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# Penalty approximation of Painlevé problem

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An elongated solid bar drops onto a rigid foundation, and Coulomb friction acts at the contact. A penalty approximation is defined, and is proved to converge to a solution of the problem.

**Keywords:** Penalty, Coulomb friction, impact

## 1. Introduction

The mathematical status of many problems in non-smooth mechanics has been greatly advanced by the work of Jean Jacques Moreau.

I have worked independently of Jean Jacques' school and set up my own techniques and approaches to problems of non-smooth mechanics; see for instance (Schatzman, 1977) and (Schatzman, 1978) for a first appearance of vector-valued measures in the context of impact mechanics. It was a great honor to be present for the celebration of Jean Jacques Moreau's eightieth birthday, and to have the opportunity to praise the immense impact he had on non smooth mechanics.

Therefore, dear Jean Jacques, I hope that you will stay with us many more years in good health and good spirits, with the sharpness of mind and the creativity that are your trademark, and may you reach your hundred and twentieth birthday.

In this article, I will report on recent results on the penalty approach to combined friction and impact. I will concentrate on a very specific problem, namely the case of a solid bar dropping onto a rigid foundation. This problem has been solved mathematically through a time-stepping

approach by David Stewart (Stewart, 1998), which is a very difficult paper.

Here, I propose to use a penalty approach, which is applied only to the non-interpenetration condition. I prove that as the stiffness in the penalty approximation tends to infinity, the solution of the system of differential equations tends to a solution of the limiting problem, in a slightly generalized sense. There is a substantial mathematical difficulty here: the passage to the limit in the Coulomb condition seems to involve the multiplication of a measure by a function which is discontinuous precisely where the measure has a Dirac mass. There is a way around this difficulty, since the details of the convergence are accessible, and therefore a relation between the velocities after impact can be deduced from the velocities before impact.

The paper is organized as follows. In Section 2, the notations are introduced, the system of differential equations and its approximation are described.

In Section 3, an auxiliary problem is stated: it is a multivalued equation of first order, involving the multivalued sign function multiplied by a variable coefficient. Complete proofs will be given elsewhere.

Section 4 gives an existence result for the approximate problem; it uses a local existence theorem through the contraction principle and an energy estimate in order to go global.

The convergence of the approximate solution is sketched in Section 5. Many interesting and intricate technical steps have been skipped, and will be given in detail in a later publication.

Finally, an appendix contains the statement of some technical results relative to the weak convergence of measures.

## 2. Notations and position of the problem

Let  $2\ell$  be the length of the bar,  $(X, Y)$  the coordinates of the center of mass,  $\theta$  the angle of the bar with the horizontal; then the coordinates of the possible contact points are  $(x, y)$  given by

$$x = X - \ell \cos \theta \text{ and } y = Y - \ell \sin \theta \quad (12.1)$$

or

$$x = X + \ell \cos \theta \text{ and } y = Y + \ell \sin \theta. \quad (12.2)$$

We concentrate on the case (12.1); both cases can occur at the same time only if  $\theta$  vanishes modulo  $\pi$ . This situation creates difficulties, which are not yet solved, and we exclude it *a priori*.

Denote by  $m$  the mass of the bar and by  $\mu$  the friction coefficient. Then, the moment of inertia of the bar about its center of mass is  $J =$

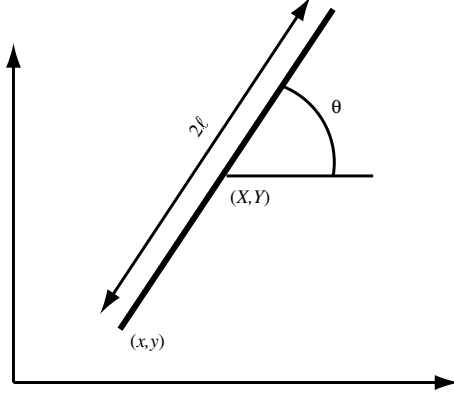


Figure 12.1. The geometry of the bar.

$m\ell^2/3$ . If  $\mathbf{R}_T$  and  $\mathbf{R}_N$  denote respectively the tangential and the normal reactions, and  $g$  denotes the gravity, the equations of motion can be written

$$m\ddot{X} = \mathbf{R}_T, \quad (12.3)$$

$$m\ddot{Y} = \mathbf{R}_N - mg, \quad (12.4)$$

$$J\ddot{\theta} = -\ell\mathbf{R}_N \cos \theta + \ell\mathbf{R}_T \sin \theta. \quad (12.5)$$

The condition of interpenetration is a Signorini condition:

$$0 \leq \mathbf{R}_N \perp y \geq 0, \quad (12.6)$$

and friction is modeled through a Coulomb law:

$$-\mathbf{R}_T \in \mu\mathbf{R}_N \text{Sign}(\dot{x}). \quad (12.7)$$

The multivalued operator Sign is given by

$$\text{Sign}(r) = \begin{cases} \{-1\} & \text{if } t < 0, \\ [-1, +1] & \text{if } t = 0, \\ \{+1\} & \text{if } t > 0. \end{cases} \quad (12.8)$$

The purpose of this article is to show the existence of a solution of the system (12.3), (12.4), (12.5), (12.6), (12.7) in an appropriate sense.

We construct an approximate solution by a penalty method *only* on the Signorini condition, that is, we replace (12.6) by

$$\mathbf{R}_N = -m \frac{y^-}{\tau^2}; \quad (12.9)$$

here  $y^-$ , the negative part of  $y$  is defined to be  $\max(-y, 0)$  and  $\tau$  is a positive parameter having the dimension of time, which will eventually tend to 0.

We consider now the multivalued differential system

$$\ddot{X} + \mu\sigma \frac{y^-}{\tau^2} = 0, \quad (12.10)$$

$$\ddot{Y} - \frac{y^-}{\tau^2} + g = 0, \quad (12.11)$$

$$\frac{\ell}{3}\ddot{\theta} + \frac{y^-}{\tau^2} \cos \theta + \mu\sigma \frac{y^-}{\tau^2} \sin \theta = 0, \quad (12.12)$$

$$\sigma \in \text{Sign}(\dot{x}). \quad (12.13)$$

This system is supplemented by Cauchy conditions on  $X$ ,  $Y$ ,  $\theta$  and their first derivatives at time 0:

$$X(0), \quad Y(0), \quad \theta(0), \quad \dot{X}(0), \quad \dot{Y}(0), \quad \dot{\theta}(0) \text{ are given.} \quad (12.14)$$

### 3. An auxiliary problem

In order to solve (12.10)–(12.14), we need to solve an auxiliary problem relative to a class of multivalued scalar equations:

**LEMMA 12.1** *Let  $\phi$  and  $\psi$  belong to  $L^1(0, T)$ , and assume moreover that  $\phi$  is almost everywhere non negative. Then, there exists a unique absolutely continuous solution of*

$$\dot{\xi} + \phi \text{Sign}(\xi) \ni \psi \quad (12.15)$$

*for all initial condition  $\xi(0) = \xi^0$ . Moreover, given  $\hat{\phi}$  and  $\hat{\psi}$  satisfying the same conditions, we denote by  $\hat{\xi}$  the solution of the analogous problem; then the following estimate holds:*

$$|\xi(t) - \hat{\xi}(t)| \leq |\xi^0 - \hat{\xi}^0| + \int_0^t (|\phi(s) - \hat{\phi}(s)| + |\psi(s) - \hat{\psi}(s)|) \, ds \quad (12.16)$$

*Principle of the proof.* The existence is clear for piecewise constant functions  $\phi$  and  $\psi$ ; one obtains the existence in general by a density argument. The uniqueness is a straightforward consequence of monotonicity, and relation (12.16) is also a consequence of monotonicity. Details will be written elsewhere.  $\square$

#### 4. Existence for the approximate problem

We will now perform some algebraic manipulations on the system (12.10)–(12.13); indeed, we infer from the definition  $x = X - \ell \cos \theta$  the identity

$$\ddot{X} = \ddot{x} - \ell \ddot{\theta} \sin \theta - \ell \dot{\theta}^2 \cos \theta. \quad (12.17)$$

We substitute now the value of  $\ell \ddot{\theta}$  given by (12.12) into (12.17), and we obtain the following equation for  $\dot{x}$ :

$$\frac{d}{dt} \dot{x} + \frac{\mu(1 + 3 \sin^2 \theta) \sigma y^-}{\tau^2} = \ell \dot{\theta}^2 \cos \theta - \frac{3y^- \sin \theta \cos \theta}{\tau^2}. \quad (12.18)$$

Therefore, if we assume that  $Y$ ,  $\theta$  and  $\dot{\theta}$  are given continuous functions, according to Lemma 12.1 there exists a unique solution of Equ. (12.18), which satisfies the Cauchy data

$$\dot{x}(0) = \dot{X}_0 + \ell \dot{\theta}_0 \sin \theta_0.$$

We denote this solution by

$$\dot{x}(t) = \mathcal{X}(Y, \theta, \dot{\theta})(t), \quad (12.19)$$

the initial data being understated, for simplicity of notation.

We are now able to prove an existence and uniqueness theorem for the penalized problem:

**THEOREM 12.2** *Given initial data (12.14), there exists a unique solution on  $[0, T]$  of (12.10)–(12.13), such that  $X, \dot{X}, Y, \dot{Y}, \theta$  and  $\dot{\theta}$  are continuous, while the second derivatives  $\ddot{X}$ ,  $\ddot{y}$  and  $\ddot{\theta}$  are bounded.*

*Proof.*

**Step 1. Existence of a local solution.** Assuming that there exists a solution, we may rewrite Equ. (12.12) by expressing  $\sigma y^-/\tau^2$  in terms of  $\ddot{x}$ ,  $\theta$  and  $Y$ ; indeed, we infer from Equ. (12.18) the following relation:

$$\mu \frac{\sigma y^-}{\tau^2} = \left( \ell \dot{\theta}^2 \cos \theta - \frac{3y^-}{\tau^2} \cos \theta \sin \theta - \ddot{x} \right) \frac{1}{1 + 3 \sin^2 \theta}.$$

We substitute this expression into Equ. (12.12), and we obtain the relation

$$\ddot{\theta} = -\frac{3y^-}{\ell \tau^2} \frac{\cos \theta}{1 + 3 \sin^2 \theta} - \frac{3 \dot{\theta}^2 \cos \theta \sin \theta}{1 + 3 \sin^2 \theta} + \frac{3}{\ell} \frac{\sin \theta}{1 + 3 \sin^2 \theta} \ddot{x}. \quad (12.20)$$

We transform the differential system satisfied by  $U$  into an integral equation; we treat the term containing  $\ddot{x}$  through an integration by parts.

Let us define now

$$U = \begin{pmatrix} Y \\ Z \\ \theta \\ \eta \end{pmatrix}, \quad U_0 = \begin{pmatrix} Y_0 \\ \dot{Y}_0 \\ \theta_0 \\ \dot{\theta}_0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } F(U) = \begin{pmatrix} F_1(U) \\ F_2(U) \\ F_3(U) \\ F_4(U) \end{pmatrix},$$

with  $F_i$  given as follows:

$$F_1(U) = Z,$$

$$F_2(U) = \frac{(Y - \ell \sin \theta)^-}{\tau^2} - g,$$

$$F_3(U) = \eta,$$

$$F_4(U) = -\frac{3(Y - \ell \sin \theta)^-}{\ell \tau^2} \frac{\cos \theta}{1 + 3 \sin^2 \theta} - \frac{3\eta^2 \cos \theta \sin \theta}{1 + 3 \sin^2 \theta}.$$

When we integrate by parts over  $[0, t]$  the  $\ddot{x}$  term in Equ. (12.20), we obtain the expression

$$\dot{x}(t) \frac{3}{\ell} \frac{\sin \theta}{1 + 3 \sin^2 \theta}(t) - \dot{x}(0) \frac{3}{\ell} \frac{\sin \theta}{1 + 3 \sin^2 \theta}(0) - \int_0^t \dot{x}(s) \frac{3}{\ell} \eta(s) \frac{d}{d\theta} \frac{\sin \theta}{1 + 3 \sin^2 \theta} ds.$$

Therefore, it is convenient to define the functions

$$\begin{aligned} \chi_0(r) &= \frac{3}{\ell} \frac{\sin r}{1 + 3 \sin^2 r}, \\ \chi(U)(t) &= \chi_0 \left( \theta(0) + \int_0^t \eta(s) ds \right), \\ \omega(U) &= -\chi'_0(\theta) \eta. \end{aligned}$$

We also need the functions, which appear naturally in the definition of  $\mathcal{X}$ :

$$\begin{aligned} \phi(U) &= \mu \frac{(1 + 3 \sin^2 \theta)(Y - \ell \sin \theta)^-}{\tau^2}, \\ \psi(U) &= \ell \eta^2 \cos \theta - \frac{3(Y - \ell \sin \theta)^- \cos \theta \sin \theta}{\tau^2}. \end{aligned}$$

Observe that  $\chi$  maps continuous functions to continuous functions, while  $F$  maps  $\mathbb{R}^4$  to itself and  $\omega$ ,  $\phi$  and  $\psi$  map  $\mathbb{R}^4$  to  $\mathbb{R}$ .

Finally, it is convenient to write  $\mathcal{X}(U)$  instead of  $\mathcal{X}(Y, \theta, \eta)$ .

Equ. (12.20) together with Cauchy data is equivalent to

$$\begin{aligned}\eta(t) &= \dot{\theta}(0) + \int_0^t F_4(U(s)) \, ds + \chi(U)(t)\mathcal{X}(U)(t) - \chi_0(\theta(0))\dot{\chi}(0) \\ &\quad + \int_0^t \mathcal{X}(U)(s)\omega(U(s)) \, ds,\end{aligned}$$

provided that we let  $\dot{\theta} = \eta$ .

At this point, we define

$$G(U)(t) = \chi(U)(t)\mathcal{X}(U)(t) - \chi_0(\theta(0))\dot{\chi}(0) + \int_0^t \omega(U)\mathcal{X}(U) \, ds.$$

It is equivalent to solve (12.10)–(12.13) with Cauchy data (12.14) and to find a fixed point of the operator  $\mathcal{T}$  defined by

$$(\mathcal{T}U)(t) = U_0 + \int_0^t F(U(s)) \, ds + e_4 G(U)(t).$$

Let  $B$  be the closed ball of radius  $r > 0$  about  $U_0$  in  $\mathbb{R}^4$ , and let  $\tilde{B}$  be  $B \times B$  deprived from its diagonal. Define for all  $r > 0$  the quantities:

$$R_1 = \max_{V \in B} \max(|F(V)|, |\phi(V)| + |\psi(V)|, |\omega(V)|),$$

$$R_2 = \max|\chi_0|,$$

$$\begin{aligned}L_1 &= \sup_{(V, V') \in B'} \frac{1}{|V - V'|} (|F(V) - F(V')|, |\phi(V) - \phi(V')| \\ &\quad + |\psi(V) - \psi(V')|, |\omega(V) - \omega(V')|),\end{aligned}$$

$$L_2 = \max|\chi'_0|,$$

$$R = \max(R_1, R_2), \quad L = \max(L_1, L_2).$$

The numbers  $R$  and  $L$  are finite, for all finite values of  $r$ .

Equip the space  $C^0([0, t_0]; \mathbb{R}^4)$  with the maximum of the sup norm in each component; the number  $t_0$  is a positive number that will be chosen later; let the set  $\mathcal{B}$  be the ball of radius  $r$  about the constant function  $U_0$  in  $C^0([0, t_0]; \mathbb{R}^4)$ . We show now that for small enough values of  $t_0$ ,  $\mathcal{T}$  is a strict contraction from  $\mathcal{B}$  to itself.

If  $U$  belongs to  $\mathcal{B}$ , we have the estimate

$$|(\mathcal{T}U)(t) - U_0| \leq tR + |G(U)(t)|;$$

Decompose  $G$  into two pieces:

$$G_1(U) = \chi(U)\mathcal{X}(U) - \chi_0(\theta_0)\dot{\chi}_0 \text{ and } G_2(U)(t) = \int_0^t \omega(U)\mathcal{X}(U) \, ds. \quad (12.21)$$



Thanks to Equ. (12.15), and by definition of  $R$ , we have

$$|\mathcal{X}(U)(t) - \dot{x}_0| \leq Rt, \quad (12.22)$$

which implies

$$|\mathcal{X}(U)(t)| \leq |\dot{x}_0| + Rt. \quad (12.23)$$

We also have, thanks to Equ. (12.16) and the definition of  $L$

$$|\mathcal{X}(U^*)(t) - \mathcal{X}(U)(t)| \leq Lt\|U^* - U\|. \quad (12.24)$$

Let us estimate now  $\chi(U)(t)$ : by definition of  $R$ ,

$$|\chi(U)(t)| \leq R \quad (12.25)$$

and by definition of  $L$ :

$$|\chi(U^*)(t) - \chi(U)(t)| \leq Lt\|U^* - U\| \text{ and } |\chi(U)(t) - \chi_0(\theta_0)| \leq Ltr. \quad (12.26)$$

We use the triangle inequality to obtain the estimates

$$\begin{aligned} |G_1(U)(t)| &\leq |\mathcal{X}(U)(t) - \dot{x}_0| |\chi(U)(t)| + |\dot{x}_0| |\chi(U)(t) - \chi_0(\theta_0)| \\ &\leq R^2 t + |\dot{x}_0| Ltr, \end{aligned}$$

thanks to (12.22), (12.25), (12.26) and

$$|G_2(U)(t)| \leq R \int_0^t (|\dot{x}_0| + Rs) \, ds = R(|\dot{x}_0| + Rt^2/2).$$

thanks to (12.23). Therefore, if

$$t_0(R + R^2 + |\dot{x}_0|Lr + R|\dot{x}_0|) + R^2 t_0^2/2 \leq r,$$

$\mathcal{T}$  maps  $\mathcal{B}$  to itself.

We move on to contractivity. It is plain that

$$|(\mathcal{T}U^* - \mathcal{T}U)(t)| \leq Lt\|U^* - U\| + |G(U^*)(t) - G(U)(t)|$$

and we reuse estimates (12.22) to (12.26) to treat the  $G$  terms:

$$\begin{aligned} |G_1(U^*)(t) - G_1(U)(t)| &\leq |\mathcal{X}(U^*)| |\chi(U^*)(t) - \chi(U)(t)| + |\chi(U)(t)| |\mathcal{X}(U^*) - \mathcal{X}(U)| \\ &\leq (|\dot{x}_0| + Rt)Lt\|U^* - U\| + RLt\|U^* - U\| \end{aligned}$$

and very similarly

$$|G_2(U^*)(t) - G_2(U)(t)| \leq \int_0^t (RLs\|U^* - U\| + L(|\dot{x}_0| + Rs)) \, ds.$$

Therefore, if

$$Lt_0(2|\dot{x}_0| + 2Rt_0 + R) < 1,$$

$\mathcal{T}$  is a strict contraction, so that the system (12.10)–(12.14) possesses a unique solution.

**Step 2. Global solution.**

We obtain an energy identity: if we multiply (12.10) by  $\dot{X}$ , (12.11) by  $\dot{Y}$  and (12.12) by  $\ell^2 \dot{\theta}$ , and we integrate over  $[0, t]$  for  $t \leq t_0$ , we get

$$\frac{1}{2} \left( \dot{X}^2 + \dot{Y}^2 + \frac{\ell^2 \dot{\theta}^2}{3} + \frac{(y^-)^2}{2\tau^2} + gY \right) \Big|_0^t + \int_0^t \mu |\dot{x}| \frac{y^-}{\tau^2} ds = 0. \quad (12.27)$$

A Gronwall inequality argument gives then

$$\max_{[0, t_0]} \max(|\dot{X}|, |\dot{Y}|, |\dot{\theta}|, |y^-/\tau|) \leq C, \quad (12.28)$$

where the number  $C$  depends neither on  $t$  nor on  $\tau$ , provided that  $Y_0 - \ell \sin \theta_0 = y(0)$  is non negative.

Therefore, the standard continuation argument shows that the solution exists up to time  $T$ .

Moreover, if we integrate (12.10) over  $[0, T]$ , we obtain immediately from the identity

$$\dot{Y}(T) - \dot{Y}_0 + gT = \int_0^T \frac{y^-}{\tau^2} dt$$

the bound

$$\max_{\tau \geq 0} \int_0^T \frac{y^-}{\tau^2} dt \leq C. \quad (12.29)$$

□

## 5. Convergence of the approximate solution

In this section, we put back the index  $\tau$  on all the quantities, which depend on  $\tau$ .

**THEOREM 12.3** *For all Cauchy data (12.14), and as long as the second end of the bar does not hit the rigid foundation, there exist three functions  $X$ ,  $Y$  and  $\theta$ , whose derivatives are of bounded variation, and which satisfy the following equations:*

$$\ddot{X}_0 + \mu \xi_0 = 0, \quad (12.30)$$

$$\ddot{Y}_0 - \eta_0 + g = 0, \quad (12.31)$$

$$\frac{\ell}{3} \ddot{\theta}_0 + \eta_0 \cos \theta_0 + \mu \xi_0 \sin \theta_0 = 0, \quad (12.32)$$

$$0 \leq Y_0 - \ell \sin \theta_0 \perp \eta_0 \geq 0, \quad (12.33)$$

$$\xi_0 = \sigma_0 \eta_0. \quad (12.34)$$

The function  $\sigma_0$  and the measures  $\eta_0$  and  $\xi_0$  satisfies the following conditions:  $\eta_0$  is non negative, according to (12.33),  $|\sigma_0| \leq 1$ ,  $\eta_0$ -almost everywhere; moreover, if the decomposition of  $\eta_0$  and  $\xi_0$  into their atomic and continuous parts are respectively

$$\eta_0 = \eta_0^a + \eta_0^c, \quad \xi_0 = \xi_0^a + \xi_0^c,$$

then

$$\xi_0^c = \sigma_0^c \eta_0^c, \quad \xi_0^a = \sigma_0^a \eta_0^a$$

and we have the relations

1 the continuous part  $\sigma_0^c$  of  $\sigma_0$  satisfies

$$\sigma_0^c \in \text{Sign}(\dot{x}_0), \quad \eta_0^c\text{-almost everywhere;}$$

2 let  $t_i$  be an atom of  $\eta_0$ ; if  $\dot{x}_0(t_i + 0)$  and  $\dot{x}_0(t_i - 0)$  have the same sign, then  $\sigma^a(t_i) = \text{sign}(\dot{x}_0(t_i \pm 0))$ ;

3 if  $\dot{x}_0(t_i - 0)$  does not vanish and  $\dot{x}_0(t_i + 0)$  vanishes or has the opposite sign to  $\dot{x}_0(t_i - 0)$ , then  $\sigma_0^a(t_i) \in [-1, 1]$ ; moreover, a precise expression of  $\sigma_0^a(t_i)$  can be given in terms of the values on the left and on the right of all the derivatives of  $X_0$ ,  $Y_0$  and  $\theta_0$  at  $t_i$ .

*Proof.* Let us write

$$\eta_\tau = \frac{y_\tau^-}{\tau^2} \quad \text{and} \quad \xi_\tau = \sigma_\tau \frac{y_\tau^-}{\tau^2}.$$

We may always extract subsequences, which will still be denoted by  $\xi_\tau$  and  $\eta_\tau$ , such that  $\xi_\tau$  and  $\eta_\tau$  converge weakly  $*$  in the space  $M^1([0, T])$  of Radon measures to their respective limits  $\xi_0$  and  $\eta_0$ .

We will identify  $\eta_\tau$  with  $\eta_\tau dt$ , and similarly for  $\xi_\tau$ . Therefore, any integral that contains  $\eta_\tau$  or  $\xi_\tau$  will contain its element of integration. This choice of notations makes life easier in the limit, when we have to integrate with respect to more or less arbitrary Radon measures.

We have to characterize now the relation between  $\eta_0$  and  $\xi_0$ . This cannot be a trivial matter since we expect that the limit of  $\dot{x}_\tau$  will change sign, and if it changes sign where  $\eta_\tau$  has an atom, then we have to pass to the limit in relation (12.13), which amounts to multiplying an atom by a function, which is discontinuous at this atom.

Therefore, it is better to use the weak formulation of (12.13). Indeed, for any given  $\tau > 0$ , Equ. (12.13) has an equivalent weak form

$$\forall z \in C^0([0, T]), \quad \int_0^T (|z + \dot{x}_\tau| - |\dot{x}_\tau|) \eta_\tau \geq \int_0^T \xi_\tau z. \quad (12.35)$$

There will be an easy part in the passage to the limit and a difficult part. Let  $\eta_0$  be decomposed into its atomic and continuous parts:

$$\eta_0 = \eta_0^a + \eta_0^c.$$

Then, according to Theorem A.3, there exists a sequence of continuous functions  $\Psi_\tau$  with values in  $[0, 1]$  such that

$$\eta_\tau \Psi_\tau \xrightarrow{*} \eta_0^a, \quad \eta_\tau(1 - \Psi_\tau) \xrightarrow{*} \eta_0^c.$$

Moreover, the support of  $\Psi_\tau$  tends to the set of atoms of  $\xi_0$ . If we consider now  $\xi_\tau \Psi_\tau$  and  $\xi_\tau(1 - \Psi_\tau)$ , we may extract subsequences such that both measures converge weakly  $*$  to their respective limits  $\xi_0^a$  and  $\xi_0^c$ .

Let us prove that these are precisely the atomic and continuous parts of  $\xi_0$ . The support of  $\xi_0^a$  is included in the set of atoms of  $\eta_0$ ; we also have the inequality

$$|\xi_0^c| \leq |\eta_0^c|; \quad (12.36)$$

therefore,  $\xi_0^c$  is continuous and the support of  $\xi_0^c$  is at most countable, which gives the desired conclusion.

We will write henceforth

$$\begin{aligned} \eta_\tau^c &= (1 - \Psi_\tau)\eta_\tau, & \eta_\tau^a &= \Psi_\tau\eta_\tau, \\ \xi_\tau^c &= (1 - \Psi_\tau)\xi_\tau, & \xi_\tau^a &= \Psi_\tau\xi_\tau; \end{aligned}$$

the reader should beware: these four measures are identified with the product of an essentially bounded function by Lebesgue's measure, and therefore, they are continuous measures. It is plainly possible for a sequence of continuous measures to converge to an atomic measure, and conversely.

The following inequality holds for all function  $\phi$  with values in  $[0, 1]$ :

$$|\dot{x}_\tau + z\phi| - |\dot{x}_\tau| \leq \phi(|\dot{x}_\tau + z| - |\dot{x}_\tau|). \quad (12.37)$$

Therefore, if we substitute  $z$  by  $1 - \Psi_\tau$  in (12.37), we can pass to the limit; indeed, let us decompose  $\dot{x}_\tau$  into two pieces:

$$\begin{aligned} \dot{x}_\tau^c(t) &= \dot{x}(0) + \int_0^t \left( \ell \dot{\theta}_\tau^2 - 3\eta_\tau^c \sin \theta_\tau \cos \theta_\tau - \mu(1 + 3\sin^2 \theta_\tau) \xi_\tau^c \right) dt, \\ \dot{x}_\tau^a &= - \int_0^t \left( \mu(1 + 3\sin^2 \theta_\tau) \xi_\tau^a + 3\sin \theta_\tau \cos \theta_\tau \eta_\tau^a \right). \end{aligned}$$

Thanks to Lemma A.2,  $\dot{x}_\tau^c$  converges uniformly to its limit

$$\dot{x}_0^c(t) = \dot{x}(0) + \int_0^t \left( \ell \dot{\theta}_0^2 dt - 3\eta_0^c \sin \theta_0 \cos \theta_0 - \mu(1 + 3\sin^2 \theta_0) \xi_0^c \right).$$

This proves the following convergence:

$$\int_0^T (|\dot{x}_\tau^c + z| - |\dot{x}_\tau^c|) \eta_\tau^c \rightarrow \int_0^T (|\dot{x}_0^c + z| - |\dot{x}_0^c|) \eta_0^c.$$

We have to estimate now

$$\int_0^T (|\dot{x}_\tau + z| - |\dot{x}_\tau| - |\dot{x}_\tau^c| + |\dot{x}_\tau^c|) \eta_\tau^c. \quad (12.38)$$

But the expression inside parentheses in (12.38) vanishes wherever  $\dot{x}_\tau^c$  is equal to  $\dot{x}_\tau$ , that is outside of the support of  $\Psi_\tau$ . Thus, we have the estimate

$$\left| \int_0^T (|\dot{x}_\tau + z| - |\dot{x}_\tau| - |\dot{x}_\tau^c| + |\dot{x}_\tau^c|) \eta_\tau^c \right| \leq \sup_t |\dot{x}_\tau^a(t)| \int 1_{\{\Psi_\tau > 0\}} \eta_\tau^a \quad (12.39)$$

and by construction the integral factor in the right hand side of (12.39) tends to 0 as  $\tau$  tends to 0. Also, the supremum of  $|\dot{x}_\tau^a|$  is bounded independently of  $\tau$ . Therefore, we have in the limit the inequality

$$\int_0^T (|\dot{x}_0^c + z| - |\dot{x}_0^c|) \eta_0^c \geq \int_0^T z \xi_0^c, \quad (12.40)$$

for all continuous function  $z$  on  $[0, T]$ . This is the weak formulation of the multivalued relation (in the sense of measure-valued differential inclusions)

$$\xi_0^c \in \eta_0^c \text{Sign}(\dot{x}_0^c). \quad (12.41)$$

There is a slightly different way of writing things; as the measure  $\xi_0^c$  is absolutely continuous with respect to the measure  $\eta_0^c$ , there exists an  $\eta_0^c$ -measurable function  $\sigma^c$ , such that

$$\xi_0^c = \sigma^c \eta_0^c.$$

In fact,  $\sigma^c$  is  $\eta_0^c$ -essentially bounded, thanks to relation (12.36). Then relation (12.41) is equivalent to

$$\sigma^c \in \text{Sign}(\dot{x}_0^c), \quad \eta_0^c\text{-almost everywhere.} \quad (12.42)$$

Let us move on to the atomic part of the measure  $\xi_0$ ; of course this will be the difficult part of the proof. Since it is highly technical, it will be only sketched.

We need more notations: define

$$\kappa = \mu(1 + 3 \sin^2 \theta), \quad \lambda = 3 \sin \theta \cos \theta. \quad (12.43)$$

In order to emphasize the dependence of  $\kappa$  and  $\lambda$  over  $\tau$ , we will affix an index  $\tau$  whenever necessary.

Then, close to an atom of  $\eta$  at  $t_i$ , the equation for  $\dot{x}_\tau$  is asymptotic to the equation

$$\frac{d\dot{x}_\tau}{dt} + (\kappa_0(t_i) \text{Sign}(\dot{x}_\tau) + \lambda_0(t_i))\eta_\tau \ni 0. \quad (12.44)$$

The comparison between (12.44) and (12.17) can be performed with the help of Lemma 12.1. The nice feature of (12.44) is that the passage to the limit is easy here, provided, of course, that we use Theorem A.3.

Then, we find the following relations between the velocity on the left of  $t_i$  and the velocity on the right of  $t_i$ , where we have deonted by  $m_i$  the mass of the atom of  $\eta_0$  at  $t_i$ :

- if  $\dot{x}_0(t_i - 0) > 0$  and  $v = \dot{x}_0(t_i - 0) - \mu m_i(\kappa_0(t_i) + \lambda_0(t_i)) > 0$ , then  $\dot{x}_0(t_i + 0) = v$ ;
- if  $\dot{x}_0(t_i - 0) > 0$ ,  $\dot{x}_0(t_i - 0) - \mu m_i(\kappa_0(t_i) + \lambda_0(t_i)) < 0$  and  $\lambda_0(t_i) - \kappa_0(t_i) > 0$ , then  $\dot{x}_0(t_i + 0) > 0$  and its value is given by

$$\dot{x}_0(t_i + 0) = \frac{(\kappa_0(t_i) - \lambda_0(t_i))(m_i + n_i)}{(\kappa_0(t_i) + \lambda_0(t_i))(m_i - n_i)} \dot{x}_0(t_i - 0),$$

where  $n_i$  denotes the mass of  $\xi_0$  at  $t_i$ ;

- if  $\dot{x}_0(t_i - 0) > 0$ ,  $\dot{x}_0(t_i - 0) - \mu m_i(\kappa_0(t_i) + \lambda_0(t_i)) < 0$  and  $\lambda_0(t_i) - \kappa_0(t_i) \leq 0$ , then  $\dot{x}_0(t_i + 0) = 0$ .

For  $\dot{x}_o(t_i - 0)$  there are analogous expressions, which are not given here for simplicity.

The question now is to define  $m_i$  and  $n_i$  from the data of the problem. In the case when  $\dot{x}$  does not change sign at  $t_i$ , we pass to the limit in the energy equality (12.27), and we find the formula

$$m_i = \frac{2\dot{Y}_0(t_i - 0) - 2\ell\dot{\theta}_0(t_i - 0)(\cos\theta(t_i) + \mu\sin\theta(t_i))}{1 + (\cos\theta(t_i) + \mu\sin\theta(t_i))^2}.$$

There are analogous, but much more complicated formulas in the two other cases. □

## Appendix: Some results on the weak \* convergence of measures

The construction of Radon measure is classical and is performed from two slightly different points of view in Dieudonné's treatise (Dieudonné, 1970) or the french original (Dieudonné, 1968) and in Malliavin's superb course book (Malliavin, 1995), with french original (Malliavin and Airault, 1994).

I will assume that the reader is familiar with the weak convergence of measures and the definition of the absolute value of a measure.

For a  $\mu$  integrable set  $A$ , we use the standard notation

$$\int_A f \mu = \int f 1_A \mu.$$

A measure  $\mu$  has an atom at  $\xi$  iff  $\langle \mu, 1_{\{\xi\}} \rangle$  does not vanish. We say that the mass of this atom is  $\langle \mu, 1_{\{\xi\}} \rangle$ . Then  $|\mu|$  also has an atom at  $x$ , and the mass of this atom is  $|\langle \mu, 1_{\{\xi\}} \rangle|$ . The basic case of a measure, which has an atom is the Dirac mass  $\delta(\cdot - \xi)$  at  $\xi \in [0, T]$ , which is defined by

$$\langle \delta(\cdot - \xi), \phi \rangle = \phi(\xi).$$

It is a classical fact that every Radon measure can be decomposed into the sum of its atomic part, of the form

$$\mu^a = \sum_{\alpha} m_{\alpha} \delta(\cdot - \xi_{\alpha}).$$

and of its continuous part, which has no atoms and is defined as

$$\mu^c = \mu - \mu^a.$$

Let us now recall Helly's theorem:

**THEOREM A.1** *Let  $(\mu_j)_{j \geq 1}$  be a weakly  $*$  convergent sequence of Radon measures, denote its limit by  $\mu_{\infty}$  and define*

$$\forall j = 1, 2, \dots, \infty, \quad F_j(x) = \int_{[0, x]} \mu_j. \quad (12.A.1)$$

*Then, there exists an at most countable set  $D \in [0, T]$  such that for all  $x \in [0, T] \setminus D$ , the sequence  $(F_j(x))_{j \in \mathbb{N}^*}$  converges to  $F_{\infty}(x)$ .*

There is a refinement of the above result when the limit  $\mu_{\infty}$  has no atomic part:

**LEMMA A.2** *Let the sequence of Radon measures  $(\mu_j)_j$  converge weakly  $*$  to a limit  $\mu_{\infty}$  without atomic part. Let  $F_j$  be defined by (12.A.1); then the convergence of  $F_j$  to  $F_{\infty}$  is uniform.*

The proof of this result will be given in a later, developed version, of this article. Its statement makes sensible the main result of the present appendix:

**THEOREM A.3** *Let  $(\mu_j)_{j \in \mathbb{N}^*}$  be a sequence of measures over  $[0, T]$ , which converges weakly  $*$  to  $\mu_{\infty}$ . Assume that  $\mu$  decomposes into its atomic part  $\mu^a$  and its continuous part  $\mu^c$ . There exists a sequence of continuous functions  $\Psi_j$  with values in  $[0, 1]$  such that the following assertions hold:*

- 1  $\Psi_j \mu_j$  converges weakly  $*$  to  $\mu^a$ ;
- 2  $(1 - \Psi_j) \mu_j$  converges weakly  $*$  to  $\mu^c$ , and in particular the integral  $\int_{[0, x]} (1 - \Psi_j) \mu_j$  converges uniformly to the integral  $\int_{[0, x]} \mu_{\infty}^c$ ;
- 3 the measure of the support of  $\Psi_j$  by  $|\mu_j|(1 - \Psi_j)$  tends to 0 as  $j$  tends to infinity.

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