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On 1-2-3 Conjecture-like problems in 2-edge-coloured graphs

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Abstract

The well-known 1-2-3 Conjecture asks whether almost all graphs can have their edges labelled with 1, 2, 3 so that any two adjacent vertices are distinguished w.r.t. the sums of their incident labels. This conjecture has attracted increasing attention over the last years, with many of its aspects of interest being investigated by several authors. In early 2023, Keusch proposed a full solution to the 1-2-3 Conjecture.

Among other aspects of interest, several works introduced and studied ways of generalising such distinguishing labellings and the 1-2-3 Conjecture to structures more general than graphs, such as digraphs and hypergraphs. In the current work, we introduce two new variants for 2-edge-coloured graphs (having negative and positive edges), in which, through labellings, pairs of adjacent vertices are considered distinguished if and only if the differences between their incident positive and negative sums are different. The difference between the two variants we introduce is that, in one of them, this distinction must be met even when considering the absolute value of these differences.

We investigate how these two variants connect, and how they relate to the original problem. For each of the two variants, we also establish upper bounds on the minimum number of consecutive labels that suffice to design a distinguishing labelling of almost any 2-edge-coloured graph. This leads us to raise some conjectures on this minimum, which, as support, we prove for some families of 2-edge-coloured graphs. We also investigate weaker versions of these conjectures, where one can choose the polarity of the edges.

Keywords: Proper labelling; 1-2-3 Conjecture; 2-edge-coloured graph; signed graph.

1. Introduction

In this work, we introduce and study generalisations of **proper labellings** and the so-called **1-2-3 Conjecture to 2-edge-coloured graphs**. To begin with, we review some of the related key concepts and notations.

Let G be a graph¹. A k -labelling ℓ of G is an assignment $\ell : E(G) \rightarrow \{1, \dots, k\}$ of labels (from $\{1, \dots, k\}$) to the edges of G . For every vertex v of G , we then compute the *sum* of labels assigned by ℓ to the edges incident to v , denoted by $\sigma(v)$. In case we have $\sigma(u) \neq \sigma(v)$ for every edge $uv \in E(G)$, or, in other words, the resulting sums form a proper vertex-colouring of G , we say ℓ is *proper*. If G admits proper labellings, then G is said *nice*, in which case we are interested in determining $\chi_\Sigma(G)$, being the smallest $k \geq 1$ such that proper k -labellings of G exist. It is known, see *e.g.* [22], that G is nice if and only if it does not contain K_2 , the complete graph on two vertices, as a connected component.

¹Throughout this work, the term *graph* refers to a simple undirected graph.

These notions were first considered by Karoński, Łuczak, and Thomason [22], who raised the following conjecture, known as the 1-2-3 Conjecture nowadays:

1-2-3 Conjecture. *If G is a nice graph, then $\chi_\Sigma(G) \leq 3$.*

Let us mention right away that a full solution to the 1-2-3 Conjecture was recently proposed by Keusch in [23]. Prior to that, the 1-2-3 Conjecture had attracted the attention of many researchers, resulting in several interesting results. For long, the best result we had was that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G , due to a clever algorithm introduced in [20] by Kalkowski, Karoński, and Pfender. It is known that one cannot do better than the 1-2-3 Conjecture in general, as there exist graphs G with $\chi_\Sigma(G) = 3$. Actually, determining whether $\chi_\Sigma(G) \leq 2$ holds for a given graph G is NP-complete [18]. The same problem for bipartite graphs, however, can be solved in polynomial time, as proved by Thomassen, Wu, and Zhang [28], which, for some time, was another main open problem in this area. Quite a lot of other works have also been dedicated to variants of the 1-2-3 Conjecture. For instance, an early evidence that the 1-2-3 Conjecture might be true was its confirmation when adjacent vertices are required to be distinguished w.r.t. their multisets of incident labels [29] or their products of incident labels [9]. Proper labellings and the 1-2-3 Conjecture have also been generalised to structures more general than graphs. For instance, several generalisations to digraphs have been considered in [2, 5, 8, 12, 19], while a generalisation to hypergraphs was introduced in [21] and further studied in [6, 16].

In the current work, we introduce generalisations of proper labellings and the 1-2-3 Conjecture to graphs in which edges are polarised, *i.e.*, are either *negative* or *positive*. In literature (see *e.g.* [27]), such graphs are sometimes called *2-edge-coloured graphs*, since their edges can be regarded as being coloured with two colours. In our context, due to our upcoming definitions and notions, for simplicity we prefer to refer to such graphs as *signed graphs*². Formally, given a graph G , by making each of its edges either negative or positive, we get a *signature* H of G . For convenience, every parameter of G we would use is used the same way in H : in particular, the notations $V(H)$, $E(H)$, $\Delta(H)$, and so on, have the same meaning as the corresponding notations directly in G .

Let us now consider a signed graph G . By any labelling ℓ of G , note that every vertex v can here be associated with two sums, $\sigma^-(v)$ and $\sigma^+(v)$, which are the sum of labels assigned to the negative edges incident to v and the sum of labels assigned to the positive edges incident to v , respectively. From here, there are thus several options for considering that any two adjacent vertices are distinguished by ℓ . In this work, we consider that ℓ is *proper* as long as $\sigma(u) \neq \sigma(v)$ for every edge $uv \in E(G)$, where, for a vertex w of G , we set $\sigma(w) = \sigma^+(w) - \sigma^-(w)$. In our opinion, this way of doing makes the most sense, as it corresponds to taking into account incident sums in full, in the sense that negative edges contribute negatively to sums while positive edges contribute positively. It can also be observed easily that these definitions generalise those in graphs (see later Observation 2.3). To finish off with the definitions, we say G is *nice* if it does not contain a signature of K_2 as a connected component, and, in case G is nice, we define $\chi_\Sigma(G)$ as the smallest $k \geq 1$ such that proper k -labellings of G exist.

As we will point out later on, it turns out that several results and proof techniques on proper labellings from literature adapt quite naturally from graphs to signed graphs (as defined above), and it is not obvious how to improve further on these. In part for these

²Be aware that, in literature, signed graphs often come up with a particular switching operation, allowing to change the polarity of edges. In the current work, we do not allow to use that operation.

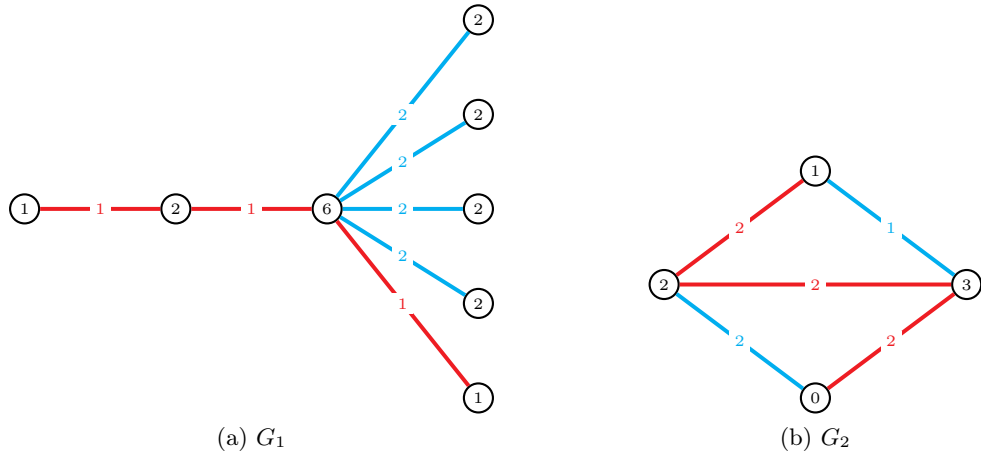


Figure 1: Two signed graphs G_1 (a) and G_2 (b). Note that, in G_1 , assigning label 1 to all edges results in a proper labelling but not in an abs-proper labelling (because of the two vertices incident to the middle edge). Meanwhile, the depicted 2-labelling of G_1 is abs-proper. Thus, $1 = \chi_\Sigma(G_1) < \chi_{|\Sigma|}(G_1) = 2$. Regarding G_2 , it can be noted that assigning label 1 to all edges does not result in a proper labelling (because of the left-most and right-most vertices being adjacent). Meanwhile, the depicted 2-labelling of G_2 is abs-proper. Thus, $\chi_\Sigma(G_2) = \chi_{|\Sigma|}(G_2) = 2$. In each picture, positive and negative edges are represented as red and blue edges. In each vertex the resulting absolute sum is indicated by the depicted labelling.

reasons, we will also investigate the following slightly different notions, which, as will be pointed out, also encapsulate the unsigned ones. For a signed graph G and a labelling ℓ of G , we say that ℓ is *abs-proper* if $|\sigma(u)| \neq |\sigma(v)|$ for every edge $uv \in E(G)$, and, assuming G is nice, we denote by $\chi_{|\Sigma|}(G)$ the smallest $k \geq 1$ such that abs-proper k -labellings of G exist. All these notions and previous ones are illustrated in Figure 1.

This work is organised as follows. In Section 2, we start by raising general remarks on all aforementioned new notions. In particular, we make more explicit the connections and discrepancies between the two new variants we introduce, and with the original one in graphs. In Section 3, we establish constant upper bounds on the parameters $\chi_\Sigma(G)$ and $\chi_{|\Sigma|}(G)$ for nice signed graphs G . This leads us to propose, in Section 4, a conjecture on the maximum value they can reach, which we then prove for some families of signed graphs. In Section 5, we investigate the effects, regarding this conjecture, of being allowed to choose the signature of a signed graph. Finally, in Section 6, we conclude by posing open questions and problems for further exploration on this topic.

2. Early remarks, and connections between the variants

As usual when dealing with a new distinguishing labelling notion, it is important to clarify which are the graphs we are dealing with. In the present case, it is important to identify which are those signed graphs G such that $\chi_\Sigma(G)$ and $\chi_{|\Sigma|}(G)$ are properly defined. Recall that, earlier on, we have introduced a notion of niceness for signed graphs; we claim it is sufficient for both parameters. For transparency, let us mention that the arguments used in the following proof are not new, as it is rather common in the field to assign labels so different that they allow for direct sum distinction (see *e.g.* in the seminal paper [22], in which the authors discuss assigning “linearly independent” labels to edges).

Proposition 2.1. *A signed graph G admits (abs-)proper labellings if and only if G is nice.*

Proof. If G contains a signature of K_2 as a connected component, then clearly G admits no proper labellings. Assume now G is nice, *i.e.*, does not contain such connected components.

We design an abs-proper labelling ℓ of G as follows. Set $E(G) = \{e_1, \dots, e_m\}$. Now, for every $i \in \{1, \dots, m\}$, set $\ell(e_i) = 2^{m+i+1}$ if e_i is positive, and $\ell(e_i) = 2^i$ otherwise. As a result, note that, for every vertex v of G , we have $|\sigma^-(v)| < 2^{m+1}$, while, if v is incident to a positive edge, then $|\sigma^+(v)| \geq 2^{m+2}$. Now, for every two adjacent vertices u and v of G :

- If u is incident to a positive edge but v is not, then

$$|\sigma(u)| = |\sigma^+(u) - \sigma^-(u)| > 2^{m+1} > |\sigma^-(v)| = |\sigma(v)|,$$

and thus u and v are not in conflict (*i.e.*, u and v do not have the same absolute sums, in the current case). Obviously we can derive a similar argument if v is incident to a positive edge while u is not.

- If none of u and v is incident to a positive edge, then $|\sigma(u)| = |\sigma^-(u)|$ and $|\sigma(v)| = |\sigma^-(v)|$. Here, so that $|\sigma(u)| = |\sigma(v)|$ it must be that the base-2 representation of $|\sigma(u)|$ is the same as that of $|\sigma(v)|$, which occurs, due to how labels have been assigned, when u and v are incident to the same edges. So we would here conclude that u and v are part of a connected component being a signature of K_2 , a contradiction.
- If both u and v are incident to a positive edge, then either $|\sigma^+(u)| = |\sigma^+(v)|$, in which case, if there is a conflict between u and v , then we would have that u and v are again incident to the same edges, which is impossible; or $|\sigma^+(u)| \neq |\sigma^+(v)|$, in which case we have $|\sigma^+(u) - \sigma^+(v)| \geq 2^{m+2}$ and $|\sigma^-(u) - \sigma^-(v)| < 2^{m+1}$, implying $(\sigma^+(u) - \sigma^+(v)) + (\sigma^-(v) - \sigma^-(u)) \neq 0$, and thus $|\sigma(u)| \neq |\sigma(v)|$.

This concludes the proof. □

The next observation establishes formally that, out of the two labelling problems we are introducing, the one involving absolute sums should be the hardest.

Observation 2.2. *If G is a nice signed graph, then $\chi_\Sigma(G) \leq \chi_{|\Sigma|}(G)$.*

Proof. This is simply because if, by a labelling of some signed graph, we have $|\sigma(u)| \neq |\sigma(v)|$ for an edge uv , then we also have $\sigma(u) \neq \sigma(v)$. □

We now establish that the two new problems we are considering are indeed generalisations of the original problem in graphs. Indeed, the following is good evidence.

Observation 2.3. *Let G be a nice graph. If H is any signature of G with all edges having the same polarity, then $\chi_\Sigma(H) = \chi_{|\Sigma|}(H) = \chi_\Sigma(G)$.*

Proof. This is because if we consider any proper labelling ℓ of G directly in H , then, for every vertex v , assuming we have $\sigma(v) = x$ in G , in H we either have $\sigma(v) = x$ (if all edges of H are positive) or $\sigma(v) = -x$ (otherwise). In both cases, in H , we also have $|\sigma(v)| = x$. From this, since ℓ is proper in G , it is abs-proper in H . □

Observation 2.3 and arguments in its proof have implications. In particular, note that if we have a proper labelling ℓ of a signed graph in which all edges have the same polarity, then ℓ is also proper in the signed graph obtained by changing the polarity of all edges (as, for every vertex, we obtain the opposite sum). Also, it is known that deciding whether $\chi_\Sigma(G) \leq 2$ holds for a given graph G is NP-complete [18], and this implies the following:

Corollary 2.4. *Deciding whether $\chi_\Sigma(G) \leq 2$ (or similarly whether $\chi_{|\Sigma|}(G) \leq 2$) holds for a given signed graph G is an NP-complete problem.*

A notable implication of Corollary 2.4 is that there exist signed graphs G satisfying $\chi_\Sigma(G), \chi_{|\Sigma|}(G) \geq 3$. More generally speaking, Observation 2.3 will also appear useful later on in this work, when discussing “bad” signed graphs w.r.t. our problems (in particular, when getting to discussing Theorems 2.9 and 2.10).

As mentioned earlier in the introductory section, labelling a signed graph in a proper way is not too distant from labelling a graph in a proper way. Indeed:

Observation 2.5. *Let H be a signature of a graph G , and ℓ' be a k -labelling of H . Consider ℓ , the $\{-k, \dots, -1, 1, \dots, k\}$ -labelling of G obtained by considering every edge e of G , and setting $\ell(e) = \ell'(e)$ if e is positive in H , and $\ell(e) = -\ell'(e)$ otherwise. Then, ℓ is proper in G if and only if ℓ' is proper in H .*

Proof. This is because $\sigma(v)$ has the same value by ℓ and ℓ' , for every vertex v of G . \square

In other words, labelling a signed graph in a proper way is equivalent to labelling its underlying graph in a proper way, but so that edges polarised positive are assigned positive labels while edges polarised negative are assigned negative labels. While this might sound rather particular at first glance, such peculiar types of proper labellings actually connect to a stronger variant of the 1-2-3 Conjecture, namely its list version, defined over the following notions. Let G be a graph, and L be an assignment of real numbers to the edges of G . An L -list-labelling ℓ of G is then a labelling of G where $\ell(e) \in L(e)$ for every edge $e \in E(G)$. Now, we define as $\text{ch}_\Sigma(G)$ the smallest $k \geq 1$ (if any) such that G admits proper L -list-labellings for every list assignment L assigning at most k labels to every edge. Due to Observation 2.5, we now derive the following immediately:

Observation 2.6. *If H is a signature of a graph G with $\text{ch}_\Sigma(G) \leq k$, then $\chi_\Sigma(H) \leq k$.*

While the notion of proper L -list-labellings is obviously much stronger than that of proper labellings, quite some dedicated, related works can be found in literature. The very first work on the topic may be attributed to Bartnicki, Grytczuk, and Niwczyk, who raised in [3] a list variant of the 1-2-3 Conjecture, stating that $\text{ch}_\Sigma(G) \leq 3$ should hold for every nice graph G . Using algebraic tools, Bartnicki, Grytczuk, and Niwczyk first confirmed this conjecture for trees, complete graphs, and complete bipartite graphs. Refinements over their method led other researchers to get great progress towards the conjecture. In particular, Cao was the first to establish a constant upper bound on the parameter in question, showing, in [14], that $\text{ch}_\Sigma(G) \leq 17$ holds for every nice graph G . More recently, Zhu improved this result, showing, in [30], that $\text{ch}_\Sigma(G) \leq 5$ holds for every nice graph G . Other results may be found in literature; for instance, the list version of the 1-2-3 Conjecture was proved to hold for dense Eulerian graphs in [25].

For a signed graph G and a vertex v of G , we denote by $d^-(v)$ the *negative degree* of v , being the number of negative edges incident to v , and by $d^+(v)$ the *positive degree* of v , being the number of positive edges incident to v . The *polarised degree* $d^\pm(v)$ of v is then the difference $d^+(v) - d^-(v)$. Now, we say that G is *locally irregular* if no two adjacent vertices of G have the same polarised degree, *i.e.*, $d^\pm(u) \neq d^\pm(v)$ for every $uv \in E(G)$. Likewise, we say G is *locally abs-irregular* if $|d^\pm(u)| \neq |d^\pm(v)|$ for every $uv \in E(G)$. As an example, note that the signed graph in Figure 1 (a) is locally irregular but not locally abs-irregular.

We remark that these notions generalise the notion of *locally irregular graphs*; then, in our context, we derive similar consequences as in the unsigned setting (see *e.g.* [4]):

Observation 2.7. *For a signed graph G , we have $\chi_{|\Sigma|}(G) = 1$ if and only if G is locally abs-irregular. Likewise, we have $\chi_\Sigma(G) = 1$ if and only if G is locally irregular.*

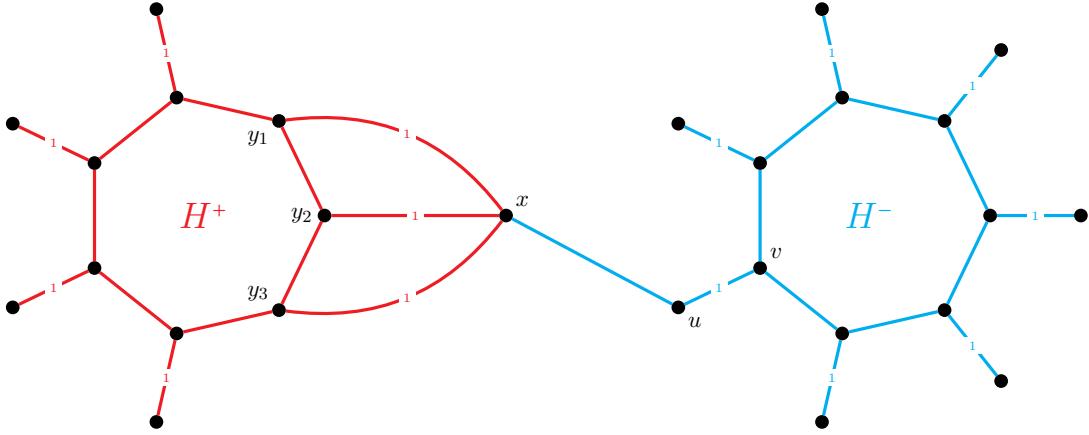


Figure 2: Illustration of the construction of G in the proof of Theorem 2.9, given together with a partial proper 2-labelling. Positive and negative edges are represented as red and blue edges.

Proof. This follows directly from the fact that, when assigning label 1 to all edges of G , we get $\sigma^-(v) = d^-(v)$ and $\sigma^+(v) = d^+(v)$ for every vertex v . \square

From Observation 2.7, we get that there are infinitely many signed graphs G with $\chi_\Sigma(G) = 1$ and even $\chi_{|\Sigma|}(G) = 1$. Obviously, due to Observation 2.3, actually when a locally irregular graph is signed so that all its edges have the same polarity, then we get a locally abs-irregular signed graph. But it is also possible to design locally irregular and locally abs-irregular signed graphs having both positive and negative edges. For instance, to design such signed graphs a useful idea is to attach pendant vertices carefully to make the negative and positive degrees rise. In particular, if uv is *e.g.* a positive edge of some signed graph where u has $d^\pm(u) = 1$, then we cannot have $d^\pm(u) = d^\pm(v)$ provided, omitting uv , the number of negative edges incident to v is not the same as the number of positive edges incident to v . A same remark can be raised when considering absolute polarised degrees. Refer *e.g.* to Figure 1 (a) for an example that can be generalised easily.

This being said, turning a graph into a locally irregular or locally abs-irregular signed graph is not easy in general. As will be exposed in later Section 5, this is actually an NP-complete problem. Another way to state this is that, given a graph G , it is not easy (unless $P = NP$) to say whether G is the support of an irregular signed graph.

Let us now add a few words regarding the possible discrepancies between the two signed graph parameters χ_Σ and $\chi_{|\Sigma|}$. First off, it is easy to generalise the signed graph in Figure 1 (a) to infinitely many examples showing the next statement is true:

Observation 2.8. *There exist arbitrarily large connected signed graphs G that satisfy $\chi_\Sigma(G) = 1 < 2 = \chi_{|\Sigma|}(G)$.*

One might argue that Observation 2.8 depicts a very peculiar situation, which, by itself, does not provide a good indication of how the two parameters χ_Σ and $\chi_{|\Sigma|}$ can differ in general. As a better argument, we prove the following:

Theorem 2.9. *There exist arbitrarily large connected signed graphs G that satisfy $\chi_\Sigma(G) = 2 < 3 \leq \chi_{|\Sigma|}(G)$.*

Proof. Consider the following construction (which generalises easily to arbitrarily large signed graphs). We use a graph construction from [10] (denoted A' there), which provides a connected (unsigned) graph H with the following properties (see Claims 1 and 2 in [10]):

- H has arbitrarily many pendant edges u_1v_1, \dots, u_kv_k (which we call “forced” below), where the u_i ’s have degree 1 and the v_i ’s are pairwise distinct;
- H is not locally irregular;
- there exist proper 2-labellings ℓ of H , and they all satisfy:
 - $\ell(u_iv_i) = 1$ for every $i \in \{1, \dots, k\}$;
 - $\sigma(v_i) = 3$ for every $i \in \{1, \dots, k\}$.

The construction, illustrated in Figure 2, now goes as follows. Start from two disjoint copies H^- and H^+ of the graph H above, where H^- has at least one forced pendant edge uv (where $d(u) = 1$), H^+ has at least three forced pendant edges x_1y_1, x_2y_2, x_3y_3 (where the x_i ’s have degree 1), and all edges of H^- are negative while all edges of H^+ are positive. Now identify all of x_1, x_2, x_3 to a single vertex x , and add the negative edge ux . The resulting signed graph is G .

We claim that $\chi_\Sigma(G) = 2$ but $\chi_{|\Sigma|}(G) > 2$. Note first that since G contains copies (H^- and H^+) of H with all edges having the same polarity, then G cannot be locally irregular (and thus G cannot be locally abs-irregular as well). For the same reason, note that any proper and abs-proper 2-labelling ℓ of G must satisfy that $\ell(uv) = 1$ and $\ell(xy_1) = \ell(xy_2) = \ell(xy_3) = 1$ (since all of $uv, x_1y_1, x_2y_2, x_3y_3$ are forced edges of H^- and H^+ , respectively). Also we must have $\sigma(v) = -3$ (and thus $|\sigma(v)| = 3$), and $\sigma(y_1) = \sigma(y_2) = \sigma(y_3) = 3$. Now, so that $\sigma(u) \neq \sigma(v)$, it must be that $\ell(ux) = 1$, which yields $\sigma(u) = -2$ (and thus $|\sigma(u)| = 2$) and $\sigma(x) = 2$. In particular, note that x cannot be in conflict with y_1, y_2, y_3 . When considering sums, note that we have totally identified what ℓ looks like, and there is no conflict contradicting it is proper. Meanwhile, note that we get $|\sigma(u)| = 2 = |\sigma(x)|$, and thus ℓ cannot be abs-proper. Thus, we have $\chi_\Sigma(G) = 2$ but $\chi_{|\Sigma|}(G) > 2$. \square

In our opinion, previous Observation 2.8 and Theorem 2.9 expose that, although they are closely related in spirit, the sum version and the absolute sum version of our problem exhibit some differences, with the latter being more difficult (in the sense that we sometimes need to consider using a larger label set) than the former. Also, we believe this justifies considering both versions, and not just the strongest one.

Regarding Observation 2.3, we would like to note also that signing the edges of a graph can sometimes make the number of needed labels rise. While this follows from earlier results on locally irregular (signed) graphs we have provided, in the next result we prove it is also true in very simple structures such as trees.

Theorem 2.10. *There are arbitrarily large trees T with a signature H such that $2 = \chi_\Sigma(T) < \chi_{|\Sigma|}(H)$.*

Proof. Recall that for every nice tree T , we have $\chi_\Sigma(T) \leq 2$ (see e.g. [15]). So it mainly suffices to show that there exist signed trees H with $\chi_{|\Sigma|}(H) > 2$, where H is a signature of a tree that is not locally irregular. To prove that arbitrarily large such H exist, and get the claim, we mostly combine two main notions. In a signed graph G , we say a pendant negative edge uv (where $d(u) = 1$) is α -forced if, in every abs-proper 2-labelling ℓ of G omitting the possible conflict between u and v , we have $|\sigma(v)| = 3$ if $\ell(uv) = 1$, and $|\sigma(v)| = 0$ if $\ell(uv) = 2$. Likewise, a pendant positive edge uv (where $d(u) = 1$) is said β -forced if, in every abs-proper 2-labelling ℓ of G omitting the possible conflict between u and v , we have $|\sigma(v)| = 1$ if $\ell(uv) = 1$, and $|\sigma(v)| = 2$ if $\ell(uv) = 2$. In the next three claims, we show how to take advantage of α -forced and β -forced edges to create larger trees with α -forced and β -forced edges, and signed trees with no abs-proper 2-labellings.

Claim 2.11. *Let G be a signed tree with an α -forced or β -forced edge uv . Then, when considering two copies G_1 and G_2 of G and merging the two copies of the edge uv (that is, identifying the copy of u in G_1 and the copy of v in G_2 , and vice versa), we get a signed tree H with no abs-proper 2-labellings.*

Proof of the claim. This is because, by the properties of G , in every abs-proper 2-labelling ℓ of H , denoting by xy the merged edge we must have $|\sigma(x)| = |\sigma(y)| = 3$ if $\ell(xy) = 1$, and $|\sigma(x)| = |\sigma(y)| = 0$ otherwise, assuming the original edge is α -forced. Either way, we have a conflict between x and y . A same conclusion is reached if the original edge is β -forced. \diamond

Claim 2.12. *Let G be a signed tree with an α -forced edge uv , where $d(u) = 1$. Then, in the signed tree H obtained from G by adding two pendant positive edges ux and uy incident to u , the edge uy is β -forced.*

Proof of the claim. Consider an abs-proper 2-labelling ℓ of H where the possible conflict between u and y might be omitted.

- If $\ell(uv) = 1$, then, by definition, we have $|\sigma(v)| = 3$. Now, so that we do not have a conflict between u and x , we must have $\ell(uy) = 2$, and, in turn, so that u and v are not in conflict we must have $\ell(ux) = 1$. As a result, we get $|\sigma(u)| = 2$ and $|\sigma(x)| = 1$.
- If $\ell(uv) = 2$, then $|\sigma(v)| = 0$. So that we do not have a conflict between u and x we must have $\ell(uy) = 1$, and, now, so that we do not have a conflict between u and v we must have $\ell(ux) = 2$. Then, $|\sigma(u)| = 1$ and $|\sigma(x)| = 2$.

Thus, uy is β -forced. \diamond

Claim 2.13. *Let G be a signed tree with a β -forced edge uv , where $d(u) = 1$. Then, in the signed tree H obtained from G by adding a pendant positive edge ux and a pendant negative edge uy incident to u , the edge uy is α -forced.*

Proof of the claim. The proof is similar to that of Claim 2.12. Consider an abs-proper 2-labelling ℓ of H , omitting possibly the conflict between u and y .

- If $\ell(uv) = 1$, then, by definition, $|\sigma(v)| = 1$. So that there is no conflict between u and x , we must have $\ell(uy) = 2$, and, now, so that u and v are not in conflict we must have $\ell(ux) = 1$. As a result, we get $|\sigma(u)| = 0$ and $|\sigma(x)| = 1$.
- If $\ell(uv) = 2$, then $|\sigma(v)| = 2$. So that there is no conflict between u and x we must have $\ell(uy) = 1$, and, next, so that we do not have a conflict between u and v we must have $\ell(ux) = 2$. Then, $|\sigma(u)| = 3$ and $|\sigma(x)| = 2$.

Thus, uy is α -forced. \diamond

We now employ Claims 2.11 to 2.13 to prove Theorem 2.10. As a starting piece, we consider G , the claw with center u and leaves v_1, v_2, v_3 , signed so that uv_1 and uv_2 are positive, while uv_3 is negative. We claim uv_3 is α -forced. Indeed, consider, omitting a possible conflict between u and v_3 , an abs-proper 2-labelling ℓ of G .

- If $\ell(uv_3) = 1$, then note that none of uv_1 and uv_2 can be assigned label 1. Indeed, if, say, we had $\ell(uv_1) = 1$, then it can be noted that we would get $|\sigma(u)| = |\sigma(v_2)|$. Thus, we must have $\ell(uv_1) = 2$ and $\ell(uv_2) = 2$. Hence, $|\sigma(v_1)| = |\sigma(v_2)| = 2$, and $|\sigma(u)| = 3$.

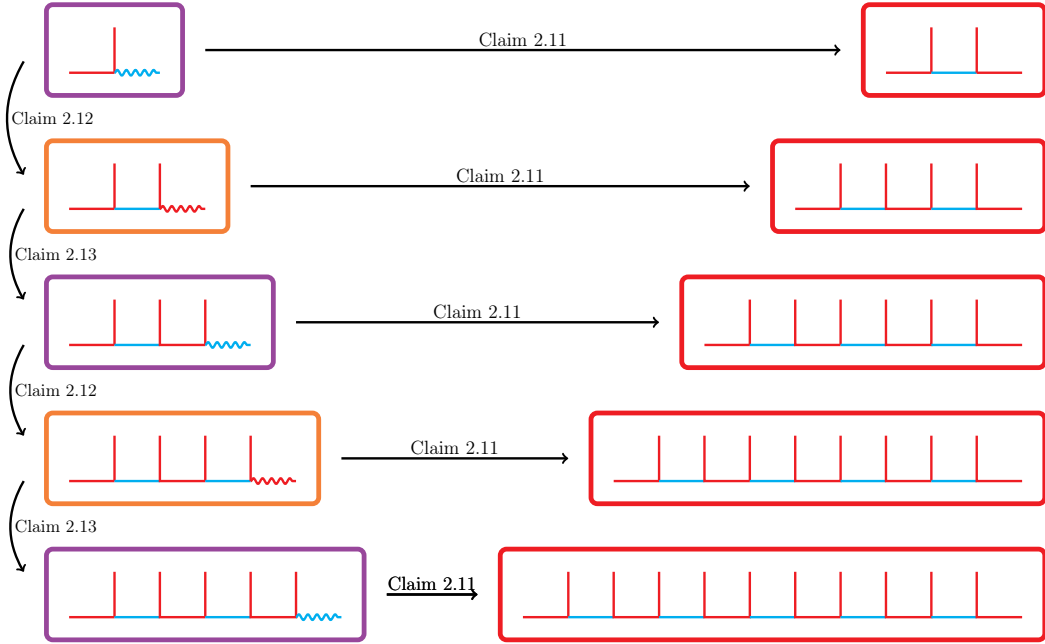


Figure 3: First steps of the construction of arbitrarily large signed trees with no abs-proper 2-labellings, described in the proof of Theorem 2.10. Any arrow joining two trees indicates which construction, from Claims 2.11 to 2.13, is applied. Signed trees in purple and orange boxes are intermediate pieces, while signed trees in red boxes have the desired property. Wiggly edges in purple boxes are α -forced edges, while those in orange boxes are β -forced edges. Positive and negative edges are represented as red and blue edges.

- Likewise, if $\ell(uv_3) = 2$, then none of uv_1 and uv_2 can be assigned label 2. So $\ell(uv_1) = \ell(uv_2) = 1$, and $|\sigma(v_1)| = |\sigma(v_2)| = 1$ while $|\sigma(u)| = 0$.

Thus, G indeed has an α -forced edge.

We now have two options. On the one hand, applying Claim 2.11, we get, from G , a signed tree H_1 with no abs-proper 2-labellings. On the other hand, applying Claim 2.12, we get, from G , a signed tree H_2 with a β -forced edge. Note that, actually, $H_1 = H_2$, which is a peculiar phenomenon due to how small G is. From H_2 , we can now apply either Claim 2.11 to create another signed tree with no abs-proper 2-labellings, or Claim 2.13 to get a signed tree H with an α -forced edge. Here again, we have two options. First, applying Claim 2.11 from H we get a signed tree I_1 with no abs-proper 2-labellings. Second, applying Claim 2.12 from H we get a signed tree I_2 with a β -forced edge. Note that, here, $I_1 \neq I_2$. Now either we apply Claim 2.11 from I_2 to get another signed tree with no abs-proper 2-labellings, or we apply Claim 2.13 from I_2 to get a signed tree I with an α -forced edge, and we again have two options. And so on.

This process can be repeated ad libitum, and after each application of Claim 2.11 we get a larger signed tree with no abs-proper 2-labellings (see Figure 3 for an illustration). The claim thus holds true. \square

3. Constant upper bounds

In this section, we establish constant upper bounds on the parameters χ_Σ and $\chi_{|\Sigma|}$ for nice signed graphs. For that, we mainly exploit tools and approaches from previous works on the 1-2-3 Conjecture, one of our goals being to investigate how they adapt to our context. As will be apparent, many of these, in particular those relying on modulo

methods (where sums are considered modulo some value), fit well when not considering absolute sums. For these, we have to tweak arguments, the price being to use more labels.

To be fully transparent, let us mention that the proof of upcoming Theorem 3.1 is inspired by the proof of Kalkowski, Karoński, and Pfender of their 1-2-3-4-5 result [20]. Also, the proofs of upcoming Theorems 3.2, 3.3, and 3.4 are inspired by modulo methods, as first used in this context by Karoński, Łuczak, and Thomason [22].

We focus first on proving a constant upper bound on $\chi_{|\Sigma|}(G)$ for every nice signed graph G (thereby also getting a constant upper bound on $\chi_{\Sigma}(G)$ by Observation 2.2). Our main result, Theorem 3.6, is that $\chi_{|\Sigma|}(G) \leq 9$ holds for every nice signed graph G . Prior to getting to the proof of this result, we first need to make a few steps.

Theorem 3.1. *If G is a nice connected signed graph with $\Delta(G) \geq 4$, then $\chi_{|\Sigma|}(G) \leq 9$.*

Proof. Let us denote by v_1, \dots, v_n the vertices of G , so that 1) every vertex v_i with $i < n$ is incident to a *forward edge* (i.e., an edge $v_i v_j$ with $i < j$), and 2) vertex v_n is incident to $\Delta(G) \geq 4$ *backward edges* (i.e., edges $v_i v_n$ with $i < n$). Such an ordering of the vertices of G exists, essentially because G is connected; one can for instance be obtained by choosing v_n as any vertex of degree $\Delta(G)$, and ordering the other vertices by reversing the order in which they are encountered during a Breadth-First Search algorithm performed from v_n .

We will design an abs-proper 9-labelling ℓ of G in the following way. Starting from all edges assigned label 5, we will process the v_i 's one by one in order, without ever coming back. Whenever considering a new v_i with $i \in \{1, \dots, n-1\}$, we will define a set $\Phi(v_i)$ of permitted sums for $\sigma(v_i)$, which, provided we indeed have $|\sigma(v_i)| \in \Phi(v_i)$, will ensure it is impossible for v_i to be involved in conflicts (w.r.t. absolute sums) with its backward neighbours. This will be achieved by defining the $\Phi(v_i)$'s so that $\Phi(v_j) \cap \Phi(v_i) = \emptyset$ for every backward neighbour v_j of v_i , and guaranteeing that $|\sigma(v_i)|$ lies in $\Phi(v_i)$ at all times. The $\Phi(v_i)$'s will also be chosen so that they include several values, so that, whenever considering a new vertex v_i , for every of its incident backward edges $v_j v_i$ it is possible to alter $\ell(v_j v_i)$ while preserving $|\sigma(v_j)| \in \Phi(v_j)$ (which most of the time will be possible because we started from all edges of G being assigned the middle label 5), thereby allowing to alter $\sigma(v_i)$ through local modifications, and thus reaching more possibilities w.r.t. the set $\Phi(v_i)$ we need to define. Once all v_i 's but v_n will be processed that way, we will get to the point where the only reason why ℓ is perhaps not abs-proper is because v_n is involved in conflicts. By performing the last local modifications to the labelling, we will get rid of these conflicts without introducing new ones, reaching a labelling of G as desired.

The formal details are as follows. Start from ℓ assigning label 5 to all edges of G . Now consider every vertex v_i of G in (v_1, \dots, v_{n-1}) in turn, following the ordering we have defined. We denote by $u_1 v_i, \dots, u_b v_i$ the $b \geq 0$ backward edges incident to v_i , and by $v_i w_1, \dots, v_i w_f$ the $f \geq 1$ forward edges incident to v_i , where (w_1, \dots, w_f) is a subsequence of (v_1, \dots, v_n) . Recall that indeed $f \geq 1$ due to how we ordered the vertices of G . Also, $d(v_i) = b + f \geq 1$.

As mentioned earlier, we need to guarantee that ℓ has certain properties; more precisely, we assume every vertex v_j with $j < i$ has been treated so that the following properties hold:

1. We have defined a positive value $\phi(v_j)$ and a set $\Phi(v_j) = (\phi(v_j), \phi(v_j) + 4)$, where $\phi(v_j) \bmod 8$ lies in $\{0, 1, 2, 3\}$ (and, thus, $\phi(v_j) + 4 \bmod 8$ lies in $\{4, 5, 6, 7\}$).
2. We have $\sigma(v_j) \in \{-(\phi(v_j) + 4), -\phi(v_j), \phi(v_j), \phi(v_j) + 4\}$, and, thus, $|\sigma(v_j)| \in \Phi(v_j)$.
3. For every backward neighbour v_k of v_j , we have $\Phi(v_k) \cap \Phi(v_j) = \emptyset$.

4. For every forward edge $v_j v_k$ incident to v_j (including $v_j v_i$), we can, regardless of the polarity of $v_j v_k$, alter $\ell(v_j v_k)$ by one of -4 or $+4$ while preserving Property 2.
5. When dealing with v_j , the only label modifications we performed are around v_j . More precisely, we may have modified the labels assigned to incident backward edges (while preserving Property 3 for the backward neighbours), and the label assigned to the forward edge $v_j v_k$ with minimum $k > j$ (while preserving Property 2 for v_j).

Properties 2 and 3 above guarantee that v_j cannot be involved (w.r.t. absolute sums) in a conflict with any of its backward neighbours. Properties 1 and 4 are essentially here to guarantee that, later on, whenever dealing with a forward neighbour v_i of v_j , we can alter ℓ locally to modify $\sigma(v_i)$ without introducing new conflicts, thereby getting more possibilities for defining the desired $\phi(v_i)$ and $\Phi(v_i)$. In particular, when dealing with v_i , by Property 4 and how $\Phi(v_j)$ was defined we can either decrease or increase $\ell(v_j v_i)$ by 4 without introducing new conflicts, and keep that v_j satisfies Property 2. We call this performing a *valid change* onto $v_j v_i$. This valid change is either *decreasing* if it does decrease $\sigma(v_i)$, or *increasing* otherwise. Finally, Property 5 above guarantees that, when dealing with v_i , every incident backward edge can indeed be subject to a valid change.

So, assume we are, in the process, now dealing with a new v_i .

- If $b = 0$, then v_i is only incident to forward edges. Assume $|\sigma(v_i)| = x$. If $x \bmod 8$ lies in $\{0, 1, 2, 3\}$, then we just set $\phi(v_i) = x$, which defines $\Phi(v_i)$ as $(x, x + 4)$, and we are done with v_i (Properties 1 to 5 are indeed fulfilled). Now, if $x \bmod 8$ lies in $\{4, 5, 6, 7\}$, then we set $\phi(v_i) = x - 4$, and we are also done. In particular, in both cases, we still have $\ell(v_i w_j) = 5$ for every forward edge incident to v_i , and we do have $\sigma(v_i) \in \{-(\phi(v_i) + 4), -\phi(v_i), \phi(v_i), \phi(v_i) + 4\}$ (and, thus, $|\sigma(v_i)| \in \Phi(v_i)$).
- Now assume $b \geq 1$. For every backward edge $u_j v_i$ recall that we can perform a valid change, being either decreasing or increasing (from $\sigma(v_i)$'s point of view). We start by performing (decreasing) valid changes backwards so that $\sigma(v_i)$ is minimised. We also change $\ell(v_i w_1)$ to either 1 or 9 so that $\sigma(v_i)$ is minimised even further. Let us denote by a the resulting value as $\sigma(v_i)$. Note that, through increasing $\ell(v_i w_1)$ to any value in $\{2, 3, 4\}$ (if $\ell(v_i w_1) = 1$) or to any value in $\{6, 7, 8\}$ (if $\ell(v_i w_1) = 9$), performing at most b of the (all increasing) b valid changes backwards, and modifying $\ell(v_i w_1)$ with some value, we can get $\sigma(v_i)$ to take any value in $\{a, \dots, a + 4b + 8\}$, which set has cardinality $4b + 9$. Now, note that the set

$$S = \{-(\phi(u_j) + 4), -\phi(u_j), \phi(u_j), \phi(u_j) + 4 : u_j \text{ is a backward neighbour of } v_i\}$$

has cardinality at most $4b$, and its at most $4b$ values are forbidden for $\sigma(v_i)$ (and thus as $\phi(v_i)$ and $\phi(v_i) + 4$). Due to Properties 4 and 5 we need to preserve, note also that we cannot define, as $\phi(v_i) + 4$, a value in $\{|a|, |a| + 1, |a| + 2, |a| + 3\}$ if the corresponding value (a , $a + 1$, $a + 2$, or $a + 3$) modulo 8 lies in $\{4, 5, 6, 7\}$ (as, to be reached as $\sigma(v_i)$, it requires to have $\ell(v_i w_1) \in \{1, 2, 3, 4\}$ (if $v_i w_1$ is positive) or $\ell(v_i w_1) \in \{6, 7, 8, 9\}$ (otherwise), making it impossible, when treating w_1 later on, to perform a valid (decreasing) change onto $v_i w_1$). Likewise, we cannot define as $\phi(v_i)$ a value in $\{|a| + 4b + 5, |a| + 4b + 6, |a| + 4b + 7, |a| + 4b + 8\}$ if the corresponding value lies in $\{0, 1, 2, 3\}$ modulo 8. These facts forbid at most eight more values as $\phi(v_i)$. There is thus at least one value x in $\{a, \dots, a + 4b + 8\}$ such that none of $-(x + 4)$, $-x$, x , and $x + 4$ lies in S , and that can be reached as $\sigma(v_i)$ through performing valid (increasing) changes backwards, and possibly altering $\ell(v_i w_1)$ in such a way that it

will be possible to perform a valid change onto $v_i w_1$ later on, when treating w_1 . So we set $\phi(v_i) = |x|$ or $\phi(v_i) + 4 = |x|$ (depending on the value of x modulo 8), define $\Phi(v_i)$ accordingly, and apply the corresponding label modifications around v_i so that $|\sigma(v_i)| \in \Phi(v_i)$. Once this is performed, note that Properties 1 to 5 are fulfilled for v_i and all u_j 's. Also, note that we still have $\ell(u_i w_j) = 5$ for every $j \in \{2, \dots, f\}$.

Once we have treated every vertex v_i with $i \in \{1, \dots, n-1\}$ this way, then, since Properties 2 and 3 have been preserved, for every edge $v_i v_j$ with $i, j < n$, we must have $|\sigma(v_i)| \neq |\sigma(v_j)|$. Thus, if v_n is not involved in any conflict (w.r.t. absolute sums), then ℓ is abs-proper and we are done. Otherwise, we get rid of these conflicts, without introducing new ones, in the following way. Recall that, onto every backward edge $u_i v_n$ incident to v_n , a valid change can be performed. We start by considering every backward edge $u_i v_n$ incident to v_n , and, if $|\sigma(u_i)| = \phi(u_i) + 4$, then we perform a valid change onto $u_i v_n$. This way, we can now assume we have $|\sigma(u_i)| = \phi(u_i)$, and thus $\sigma(u_i) \bmod 8 \in \{0, 1, 2, 3\}$, for every $i \in \{1, \dots, b\}$. Again, if v_n is not involved in conflicts, then we are done. So let us assume this is not the case. In particular, this means $|\sigma(v_n)|$ modulo 8 lies in $\{0, 1, 2, 3\}$.

We need to analyse the effects of at least four valid changes that we can perform backward at v_n . Note that any of these valid changes modifies $\sigma(v_n)$ by -4 or $+4$, which we regard similarly as decrementing $\sigma(v_n)$ by 1 four times, or incrementing $\sigma(v_n)$ by 1 four times. When performing these four decrements or increments, if $\sigma(v_n)$ gets closer to 0 at some point, then the valid change is said *tightening*. Otherwise, it is said *releasing*. As an example, note that if $\sigma(v_n) = 1$ and, through performing a valid change onto $u_i v_n$ for some i , we get $\sigma(v_n) = -3$, then that valid change is considered tightening (although 3 is more distant to 0 than 1 is; what matters is that we “went through” 0). Another way to state this is that if a valid change decreases $\sigma(v_n)$ by 4, then it is considered tightening if $\sigma(v_n) - 1$ is closer to 0 than $\sigma(v_n)$ is. Otherwise, it is considered releasing. Things are defined analogously when a valid change increases $\sigma(v_n)$ by 4.

We consider a few cases:

- For every backward neighbour u_i of v_n , we have $|\sigma(u_i)| = |\sigma(v_n)| = x$.

In that case, let $k \geq 0$ be the number of releasing valid changes. If $k \geq 2$, then assume $u_1 v_n$ and $u_2 v_n$ are two edges onto which releasing valid changes can be applied, and apply them. As a result, recall we still have $|\sigma(u_1)| \in \Phi(u_1)$ and $|\sigma(u_2)| \in \Phi(u_2)$, and, actually, $|\sigma(u_1)| = \phi(u_1) + 4$ and $|\sigma(u_2)| = \phi(u_2) + 4$, which both values lie in $\{4, 5, 6, 7\}$ modulo 8. Meanwhile, we still have $|\sigma(u_i)| = x$ for every $i \geq 3$, while, now, $|\sigma(v_n)| = x + 8$, which lies in $\{0, 1, 2, 3\}$ modulo 8. Thus, the resulting ℓ is abs-proper. Now assume $k \leq 1$. Since $d(v_n) \geq 4$, we can thus assume $u_1 v_n$, $u_2 v_n$, and $u_3 v_n$ are three edges onto which tightening valid changes can be performed.

- If $x \geq 4$, then apply a valid change to $u_1 v_n$. As a result, we now have $|\sigma(u_1)| = x + 4$, while we still have $|\sigma(u_i)| = x$ for every $i \geq 2$. Meanwhile, by the definition of a tightening change, now we have $|\sigma(v_n)| = x - 4$. Thus, there are no conflicts.
- If $x = 3$, then apply a valid change to $u_1 v_n$. Here, we get $|\sigma(u_1)| = x + 4 = 7$ while we still have $|\sigma(u_i)| = x = 3$ for every $i \geq 2$, but now $|\sigma(v_n)| = |x - 4| = 1$. So there cannot be any conflict. Note that such arguments also apply when $x = 1$.
- If $x = 2$, then apply a valid change to all of $u_1 v_n$, $u_2 v_n$, and $u_3 v_n$. As a result, we get $|\sigma(v_n)| = x + 8 = 10$. Meanwhile, we have $|\sigma(u_1)| = |\sigma(u_2)| = |\sigma(u_3)| = x + 4 = 6$, while we still have $|\sigma(u_i)| = x = 2$ for every $i \geq 4$. Thus, here as well there are no conflicts remaining.

– If $x = 0$, then note that, actually, none of the valid changes backwards can be tightening. So this case cannot occur.

- There is a backward neighbour u_i of v_n with $|\sigma(u_i)| \neq |\sigma(v_n)| = x$.

We denote by S the set of u_i 's with the same absolute sum as v_n , and by D the set of all other u_i 's, *i.e.*, with different absolute sums. So, here, D is not empty; throughout, we assume w.l.o.g. that $u_1 \in D$. Note that if the valid change we can perform onto u_1v_n is releasing, then we are done when performing this valid change (since, then, only u_1 and v_n have absolute sum in $\{4, 5, 6, 7\}$ modulo 8, while we know $|\sigma(u_1)| \neq |\sigma(v_n)|$). Thus, we can assume that, for every $u_i \in D$, the valid change to u_iv_n is tightening. We now consider a few more cases. As before, note that we cannot have $x = 0$ now that we have identified that the valid change of u_1v_n is tightening.

– If $x \geq 4$, then applying the valid change to u_1v_n yields $|\sigma(u_1)| = x+4$ and $|\sigma(v_n)| = x-4$, and these two absolute sums lie in $\{4, 5, 6, 7\}$ modulo 8. Meanwhile, recall that all other u_i 's have absolute sum in $\{0, 1, 2, 3\}$ modulo 8. Thus, we have no conflict if, initially, prior to performing the valid change to u_1v_n , we had $\phi(u_1) \neq x-8$. So, now, come back to the situation where $|\sigma(u_i)| = \phi(u_i)$ for every $i \in \{1, \dots, b\}$. By the previous remarks, we can now assume that all $\phi(u_i)$'s lie in $\{x, x-8\}$ (either $|\sigma(u_i)| = |\sigma(v_n)| = x$, or $|\sigma(u_i)| = x-8$). Since the current labelling is not abs-proper, we can assume v_n is in conflict with at least one of the u_i 's.

- * If, say, $|\sigma(u_2)| = x$ and the valid change we can perform to u_2v_n is tightening, then we are done when performing that valid change, since this yields $|\sigma(u_2)| = x+4$ and $|\sigma(v_n)| = x-4$, which two values are different (since $x \neq 0$), while all other u_i 's have absolute sums with a different value modulo 8.

- * If, say, u_2 is the only vertex u_i with $|\sigma(u_i)| = x$, then, since the previous case does not apply, recall the valid change we can perform to u_2v_n is releasing. In that case, we perform a valid change to both u_1v_n and u_2v_n . Recall that, since the valid change to u_1v_n was tightening, we get $|\sigma(v_n)| = x$. Meanwhile, $|\sigma(u_1)| = x-4$ while $|\sigma(u_2)| = x+4$. Also, by hypothesis, no other u_i has $|\sigma(u_i)| = x$. Thus, there are no conflicts remaining.

- * The last case to consider is when, say, u_2 and u_3 satisfy $|\sigma(u_2)| = |\sigma(u_3)| = x$. Recall that the valid changes to u_2v_n and u_3v_n are both releasing (as otherwise a previous case would apply). In that case, we apply both valid changes. This yields $|\sigma(v_n)| = x+8$, and $|\sigma(u_2)| = |\sigma(u_3)| = x+4$, while all other u_i 's have absolute sum in $\{x, x-8\}$. Thus, the labelling is abs-proper.

– Assume now $x \in \{1, 3\}$, and set $|D| = k \geq 1$. Recall that, for every $u_i \in D$, the valid change to u_iv_n is tightening.

- * If $k = 1$, then apply the valid change to u_1v_n . As a result, $|\sigma(u_1)| = x+4 \geq 4$, and, since $x \in \{1, 3\}$, we get $|\sigma(v_n)| \in \{1, 3\} \setminus \{x\}$. Meanwhile, we have $|\sigma(u_i)| = x$ for all $i \geq 2$.

- * If $k \geq 3$ and k is odd, then we perform every (tightening) valid change to u_iv_n for $u_i \in D$. As a result, note that $|\sigma(v_n)|$ changed (since $k \geq 3$), and that the resulting $|\sigma(v_n)|$ lies in $\{0, 1, 2, 3\}$ modulo 8. Actually, $|\sigma(v_n)| \geq x+6$. Meanwhile, for every $u_i \in D$, we have $|\sigma(u_i)| = \phi(u_i) + 4$, and $|\sigma(u_i)|$ lies in $\{4, 5, 6, 7\}$ modulo 8. For every $u_i \in S$, we still have $|\sigma(u_i)| = x$.

- * The last case is when $k \geq 2$ and k is even. In that case, we perform a valid change to every $u_i v_n$ such that $u_i \in D \setminus \{u_1\}$. So, just as in the previous case, we perform an odd number of valid changes. As a result, the absolute sum of v_n lies in $\{0, 1, 2, 3\}$ modulo 8, and it is different from x (even if $k = 2$, since $x \in \{1, 3\}$). Since u_1 is the only neighbour of u_i with an absolute sum in $\{0, 1, 2, 3\}$ modulo 8 being not x , if there is a conflict then it must be between u_1 and v_n . Note that we would have ended up with the same conclusion if we had performed the previous valid changes to any $k - 1$ edges $u_i v_n$ where $|\sigma(u_i)| \neq x$, as otherwise we would be done. So, initially, all $u_i \in D$ can be assumed to have the same absolute sum, which is, actually, $|x - 4| + 4(k - 2)$. Note that this is at least 4 if $k > 2$.
 - If $k > 2$, then, from the initial situation, we perform a (tightening) valid change to $u_1 v_n$ only. As a result, recall that $|\sigma(v_n)| \neq x$ since $x \in \{1, 3\}$, so v_n is not in conflict with any $u_i \in S$. Meanwhile, since $|\sigma(v_n)| \in \{1, 3\}$, we cannot have a conflict between v_n and the other k u_i 's, since their absolute sums are at least 4.
 - If $k = 2$, then, since $b \geq 4$, then, say, u_3 exists and $|\sigma(u_3)| = x$. Assume w.l.o.g. that $|\sigma(u_2)| \neq x$. Here, we perform valid changes to $u_1 v_n$, $u_2 v_n$, and $u_3 v_n$. As a result, we get $|\sigma(v_n)| \in \{|x - 4|, |x - 4| + 8\}$, depending on whether the valid change to $u_3 v_n$ was releasing or tightening. Regardless, note that $|\sigma(v_n)|$ lies in $\{0, 1, 2, 3\}$ modulo 8. Thus, v_n cannot be in conflict with any of u_1 , u_2 , and u_3 . Also, since $x \in \{1, 3\}$ modulo 8, we have $|\sigma(v_n)| \neq x$, implying v_n cannot be in conflict with any $u_i \in S$.
- Assume last $x = 2$. If, say, u_2 is another vertex $u_i \in D$, then we perform both (tightening) valid changes to $u_1 v_n$ and $u_2 v_n$. As a result, we get $|\sigma(v_n)| = x + 4 = 6$, while $|\sigma(u_1)|$ and $|\sigma(u_2)|$ are not 6 (as otherwise v_n would have been in conflict with u_1 and u_2 initially), and all other u_i 's have absolute sum in $\{0, 1, 2, 3\}$ modulo 8. So, we may assume u_1 is the only neighbour of v_n with a distinct absolute sum.
 - * If $\phi(u_1) \neq 10$, then, since $b \geq 4$, vertices u_2 , u_3 , and u_4 exist, and they have absolute sum x . If, say, the valid changes to $u_2 v_n$ and $u_3 v_n$ are releasing, then, by performing them, we get $|\sigma(v_n)| = x + 8 = 10$, while $|\sigma(u_1)| \neq 10$, $|\sigma(u_2)| = |\sigma(u_3)| = x + 4 = 6$, and $|\sigma(u_i)| = x$ for every $i \geq 4$. Otherwise, *i.e.*, say, the valid changes to $u_2 v_n$ and $u_3 v_n$ are tightening, and by performing the valid changes to $u_1 v_n$, $u_2 v_n$, and $u_3 v_n$ we get $|\sigma(v_n)| = x + 8 = 10$, while u_1 , u_2 , and u_3 have absolute sum in $\{4, 5, 6, 7\}$ modulo 8, and $|\sigma(u_i)| = x$ for every $i \geq 4$.
 - * If $\phi(u_1) = 10$, then, again, we consider u_2 , u_3 , and u_4 . If, say, the valid changes to $u_2 v_n$, $u_3 v_n$, and $u_4 v_n$ are releasing, then, by performing them, we get $|\sigma(v_n)| = x + 12 = 14$, while u_2 , u_3 , and u_4 have absolute sum $x + 4 = 6$, vertex u_1 has absolute sum 10, and $|\sigma(u_i)| = x$ for every $i \geq 5$. Likewise, if, say, the valid changes to $u_2 v_n$ and $u_3 v_n$ are tightening, then, by performing valid changes to $u_1 v_n$, $u_2 v_n$, and $u_3 v_n$, we get $|\sigma(v_n)| = 10$, while $|\sigma(u_1)| = 14$, $|\sigma(u_2)| = |\sigma(u_3)| = x + 4 = 6$, and $|\sigma(u_i)| = x$ for every $i \geq 4$. Now, if none of the previous cases applied, then, since $b \geq 4$, it must be that $b = 4$, and that, say, the valid change to $u_2 v_n$ is tightening while the valid change to $u_3 v_n$ and $u_4 v_n$ are releasing. In that case, we perform valid changes to $u_1 v_n$, $u_2 v_n$, $u_3 v_n$, and $u_4 v_n$. As a result, we get $|\sigma(v_n)| = x = 2$, while $|\sigma(u_1)| = 14$, and $|\sigma(u_2)| = |\sigma(u_3)| = |\sigma(u_4)| = x + 4 = 6$.

Thus, in all cases, we can get rid of the conflicts involving v_n without introducing new ones, and end up with an abs-proper 9-labelling of G . \square

We now adapt modulo methods for classes of signed graphs that can have their vertex set partitioned into a few number of independent sets. We consider bipartite signed graphs first, before focusing on 3-chromatic ones.

Theorem 3.2. *Let G be a connected bipartite signed graph with bipartition A, B . If $|A|$ is even, then $\chi_{|\Sigma|}(G) \leq 2$.*

Proof. This follows from the fact that this holds in unsigned graphs. Namely, it is known, see *e.g.* [15], that, omitting the signature, there is a 2-labelling ℓ of G such that all vertices in A have an odd sum, while all vertices in B have an even sum. Clearly, these properties are preserved in the signed G , and when absolute sums are considered. In other words, ℓ also stands as an abs-proper 2-labelling of the signed G . \square

Previous Theorem 3.2 allows us to prove the following:

Theorem 3.3. *If G is a nice connected bipartite signed graph, then $\chi_{|\Sigma|}(G) \leq 3$.*

Proof. Let A, B denote the bipartition of G . We can assume both $|A|$ and $|B|$ are odd, as otherwise the result would follow from Theorem 3.2. Also, since G is nice, we may assume $|B| \geq 3$. We get the result by designing a 3-labelling ℓ of G such that all vertices in A have absolute sum congruent to 0 modulo 3, while all vertices in B have absolute sum congruent to 1 or 2 modulo 3. Such a labelling is clearly abs-proper since G is bipartite.

We obtain ℓ through arguments that are fairly classical in the field (see *e.g.* [22]), which consist, starting from an initial labelling, in repeatedly fixing pairs of vertices that do not have the desired sums, without introducing new defecting vertices, until most vertices fulfil the desired sum properties. One has to be careful, however, as, since we are dealing with absolute sums, the fixing mechanisms are not as easy to deal with in our context.

Start from ℓ assigning label 3 to all edges of G . Then, for now, all vertices of G have absolute sum congruent to 0 modulo 3, and, thus, in A all vertices fulfil the desired sum property, while in B no vertex meets the desired property. We fix vertices of B in pairs, repeating the following fixing procedure. Consider two vertices v_1 and v_p of B with absolute sum congruent to 0 modulo 3 that are at minimum distance in G . Consider $P = v_1 v_2 \dots v_p$, a path of G ; note that every v_i with i odd (including v_1 and v_p) lies in B while every v_i with i even lies in A . By the choice of v_1 and v_p , every $v_i \in \{v_3, \dots, v_{p-2}\}$ with odd i currently has an absolute sum congruent to 1 or 2 modulo 3. Our goal is to modify labels assigned by ℓ along P so that we get $|\sigma(v_1)|$ and $|\sigma(v_p)|$ lying in $\{1, 2\}$ modulo 3, while, for every $v_i \in \{v_2, \dots, v_{p-1}\}$, the absolute sum of v_i is a multiple of 3 if i is even, or is congruent to 1 or 2 modulo 3 otherwise, if i is odd.

Since $|\sigma(v_1)|$ is congruent to 0 modulo 3, by changing the label $\ell(v_1 v_2)$ arbitrarily we get that $|\sigma(v_1)|$ lies in $\{1, 2\}$ modulo 3, as desired. Note that this changes $|\sigma(v_2)|$ modulo 3, which is not congruent to 0 any more. We claim we can modify $\ell(v_2 v_3)$ in turn so that $|\sigma(v_2)|$ gets congruent to 0 modulo 3 again. Indeed, note that $|\sigma(v_2)| \neq 0$. If $\ell(v_2 v_3) = 2$, then note that by either subtracting or adding 1 to $\ell(v_2 v_3)$ it must be that $|\sigma(v_2)|$ gets congruent to 0 modulo 3 for one of the two options. Otherwise, if, say, $\ell(v_2 v_3) = 1$ (the case where $\ell(v_2 v_3) = 3$ can be treated analogously), then by adding 1 or 2 to $\ell(v_2 v_3)$ it must be again that $|\sigma(v_2)|$ gets congruent to 0 modulo 3 at some point. Indeed, when adding 1 to $\ell(v_2 v_3)$ twice, either $|\sigma(v_2)|$ decreases or increases by 1 twice, or we get $|\sigma(v_2)| = 0$ when adding 1 for the first time and $|\sigma(v_2)|$ is a multiple of 3, as desired.

If $p = 3$, then note that, since we necessarily changed $\ell(v_2v_3)$ when dealing with v_2 , and v_2v_3 is the only edge of P incident to v_3 , necessarily the absolute sum of v_3 changed, and it is now congruent to 1 or 2 modulo 3, as desired. Otherwise, we had that $|\sigma(v_3)|$ was not a multiple of 3 prior to altering $\ell(v_2v_3)$, while, now, it may have this property or not. Regardless, we change $\ell(v_3v_4)$ to force $|\sigma(v_4)|$ to change modulo 3, which is an important part of the process to end up correctly. We proceed as follows:

- If $|\sigma(v_3)|$ is a multiple of 3, then by changing $\ell(v_3v_4)$ to any value, we get that $|\sigma(v_3)|$ is congruent to 1 or 2 modulo 3. Also, $|\sigma(v_4)|$ is not a multiple of 3 any more.
- If $|\sigma(v_3)|$ is congruent to 1 modulo 3, then we go as follows. If $\ell(v_3v_4) = 2$, then, since $|\sigma(v_3)| \geq 1$, through either subtracting or adding 1 to $\ell(v_3v_4)$ we can get $|\sigma(v_3)|$ to be congruent to 2 modulo 3. Now, if $\ell(v_3v_4) = 1$ (the case where $\ell(v_3v_4) = 3$ can be treated similarly), then by adding 1 or 2 to $\ell(v_3v_4)$, either we do not get $|\sigma(v_3)| = 0$ at any point and thus $|\sigma(v_3)|$ gets congruent to 2 modulo 3 at some point, or we get $|\sigma(v_3)| = 0$ when adding 1 to $\ell(v_3v_4)$, in which case when adding 1 to $\ell(v_3v_4)$ again $|\sigma(v_3)|$ gets back congruent to 1 modulo 3, but the situation is different here since $|\sigma(v_4)|$ changed and is no more a multiple of 3 (since $\sigma(v_4)$ was altered by 1 or 2). Thus, in all cases, we get that both $|\sigma(v_3)|$ and $|\sigma(v_4)|$ lie in $\{1, 2\}$ modulo 3.
- If $|\sigma(v_3)|$ is congruent to 2 modulo 3, then we can proceed as in the previous case.

These arguments repeat along P ; namely, when modifying some $\ell(v_{i-1}v_i)$, then either

- $v_i \in A$ and we lost the fact that $|\sigma(v_i)|$ is a multiple of 3, but we can fix this through modifying $\ell(v_iv_{i+1})$, or
- $v_i \in B \setminus \{v_p\}$ and, regardless of whether $|\sigma(v_i)|$ is as desired modulo 3, we can modify $\ell(v_iv_{i+1})$ so that $|\sigma(v_i)|$ is as desired and $|\sigma(v_{i+1})|$ is no longer a multiple of 3, or
- $v_i = v_p$ and we lost the fact that $|\sigma(v_i)|$ is a multiple of 3.

In particular, once the whole process ends, we get that all vertices of P satisfy the desired absolute sum properties, as required.

By repeating the fixing process above with pairs of vertices of B , we can get to a situation where, by ℓ , all vertices in A have absolute sum multiple of 3, all vertices in B but some b have absolute sum congruent to 1 or 2 modulo 3, and $|\sigma(b)| \equiv 0 \pmod{3}$. Recall that $|B| \geq 3$; let thus bab' be a path of length 2 of G . Then, $a \in A$ and $b' \in B$, and so $|\sigma(a)|$ is a multiple of 3, while $|\sigma(b')|$ is not. By similar arguments as earlier, note that it is possible to change $\ell(b'a)$ so that $|\sigma(b')|$ remains congruent to 1 or 2 modulo 3, while $|\sigma(a)|$ loses it is a multiple of 3. Then, by changing $\ell(ab)$, by similar arguments again we can make sure $|\sigma(a)|$ gets a multiple of 3 again, which changes $|\sigma(b)|$ and guarantees it is not a multiple of 3 any more. Then, by the resulting ℓ , all vertices of A and B satisfy the desired properties, and ℓ is thus abs-proper. \square

We now consider 3-chromatic signed graphs. The next proof stands as a good illustration of why the modulo method cannot be used in a straight way when considering absolute sums. The price here for this approach to work, is the introduction of two additional labels.

Theorem 3.4. *If G is a connected 3-chromatic signed graph, then $\chi_{|\Sigma|}(G) \leq 5$.*

Proof. Let A, B, C denote the tripartition of G . We design a 5-labelling ℓ of G where all vertices in A have absolute sum congruent to 0 modulo 5, all vertices in B have absolute sum congruent to 2 or 3 modulo 5, while all vertices in C have absolute sum congruent to 1 or 4 modulo 5. Note that, for a vertex v in any labelled signed graph, if $|\sigma(v)|$ modulo 5 lies in any S of the sets $\{0\}$, $\{2, 3\}$, and $\{1, 4\}$, then $\sigma(v)$ also lies in S modulo 5 (to see this is true, just consider any two $x, y \in \{0, 1, 2, 3, 4\}$ and compare $x - y$ and $y - x$). For this reason, for the sake of simplicity we reason in terms of (non-absolute) sums.

We start by assigning label 5 to all edges of G , and denote by ℓ the resulting labelling. As a result, all vertices in A have their sum fulfilling the desired property, while no vertex in $B \cup C$ does. Similarly as in the proof of Theorem 3.3, our goal is to modify ℓ iteratively through several steps of a fixing procedure, so that, after each step, some vertices have been fixed, without introducing new problems.

We first consider vertices of B . Let $b \in B$ be any vertex such that $\sigma(b)$ is not congruent to any value in $\{2, 3\}$ modulo 5. Since G is not bipartite, there must be an odd-length cycle Q . Then, from Q , since G is connected we deduce that there must be an odd-length closed walk $W = v_0 \dots v_k$ starting and ending at b (thus $v_0 = v_k = b$) in G . For instance, W can be obtained by traversing any path P from b to any vertex of Q (note that P may be empty), then going along Q once or twice, and then going back to b via P . Now we modify the labels assigned by ℓ to the edges of W as follows. For every edge $v_i v_{i+1}$ of W ,

- if i is even, then we modify $\ell(v_i v_{i+1})$ so that its contribution to $\sigma(v_i)$ and $\sigma(v_{i+1})$ increases by 1 modulo 5;
- otherwise, if i is odd, then we modify $\ell(v_i v_{i+1})$ so that its contribution to $\sigma(v_i)$ and $\sigma(v_{i+1})$ decreases by 1 modulo 5.

For every vertex v of G not in W , note that $\sigma(v)$ was not altered by these label modifications. For every vertex v_i in W that is not b , note that $\sigma(v_i)$ has the same value modulo 5 as before we performed the modifications, since $\sigma(v_i)$ was increased and decreased by 1 modulo 5 the same number of times. Now, since W has odd length, note that $\sigma(b)$ was increased twice (modulo 5) by the modifications we performed. Thus, since initially $\sigma(b)$ was congruent to 0 modulo 5, now $\sigma(b)$ is congruent to 2 modulo 5, as desired.

By repeating this process for every vertex b of B , we get to the point where, by the resulting ℓ , all vertices in $A \cup C$ have sum congruent to 0 modulo 5, while all vertices in B have sum congruent to 2 or 3 modulo 5. To now fix the vertices of C , we essentially repeat the same process as for the vertices of B : for every vertex $c \in C$ we need to fix, we consider an odd-length closed walk W originating from c , and we again modify the labels assigned to its edges, but so that their contribution to the two incident vertices is increased (for the edges at even distance from c) or decreased (for the others) by 2 modulo 5. As a result, $\sigma(c)$ becomes congruent to 4 modulo 5, while, modulo 5, the sum of every other vertex of G is not altered. By repeating this procedure for every faulty $c \in C$, eventually ℓ has the desired properties, and we are done. \square

Before getting to the proof of Theorem 3.6, we need a last ingredient.

Theorem 3.5. *If G is any signature of K_4 , then $\chi_{|\Sigma|}(G) \leq 3$.*

Proof. It suffices to provide an abs-proper 3-labelling for every signature of K_4 . Actually, note that we only need to consider signatures that are pairwise non-isomorphic. Also, recall that if we have an abs-proper 3-labelling of a signed graph, then by changing the polarity of every edge while keeping the same labelling, we also obtain an abs-proper 3-labelling for

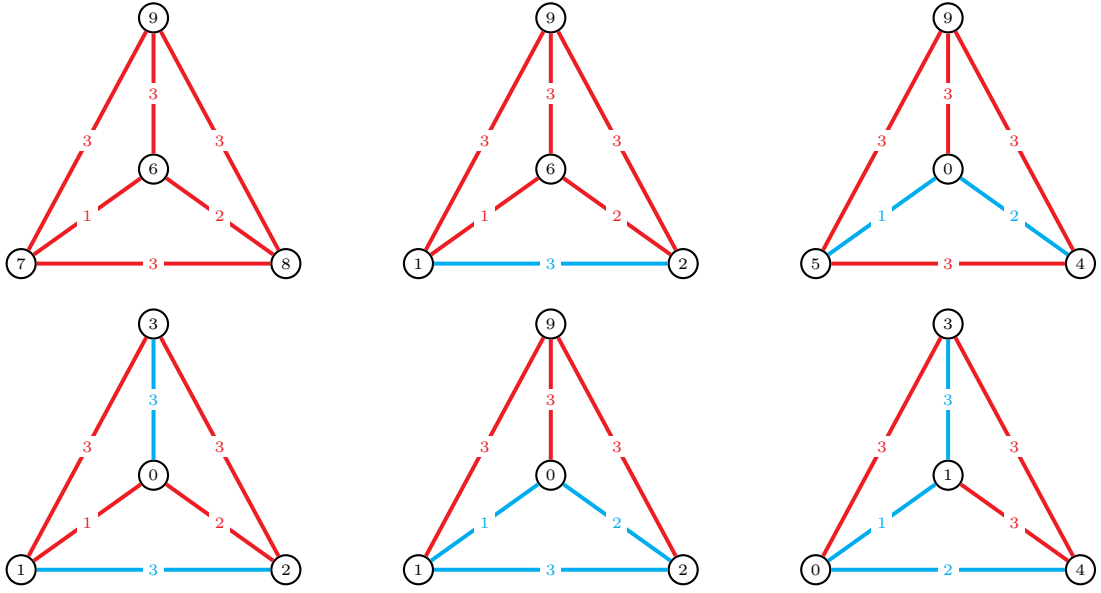


Figure 4: Abs-proper 3-labellings of all canonical signatures of K_4 . In each picture, positive and negative edges are represented as red and blue edges. In each vertex the resulting absolute sum is indicated by the depicted labelling.

this signature. It turns out that, for K_4 , there are only six canonical signatures to consider, namely those in Figure 4. To be convinced of this, note that, by the previous arguments, we may focus on signatures of K_4 having at least three positive edges. Then, when considering signatures of K_4 having $x \in \{3, 4, 5, 6\}$ positive edges, the canonical signatures are those for which the $6 - x$ negative edges induce signed graphs that are pairwise non-isomorphic. For $x = 6$, there is only one possible signature. For $x = 5$, there is only one negative edge, and thus only one canonical signature. For $x = 4$, the two negative edges are either independent or adjacent, and thus there are two canonical signatures to consider. For $x = 3$, the three negative edges form either a triangle or a path of length 3. Thus, as claimed, there are only six canonical signatures of K_4 to consider. For each of these, we have provided, in Figure 4, an abs-proper 3-labelling. \square

We are now ready for our main result in this section.

Theorem 3.6. *If G is a nice connected signed graph, then $\chi_{|\Sigma|}(G) \leq 9$.*

Proof. This follows from the previous results we proved. If $\Delta(G) \geq 4$, then we deduce the result from Theorem 3.1. Otherwise, if G is subcubic, then either G is bipartite and we get the result from Theorem 3.3, or G is either 3-chromatic or a signature of K_4 (by Brooks' Theorem [13]). In the former case, the result follows from Theorem 3.4, while, in the latter one, it follows from Theorem 3.5. \square

We end up this section by illustrating a fact we mentioned earlier, being that, for proper labellings of signed graphs, certain graph labelling tools and techniques, in particular those based on the modulo method, apply right away, without additional care. For instance:

Theorem 3.7. *If G is a connected 3-chromatic signed graph, then $\chi_{|\Sigma|}(G) \leq 3$.*

Proof. The proof is similar to that *e.g.* in [22]. Assume the tripartition of G is A, B, C . To be done, it suffices to design a 3-labelling of G where all vertices in A have sum congruent to 0 modulo 3, all vertices in B have sum congruent to 2 modulo 3, and all vertices in

C have sum congruent to 1 modulo 3. Start from all edges assigned label 3, so that all vertices in A satisfy the desired property while none of those in $B \cup C$ does. Then, similarly as in the proof of Theorem 3.4, we can consider the vertices of B and C one by one, and modify labels along closed walks of odd length to fix their sums modulo 3 without altering other sums modulo 3. Once all vertices of $B \cup C$ are treated, the resulting 3-labelling of G is then proper, and the result thus follows. \square

Likewise, the algorithm from [20] adapts in a straight way to proper labellings of signed graphs, to get a similar result. We voluntarily do not enter too much into the details; indeed, a full, thorough proof would be very close to the lines of [20], and would be very redundant of the proof of Theorem 3.1, which is already an adaptation of the proof of the main result in [20]. Let us mention also that the upcoming result also follows from Zhu's result in [30] that $\text{ch}_\Sigma(G) \leq 5$ holds for every nice graph G , by Observation 2.5. For all these reasons, we restrict to explaining why arguments from [20] adapt to our context.

Theorem 3.8. *If G is a nice signed graph, then $\chi_\Sigma(G) \leq 5$.*

Sketch of proof. The main argument for the algorithm in [20] to work, is that every edge is initially assigned a label, 3, in a set of labels, $S = \{1, 2, 3, 4, 5\}$, that can be decreased by 2 to another label, 1, in S , and similarly be increased by 2 to another label, 5, in S . Also, label 3 can be increased by 1 to another label, 4, in S , which itself can be decreased by 2 to another label, 2, in S ; and label 3 can be decreased by 1 to another label, 2, in S , which itself can be increased by 2 to another label, 4, in S . From this, it is not too complicated to see that the algorithm can be performed from start to end even when edges are polarised, since, essentially, labels in $\{-5, -4, -3, -2, -1\}$ share the same connections as those in S . \square

4. A conjecture and some results

At this point, recall that we have proved that (abs-)proper labellings of signed graphs encapsulate proper labellings of graphs (Observation 2.3), and that, in signed graphs, the parameters $\chi_{|\Sigma|}$ and χ_Σ are bounded above by constants (Theorems 3.6 and 3.8). For these reasons, and because we were not able to construct signed graphs refuting it, we believe it might make sense to raise the following generalisation of the 1-2-3 Conjecture:

Conjecture 4.1. *For every nice signed graph G , we have $\chi_\Sigma(G) \leq \chi_{|\Sigma|}(G) \leq 3$.*

Recall that Conjecture 4.1, if true, would be best possible, both regarding abs-proper labellings and proper labellings (by Observation 2.3 and the fact that graphs G with $\chi_\Sigma(G) = 3$ exist). Also, recall that some of the results from Section 3 already establish that Conjecture 4.1 holds for certain classes of signed graphs. In particular, the conjecture holds for bipartite signed graphs by Theorem 3.3.

In what follows, we provide a bit more support to Conjecture 4.1 by showing it holds for two classes of signed graphs, namely signed graphs with low maximum degree and sparse signed graphs. We start off with the former class.

Theorem 4.2. *If G is a nice signed graph with $\Delta(G) \leq 2$, then $\chi_{|\Sigma|}(G) \leq 3$.*

Proof. We can assume G is connected. Then G is either a signed path or cycle. If G is bipartite, then the result follows from Theorem 3.3. Thus, it remains to consider when G is a signed odd-length cycle. Assume the consecutive vertices of G are v_1, \dots, v_n, v_1 for some $n \geq 3$ odd. If all edges of G have the same polarity, then, by Observation 2.3, we have

$\chi_{|\Sigma|}(G) \leq 3$, since odd-length cycles admit proper 3-labellings (see *e.g.* [15]). Thus, we can assume G has both positive and negative edges. Actually, since $n \geq 3$, we even know that, w.l.o.g., G contains a positive edge adjacent both to a positive edge and to a negative edge. In what follows, we assume v_1v_n and v_nv_{n-1} are positive while $v_{n-1}v_{n-2}$ is negative.

We consider two cases:

- $n \equiv 3 \pmod{4}$.

If $n = 3$, then we know what the actual signature of G is; here, we get an abs-proper 3-labelling when assigning label 3 to v_1v_2 , label 1 to v_2v_3 , and label 2 to v_3v_1 .

When $n \geq 7$, consider the 2-labelling of G obtained by traversing the consecutive edges $v_nv_1, v_1v_2, v_2v_3, \dots, v_{n-1}v_n$ following that order, and, as going along, assigning labels following the pattern $1, 1, 2, 2, 1, 1, 2, 2, \dots$. Then it can be checked that, for every $i \in \{1, \dots, n-2\}$, there is no conflict between v_i and v_{i+1} . This is because every such v_i has absolute sum in $\{1, 3\}$ if i is even, and in $\{0, 2, 4\}$ otherwise. Now, due to the length of G and the polarity of the edges incident to v_n , it can be checked that $v_{n-1}v_n$, which is positive, is assigned label 2, while v_nv_1 , which is positive, is assigned label 1. Then $|\sigma(v_n)| = 3$, while $|\sigma(v_1)| \in \{0, 2\}$ and $|\sigma(v_{n-1})| = 1$ (recall $v_{n-1}v_{n-2}$ is assumed negative). Thus the resulting 2-labelling of G is abs-proper.

- $n \equiv 1 \pmod{4}$.

We proceed similarly as in the previous case, but 2-labelling edges following the ordering $v_{n-1}v_n, v_nv_1, v_1v_2, \dots, v_{n-2}v_{n-1}$. For similar reasons, the resulting 2-labelling of G is abs-proper. In particular, due to the length of G , it can be checked that $v_{n-2}v_{n-1}$ and $v_{n-1}v_n$ are both assigned label 1, and thus $|\sigma(v_{n-1})| = 0$. Meanwhile, $|\sigma(v_n)| = 2$, while $|\sigma(v_{n-2})|$ is odd. Thus, we are done again.

This concludes the proof. □

We can go a bit beyond Theorem 4.2 by noting that there is an interesting way to identify whether $\chi_{|\Sigma|}(G) \leq 2$ holds for a signed cycle G . Recall that deciding whether $\chi_{|\Sigma|}(G) = 1$ holds can be done easily, since it suffices to check whether G is locally abs-irregular. Regarding 2-labellings of G , an interesting property is that, if v_{i-1}, v_i, v_{i+1} , and v_{i+2} are four consecutive vertices of G such that $v_{i-1}v_i$ and $v_{i+1}v_{i+2}$ have distinct polarities, then, regardless of the labels assigned to the three edges, we cannot get a conflict between v_i and v_{i+1} . Meanwhile, if $v_{i-1}v_i$ and $v_{i+1}v_{i+2}$ have the same polarity, then they must be assigned distinct labels. So our 2-labelling problem can be modelled as a proper 2-vertex-colouring problem. Indeed, from G , build the constraint graph H having a vertex v_e for each edge e of G , in which two vertices v_e and v_f are joined by an edge if and only if, in G , edges e and f have the same polarity and are at distance 2. Then, by the arguments above, it is easy to observe that G admits abs-proper 2-labellings if and only if H admits proper 2-vertex-colourings, which can be determined easily since $\Delta(H) \leq 2$.

Recall that the *average degree* of a graph G is the quantity $\frac{2|E(G)|}{|V(G)|}$, while the *maximum average degree* $\text{mad}(G)$ of G is the maximum average degree over all subgraphs of G . We now turn to approaching Conjecture 4.1 further for nice signed graphs that are somewhat sparse, *i.e.*, have maximum average degree less than 3. That is, although we do not prove Conjecture 4.1 in full for these signed graphs, still we improve upon Theorem 3.6 for them. Actually, we note that previous Theorems 3.3 and 3.4 provide a bound that is actually better than the one to be proved, since graphs with maximum average degree less than 3 are 2-degenerate and thus 3-colourable. We believe, however, that the proof arguments we

develop below are of interest, as they rely on other tools and techniques, such as algebraic ones, which could be of interest for further study on the topic. Still, partly for these reasons, we voluntarily move most of the proof in an appendix section.

Theorem 4.3. *If G is a nice signed graph with $\text{mad}(G) < 3$, then $\chi_{|\Sigma|}(G) \leq 7$.*

Proof. We prove the result through the so-called *discharging method*. That is, assuming the claim is wrong, we consider a nice signed graph G with $\text{mad}(G) < 3$ but $\chi_{|\Sigma|}(G) > 7$, that is a minimal (in terms of $|V(G)| + |E(G)|$) counterexample to the claim. Our main goal is to prove that G cannot exist. For that, we proceed in two steps. First, we prove that, G being a minimal counterexample, it cannot contain certain configurations, called *reducible configurations*, being, essentially, subgraphs with vertices of certain small degrees adjacent to each other in some fashion. The fact that G cannot contain such configurations is essentially proved by showing that if G contained any of them, then we could remove some structure from G , end up with a graph H satisfying $\text{mad}(H) < 3$, being smaller than G , and thus admitting an abs-proper 7-labelling, which, due to the structure removed, could be extended to one of G , a contradiction. Second, assuming G cannot contain a certain set of reducible configurations, we then get to contradicting the fact that $\text{mad}(G) < 3$, essentially because, the maximum average degree of G being small, it must contain sparse configurations. This contradiction is obtained through leading a certain *discharging process*, which we will define more thoroughly later on.

Assume thus that G is a minimal counterexample to the claim. In particular, we can assume G is connected. We start off by proving G cannot contain certain reducible configurations. In what follows, for any $k \geq 1$, a k -vertex refers to a vertex with degree k , a k^- -vertex to a vertex with degree at most k , and a k^+ -vertex to a vertex with degree at least k . If a vertex u is adjacent to a k -vertex, k^- -vertex, or k^+ -vertex v , then v is called a k -neighbour, k^- -neighbour, or k^+ -neighbour, respectively, of u . A *pendant triangle* uvw incident to u is a triangle containing two 2-neighbours v and w of u joined by an edge. A 3-vertex is said *weak* if it has a 1-neighbour. In particular, note that if u is a 1-vertex adjacent to a 3-vertex v , then v is weak; we say these two vertices are *associated*.

Claim 4.4. *G cannot contain any of the following configurations:*

- (C1) *For any $k \geq 2$, a k -vertex with $k - 1$ 1-neighbours.*
- (C2) *A 5^- -vertex with a 1-neighbour and a 3^- -neighbour.*
- (C3) *A k -vertex incident to α pendant triangles, and having β weak 3-neighbours and one 2^- -neighbour being not part of a pendant triangle, such that $\alpha + \beta \geq 1$ and $k \leq 2\alpha + \beta + 3$.*
- (C4) *A k -vertex incident to α pendant triangles, and having β weak 3-neighbours and γ 2^- -neighbours being not part of a pendant triangle, such that $\gamma \geq 2$ and $k \leq 2\alpha + \beta + \gamma + 2$.*
- (C5) *A k -vertex incident to α pendant triangles and having β weak 3-neighbours, such that $\alpha + \beta \geq 1$ and $k \leq 2\alpha + \beta + 2$.*
- (C6) *A 2-vertex with a 2-neighbour and a 3^- -neighbour.*
- (C7) *A 2-vertex with two 3-neighbours.*

Proving Claim 4.4 is a bit tedious and demanding, as it requires to introduce several intermediate results and tools, for the sake of keeping the current proof legible, we refer the reader to the appendix for a thorough proof.

Given that G cannot contain any of Configurations (C1) to (C7), we now prove it cannot exist. This is done through a certain discharging process (see below for a thorough explanation), making use of the following result:

Theorem 4.5 (see e.g. [11]). *Let G be a graph, m be some value, and V_1, V_2 be any partition of $V(G)$. Let also ω be a charge function where $\omega(v) = d(v) - m$ for every $v \in V(G)$. If there is a discharging process resulting in a charge function ω^* where*

- $\omega^*(v) \geq 0$ for every $v \in V_1$, and
- $\omega^*(v) \geq \omega(v) + d_{V_1}(v)$ for every $v \in V_2$,

then $\text{mad}(G) \geq m$.

For every vertex v of G , we set $\omega(v) = d(v) - 3$. So that Theorem 4.5 can be used, we also define V_1 as the set of all 2^+ -vertices of G , and V_2 as the set of all 1 -vertices. Our goal, now, is to define discharging rules, *i.e.*, a way to move charges from vertices to vertices, such that eventually every vertex v has charge $\omega^*(v)$, so that:

- if $v \in V_1$, then $\omega^*(v) \geq 0$; and
- if $v \in V_2$ (thus $d(v) = 1$), then $\omega^*(v) \geq -1$.

In the second item above, recall that if $v \in V_2$, then, since G is nice, its unique neighbour cannot be a 1 -vertex, so we have $d(v) = 1 = d_{V_1}(v)$, and, thus, $\omega(v) + d_{V_1}(v) = -1$. Thus, if ω^* satisfies all conditions above, then Theorem 4.5 applies, and we get a contradiction to the fact that G has maximum average degree less than 3.

The discharging process we run consists in applying the following two rules:

(R1) Every 4^+ -vertex sends 1 to every of its 2^- -neighbours.

(R2) Every 4^+ -vertex sends 1 to every 1 -vertex associated with one of its weak 3 -neighbours. ■

After applying the two rules above from the initial charge function ω , every vertex v gets a resulting charge $\omega'(v)$. We analyse the eventual charge function ω' , our goal being to prove that $\omega'(v) \geq \omega^*(v)$ holds for every vertex v of G . In what follows, we consider any vertex v , and apply arguments based on its degree $d(v)$.

- $d(v) = 1$.

Recall that $\omega(v) = -2$ and $\omega^*(v) = -1$. Note that v did not send any charge through Rules (R1) and (R2). Now, since G is nice and G does not contain Configuration (C1), the unique neighbour u of v is a 3^+ -vertex.

- If u is a 4^+ -vertex, then u sent 1 to v by Rule (R1). Thus, in that case, we have $\omega'(v) = \omega(v) + 1 = -2 + 1 = -1 = \omega^*(v)$.
- Otherwise, if u is a 3 -vertex, then u is weak. Since G does not contain Configuration (C2), it must be that u neighbours a 4^+ -vertex w . Then w sent 1 to v by Rule (R2), and we have $\omega'(v) \geq \omega(v) + 1 = -2 + 1 = -1 = \omega^*(v)$.

- $d(v) = 2$.

Recall that $\omega(v) = -1$ and $\omega^*(v) = 0$, and note, again, that v did not send any charge through Rules (R1) and (R2). Since G does not contain Configuration (C1), v cannot have a 1 -neighbour. Actually, since G does not contain Configurations (C6) and (C7), it must be that one neighbour of v is a 4^+ -vertex, which sent 1 to v by Rule (R1). Thus, we here have $\omega'(v) \geq \omega(v) + 1 = -1 + 1 = 0 = \omega^*(v)$.

- $d(v) = 3$.

Note that v neither sent nor received any charge through Rules (R1) and (R2). Thus, we have $\omega'(v) = \omega(v) = 0 = \omega^*(v)$.

- $d(v) \geq 4$.

Recall that $\omega(v) = d(v) - 3$ and $\omega^*(v) = 0$, and note that v did not receive any charge through Rules (R1) and (R2). Note that if v is adjacent to a weak 3-vertex u , then u has at most one 1-neighbour w , as otherwise G would contain Configuration (C1); by Rule (R2), v sent 1 to w . More generally speaking, v thus sent 1 for each of its weak 3-neighbours. By Rule (R1), v also sent 1 to each of its 2^- -neighbours. However, looking closely at the conditions on α , β , and γ in Configurations (C3) to (C5), since G does not contain any of these configurations, at least three neighbours of v must be neither 2^- -vertices nor weak 3-vertices. In other words, there are at least three neighbours for which v did not send any charge. Thus, we deduce $\omega'(v) \geq \omega(v) - (d(v) - 3) = d(v) - 3 - (d(v) - 3) = 0 = \omega^*(v)$.

Thus, we have $\omega'(v) \geq \omega^*(v)$ for every vertex v of G , and by earlier arguments we deduce that G cannot exist. There is thus no counterexample to the claim, which holds. \square

5. Choosing graph signatures

Another way to progress towards Conjecture 4.1 could be to prove the weaker result that any nice graph G admits a signature H with $\chi_\Sigma(H) \leq \chi_{|\Sigma|}(H) \leq 3$. However, we can already establish that this holds, given Observation 2.3 and the fact that Keusch proposed a proof of the 1-2-3 Conjecture in [23]. So, to sum up, given any nice graph G , by making all edges the same polarity, we get a signature H with $\chi_\Sigma(H) \leq \chi_{|\Sigma|}(H) \leq 3$, as desired.

This being said, this solution is not too satisfying, given that it relies mainly on invoking an existing result as is. Fortunately, there is an interesting way to go over this, by just remarking that the number of canonical signatures of a graph is, in general, large, and thus that it may be plausible that the following refined conjecture holds:

Conjecture 5.1. *Every nice graph G has a signature H with $\chi_\Sigma(H) \leq \chi_{|\Sigma|}(H) \leq 2$.*

Due to Observation 2.3, recall that Conjecture 5.1 holds for all graphs G with $\chi_\Sigma(G) \leq 2$, graphs which cannot be characterised easily (unless $P = NP$, see *e.g.* [18]). By previous remarks and observations in Section 2, be aware also that Conjecture 5.1 can be interpreted from the unsigned point of view (through, notably, the lens of Observation 2.5); we discuss this point further in the concluding section.

In the rest of this section, we give further evidence that Conjecture 5.1 might hold true, by proving it holds for a few classes of graphs. We also prove in later Theorem 5.6 that, unless $P = NP$, there is no good characterisation of graphs G admitting a locally irregular signature H , thus with $\chi_\Sigma(H) = 1$. In other words, it is not easy to say when we can do even better than what is conjectured in Conjecture 5.1. The other way round, there are many graphs showing Conjecture 5.1, if true, would be best possible.

Before starting off, we show that, through playing again with ideas from [20], we can get very close to Conjecture 5.1, without using the straight, unsatisfying arguments resulting from Keusch's solution of the 1-2-3 Conjecture.

Theorem 5.2. *Every nice graph G has a signature H with $\chi_\Sigma(H) \leq 4$.*

Proof. This follows from straight modifications to the proof of the 1-2-3-4-5 result from [22], which relies, recall, on an algorithm which goes along the lines of the derived one we designed in the proof of Theorem 3.1. Recall that k -labelling a signed graph is sort of similar to labelling the underlying graph with two sets of labels, $\{1, \dots, k\}$ (corresponding to labels assigned to positive edges) and $\{-1, \dots, -k\}$ (corresponding to labels assigned to negative edges). Thus, k -labelling a signed graph in which we can alter the signature is sort of equivalent to labelling the underlying graph with $\{-k, \dots, -1, 1, \dots, k\}$. So, to show the statement, we can equivalently show that every nice graph admits a proper labelling assigning labels in a subset of $\{-4, -3, -2, -1, 1, 2, 3, 4\}$.

Looking at the proof from [22], due to the arguments above on how the algorithm works, it is not too hard to check that it also provides a proof that every nice graph admits a proper labelling assigning labels in $S = \{-1, 1, 2, 3, 4\}$. Without entering too much into the details, here is how the algorithm can be modified:

- Initially, all edges are now assigned label 1.
- Labels assigned to forward edges can be modified from 1 to either 2 or 4. Note that modifying a label this way changes the parity of the sum of any incident vertex.
- Labels assigned to backward edges can be decreased or increased by 2, provided we follow the restrictions imposed by valid changes. In particular, note that subtracting or adding 2 to label 1 yields a label in S (-1 or 3). Regarding label 2, adding 2 to it yields a label in S (4), while, regarding label 4, subtracting 2 yields a label in S (2).
- Then, when treating a vertex v_i (with $i < n$), we can perform valid changes backwards to modify $\sigma(v_i)$. Each of these combinations of valid changes does not alter the parity of $\sigma(v_i)$. In case we need to open up more possibilities (so that $\Phi(v_i)$ can be defined properly), we can add either 1 or 3 to the label assigned to the shortest forward edge, modifying the parity of $\sigma(v_i)$. All the options granted by these possibilities are sufficient, w.r.t. the constraints we have due to the backward neighbours.
- The last vertex, v_n , can be treated similarly as in the original algorithm. The important point is indeed that each of its incident edges provides a valid change.

With all these arguments, it can be checked that the algorithm from [22] provides a proper S -labelling of G , which yields a signature H and a proper 4-labelling of H . \square

We now prove Conjecture 5.1 for a few more classes of nice graphs, namely complete graphs, bipartite graphs, and graphs with maximum degree 2.

Theorem 5.3. *Every nice complete graph G has a signature H with $\chi_\Sigma(H) \leq \chi_{|\Sigma|}(H) \leq 2$.*

Proof. Denote by v_1, \dots, v_n the vertices of G . We consider a few cases, in each of which we design both a signature and an abs-proper 2-labelling ℓ of it.

- $n = 4k$ for some $k \geq 1$.

For every $1 \leq i \leq k$, we set $v_{k+i}v_{2k+1}$ positive and set $\ell(v_{k+i}v_{2k+1}) = 2$. Likewise, we set $v_{k+i}v_{k+j}$ positive and $\ell(v_{k+i}v_{k+j}) = 2$ for every $1 \leq i < j \leq k$, and we set $v_i v_{k+1-j}$ positive and $\ell(v_i v_{k+1-j}) = 2$ for every $1 \leq j < i \leq k$. Finally, we set $v_{3k+i}v_{3k+j}$ negative and $\ell(v_{3k+i}v_{3k+j}) = 1$ for every $1 \leq i < j \leq k$, and $v_{2k+i}v_{3k+1-j}$ negative and $\ell(v_{2k+i}v_{3k+1-j}) = 1$ for every $1 \leq j < i \leq k$. All remaining edges are set positive and assigned label 1 by ℓ .

It can be checked that we have the following, for every $i \in \{1, \dots, k\}$:

- v_i is incident to i positive edges assigned label 2; thus, $|\sigma(v_i)| = n - 1 + i$.
- v_{k+i} is incident to $k+i$ positive edges assigned label 2; thus, $|\sigma(v_{k+i})| = n - 1 + i + k$.
- v_{2k+i} is incident to i negative edges assigned label 1 and one positive edge assigned label 2; thus, $|\sigma(v_{2k+i})| = n - 2i$.
- v_{3k+i} is incident to $k - 1 + i$ negative edges assigned label 1; thus, $|\sigma(v_{3k+i})| = n - 2(k - i + 1) - 1$.

Note that for all $i, j \in \{1, \dots, k\}$, $|\sigma(v_{2k+i})|$ and $|\sigma(v_{3k+j})|$ have different parity. Since all these absolute sums are pairwise distinct, the resulting 2-labelling on the resulting signature of G is abs-proper. In particular, the v_i 's with index in $\{1, \dots, 2k\}$ have absolute sum in $\{n, \dots, n + 2k - 1\}$.

- $n = 4k + 1$ for some $k \geq 1$.

It suffices to apply the same process as in the previous case for vertices v_1, \dots, v_{4k} , and then to set all edges incident to v_{4k+1} positive and assign label 1 to them. The resulting 2-labelling is then abs-proper, since we get $|\sigma(v_{4k+1})| = n - 1$, while no other vertex has this absolute sum. Meanwhile, all vertices other than v_{4k+1} got their absolute sums increased by 1, so they cannot be in conflict.

- $n = 4k + 2$ for some $k \geq 1$.

We proceed similarly as in the previous case for vertices v_1, \dots, v_{4k+1} , and, last, set all edges incident to v_{4k+2} positive with label 2. As a result, all other vertices get their absolute sum increased by 2. Meanwhile, we get $|\sigma(v_{4k+2})| = 2n - 2$, which is the absolute sum of no other vertex. Thus, we get an abs-proper 2-labelling.

- $n = 4k + 3$ for some $k \geq 0$.

If $k = 0$, then G is actually a cycle, in which case it is easily observed that the result holds (see *e.g.* the proof of later Theorem 5.5 for formal arguments). When $k \geq 1$, apply the same process as in the previous case, for vertices v_1, \dots, v_{4k+2} . It can be observed that, for every $i \in \{1, \dots, 2k\} \cup \{4k + 1, 4k + 2\}$, we have $|\sigma(v_i)| \geq n$, and $1 < |\sigma(v_i)| < n$ for every $i \in \{2k + 1, \dots, 4k\}$. We here set positive with label 1 every edge incident to v_{4k+3} going to any v_i with $i \in \{1, \dots, 2k\} \cup \{4k + 1\}$, while we set negative with label 1 all other edges incident to v_{4k+3} . As a result $|\sigma(v_{4k+3})| = 0$, while no other vertex has this absolute sum. Meanwhile, we have increased by 1 the absolute sum of every v_i with “large” absolute sum, while we decreased by 1 the other absolute sums. Thus, it can be checked there are no conflicts.

We have covered all possible cases, so the proof is complete. □

Theorem 5.4. *Every nice bipartite graph G has a signature H with $\chi_\Sigma(H) \leq \chi_{|\Sigma|}(H) \leq 2$.*

Proof. We can assume G is connected. If $\chi_\Sigma(G) \leq 2$, then, by Observation 2.3, we get a desired signature of G when making all edges the same polarity. So assume now $\chi_\Sigma(G) = 3$. Then, according to [28], it must be that G is an *odd multi-cactus* (a particular class of generalised polygon trees, studied earlier in this context in [17]), a graph that, in brief, is obtained from an initial cycle by repeatedly attaching new cycles onto edges in a tree-like fashion. For now, let us just insist on the fact that the two parts of the bipartition of G must have odd cardinality (as otherwise Theorem 3.2 would apply anyway), and that G necessarily contains six vertices u, v_1, v_2, v_3, v_4, w , where all v_i 's have degree 2, and such

that uv_1 , v_1v_2 , v_2v_3 , v_3v_4 , and v_4w are edges. Also, $d(u), d(w) \geq 2$, and $G - \{v_2, v_3\}$ is connected. See *e.g.* [7] for more details.

Consider the graph $G' = G - \{v_2, v_3\}$. Note that G' is a (nice) bipartite graph in which both parts of the bipartition, A', B' , have even cardinality. Assume w.l.o.g. that $v_1 \in B'$ and $v_4 \in A'$. By Theorem 3.2, there is a (proper) 2-labelling of G' through which all vertices in A' have odd sum while all vertices in B' have even sum. By Observation 2.3, by setting all edges of G' positive and keeping the same labelling, we obtain a signature of G' and a labelling with the same properties. In particular, by that labelling $|\sigma(v_1)| = 2$ (since v_1 has degree 1 in G' and $v_1 \in B'$; this means uv_1 is assigned label 2) and $|\sigma(v_4)| = 1$ (since v_4 has degree 1 in G' and $v_4 \in A'$; this means wv_4 is assigned label 1), while $|\sigma(u)|$ is odd and $|\sigma(w)|$ is even.

We extend this signature and this labelling to the whole of G , by making v_1v_2 positive, making v_2v_3 and v_3v_4 negative, and assigning label 2 to v_1v_2 and label 1 to both v_2v_3 and v_3v_4 . As a result, we get $|\sigma(v_1)| = 4$, which is still even (and thus there is no conflict between u and v_1). Similarly, we get $|\sigma(v_4)| = 0$ while w is incident only to positive edges, hence $|\sigma(w)| \geq 2$ and v_4 and w cannot be in conflict. Also, $|\sigma(v_2)| = 1$ and $|\sigma(v_3)| = 2$, implying there cannot be any conflict between adjacent v_i 's. Thus, we have reached a signature of G and an abs-proper 2-labelling of it. \square

Theorem 5.5. *Every nice graph G with $\Delta(G) \leq 2$ has a signature H with $\chi_\Sigma(H) \leq \chi_{|\Sigma|}(H) \leq 2$.*

Proof. We can assume G is connected. If G is bipartite, then the result follows from Theorem 5.4. Otherwise, if G is an odd-length cycle, then we can consider any signature of G having a positive edge adjacent to both a negative edge and a positive edge. In any such signature of G we can then apply the arguments in the proof of Theorem 4.2 to get an abs-proper 2-labelling. \square

For the rest of this section, we focus on graphs admitting locally irregular signatures. Our goal is to prove that they cannot be characterised easily, unless $P = NP$.

Theorem 5.6. *Deciding whether a graph admits locally irregular signatures is NP-complete.*

Before getting to the proof of Theorem 5.6, we first need some preparation. Namely, we need to introduce several constructions and gadgets to be used in the proof. These constructions are illustrated in Figures 5 and 6.

Given a graph G and a vertex u of G , by *attaching a triangle uvw* at u we mean adding a 3-cycle uvw at u , where v and w are new vertices. Then:

Observation 5.7. *Let H be the graph obtained from a graph G by attaching a triangle uvw at some vertex u . Then, in every locally irregular signature of H , edges uv and uw must have different polarities. Furthermore, $\{ |d^\pm(v)|, |d^\pm(w)| \} = \{0, 2\}$. Also, omitting conflicts involving vertices of G , there are locally irregular signatures of H .*

Proof. In every locally irregular signature of H , note that if uv and uw have the same polarity, then, regardless of the polarity of vw , necessarily $d^\pm(v) = d^\pm(w)$, a contradiction. Now, assuming, w.l.o.g., that uv is positive while uw is negative, if vw is positive then $d^\pm(v) = 2$ and $d^\pm(w) = 0$, while if vw is negative then $d^\pm(v) = 0$ and $d^\pm(w) = -2$. \square

In what follows, a *chair* will refer to a graph obtained starting from a path uvw of length 2 by attaching a triangle at v . This graph has the following properties:

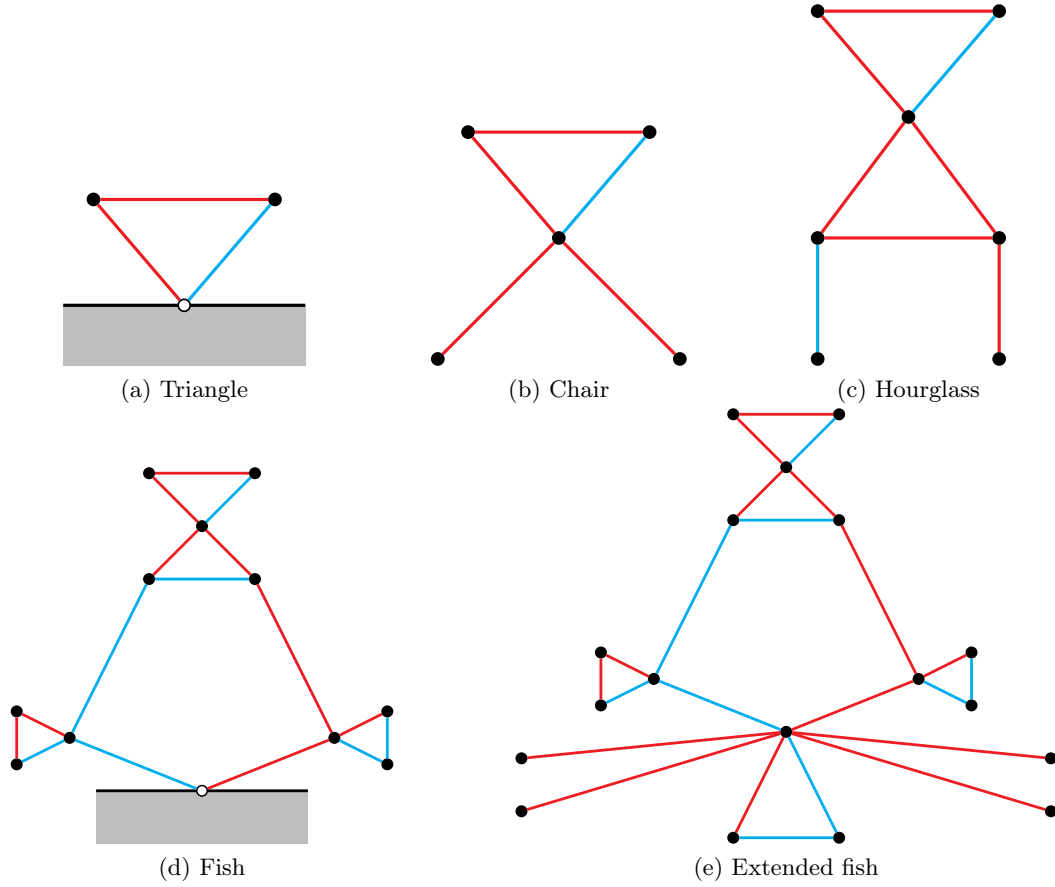


Figure 5: Constructions and gadgets used in the proof of Theorem 5.6, together with (part of) a possible locally irregular signature. In attaching constructions, the structure is attached at the white vertex, with the original graph represented by the gray area. In each picture, positive and negative edges are represented as red and blue edges.

Observation 5.8. *Let G be a chair obtained from a path uvw by attaching a triangle vab at v . Then, in every locally irregular signature of G , edges uv and vw must have the same polarity. Furthermore, $d^\pm(v) \in \{-2, 2\}$. Also, there are locally irregular signatures of G .*

Proof. By Observation 5.7, in every locally irregular signature of G , edges va and vb must have different polarities, and thus their contribution to $d^\pm(v)$ is 0. Also, we have $\{|d^\pm(a)|, |d^\pm(b)|\} = \{0, 2\}$. Thus, so that v is not in conflict with any of a and b , it must be that uv and vw have the same polarity, chosen so that $d^\pm(v)$ does not lie in $\{d^\pm(a), d^\pm(b)\}$. Then $d^\pm(v)$ lies in $\{-2, 2\}$. Last, note that $d^\pm(u), d^\pm(w) \in \{-1, 1\}$. \square

Now, the *hourglass* is the graph obtained from the chair with pendant edges vu and vw by adding the edges uw , ux , and wy , where x and y are two new (degree-1) vertices.

Observation 5.9. *Let G be an hourglass obtained from a chair with pendant edges vu and vw by adding edges uw , ux , and wy . Then, in every locally irregular signature of G , edges ux and wy must have different polarities. Furthermore, $d^\pm(u)$ and $d^\pm(w)$ must both be odd. Also, there are locally irregular signatures of G .*

Proof. In every locally irregular signature of G , by Observation 5.8 we have, w.l.o.g., that vu and vw are positive, and $d^\pm(v) = 2$. Now, if ux and wy had the same polarity, then we would have $d^\pm(u) = d^\pm(w)$, a contradiction. So, say, ux is positive while wy is negative.

By now having uw positive, we get $d^\pm(u) = 3$, $d^\pm(w) = 1$, $d^\pm(x) = 1$, and $d^\pm(y) = -1$, so no two adjacent vertices have the same polarised degree. Regarding $d^\pm(u)$ and $d^\pm(w)$, note that both values must be odd anyway, since, in any signature of a graph, clearly any even-degree vertex must have even polarised degree, and similarly any odd-degree vertex must have odd polarised degree. Thus, the claim holds. \square

Given a graph G and a vertex u of G , by *attaching a fish $uvwxyu$* at u we mean adding the following structure to G . Add first an hourglass to the graph, with pendant edges wv and xy , where we have that wx is an edge. Then add the edges vu and yu , and finally attach both a triangle at v and a triangle at y .

Observation 5.10. *Let H be the graph obtained from a graph G by attaching a fish $uvwxyu$ at some vertex u . Then, in every locally irregular signature of H , edges wv and xy must have different polarities. Furthermore, $\{d^\pm(v), d^\pm(y)\} = \{-2, 2\}$. Also, omitting conflicts involving vertices of G , there are locally irregular signatures of H .*

Proof. Consider a locally irregular signature of H . By Observation 5.9, by how H was obtained from G through adding an hourglass, we must have, say, that wv is positive and xy is negative. Now note that the structure around v forms a chair, and similarly for the structure around y . By Observation 5.8, since wv is positive and xy is negative, it must be that vu is positive while yu is negative. Also, $d^\pm(v) = 2$ while $d^\pm(y) = -2$. The last part of the claim follows from Observations 5.8 and 5.9, which guarantee the chairs and hourglass in H can be signed as desired. In particular, note that vertices of even degree and vertices of odd degree in H cannot have the same polarised degree in a signature. In particular, this implies v and w cannot be in conflict, and similarly for x and y . \square

Now, the *extended fish* is the graph obtained as follows. Start from a vertex u , and attach both a triangle and a fish at u . Then add four edges uv, uw, ux, uy , where v, w, x, y are new (degree-1) vertices.

Observation 5.11. *Let G be an extended fish with pendant edges uv, uw, ux, uy . Then, in every locally irregular signature of G , edges uv, uw, ux, uy must have the same polarity. Furthermore, $d^\pm(u) \in \{-4, 4\}$. Also, there are locally irregular signatures of G .*

Proof. Consider a locally irregular signature of G . By Observation 5.7, in the triangle attached at u the two edges incident to u must have distinct polarities, and u , in that triangle, is adjacent to a vertex with polarised degree 0 (while the other vertex can essentially be of any polarised degree in $\{-2, 2\}$). By Observation 5.10, for the fish attached at u the two edges incident to u must have distinct polarities, and u is adjacent to a vertex with polarised degree -2 and to one with polarised degree 2. At this point, note that the four edges incident to u being part of the attached triangle and fish bring, in total, 0 to $d^\pm(u)$. Since the degree of u in G is even, note that, in the signature, necessarily $d^\pm(u)$ is even. Now, because we cannot have $d^\pm(u) \in \{-2, 0, 2\}$, we get that all of uv, uw, ux, uy must have the same polarity so that $d^\pm(u) \in \{-4, 4\}$. \square

For any $k \geq 1$, the k -*line* is obtained as follows. Start from k copies of the extended fish, where, for each i th copy, the four pendant edges are $u_i v_i, u_i w_i, u_i x_i, u_i y_i$. Now, for every $i \in \{1, \dots, k-1\}$, identify vertices y_i and v_{i+1} to a single vertex $z_{i,i+1}$, and attach a triangle at $z_{i,i+1}$. We prove the following:

Observation 5.12. *Let G be a k -line for some $k \geq 1$, with main path $u_1 z_{1,2} \dots z_{k-1,k} u_k$ such that u_i is adjacent to two pendant vertices w_i and x_i for every $i \in \{1, \dots, k\}$, vertex*

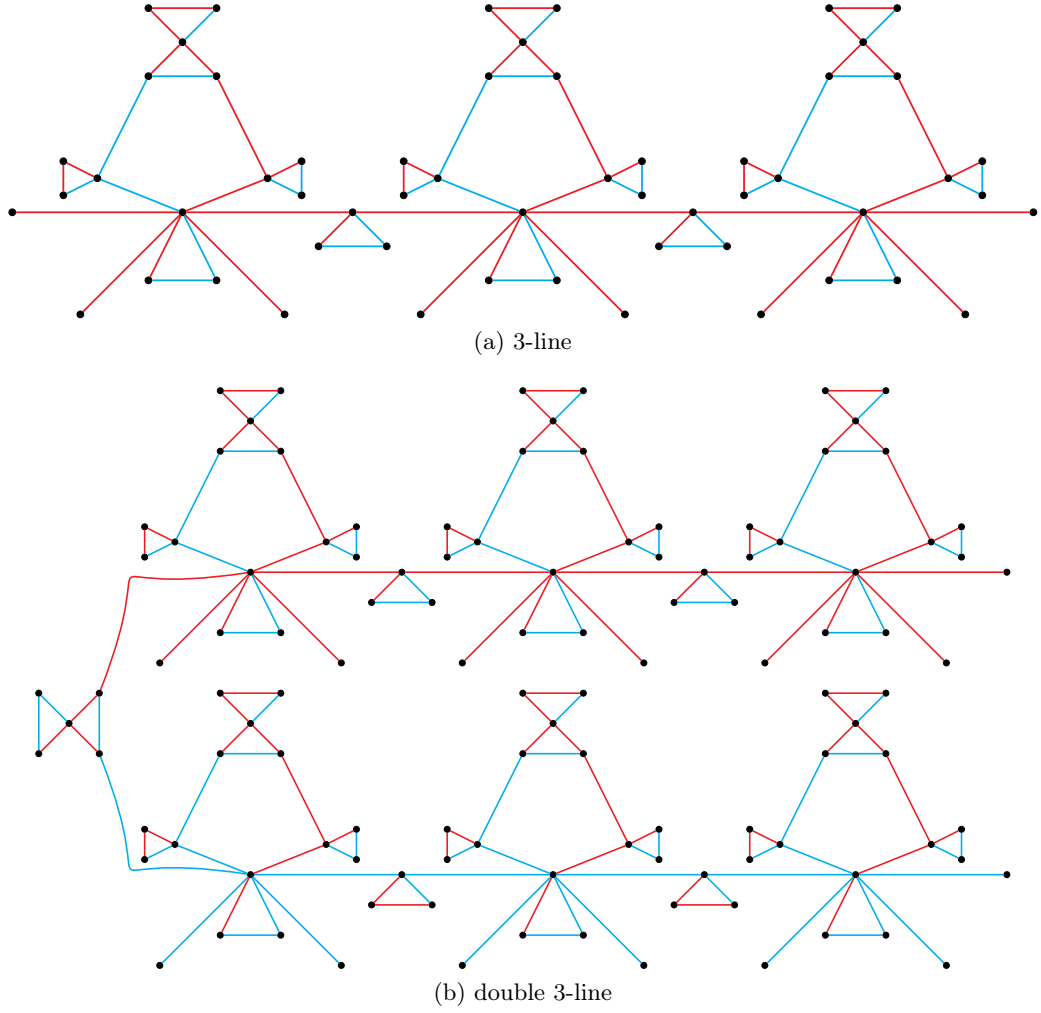


Figure 6: Gadgets used in the proof of Theorem 5.6, together with a locally irregular signature. In each picture, positive and negative edges are represented as red and blue edges.

u_1 is adjacent to another pendant vertex v_1 , and vertex u_k is adjacent to another pendant vertex y_k . Then, in every locally irregular signature of G , all $u_i v_i$'s, $u_i w_i$'s, $u_i x_i$'s, $u_i y_i$'s, $u_i z_{i,i+1}$'s, and $z_{i,i+1} u_{i+1}$'s have the same polarity. Furthermore, for every $i \in \{1, \dots, k\}$, we have $d^\pm(u_i) \in \{-4, 4\}$. Also, there are locally irregular signatures of G .

Proof. Consider a locally irregular signature of G . Recall that, by construction, for every $i \in \{1, \dots, k\}$, vertex u_i belongs to an extended fish. From Observation 5.11 we get that all existing edges in $\{u_i v_i, u_i w_i, u_i x_i, u_i y_i, u_i z_{i,i+1}, z_{i-1,i} u_i\}$ have the same polarity, and $d^\pm(u_i) \in \{-4, 4\}$. Furthermore, by Observation 5.7, for every $z_{i,i+1}$ the two incident edges from the attached triangle bring 0 to $d^\pm(z_{i,i+1})$, while, in that triangle, $z_{i,i+1}$ is adjacent to a vertex with polarised degree 0. Now observe that if, for some i , we have, say, that $u_i z_{i,i+1}$ is positive while $z_{i,i+1} u_{i+1}$ is negative, then $d^\pm(z_{i,i+1}) = 0$, and we get a conflict (recall Observation 5.8, since the structure around $z_{i,i+1}$ forms a chair). Meanwhile, if both edges have the same polarity, then $d^\pm(z_{i,i+1}) \in \{-2, 2\}$, implying $z_{i,i+1}$ is in conflict with neither u_i nor u_{i+1} . Also, the unique edge of the triangle attached at $z_{i,i+1}$ not incident to $z_{i,i+1}$ can be polarised so that there is no inner conflict. Then the claim follows. \square

Last, for any $k \geq 1$, the *double k -line* is obtained as follows. Start from an hourglass with pendant edges ux and wy (where uw is an edge, and x and y are degree-1 vertices).

Now add a k -line to the graph, and, assuming one of its pendant edges is ab (where a has degree 1 and b is incident to three pendant edges), identify the edges ux and ab , that is, identify x and b and identify u and a . Similarly, add another k -line to the graph, and identify one of its similar pendant edges and wy .

Observation 5.13. *Let G be a double k -line for some $k \geq 1$, obtained as above from an hourglass with pendant edges ux and wy , by merging ux and a pendant edge of a k -line L_1 , and by merging wy and a pendant edge of a k -line L_2 . Then, in every locally irregular signature of G , all pendant edges of L_1 have the same polarity, all pendant edges of L_2 have the same polarity, and these two polarities are different. Furthermore, for every pendant edge zz' where z' is the degree-1 vertex, we have $d^\pm(z) \in \{-4, 4\}$. Also, there are locally irregular signatures of G .*

Proof. Consider a locally irregular signature of G . By Observation 5.9, since ux and wy are part of an hourglass, it must be, say, that ux is positive while wy is negative. Now, since ux is part of a k -line and ux is positive, we get, by Observation 5.12, that all pendant edges of L_1 must be positive. Furthermore, we have $d^\pm(x) = 4$, and so we have no conflict between u and x . Likewise, for every pendant edge zz' of L_1 where z' has degree 1, we have $d^\pm(z) = 4$. We derive the same conclusions regarding L_2 . In particular, since wy is part of a k -line and wy is negative, all pendant edges of L_2 must be negative, we have $d^\pm(y) = -4$, and thus there is no conflict between w and y . Also, for every pendant edge zz' of L_2 where z' has degree 1, we have $d^\pm(z) = -4$. Thus the claim holds. In particular, the last, existential part follows from previous observations. Let us remind also that vertices of even degree and vertices of odd degree in G cannot have the same polarised degree in any signature. This guarantees, in particular, that u and w (which have degree 3 in G) cannot be in conflict with their neighbours not in the hourglass (since they have degree 8). \square

We are finally ready for the proof of Theorem 5.6.

Proof of Theorem 5.6. Since the problem is clearly in NP, let us focus on proving its NP-hardness. We do it by presenting a reduction from the CUBIC MONOTONE 1-IN-3 SAT problem, which was indeed proved to be NP-hard [26]. An instance of this problem is a 3CNF formula F over clauses C_1, \dots, C_m and variables x_1, \dots, x_n , where every clause contains exactly three distinct (positive) variables and every variable appears in exactly three (distinct) clauses. The question is whether F can be satisfied in a 1-in-3 way, that is, whether there is a truth assignment to the variables such that every clause contains exactly one true variable.

From an instance F of CUBIC MONOTONE 1-IN-3 SAT, we construct, in polynomial time, a graph G such that F is 1-in-3 satisfiable if and only if G admits a locally irregular signature. The construction of G goes as follows:

- We start from the cubic bipartite graph modelling the structure of F . That is, we add a *clause vertex* c_i for every clause C_i in F , a *variable vertex* x_i for every variable x_i in F , and a *formula edge* $c_j x_i$ whenever, in F , variable x_i appears in clause C_j .
- Then, we add to the graph a double $2m$ -line L . We denote by L^- and L^+ the two k -lines that were used in the construction of L .
- To every clause vertex c_i , we first attach a fish $F(c_i)$. We also pick two pendant edges $a(c_i)a'(c_i)$ and $b(c_i)b'(c_i)$ of L^+ , where $a(c_i)$ and $b(c_i)$ have degree 1 and $a'(c_i) \neq b'(c_i)$, and one pendant edge $c(c_i)c'(c_i)$ of L^- , where $c(c_i)$ has degree 1, and identify c_i and each of $a(c_i)$, $b(c_i)$, and $c(c_i)$. Since the pendant edges of L^- and L^+ are attached at $2m$ distinct vertices, this whole procedure can be performed properly.

- Last, to every variable vertex x_i , we attach a pendant edge $x_i x'_i$ (where x'_i is a new vertex), a triangle, and a fish. Note that, together with the three formula edges incident to x_i , these structures actually form an extended fish.

Clearly, the construction of G is achieved in polynomial time.

We claim that we have the desired equivalence between F and G . To confirm this, let us analyse how a locally irregular signature of G should behave.

- By Observation 5.13, in L , all pendant edges in L^- , including those attached to clause vertices, must have the same polarity, say they are negative. Also, if zz' is such an edge where z' has/had degree 1, then $d^\pm(z) = -4$. Likewise, all pendant edges in L^+ must be positive, and if zz' is such an edge where z' has/had degree 1 then $d^\pm(z) = 4$.
- By Observation 5.10, the fish $F(c_i)$ attached to any clause vertex c_i brings 0 to $d^\pm(c_i)$, and, in that fish, c_i is adjacent both to a vertex with polarised degree -2 and to a vertex with polarised degree 2 . Furthermore, due to previous arguments, two edges incident to c_i going to L are positive, while one edge going to L is negative.

As a result, if we denote by $\alpha \in \{0, 1, 2, 3\}$ the number of positive formula edges incident to c_i and by $\beta \in \{0, 1, 2, 3\}$ the number of negative ones ($\alpha + \beta = 3$), then it cannot be that:

- $\alpha = 3$ and $\beta = 0$, as we would get $d^\pm(c_i) = 4$;
- $\alpha = 2$ and $\beta = 1$, as we would get $d^\pm(c_i) = 2$; or
- $\alpha = 0$ and $\beta = 3$, as we would get $d^\pm(c_i) = -2$.

Thus, the only possible option is to have $\alpha = 1$ and $\beta = 2$, which yields $d^\pm(c_i) = 0$.

- As mentioned earlier, around every variable vertex x_i there is actually an extended fish. By Observation 5.11, the three formula edges and the pendant edge $x_i x'_i$ incident to x_i must thus have the same polarity. Furthermore, $d^\pm(x_i) \in \{-4, 4\}$.
- Note, then, that a clause vertex and an adjacent variable vertex cannot be in conflict.

To see now that the equivalence holds, consider, by a locally irregular signature of G , assuming the pendant edges of L^+ are positive (while those of L^- are negative), that having a formula edge $c_j x_i$ positive models, by a truth assignment to the variables of F , that variable x_i brings truth value *true* to C_j , while it brings truth value *false* otherwise. Then, as noticed above, it must be that every clause vertex is incident to exactly one positive formula edge, which thus models that a clause is considered satisfied if and only if it contains only one true variable. Meanwhile, all three formula edges incident to a variable vertex must have the same polarity, which models that a variable brings the same truth value to every clause that contains it.

From these arguments, a 1-in-3 truth assignment to the variables of F can be deduced from a locally irregular signature of G , and *vice versa*. This concludes the proof. \square

Through the exact same reduction scheme as in the proof of Theorem 5.6, but with slightly different gadgets and constructions, it is worth mentioning we could also prove that it is NP-complete to determine whether a given graph admits a locally abs-irregular signature. As this would be a bit overwhelming to the reader, we omit a proof.

6. Conclusion

In this work, we have introduced two generalisations of the 1-2-3 Conjecture to signed graphs. As mentioned in Section 2, these two variants indeed encapsulate the original one, which we believe is of interest now that Keusch came up with a full proof of the 1-2-3 Conjecture [23]. We thus raised Conjecture 4.1 as a new main conjecture, supported by the facts that we have not been able to come up with signed graphs G satisfying $\chi_{|\Sigma|}(G) > 3$, and that the parameter $\chi_{|\Sigma|}$ is bounded above by a constant (recall Theorem 3.6).

Following our investigations, we believe many aspects of interest could be worth considering further; for instance:

- Our main bound in Theorem 3.6 was obtained by adapting the proof of the 1-2-3-4-5 result from [20], which is a bit outdated now that a 1-2-3 result was provided in [23]. It could be interesting to investigate whether the arguments and tools from that latter work could be adapted to our context. More generally speaking, it would be interesting to understand further to what extent other tools and techniques for proper labellings adapt to our context. Recall, for instance, that it is far from clear regarding modulo arguments and abs-proper labellings (refer *e.g.* to Section 3). Likewise, it seems to us that our proof of Theorem 3.6 cannot be improved upon in general, so it might be necessary to introduce new ideas to go further with the ideas from [20].
- Towards Conjecture 4.1, it could be interesting to consider other classes of signed graphs. In particular, Theorem 4.3 could be improved through either decreasing the bound or considering signed graphs with larger maximum average degree. One could also wonder about signatures of other classical families of graphs, such as complete graphs, or graphs with bounded maximum degree. Such concerns could also be considered regarding Conjecture 5.1, which is another interesting problem.
- It could be interesting to investigate further the discrepancies between abs-proper and proper labellings of signed graphs. Through Observation 2.8 and Theorem 2.9, recall that we proved that the two associated parameters can differ by 1. Due to Conjecture 4.1, we would be interested in knowing, for instance, whether there are signed graphs G with $\chi_{\Sigma}(G) = 1 < 3 = \chi_{|\Sigma|}(G)$, or, in other words, whether the difference between the two parameters can be 2.
- Recall that we have exhibited, in Theorem 2.10, examples of signed trees requiring all of labels 1, 2, 3 in their abs-proper labellings, thereby showing that signing a nice tree might make the number of needed labels by a proper labelling rise. As far as we can tell, the family of signed trees with maximum degree 3 we have provided (which is better visualised through Figure 3) might be the only one. We would be interested in knowing if there are other such signed trees.
- Recall that, similarly as in the unsigned context, bipartite signed graphs can be 3-labelled properly (by Theorem 3.3), and that this is best possible (due to Observation 2.3). By a result from [28], there is a good characterisation of bipartite graphs G with $\chi_{\Sigma}(G) = 3$. We wonder whether it can be adapted to our context. In particular, we were not able to design an NP-completeness proof in order to show that these signed graphs cannot be characterised easily.

Another appealing point, is the interpretation of Conjecture 5.1 from the unsigned point of view. Indeed, due to remarks we raised in Section 2 and to Observation 2.5, note that we could equivalently wonder about the following:

Conjecture 6.1. *Every nice graph admits a proper $\{-2, -1, 1, 2\}$ -labelling.*

Conjecture 6.1 relates to several investigations on variants of the 1-2-3 Conjecture, where, in the line of its list variant, one considers particular restrictions over the label set. Such restrictions are of particular interest, as they allow to go beyond the scope of the 1-2-3 Conjecture, by wondering about *e.g.* the importance of assigning consecutive labels $1, \dots, k$, of assigning strictly positive labels only, of assigning real labels, etc. The list version of the 1-2-3 Conjecture apart, such questions have been considered in a few works of the literature, such as in [17, 24] regarding the assignment of real labels.

Note too that the case of Conjecture 5.1 for absolute values could also be investigated under the unsigned formalism, leading to a dedicated variant of Conjecture 6.1. Namely, such a conjecture, which would perhaps be a bit less natural, would ask whether all nice graphs admit $\{-2, -1, 1, 2\}$ -labellings that are proper, when considering the absolute values of the resulting sums.

Many other directions for further work on the topic could also be worth investigating. For instance, there are many variants of the 1-2-3 Conjecture, such as its list and total variants, and one could investigate how these generalise to signed graphs. Also, as described in the introductory section, there are actually plenty of ways to generalise proper labellings to signed graphs. Thus, one could, as well, investigate other variations, to see whether interesting problems and notions result.

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Appendix: Proof of Claim 4.4

We here provide a thorough proof of Claim 4.4. We prove that G cannot contain any of Configurations (C1) to (C7) following that order. Thus, whenever proving that G cannot contain some Configuration (Ci), it is assumed that G cannot contain the previous ones (C1), ..., (Ci-1). We start by proving an easy lemma, and raising an obvious consequence.

Lemma 6.2. *For any $a, b \in \mathbb{Z}$ with $a \neq b$, there is at most one $c \in \mathbb{Z}$ such that $|a + c| = |b + c|$.*

Proof. If $a + c$ and $b + c$ have the same sign, then $|a + c| - |b + c| = a - b$. Thus, $|a + c| = |b + c|$ if and only if $a = b$, which violates one of our hypotheses on a and b . Now, if $a + c$ and $b + c$ are of different signs, then $|a + c| - |b + c| = |a + b + 2c|$. Thus, $|a + c| = |b + c|$ if and only if $c = \frac{-(a+b)}{2}$ and we have $\frac{-(a+b)}{2} \in \mathbb{Z}$. \square

Corollary 6.3. *Let G be a signed graph, uv be an edge of G , and consider $H = G - \{uv\}$ along with a labelling ℓ of H . If $\sigma(u) \neq \sigma(v)$ (by ℓ in H), then, when extending ℓ to G (thus to uv), there is at most one label yielding $|\sigma(u)| = |\sigma(v)|$ for the resulting labelling of G . Furthermore, if by an extension of ℓ to G we do get $|\sigma(u)| = |\sigma(v)|$, then, in H , we must have $\sigma(u)$ and $\sigma(v)$ being of the same parity by ℓ .*

In the proof of Claim 4.4, we often use the fact that if u and v are two adjacent vertices in a signed graph, and we have a partial labelling such that all edges incident to v are labelled and we have to extend the labelling to edges incident to u , then, to guarantee $|\sigma(u)| \neq |\sigma(v)|$, it suffices to assign labels so that $\sigma(u) \notin \{-\sigma(v), \sigma(v)\}$.

We start by proving that G cannot contain Configurations (C1) and (C2).

Proof of Configuration (C1). Set $k \geq 2$. Assume G contains a k -vertex v adjacent to $k - 1$ 1-vertices u_1, \dots, u_{k-1} , and denote by w the last neighbour of v . We may assume that $d(w) \geq 2$, as otherwise G would be a signed star and we would get a contradiction from e.g. Theorem 3.3. In particular, this implies that the graph $H = G - \{u_1, \dots, u_{k-1}\}$ is nice. Since $\text{mad}(H) < 3$ and H is smaller than G , there is an abs-proper 7-labelling ℓ of H , which we wish to extend to G , to get a contradiction. W.l.o.g., we may assume vw is positive.

Assume first $k = 2$. By Corollary 6.3, we get that there is at most one label in $\{1, \dots, 7\}$ which, when assigned to vu_1 , yields $|\sigma(v)| = |\sigma(u_1)|$. Among the at least six other labels, we must assign to vu_1 one that does not yield $|\sigma(v)| = |\sigma(w)|$. Since this is achieved through guaranteeing $\sigma(v) \notin \{-\sigma(w), \sigma(w)\}$, we have enough labels in hand to avoid this. Thus, there is a proper extension of ℓ to G .

Assume second that $k = 3$. Assign any label to vu_1 so that $\sigma(v) \neq 0$. Again, by Corollary 6.3, there is at most one label in $\{1, \dots, 7\}$ which, when assigned to vu_2 , yields $|\sigma(v)| = |\sigma(u_2)|$. Through assigning the other at least six labels to vu_2 , note that we can modify $\sigma(v)$ in six possible ways. In particular, one of these ways must yield the resulting $\sigma(v)$ does not lie in $\{-\sigma(u_1), \sigma(u_1), -\sigma(w), \sigma(w)\}$. This results in an abs-proper 7-labelling of G .

Now assume $k \geq 4$. Since we are assigning labels in $\{1, \dots, 7\}$, then, when extending ℓ to the vu_i 's, the u_i 's will get sums in $\{-7, \dots, -1, 1, \dots, 7\}$. Also, to make sure we do not

get a conflict between v and w , it suffices to guarantee $\sigma(v) \notin \{-\sigma(w), \sigma(w)\}$. Meanwhile, by labelling the vu_i 's we reach $6(k-1) + 1 = 6k - 5$ values as $\sigma(v)$, while the number of forbidden values is at most $14 + 2 = 16$. Since $k \geq 4$, we can thus 7-label the vu_i 's without creating conflicts, and get an abs-proper 7-labelling of G . \square

Proof of Configuration (C2). Assume G contains a 5^- -vertex v adjacent to a 1-vertex u and to a 3^- -vertex w . Set $H = G - \{vu, vw\}$. We claim H must be nice. Indeed, if H contains a signed K_2 containing v , then it must be that, in G , vertex v is a 3-vertex with two 1-neighbours, and thus that G contains Configuration (C1). Otherwise, if H contains a signed K_2 containing w , then, in G , it must be that w is a 2-vertex with a 1-neighbour, and, again, that G contains Configuration (C1). Thus, H can indeed be assumed to be nice, and, by minimality of G , it admits an abs-proper 7-labelling ℓ which we wish to extend to one of G to get a contradiction.

We start by assigning a label to vw without creating any conflict between w and its neighbours different from v . To be safe, note that it suffices to label vw so that, denoting the at most two other neighbours of w by x_1 and x_2 , we eventually get $\sigma(w) \notin \{-\sigma(x_1), \sigma(x_1), -\sigma(x_2), \sigma(x_2)\}$. Since, through assigning a label in $\{1, \dots, 7\}$ to vw , we can alter $\sigma(w)$ in seven possible way, at least three values in $\{1, \dots, 7\}$ can be assigned in vw while avoiding these conflicts. We assign to vw any of these at least three values that also guarantees we get $\sigma(v) \neq 0$; clearly, such exists. Additionally, we can also guarantee $\sigma(v) \notin \{-4, 4\}$, since $\{1, \dots, 7\}$ is a set of cardinality 7 (while $4 - (-4) + 1 = 9$).

It now remains to label uv . When labelling this edge, we must guarantee v does not get in conflict with w , with its at most three neighbours in H , and with u . Regarding the latter conflict, by Corollary 6.3 we know at least six values in $\{1, \dots, 7\}$ can safely be assigned to uv . If at least six of these values result, as $\sigma(v)$, in at least six distinct values with pairwise distinct absolute values (as a particular case, note that this is fulfilled if they all have the same sign), then one of them can be assigned to uv . Indeed, in that case, w and the other (at most three) neighbours of v in H each forbids one value as $\sigma(v)$, and thus they forbid at most four of these at least six values. Likewise, note that if, prior to labelling uv , we have that $\sigma(v)$ is odd, then, actually, by Corollary 6.3 no conflict can arise between u and v when labelling uv . In that case, the previous arguments would also work w.r.t. to v and its other neighbours if we only have a set of five labels we can assign to uv .

We now describe formally how uv can be labelled. If $\sigma(v)$ and uv are both positive (or both negative), then note that assigning any label in $\{1, \dots, 7\}$ to uv results in $\sigma(v)$ being positive, and thus we can reach seven values as $\sigma(v)$ all of which have the same sign, in which case we can extend the labelling as explained above. So now assume, w.l.o.g., that uv is negative and $\sigma(v)$ is strictly positive prior to labelling uv . Recall that we have assumed $\sigma(v) \notin \{-4, 4\}$, which is why we do not have to consider the case where $\sigma(v) = 4$ in what follows.

- If $\sigma(v) \geq 6$, then, when assigning labels in $\{1, \dots, 7\}$ to uv we can alter the absolute sum of v to any of $\sigma(v) - 1, \dots, \sigma(v) - 6$, which are six distinct values all of which have the same sign. Thus, we can extend ℓ as previously.
- If $\sigma(v) = 5$, then we can alter the absolute sum of v to any of $4, 3, 2, 1, 0$, thus to only five distinct values with the same sign, which here is sufficient, by arguments above, since $\sigma(v)$ is odd.
- If $\sigma(v) = 3$, then we can alter the absolute sum of v to any of $0, -1, -2, -3, -4$, and so we can be done as earlier. Again, five values suffice here, since $\sigma(v)$ is odd.

- If $\sigma(v) \in \{1, 2\}$, then we can alter the absolute sum of v to any of $0, -1, -2, -3, -4, -5$, thus to six possible values. Again, this is sufficient.

In all cases, we can thus reach an abs-proper 7-labelling of G , a contradiction. \square

Before proceeding with the rest of the proof, we need to introduce two lemmas first. These deal with the fact that if some certain sparse structures are incident to some vertex, then, when extending a partial labelling, there are several possibilities around that vertex.

Lemma 6.4. *Let G be a signed graph containing a vertex v with two neighbours u and w , where v is a 3-vertex, u is a 1-vertex, and vw is positive. Assume further $H = G - \{vu, vw\}$ admits an abs-proper 7-labelling ℓ . Then, we can extend ℓ to a 7-labelling of G where v is not involved in conflicts, and there are six distinct labels that can be assigned to vw while achieving this.*

Proof. We first extend ℓ to vw so that the resulting $\sigma(v)$ is not 0. Since we are assigning labels in $\{1, \dots, 7\}$, there are at least six values we can assign to vw while achieving these conditions, which is one of the requirements of the claim. Assuming now vw is labelled, we need to prove that we can label vu so that v is not involved in any conflict. When assigning a label in $\{1, \dots, 7\}$ to vu , we need to guarantee that v does not get in conflict with w , with its third neighbour x different from u and w , and with u . To guarantee the former two conflicts are avoided, it is sufficient to guarantee the eventual sum of v does not lie in $\{-\sigma(w), \sigma(w), -\sigma(x), \sigma(x)\}$, and this forbids at most four values in $\{1, \dots, 7\}$ as the label of uv . Likewise, there is at most one value in $\{1, \dots, 7\}$ that can yield the former conflict, by Corollary 6.3. Thus, at least two values in $\{1, \dots, 7\}$ can be assigned to uv , without creating any conflicts between v and its neighbours. The labelling can thus be extended, and the claim holds. \square

Lemma 6.5. *Let G be a signed graph containing a vertex u incident to a pendant triangle $uvwu$. Assume further $H = G - \{v, w\}$ admits an abs-proper 7-labelling ℓ . Then, we can extend ℓ to a 7-labelling of G where v and w are not involved in conflicts, and this can be achieved through altering $\sigma(v)$ in at least eleven ways.*

Proof. Recall that v and w have degree 2, by the definition of a pendant triangle. We extend ℓ to G so that the label assigned to uv is different from that assigned to uw . Each of uv and uw can be assigned any label in $\{1, \dots, 7\}$. If uv and uw have the same polarity, then, through labelling uv and uw with distinct labels, we can alter $\sigma(v)$ by any value in $\{3, \dots, 13\}$, which set has cardinality 11. Now, if uv and uw have distinct polarities, then we can alter $\sigma(v)$ by any value in $\{-6, \dots, -1, 1, \dots, 6\}$, which set has cardinality 12. To be done, it now remains to prove that, whatever of these pairs of labels we assign to uv and uw , we can label vw so that v and w are not involved in conflicts. Assume thus uv and uw have been labelled with distinct labels, and thus that $\sigma(v)$ is fixed. Then, at this point, we have $\sigma(v) \neq \sigma(w)$. By Corollary 6.3, there is thus at most one value in $\{1, \dots, 7\}$ which, when assigned to vw , yields a conflict between v and w . Now, so that we do not get a conflict with u , it suffices to guarantee $\sigma(v), \sigma(w) \notin \{-\sigma(u), \sigma(u)\}$, which forbids at most four other values from $\{1, \dots, 7\}$. Thus, there are at least two values that can be assigned to vw so that there is no conflict between u , and v and w . \square

We now proceed with Configurations (C3), (C4), and (C5). Recall that, in these, it is granted for the central k -vertex that most of its neighbourhood is part of the configuration. Actually, at most two of its neighbours are not part of the configuration.

Proof of Configuration (C3). Assume G contains a k -vertex v being incident to α pendant triangles $vu_1u'_1v, \dots, vu_\alpha u'_\alpha v$, and being adjacent to β weak 3-vertices w_1, \dots, w_β and to a 2^- -vertex x being not part of the pendant triangles; where $\alpha + \beta \geq 1$ and $k \leq 2\alpha + \beta + 3$. Recall that, because G cannot contain Configuration (C1), each w_i is adjacent to exactly one 1-vertex, which we denote by w'_i . Let H be the signed graph obtained from G by removing all u_i 's, all u'_i 's, the edge vx , as well as all uw_i 's and $w_iw'_i$'s. Note that H is nice. Indeed, in H any w_i cannot be part of a signed K_2 : since w_i 's is the sole 1-neighbour of w_i in G , this would mean w_i would have to be adjacent to another w_j or to x , in which case we would deduce G contains Configuration (C2). For the same reasons, x cannot be part of a signed K_2 . Last, v is either isolated in H , or it is adjacent to 3^+ -vertices. Thus we can assume H is nice, and that there is, by minimality of G , an abs-proper 7-labelling ℓ of H we wish to extend to one of G , thereby getting a contradiction.

We start by extending ℓ to the vu_i 's, vu'_i 's, and vw_i 's as follows.

- If $\beta \geq 1$, then, for every $i \in \{1, \dots, \alpha\}$, we assign label 1 to vu_i and label 2 to vu'_i , and, for every $i \in \{1, \dots, \beta - 1\}$, any label to vw_i so that $\sigma(w_i) \neq 0$.
- Otherwise, if $\beta = 0$, then, for every $i \in \{1, \dots, \alpha - 1\}$, we assign label 1 to vu_i and label 2 to vu'_i .

Now:

- If $\beta \geq 1$, then vw_β has not been labelled. Here, we assign a label to vw_β so that $\sigma(v)$ and $\sigma(x)$ have different parities, and so that $\sigma(w_\beta) \neq 0$. This is possible, since the latter constraint forbids at most one value in $\{1, \dots, 7\}$ to be assigned, while the former one forbids at most four (since we assign labels in $\{1, \dots, 7\}$).
- Otherwise, if $\beta = 0$, then vu_α and vu'_α have not been labelled. In this case, we assign label 1 to vu_α , and label either 2 or 3 to vu'_α so that $\sigma(v)$ and $\sigma(x)$ have different parities. Again, this is possible, since, depending on the polarities of vu_α and vu'_α , assigning label 2 to vu'_α , together with label 1 assigned to vu_α , alters $\sigma(v)$ by either $-3, -1, 1$, or 3 , while assigning label 3 to vu'_α alters $\sigma(v)$ by either $-4, -2, 2$, or 4 . Thus, the two options do not affect the parity of $\sigma(v)$ the same way.

At this point, note that vx is the only edge incident to v that remains to be labelled. Since $\alpha + \beta \geq 1$, note that, currently, $\sigma(v)$ and $\sigma(x)$ have different parities. By Corollary 6.3, whatever label we assign to vx we cannot get a conflict between v and x . We assign a label to vx so that x is not involved in any conflict. Since x is a 2^- -vertex in G , there is at most one neighbour x' of x in H . Clearly, if x' exists, then we can assign a label in $\{1, \dots, 7\}$ to vx so that $\sigma(x) \notin \{-\sigma(x'), \sigma(x')\}$. Otherwise, if x' does not exist, then we assign any label to vx . In both cases, x is not involved in any conflict. Thus, w.r.t. x , at least five labels can be assigned to vx . Recall that v also has at most two neighbours in H ; so that we do not get a conflict between v and these, there might be up to four more values that cannot be assigned to vx . Thus, in total, there is at least one value in $\{1, \dots, 7\}$ we can freely assign to vx without raising conflicts involving x or v .

It now remains to label all edges $u_iu'_i$ and $w_iw'_i$. By Lemmas 6.4 and 6.5, this can be done without introducing conflicts. So we get an abs-proper 7-labelling of G . \square

For Configuration (C4), we need an algebraic tool.

Theorem 6.6 (Combinatorial Nullstellensatz [1]). *Let \mathbb{F} be an arbitrary field, and $P = P(X_1, \dots, X_p)$ be a polynomial in $\mathbb{F}[X_1, \dots, X_p]$. Suppose that the coefficient of a monomial*

$X_1^{k_1} \dots X_p^{k_p}$, where every k_i is a non-negative integer, is non-zero in P and the degree of P equals $\sum_{i=1}^p k_i$. If S_1, \dots, S_p are subsets of \mathbb{F} with $|S_i| > k_i$ for every $i \in \{1, \dots, p\}$, then there are $z_1 \in S_1, \dots, z_p \in S_p$ so that $P(z_1, \dots, z_p) \neq 0$.

Proof of Configuration (C4). Assume G contains a k -vertex v being incident to α pendant triangles $vu_1u'_1v, \dots, vu_\alpha u'_\alpha v$, and adjacent to β weak 3-vertices w_1, \dots, w_β and to $\gamma \geq 2$ 2-vertices x_1, \dots, x_γ being not part of pendant triangles. Similarly as in the proof of Configuration (C3), we can assume every w_i is adjacent to a unique 1-vertex w'_i . Also, for every $i \in \{1, \dots, \gamma\}$, we denote by x'_i the neighbour of x_i different from v (if it exists).

Here, we consider H , the signed subgraph obtained from G by removing all u_i 's, u'_i 's, vu_i 's, $w_i w'_i$'s, and vx_i 's. As in the proof of Configuration (C3), H can be assumed nice; by minimality of G , there is thus an abs-proper 7-labelling ℓ of H we aim to extend to G .

We start by extending ℓ by assigning label 1 to all vu_i 's, label 2 to all vu'_i 's, and, for every $i \in \{1, \dots, \beta\}$, any label to vw_i so that $\sigma(w_i) \neq 0$. Our goal, now, is to show we can label the vx_i 's so that, in G , there is no conflict involving v and its at most two neighbours in H and the x_i 's, nor between any x_i and x'_i (if it exists). For that, we make use of the Combinatorial Nullstellensatz; we thus need to model the situation as a polynomial P .

For every $i \in \{1, \dots, \gamma\}$, let X_i be a variable associated with the label to be assigned to vx_i . Also, if x'_i exists, then we set $l_i = \ell(x_i x'_i)$ if $x_i x'_i$ is positive, and we set $l_i = -\ell(x_i x'_i)$ otherwise; while, if x'_i does not exist, then we set $l_i = 0$. Similarly, we set $s_i = |\sigma(x'_i)|$ if x'_i exists, and $s_i = 0$ otherwise. Recall that v also has at most two 3⁺-neighbours v_δ and v_ϵ in H ; we denote by s_δ and s_ϵ their absolute sums by ℓ (where these values are set as 0 in case some neighbours do not exist). Last, denote by d the current value of $\sigma(v)$.

Each of the labelling constraints is to be represented by factors in P . Namely:

- For every $i \in \{1, \dots, \gamma\}$, the constraint between x_i and x'_i is represented by two factors, $((X_i + l_i) - (s_i))$ and $((X_i + l_i) + (s_i))$.
- For every $i \in \{1, \dots, \gamma\}$, the constraint between v and x_i is represented by two factors, $((d + X_1 + \dots + X_\gamma) - (X_i + l_i))$ and $((d + X_1 + \dots + X_\gamma) + (X_i + l_i))$.
- The constraint between v and v_δ is represented by two factors, $((d + X_1 + \dots + X_\gamma) - (s_\delta))$ and $((d + X_1 + \dots + X_\gamma) + (s_\delta))$.
- The constraint between v and v_ϵ is represented by two factors, $((d + X_1 + \dots + X_\gamma) - (s_\epsilon))$ and $((d + X_1 + \dots + X_\gamma) + (s_\epsilon))$.

So, P is the product of all these factors. If we can find z_1, \dots, z_γ such that $|z_1|, \dots, |z_\gamma| \in \{1, \dots, 7\}$, every z_i is positive if and only if vx_i is positive, and $P(z_1, \dots, z_\gamma) \neq 0$, then we deduce that, by assigning label $|z_i|$ to every vx_i we get an extension of the labelling to G where we have $\sigma(x_i) \notin \{-\sigma(x'_i), \sigma(x_i)\}$ for every $i \in \{1, \dots, \gamma\}$, and, for every neighbour y of v for which the constraint is modelled above, we have $\sigma(v) \notin \{-\sigma(y), \sigma(y)\}$. In particular, this means the x_i 's and v cannot be in conflict with its neighbours in G (saved the u_i 's, the u'_i 's, and the w_i 's, which we will deal with eventually).

So that we can apply the Combinatorial Nullstellensatz, we need to consider a well-chosen monomial of maximum degree from the expansion of P . Note that P contains $4\gamma + 4$ factors, and, in these, the variable coefficients are strictly positive, and every variable exponent is 1. From this, we get that the degree of P is $4\gamma + 4$, and that a monomial of maximum degree in the expansion of P is obtained by picking one variable in each factor. In particular, recall that d , the l_i 's, and the s_i 's are constant terms.

So, to label the remaining edges incident to v properly, by the Combinatorial Nullstellensatz it suffices to show there is a monomial $\prod_{i=1}^{\gamma} X_i^{k_i}$ where $\sum_{i=1}^{\gamma} k_i = 4\gamma + 4$ and all k_i 's are at most 6. Since the variable coefficients are positive, it suffices to show that, by picking one variable in each factor of P , we can obtain such a monomial where all k_i 's are at most 6. One way to proceed, for instance, is as follows.

- For every $i \in \{1, \dots, \gamma\}$, pick variable X_i in both $((X_i + l_i) - (s_i))$ and $((X_i + l_i) + (s_i))$.
- For every $i \in \{1, \dots, \gamma\}$, pick variable X_{i+1} in $((d + X_1 + \dots + X_{\gamma}) - (X_i + l_i))$, where X_1 is regarded as $X_{\gamma+1}$, and pick variable X_i in $((d + X_1 + \dots + X_{\gamma}) + (X_i + l_i))$.
- Pick variable X_1 in both $((d + X_1 + \dots + X_{\gamma}) - (s_{\delta}))$ and $((d + X_1 + \dots + X_{\gamma}) + (s_{\delta}))$.
- Pick variable X_2 in both $((d + X_1 + \dots + X_{\gamma}) - (s_{\epsilon}))$ and $((d + X_1 + \dots + X_{\gamma}) + (s_{\epsilon}))$.

This way of picking variables in the factors of P results in monomial $\prod_{i=1}^{\gamma} X_i^{k_i}$, where $k_1 = k_2 = 6$ while $k_3 = \dots = k_{\gamma} = 4$. Besides, this monomial is of maximum degree, $4\gamma + 4$, and, as mentioned above, it is of non-zero coefficient. As explained earlier, by the Combinatorial Nullstellensatz, we can assign values in $\{1, \dots, 7\}$ to variables in $\{z_1, \dots, z_{\gamma}\}$ representing positive edges and values in $\{-7, \dots, -1\}$ to variables in $\{z_1, \dots, z_{\gamma}\}$ representing negative edges, such that $P(z_1, \dots, z_{\gamma}) \neq 0$, and, due to how P models the constraints around v and the x_i 's, we deduce we can assign labels in $\{1, \dots, 7\}$ to the vx_i 's (essentially, we just assign label z_i to each edge vx_i) so that the x_i 's are not in conflict with their neighbours, and, the u_i 's, u_i' 's, and w_i 's apart, v also is not in conflict with its neighbours.

Now, for every $i \in \{1, \dots, \alpha\}$, we can label the edge $u_i u_i'$ so that u_i and u_i' are not involved in conflicts, by Lemma 6.5. Similarly, for every $i \in \{1, \dots, \beta\}$ we can label $w_i w_i'$ so that w_i is not involved in any conflict, by Lemma 6.4. Eventually, this results in an abs-proper 7-labelling of G . \square

Proof of Configuration (C5). The proof goes similarly as the previous ones for Configurations (C3) and (C4). Assume G contains a k -vertex v incident to α pendant triangles $vu_1 u_1' v, \dots, vu_{\alpha} u_{\alpha}' v$, and adjacent to β weak 3-vertices w_1, \dots, w_{β} , where each w_i is adjacent to a unique 1-vertex w_i' (this neighbour is indeed unique, as, again, otherwise G would contain Configuration (C1)). We consider H the signed graph obtained from G by removing the u_i 's and the u_i' 's, as well as all vw_i 's and $w_i w_i'$'s. By similar arguments as earlier, H is nice and thus admits an abs-proper 7-labelling ℓ we wish to extend to G .

Since $\alpha + \beta \geq 1$, we suppose $\beta > 0$, as the case where $\beta = 0$ and $\alpha \geq 1$ can be treated analogously. We start by assigning label 1 to every vu_i , label 2 to every vu_i' , and any labels to $vw_1, \dots, vw_{\beta-1}$ so that $\sigma(w_i) \neq 0$ for every i . By Lemma 6.4, we get that we can label vw_{β} in six different ways while preserving $\sigma(w_i) \neq 0$ for every i . We label vw_{β} so that v is not in conflict with its at most two neighbours, say x and x' , that are neither one of the u_i 's, u_i' 's, and w_i 's. So we want $\sigma(v) \notin \{-\sigma(x), \sigma(x), -\sigma(x'), \sigma(x')\}$, which can be achieved since there are at least six ways to label vw_{β} , thus to alter $\sigma(v)$, while still guaranteeing $\sigma(w_{\beta}) \neq 0$.

So, we get to the point where all edges incident to v are labelled, and, as before, according to Lemmas 6.4 and 6.5 we can label the remaining edges so that no conflicts arise. Eventually we thus get an abs-proper 7-labelling of G . \square

Proof of Configuration (C6). Assume G contains a 2-vertex v adjacent to both a 2-vertex u and a 3-vertex w . Since G cannot contain Configuration (C1), we can assume w actually has degree 2 or 3. In this case, we consider $H = G - v$. Note that H must be nice, as

otherwise either u would have a 1-neighbour in G , or w would be, in G , a 2-vertex with a 1-neighbour, and in both cases we would deduce that G contains Configuration (C1). By minimality of G , there is thus an abs-proper 7-labelling ℓ of H , which, by the arguments below, we extend to G to get a contradiction.

When labelling vw , we must guarantee that w does not get in conflict with its at most two neighbours in H . For that, it suffices to guarantee that we do not get $\sigma(w) \in \{-\sigma(w_1), -\sigma(w_2), \sigma(w_1), \sigma(w_2)\}$, where w_1 and w_2 denote the at most two possible neighbours of w in H . Thus, due to this constraint, at most four values in $\{1, \dots, 7\}$ cannot be assigned to vw . There are thus at least three values we can safely assign to vw , and thus there is at least one value we can assign that also guarantees we do not get $\sigma(u) = \sigma(v)$, for the resulting sums of u and v . We assign one such label to vw .

It now remains to label uv . By Corollary 6.3, since we currently have $\sigma(u) \neq \sigma(v)$, there is at most one value in $\{1, \dots, 7\}$ which, when assigned to uv , yields a conflict between u and v . Also, at most two values in this set yield a conflict between u and its unique neighbour in H , and, similarly, at most two values yield a conflict between v and w . Thus, at most five values in $\{1, \dots, 7\}$ cannot be assigned to uv , and there is thus a safe value we can freely assign. As a result, we end up with an abs-proper 7-labelling of G . In particular, note that these arguments apply even if uw is an edge of G (the main difference being that, when labelling vw , technically we do not need to worry about a possible conflict between u and w at this point, as this conflict is handled later on, when labelling uw). \square

We now deal with the last configuration.

Proof of Configuration (C7). Assume G contains a 2-vertex u with two 3-neighbours v and w . Set $H = G - u$. Since v and w both have degree 2 in H , necessarily H is nice. There is thus an abs-proper 7-labelling ℓ of H , which we extend to G to get a contradiction.

We first deal with the case where uv and uw have the same polarity. Similarly as in the proof of Configuration (C6), there are at least three labels in $\{1, \dots, 7\}$ that can be assigned to uv so that v is not in conflict with any of its two neighbours in H . We assign to uv one of these labels guaranteeing $\sigma(u) \neq \sigma(w)$. Now, since uv and uw have the same polarity, then, whatever label we assign to uw , we will get the same sign as $\sigma(u)$. Thus, when labelling uw , there is only one label which, when assigned, would yield $\sigma(u) = \sigma(v)$. Regarding the two neighbours of w in H , to avoid any conflict the eventual $\sigma(w)$ should avoid at most four values. Last, by Corollary 6.3, since, at this point, we have $\sigma(u) \neq \sigma(w)$, there is at most one label in $\{1, \dots, 7\}$ which, when assigned to uw , would result in $\sigma(u) = \sigma(w)$. Thus, in total, at most six labels in $\{1, \dots, 7\}$, when assigned to uw , would yield a conflict. So there is one label we can safely assign to uw to be done. Similarly as in the proof of Configuration (C6), note that these arguments apply regardless of whether v and w are adjacent or not.

From now on, we can thus assume uv and uw have distinct polarities. We start by additionally assuming that vw is an edge of G , and for this subcase, we will consider $H = G - \{u, vw\}$. W.l.o.g. we can assume vw is positive. In that particular case, note that v and w both have only one neighbour in H , which we denote by v' and w' , respectively. Note that, through labelling vu and vw , we can alter $\sigma(v)$ by any value in $\{2, \dots, 14\}$, which set has cardinality 13. To avoid any conflict between v and v' , at most two of these alterations are forbidden, which leaves eleven alterations. Among these possible alterations, at least one alters $\sigma(v)$ by an even amount, and at least one modifies it by an odd amount. So that we alter $\sigma(v)$ by an even amount, note that the labels assigned to vu and vw must have the same parity, while they must have distinct parity to alter $\sigma(v)$ by an odd amount. This all implies we can label vu and vw so that v is not in conflict with v' , and

Configuration		Assigned labels		Resulting sums		
$ \sigma(w_1) $	$ \sigma(w_2) $	$\ell(uv)$	$\ell(uw)$	$ \sigma(v) $	$ \sigma(u) $	$ \sigma(w) $
2	3	3	3	1	0	1
1	2	3	1	1	2	3
1	3	3	6	1	3	2

Table 1: Ways to extend the abs-proper 7-labelling in the proof of Configuration (C7), when $\{|\sigma(v_1)|, |\sigma(v_2)|\} = \{2, 3\}$, $\sigma(v) = -4$, and $\sigma(w) = 4$. For values of $|\sigma(w_1)|$ and $|\sigma(w_2)|$, are given labels $\ell(uv)$ and $\ell(uw)$ we can assign to uv and uw , as well as the resulting absolute sums of u , v , and w .

the resulting $\sigma(u)$ and $\sigma(w)$ have distinct parities. Then, by Corollary 6.3, whatever label we assign to uw , we cannot get a conflict between u and w . So, when labelling uw , we only need to make sure w does not get in conflict with w' , and v , which now has its sum fixed, does not get in conflict with either u or w . So we need to guarantee that, eventually, $\sigma(w) \notin \{-\sigma(w'), \sigma(w'), -\sigma(v), \sigma(v)\}$ and $\sigma(u) \notin \{-\sigma(v), \sigma(v)\}$. So at most six values in $\{1, \dots, 7\}$ cannot be assigned to uw , and there is thus one value we can safely assign to get an abs-proper 7-labelling of G .

It remains to consider when vw is not an edge of G . Assume w.l.o.g. uv is positive and uw is negative. We denote by v_1 and v_2 the two neighbours of v in H , and by w_1 and w_2 the two neighbours of w . Similarly as earlier, we can assign a label in $\{1, \dots, 7\}$ to uv so that v is in conflict with neither v_1 nor v_2 , and $\sigma(u) \neq \sigma(w)$. Let us now analyse the number of values in $\{1, \dots, 7\}$ we cannot assign to uw . Because of the possible conflict between u and v , up to two values might be forbidden. Likewise, up to four values might be forbidden because of the possible conflicts between w , and w_1 and w_2 . Furthermore, since $\sigma(u) \neq \sigma(w)$, up to one value might be forbidden due to the possible conflict between u and w . Thus, up to seven values in $\{1, \dots, 7\}$ might be forbidden, and it might be that all of them indeed are (but this occurs only when the seven possible conflicts forbid exactly one value each).

Recall, however, that, through labelling uv with a label in $\{1, \dots, 7\}$, we can alter $\sigma(w)$ in seven possible ways (by any value in $\{-7, \dots, -1\}$). We claim that, for w_1 and w_2 to bring exactly four constraints, we must have $|\sigma(w_1)|, |\sigma(w_2)| \in \{1, 2, 3\}$. Indeed, if, say, $|\sigma(w_1)| = k$ for some $k \geq 1$, then, for w_1 to forbid exactly two labels as uw , there must be $i, j \in \{-7, \dots, -1\}$ such that $\sigma(w) + i = k$ and $\sigma(w) + j = -k$. So, $i - j = 2k$, where $|i - j| \leq 6$, implying $k \leq 3$. Furthermore, so that w_1 and w_2 bring exactly four constraints, it must be that $|\sigma(w_1)| \neq |\sigma(w_2)|$. By symmetry, by ℓ we can assume v satisfies the same relationship with its two neighbours v_1 and v_2 .

Let us start from ℓ again, that is, assume uv is not labelled. We claim that, due to all the information we have revealed, we must have $\sigma(v) < 0$ and $\sigma(w) > 0$. Indeed, if $\sigma(v) \geq 0$, then $\sigma(v) + i > 0$ for every $i \geq 1$, and, because uv is positive, necessarily through labelling uv the eventual sum of $\sigma(v)$ remains positive, implying that v_1 and v_2 actually forbid only one value each for labelling uv . Since uw is negative, a similar argument applies regarding w . Thus, we can indeed assume $\sigma(v) < 0$ and $\sigma(w) > 0$.

We now analyse the possible absolute sums that the neighbours of v and w can have, and come up with a way to label uv and uw in every case. Recall that we only need to make sure v does not get in conflict with v_1 and v_2 , vertex w does not get in conflict with w_1 and w_2 , and u gets in conflict with neither v nor w .

- Assume first $\{|\sigma(v_1)|, |\sigma(v_2)|\} = \{2, 3\}$. For v_1 and v_2 to forbid exactly four values

Configuration		Assigned labels		Resulting sums		
$ \sigma(w_1) $	$ \sigma(w_2) $	$\ell(uv)$	$\ell(uw)$	$ \sigma(v) $	$ \sigma(u) $	$ \sigma(w) $
1	3	2	2	2	0	2
1	2	2	1	2	1	3

Table 2: Ways to extend the abs-proper 7-labelling in the proof of Configuration (C7), when $\{|\sigma(v_1)|, |\sigma(v_2)|\} = \{1, 3\}$, $\sigma(v) = -4$, and $\sigma(w) = 4$. For values of $|\sigma(w_1)|$ and $|\sigma(w_2)|$, are given labels $\ell(uv)$ and $\ell(uw)$ we can assign to uv and uw , as well as the resulting absolute sums of u , v , and w .

Configuration	Assigned labels		Resulting sums		
$\sigma(v)$	$\ell(uv)$	$\ell(uw)$	$ \sigma(v) $	$ \sigma(u) $	$ \sigma(w) $
-3	6	6	3	0	3
-4	1	1	3	0	3
-5	1	1	4	0	4

Table 3: Ways to extend the abs-proper 7-labelling in the proof of Configuration (C7), when $|\sigma(v_1)| = |\sigma(w_1)| = 1$ and $|\sigma(v_2)| = |\sigma(w_2)| = 2$. For values of $\sigma(v)$, are given labels $\ell(uv)$ and $\ell(uw)$ we can assign to uv and uw , as well as the resulting absolute sums of u , v , and w .

as the label of uv , by arguments above there must be $i, j \in \{1, \dots, 7\}$ such that $\sigma(v) + i = 3$ and $\sigma(v) + j = -3$. Thus, $i - j = 6$, implying $i = 7$, $j = 1$, and $\sigma(v) = -4$. Then the labels in $\{1, \dots, 7\}$ that can be assigned to uv without raising a conflict between v , and v_1 and v_2 are 3, 4, 5. By assigning label 4 to uv , we thus get $\sigma(v) = 0$. If this leads to $\sigma(u) \neq \sigma(w)$, then there is one label in $\{1, \dots, 7\}$ we can safely assign to uw , since the possible conflicts between w , and w_1 and w_2 forbid up to four values, the possible conflict between u and v forbids at most one, and that between u and w forbids at most one too by Corollary 6.3. If assigning label 4 to uv results in $\sigma(u) = \sigma(w)$, then $\sigma(w) = 4$. It can now be observed that there is always a way to extend the labelling properly. We gather, in Table 1, ways to do so, depending on the faced configurations. In these, we assume $|\sigma(w_1)| < |\sigma(w_2)|$. In each case, it can be checked we indeed get an abs-proper 7-labelling of G .

- Assume now that the previous case does not apply, and that, similarly, we do not have $\{|\sigma(w_1)|, |\sigma(w_2)|\} = \{2, 3\}$.
 - Assume first $\{|\sigma(v_1)|, |\sigma(v_2)|\} = \{1, 3\}$. For the same reasons as earlier, we can suppose $\sigma(u) = -4$. Then the labels in $\{1, \dots, 7\}$ we can freely assign to uv (w.r.t. the possible conflicts between v , and v_1 and v_2), are 2, 4, 6. If $\sigma(w) \neq 4$, then we are done as previously. Otherwise, *i.e.*, $\sigma(w) = 4$, then, again, depending on the situation there is always a way to extend the labelling. We provide such ways in Table 2. Recall we do not have to consider situations where $\{|\sigma(w_1)|, |\sigma(w_2)|\} = \{2, 3\}$.
 - At this point, we can thus assume none of v_1 , v_2 , w_1 , and w_2 has absolute sum 3. W.l.o.g., we may assume $|\sigma(v_1)| = |\sigma(w_1)| = 1$ and $|\sigma(v_2)| = |\sigma(w_2)| = 2$. So that, upon labelling uv with labels from $\{1, \dots, 7\}$, the sum of v can reach both -2 and 2 , there are $i, j \in \{1, \dots, 7\}$ such that $\sigma(v) + i = -2$ and $\sigma(v) + j = 2$. Thus, $j - i = 4$, implying $i \in \{1, 2, 3\}$, and, thus, $\sigma(v) \in \{-3, -4, -5\}$. Now, as previously, for every possible resulting value of $\sigma(v)$, if $|\sigma(v)| \neq |\sigma(w)|$, assigning

label $|\sigma(v)|$ to uv yields $\sigma(v) = 0$ (and thus v is in conflict with neither v_1 nor v_2) and $\sigma(u) \neq \sigma(w)$, and, again, counting arguments show there is a label we can properly assign to uw . Thus, we can last assume $\sigma(w) = -\sigma(v)$. By now analysing all possible cases, we can extend the labelling properly to the whole of G , see Table 3.

This concludes the proof. □