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# On optimality conditions for multivariate Chebyshev approximation and convex optimization

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## Abstract

We review state-of-the-art optimality conditions of multivariate Chebyshev approximation, including, from oldest to newest, Kirchberger's kernel condition, Kolmogorov criteria, Rivlin and Shapiro's annihilating measures. These conditions are then re-interpreted using the optimality conditions of convex optimization, subdifferential and directional derivative. Finally, this new point of view is used to derive new optimality conditions for the following problems: First for the multivariate Chebyshev approximation with a weight function. Second, the approximation problem proposed by Arzelier, Bréhard and Joldes (26th IEEE Symposium on Computer Arithmetic 2019) consisting in minimizing the sum of both the polynomial approximation error and the first order approximation of the worst case evaluation error of the polynomial in Horner form.

**Keywords:** Chebyshev approximation problem, multivariate approximation, alternation conditions, convex optimization, Newton method

## 1 Introduction

Given  $f : X \rightarrow \mathbb{R}$  continuous, we consider the Chebyshev approximation problem

$$\min_{a \in \mathbb{R}^n} \max_{x \in X} \left| \sum_{i=1}^n a_i \phi_i(x) - f(x) \right|, \quad (1)$$

where basis functions  $\phi_i : X \rightarrow \mathbb{R}$  are continuous. The domain  $X$  is assumed to be compact, so that the maximum is well defined. Typically,  $X$  will be a closed bounded subset of  $\mathbb{R}^m$ , the problem (1) consisting in approximating a function of  $m$  variables. The basis function vector  $\phi : X \rightarrow \mathbb{R}^n$  is defined by  $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$ , and  $p(x) = \phi(x)^T a = \sum_{i=1}^n a_i \phi_i(x)$  will be called a generalized polynomial. We define the error  $e(a, x) = \phi(x)^T a - f(x)$  and

$$m(a) = \max_{x \in X} |e(a, x)|, \quad (2)$$

so that the Chebyshev approximation problem is  $\min_{a \in \mathbb{R}^n} m(a)$ . It is well known that this problem has at least one optimal solution provided that  $X$  contains  $n$  points  $x_1, \dots, x_n$  such that the vectors  $\phi(x_i)$  are linearly independent, this condition being assumed. For simplicity, we suppose that  $\min_{a \in \mathbb{R}^n} m(a) > 0$ , the other case being trivial and leading to vacuous optimality conditions.

Optimality conditions play a central role in understanding an optimization problem, as well as in finding formally or numerically optimal solutions. The first optimality condition for Chebyshev approximation problems is the celebrated equioscillation theorem of Chebyshev, which characterizes optimal solution for basis functions satisfying the Haar condition. Its scope is restricted to approximation of univariate function: Mairhuber [2] proved that the Haar condition cannot hold for multivariate approximation. Nevertheless, several conditions have been proposed in the context of multivariate approximation. State-of-the-art optimality conditions for multivariate Chebyshev approximation problems are reviewed in Section 2, including signatures based conditions, like Kolmogorov criteria and Rivlin and Shapiro annihilating measures, and basis function vector based optimality conditions, like Kirchner kernel condition and Cheney's convex hull condition. In the context of nonsmooth convex optimization, the objective function  $m(a)$  is called a pointwise supremum. Related optimality conditions are reviewed in Section 3, and their relationship with the previous optimality conditions is shown. Finally in Section 4, some new optimality conditions are proposed for two classes of approximation problems: First in Section 4.1, multivariate approximation problems with weight function, e.g., relative approximation error. Second in Section 4.2, approximation problems introduced by Arzelier, Bréhard and Joldes in [3], which consists in minimizing the sum of the approximation error and the first order approximation of worst case the polynomial evaluation in Horner form. This newly introduced optimality condition is illustrated inside a Newton operator applied on a numerical example. In both cases, convex optimization optimality conditions are used to derive these new optimality conditions in a simple way, in particular taking advantage of standard subdifferential calculation rules.

## 2 Review of optimality conditions for multivariate Chebyshev optimization

Optimality conditions of Chebyshev approximation problems rely on the extreme points of the error  $e(a, x) = \phi(x)^T a - f(x)$ , the set of these extreme points being denoted by  $\text{ext}(a) = \{x \in X : |e(a, x)| = m(a)\}$ . Optimality conditions belong to

two classes: the first class relies on signatures, which associate the sign of the error to each extreme point, the second class relies on basis function vectors evaluated at extreme points. Some conditions actually apply to approximation problems in complex variables, namely Kolmogorov criterion, Rivlin and Shapiro's annihilating measure condition and Cheney's convex hull condition, but we are concerned here only by their real counterparts.

## 2.1 Conditions based on signatures

A signature is subsets of  $X \times \{-1, 1\}$ , which associates a sign to elements of a subset of  $X$ , called the support of the signature. There is a special signature  $\Sigma(a)$  associated to a generalized polynomial  $p(x) = \phi(x)^T a$ , which associates the sign of the error to its extrema:

$$\Sigma(a) = \left\{ \left( x, \frac{e(a,x)}{|e(a,x)|} \right) \in X \times \{-1, 1\} : x \in \text{ext}(a) \right\}. \quad (3)$$

We say that  $\Sigma \subseteq X \times \{-1, 1\}$  is a signature of the generalized polynomial if  $\Sigma \subseteq \Sigma(a)$ .

Signatures of generalized polynomial are of great significance because they contain enough information to characterize exactly the optimality of Chebyshev approximation problems. The criteria that allows deciding if a signature correspond to an optimal solution of the associated Chebyshev problem are of great theoretical and practical importance. For exemple, in the context of univariate Chebyshev optimization, the equioscillation theorem says that signatures containing  $n + 1$  extreme points associated to oscillating signs correspond to optimal solutions. The two main criteria for such signatures in the context of multivariate Chebyshev approximation are Kolmogorov criterion and Rivlin and Shapiro's annihilating measures, presented in the next two subsections. Rivlin and Shapiro have proved that both criteria are equivalent [4, Remark 1 and Remark 2 page 677], hence we can use Rivlin and Shapiro's extremal signature denomination for a signature that satisfies one of them. Finally, the last subsection briefly presents the case of approximation by affine functions, which enjoys a simple signature based optimality condition.

### 2.1.1 Kolmogorov criterion

The most fundamental optimality condition is the so-called Kolmogorov criterion [5]. We say that a generalized polynomial  $p(x) = \phi(x)^T a$  is synchronized with a signature  $\Sigma$  if, and only if, it has the same sign as the signature on its support. Then Kolmogorov criterion says that the generalized polynomial  $\bar{p}(x) = \phi(x)^T \bar{a}$  is optimal if and only if its signature  $\Sigma(\bar{a})$  prevents synchronization with any generalized polynomial:

**Theorem 1** (Kolmogorov criterion). *The generalized polynomial  $\bar{p}(x) = \phi(x)^T \bar{a}$  is optimal for (1) if and only if no generalized polynomial  $p(x) = \phi(x)^T a$  satisfy*

$$\forall (x, s) \in \Sigma(\bar{a}), \quad s p(x) > 0. \quad (4)$$

Many authors use the statement (4) of Kolmogorov criteria, e.g.: Rice [1, page 446] says that extreme points satisfying this statement are isolable; Shapiro [6, Lemma 2.2.1 page 7] proves Kolmogorov's criterion; Powel proves the same criterion in [7, Equation (7.6) page 74] and uses it to prove Chebyshev's alternation theorem. An

equivalent statement of Kolmogorov criteria is that  $\min_{x \in \text{ext}(\bar{a})} e(\bar{a}, x) p(x) \leq 0$  holds for all generalized polynomials, e.g. [8], or again equivalently

$$\max_{x \in \text{ext}(\bar{a})} e(\bar{a}, x) p(x) \geq 0 \quad (5)$$

holds for all generalized polynomials, e.g. [9].

Identifying signature patterns preventing any synchronization with them allows deriving practical optimality conditions. For example, in the special case of univariate polynomial approximation, signature that prevent synchronization with any polynomial of degree  $n - 1$  are exactly those containing  $n + 1$  points with alternating sign (a degree  $n - 1$  polynomial with  $n + 1$  alternating signs has  $n$  zeros and is therefore null, preventing any synchronization (4)), leading to Chebyshev equioscillation theorem. Chebyshev equioscillation theorem is extended to univariate generalized polynomial with basis functions satisfying the Haar condition using the same argument.

In the case of multivariate approximation, identifying signature patterns preventing the synchronization (4) is more difficult. Shapiro [10] calls these patterns Chebyshev patterns, and he uses the Euler-Jacobi formula to deduce some signature patterns preventing synchronization. E.g., with  $X \subseteq \mathbb{R}^2$  a signature having  $2n$  points on an ellipse with alternating signs when traveling around the ellipse does prevent any synchronization with any polynomial of degree  $n - 1$ . Gearhart [11] deduces from this condition the optimal approximation over the unit disk of two variables monomials  $x_1^n x_2^m$  by polynomials of degree  $n + m - 1$ , extending to multivariate the ways and means of Chebyshev polynomials.

### 2.1.2 Rivlin and Shapiro's annihilating measures

Rivlin and Shapiro [4, Theorem 2 page 678] (see also the lecture notes [6, Main Theorem page 14]) have proposed another characterization of optimality based on signatures:

**Theorem 2** (Rivlin and Shapiro annihilating measure condition). *The generalized polynomial  $\bar{p}(x) = \phi(x)^T \bar{a}$  is optimal for (1) if and only if it has a finite signature  $\{(x_1, s_1), \dots, (x_m, s_m)\} \subseteq \Sigma(\bar{a})$  for which there exists  $c_i > 0$  such that*

$$\sum_{i=1}^m c_i s_i p(x_i) = 0 \quad (6)$$

*holds for all generalized polynomial  $p(x) = \phi(x)^T a$ .*

It is convenient to interpret the  $c_i$  as a measure whose support is the finite subset of extreme points, and a measure satisfying (6) for all generalized polynomial is said to annihilate the subspace of the generalized polynomial. As said previously, the existence of the annihilating measure is equivalent to Kolmogorov criterion. It is less intuitive but brings more information: for example, as shown in [12, proof of Theorem 1], a direct consequence of the existence of this annihilating measure is that all optimal

generalized polynomial have the same extreme points<sup>1</sup>, while this fact seems hard to prove directly from Kolmogorov criterion.

The relationship between annihilating measures and Chebyshev equioscillation theorem is technical, see [4, page 681 and 682], and not presented here. A relationship between all conditions, including annihilating measures condition, is described in Section 3.2.3 below.

### 2.1.3 Approximation by affine functions

Approximation by affine functions enjoys an elegant optimality condition based on signatures. This line was started by Collatz [13], recognized by many authors in spite of the difficulty of obtaining the original texts nowadays. The classification was further refined by [14], and the final formulation is given by Rivlin and Shapiro [4, Remark page 697] as a special case of their intersecting convex hull condition [4, Theorem 4], which presented in Section 2.2 below: with  $\phi(x) = (1, x_1, \dots, x_n)$ , the affine function  $\bar{p}(x) = \phi(x)^T \bar{a}$  is an optimal Chebyshev approximation of  $f(x)$  if and only if the convex hull of  $\text{ext}^+(\bar{a})$  intersects the convex hull of  $\text{ext}^-(\bar{a})$ , where  $\text{ext}^\pm(\bar{a})$  contain extreme points with positive and negative error respectively.

Rivlin and Shapiro [4, Problem 4 page 697] deduce from this condition an elegant formal solution to the Chebyshev approximation problem that consists in finding the best affine approximation of  $f(x) = \sum_{i=1}^m x_i^2$  on an arbitrary compact convex set  $X \subseteq \mathbb{R}^m$ . To this end, one needs only the minimal radius sphere circumscribing  $X$ , say with center  $c \in \mathbb{R}^m$  and radius  $r > 0$ . Then the best affine Chebyshev approximation is

$$p(x) = \sum_{i=1}^m x_i^2 - \sum_{i=1}^m (x_i - c_i)^2 + \frac{1}{2}r^2, \quad (7)$$

which is affine because quadratic terms cancel each other exactly.

## 2.2 Conditions based on basis function vectors

The following conditions need not only the error extreme points  $x \in \text{ext}(a)$  and their error sign, but also the basis functions evaluation  $\phi(x)$  at extreme points. The advantage of using this additional information is the easiness of checking them with respect to previous signature based conditions: they consist in checking if 0 is in the convex hull of a finite set of vectors, or checking if a matrix has a kernel vector with some sign pattern. The two classes of conditions are strongly related to each other: indeed, 0 is in the convex hull of some vectors if and only if the matrix whose columns are these vectors has a non trivial kernel vector with non-negative components.

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<sup>1</sup>Consider two optimal generalized polynomials  $p_0(x) = a_0^T \phi(x)$  and  $p_1(x) = a_1^T \phi(x)$  with maximal error  $m$ , and the annihilating measure  $c$  for  $p_0$ , which annihilates in particular  $p_0 - p_1$ :  $\sum_{i=1}^m c_i s_i (p_0(x_i) - p_1(x_i)) = 0$ . So by adding  $0 = c_i s_i f(x_i) - c_i s_i f(x_i)$  we have  $\sum_{i=1}^m c_i (s_i (p_0(x_i) - f(x_i)) - s_i (p_1(x_i) - f(x_i))) = 0$ . Note that  $s_i (p_0(x_i) - f(x_i)) = m$  because  $x_i$  is an extreme point for  $p_0(x)$ , and  $|s_i (p_1(x_i) - f(x_i))| \leq m$  because  $p_1(x)$  is optimal, so their difference is non-negative, which together with  $c_i > 0$  prove that the difference is actually null. As a consequence  $p_0(x_i) = p_1(x_i) = m$  so  $x_i$  are extreme points for  $p_1(x)$  too.

### 2.2.1 Kirchberger's kernel condition

The following condition is granted to the 1903 work of Kirchberger [15] by Watson's historical paper [16], where the statement is presented for multivariate discrete problems and to continuous univariate problems. On the other hand, the historical book [17] credits to Kirchberger some general multivariate conditions but does not show any condition explicitly. Kirchberger's old German style paper [15] did not allow the author to clarify the situation. The same statement appears in [18], presented as a straightforward reformulation of Cheney's Characterization Theorem (see Section 2.2.2 below), but again restricted to univariate problems, although Cheney's Characterization Theorem actually applies to multivariate problems (see next section).

The modern statement [16, 18] uses a Haar matrix evaluated at some points in  $X$ :

$$H(x_1, \dots, x_k) = (\phi(x_1) \ \phi(x_2) \ \cdots \ \phi(x_k)) \in \mathbb{R}^{n \times k}. \quad (8)$$

In the special case of univariate polynomial approximation, i.e.,  $\phi(x) = (1, x, x^2, \dots, x^{n-1})$ , the Haar matrix becomes a Vandermonde matrix

$$H(x_1, \dots, x_k) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & \cdots & x_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_k^{n-1} \end{pmatrix} \in \mathbb{R}^{n \times k}. \quad (9)$$

Then, a generalized polynomial  $\bar{p}(x) = \phi(x)^T \bar{a}$  is optimal if and only if the Haar matrix  $H(x_1, \dots, x_k)$  evaluated at some extreme points  $x_i \in \text{ext}(\bar{a})$  has a non-trivial kernel vector  $u \in \mathbb{R}^k$  whose component signs match the error signs:

**Theorem 3** (Kirchberger's condition). *The generalized polynomial  $\bar{p}(x) = \phi(x)^T \bar{a}$  is optimal for (1) if and only if there exist  $x_1, \dots, x_k \in \text{ext}(\bar{a})$  such that*

$$\exists u (\neq 0) \in \mathbb{R}^k, \ H(x_1, \dots, x_k) u = 0, \ e(a, x_i) u_i \geq 0 \text{ for all } i \in \{1, \dots, k\}. \quad (10)$$

Obviously, extreme points for which  $u_i = 0$  are useless in the characterization. More precisely, as noted in [8, 18], the condition can be sharpened: the set of extreme points used in (13) can always be selected so that it is minimal, in the sense that no strict subset satisfies the same condition. In this case, we have that  $H(x_1, \dots, x_k)$  is rank  $k - 1$ , i.e., has a kernel of dimension 1, and therefore  $k \leq n + 1$ ,  $u_i > 0$  and  $e(a, x_i) u_i > 0$  hold for all  $i \in \{1, \dots, k\}$ .

The next example shows how this condition can be used to find the formal solution of some simple multivariate Chebyshev approximation problem consisting of approximating a quadratic function by affine functions. It is less general than Rivlin and Shapiro's formal solution given in Section 2.1.3, but Kirchberger's condition is far easier to apply in this example.

**Example 1.** *Let  $X = [0, 1]^m \subseteq \mathbb{R}^m$ ,  $f(x) = x_1^2 + \cdots + x_m^2$  and  $\phi(x) = (1, x_1, \dots, x_m)$ , so that we approximate  $f$  by affine functions  $\phi(x)^T a = a_0 + a_1 x_1 + \cdots + a_m x_m$ . Let  $\bar{p}(x) = \phi(x)^T \bar{a}$  be the optimal polynomial. The error function  $e(\bar{a}, x) = \phi(x)^T \bar{a} - f(x)$*

is concave, therefore the error minimizers must be at the corners of  $X$ . Because of the symmetry of  $f$ , we guess that all corners will be extreme points for the optimal approximation:  $e(\bar{a}, 0) = a_0 = -m^*$ , where  $m^* = m(\bar{a})$ , and  $e(\bar{a}, u_i) = a_0 + a_i - 1 = -m^*$ , where  $u_i$  is the  $i^{\text{th}}$  basis vector, from which we deduce that  $a_1 = a_2 = \dots = a_m = 1$ . For finding  $a_0 = -m^*$ , we guess that the last extreme point  $x^*$  is a maximizer in the interior of  $X$ . Hence  $\nabla_x e(\bar{a}, x^*) = 0$  from which we deduce  $a_i - 2x_i^* = 0$  so  $x_i^* = \frac{1}{2}$  and eventually  $a_0 = -m^* = -e(\bar{a}, x^*) = -a_0 - \frac{m}{2} + \frac{m}{4}$ , and finally  $a_0 = -\frac{m}{8}$ .

We now use Kirchberger's condition to check if  $\bar{p}(x) = -\frac{m}{8} + x_1 + \dots + x_n$  is optimal. We consider the Haar matrix evaluated for the  $2^n + 1$  extreme points, e.g., for  $n = 3$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \frac{1}{2} \end{pmatrix}. \quad (11)$$

We can observe and generalize to arbitrary  $n$  that each line contains exactly  $\frac{2^n}{2}$  zeros and as many ones, and  $\frac{1}{2}$  at the end. As a consequence, the vector  $(-1, \dots, -1, 2^n)$  is a kernel vector, which satisfies  $u_i e(\bar{a}, x_i) \geq 0$  and proves that  $\bar{a}$  is optimal.

Following Cheney's argument [19, page 75], Chebyshev equioscillation theorem is deduced from Kirchberger's condition using the following lemma, which is a kernel reformulation of the convex hull lemma [19, Lemma page 74]:

**Lemma 4.** *With  $X = [\underline{x}, \bar{x}] \subseteq \mathbb{R}$  and the Haar condition holds for  $\phi$ , any Haar matrix  $H(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n \times (n+1)}$  with  $x_i < x_{i+1}$  has a one dimensional kernel with a kernel vector  $0 \neq u \in \mathbb{R}^{n+1}$  satisfying  $u_i u_{i+1} < 0$ .*

Indeed in view of Lemma 4,  $\underline{x} \leq x_1 < \dots < x_{n+1} \leq \bar{x}$  are extreme points satisfying  $e(a, x_i) e(a, x_{i+1}) < 0$  if and only if  $H(x_1, \dots, x_k)$  has a kernel vector  $0 \neq u$  satisfying  $e(a, x_i) u_i > 0$ .

### 2.2.2 Cheney's convex hull condition

Cheney's Characterization Theorem [19, page 73] is a convex hull condition: a generalized polynomial  $\bar{p}(x) = \phi(x)^T \bar{a}$  is optimal if and only if  $0 \in \text{conv}\{e(\bar{a}, x)\phi(x) : x \in \text{ext}(\bar{a})\}$ . Here  $\text{ext}(\bar{a})$  could be infinite, but by Carathéodory theorem only a finite set of extreme points is actually necessary. We emphasize here the equivalent kernel condition for further comparison to other conditions. It involves the matrix

$$C(x_1, \dots, x_k) = (e(a, x_1)\phi(x_1) \ e(a, x_2)\phi(x_2) \ \dots \ e(a, x_k)\phi(x_k)) \in \mathbb{R}^{n \times k}. \quad (12)$$

The kernel version of Cheney's condition is that the matrix  $C(x_1, \dots, x_k)$  evaluated at some extreme points  $x_i \in \text{ext}(\bar{a})$  has a non-trivial kernel vector  $u \in \mathbb{R}^k$  with non-negative components:

**Theorem 5** (Cheney's condition). *The generalized polynomial  $\bar{p}(x) = \phi(x)^T \bar{a}$  is optimal for (1) if and only if there exist  $x_1, \dots, x_k \in \text{ext}(\bar{a})$  such that*

$$\exists u(\neq 0) \in \mathbb{R}^k, \ C(x_1, \dots, x_k) u = 0, \ u_i \geq 0 \text{ for all } i \in \{1, \dots, k\}. \quad (13)$$

Its connection to Kirchberger's connection is obvious: the columns of  $C(x_1, \dots, x_k)$  are the same as the columns of  $H(x_1, \dots, x_k)$  but multiplied by  $e(a, x_i)$ , hence the components of the kernel vectors are also multiplied by  $e(a, x_i)$ : indeed, with  $E \in \mathbb{R}^{k \times k}$  the diagonal matrix whose entries are  $e(a, x_i)$ , we have  $C(x_1, \dots, x_k) = H(x_1, \dots, x_k) E$  and  $0 = C(x_1, \dots, x_k)u = (H(x_1, \dots, x_k) E)u = H(x_1, \dots, x_k)(Eu)$ . As a consequence, non-negativeness of the components of a kernel vector of  $C(x_1, \dots, x_k)$  is equivalent to a kernel vector of  $H(x_1, \dots, x_k)$  having the same sign of the error for the last.

Rivlin and Shapiro proved that this condition is necessary [4, Theorem 1 page 672]. As said previously, Osborn and Watson identifies Cheney's condition with Kirchberger's condition in [18].

### 2.2.3 Rivlin and Shapiro's intersecting convex hull condition

In typical the case where the set of basis functions includes a constant function, Rivlin and Shapiro [4, Theorem 4] used their annihilating measure condition to prove the following more practical condition: the generalized polynomial  $p(x) = \phi(x)^T a$  is optimal if and only if

$$\text{conv}\{\phi(x) : x \in \text{ext}^+(a)\} \cap \text{conv}\{\phi(x) : x \in \text{ext}^-(a)\} \neq 0, \quad (14)$$

where as previously  $\text{ext}^\pm(\bar{a})$  contain extreme points with positive and negative error respectively. This condition was rediscovered by Sukhorukova et al. [20, Theorem 2].

## 3 Convex optimization optimality conditions

In this section, the optimality conditions of convex optimization is applied to Chebyshev approximation, leading to two conditions given in Section 3.1. Their relationships to previously presented classical conditions are shown Section 3.2.

### 3.1 Convex optimization optimality conditions for Chebyshev approximation

The function  $m : \mathbb{R}^n \rightarrow \mathbb{R}$  in (2) is called a pointwise supremum, and it's properties are well known in the context of nonsmooth convex optimization. The reader is referred to [21, 22] for an introduction to convex optimization ([21, Section 5.3 page 198] includes some basic fact about uniform approximation in the framework of convex optimization, including a simplified version of Chebyshev equioscillation theorem). The function  $e(a, x)$  is linear with respect to  $a$  and therefore  $|e(a, x)|$  is convex and so is  $m(a)$ , which is therefore continuous. Subgradients generalize gradients in the sense that they provide affine underestimators of convex functions:  $g \in \mathbb{R}^n$  is a subgradient of  $m$  at  $\bar{a}$  if and only if  $m(a) \leq m(\bar{a}) + g^T(a - \bar{a})$ . The subdifferential of  $m$  at  $a$ , denoted by  $\partial m(a)$ , is the set of all subgradients at  $a$ , which is nonempty, compact and convex. The subdifferential of the pointwise supremum  $m(a)$  enjoys a simple explicit expression:

$$\partial m(a) = \text{conv}\{\partial_a |e(a, x)| : x \in \text{ext}(a)\}. \quad (15)$$

**Remark.** Formulas for the subdifferential of a pointwise maximum dates from the 70's, and are now textbook classic. Several versions exist with different assumptions, the one used here, [21, Theorem 4.4.2 page 189], is the simplest and is restricted to real-valued convex functions defined in  $\mathbb{R}^n$ , and requires only that  $X$  is compact and the functions  $x \mapsto |e(a, x)|$  is upper semi-continuous for all  $a \in \mathbb{R}^n$ . Several extensions exists, where the convex function has values in  $\mathbb{R}^n \cup \{+\infty\}$  [23, Proposition A22 page 154], encoding a domain for the function, and to infinite dimensional spaces. See [21, page 247] for some historical details about this formula.

Since we assume  $\min_{a \in \mathbb{R}^n} m(a) > 0$ , we have that  $|e(a, x)|$  differentiable for all  $a \in \mathbb{R}^n$  and  $x \in \text{ext}(a)$ , and therefore  $\partial e(a, x) = \{\epsilon(a, x) \phi(x)\}$  with  $\epsilon(a, x) = \text{sign}(e(a, x)) \in \{-1, 1\}$ . Finally we have

$$\partial m(a) = \text{conv}\{\epsilon(a, x) \phi(x) : x \in \text{ext}(a)\}. \quad (16)$$

The vectors  $\epsilon(a, x) \phi(x)$  for extreme points  $x \in \text{ext}(a)$  are subgradients, of special kind since they generate all other subgradient by convex combinations.

The subdifferential gives rise to two equivalent characterizations of optimality. Firstly,  $\bar{a}$  is a minimizer of  $m(a)$  if and only if  $0 \in \partial m(\bar{a})$ . The directional derivative  $m'(a, u)$  of  $m$  in the direction  $u \in \mathbb{R}^n$  gives rise to a second optimality condition. It can be computed using the subdifferential [21, Theorem 4.4.2 page 189] by

$$m'(a, u) = \max\{u^T g : g \in \partial m(a)\}. \quad (17)$$

Using (16) and (17), and noting that a linear function has the same extrema over a set and over its convex hull, we obtain the following expression of the subdifferential:

$$m'(a, u) = \max\{\epsilon(a, x) u^T \phi(x) : x \in \text{ext}(a)\}. \quad (18)$$

Finally, the second classical optimality condition is that  $\bar{a}$  is a minimizer of  $m(a)$  if and only if  $m'(\bar{a}, u) \geq 0$  for all directions  $u \in \mathbb{R}^n$ . To summarize we have:

**Proposition 6.** *Let  $m : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. For an arbitrary  $\bar{a} \in \mathbb{R}^n$ , the following three conditions are equivalent:*

- (1)  $\forall a \in \mathbb{R}^n, m(a) \geq m(\bar{a})$ ;
- (2)  $0 \in \partial m(\bar{a})$ ;
- (3)  $\forall u \in \mathbb{R}^n, m'(\bar{a}, u) \geq 0$ .

The kernel condition corresponding to zero in (16) involves the matrix

$$G(x_1, \dots, x_k) = (\epsilon(a, x_1)\phi(x_1) \ \epsilon(a, x_2)\phi(x_2) \ \dots \ \epsilon(a, x_k)\phi(x_k)) \in \mathbb{R}^{n \times k}, \quad (19)$$

where  $x_k \in \text{ext}(a)$ , so that its columns are subgradients at  $a$ . Then  $\bar{a}$  is optimal if and only if there exist extreme points  $x_1, \dots, x_k \in X$  such that the matrix  $G(x_1, \dots, x_k)$  has a nontrivial kernel vector with non-negative components.

Finally, the following proposition is a stronger version of Proposition 6, which shows some characterization of strong uniqueness in the general framework of convex

optimization. The statement is folklore for experts of the field, the proof provided in appendix for completeness.

**Proposition 7.** *Let  $m : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. For arbitrary  $\bar{a} \in \mathbb{R}^n$  and  $r \geq 0$ , the following three conditions are equivalent*

- (1)  $\forall a \in \mathbb{R}^n, m(a) \geq m(\bar{a}) + r \|a - \bar{a}\|_2$ ;
- (2)  $B_r \subseteq \partial m(\bar{a})$ , where  $B_r$  is the radius  $r$  ball;
- (3)  $\forall u \in \mathbb{R}^n, m'(\bar{a}, u) \geq r \|u\|_2$ .

## 3.2 Relationship to classical optimality conditions

We show how the subgradient conditions allows reformulating advantageously previously presented classical conditions, firstly Kirchberger and Cheney's conditions which are very close to each other, then Kolmogorov criterion, Rivlin and Shapiro's annihilating measure conditions, and finally the intersecting convex hull condition.

### 3.2.1 Kirchberger and Cheney's conditions

Subgradient, Kirchberger and Cheney's conditions are all about the kernel vector of a matrix whose columns are proportional to  $\phi(x)$  for some extreme points  $x \in \text{ext}(a)$ . It is easy to see that they are all equivalent: matrices differing by constant factor on their columns have respective kernel vectors whose components are multiplied by the same constants. Formally, kernel vectors  $u, v, w \in \mathbb{R}^k$  of  $C(x_1, \dots, x_k)$ ,  $G(x_1, \dots, x_k)$  and  $H(x_1, \dots, x_k)$  satisfy respectively

$$\sum_{i=1}^k u_i \left( e(a, x_i) \phi(x_i) \right) = 0, \quad \sum_{i=1}^k v_i \left( \epsilon(a, x_i) \phi(x_i) \right) = 0 \quad \text{and} \quad \sum_{i=1}^k w_i \phi(x_i) = 0. \quad (20)$$

The kernels of these matrices are obviously related by  $u_i e(a, x_i) = v_i \epsilon(a, x_i) = h_i$ , from which follows that having a positive kernel vector is equivalent for  $C(x_1, \dots, x_k)$  and  $G(x_1, \dots, x_k)$ , while this is equivalent to  $H(x_1, \dots, x_k)$  having a kernel vector whose components sign match the sign of the corresponding errors.

Although being so close to each other, the subgradient condition presents the advantage that they are related to affine underestimators. For example they easily lead to a sufficient condition for the uniqueness of the minimizer: if  $G(x_1, \dots, x_k)v = 0$  with  $v_i \geq 0$  for some  $x_1, \dots, x_k \in \text{ext}(\bar{a})$  then  $\bar{a}$  is a minimizer and

$$m(a) \geq \max_{i \in \{1, \dots, k\}} m(\bar{a}) + g_i^T (a - \bar{a}) = m(\bar{a}) + l(a - \bar{a}), \quad (21)$$

where  $g_i = \epsilon(\bar{a}, x_i) \phi(x_i)$  the columns of  $G(x_1, \dots, x_k)$  and hence subgradients of  $m$  at  $\bar{a}$ , and where  $l(a - \bar{a}) = \max_{i \in \{1, \dots, k\}} g_i^T (a - \bar{a})$ . This finite pointwise linear maximum lower bound can be further characterized if we make the additional assumptions that  $G(x_1, \dots, x_k)$  is full rank (which implies that  $k \geq n + 1$ ) and that  $v_i > 0$  for all  $i \in \{1, \dots, k\}$ : since  $v_i > 0$  Stiemke's alternative theorem [24] applies and proves that for an arbitrary  $(a - \bar{a})$  if  $g_i^T (a - \bar{a}) \leq 0$  holds for all  $i \in \{1, \dots, k\}$  then in fact  $g_i^T (a - \bar{a}) = 0$  holds for all  $i \in \{1, \dots, k\}$ . Now since  $G(x_1, \dots, x_k)$  is full rank, there

are  $n$  linearly independent vectors  $g_i$  and we conclude that  $(a - \bar{a}) = 0$ . We have proved that  $g_i^T(a - \bar{a}) \leq 0$  cannot hold for all  $i \in \{1, \dots, k\}$  for  $(a - \bar{a}) \neq 0$ . Hence  $g_i^T(a - \bar{a}) > 0$  must hold for some  $i \in \{1, \dots, k\}$ , and  $m(a) > m(\bar{a})$ . Since this holds for all  $a \neq \bar{a}$ , the minimizer is unique. In fact, it is not hard to prove that the unique minimizer is furthermore strongly unique: the maximum is homogenous for positive scalar hence we have

$$l(a - \bar{a}) = \|a - \bar{a}\|_1 l\left(\frac{a - \bar{a}}{\|a - \bar{a}\|_1}\right) \geq \alpha \|a - \bar{a}\|_1 \quad (22)$$

with  $\alpha = \min_{\|a - \bar{a}\|_1 = 1} l(a - \bar{a})$ . We have just proved that  $l(a - \bar{a}) > 0$  for all  $a - \bar{a} \neq 0$ , so  $\alpha > 0$  since the unit ball is compact. Finally,  $m(a) \geq m(\bar{a}) + \alpha \|a - \bar{a}\|_1$  for some  $\alpha > 0$ , which proves the strong uniqueness. A similar strong uniqueness sufficient condition was given in [8, Theorem 3], the proof here based on subgradient being arguably simpler.

Subgradients can also be useful in practical algorithms, e.g., for finding descent directions or in subgradients algorithms. Interestingly, subgradient algorithms with memory, like Kelly's Method [22, Section 3.2.2 page 226], applied to the minimization of  $m(a)$  are closely related to Remez's algorithm and the inner .

### 3.2.2 Kolmogorov criterion

Comparing Kolmogorov criterion (5) and subdifferential expression of the directional derivative (18) we see that they are closely related: for an arbitrary  $p(x) = u^T \phi(x)$  we have

$$\max_{x \in \text{ext}(\bar{a})} e(\bar{a}, x) p(x) = m(\bar{a}) \max_{x \in \text{ext}(\bar{a})} \epsilon(\bar{a}, x) p(x) \quad (23a)$$

$$= m(\bar{a}) \max_{x \in \text{ext}(\bar{a})} u^T (\epsilon(\bar{a}, x) \phi(x)) \quad (23b)$$

$$= m(\bar{a}) \max_{g \in \partial m(\bar{a})} u^T g \quad (23c)$$

$$= m(\bar{a}) m'(\bar{a}, u). \quad (23d)$$

Since  $m(\bar{a}) > 0$  by assumption, Kolmogorov criterion, i.e.,  $(\max_{(x,s) \in \Sigma} s p(x)) \geq 0$  holds for all generalized polynomial  $p(x)$ , says exactly that the directional derivative is non-negative in all directions, the standard optimality condition of convex optimization.

Considering now the strong optimality conditions given by Proposition 7, we have

$$\forall a \in \mathbb{R}^n, m(a) \geq m(\bar{a}) + r \|a - \bar{a}\|_2 \quad (24)$$

if and only if  $\forall u \in \mathbb{R}^n, m'(\bar{a}, u) \geq r \|u\|_2$ . By multiplying both sides of this latter inequality by  $m(\bar{a})$  we obtain that (24) holds if and only if

$$\forall u \in \mathbb{R}^n, \max_{x \in \text{ext}(\bar{a})} e(\bar{a}, x) u^T \phi(x) \geq r m(\bar{a}) \|u\|_2, \quad (25)$$

which is the generalized Kolmogorov criterion for strong uniqueness given in [25, Theorem 5], here with the 2-norm.

### 3.2.3 Rivlin and Shapiro's annihilating measures

Simple computations show that Rivlin and Shapiro's annihilating measure condition is closely related to the subgradient condition: the annihilating measure condition is that the generalized polynomial  $\bar{p}(x) = \phi(x)^T a$  has a signature  $\{(x_1, s_1), \dots, (x_m, s_m)\} \subseteq \text{ext}(a) \times \{0, 1\}$  for which there exist  $c_1, \dots, c_m > 0$  such that

$$\forall a \in \mathbb{R}^n, \sum_{i=1}^m c_i s_i \phi(x_i)^T a = 0. \quad (26)$$

Now, one has

$$\sum_{i=1}^m c_i s_i \phi(x_i)^T a = \left( \sum_{i=1}^m c_i s_i \phi(x_i) \right)^T a \quad (27)$$

$$= (G(x_1, \dots, x_m)c)^T a, \quad (28)$$

which is null for all  $a \in \mathbb{R}^n$  if and only if  $G(x_1, \dots, x_m)c = 0$ . Hence, the measure weights  $c_i$  in the annihilating measure condition are exactly the components of the kernel vector in the subgradient kernel condition.

### 3.2.4 Rivlin and Shapiro's intersecting convex hull condition

The intersecting convex hull condition of Rivlin and Shapiro can be obtained by a simple rewriting of the zero in subgradient convex hull condition that can be performed when one basis function is constant. Without loss of generality, we suppose that  $\phi_1(x) = 1$ . The subgradient convex hull condition  $\sum_{i=1}^m \lambda_i \epsilon(\bar{a}, x_i) \phi(x_i) = 0$ , for some nonnegative  $\lambda_i$  not all zero, can be rewritten so that contributions of positive and negative errors are grouped together:

$$\sum_{i \in I_+} \lambda_i \phi(x_i) = \sum_{i \in I_-} \lambda_i \phi(x_i), \quad (29)$$

with  $I_{\pm} = \{i : \epsilon(\bar{a}, x_i) = \pm 1\}$ . Since  $\phi_1(x) = 1$  we must have  $\sum_{i \in I_+} \lambda_i = \sum_{i \in I_-} \lambda_i = \frac{1}{2} \|\lambda\|_1$ , and dividing both sides of (29) by  $\frac{1}{2} \|\lambda\|_1$  shows it expresses exactly that  $\text{conv}\{\phi(x_i) \in \mathbb{R}^n : i \in I_+\}$  and  $\text{conv}\{\phi(x_i) \in \mathbb{R}^n : i \in I_-\}$  have a nonempty intersection.

## 4 New optimality conditions

### 4.1 Multivariate Chebyshev approximation with weight function

We consider the weighted Chebyshev approximation problem  $\min_{a \in \mathbb{R}^n} m_w(a)$  with

$$m_w(a) = \max_{x \in X} |w(x)(\phi(x)^T a - f(x))|, \quad (30)$$

where the weight function  $w(x)$  is supposed positive. As previously, we suppose for simplicity that the minimum is not zero. Extreme points  $\text{ext}_w(a)$  are now extrema of the weighted error  $w(x)(\phi(x)^T a - f(x))$ , and signatures associate the sign of the weighted error to its extrema. The following theorem shows optimality with a weight function is characterized by usual extremal signatures, the framework of convex analysis offering a simple proof of this statement.

**Theorem 8.** *The generalized polynomial  $\bar{p} = \phi(x)^T \bar{a}$  is an optimal solution to the weighted Chebyshev problem if and only if it has an extremal signature, in the usual sense of Kolmogorov.*

*Proof.* Since  $w(x)$  is nonnegative,  $m_w(a)$  is a pointwise supremum of convex functions, hence convex. Its subdifferential is

$$\partial m_w(a) = \text{conv} \{ \partial [ |w(x)(\phi(x)^T a - f(x))| ] : x \in \text{ext}_w(a) \} \quad (31)$$

$$= \text{conv} \{ w(x)\epsilon(a, x)\phi(x) : x \in \text{ext}_w(a) \}, \quad (32)$$

the second equality holds by standard computation rules of subgradients. We finally see that the directional derivative  $m'_w(a, u) = \max_{x \in \text{ext}_w(a)} w(x)\epsilon(a, x)\phi(x)^T u$  is nonnegative if and only if  $\max_{x \in \text{ext}_w(a)} \epsilon(a, x)\phi(x)^T u$  is nonnegative, which is Kolmogorov criterion.  $\square$

In the context of univariate approximation, Dunham [26] shows that equioscillation is necessary and sufficient for nonlinear Chebyshev approximation problems with more general ordering functions, provided that the nonlinear approximation problem enjoys an equioscillation characterization of its optimal solutions. Generalizing Theorem 8 in such directions may be possible by using Clarke generalized gradients [27], which enjoy a formula for pointwise supremum functions similar to the one of subgradients [27, Theorem 2.1 page 251].

### 4.2 Optimizing with polynomial evaluation error

In this section, we consider univariate polynomial approximation with  $\phi : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}^n$  and  $\phi(x) = (1, x, x^2, \dots, x^{n-1})$ . Remind that  $n$  is the number of functions in the basis, hence  $\phi(x)^T a = \sum_{i=1}^n a_i x^{i-1}$ , which is an unusual way of writing a polynomial but will be a useful notation below. Arzelier, Bréhard and Joldes [3] proposed to minimize the evaluation error of a polynomial together with its approximation error. We restrict our attention to evaluation of polynomials using the Horner form, whose worst case evaluation error can be approximated to first order by  $u \sum_{j=1}^n c_j |\sum_{i=j}^n a_i x^{i-1}|$ , where

$u = 2^{-p}$ ,  $p \in \mathbb{N}$  being the precision of the floating point number format, i.e., number of bits of the mantissa [28], and  $c_1 = c_n = 1$  and  $c_j = 2$  otherwise. This can be written in matrix form  $u \sum_{j=1}^n |(E_j \phi(x))^T a|$  with  $E_j$  the diagonal matrix with zero on  $j - 1$  first diagonal entries, and  $c_j$  in the other entries. For example, with  $n = 4$  we have

$$E_1 = I, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (33)$$

For writing convenience, let  $e_0 = \phi(x)^T a - f(x)$ , and  $e_i(a, x) = (E_i \phi(x))^T a$  for  $i \in \{1, \dots, n\}$ , and  $e(a, x) = (e_0(a, x), \dots, e_n(a, x)) \in \mathbb{R}^{n+1}$ . The approximation problem consists in minimizing worst case of the approximation error added to the linearized worst case evaluation error:

$$m_u(a) = \max_{x \in [\underline{x}, \bar{x}]} \left( |e_0(a, x)| + u \sum_{j=1}^n |e_j(a, x)| \right). \quad (34)$$

For writing convenience, we define  $E_u(a, x) = |e_0(a, x)| + u \sum_{j=1}^n |e_j(a, x)|$ . Extreme points associated to the polynomial  $p(x) = \phi(x)^T a$  are now defined as  $\text{ext}(a) = \{x \in [\underline{x}, \bar{x}] : E_u(a, x) = m_u(a)\}$ . In this context, some errors  $e_i(a, x)$  may turn out to be zero for some extreme points. We need to define signatures  $\Sigma(a) \subseteq \mathbb{R} \times \mathbb{R}^{n+1}$  that include signs of each error  $e_i(a, x)$ , and so that if an error is zero then the corresponding extreme point appears twice with each sign:

$$(x, s) \in \Sigma(a) \iff x \in \text{ext}(a) \text{ and } \forall i \in \{0, \dots, n\}, s_i e_i(a, x) \geq 0, \quad (35)$$

e.g., if  $x \in \text{ext}(a)$  has  $k$  zero errors then it will appear  $2^k$  times inside  $\Sigma(a)$ , with different signs for each zero error. Finally, for  $(x, s) \in \Sigma(a)$  we define

$$\psi(x, s) = \nabla \left( s_0 e(a, x) + u \sum_{j=1}^n s_j e_j(a, x) \right) \quad (36)$$

$$= s_0 \phi(x) + u \sum_{j=0}^n s_j E_j \phi(x). \quad (37)$$

The following theorem is a kernel optimality condition for problem of this section.

**Theorem 9.** *The polynomial  $\bar{p}(x) = \phi(x)^T \bar{a}$  is a minimizer of  $m_u(a)$  if and only if there exists a finite signature  $\{(x_1, s_1), \dots, (x_m, s_m)\} \subseteq \Sigma(\bar{a})$ , where  $s_i = (s_{i0}, \dots, s_{in}) \in \{-1, 1\}^{n+1}$ , such that the matrix whose columns are  $\psi(x_i, s_i)$  has a nonzero kernel vector with non-negative components.*

*Proof.* The subdifferential of  $m_u(\bar{a})$  is computed using standard rules:

$$\partial m_u(\bar{a}) = \text{conv} \left\{ \partial \left( |e_0(\bar{a}, x)| + u \sum_{j=1}^n |e_j(\bar{a}, x)| \right) : x \in \text{ext}(\bar{a}) \right\} \quad (38)$$

$$= \text{conv} \left\{ \partial |e_0(\bar{a}, x)| + u \sum_{j=1}^n \partial |e_j(\bar{a}, x)| : x \in \text{ext}(\bar{a}) \right\} \quad (39)$$

$$= \text{conv} \left\{ s_0 \nabla e(a, x) + u \sum_{j=1}^n s_j \nabla e_j(a, x) : (x, s) \in \Sigma(\bar{a}) \right\}. \quad (40)$$

The last expression gives rise to  $\partial m_u(\bar{a}) = \text{conv} \{ \psi(x, s) : (x, s) \in \Sigma(\bar{a}) \}$ . Finally, by Carathéodory's theorem, zero in the convex hull of this subdifferential is equivalent to zero in the convex hull of finitely many generators, which is the statement.  $\square$

We now use the two-step discretization-and-Newton method proposed by Hettich [29] with the optimality condition of Theorem 9 for the Newton step. The two-step method needs a local version of the optimality condition encoded in the form of a system of equations, and an initial iterate close enough to the solution so that the number of extreme points and their position with respect to the boundary of the domain are guessed correctly.

We use a local necessary condition of Theorem 9 in the form of a system of equations. Variables are  $x_1, \dots, x_k \in \mathbb{R}$ ,  $a_1, \dots, a_n \in \mathbb{R}$  and  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ , where  $k$  is fixed to the number of thought extrema from the initial iterate. The first group of  $k$  constraints encodes local extremality of each extreme point. For simplicity, we now assume that no error is zero so that local extremality can be characterized using derivatives (this assumption needs to be confirmed on the initial iterate). Local extremality is then expressed by

$$x_1 = \underline{x} \text{ or } \frac{\partial}{\partial x} E_u(a, x_1) = 0 \quad (41)$$

$$\frac{\partial}{\partial x} E_u(a, x_i) = 0 \text{ for } i \in \{2, \dots, k-1\} \quad (42)$$

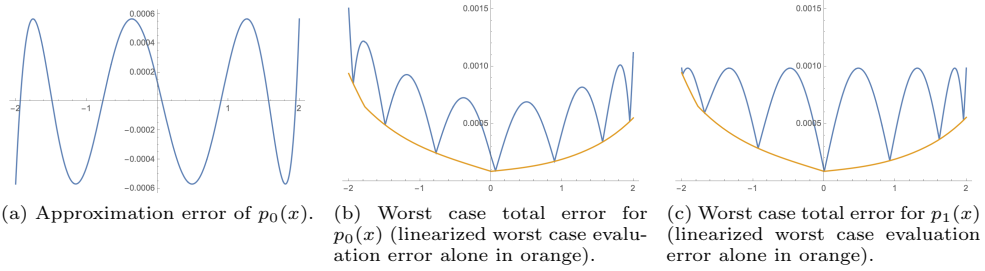
$$x_m = \underline{x} \text{ or } \frac{\partial}{\partial x} E_u(a, x_k) = 0, \quad (43)$$

where the disjunctions for the first and last extreme points depends whether they are guessed to lie on the boundary or inside the domain (see Example 2 below). The second group of  $k-1$  constraints encodes that extreme points need to have the same total error value:

$$E_u(a, x_i) = E_u(a, x_{i+1}) \text{ for } i \in \{1, \dots, k-1\}, \quad (44)$$

which is differentiable with respect to  $x$ , accordingly to the assumption that no error is zero. The third group of  $n+1$  constraints encodes the kernel condition

$$\sum_{i=1}^k \lambda_i \psi(x_i, \epsilon(a, x_i)) = 0 \text{ and } \sum_{i=1}^k \lambda_i = 1, \quad (45)$$



where  $\epsilon(a, x_i) = (\text{sign } e_0(a, x), \dots, \text{sign } e_n(a, x)) \in \{-1, 1\}^{n+1}$ , which is well defined by the assumption that no error is zero, and where a linear normalization can be used because kernel vectors are expected to be non-negative. We finally end with a square system of dimension  $2k + n$ , which is a local version of Theorem (9).

The two-step process is illustrated on the case of Example 3 of [3].

**Example 2.** *The Airy function is to be approximate by a polynomial of degree 6 on the interval  $[-1, 1]$ . We first approximate it in the usual Chebyshev sense. Using a sample of 81 equidistant points (with sample distance 0.05) and solving the corresponding finite linear problem, we obtain the polynomial*

$$p_0(x) = 0.00173x^6 - 0.0026x^5 - 0.02068x^4 + 0.06367x^3 - 0.00088x^2 - 0.26085x + 0.35516, \quad (46)$$

where coefficients are rounded to  $10^{-5}$ . The error function  $p_0(x) - f(x)$  is shown in Figure 1a, where it is seen to approximately equioscillate. Figure 1b shows the linearized worst case evaluation error in orange, and the sum of the two. In order to apply Newton's method, we need initial guesses for the polynomial, the extreme points and the kernel vector. For the first two, we use  $p_0(x)$  and approximations of its extreme points given in the following table:

	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$	$x_8$
$p_0(x) - f(x)$	-2.	-1.7943	-1.1847	-0.3875	0.4998	1.2803	1.8159	2.

We clearly see that the first and last extreme points lie on the boundary of the interval, and we guess so for the optimal solution. The corresponding approximate subgradient matrix is

$$\begin{pmatrix} -0.9998 & 1.0002 & -0.9998 & 1.0002 & -0.9998 & 1.0002 & -0.9998 & 1.0002 \\ 2.0005 & -1.7939 & 1.1838 & -0.3878 & -0.4999 & 1.28 & -1.8163 & 1.9995 \\ -4.0029 & 3.2173 & -1.4032 & 0.1502 & -0.2497 & 1.6395 & -3.2966 & 4.001 \\ 8.0098 & -5.7702 & 1.6631 & -0.0582 & -0.1247 & 2.1 & -5.9832 & 8.0059 \\ -16.0273 & 10.3487 & -1.9712 & 0.0225 & -0.0624 & 2.6872 & -10.8701 & 16.0039 \\ 32.0391 & -18.5782 & 2.3342 & -0.0087 & -0.0312 & 3.4387 & -19.729 & 32.0234 \\ -64.0625 & 33.3439 & -2.7646 & 0.0034 & -0.0156 & 4.4035 & -35.8165 & 64.0625 \end{pmatrix}, \quad (47)$$

whose kernel vector  $\lambda \approx (0.0597, 0.1188, 0.128, 0.14, 0.1476, 0.1563, 0.1648, 0.0848)$  can be used as an initial iterate for the Newton method. With these initial guesses, the

*Newton method converges to the polynomial*

$$p_1(x) = 0.0018x^6 - 0.00277x^5 - 0.02113x^4 + 0.06447x^3 - 0.00027x^2 - 0.26164x + 0.35504, \quad (48)$$

*where coefficients are rounded to  $10^{-5}$ , in 5 iterations with residual norm  $10^{-14}$ . Figure 1c shows the sum of the worst case errors for this new polynomial. All figures are coherent with the results from [3].*

## Appendix A Proof of proposition 7

**Proposition.** Let  $m : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. Then, for arbitrary  $\bar{a} \in \mathbb{R}^n$  and  $r \geq 0$ , the following three conditions are equivalent

- (1)  $\forall a \in \mathbb{R}^n, m(a) \geq m(\bar{a}) + r \|a - \bar{a}\|_2$ ;
- (2)  $B_r \subseteq \partial m(\bar{a})$ ;
- (3)  $\forall u \in \mathbb{R}^n, m'(\bar{a}, u) \geq r \|u\|_2$ .

*Proof.* (2)  $\Rightarrow$  (1): We have for all subgradient  $g \in B_r$ ,  $m(a) \geq m(\bar{a}) + g^T(a - \bar{a})$ . Therefore,  $m(a) \geq m(\bar{a}) + \max_{g \in B_r} g^T(a - \bar{a}) = m(\bar{a}) + \|\bar{g}\|_2 \|a - \bar{a}\|_2 = r \|a - \bar{a}\|_2$  for  $\bar{g}$  radius  $r$  aligned with  $(a - \bar{a})$ , using Cauchy-Schwarz inequality. (1)  $\Rightarrow$  (3): we have  $m'(\bar{a}, u) = \lim_{t \rightarrow 0^+} \frac{1}{t} (m(\bar{a} + tu) - m(\bar{a})) \geq \lim_{t \rightarrow 0^+} \frac{1}{t} (m(\bar{a}) + r \|tu\|_2 - m(\bar{a})) = r \|u\|_2$ .  $\neg(2) \Rightarrow \neg(3)$ : Let  $\bar{g} \in B_r$  with  $\bar{g} \notin \partial m(\bar{a})$ . Since  $\partial m(\bar{a})$  is compact, the separating hyperplane theorem proves the existence of  $u \in \mathbb{R}^n$  such that  $\forall g \in \partial m(\bar{a}), u^T g < u^T \bar{g}$ . Finally,  $m'(\bar{a}, u) = \max_{g \in \partial m(\bar{a})} u^T g < u^T \bar{g} \leq r \|u\|_2$ , which is  $\neg(3)$ .  $\square$

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