delta > 0$: then $W^s(\Omega) = \bigcup_{n \geq 0} f^{-n}(W_{\delta}^s(\Omega))$. By Baire's theorem, there exists n such that $f^{-n}(W_{\delta}^s(\Omega))$ has non-empty relative interior in Λ , hence so does $W_{\delta}^s(\Omega)$, and we conclude that for some i_0 , $W_{\delta}^s(\Omega_{i_0})$ has relative

non-empty interior in Λ . Let us show that $\Lambda' = \Omega_{i_0}$ satisfies the requirements of the proposition.

If ℓ is the least integer such that $f^\ell(\Lambda') = \Lambda'$, the fact that $f^\ell|_{\Lambda'}$ is topologically mixing follows from the Spectral Decomposition Theorem. Since Λ' has local product structure and $W_\delta^s(\Lambda')$ has relative non-empty interior in Λ , we see that there exists a relatively open subset U in Λ' such that for any $x_0 \in U$, a neighborhood of x_0 in $J^u(x_0)$ is contained in Λ' . Since $f^\ell|_{\Lambda'}$ is topologically transitive we may assume that x_0 has a dense orbit under f^ℓ . So if $y \in \Lambda'$ is arbitrary we can find a sequence (n_j) such that $f^{\ell n_j}(x_0) \rightarrow y$. By expansion in the unstable direction, there exists a uniform $\delta > 0$ such that for every j , $f^{\ell n_j}(\Lambda') = \Lambda'$ contains a δ -neighborhood of $f^{\ell n_j}(x_0)$ in $J^u(f^{\ell n_j}(x_0))$, so by local product structure we conclude that a neighborhood of y in $J^u(y)$ is contained in Λ' . On the other hand since Λ' is closed it is also relatively closed in unstable manifolds. This shows that Λ' is saturated by unstable components.

Let us show that for every periodic point $p \in \Lambda'$, $\overline{J^u(p)} = \Lambda'$. Let $N = \ell m$ be the period of p . Since $f^\ell|_{\Lambda'}$ is topologically mixing, $f^{\ell m}|_{\Lambda'}$ is topologically transitive, so there exists y arbitrary close to p such that $(f^{\ell mn}(y))_{n \geq 0}$ is dense in Λ' . Let y' be the projection of y in $W_{\text{loc}}^u(p)$ under stable holonomy. By local product structure, y' belongs to $J^u(p)$, and $y' \in W^s(y)$ so $(f^{\ell mn}(y'))$ is dense, too. Since all these points belong to $J^u(p)$, we conclude that $J^u(p)$ is dense in Λ' , as asserted.

For p as above, since $J^u(p)$ is leafwise unbounded, it must accumulate non-trivially in Λ' . More precisely, there exists $x \in \Lambda'$ and a sequence of points $x_n \in J^u(p)$, with $x_n \notin W_{\text{loc}}^u(x)$ and $x_n \rightarrow x$. Note that by local product structure, $W_{\text{loc}}^u(x_n) \cap \Lambda'$ corresponds to $W_{\text{loc}}^u(x) \cap \Lambda'$ under local stable holonomy. Now as before there exists $y' \in W_{\text{loc}}^u(p) \cap \Lambda'$ whose orbit is dense in Λ' . Thus any $z \in \Lambda'$ is the limit of $f^{n_j}(y')$ for some subsequence n_j . But $f^{n_j}(y')$ is an accumulation point of $W_{\text{loc}}^s(f^{n_j}(y')) \cap \Lambda'$, so the same holds for z , and we conclude that Λ' is transversally perfect in the stable direction, hence it is transversally a Cantor set. \square

7.2. Basins and solenoids. Assume that f has an attracting cycle $\{a_1, \dots, a_q\}$ of exact period q . We denote by \mathcal{B} its basin of attraction, which is made of k connected components \mathcal{B}_i biholomorphic to \mathbb{C}^2 . For every i we can write $\mathcal{B}_i \cap \mathbb{B}$ as the (at most) countable union $(\mathcal{B}_{i,j})_{j \geq 0}$ of its components, with $a_i \in \mathcal{B}_{i,0}$. We refer to these open sets as basin components and to $\mathcal{B}_{i,0}$ as the *immediate basin* of a_i . Note that if we replace f by f^q , the basin of attraction of a_i is now made of a single component, but $\mathcal{B}_{i,0}$ is unchanged.

By definition a *Jordan star* in $U \subset \mathbb{C}$ is a finite union of simple Jordan arcs in U , intersecting at a single point.

Theorem 7.2. *Let f be dissipative and hyperbolic with a disconnected and stably totally disconnected Julia set. Suppose that f admits an attracting fixed point with immediate basin \mathcal{B}_0 . Then:*

- (i) $\partial \mathcal{B}_0$ is a properly immersed topological submanifold of dimension 3, which intersects any global unstable transversal in finitely many Jordan domains.
- (ii) $\bigcap_{n \geq 0} \partial \mathcal{B}_0$ is a quasi-solenoid, whose unstable slices are Jordan stars. In particular there is a (saddle) periodic point in $\partial \mathcal{B}_0$.

We can be more precise about the structure of $\partial\mathcal{B}_0$: locally it is homeomorphic to the product of a 2-disk by a Jordan star. The proof of the theorem shows that if the components of $\mathcal{B}_0 \cap \Delta^u$ have disjoint closures, then these stars are reduced to Jordan arcs, that is, $\partial\mathcal{B}_0$ is a topological submanifold.

The following basic fact is crucial for the proof.

Lemma 7.3. *The stable lamination \mathcal{W}^s respects basin boundaries. That is, if $x \in J^+$ belongs to the boundary of an attracting basin \mathcal{B} , then so does its image under stable holonomy.*

Proof. This follows readily from the existence of a local extension of the stable lamination (Lemma 5.7): indeed if a leaf of the extended foliation joined a point from $\text{Int}(K^+)$ to a point of $(K^+)^c$, it would have to intersect J^+ . (See also [16], Step 3 of the proof of the main theorem, for an alternate argument without extending the stable lamination.) \square

Proof of Theorem 7.2. Fix a global unstable transversal Δ^u . Since every semi-local stable manifold intersects Δ^u , $\mathcal{B}_0 \cap \Delta^u$ is non-empty, and by the Maximum Principle each of its connected components is a topological disk. Pick such a connected component Ω_0 . By the John-Hölder property (Theorem 3.10), $\partial\Omega_0$ is locally connected, and by the Maximum Principle again there is no cut point, and it follows that Ω_0 is a Jordan domain (see [40, Thm 2.6]).

If the diameter of Ω_0 is small then, by Remark 5.16, enlarging \mathbb{B} if necessary the saturation $\widehat{\partial\Omega_0}$ of $\partial\Omega_0$ by semi-local stable leaves is topologically a product and we infer that $\widehat{\partial\Omega_0} \cap \Delta^u$ has finitely many components. Otherwise the diameter is large and by the same remark, every component of $\widehat{\partial\Omega_0} \cap \Delta^u$ has a large diameter. Then the finiteness of the number of such components follows from the John-Hölder property of $W^u(x) \setminus K^+$, Proposition 5.17, and the finiteness statement for interior components in Lemma 3.9.

By the Maximum Principle, if Ω_0 and Ω_1 are two components of $\mathcal{B}_0 \cap \Delta^u$ such that $\overline{\Omega_0} \cap \overline{\Omega_1} \neq \emptyset$, then $\overline{\Omega_0} \cap \overline{\Omega_1}$ is a single point. Indeed if this set contained two distinct points z and z' , by using crosscuts of $\overline{\Omega_0}$ and $\overline{\Omega_1}$ ending at z and z' we could construct a Jordan domain U with $\partial U \subset \overline{\Omega_0} \cup \overline{\Omega_1}$, and U would be contained in the Fatou set, a contradiction. Create a plane graph from $\mathcal{B}_0 \cap \Delta^u$ whose vertices are its components and edges are added when two components touch. The Maximum Principle again shows that this graph is a finite union of trees. Since the stable holonomy respects $\partial\mathcal{B}_0$ and $\partial\mathcal{B}_0$ is obtained from $\partial\mathcal{B}_0 \cap \Delta^u$ by saturating by stable manifolds, the description of $\partial\mathcal{B}_0$ as a properly immersed topological submanifold of dimension 3 follows.

The proof of the second item of the theorem is similar to that of Theorem 6.3. First, $\partial\mathcal{B}_0$ is connected: the argument is identical to that of Lemma 2.3. Then, for every $x \in \partial\mathcal{B}_0 \cap J^-$, there are only finitely many components of $\mathcal{B}_0 \cap D^u(x, 1)$ (resp. $\partial\mathcal{B}_0 \cap D^u(x, 1)$) intersecting $D^u(x, 1/2)$. Indeed, observe first that it is enough to prove this in $D^u(x, r)$ for some uniform r . By the uniform boundedness of the degree of semi-local stable manifolds in \mathbb{B} , there is a uniform r such that $D^u(x, r)$ can be pushed to Δ^u by stable holonomy, and the applying item (i) of the theorem completes the argument. From this point we proceed exactly as in Theorem 6.3. The existence of a periodic point in $\partial\mathcal{B}_0$ follows from general hyperbolic dynamics (see the comments after Proposition 7.1). \square

Remark 7.4. It follows from this description that if $x \in \Lambda$ lies at the boundary of \mathcal{B}_0 , then in $W^u(x)$, x belongs to the boundary of a component Ω of $\mathcal{B}_0 \cap W^u(x)$. In particular, Ω is a Fatou disk contained in $\text{Comp}_K(x)$.

Remark 7.5. We do not know whether components of $\mathcal{B}_0 \cap \Delta^u$ can actually bump into each other, or equivalently if $\bigcap_{n \geq 0} \partial \mathcal{B}_0$ does contain stars. If bumping occurs, let E be the finite set of points at which the closures of the components of $\mathcal{B}_0 \cap \Delta^u$ touch each other. Then $W_{\mathbb{B}}^s(E)$ is a finite union of vertical submanifolds, and $f(W_{\mathbb{B}}^s(E)) \subset W_{\mathbb{B}}^s(E)$. It follows that $\bigcap_{n \geq 0} f^n(W_{\mathbb{B}}^s(E))$ is a finite set of periodic points, and for any other point x in the limiting quasi solenoid $\Lambda := \bigcap_{n \geq 0} \partial \mathcal{B}_0$, $\Lambda \cap W_{\text{loc}}^u(x)$ is a Jordan arc. Thus, roughly speaking, Λ has the structure of finitely many solenoids attached at periodic “junction” points.

7.3. Branched Julia set model. Let Λ be a quasi-solenoidal component of J , and without loss of generality assume that Λ is fixed. Let $J_{\mathbb{B}}^+(\Lambda)$ be its connected component in $J_{\mathbb{B}}^+$ and consider its intersection $D := J_{\mathbb{B}}^+(\Lambda) \cap \Delta^u$ with some unstable transversal, which is made of finitely many thick components. Introduce a relation \sim on D by $x \sim y$ if and only if $W_{\mathbb{B}}^s(x) = W_{\mathbb{B}}^s(y)$, where by definition $W_{\mathbb{B}}^s(x) = \bigcap_{\varepsilon > 0} W_{(1+\varepsilon)\mathbb{B}}^s(x)$. Equivalently $x \sim y$ iff $\overline{W_{\mathbb{B}}^s(x)} \cap \overline{W_{\mathbb{B}}^s(y)} \neq \emptyset$: concretely, this means that x and y are related when they are connected by a stable manifold which is tangent to $\partial \mathbb{B}$. This defines a closed equivalence relation on D . We denote by $\tilde{D} := D / \sim$ the quotient topological space, which is compact (and Hausdorff) and by $\pi : D \rightarrow \tilde{D}$ the natural projection. Since $f(W_{\mathbb{B}}^s(x)) \subset W_{\mathbb{B}}^s(f(x))$, f descends to the quotient $\tilde{D} := D / \sim$ to a well defined continuous map \tilde{f} .

Geometrically \tilde{D} has to be thought of as a *branched Julia set*, lying on the branched surface –in the sense of Williams [45]– obtained by collapsing the semi-local stable leaves of the extended stable lamination. Then \tilde{f} is expanding on the plaques of this branched manifold³, and its iterates are uniformly quasiconformal wherever defined, since they are obtained by iterating f and projecting along the stable lamination. Observe that f is not necessarily surjective, since for every $x \in D$, $f^n(x)$ eventually belongs to $W_{\mathbb{B}}^s(\Lambda)$, which may be smaller than $J_{\mathbb{B}}^+(\Lambda)$ (cf. Figure 1). On the other hand by the last assertion of Theorem 6.4, there exists a uniform N such that $f^N(J_{\mathbb{B}}^+(\Lambda)) \subset W_{\mathbb{B}}^s(\Lambda)$. It follows that the sequence $\bigcap_{0 \leq k \leq n} \tilde{f}^k(\tilde{D})$ is stationary for $n \geq N$ and that $\tilde{D}' := \pi(W_{\mathbb{B}}^s(\Lambda) \cap \Delta^u)$, is an invariant, closed, and plaque-open subset of \tilde{D} on which \tilde{f} is surjective.

Proposition 7.6. *With the above definitions, the dynamical system (Λ, f) is topologically conjugate to the natural extension of (\tilde{D}, \tilde{f}) (or equivalently (\tilde{D}', \tilde{f})).*

Proof. Indeed define $h : \varprojlim(\tilde{D}, \tilde{f}) \rightarrow \Lambda$ by $h((\tilde{x}_n)_{n \in \mathbb{Z}}) = \bigcap_{n \geq 0} f^n(W_{\mathbb{B}}^s(x_{-n}))$, whose inverse is $y \mapsto h^{-1}(y) = ((W_{\mathbb{B}}^s(f^n(y)))_{n \in \mathbb{Z}}$. \square

³Here by plaque we mean one of the finitely many overlapping disks which make up a local chart of a branched manifold, see [45, Def. 1.0]

8. NON-DIVERGENCE OF HOLONOMY AND APPLICATIONS

8.1. The NDH property. We say that the property of *Non-Divergence of Holonomy* (NDH) holds if for every pair of points $x, y \in J$ such that y belongs to $W^s(x)$, the stable holonomy, which is locally defined from a neighborhood of x in $W^u(x)$ to a neighborhood of y in $W^u(y)$, can be continued along any path contained in $J^u(x)$.

Remark 8.1.

- (1) The stable holonomy $h : W^u(x) \rightarrow W^u(y)$ is independent of the choice of a path c from x to y in $W^s(x)$ because $W^s(x)$ is simply connected.
- (2) An unstable component $J^u(x)$ is typically *not* simply connected (since it may enclose the trace of an attracting basin on $W^u(x)$). So even if the stable holonomy from x to y admits an extension along continuous paths, it does not generally yield a well-defined map from $J^u(x)$ to $J^u(y)$.

We do not know any example where the NDH property fails. An analogue of this property was studied in the context of the classification of Anosov diffeomorphisms, where it is expected to be a crucial step in the classification program. It was established in the two dimensional case in [20] (see also [10, 32] for related results).

Back to automorphisms of \mathbb{C}^2 , we have the following simple criterion:

Proposition 8.2. *A sufficient condition for the NDH property is that the stable lamination W^s of J^+ is transverse to $\partial\mathbb{B}$ (No Tangency condition, NT).*

Proof. Assume that the No Tangency condition holds and let $x, y \in J$ be such that y belongs to $W^u(x)$. Replacing x and y by $f^k(x)$ and $f^k(y)$ for some positive k , we may assume that $y \in W_{\mathbb{B}}^s(x)$. There is a germ of stable holonomy h sending a neighborhood of x in $J^u(x)$ to some neighborhood of $y \in J^u(y)$. Let $\gamma : [0, 1] \rightarrow J^u(x)$ be a continuous path: we have to show that h can be continued along γ . For this, introduce $E \subset [0, 1]$ the set of parameters t such that h can be continued along $\gamma|_{[0,t]}$ and $h(\gamma(t)) \in W_{\mathbb{B}}^s(\gamma(t))$. Obviously, E is a relatively open subinterval of $[0, 1]$ containing 0, and the proof will be complete if we show that E is closed. Thus, assume that $(t_n) \in E^{\mathbb{N}}$ is an increasing sequence converging to t_{∞} , and let y_{∞} be any cluster value of the sequence $(h(\gamma(t_n)))$. The main observation is that since W^s is transverse to $\partial\mathbb{B}$, $W_{\mathbb{B}}^s(\gamma(t_n))$ converges to $W_{\mathbb{B}}^s(\gamma(t_{\infty}))$ in the Hausdorff topology, with multiplicity 1, or equivalently in the C^1 topology. Furthermore, by the uniform boundedness of the vertical degree, there is a uniform L such that for every n , there is a path of length at most L joining $\gamma(t_n)$ to $h(\gamma(t_n))$ in $W^s(\gamma(t_n))$. It follows that the assignment $\gamma(t_n) \mapsto h(\gamma(t_n))$ is equicontinuous. Let y_{∞} be a cluster value of $(h(\gamma(t_n)))$. The equicontinuity property shows that $h(\gamma(t_n))$ actually converges to y_{∞} , and also that the the points $h(\gamma(t_n))$ belong to the same local plaque of the unstable lamination, which must thus coincide with $W_{\text{loc}}^u(y_{\infty})$. From this we conclude that h extends to a neighborhood of $\gamma(t_{\infty})$, with $h(\gamma(t_{\infty})) = y_{\infty}$, and we are done. \square

One may argue that the NT condition is not intrinsic since it depends on the choice of the bidisk \mathbb{B} . To get around this issue we may consider the following variant:

(NT_G) there exists $R > 0$ such that the stable foliation admits no tangency with the hypersurface $\{G^- = R\}$.

Note that the level set $\{G^- = R\}$ is smooth near J^+ for every $R > 0$: indeed by the local structure of G^- near infinity this is the case when R is large, and then we use invariance to propagate this property to all $R > 0$. Arguing exactly as in the previous proposition shows that the NT_G property implies NDH.

Using this idea also enables us to understand more precisely how the NDH property may fail. If x and y are two points in J with $y \in W^s(x)$, define the *Green distance*

$$d_G(x, y) := \inf_{c: x \rightarrow y} \max(G^-|_c)$$

where the infimum runs over the set of continuous paths $c : [0, 1] \rightarrow W^s(x)$ joining x to y . Since $W^s(x) \cap J$ is totally disconnected, this indeed defines an ultrametric on $W^s(x) \cap J$, which is uniformly contracted by f : $d_G(f(x), f(y)) = d^{-1}d_G(x, y)$. It provides an intrinsic way of measuring how far we need to go in \mathbb{C}^2 to connect two unstable components by stable manifolds. Arguing exactly as in Proposition 8.2 shows:

Proposition 8.3. *Let $x, y \in J$ with $y \in W^s(x)$ and denote by h the germ of stable holonomy $h : W_{\text{loc}}^u(x) \rightarrow W_{\text{loc}}^u(y)$. Let $\gamma : [0, 1] \rightarrow J^u(x)$ be a continuous path and assume that h can be continued along $\gamma([0, t^*))$. Then h admits an extension to t^* if and only if $d_G(\gamma(t), h(\gamma(t)))$ is bounded as $t \rightarrow t^*$.*

8.2. No queer components.

Theorem 8.4. *Let f be dissipative and hyperbolic, with a disconnected and stably totally disconnected Julia set. Assume further that the NDH property holds. Then any non-trivial periodic component of K contains an attracting point.*

Proof. We argue by contradiction: assume that Λ is a component of K which does not contain any attracting periodic point. Let C be the component of Λ in $K^+ \cap \mathbb{B}$. Our hypothesis implies that C has empty interior, so C is a component of $J^+ \cap \mathbb{B}$ (and Λ is a component of J). Fix an unstable transversal Δ^u and let E be a component of $C \cap \Delta^u$, which must have empty interior in Δ^u by Lemma 2.1. Thus E is a locally connected continuum with empty interior, that is, a dendrite.

Lemma 8.5. *For every $x \in E$, $W^s(x) \cap E = \{x\}$.*

Assuming this lemma for the moment, let us complete the proof. By the expansion in the unstable direction, for every $x \in E$, there exists $\delta_1 > 0$ such that for every $n \geq 0$, $f^n(E)$ is not relatively compact in $D^u(f^n(x), \delta_1)$, and by the John-Hölder property, there exists $\delta_2 > 0$ such that any two components of $f^n(E)$ in $D^u(f^n(x), \delta_1)$ intersecting $D^u(f^n(x), \delta_1/2)$ are δ_2 -separated. Fix a covering of J by unstable flow boxes. By the product structure of J , there exists $\varepsilon > 0$ such that if $y, z \in f^n(E)$ are ε -close in \mathbb{C}^2 but not on the same unstable plaque, then the components $\text{Comp}_{f^n(E) \cap D^u(y, \delta_1)}(y)$ and $\text{Comp}_{f^n(E) \cap D^u(z, \delta_1)}(z)$ are related by local stable holonomy. Finally, by expansion along the unstable direction and the previous separation property, $f^n(E)$ cannot be contained in boundedly many unstable plaques as $n \rightarrow \infty$. Thus, for sufficiently large n we can find two points in $f^n(E)$ which are ε -close in \mathbb{C}^2 but not on the same unstable plaque,

so there exists $y \in f^n(E)$ such that $W_{\text{loc}}^s(y)$ intersects $f^n(E)$ in another point. This contradicts Lemma 8.5 and we are done. \square

Proof of Lemma 8.5. Assume that $W^s(x) \cap E$ contains another point $y \neq x$. Then the stable holonomy defines a germ of homeomorphism $h : E \cap U_x \rightarrow E \cap U_y$, where U_x is some neighborhood of x (resp. y). By the NDH property, h can be continued along paths in E . Since E is simply connected, this extends to a globally defined map $h : E \rightarrow E$, sending x to y , which is a local homeomorphism, hence a covering, so again using the fact that E is simply connected, we conclude that h is a homeomorphism.

It is a classical fact that any continuous self-map of E admits a fixed point. For the reader's convenience let us include the argument. View E as a subset of the plane. Then, by the Carathéodory theorem, the Riemann map $\mathbb{C} \setminus \mathbb{D} \rightarrow \mathbb{C} \setminus E$ extends to a continuous and surjective map $\partial\mathbb{D} \rightarrow \partial E = E$. From this we can construct a topological disk $U \supset E$ and a retraction $r : \bar{U} \rightarrow E$: indeed take the disk bounded by some equipotential and define r as collapsing each external ray to its endpoint. Now let $g = h \circ r$. Since g maps \bar{U} into itself, by the Brouwer fixed point theorem it admits a fixed point x_0 . Finally, since $g(\bar{U}) \subset E$, x_0 belongs to E , so $g(x_0) = h(r(x_0)) = h(x_0) = x_0$.

To conclude the proof we show that the existence of such a fixed point contradicts the hyperbolicity of f . For this, fix a continuous path $(x_t)_{t \in [0,1]}$ joining x_0 to $x_1 := x$ and let $t^* = \max\{t \in [0,1], h(x_t) = x_t\}$, which satisfies $0 \leq t^* < 1$. As $t > t^*$ tends to t^* , we see that the two point set $\{x_t, h(x_t)\}$ collapses to $\{x_{t^*}\}$. This means that there is a tangency between the stable lamination and Δ^u at x_{t^*} , which is the desired contradiction. \square

Remark 8.6. With notation as in the proof of the theorem, it is not difficult to deduce from the proof that for every $\delta > 0$, for $n \geq n(\delta)$ there exists a non-trivial simple closed curve contained in $W_\delta^s(f^n(E))$. So by the last assertion of Theorem 6.4, there is a non-trivial simple closed curve contained in $W_\delta^s(\Lambda)$. Without the NDH property, we cannot exclude a situation where these simple closed curves do not enclose an attracting basin. We may qualify these dendrites and their limit sets as *queer components* of J . So Theorem 8.4 asserts that under the NDH property, *queer components of J do not exist*.

8.3. Topological mixing.

Theorem 8.7. *If the NDH property holds, if Λ is a quasi-solenoidal component of period k , then $f^k|_\Lambda$ is topologically mixing. In particular Λ is transversally a Cantor set.*

Proof. Without loss of generality we may assume $k = 1$. We resume Proposition 7.1 and its proof. Let Λ' be as in Proposition 7.1, and let us show that $\Lambda' = \Lambda$. Since Λ' is saturated in the unstable direction, $W^s(\Lambda')$ is relatively open in Λ . The NDH property shows that if $y \in W^s(\Lambda')$, then $J^u(y) \subset W^s(\Lambda')$: indeed the set of points $z \in J^u(y)$ such that $z \in W^s(\Lambda')$ is open because $W^s(\Lambda')$ is relatively open, and since $J^u(y)$ is arcwise connected, the NDH property implies that it is closed as well. Thus by the local product structure of Λ , we conclude that $W^s(\Lambda')$ is relatively closed in Λ , and by connectedness we conclude that $W^s(\Lambda') = \Lambda$.

Fix a small $\delta > 0$. By Baire's theorem, we infer that $f^{-n}(W_\delta^s(\Lambda'))$ has non-empty relative interior in Λ for large n , hence so does $W_\delta^s(\Lambda')$ by invariance. Arguing as in Proposition 7.1, we see that by topological transitivity, $W_\delta^s(\Lambda')$ is actually relatively

open in Λ . Therefore $\bigcup_{n \geq 0} f^{-n}(W_\delta^s(\Lambda'))$ is an open cover of Λ and by compactness we conclude that Λ is contained in $\bigcup_{0 \leq n \leq n_0} f^{-n}(W_\delta^s(\Lambda'))$ for some n_0 . and since $f^{n_0}(\Lambda) = \Lambda$ we finally deduce that $\Lambda \subset W_\delta^s(\Lambda')$. Since δ was arbitrary, $\Lambda \subset \Lambda'$, and we are done. \square

Remark 8.8. A similar argument shows that under the NDH property, the quasi-solenoids obtained as limit sets of basin boundaries in Theorem 7.2 are transitive.

As a consequence of transitivity we can be more precise about the topological structure of periodic components of K .

Proposition 8.9. *Let f be dissipative and hyperbolic, with a disconnected and stably totally disconnected Julia set. Assume further that the NDH property holds. Then for any non-trivial component D of K , $D \cap \text{Int}(K^+)$ is dense in D . Equivalently, for any $x \in D$, $D \cap W^u(x)$ is the closure of its interior for the intrinsic topology.*

Proof. The equivalence between the two assertions follows from Lemma 2.1, Lemma 7.3, and the local product structure. Let D be as in the statement of the proposition and C be its component in $K^+ \cap \mathbb{B}$. Let also Λ the unique component of J contained in D (Proposition 6.6). Without loss of generality we may assume that D (hence C and Λ) is fixed by f . By Theorem 8.4 D contains an attracting periodic point a , so the immediate basin \mathcal{B}_0 of a is contained in C . By Theorem 7.2, $\partial \mathcal{B}_0$ contains a saddle periodic point p , which must belong to Λ (indeed by Lemma 2.3 and Theorem 6.3, $\Lambda = \bigcap_{n \geq 0} f^n(\partial C)$). The topological mixing of $f|_\Lambda$ (Theorem 8.7) classically implies that $W^s(p) \cap \Lambda$ is dense in Λ . Indeed let U be a product neighborhood of p in Λ , and V be an arbitrary open subset of Λ . Then for sufficiently large $q \geq 0$ there exists $y_q \in V$ such that $f^q(y_q) \in U$. Since Λ has local product structure $[f^q(y_q), p] := W_{\text{loc}}^u(f^q(y_q)) \cap W_{\text{loc}}^s(p)$ belongs to Λ , hence increasing n again if needed, $z_q := f^{-q}([f^q(y_q), p])$ is a point in $W^s(p) \cap V$.

To conclude from this point, we observe that by Remark 7.4 (applied to $f^{-q}(\mathcal{B}_0)$) z_q belongs to the boundary of a component Ω of $W^u(z_q) \cap f^{-q}(\mathcal{B}_0)$ contained in D , and we are done. \square

8.4. Concluding remarks. The non-existence problem for queer components bears some similarity with another well-known open problem: the non-existence of Herman rings for complex Hénon maps (see [4] for an early account). Indeed assume that f admits a Herman ring, that is, a Fatou component Ω biholomorphic to the product of an annulus times \mathbb{C} . More precisely there exists a biholomorphism $h : \Omega \rightarrow A \times \mathbb{C}$, where A is a standard annulus, which conjugates f to $(x, y) \mapsto (e^{i\theta}x, \delta y)$, $|\delta| < 1$. Assume further that J is disconnected, and fix an unstable transversal Δ^u (recall that its existence does not require f to be hyperbolic). Then if C is an invariant circle in A , f admits an invariant ‘‘cylinder’’ $\mathcal{C} = h^{-1}(C \times \mathbb{C})$. Any component of $\mathcal{C} \cap \Delta^u$ is a piecewise smooth immersed curve, and a contradiction would follow if we can show that it bounds a disk in Δ^u (since by the maximum principle this disk would be a Fatou disk, whose normal limits would fill up the annulus). In other words, if f admits a Herman ring, $\mathcal{C} \cap \Delta^u$ is a countable union of dendrites whose saturation under the stable foliation of \mathcal{C} bounds a disk, but not a holomorphic disk (compare with Remark 8.6). Note however that a limitation to the analogy between the two problems is that the NDH property holds trivially in the Herman ring case, so the difficulty is of a different nature.

APPENDIX A. THE CORE OF A QUASI-SOLENOID

In this Appendix, we sketch the construction of the *core* of a quasi-solenoidal component, which should intuitively be understood as the space obtained from this component after removing all “bounded decorations” in unstable manifolds. Initially designed as a potential tool to prove the non-existence of queer quasi-solenoids, it also gives interesting information on the combinatorial structure of tame ones. It would be interesting to compare it with other constructions such as Ishii’s Hubbard trees (see [27]). We keep the setting as in the previous sections, that is f is a uniformly hyperbolic dissipative Hénon map, with a disconnected and stably totally disconnected Julia set.

A.1. Number of accesses. The discussion in this paragraph is reminiscent from [7, §7], which deals with the connected case. Pick $x \in J$. For any $R > 0$, define $N^u(x, R)$ to be the number of connected components Ω of $D^u(x, R) \setminus J$ such that $x \in \bar{\Omega}$. Since $K \cap D^u(x, R)$ has the John-Hölder property, Corollary 3.3 implies that $N^u(x, R) < \infty$. Thus, $R \mapsto N^u(x, R)$ is a integer-valued non-increasing function which drops when two components of $D^u(x, R) \setminus J$ merge. The limit

$$N_{\text{loc}}^u(x) := \lim_{R \rightarrow 0} N^u(x, R)$$

is the number of local accesses to x , and

$$N^u(x) := \lim_{R \rightarrow \infty} N^u(x, R)$$

is the number of connected components of $W^u(x) \setminus J$. Note that if $J^u(x)$ is bounded then $N^u(x) = 1$, so this notion is interesting only when x belongs to a quasi-solenoidal component.

We can also restrict to counting accesses from infinity, that is components of $D^u(x, R) \setminus K^+$, and we obtain corresponding numbers $N_{\infty}^u(x, R)$, $N_{\infty, \text{loc}}^u(x)$ and $N_{\infty}^u(x)$. We have that $N_{\infty}^u(x) \leq N^u(x)$ (and similarly for the other quantities), and, since every point of J is accessible from infinity, $N_{\infty}^u(x) \geq 1$.⁴

Lemma A.1. N^u (resp. N_{∞}^u) is upper semicontinuous on J , that is, for any $k \geq 1$, $\{x, N^u(x) \geq k\}$ is closed.

Proof. We deal with N^u , the proof for N_{∞}^u is similar. It is enough to assume that $k \geq 2$. By the local product structure of J , it is enough to study the semi-continuity of $x \mapsto N^u(x)$ separately along stable and unstable manifolds. Let us start by studying this semicontinuity along a local stable transversal. We have to prove that $\{x, N^u(x) < k\}$ is open. Indeed assume that there are $j < k$ accesses to x in $W^u(x) \setminus J$. This means that for large R , $D^u(x, R) \setminus J$ has j connected components accumulating at x . If $x' \in W^s(x)$ then the local stable holonomy between $W_{\text{loc}}^u(x)$ and $W_{\text{loc}}^u(x')$ is a homeomorphism, which locally preserves the number of components of $W_{\text{loc}}^u(x) \setminus J$. In addition if x' is sufficiently close to x , this holonomy is defined in $D^u(x, R)$. Indeed for this it is enough

⁴The John-Hölder property of the basin of infinity directly guarantees the finiteness of $N_{\infty, \text{loc}}^u(x)$, but not that of $N_{\text{loc}}^u(x)$ (see Remark 3.11). This property can actually be salvaged as follows: if for small R , $N^u(x, R)$ is large, then for some $k \gg 1$, $N^u(f^k(x), 1)$ is large, and projecting to some fixed transversal yields a contradiction.

to iterate backwards until $f^{-n}(D^u(x, R))$ is contained in the domain of the extended stable lamination. Therefore, there is a large domain D' in $W^u(x')$ such that $D' \setminus J$ has j connected components accumulating on x' . Since the number of components may drop when enlarging this disk further, we conclude that $N^u(x') \leq j$.

Now we work inside a given unstable manifold. Let R be such that $N^u(x, s) = N^u(x) = j$ for $s \geq R - 1$. By the Hölder-John property, for $R' < R$, $D^u(x, R) \setminus J$ admits finitely many components intersecting $D^u(x, R')$. So if $N^u(x) = j$, there is some $0 < \varepsilon < 1$ such that only j of these components reach $D^u(x, \varepsilon)$, and we conclude that for $x' \in D^u(x, \varepsilon)$, $N^u(x', R - 1) \leq j$, hence $N^u(x') \leq j$, as asserted. \square

Since f acts linearly on unstable parameterizations, $N^u(x, R) = N^u(f(x), \lambda^u R)$, and we obtain:

Corollary A.2. *If $N_{\text{loc}}^u(x) \geq k$ then for any $y \in \omega(x)$, $N^u(y) \geq k$.*

An argument similar to that of the second part of Lemma A.1 implies (compare [7, pp. 490-491]):

Lemma A.3. *For any $R > 0$ and any $x \in \Lambda$, the set $\{y \in W^u(x), N^u(y, R) \geq 3\}$ is discrete for the intrinsic topology.*

Proposition A.4. *The set $\{x \in J, N^u(x) \geq 3\}$ is a finite set of saddle periodic points.*

Proof. By Lemma A.3, the set $\{x \in J, N^u(x, R) \geq 3\}$ is contained in a countable union of local stable manifolds. Since any point in J can be joined to a given unstable transversal Δ^u by a stable path of uniform length, by taking small enough R we infer that the projection of this set to Δ^u is actually finite. Therefore, the set $\{x \in \Lambda, N^u(x) \geq 3\}$ is a closed invariant set contained in a finite union of semi-local stable manifolds, so it is finite. \square

A.2. Definition(s) and properties of the core. Let Λ be a quasi-solenoidal component of J . There are several possible definitions for the core of Λ . It is unclear for the moment which choice is the most appropriate. We define:

- ▶ $\text{Core}(\Lambda) = \{x \in \Lambda, N^u(x) \geq 2\}$
- ▶ $\text{Core}'(\Lambda) = \omega(\{x \in \Lambda, N_{\text{loc}}^u(x) \geq 2\})$

By Corollary A.2 we have the inclusion $\text{Core}'(\Lambda) \subset \text{Core}(\Lambda)$, and it is an open problem whether equality holds. It is obvious from the definition that $\text{Core}(\Lambda)$ (resp. $\text{Core}'(\Lambda)$) is invariant and Lemma A.1 implies that it is closed. Hence it is a closed hyperbolic set. Another natural open question is whether $\text{Core}(\Lambda)$ is connected.

The core of the Julia set is the union of the cores of its finitely many quasi-solenoidal components. If $x \in J$ is any point such that $W^u(x) \setminus J$ has several local accesses, then $\omega(x) \subset \text{Core}(J)$.

We say that $x \in \text{Core}(\Lambda)$ is *regular* if $N^u(x) = 2$ and *singular* otherwise. Recall that the singular set is a finite set of periodic points. Note that if x belongs to the core, then $J^u(x)$ disconnects $W^u(x)$.

Conjecture A.5. *$\text{Core}(\Lambda)$ has local product structure near any regular point, and is locally the product of a Jordan arc by a totally disconnected set.*

On the other hand, $\text{Core}(\Lambda)$ does not have local product structure in the neighborhood of any of its singular points, unless it is locally contained in a single unstable manifold. So the structure of the core should be that of a union of solenoids joined at finitely many branch points. It seems that in the example described in [26, Thm 4.23], one quasisolenoidal component has a core made of two solenoids attached at a fixed saddle point.

Note that if Λ is not a queer component, that is the associated component of K contains an attracting periodic point, then the solenoid at the boundary of the immediate basin, constructed in § 7.2, is contained in the core. Indeed it is obtained by taking limits of Jordan arcs locally separating an attracting basin from the basin of infinity. So the topological structure of the core should give an account how these various basins are organized and attached to each other in Λ (compare with the Hubbard tree in one-dimensional dynamics).

Finally, we may also define $\text{Core}_\infty(\Lambda) = \{x \in \Lambda, N_\infty^u(x) \geq 2\}$. (If Λ is a queer component, then $\text{Core}_\infty(\Lambda) = \text{Core}(\Lambda)$.) We expect that $\text{Core}_\infty(\Lambda)$ is a finite set. Indeed, if not, it should contain a Jordan arc such that every point is accessible from both sides by the basin of infinity, and such arcs should not exist. Indeed, iterating forward, and arguing as in Theorem 8.4, a large iterate of this arc must spiral and come close to itself, hence, projecting to an unstable transversal, this would cut out a Fatou disk, and we conclude that one side of the arc is contained in an attracting basin.

APPENDIX B. CONTINUITY OF AFFINE STRUCTURE

Here we present the following mild generalization of a theorem by Étienne Ghys [22]. Recall that the ratio of a triple $(u, v, w) \in \mathbb{C}^3$ is $\frac{u-v}{u-w}$.

Theorem B.1. *Let $\psi : \mathbb{C} \rightarrow \mathbb{C}^2$ be an injective holomorphic immersion, and $L = \psi(\mathbb{C})$. Assume that (L_n) is a sequence of immersed complex submanifolds converging to L in the following sense: if $K \Subset L$ is any relatively compact subset (relative to the leafwise topology), then L_n contains a graph over a neighborhood of K for large n , that is there exists a neighborhood $N(K)$ of K in L and a sequence of injective holomorphic maps $\pi_n : N(K) \rightarrow L_n$ such that $\pi_n(x) \rightarrow x$ for every x . Assume further that for every n , L_n is biholomorphic to \mathbb{C} .*

Then the affine structures on the L_n converge to that of L in the following sense: for any compact set $K \Subset L$ as above and any triple $(x, y, z) \in K^3$, if $(x_n, y_n, z_n) \in \pi_n(N(K))$ are close to $(\pi_n(x), \pi_n(y), \pi_n(z))$ and converge to (x, y, z) , then the corresponding ratios converge as well.

The point of this statement is to emphasize that there is no need in Ghys' theorem to work with the leaves of a Riemann surface lamination. Also, compactness of the ambient space is not required. The theorem is certainly not written in its most general form: one might assume more generally that

- ▶ the π_n are $(1 + \varepsilon_n)$ quasi-conformal for some $\varepsilon_n \rightarrow 0$;
- ▶ L and the L_n are parabolic Riemann surfaces instead of copies of \mathbb{C} .

The adaptation is left to the reader. Notice also that any submanifold V of a Stein manifold admits a neighborhood W endowed with a holomorphic retraction $W \rightarrow V$

(see [42, Cor. 1]). Therefore our convergence assumption essentially means that L_n converges to L with multiplicity 1.

Proof. We follow [22, §4] closely. Pick a triple of distinct points (x, y, z) in L and R_0 such that $\psi(D(0, R_0))$ contains x, y, z . For $\alpha \in L$ let $\tilde{\alpha} = \psi^{-1}(\alpha)$. Without loss of generality we may assume $R_0 = 1$. Let R be a large positive number to be determined. For $n \geq n(R)$, π_n is well defined in $\psi(D(0, R))$. Let $(x_n, y_n, z_n) \in \pi_n(D(0, 1))^3$ converging to (x, y, z) , and fix $\varepsilon > 0$. Then by assumption $(\pi_n^{-1}(x_n), \pi_n^{-1}(y_n), \pi_n^{-1}(z_n))$ converges to (x, y, z) for the leafwise topology in L . Let $\psi_n : \mathbb{C} \rightarrow L_n$ be any parameterization, and let $\tilde{x}_n = \psi_n^{-1}(x_n)$, $\tilde{y}_n = \psi_n^{-1}(y_n)$ and $\tilde{z}_n = \psi_n^{-1}(z_n)$. Without loss of generality we may assume $\tilde{x}_n = 0$. We have to show that for large n , the ratio of $(\tilde{x}_n, \tilde{y}_n, \tilde{z}_n)$ is close to that of $(\tilde{x}, \tilde{y}, \tilde{z})$.

By assumption $h_n := \psi_n^{-1} \circ \pi_n \circ \psi : D(0, R) \rightarrow \mathbb{C}$ is an injective holomorphic map. By renormalizing ψ_n we may assume that $h_n'(0) = 1$ (we use $L_n \simeq \mathbb{C}$ precisely here). Then by the Koebe distortion theorem, h_n is almost affine in $D(0, 1)$, that is, it distorts the ratios of points in $D(0, 1)$ by some small amount $\varepsilon(R)$. Fix R so large that $\varepsilon(R) < \varepsilon$. In particular for $n \geq n(R)$ we get that

$$\left| \frac{h_n(\tilde{x}) - h_n(\tilde{y})}{h_n(\tilde{x}) - h_n(\tilde{z})} - \frac{\tilde{x} - \tilde{y}}{\tilde{x} - \tilde{z}} \right| \leq \varepsilon.$$

Now for $\alpha \in K$, $h_n(\tilde{\alpha})$ is the parameter in \mathbb{C} corresponding to $\pi_n(\alpha) \in L_n$, so $\tilde{\alpha}_n$ is close to $h_n(\tilde{\alpha})$ in \mathbb{C} and for large n we also get that

$$\left| \frac{h_n(\tilde{x}) - h_n(\tilde{y})}{h_n(\tilde{x}) - h_n(\tilde{z})} - \frac{\tilde{x}_n - \tilde{y}_n}{\tilde{x}_n - \tilde{z}_n} \right| \leq \varepsilon,$$

and we are done. \square

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